Universal fields of fractions: their orderings and determinants

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Abstract

We are concerned with two problems. Firstly, given a ring R and an epic R-field K, under what conditions can K be fully ordered? Epic R-fields can be constructed in terms of matrices over R; this makes it natural, in describing full orders on K, to consider matrix cones over R rather than ordinary cones of elements of K. Essentially, a matrix cone over R, associated with a given ordering of K consists of all square matrices which either become singular or have positive Dieudonné determinant over K. We give necessary and sufficient conditions in terms of matrix cones for (i) an epic R-field to be orderable, (ii) a full order on R to be extendible to a field of fractions of R and (iii) for such an extension to be unique.

The second problem is finding $K_1(U(R))$, where R is is a Sylvester domain and U(R) denotes its universal field of fractions. Let R be a Sylvester domain and let Σ be the monoid of full matrices over R. We show that $K_1(U(R))$ is naturally isomorphic to $\mathfrak{a}(\Sigma)$, the universal abelian group of Σ. The inclusion $R \subseteq U(R)$ induces a map $K_1(R) \rightarrow K_1(U(R))$; we also prove that if R is a fully atomic semifir (e.g. if R is a fir) then

$$K_1(U(R)) \cong \overline{K_1(R)} \times D(R),$$

where $\overline{K_1(R)}$ denotes the image of $K_1(R)$ in $K_1(U(R))$ and D(R) is the free abelian group on the set of equivalence classes of stably associated matrix atoms over R.

Contents

Introduction

Let R be a ring; by an epic R-field we mean a (not necessarily commutative) field K, together with a homomorphism $\alpha : R \to K$ which is epic in the category of rings. When R is commutative K can be characterized by $\ker\alpha$. For general rings this no longer holds; however P.M. Cohn has shown that K can be characterized and constructed in terms of the set of matrices over R which become non-singular over K. Thus every statement made about an epic R-field can be expressed equivalently by a statement about matrices over R.

In the first two sections of Chapter 1 we recall the necessary background material, with proofs whenever practicable. These sections have been included partly to make the thesis more self-contained and partly because the notation of [4], the standard reference, has somewhat changed. In §1.3 the universal fields of fractions (see §1.2 for the definition) of certain free rings are constructed as a subfield of an ultrapower of a field.

In Chapter 2 we study orderings of epic R-fields. Our most important tools are matrix cones; these are analogues of the positive cone associated with a partial order on a ring. We formulate Szele's theorem, in terms of matrix cones, which states that a field can be fully ordered if and only if it is formally real. We also discuss the problem of extending a full order on a ring R to an epic R-field containing R. Two sections have been included which deal with corresponding problems for monoids and groups.

In Chapter 3 we study the abelianized multiplicative group of the universal field of fractions of a Sylvester domain R (see §1.2 for the definition). It turns out that this abelian group can be obtained as the universal abelian group of the monoid of full matrices over R. For certain subclasses of Sylvester domains more explicit results are proved; these depend on Cohn's unique factorization theorem for full matrices over a fully atomic semifir.

We remark that a more detailed introduction precedes each chapter.

Theorems etc. are credited as they occur, most of the uncredited results are original except those of sections 1.1 and 1.2 which have been collected from [4], [5] and [6].

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Terminology and conventions

Mappings are written on the right with the exception of the determinant map, thus the image of $a \in A$ under the mapping $f : A \rightarrow B$ is written $(a)f$ or a^f. The image of A under f is denoted by A^f.

All rings are associative with a unit element 1. Let R be a ring, we put R^x for the set $R \setminus \{0\}$. R is called an integral domain if R^x is a monoid under multiplication. A non-zero element of R is said to be an atom if it is not a unit and cannot be written as a product of two non-units. An integral domain is called atomic if every non-zero element which is not a unit can be decomposed as a product of atoms. The group of units of R is denoted by G(R). We write ${}^mR^n$ for the set of mxn matrices over R; ${}^1R^n$ is abbreviated to R^n and ${}^mR^1$ to ${}^mR$. We write R_n or $M_n(R)$ for ${}^nR^n$. Thus R_n is the ring of nxn matrices over R; we put $GL_n(R)$ for $G(R_n)$ and $E_n(R)$ for the subgroup of $GL_n(R)$ generated by the elementary matrices. R is said to be weakly finite if for each n $GL_n(R)$ is closed under factorizations in R_n. Thus R is weakly finite if and only if for each n and any $A, B \in R_n$ we have

$$AB = I_n \implies BA = I_n$$

(cf. [4; pp.6-7]). We define M(R) as the direct limit:

$$M(R) = \varinjlim R_n,$$

where R_n is embedded in R_{n+1} by the rule

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly, let $GL(R) = \varinjlim GL_n(R)$ and $E(R) = \varinjlim E_n(R)$. For each n we have

$$R_n \supseteq GL_n(R) \supseteq E_n(R);$$

in consequence:

$$M(R) \supseteq GL(R) \supseteq E(R).$$

Let $A \in R_n$; we shall usually write A for the image of A in M(R) (thus identifying A, $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ etc.) and so if $B \in R_n$ is another square matrix over R, by AB we mean

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \in M(R).$$

Accordingly, M(R) will also denote the set $\bigcup_n R_n$. Similar conventions apply to GL(R) and E(R).

Let G be a group; the derived group G' of G is defined as the (normal) subgroup of G generated by the set $\{ (a,b) \in G \mid a, b \in G \}$, where $(a,b) = a^{-1}b^{-1}ab$. The derived group of G is also called the commutator subgroup of G. The Whitehead lemma states that for any ring R

$$GL(R)' = E(R)$$

(cf. [14; Lemma 3.1]).

Two matrices A and B over R are said to be <u>associated</u> if there exist invertible matrices U and V over R, such that $UAV = B$. If in addition $U, V \in E(R)$, A and B are said to be E-associated. When $\begin{pmatrix} A & 0 \\ 0 & I_r \end{pmatrix}$ and $\begin{pmatrix} B & 0 \\ 0 & I_s \end{pmatrix}$ are associated (or E-associated) for some r and s we say that A and B are stably associated (or stably E-associated). The diagonal sum of A and B is defined as follows:

$$A \dotplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

We write $\overset{n}{\dotplus} A$ for the expression $A \dotplus \ldots \dotplus A$ (n times). Assume now that $A, B \in {}^mR^n$ and further that A and B agree except possibly in one row or column, say the first column. Write $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, A_2, \ldots, A_n)$. We define the determinantal sum of A and B with respect to the first column as

$$A \nabla B = (A_1 + B_1, A_2, \ldots, A_n).$$

The determinantal sum with respect to other columns and rows is defined analogously.

We shall use the term 'field' to mean 'not necessarily commutative division ring' and add the adjective 'commutative' when needed. Let R be an integral domain; R is called a right Ore domain if for every pair of elements $a, b \in R^x$ we have $as = bt$ for some $s, t \in R^x$. Left Ore domains are defined analogously. It is well-known that a left or right Ore domain has a field of quotients which can be obtained by localizing at R^x. Similarly, a monoid M is said to be left Ore if for every pair of elements $m, n \in M$ there exist s and t in M, such that $sm = tn$. Let M be a left Ore cancellation monoid; then M has a group of quotients in which every element is of form $a^{-1}b$, a, $b \in M$. As an example of a right Ore domain we mention the skew polynomial ring $k[x;\alpha]$, where k is a field, $\alpha \in \operatorname{End} k$ and multiplication in $k[x;\alpha]$ is based on the commutation rule

$$ax = xa^{\alpha} \quad \text{for all} \quad a \in k$$

(cf. [4; §0.8 and §8.3]).

A ring R in which every finitely generated left ideal (or equivalently: right ideal) is free, of unique rank is said to be a <u>semifir</u>; if all left and right ideals of R are free, of unique rank R is called a <u>fir</u>. For instance, fields and principal ideal domains are firs; an example of a semifir which is not a fir is the free power series ring $k\langle\langle X\rangle\rangle$, where k is a field, X is a set and $|X|>1$. One way of constructing firs is the following. Let K be a field; by a <u>K-ring</u> R we mean a ring with a homomorphism of K into R. Clearly, such a homomorphism is injective. Let $\{R_i\}$ be a family of K-rings and assume for simplicity that $K \subseteq R_i$ for all i. The push-out of the family of inclusions $K \subseteq R_i$ in the category of rings is called the <u>coproduct</u> of the R_i over K, it is denoted by $\coprod_K R_i$ (cf. [5; Ch.5]). Now Theorem 5.3.2 of [5] states that the coproduct of a family of firs over a field is a fir. In particular, the coproduct of a family of fields over a common subfield is a fir.

1

Homomorphisms of rings into fields

Let R be a ring, K a field and assume we have a homomorphism

$$\alpha : R \to K. \qquad (1)$$

In what follows we shall ask certain questions about K and try to answer them using information on R. Clearly we have to restrict ourselves to situations where K is not "too large" with respect to R, or rather R^{α}; we shall assume that K is generated, as a field, by the image of R. This is equivalent to α being an epimorphism in the category of rings. Let α in (1) be an epimorphism in <u>Rg</u>, then K, together with α, or simply (1), is called an <u>epic R-field</u>. If α is injective we also say that K is a <u>field of fractions</u> of R. When R is commutative K can be characterized up to isomorphism by the prime ideal $\ker\alpha$; more precisely: K is isomorphic to the field of quotients of the integral domain $R/\ker\alpha$. In contrast, if R is non-commutative we find that $\ker\alpha$ is not sufficient to describe K: there exist non-isomorphic epic R-fields $\alpha : R \to K$ and $\beta : R \to L$ such that $\ker\alpha=\ker\beta$ (cf. [5, pp.15-16 and p.22]). It turns out that an epic R-field can be characterized and constructed in terms of the set of matrices over R which become non-singular over K.

In section 1 we recall the basic notions in a more general context and then specialize to epic R-fields in §2 where we also discuss a few properties of Sylvester domains, a class of rings which have a universal field of fractions. In section 3 we construct the universal field of fractions of a free algebra (and certain related rings) as a subfield of an ultrapower of a field and use this method to prove some facts about the universal field of fractions of a free algebra.

1.1 Σ-inverting homomorphisms, the Σ-rational closure of a ring

Let R be a ring and let Σ be a set of square matrices over R. A homomorphism $f : R \to S$ is said to be Σ-inverting if $A^f \in GL(S)$ for all $A \in \Sigma$. A set Σ of square matrices over R is called multiplicative if $1 \in \Sigma$ and

$$A, B \in \Sigma \quad \Rightarrow \quad \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \text{ and } \begin{pmatrix} A & 0 \\ D & B \end{pmatrix} \in \Sigma$$

for all matrices C and D of the appropriate size. For instance, let $f : R \to S$ be a homomorphism of rings and denote by $\Sigma(f)$ the set of square matrices over R which become invertible over S; then $\Sigma(f)$ is multiplicative and f is $\Sigma(f)$-inverting. Let R be a ring and Σ a subset of M(R); we define the universal Σ-inverting ring R_Σ by the following universal property:

-we have a Σ-inverting homomorphism $\lambda : R \to R_\Sigma$, such that any other Σ-inverting homomorphism $f : R \to S$ factors through λ uniquely.

To construct R_Σ explicitly, for each nxn matrix (a_{ij})

in Σ adjoin n^2 new elements $\{a'_{ij}\}$ to R with defining relations

$$(a_{ij})(a'_{ij})=(a'_{ij})(a_{ij})=I_n.$$

We put λ for the natural homomorphism $R \to R_\Sigma$. Thus R_Σ is generated by R^λ and the entries of the inverses of all matrices of form A^λ, $A\in\Sigma$. Let $f : R \to S$ be a Σ-inverting homomorphism; it is clear that for the diagram

$$\begin{tikzcd} & R_\Sigma \arrow[d] \\ R \arrow[ur, "\lambda"] \arrow[r, "f"'] & S \end{tikzcd}$$

to commute a generator a'_{kl}, where $(a'_{ij})=(a^\lambda_{ij})^{-1}$ and $(a_{ij})\in\Sigma$, has to be mapped onto the kl^{th} entry of $(a^f_{ij})^{-1}$ over S. Hence a homomorphism, if it exists, must be unique and its existence follows by universal algebra.

Let $f : R \to S$ be a Σ-inverting homomorphism; the set $R_\Sigma(S)$ of entries of matrices $(A^f)^{-1}$, where $A\in\Sigma$, is called the <u>Σ-rational closure</u> of R (under f in S). When Σ is multiplicative we have the following

<u>Theorem 1.1</u>. Let R be a ring and Σ a multiplicative set of matrices over R. Let $f : R \to S$ be a Σ-inverting homomorphism. Then $R_\Sigma(S)$ is a subring of S containing R^f and for any $x\in S$ the following conditions are equivalent

(i) $x\in R_\Sigma(S)$,

(ii) x is a component of the solution u' over S of a matrix equation

$$A^f u'+a^f=0, \qquad (2)$$

where $A\in\Sigma$ and a is a column over R.

Proof. This is part of Theorem 7.1.2 of [4].

Following [6] we shall standardize systems of form as in (2); to do this we shall have to put a further restriction on Σ. We start with a homomorphism $f : R \rightarrow S$ and write $\Sigma(f)=\{A\in M(R) \mid A^f \in GL(S)\}$. Then $GL(R)^f \subseteq \Sigma(f)$ and $\Sigma(f)$ is multiplicative. Furthermore $\Sigma(f)$ is closed under multiplication in the usual sense. Let $A\in {}^nR^{n+1}$ and put $A=(A_0,A_1,\ldots,A_n)$ where A_i denotes the i^{th} column of A. Further write A_* for the $n\times(n-1)$ matrix $(A_1,A_2,\ldots,A_{n-1})$. Now let $p\in S$ and assume that

$$A^f \begin{pmatrix}1\\u\\p\end{pmatrix} =0 \qquad (3)$$

where $u\in {}^{n-1}S$. If $(A_*,A_n)\in\Sigma(f)$ we say that (3) is an admissible system for p and A is an admissible matrix for p. Let (3) be an admissible system for p, then

$$A_0^f+(A_*,A_n)\begin{pmatrix}u\\p\end{pmatrix}=0$$

and hence, by Theorem 1.1, $p\in R_\Sigma(S)$. Conversely, if p is in the $\Sigma(f)$-rational closure of R then a system like (2) exists in which p is a component of u'. Since $\Sigma(f)$ is closed under multiplication by an invertible matrix over R,we may assume that p is the last component of u'. Then

$$(-a,A)^f\begin{pmatrix}1\\u\\p\end{pmatrix}=0,$$

where $\begin{pmatrix}u\\p\end{pmatrix}=u'$, is an admissible system for p. We deduce that the set of elements of S, for which admissible systems exist, and $R_{\Sigma(f)}(S)$ coincide.

We can rewrite (3) as follows:

$$(A_*, A_n)^f \begin{pmatrix} I_{n-1} & u \\ 0 & p \end{pmatrix} = (A_*, -A_o)^f. \tag{4}$$

We say that (A_*, A_n) is the <u>denominator</u> and $(A_*, -A_o)$ is the <u>numerator</u> of p in (3). Inspecting (4) we find that p is a unit in S if and only if the numerator of p is in $\Sigma(f)$. If this is so then

$$(A_n, A_*, A_o)\begin{pmatrix} 1 \\ up^{-1} \\ p^{-1} \end{pmatrix} = 0$$

is an admissible system for p^{-1}. Let q be another element of the $\Sigma(f)$-rational closure of R and let

$$(B_o, B_*, B_m)\begin{pmatrix} 1 \\ v \\ q \end{pmatrix} = 0$$

be an admissible system for q. Then

$$\begin{pmatrix} A_o & A_* & A_n & 0 & 0 \\ 0 & 0 & B_o & B_* & B_m \end{pmatrix}\begin{pmatrix} 1 \\ u \\ p \\ vp \\ qp \end{pmatrix} = 0 \tag{5}$$

and $\begin{pmatrix} A_* & A_n & 0 & 0 \\ 0 & B_o & B_* & B_m \end{pmatrix}$ is in $\Sigma(f)$ since $\Sigma(f)$ is multiplicative. Thus (5) is an admissible system for qp. Similarly:

$$\begin{pmatrix} A_o & A_* & A_n & 0 & 0 \\ B_o & 0 & B_m & B_* & B_m \end{pmatrix}\begin{pmatrix} 1 \\ u \\ p \\ v \\ q-p \end{pmatrix} = 0$$

is an admissible system for q-p.

We shall need to establish that systems for NxN matrices, analogous to (3), exist. Let f, R and S be as before and put $T=R_{\Sigma(f)}(S)$. Let $P \in T_N$ and consider the system

$$(A_0, A_*, A_n)^f (I_N, X_*^T, P)^T = 0, \tag{6}$$

where A_0, $A_n \in {}^{nN}R^N$, $A_* \in {}^{nN}R^{(n-1)N}$, $X_* \in {}^{(n-1)N}T^N$ and further (A_*, A_n) is in $\Sigma(f)$. Then (6) is called an <u>admissible</u> system for P of order n. As in the 1x1 case (A_*, A_n) is called the <u>denominator</u> and $(A_*, -A_0)$ the <u>numerator</u> of P in (6); further $P \in GL(S)$ if and only if $(A_*, -A_0) \in \Sigma(f)$.

<u>Lemma 1.2</u>. Let Σ be a set of square matrices over a ring R. Then for each n there is a natural isomorphism

$$M_n(R_\Sigma) \cong (M_n(R))_{\Sigma'}$$

where $\Sigma' = \left\{ A^{\dotplus n} \in M_n(R) \;\middle|\; A \in \Sigma \right\}$. (Here $A^{\dotplus n} = A \dotplus A \dotplus \ldots \dotplus A$, n times, can be viewed as a matrix over $M_n(R)$.)

<u>Proof</u>. Clearly $(M_n(R))_{\Sigma'}$ is an nxn matrix ring, say $(M_n(R))_{\Sigma'} = M_n(S)$. Then S contains the image of R and under the natural map $R \rightarrow S$ every element of Σ' becomes invertible, hence so does every element of Σ. It follows that we have a homomorphism $R_\Sigma \rightarrow S$ and thus we also obtain $\alpha : M_n(R_\Sigma) \rightarrow M_n(S) = (M_n(R))_{\Sigma'}$. Furthermore $M_n(R) \rightarrow M_n(R_\Sigma)$ is Σ'-inverting, hence there is a map $\beta : M_n(S) \rightarrow M_n(R_\Sigma)$. It is clear that α and β are mutually inverse.

<u>Proposition 1.3</u>. Let $f : R \rightarrow S$ be a homomorphism of

rings; set $\Sigma=\Sigma(f)$ and write T for the Σ-rational closure of R in S. Then for any N and $P\in T_N$ there exists an admissible system over R for P.

<u>Proof</u>. We have the commuting triangle

$$\begin{tikzcd} & R_\Sigma \arrow[d, "f'"] \\ R \arrow[ur, "\lambda"] \arrow[r, "f"'] & S \end{tikzcd}$$

where f' is onto T. Observe that $\Sigma=\Sigma(\lambda)$; it will suffice to verify the claim for λ. For let $P'\in M_N(R_\Sigma)$ be such that $(P')^{f'}=P$; on applying f' to any admissible system for P' we obtain an admissible system for P.

By the above lemma we may identify $M_N(R_\Sigma)$ and $(M_N(R))_{\Sigma'}$. Denote by $\hat{\Sigma}$ the set of square matrices over $M_N(R)$ which become invertible over $(M_N(R))_{\Sigma'}$; then $\hat{\Sigma}\subseteq\Sigma$. Now P is an element of $(M_N(R))_{\Sigma'}$ and further it is in the $\hat{\Sigma}$-rational closure of $M_N(R)$ so, by Theorem 1.1, there is an admissible system over $M_N(R)$ for P whose denominator is in $\hat{\Sigma}$ and hence in Σ. This is the required system.

<u>1.2 The category of epic R-fields and specializations</u>

Let R be a ring and let $\alpha : R \to K$ be an epic R-field. We shall write Σ_K for $\Sigma(\alpha)$. The complement of Σ_K in M(R) is called the <u>singular kernel</u> of α (or K), it is denoted by $\mathcal{P}_K$. We first show that Σ_K characterizes K up to isomorphism. Form the universal Σ_K-inverting ring; we have the commutative diagram

$$\begin{tikzcd} & R_{\Sigma_K} \arrow[d, "\alpha'"] \\ R \arrow[ur, "\lambda_K"] \arrow[r, "\alpha"'] & K \end{tikzcd} .$$

We claim that the set of non-units of R_{Σ_K} is precisely the kernel of α'. If this is so then R_{Σ_K} is a local ring with maximal ideal $\ker\alpha'$ and residue class field K. To verify the claim we only have to show that if $p \in R_{\Sigma_K}$ is not a unit then $p^{\alpha'}=0$. By earlier results we can find a relation over R_{Σ_K}

$$(A_*,A_n)^{\lambda_K} \begin{pmatrix} I_n & u \\ 0 & p \end{pmatrix} = (A_*,-A_o)^{\lambda_K}, \qquad (7)$$

where $(A_*,A_n) \in \Sigma_K$ and $(A_*,-A_o) \notin \Sigma_K$ since p would be a unit otherwise. Applying α' to (7) we obtain a relation over K and $(A_*,-A_o)^{\lambda_K\alpha'}=(A_*,-A_o)^{\alpha}$ is not invertible. Over a field this implies that $p^{\alpha'}=0$. It follows now that if K and L are epic R-fields and $\Sigma_K=\Sigma_L$ then $K \cong L$.

P. M. Cohn has found necessary and sufficient conditions for a set Σ of matrices over R to coincide with Σ_K for some epic R-field K. In fact, he determined under what conditions a set of square matrices would be the singular kernel of some epic R-field. To recall Cohn's theorem we have to make a few definitions.

Let R be a ring and let $A \in {}^mR^n$. By the <u>rank</u> of A we understand the least integer r, such that A can be written as a product of an mxr and an rxn matrix over R (cf. [4; p.195]). The rank of A is denoted by $\rho(A)$. Over a field this definition coincides with the usual notion of the rank of a matrix. A homomorphism $f : R \rightarrow S$ of rings is said to be <u>rank-preserving</u> if $\rho(A)=\rho(A^f)$ for every matrix A over R. Clearly, a necessary condition for a matrix A over R to become invertible over any epic R-field is that A should be square and, assuming $A \in R_n$, the rank of A should be n. Such a matrix A is said to be <u>full</u>. It follows that the singular kernel of any epic R-field must contain all non-full matrices. A set $\mathcal{P}$ of square matrices over R is called a <u>prime matrix ideal</u> if it satisfies the following conditions:

1. $\mathcal{P}$ contains all non-full square matrices,
2. $A, B \in \mathcal{P} \Longrightarrow A \nabla B \in \mathcal{P}$, whenever the deteminantal sum makes sense,
3. $A \in \mathcal{P} \Longrightarrow A \dotplus B \in \mathcal{P}$ for all $B \in M(R)$,
4. $A \dotplus 1 \in \mathcal{P} \Longrightarrow A \in \mathcal{P}$,
5. $\mathcal{P} \neq M(R)$,
6. $A \dotplus B \in \mathcal{P} \Longrightarrow A \in \mathcal{P}$ or $B \in \mathcal{P}$.

It is easy to see that the singular kernel of an epic R-field satisfies the above conditions. Theorem 7.5.3 of [4] states the converse: given a prime matrix ideal $\mathcal{P}$ over a ring R one can construct an epic R-field with precise singular kernel $\mathcal{P}$.

Let R be a ring; the set of epic R-fields can be made into a category, $\mathcal{F}_R$, whose morphisms are called R-specializations (cf. [5; pp.73-74]). Let $\alpha : R \rightarrow K$ and $\beta : R \rightarrow L$ be epic R-fields; essentially, an R-specialization from K to L is a surjective homomorphism $f : K_o \rightarrow L$, where K_o is a local subring of K containing R^α, such that the diagram

$$\begin{tikzcd} & K_o \arrow[dd, "f"] \\ R \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & \\ & L \end{tikzcd}$$

commutes. Clearly, L is then isomorphic to the residue class field of K_o. The category of epic R-fields and specializations can be described as follows. Theorem 7.2.3 of [4] states that there is a specialization $K \rightarrow L$ between epic R-fields if and only if $\mathcal{P}_K \subseteq \mathcal{P}_L$, and further, that such a specialization must be unique. Thus the category of epic R-fields and R-specializations is equivalent to the set of prime matrix ideals over R partially ordered by inclusion. It is clear now that $\mathcal{F}_R$ has an initial object if and only if there is a unique minimal prime matrix ideal over R. Assume that U is an initial object of $\mathcal{F}_R$ and further that U is a field of fractions of R; then U is called the <u>universal field of fractions</u> of R and it is unique up to isomorphism.

We have seen that every prime matrix ideal over a ring R contains all the non-full matrices in M(R). A ring for which the set of non-full square matrices form a prime matrix ideal is called a <u>Sylvester domain</u>. Let R be a Sylvester domain and denote by $\mathcal{P}_o$ the prime matrix ideal consisting of all non-full square matrices over R. Then $\mathcal{P}_o$ is the unique, minimal prime matrix ideal over R; the epic R-field with singular kernel is denoted by U(R). Clearly U(R) is an initial object in $\mathcal{F}_R$. Let a be a non-zero element of R; then a is full considered as a 1x1 matrix over R and hence a becomes invertible over U(R). It follows that $R \subseteq U(R)$ and hence U(R) is the universal field of fractions of R. We note that not only Sylvester domains have a universal field of fractions, e.g. for every commutative integral domain its field of quotients is also its universal field of fractions, but not every commutative integral domain is a Sylvester domain (cf. [9; Thm.6]).

Sylvester domains have been introduced and studied in [9]; for us it will suffice that for a ring R the following conditions are equivalent

(a) R is a Sylvester domain,

(b) there is an epic R-field over which every full matrix of M(R) becomes invertible,

(c) R has a rank-preserving homomorphism into a field (cf. [9; Thm. 3]). We note that the field referred to in (b) and (c) is U(R).

A notion which arises naturally in connection with Sylvester domains is that of an honest homomorphism. A homomorphism of rings is said to be <u>honest</u> if it keeps full matrices full. Let $f : R \to S$ be an honest homomorphism of Sylvester domains. Composing f with the inclusion $S \subseteq U(S)$ we obtain a homomorphism $R \to U(S)$. Since full matrices remain full under f and their image becomes invertible over U(S) we find that the subfield of U(S) generated by R^f is U(R). We have shown

<u>Theorem 2.1</u>. ([5; Thm. 4.3.3]) Let $f : R \to S$ be an honest homomorphism of Sylvester domains. Then f induces a (unique) embedding $f' : U(R) \to U(S)$, such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ U(R) & \xrightarrow[f']{} & U(S) \end{array}$$

commutes.

A very important subclass of Sylvester domains is the class of semifirs. The fact that a semifir is a Sylvester domain is proved e.g. in [4; p.283].

1.3 The universal field of fractions of a free algebra

First we briefly recall the notion of a free E-ring. Let E be a field: by an E-ring R we mean a ring with a homomorphism of E into R. Let R_1 and R_2 be E-rings, a homomorphism $R_1 \to R_2$ is said to be an E-ring homomorphism if the diagram

$$\begin{array}{ccc} & \nearrow & R_1 \\ E & & \downarrow \\ & \searrow & R_2 \end{array}$$

commutes. Let K be a subfield of E; the free E-ring on a set X over K, $E_K\langle X\rangle$, is defined by the following universal property:

- $E_K\langle X\rangle$ is an E-ring generated by E and X such that X centralizes K and any K-centralizing map $X \to R$ into an E-ring R can be extended to a unique E-ring homomorphism of $E_K\langle X\rangle$ into R.

When E=K, $E_K\langle X\rangle$ is abbreviated to $K\langle X\rangle$ and if K is commutative, $K\langle X\rangle$ is called the free associative K-algebra on X. $E_K\langle X\rangle$ can be obtained as the coproduct:

$$E_K\langle X\rangle = E \bigsqcup_K \left(\bigsqcup_K K[x]\right),$$

where x runs through X, and so $E_K\langle X\rangle$, being a coproduct of firs, is a fir by Theorem 5.3.2 of [5] (cf. [5; pp.111-112]). Hence $E_K\langle X\rangle$ is a Sylvester domain and so possesses a universal field of fractions which is denoted by $E_K\langle\!\langle X\rangle\!\rangle$. We write $K\langle\!\langle X\rangle\!\rangle$ for $U(K\langle X\rangle)$.

Let E be a field with a subfield K and let X be a set; an E-ring homomorphism of $E_K\langle X\rangle$ into E is called an <u>evaluation</u>. Let f be an evaluation, by the universal property of $E_K\langle X\rangle$ f is uniquely determined by its action on X. Notice that E is an epic $E_K\langle X\rangle$-field with respect to f; consequently f extends to a specialization $E_K\langle\!\langle X\rangle\!\rangle \longrightarrow E$ whose domain is the $\Sigma(f)$-rational closure of $E_K\langle X\rangle$ in $E_K\langle\!\langle X\rangle\!\rangle$. The following is a basic result on evaluations.

<u>Theorem 3.1</u>. (Specialization lemma, [5; Lemma 6.3.1]) Let E be a field with centre k and assume that (i) k is infinite and (ii) $[E:k]=\infty$. Then for any full matrix A over $E_k\langle X\rangle$, there exists an evaluation $f : E_k\langle X\rangle \longrightarrow E$, such that A^f is non-singular over E.

The specialization lemma can sometimes applied even if the hypotheses of Theorem 3.1 are not satisfied. Let D be a field with centre C and assume that D and C satisfy the hypotheses of the specialization lemma. Let E be a subfield of D and put $k = C\cap E$. Then we have a natural map $E_k\langle X\rangle \rightarrow D_C\langle X\rangle$, obtained by the coproduct property of $E_k\langle X\rangle$. If this map is honest, every full matrix over $E_k\langle X\rangle$ remains full over $D_C\langle X\rangle$ and hence can be evaluated so that it becomes non-singular over D.

We have seen that every evaluation induces a specialization. Suppose E and k satisfy the hypotheses of the specialization lemma and let $p \in E_k\langle\!\langle X\rangle\!\rangle$. Further let (A_o, A_*, A_n) be a matrix over $E_k\langle X\rangle$, admissible for p. By Theorem 3.1 we can choose an evaluation a so that $(A_*, A_n)^a$ is non-singular over E and then p is in the domain of the specialization, say α, induced by a. If $p \neq 0$ then $(A_*, -A_o)$ is full over $E_k\langle X\rangle$ and hence, choosing a so that $((A_*, A_n) \dotplus (A_*, -A_o))^a$ is non-singular, we have $p^\alpha \neq 0$. Moreover, let $\{p_i\}$ be a finite family of non-zero elements of $E_k\langle\!\langle X\rangle\!\rangle$ and for each i let $(A_o^{(i)}, A_*^{(i)}, A_{n_i}^{(i)})$ be admissible for p_i; we can choose a so that

$$\left(\dotplus_i ((A_*^{(i)}, A_{n_i}^{(i)}) \dotplus (A_*^{(i)}, -A_o^{(i)}))\right)^a$$

is non-singular over E. The domain of α will then contain all the p_i and $p_i^\alpha \neq 0$ for each i. Hence we obtain

Theorem 3.2. Let E be a field with centre k and let X be a set. Assume that (i) k is infinite and (ii) $[E:k] = \infty$. Then for each finite family of non-zero elements of $E_k\langle\!\langle X\rangle\!\rangle$ there is a subring S of $E_k\langle\!\langle X\rangle\!\rangle$, containing $E_k\langle X\rangle$ and $\{p_i\}$, and an E-ring homomorphism α of S into E, such that $p_i^\alpha \neq 0$ for all i.

We remark that the above theorem is obtained from the specialization lemma in much the same way as Theorem 7.2.7 of [5]. (Note that the statement of this theorem is incorrect. However we are only concerned with the special case $C = k$ for which the stated form holds.)

Let E be a field with centre k and let X be a set.

Put $R = E_k\langle X\rangle$ and $U = E_k\langle\!\langle X\rangle\!\rangle$. Since $\operatorname{ctr} E = k$ every map $X \rightarrow E$ induces an evaluation $R \rightarrow E$; thus the set of evaluations on R is just E^X. Let A be a full matrix over R; by the <u>non-singularity support</u> of A we understand the set

$$s(A) = \left\{ f \in E^X \;\middle|\; A^f \in GL(E) \right\}.$$

Assume now that k is infinite and $[E:k] = \infty$; Theorem 3.1 ensures that s(A) is non-empty. For any other full matrix B over R, $A \dotplus B$ is also full and

$$s(A) \cap s(B) = s(A \dotplus B),$$

as is easily checked. Hence the family $\{ s(A) \subseteq E^X \mid$ A is full over R$\}$ has the finite intersection property and therefore it is contained in some ultrafilter of $P(E^X)$. Let $\mathcal{F}$ be such an ultrafilter; we construct

$$E^{(E^X)}/\mathcal{F},$$

an ultrapower of E. This is a field, say K, and clearly K contains copies of E. Let α be an element of $E^{(E^X)}$, the image of α in K will be denoted by $\bar{\alpha}$. First we show that R embeds canonically into K and then prove that the subfield of K, generated by R, is U. Let a be an element of R, we shall identify a with $\bar{\alpha} \in K$, where $\alpha(f) = a^f$ for all $f \in E^X$. The correspondence $a \mapsto \alpha$ is clearly a homomorphism, it remains to verify that it is one-to-one. Assume $a \neq 0$, then a is full considered as a 1x1 matrix over R and hence $s(a) \in \mathcal{F}$. On the other hand:

$$\overline{\alpha} = 0 \iff T = \{f \in E^X \mid \alpha(f) = 0\} \in \mathcal{F}.$$

But $T = E^X \setminus s(a)$ and $s(A) \in \mathcal{F}$, hence $T \notin \mathcal{F}$ and thus $\overline{\alpha} \neq 0$. Now let A be any full matrix over K; we construct the inverse of A over K which will prove that the subfield of K generated by R is U. For each $f \in s(A)$, A^f is invertible over E; we put

$$(A^f)^{-1} = (b_{ij}^{(f)}), \quad b_{ij}^{(f)} \in E.$$

Define $\beta_{ij} \in E^{(E^X)}$ as follows:

$$\beta_{ij}(f) = \begin{cases} b_{ij}^{(f)} & \text{if } f \in s(A) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_k \alpha_{ik}(f)\beta_{kj}(f) = \delta_{ij}$$

for all $f \in s(A)$ and so $(\overline{\beta}_{ij})$ is the required inverse. This construction allows us to prove

<u>Lemma 3.3.</u> Let D be a field with centre C and let X be a set. Further let E be a subfield of D and put $k = E \cap C$. Assume that

(i) C is infinite,

(ii) $[D:C] = \infty$,

(iii) the natural map $E_k\langle X\rangle \to D_C\langle X\rangle$ is honest.

Then $E_k\langle\!\langle X\rangle\!\rangle$ can be embedded in an ultrapower of D.

<u>Proof</u>. We have seen that $D_C\langle\!\langle X\rangle\!\rangle$ can be embedded in an ultrapower of D. This proves the assertion since by Theorem 2.1 $E_k\langle\!\langle X\rangle\!\rangle$ embeds in $D_C\langle\!\langle X\rangle\!\rangle$.

To demonstrate the scope of the lemma we show that if $\operatorname{ctr} E = k$, we can find an extension D of E, such that the hypotheses of the lemma are satisfied. Suppose first that $[E:k] < \infty$ and k is infinite. Set $D = E(t)(y;\alpha)$, where α is the endomorphism of $E(t)$ induced by $t \mapsto t^2$. Using Laurent series it is easy to see that $\operatorname{ctr} D = k$. Furthermore $[D:k] = \infty$ and the natural map $E_k\langle X\rangle \rightarrow D_k\langle X\rangle$ is honest by Proposition 5.4.2, Corollary of [5]. If k is finite, embed E in $E(t)$ whose centre is $k(t)$. Again, this can be checked using Laurent series. Now the natural map $E_k\langle X\rangle \rightarrow E(t)_{k(t)}\langle X\rangle$ is honest by Lemma 6.3.4 of [5] and this reduces the situation to the previous case.

Let $\mathcal{L}$ be a language for fields. Recall that a property P of fields is said to be a first-order property if there is an $\mathcal{L}$-sentence (i.e. an $\mathcal{L}$-formula without free variables) σ such that, for any field F,

$$F \text{ has property } P \iff \sigma \text{ holds in } F.$$

Let us call an $\mathcal{L}$-sentence σ <u>universal</u> if

$$\sigma = \forall x_1,\ldots,x_n(\varphi),$$

where $n \geq 1$ and φ is an $\mathcal{L}$-formula with no quantifiers. We can now prove

<u>Theorem 3.4</u>. Let D be a field with centre C and let X be a set. Further let E be a subfield of D and put $k = E \cap C$. Assume that

(i) C is infinite,

(ii) $[E:C] = \infty$,

(iii) the natural map $E_k\langle X\rangle \longrightarrow D_C\langle X\rangle$ is honest.

Then $E_k\langle X\rangle$ has every first-order property of D which can be expressed by a universal $\mathcal{L}$-sentence.

Proof. We know from Lemma 3.3 that $E_k\langle X\rangle$ can be embedded in a field K, which is an ultrapower of D. Now by Łoś' theorem D and K have the same first-order properties. This verifies the theorem because a first-order property expressed by a universal $\mathcal{L}$-sentence is clearly inherited by subfields.

The above theorem has some useful applications. For instance, consider the universal $\mathcal{L}$-sentence

$$\sigma = \forall x,y,t,u(\varphi(x,y,t,u)),$$

where

$$\varphi(x,y,t,u) = (xy \neq yx) \Rightarrow ((xt = tx \wedge \overset{xu}{\cancel{tz}} = \overset{ux}{\cancel{zt}}) \Rightarrow tu = ut).$$

It is easy to see that σ expresses a first-order property, say CC, of fields: the centralizer of every non-central element is commutative. Let k be a commutative field and let X be a set. We shall verify that $k\langle X\rangle$ has the property CC. First we have to establish

Proposition 3.5. Let D be a field, infinite dimensional over its centre, say k, and assume that k is infinite. Then $\operatorname{ctr} D_k\langle X\rangle = k$.

Proof. Put $U = D_k\langle X\rangle$. Let $a \in \operatorname{ctr} U$; then, in particular, a centralizes D and hence any specialization $U \to D$, which is defined on a, maps a into k. Assume that $a \notin k$. Consider the field $V = U_k\langle y\rangle = D_k\langle X \cup \{y\}\rangle$. Then clearly $ay \neq ya$; by Theorem 3.2 we can choose a specialization $s : V \to D$ which maps $ay - ya$ onto a non-zero element of D. This implies that $a^s \notin k$. Restricting s to the intersection of its domain with U we obtain a specialization $U \to D$ which maps a outside k, a contradiction. Hence $\operatorname{ctr} U \subseteq k$, the reverse inclusion is obvious. ~~can be verified similarly.~~

Let $T = \{t_i\}_{i \in \mathbb{Z}}$ be a set of commuting indeterminates and let φ be the shift-automorphism of $k(T)$, induced by the rule

$$t_i \longmapsto t_{i+1} \quad \text{for all} \quad i \in \mathbb{Z}.$$

Set $D = k(T)(y;\varphi)$; using Laurent series it is not hard to show that $\operatorname{ctr} D = k$ and D has the property CC. We know from the above proposition that $\operatorname{ctr} D_k\langle X\rangle = k$, provided k is infinite. Then from Theorem 3.4 we can deduce that elements of $D_k\langle X\rangle$, outside k, have commutative centralizers. Since $k\langle X\rangle \subseteq D_k\langle X\rangle$, the same is true of $k\langle X\rangle$. When k is infinite this proves

<u>Proposition 3.6</u>. Let k be a commutative field and let X be a set. Then the centralizer of every element of $k\langle X\rangle$, outside k, is commutative.

<u>Proof</u>. It remains to verify the claim when k is finite. Set $V = k(t)\langle X\rangle$; we know that $k\langle X\rangle \subset V$ and elements of V outside $k(t)$ have commutative centralizers. Let $a \in k\langle X\rangle \setminus k$, then $a \in V \setminus k(t)$ and so the centralizer of a in V, hence also in $k\langle X\rangle$, is commutative.

We easily obtain the

<u>Corollary</u>. Let k be a commutative field and let X be a set. Then $\operatorname{ctr} k\langle X\rangle = k$.

We note that Proposition 3.5 and Proposition 3.6, Corollary are special cases of Theorem 4.7 (iv) of [7].

Proposition 3.6 is also known; it was first proved by P.M. Cohn.

To close this section we outline a result due to J. Lewin which states that if k is a commutative field, $k\langle X\rangle$ can be realized as a subfield of a Malcev-Neumann field $k((F))$, where F is the free group on X. Let k be a field and let G be a group with a full order $\leq$. Denote by k^G the set of all mappings from G to k; then k^G is a k-space. Let $a \in k^G$, the support of a is defined as

$$\mathrm{supp}(a) = \{ g \in G \mid g^a \neq 0 \} .$$

Let $k((G, \leq))$ be the subspace of k^G consisting of all mappings with well-ordered support (with respect to $\leq$). Let $a \in k((G, \leq))$; then a can be represented as a power series:

$$a = \sum_{g \in G} a_g g , \qquad (a_g \in k)$$

where $a_g = g^a$. It is proved e.g. in [11] that $k((G, \leq))$ admits multiplication (cf. [11; Ch.VIII, §5]). Let

$$b = \sum_{h \in G} b_h h$$

be another element of $k((G, \leq))$; the product of a and b is defined as follows:

$$a\,b = \sum_{f \in G} \Big(\sum_{gh=f} a_g b_h \Big) f.$$

Thus $k((G, \leq))$ becomes a ring; Theorem 10 of [11; p.137] states that it is in fact a field. Assume now that k is commutative. Let X be a set, write S for the free monoid and F for the free group on X. It is well-known that F can be fully ordered (cf. §2.1 below); let $\leq$ be a full order on F. Then the free algebra $k\langle X\rangle$ can be identified with the subelgebra of $k((F,\leq))$ consisting of all elements whose support is contained in S and is finite. Thus the subfield, say U, of $k((F,\leq))$ generated by $k\langle X\rangle$ is a field of fractions of $k\langle X\rangle$. Lewin has shown that $U = k\langle\!\langle X\rangle\!\rangle$ (cf. [12; Thm.2, p.343]). We note that this gives another proof of Proposition 3.6, Corollary (cf. [12; Thm.4, p.343]).

2

Orderings of epic R-fields

Any ordered ring R is a domain: if it is also an Ore ring, R then has a field of quotients to which the ordering of R can be uniquely extended by putting

$$ab^{-1} \geq 0 \quad \text{if} \quad ab \geq 0 \qquad (0 \neq b,\ a \in R)$$

(cf. [11; Thm.3, p.109]). We may ask how general rings compare with Ore rings as regards the above properties of the latter; in particular the following questions arise:

- Is every ordered ring embeddable in a field?
- If K is a field of fractions of of an ordered ring R, can the ordering of R be extended to K?
- If so, is the extension unique?

Let M be a monoid; by a <u>group of fractions</u> of M we understand a group, say G, together with an embedding of M into G which is an epimorphism in the category of monoids. This amounts to saying that G is generated as a group by the image of M. Analogously to rings, one may ask whether or not an ordering of a monoid can be extended to a group of fractions. Apart from their intrinsic interest we shall find monoids and their groups of fractions useful in constructing examples of ordered rings. For instance, in section 2 we exhibit an ordered monoid M with no groups of fractions. By taking the semigroup algebra kM over any ordered field k we obtain an ordered ring with no fields of fractions.

We start with monoids and groups. In §1 we show that if M is a monoid with a full order $\leq$ then this ordering extends to the universal group of M if and only if it can be extended to a group of fractions of M. A somewhat stronger statement is then proved for free monoids and groups and we also give an example of an ordering of the free monoid on x and y which can be extended to to the free group on x and y in an infinite number of distinct ways. As we have mentioned, in section 2 we construct an example of an ordered monoid which cannot be embedded in a group. Sections 3-6 treat rings. In §3 we recall some basic results on orderings of fields while in §4 we show how to express an ordering of a field using the Dieudonné determinant. In section 2 of Chapter 1 we have sketched how epic R-fields can be characterized in terms of matrices over R. This makes it natural, in describing full orders on an epic R-field K, to consider matrix cones over R rather than ordinary cones of elements of K. Matrix cones are introduced in §5: essentially a matrix cone over R, associated with a full order on K, consists of all the square matrices (over R) which either become singular or have a positive Dieudonné determinant over K. We prove necessary and sufficient conditions, in terms of matrix cones, (i) for an epic R-field to be orderable, (ii) for a full order on R to be extendible to a field of fractions of R and (iii) for such an extension to be unique. We apply these results to show that if E is an ordered field with centre k then the ordering of E can be extended to $E_k\langle X\rangle$. In section 6 examples are given which demonstrate that the above questions can be answered in the negative.

2.1 The extension of a full order on a monoid to a group of fractions

A semigroup S is said to be partially orderered by the relation $\leq$ if

S1. S is partially ordered under $\leq$ as a set and

S2. $a \leq b \implies ac \leq bc$ and $ca \leq cb$ for all $c \in S$.

If, in addition, $\leq$ totally orders S as a set we say that S is (fully, totally) ordered by $\leq$. An order-preserving homomorphism of partially ordered semigroups is called an o-homomorphism.

A group is said to be partially ordered by the relation $\leq$ if it is partially ordered by $\leq$ as a semigroup. Similarly, by a (full, total) ordering of a group G we understand an ordering of G as a semigroup. Suppose G is a partially ordered group under $\leq$, the set

$$P=\left\{x \in G \;\middle|\; x \geq 1\right\}$$

is called the positive cone associated with $\leq$. Set $P^{-1}=\left\{x \in G \;\middle|\; x^{-1} \in P\right\}$; it is easy to see that P and P^{-1} satisfy

C1. $P \cap P^{-1}=1$,

C2. $PP \subseteq P$,

C3. $xPx^{-1} \subseteq P$ for all $x \in G$.

Suppose now that P is a subset of a group G for which C1-C3 hold; it is easy to see that P is then the positive cone of a unique partial order on G which is defined as follows :

$$a \leq b \text{ if and only if } ba^{-1} \in P.$$

Moreover $\leq$ is total if and only if

C4. $P \cup P^{-1} = G$.

It follows that we can identify a partial ordering of a group with its associated positive cone. The conditions C2 and C3 express that P is a normal subsemigroup of G; thus we find

Proposition 1.1. Let G be a group. A normal subsemigroup P of G is the positive cone of some partial ordering of G if and only if $P \cap P^{-1} = 1$.

Let M be a monoid with a full order $\leq$ and let G be a group of fractions of M; by an extension of $\leq$ to G we mean an ordering of G with respect to which $M \rightarrow G$ is an o-embedding. Our primary interest here is to find conditions under which $\leq$ extends to a full order on G. Suppose that $\leq'$ is such an extension, the fact that $\leq'$ extends $\leq$ can be expressed by saying

$$1 \leq' ba^{-1} \quad \text{whenever} \quad a \leq b \qquad (a,\ b \in M).$$

Hence the normal subsemigroup of G generated by all elements of G of form ba^{-1}, where $a \geq b$ and $a,\ b \in M$, must be contained in the positive cone associated with $\leq'$. We introduce the following notation. Let H be a group and let A be a subset of H. The normal subsemigroup of H generated by A will be denoted by $S_H(A)$. It is easy to see that

$$S_H(A) = \left\{ h \in H \;\middle|\; h = \prod_{i=0}^{n} x_i^{-1} a_i x_i;\ x_i \in H,\ a_i \in A,\ n > 0 \right. .$$

The normal subsemigroup $S_G(A)$, where

$$A = \left\{ ba^{-1} \in G \;\middle|\; a,\ b \in M;\ a \leq b \right\},$$

is denoted by $P_G(\leq)$. We shall need the following

Theorem 1.2. ([11; Thm.1, p.34]) A partial order P on a group G can be extended to a full order if and only if for each finite set of elements $\{a_1,\ldots,a_n\}$ of G, the signs $\varepsilon_1,\ldots,\varepsilon_n$ ($\varepsilon_i = 1$ or -1) can be chosen so that

$$P \cap S_G(a_1^{\varepsilon_1},\ldots,a_n^{\varepsilon_n}) = \emptyset.$$

Putting Proposition 1.1 and Theorem 1.2 together we obtain the following result:

Theorem 1.3. Let M be a monoid with a full order $\leq$ and assume that G is a group of fractions of M. Then $\leq$ can be extended to a full order on G if and only if

(a) $P_G(\leq) \cap (P_G(\leq))^{-1} = \emptyset$ and

(b) for every finite set of elements $\{a_1,\ldots,a_n\}$ of G the signs ε_i can be chosen so that

$$P_G(\leq) \cap S_G(a_1^{\varepsilon_1},\ldots,a_n^{\varepsilon_n}) = \emptyset.$$

The groups of fractions of a monoid M are the objects of a category, $\mathcal{G}_M$, whose morphisms are defined as follows. Let G_1 and G_2 be groups of fractions of M with embeddings α_i of M into G_i. A morphism $G_1 \to G_2$ in $\mathcal{G}_M$ is a group homomorphism of G_1 into G_2 which makes the diagram

$$\begin{tikzcd} & G_1 \arrow[d] \\ M \arrow[ur, "\alpha_1"] \arrow[r, "\alpha_2"'] & G_2 \end{tikzcd}$$

commute. It is clear that such a homomorphism is surjective and there is at most one morphism in $\mathcal{G}_M$ between G_1 and G_2. We shall prove

Theorem 1.4. Let M be a monoid with a full order $\leq$ and assume that G_1 and G_2 are groups of fractions of M. Assume further that we have a morphism, say ϕ, between G_1 and G_2 in $\mathcal{G}_M$. If $\leq$ extends to a full order on G_2 then it can also be extended to a full order on G_1.

First we verify the

Lemma. Let M, G_1, G_2, $\leq$ and ϕ be as in the statement of the theorem. Then

(i) $S_{G_1}(A)^{\phi} = S_{G_2}(A^{\phi})$ for any subset A of G_1,

(ii) $(P_{G_1}(\leq)\setminus\{1\})^{\phi} = P_{G_2}(\leq)\setminus\{1\}$.

Proof. (i) is obvious.

(ii). Put $A_i = \{ba^{-1} \in G_i \mid a, b \in M, a \leq b\}$; since both G_1 and G_2 are groups of fractions of M we have $A_1^{\phi} = A_2$. Now by definition $P_{G_1}(\leq) = S_{G_1}(A_1)$ and hence from (i) it follows that

$$P_{G_1}(\leq)^{\phi} = P_{G_2}(\leq).$$

It remains to verify that $g \in P_{G_1}(\leq) \cap \ker \phi$ implies that $g = 1_{G_1}$. Assume that

$$g = \prod x_i^{-1} b_i a_i^{-1} x_i ,$$

where $x_i \in G_1$, a_i, $b_i \in M$ and $a_i \leq b_i$; then

$$g^{\phi} = \prod (x_i^{\phi})^{-1} b_i a_i^{-1} x_i^{\phi}.$$

Now it is easy to see that $g^{\phi} > 1_{G_2}$ unless $a_i = b_i$ for all i in which case $g = 1_{G_1}$.

Proof of Theorem 1.4. We verify conditions (a) and (b) of Proposition 1.3 for G_1 exploiting that, by hypothesis, these conditions are satisfied by G_2.

(a). Suppose that $g \in P_{G_1}(\leq) \cap P_{G_1}(\leq)^{-1}$; it follows that $g^\phi \in P_{G_2}(\leq) \cap P_{G_2}(\leq)^{-1}$. By hypothesis $\leq$ extends to a full order on G_2, hence $g^\phi = 1_{G_2}$ and by (ii) of the above lemma we deduce that $g = 1_{G_1}$.

(b). Let $a_1, \ldots, a_n$ be elements of G_1. We have to find signs $\varepsilon_i = 1$ or -1 such that

$$P_{G_1}(\leq) \cap S_{G_1}(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) = \emptyset. \tag{1}$$

Fix $\varepsilon_1, \ldots, \varepsilon_n$; from the lemma we know that

$$\left\{ \begin{array}{l} \left(P_{G_1}(\leq) \cap S_{G_1}(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) \right)^\phi \\ \subseteq P_{G_2}(\leq) \cap S_{G_2}((a_1^\phi)^{\varepsilon_1}, \ldots, (a_n^\phi)^{\varepsilon_n}) . \end{array} \right. \tag{2}$$

Since G_2 satisfies (b), for some choice of $\varepsilon_1, \ldots, \varepsilon_n$ we have

$$P_{G_2}(\leq) \cap S_{G_2}((a_1^\phi)^{\varepsilon_1}, \ldots, (a_n^\phi)^{\varepsilon_n}) = \emptyset$$

and (2) shows that for this choice (1) holds.

<u>Corollary</u>. Let M be a monoid with a full order and assume that the universal group of M is a group of fractions of M. Then $\leq$ extends to the universal group if and only if it can be extended to a group of fractions.

<u>Proof</u>. The universal group of M is an initial object of $\mathcal{G}_M$; thus the assertion is an immediate consequence of the theorem.

The following example demonstrates that a full order on a monoid M need not be extendible to a group of fractions G of M. However, G is not the universal group in the example; it would be interesting to know whether or not a full order on a monoid can always be extended to its universal group, provided the latter is a group of fractions of the monoid.

Example 1.5. Let S be the free monoid on x and y; as is well-known the elements x, yx and y^2x are free in the submonoid S' of S generated by them. Further the free group F on x and y is a group of fractions of S'. Order S' lexicographically so that

$$1 < yx < x < y^2x.$$

Now if $\leq$ were extendable to F we would have

$$y < 1 < y^2$$

which is impossible.

Next we strengthen Theorem 1.4,Corollary for free monoids and then apply the obtained result to discuss two examples. The assertion to be used is an easy consequence of Theorem 3.4 of [17] which we state in a slightly different form.

Theorem 1.6. Let H be an ordered group and let $\phi : F \rightarrow H$ be a group epimorphism where F is a free group. Then F can be ordered so that ϕ is an o-epimorphism.

Corollary. Let S be the free monoid and F the free group on a set X. Let H be a group of fractions of S and denote by ϕ the group epimorphism which makes the diagram

$$\begin{tikzcd}[row sep=small] & F \arrow[dd, "\phi"] \\ S \arrow[ur] \arrow[dr] & \\ & H \end{tikzcd}$$

commute. Assume that $\leq$ is a total order on S and further that $\leq$ extends to a full order on H. Then $\leq$ can be extended to a full order on F so that ϕ is an o-epimorphism. Moreover, if there exist full orderings of H which agree on S then there exist distinct orderings of F which also agree on S.

In the following two examples we use some elementary results on ordered rings; for the definitions the reader is referred to section 3 below while references to theorems etc. are given in the text.

Example 1.7. In [15] Moufang has defined a full order on the free metabelian group H on x and y and has shown further that the submonoid S of H generated by x and y is free, that is, H is a group of fractions of the free monoid on x and y. We shall present Moufang's method of ordering H and show that it leads to an infinite family of distinct full orders on H which all agree on S. We then use Theorem 1.6, Corollary to lift these orderings to F.

First we need a couple of lemmas.

Lemma 1.8. Let M be a left Ore monoid with a full order $\leq$ and let G be the group of quotients of M. Then $\leq$ can be extended to G in a unique way.

Proof. Every element of G is of form $b^{-1}a$, where $a, b \in M$. It is a straightforward exercise to verify that

$$b^{-1}a \leq 1 \quad \text{if and only if} \quad a \leq b \quad (a, b \in M)$$

is the required ordering of G.

Lemma 1.9. Let G be a group with a full order $\leq$ and let α be an endomorphism of G. Write G_1 for the extension of G by $\mathbb{Z}$ obtained by adjoining x to G with the commutation formula

$$xg=g^{\alpha}x \qquad \text{for all } g\in G.$$

Then $\leq$ can be extended to a full order on G_1 if and only if α is order-preserving.

Proof. If α is not an o-endomorphism we can find $g\in G$ such that $g>1$ but $g^{\alpha}<1$. Then $gx>x$. On the other hand $gx=xg^{\alpha}$ and $g^{\alpha}<1$ imply that $gx<x$, a contradiction. To prove the converse it will suffice to verify that $\leq$ can be extended to the submonoid M of G_1 generated by G and x. For M is a left Ore monoid with G_1 as its group of quotients and by Lemma 1.8 any full order on M can be extended to G_1. Now every element of M can be written uniquely in normal form as follows:

$$gx^i \qquad (i\in\mathbb{N},\ g\in G).$$

We define the extension $\leq'$ of $\leq$ to M by putting

$$gx^i <' hx^j \qquad \text{if} \quad \begin{cases} i>j & \text{or} \\ i=j \text{ and } g>h. \end{cases}$$

Using the commutation formula it is easy to verify that $\leq'$ is the required extension of $\leq$.

To construct the free metabelian group on x and y consider first the free abelian group A on the symbols $u^{a^ib^j}$, $i,j\in\mathbb{Z}$. Write

$$u^{-a^ib^j} \quad \text{for} \quad \left(u^{a^ib^j}\right)^{-1}$$

and

$$u^{(\varepsilon_1 a^ib^j+\varepsilon_2 a^kb^l)} \quad \text{for} \quad u^{\varepsilon_1 a^ib^j} u^{\varepsilon_2 a^kb^l}$$

where $\varepsilon_i=1$ or -1 $(i=1,2)$. It is clear that every element of A can be represented as follows: u^{ψ}, $\psi\in\mathbb{Z}[a,a^{-1},b,b^{-1}]$, and conversely, every such expression gives rise to a unique element of A. Let $\preceq$ be any full order on the ring $\mathbb{Z}[a,a^{-1},b,b^{-1}]$; we order A by putting

$$u^{\psi}\geq 1 \quad \text{if} \quad \psi\succeq 0 \qquad (\psi\in\mathbb{Z}[a,a^{-1},b,b^{-1}]).$$

Assume now that $\preceq$ satisfies the following conditions:

$$1 \succ a\mathbb{N}[a,b] + b\mathbb{N}[a,b], \tag{3}$$

where $\mathbb{N}[a,b]$ denotes the subsemiring of $\mathbb{Z}[a,a^{-1},b,b^{-1}]$ generated by 1, a, b, and

$$a\succ 0, \quad b\succ 0. \tag{4}$$

Then the automorphism α of A induced by

$$u^{a^ib^j}\longmapsto u^{a^{i+1}b^j}$$

is easily seen to be order-preserving. By Lemma 1.9 the extension G of A by $\mathbb{Z}$ obtained by adjoining x to A with the commutation formula

$$xu^{\psi}=(u^{\psi})^{\alpha}x=u^{\psi a}x$$

can be ordered so as to extend $\leq$. Explicitly:

$$u^{\psi}x^i \geq 1 \quad \text{if} \quad \begin{cases} i>0 \text{ or} \\ i=0 \text{ and } \psi\succeq 0, \end{cases} \quad (i\in\mathbb{Z}).$$

Consider now the automorphism β of G induced by

$$u^{a^ib^j} \longmapsto u^{a^ib^{j+1}} \quad \text{and} \quad x \longmapsto ux;$$

that β is indeed an automorphism can be checked directly using the normal form for elements of G. Furthermore β is order-preserving and consequently the group H, obtained by adjoining y to G with the commutation formulae

$$yu^{\Psi}=(u^{\Psi})^{\beta}=u^{\Psi b}y \quad \text{and} \quad yx=x^{\beta}y=uxy,$$

can be ordered setting

$$u^{\Psi}x^iy^j \geq 1 \quad \text{if} \quad \begin{cases} j>0 \text{ or} \\ j=0 \text{ and } i>0 \text{ or} \\ j=0 \text{ and } i=0 \text{ and } \Psi \succcurlyeq 0 \end{cases} \qquad (i,j\in\mathbb{Z}).$$

Notice that in H we have $u=xyx^{-1}y^{-1}$. Furthermore

$$u^ax=xu=xxyx^{-1}y^{-1}=x(xyx^{-1}y^{-1})x^{-1}x$$

so by cancellation we find that $u^a=xux^{-1}$. Applying a similar argument and induction one can show that

$$u^{a^ib^j}=x^iy^ju(x^iy^j)^{-1} \qquad \text{for all } i,\ j\in\mathbb{Z}.$$

Moreover $(u^{\Psi},u^{\Phi})=1$ for all $\Phi,\Psi\in\mathbb{Z}[a,a^{-1},b,b^{-1}]$. To sum up: H is a metabelian group generated by x and y with commutation formula

$$y^jx^i=u^{p_iq_j}x^iy^j \qquad (i,\ j\in\mathbb{N}),$$

where

$$p_i=1+a+\ldots+a^{i-1} \text{ if } i>0 \quad \text{and} \quad p_o=0,$$

$$q_j=1+b+\ldots+b^{j-1} \text{ if } j>0 \quad \text{and} \quad q_o=0,$$

(cf. [15; p.204]). It is not hard to see that, in fact, H is the free metabelian group on x and y. Elements of H can be put in a unique normal form:

$$u^{\psi}x^{i}y^{j} \qquad (\psi \in \mathbb{Z}[a,a^{-1},b,b^{-1}];\ i,\ j \in \mathbb{Z})$$

and $\leq$ is given by

$$u^{\psi}x^{i}y^{j} \leq u^{\chi}x^{m}y^{n} \quad \text{if} \quad \begin{cases} j<n \text{ or} \\ j=n \text{ and } i<m \text{ or} \\ j=n \text{ and } i=m \text{ and } \psi \preceq \chi . \end{cases}$$

Denote by S the submonoid of H generated by x and y; Moufang has shown that S is free on x and y (cf. [15; §1]). We claim that all the orderings of H obtained, as described above, from an ordering $\preceq$ of $\mathbb{Z}[a,a^{-1},b,b^{-1}]$ which satisfies (3) and (4), agree on S. Let s_1 be an element of S; in word-form s_1 can be written as follows:

$$s_1=y^{m_1}x^{n_1}y^{m_2}x^{n_2}\dots y^{m_t}x^{n_t},$$

where $t\geq 0$, m_i, $n_j>0$ except possibly m_1 and n_t. Bringing s_1 to normal form in H we find

$$s_1=u^{\psi}x^{\sum m_i}y^{\sum n_i},$$

where

$$\begin{cases} \psi=q_{m_1}p_{n_1}+a^{n_1}q_{m_1+m_2}p_{n_2}+\dots \\ \quad +a^{(n_1+\dots n_{t-1})}q_{(m_1+\dots m_t)}p_{n_t} \end{cases} \tag{5}$$

(cf. 15; p.204). Let s_2 be another element of S and suppose

$$s_2 = y^{m_1'} x^{n_1'} y^{m_2'} x^{n_2'} \ldots y^{m_{t'}'} x^{n_{t'}'}, \quad t' \geq 1.$$

Thus in normal form we have:

$$s_2 = u^{\Psi'} x^{\Sigma m_i'} y^{\Sigma n_i'},$$

where

$$\begin{cases} \Psi' = q_{m_1'} p_{n_1'} + a^{n_1'} q_{m_1'+n_1'} p_{n_2'} + \ldots \\ \qquad + a^{(n_1'+\ldots+n_{t-1}')} q_{(m_1'+\ldots+m_t')} p_{n_{t'}'} . \end{cases} \tag{6}$$

Clearly, if $\sum^{t} m_i \neq \sum^{t'} m_i'$ or $\sum^{t} n_i = \sum^{t'} n_i'$ then whether $s_1 < s_2$ or $s_2 < s_1$ does not depend on $\precsim$. Assume therefore that $\sum^{t} m_i = \sum^{t'} m_i'$ and $\sum^{t} n_i = \sum^{t'} n_i'$; we may also assume withot loss of generality that s_1 and s_2 begin with different symbols in word-form, say $m_1 \neq 0$, $m_1' = 0$ and $n_1' \neq 0$. It follows then that $n_1 \neq 0$. Now $q_{m_1} p_{n_1} = 1 + \chi$, where $\chi \in a\mathbb{N}[a,b] + b\mathbb{N}[a,b]$, and $q_{m_1'} p_{n_1'} = q_o p_{n_1'} = 0$. Inspecting (5) and (6) we find that $\Psi \in 1 + a\mathbb{N}[a,b] + b\mathbb{N}[a,b]$ while $\Psi' \in a\mathbb{N}[a,b] + b\mathbb{N}[a,b]$. Hence $\Psi \succ \Psi'$, by (3), and so $s_2 < s_1$. Thus, considering s_1 and s_2 in word-form, we have

$$s_2 < s_1 \iff \begin{cases} \sum^{t'} n_i' < \sum^{t} n_i \quad \text{or} \\ \sum^{t'} n_i' = \sum^{t} n_i \quad \text{and} \quad \sum^{t'} m_i' < \sum^{t} m_i \quad \text{or} \\ \sum^{t'} n_i' = \sum^{t} n_i \quad \text{and} \quad \sum^{t'} m_i' = \sum^{t} m_i \\ \text{and the first symbol, in which} \\ s_1 \text{ and } s_2 \text{ differ, is } x, \end{cases}$$

which proves the claim.

Distinct orderings of $\mathbb{Z}[a,a^{-1},b,b^{-1}]$ satisfying (3) and (4) give rise to distinct orderings of H which, as we have seen, all agree on S. Further, by Theorem 1.6, Corollary, these orderings lift to distinct full orders on the free group F on x and y. In consequence: to verify that $\leq$, restricted to S, can be extended to F in an infinite number of ways it will suffice to find an infinite number of distinct orderings of $\mathbb{Z}[a,a^{-1},b,b^{-1}]$ which satisfy (3) and (4). These can be obtained as follows. Let C be the free abelian group on a and b and let ϱ and σ be non-zero real numbers. Order C by putting

$$a^m b^n \overset{\varrho,\sigma}{\geqq} 1 \quad \text{if} \quad \begin{cases} \varrho m+\sigma n > 0 & \text{or} \\ \varrho m+\sigma n = 0 \text{ and } \varrho m \geq 0. \end{cases}$$

Set $\varrho=1$ and let σ run through the positive natural numbers. Then $\{\overset{1,\sigma}{\leq}\}_{\sigma\in\mathbb{N}^+}$ is a family of distinct orderings of C and furthermore for each $\sigma\in\mathbb{N}^+$ we have

$$a^m b^n \overset{1,\sigma}{>} 1 \qquad (m,\ n \geq 0 \text{ but not } m=n=0). \quad (7)$$

Now $\mathbb{Z}[a,a^{-1},b,b^{-1}]$ is just the group algebra $\mathbb{Z}C$ and for each $\sigma\in\mathbb{N}^+$ we can define a full order $\overset{1,\sigma}{\preceq}$ on $\mathbb{Z}C$ by setting

$$\sum_{i=1}^{r} z_i c_i \overset{1,\sigma}{\succ} 0 \ (0\neq z_i\in\mathbb{Z},\ c_i\in C,\ c_1 \overset{1,\sigma}{<} \dots \overset{1,\sigma}{<} c_r) \quad \text{if} \quad z_1 > 0.$$

From this definition and (7) it follows that for every $\sigma\in\mathbb{N}^+$ $\overset{1,\sigma}{\preceq}$ satisfies (3) and (4) as required.

Next we apply Theorem 1.6, Corollary to show that the lexicographic ordering of the free monoid on two generators can be extended to the free group on two generators.

Example 1.10. Let K be a field with a full order $\leq$. We order the polynomial ring $K[t]$ as follows:

$$\sum_{i=0}^{n} \alpha_i t^i \;'> 0 \quad \text{if} \quad \alpha_n > 0 \qquad (\alpha_i \in K) \tag{8}$$

and then extend $\leq'$ to $K(t)$ by putting

$$fg^{-1} \;'> 0 \quad \text{if} \quad fg \;'> 0 \qquad (f, 0 \neq g \in K[t]),$$

(cf. Proposition 3.1 below). The endomorphism σ of $K(t)$ induced by $t \mapsto t^2$ is orderpreserving, hence $\leq'$ can be extended to the skew polynomial ring $R = K(t)[x;\sigma]$ by putting

$$\sum_{i=0}^{n} x^i q_i \;'> 0 \quad \text{if} \quad q_n \;'> 0 \qquad (q_i \in K(t)) \tag{9}$$

(cf. Proposition 3.1 below). Finally, we extend $\leq'$ to the field of quotients D of R:

$$r_1 r_2^{-1} \;'\geq 0 \text{ if } r_1 r_2 \;'\geq 0 \qquad (r_1, 0 \neq r_2 \in R).$$

J.L. Fisher has shown in [10] that the subalgebra of R generated by x and xt over k is free (on x and xt). It follows that the submonoid S of R^x generated by x and xt is also free. Further, $x \;'> 0$ and $xt \;'> 0$ so $\leq'$ orders S as a monoid. We claim that $\leq'$, restricted to S, is the lexicographic order on S with $x <' xt$. If this is so, the lexicographic ordering of S can be extended to the free group on two generators since $\leq'$ extends to a group of fractions of S, namely the subgroup of D^x generated by x and xt. Note that it is essential here that x and xt should be positive with respect to $<'$. Let u be an element of S; then u is a word in x and xt and so we can write $u=\alpha_1\alpha_2\cdots\alpha_n$, where $\alpha_i=x$ or xt for all $i=1,\dots,n$. As an element of R, u has a normal form which can be obtained by repeated applications of the commutation formula $tx=xt^2$. Thus in normal form we have

$$u=x^n t^i,$$

where $n=\deg_x(u)=l(u)$, the length of u, and $i\in\mathbb{N}$. It is easy to check that if $\alpha_1=\dots=\alpha_r=x$ and $\alpha_{r+1}=xt$, $0\leq r\leq n-1$, then

$$2^{n-(r+1)}\leq i\leq\sum_{j=1}^{n-(r+1)}2^j = 2^{n-r}-1 < 2^{n-r} \qquad (10)$$

Let v be another element of **S** and suppose that in normal form

$$v=x^m t^{i'}.$$

If $l(u)< l(v)$ then $\deg_x(u)=n< m=\deg_x(v)$ and from (9) we deduce that $u<' v$. Suppose now that $l(u)=l(v)$ and $u\neq v$; then for some $1\leq s\leq n$ we have

$$u=\alpha_1\cdots\alpha_{s-1}\,x\alpha_{s+1}\cdots\alpha_n,$$

$$v=\alpha_1\cdots\alpha_{s-1}\,xt\,\beta_{s+1}\cdots\beta_n.$$

To prove that $\leq'$ is the lexicographic ordering of S we have to verify that $u<' v$. Without loss of generality we may assume that s=1. It follows from (10) that

$$i\geq 2^{n-1}> i'.$$

Clearly $u <' v$ if and only if $v-u=x^n(t^i-t^{i'}) '> 0$ and by (9) this is equivalent to $t^i-t^{i'} '> 0$. Now (8) shows that this is the case precisely when $i > i'$ and hence $u <' v$.

2.2 An ordered monoid which cannot be embedded into a group

In [13] Malcev gave a necessary condition for a cancellation semigroup to be embeddable in a group and constructed a cancellation monoid S which does not satisfy this condition. We shall order a submonoid of S which still does not satisfy Malcev's condition and thus obtain an ordered monoid with no groups of fractions.

Let

$$A_1=\{a,b\},\ A_2=\{c,d\},\ A_3=\{x,y\},\ A_4=\{u,v\}$$

and put $B_1=A_1 \cup A_2$, $B_2=A_3 \cup A_4$. Denote by S the monoid generated by $X=B_1 \cup B_2$ with defining relations

$$ax=by, \quad cx=dy \quad \text{and} \quad au=bv. \tag{11}$$

Let F be the free monoid on X; S can be obtained from F by factoring out the congruence, say $\sim$, generated by the above relatins. Let w_1 and w_2 be elements of F; then $w_1 \sim w_2$ if and only if w_1 can be transformed into w_2 using the relations given under (11). The congruence class of $w \in F$ will be denoted by $\overline{w}$, thus $\overline{w} \in S$. It is easy to see that no transformation can affect the length of w; we put $l(\overline{w})=l(w)$.

Order F lexicographically stipulating

$$a < b < c < d < y < v < x < u.$$

For each $w \in F$, $\overline{w}$ contains only a finite number of elements of F and consequently has a least element which we denote by w_o and which is said to be in minimal form in $\overline{w}$. Suppose $w = \alpha_1 \alpha_2 \ldots \alpha_n$, $\alpha_i \in X$; w_o can be obtained from w by performing all possible transformations

$$by \mapsto ax, \quad dy \mapsto cx, \quad bv \mapsto au$$

on w (e.g. if $\alpha_i = b$ and $\alpha_{i+1} = y$, replace $\alpha_i \alpha_{i+1}$ by ax). Moreover these transformations can be made in any order. For let $0 \leq i < n$; then $\alpha_i \alpha_{i+1}$ can be transformed only if $\alpha_i \in B_1$ and $\alpha_{i+1} \in B_2$. It follows that only at most one of $\alpha_i \alpha_{i+1}$ and $\alpha_{i+1} \alpha_{i+2}$ can be transformed. In consequence: a transformation on $\alpha_i \alpha_{i+1}$ does not affect possible transformations on $\alpha_j \alpha_{j+1}$ where $i \neq j$. Define a relation $\preceq$ on S by putting

$$\overline{w} \preceq \overline{z} \quad \text{if} \quad w_o \leq z_o.$$

It is easy to see that $\preceq$ totally orders S as a set; we shall next prove that $\preceq$ is preserved by multiplication on the right, that is,

$$\overline{w} \preceq \overline{z} \implies \overline{wt} \preceq \overline{zt} \quad \text{for all } t \in S.$$

This is obvious if $l(w) \neq l(z)$, assume therefore that $l(w) = l(z) = n$. Without loss of generality we may assume that w and z are in minimal form and further that $t = \gamma \in X$. Notice that wt is either in minimal form or can be brought to minimal form by a single transformation on $\alpha_n\gamma$, where α_n denotes the last symbol of w. The same argument applies to zt. Thus it suffices to show that

$$\bar{\alpha} \prec \bar{\beta} \Rightarrow \bar{\alpha}\bar{\gamma} \prec \bar{\beta}\bar{\gamma} \quad (\alpha, \beta, \gamma \in X),$$

or equivalently:

$$\alpha < \beta \Rightarrow (\alpha\gamma)_o < (\beta\gamma)_o \quad \text{for all } \gamma \in X. \qquad (12)$$

We have the following cases to consider:

Case 1. $\alpha \in B_2$,

Case 2. $\alpha \in A_2$,

Case 3. $\alpha \in A_1$.

In the first case we have that $\beta \in B_2$ since $\alpha < \beta$. Then no transformation can be performed on $\alpha\beta$ and $\beta\gamma$ and hence $(\alpha\gamma)_o = \alpha\gamma < \beta\gamma = (\beta\gamma)_o$. In case 2 $\beta \notin A_1$ since $\alpha < \beta$. Given that $\beta \in B_2$, $\beta\gamma$ cannot be reduced and so (12) holds. If $\beta \in A_2$, the only possibility is $\alpha = c$, $\beta = d$. No transformation can reduce $c\gamma$ and $d\gamma$ can only be reduced if $\gamma = y$. Then

$$(\alpha\gamma)_o = (cy)_o = cy < cx = (dy)_o = (\beta\gamma)_o$$

which proves (12). Finally, in case 3, if $\beta \notin A_1$ then (12) easily follows. Suppose $\beta \in A_1$, then $\alpha = a$ and $\beta = b$ and hence $\beta\gamma$ can only be decreased only if $\gamma = y$ in which case

$$(\alpha\gamma)_o = (ay)_o = ay < ax = (by)_o = (\beta\gamma)_o.$$

This completes the proof of (12)

The following example shows that $\preceq$ is not compatible with multiplication on the left. We have $\overline{a} \prec \overline{y}$ while $\overline{b}\ \overline{a} \succ \overline{b}\ \overline{y}$ since

$$(ba)_o = ba \succ ax = (by)_o.$$

We can however restrict ourselves to a submonoid of S in which this cannot occur. Put

$$S' = \left\{ \overline{w} \in S \;\middle|\; w = \alpha_1 \ldots \alpha_n,\ \alpha_1 \notin B_1 \right\};$$

the defining relations of S show that S' is well-defined as a set and it is easy to see that S' is, in fact, a submonoid of S. Furthermore S' does not satisfy Malcev's condition (cf. [13; p.687]) since in S' we have

$$\overline{xa}\ \overline{x} = \overline{xb}\ \overline{y},$$
$$\overline{xc}\ \overline{x} = \overline{xd}\ \overline{y},$$
$$\overline{xa}\ \overline{u} = \overline{xb}\ \overline{v}$$

but $\overline{xc}\ \overline{u} \neq \overline{xd}\ \overline{v}$; hence S' cannot be embedded in a group. We have seen that $\preceq$ is preserved by right multiplication in S, hence also in S'. Thus to prove that $\preceq$ fully orders S' it remains to verify that

$$\overline{w} \prec \overline{z} \;\Rightarrow\; \overline{tw} \prec \overline{tz} \quad (\overline{w},\ \overline{z},\ \overline{t} \in S'). \quad (13)$$

Again, we may restrict ourselves to $w=\alpha$, $z=\beta$, $t=\gamma$ $(\alpha, \beta, \gamma \in X)$ but now we also have $\alpha, \beta \notin B_1$ since $\overline{w}, \overline{z} \in S'$. If $\gamma \in B_2 \cup \{a,c\}$ then both $\gamma\alpha$ and $\gamma\beta$ are in minimal form so (13) holds then. The remaining possibilities are $\gamma=b$ and $\gamma=d$. Assume first that $\gamma=b$ and recall that $\overline{\alpha} \prec \overline{\beta}$; it follows that $\beta \neq y$. Now $\gamma\beta = b\beta$ can only be decreased if $\beta = v$ in which case α has to be y. Hence

$$(\gamma\alpha)_0=(by)_0=ax < au=(bv)_0=(\gamma\beta)_0$$

and so (13) follows. Assume now that $\gamma=d$; then $\gamma\beta=d\beta$ cannot be reduced at all since $\beta=y$ is excluded by the assumptions $\alpha, \beta \in B_1$ and $\bar{\alpha} \prec \bar{\beta}$. Hence $\bar{\gamma}\bar{\alpha} \prec \bar{\gamma}\bar{\beta}$ and this completes the proof.

2.3 Partial order on rings and fields

A ring R is said to be partially ordered by a relation $\leq$ if

R1. R is a partially ordered set under $\leq$,

R2. $a \leq b \implies a+c \leq b+c$ for all $c \in R$,

R3. $a \leq b$ and $c > 0 \implies ac \leq bc$ and $ca \leq cb$

$(a, b, c \in R)$.

If in addition, $\leq$ totally orders R as a set we say that R is (fully, totally) ordered by $\leq$. Thus by an ordering of R we understand a total order on R as a set which respects addition and multiplication by a positive element. Let R be a ring, partially ordered by $\leq$ and set $P=\{p \in R \mid p \geq 0\}$; then P satisfies the following conditions:

P1. $P \cap -P=0$, where $-P=\{a \in R \mid -a \in P\}$,

P2. $P+P \subseteq P$,

P3. $PP \subseteq P$.

From P1 and P2 it follows that if r and s are non-zero elements of P then $r+s \neq 0$. Hence the above conditions imply that P is a conical semiring: it is called the positive cone associated with $\leq$. Conversely, let P be a conical subsemiring of R and set

$$a \leq b \text{ if } b-a \in P \qquad (a, b \in R).$$

Then $\leq$ is a partial order on R with associated positive cone P and, further, $\leq$ is a full order on R if and only if

P4. $P \cup -P = R$

in which case we say that P is a <u>total</u> cone over R. The above correspondence between partial orders on R and conic subsemirings of R is bijective and this justifies identifying a partial order on a ring with its associated positive cone.

An order-preserving homomorphism of partially ordered rings is called an o-homomorphism. Let R_1, R_2 be rings with partial orders $\leq^1$, $\leq^2$ respectively and let f be a homomorphism of R_1 into R_2; then f is an o-homomorphism if and only if

$$0 \leq^1 a \Rightarrow 0 \leq^2 a^f \qquad (a \in R_1).$$

If f is an o-embedding we also say that $\leq^2$ extends $\leq^1$. We can now prove the analogue of Lemma 1.9:

<u>Proposition 3.1</u>. Let K be a field with a full order $\leq$ and let $\alpha \in \mathrm{End}K$. Then $\leq$ can be extended to $K(x;\alpha)$ if and only if α is an o-endomorphism.

<u>Proof</u>. Assume first that α is not order-preserving. Then we can find an element a of K such that $a > 0$ and $a^\alpha < 0$. It follows that ax and xa^α must have different signs but the commutation rule in $K(x;\alpha)$ implies that $ax = xa^\alpha$ and hence $\leq$ cannot be extended to $K(x;\alpha)$. Conversely, assume that α is an o-endomorphism; it will suffice to show that $\leq$ can be extended to the skew

polynomial ring $R=K[x;\alpha]$. For R is a right Ore domain and every full order on R can be extended to its field of quotients, $K(x;\alpha)$, by Theorem 3 of [(11; p.109].
An extension $\leq'$ of $\leq$ to R is defined as follows: every non-zero element f of R is of form

$$f=\sum_{i=i_o}^{n} x^i a_i \qquad (a_i \in K,\ a_n \neq 0,\ a_{i_o} \neq 0)$$

and we put

$$f\ '> 0 \quad \text{if} \quad a_n > 0.$$

It is straightforward to verify that $<'$ is a full order on R which extends $\leq$. We note that another extension of $\leq$ to R, and hence to $K(x;\alpha)$, can be defined by setting

$$f\ ,> 0 \quad \text{if} \quad a_{i_o} > 0.$$

Let K be a field; we define a subset, S(K), of K as follows:

$$S(K)=\left\{ a \in K \ \middle|\ a=\sum_i a_{i_1}^2 \cdots a_{i_{n(i)}}^2,\ a_i \in K^x \right\}.$$

A partial order, with associated positive cone P is said to be <u>square-positive</u> if $S(K) \subseteq P$. Every full order on a field is square-positive, as is easily checked. In the remainder of this section we prove two basic results on orderings of fields which can be found e.g. in [11; Ch.VII., §2]; we derive these results by a slightly different route based on the commutative case. First we need a couple of lemmas.

<u>Lemma 3.1</u>. Let K be a field and denote by $K^{x'}$ the deri-

ved group of the multiplicative group of K. Then $K^{\times\prime} \subseteq S(K)$.

Proof. Let a, $b \in K^{\times}$; we have

$$(a,b)=a^{-1}b^{-1}ab=a^{-2}(ab^{-1})^2b^2$$

and hence the assertion follows.

Lemma 3.2. Let K be a field and assume that P is the positive cone associated with a square-positive partial order on K. Then for each $a \in K \setminus (P \cup -P)$, the semiring $\langle P,a \rangle$ generated by P and a is conical.

Proof. If P is a total cone there is nothing to prove so assume that P is not total and let $a \in K \setminus (P \cup -P)$. Let us suppose that $\langle P,a \rangle$ is not conical; then we can find elements $0 \neq p_{i_j}$ and p in P so that

$$r = \sum_i p_{i_1} a p_{i_2} a \ldots a p_{i_{n(i)}} + p = 0.$$

Now P is square- positive and hence, by Lemma 3.1, $K^{\times\prime} \subset P$. Thus, inserting suitable commutators in the summands of r, the above relation can be rewritten in the following form:

$$r = \sum_i a^{m(i)} p_i + p = 0 \qquad (0 \neq p_i,\ p \in P,\ m(i) \in \mathbb{N}).$$

Moreover, even powers of a can be absorbed in the p_i's because P is square-positive and so we obtain

$$r = a \sum_i p_i' + p = 0 \qquad (0 \neq p_i',\ p \in P).$$

Since $\sum_i p_i'$ is non-zero it follows now that $p \neq 0$. Put $q = \sum_i p_i'$. Now $aq+p=0$ and so

$$-a=pq^{-1}=pq(q^{-1})^2 \in P$$

whence $a \in -P$, in contradiction with the assumption $a \notin P \cup -P$. We deduce that $\langle P,a \rangle$ is conical.

Theorem 3.3. Let K be a field and assume that P is the positive cone associated with a square-positive partial order on K. Then for each $a \in K \setminus (P \cup -P)$ there is a total cone over K which contains P and a.

Proof. Assume that P is not total and let $a \in K \setminus (P \cup -P)$. The above lemma shows that the set S of partial orders, containing P and a, is non-empty; by Zorn's lemma we may choose a maximal element of S, say P_1. Suppose P_1 is not total. We can then apply Lemma 4.2 again to enlarge P_1 which contradicts the maximality of P_1. Hence P_1 is total and this proves the theorem.

Putting $P=S(K)$ in the statement of the theorem we obtain the

Corollary. ([11; Corollary 11, p.117]) A field K is orderable if and only if S(K) is conical.

2.4 The Dieudonné determinant

Let K be a field. Recall that the Dieudonné determinant is a homomorphism

$$\det : GL(K) \to K^{xab},$$

where K^{xab} denotes the multiplicative group of K made abelian, whose restriction to $K^x=GL_1(K)$ is just the natural surjection. Let $A \in GL(K)$; then $\det A$ is a coset of the derived group $K^{x'}$ of K^x. We shall write $\overline{p}$

for $pK^{x'}$, where $p \in K^x$. Clearly, elementary matrices over K have determinant $\overline{1}$; in fact: ker(det)=E(K). Thus det can be viewed as an isomorphism $GL(K) \to K^{xab}$ and then the inverse of det is induced by the embedding $K^x \to GL(K)$.

We shall need to extend det to singular square matrices by adjoining 0 to K^{xab} with the obvious multiplication, stipulating $\det T=0$ for every singular square matrix T over K. The resulting abelian monoid is denoted by $\overline{K}$. The following properties of det are immediate consequences of those proved on p.153 of [1] , except D4 which is Theorem 4.5 of [1; p.157]:

D1. $\det E=\overline{1}$ for all $E \in E(R)$,

D2. $\det(AB)=\det(A\dotplus B)=\det A \det B$ for all $A, B \in M(R)$,

D3. det, restricted to $GL_1(K)=K^x$, is the natural map $K^x \to \overline{K}$,

D4. $\det(A \nabla B) \subseteq \det A+\det B = \{a+b \in K \mid a \in \det A,\ b \in \det B\}$, whenever the determinantal sum makes sense.

Now let K be a field with a square-positive partial order $\leq$ and associated positive cone P. Then $K^{x'} \subseteq P$ by Lemma 3.1 and hence

$$p \geq 0 \qquad \text{if and only if} \qquad \overline{p}=pK^{x'} \subset P$$

for every p in K. This justifies writing $\overline{p} \geq 0$ whenever $p \in P$. Thus if $\leq$ is a full order on K, for any square matrix A over K we have $\det A \geq 0$ or $\det A \leq 0$. The following lemma will be crucial in the definition of matrix cones.

Lemma 4.1. Let K be a field with a square-positive partial order $\leq$. Then

$$\det A \geq 0 \text{ and } \det B \geq 0 \implies \det(A \nabla B) \geq 0 \quad (A,\ B \in M(K)),$$

if the determinantal sum is defined.

<u>Proof</u>. The assertion follows by D4 since P is closed under addition.

<u>2.5. Full order on epic R-fields</u>

Let R be a ring and assume that $\alpha : R \longrightarrow K$ is an epic R-field. As we have seen in §2 of Chapter 1, given an element p of K we can find an admissible system

$$(A_0, A_*, A_n)^{\alpha} \begin{pmatrix} 1 \\ u \\ p \end{pmatrix} = 0$$

for p whence we can obtain the relation

$$(A_*, A_n)^{\alpha} \begin{pmatrix} I_n & u \\ 0 & p \end{pmatrix} = (A_*, -A_0)^{\alpha}, \tag{14}$$

where $(A_*, A_n) \in \Sigma_K$ and

$$(A_*, -A_0) \in \Sigma_K \iff p \neq 0.$$

Assume $p \neq 0$, on taking determinants in (14) we find that

$$\bar{p} = \det \begin{pmatrix} I_n & u \\ 0 & p \end{pmatrix} = \det((A_*, A_n)^{\alpha})^{-1} \det(A_*, -A_0)^{\alpha}.$$

When K is fully ordered, say by $\leq$, this implies that

$$p \geq 0 \iff \bar{p} \geq 0 \iff \det((A_*, A_n) \dagger (A_*, -A_0))^{\alpha} \geq 0.$$

This motivates the following definition. Let Π be a subset of M(R) and set $-\Pi = \{A \in M(R) \mid -1 \dagger A \in \Pi\}$, $\mathcal{P} = \Pi \cap -\Pi$ and $\Pi^+ = \Pi \setminus \mathcal{P}$. Then Π is called a <u>matrix</u>

<u>cone</u> if the following hold:

M1. $A, B \in \Pi \Rightarrow A \dotplus B \in \Pi$,

M2. $A, B \in \Pi \Rightarrow A \nabla B \in \Pi$, whenever the determinantal sum is defined,

M3. $E, E' \in E_n(R)$ and $A \in \Pi \cap R_n \Rightarrow EAE' \in \Pi$,

M4. Π contains all non-full square matrices over R,

M5. $E(R) \subseteq \Pi^{+}$,

M6. $A \in \mathcal{P} \Rightarrow A \dotplus C \in \Pi$ for all $C \in M(R)$,

M7. $P \dotplus Q \in \mathcal{P} \Rightarrow P \in \mathcal{P}$ or $Q \in \mathcal{P}$.

If in addition, Π satisfies

M8. $A \dotplus A \in \Pi$ for all $A \in M(R)$

then Π is said to be a <u>square-positive</u> matrix cone over R.

By a <u>matrix semicone</u> over R we understand a subset of M(R) which satisfies M1-M4. Matrix semicones are useful because, in contrast to matrix cones, every subset A of M(R) generates a matrix semicone consisting of all matrices which can be obtained from elements of A and non-full matrices by repeated operations $\dotplus$, ∇ and multiplication by elementary matrices over R.

We return to matrix cones.

<u>Lemma 5.1</u>. Let Π be a matrix cone over a ring R. Then $\mathcal{P} = \Pi \cap -\Pi$ is a prime matrix ideal.

<u>Proof</u>. The individual prime matrix axioms (see Chapter 1, §2) immediately follow from the matrix cone axioms. Let us verify 2, viz.

$$A, B \in \mathcal{P} \Rightarrow A \nabla B \in \mathcal{P}.$$

By assumption we have $A, B, -1 \dotplus A, -1 \dotplus B \in \Pi$ and so $A \nabla B$ and $-1 \dotplus (A \nabla B) = (-1 \dotplus A) \nabla (-1 \dotplus B)$ are in $\overset{\Pi}{\mathcal{P}}$ by M2.

Thus $A\nabla B\in\mathcal{P}$ by definition.

Let Π be a matrix cone; $\Pi\cap-\Pi$ is called the prime matrix ideal, or singular kernel, associated with Π. We note that the set of matrix cones over R with given associated singular kernel is partially ordered by inclusion; it is closed under unions of chains (and under any intersection), hence every matrix cone is contained in a maximal one by Zorn's lemma.

Next we prove some elementary properties of matrix cones.

Lemma 5.2. Let Π be a matrix cone over a ring R and let A, B, C, $D\in M(R)$. Then

(i) $A\in\Pi^+$ and $B\in\Pi^+ \Rightarrow A\oplus B\in\Pi^+$,

(ii) $A\in\Pi^+ \Rightarrow EAE'\in\Pi^+$ for any $E, E'\in E(R)$,

(iii) $A\in\Pi^+$ and $B\in\Pi \Rightarrow A\nabla B\in\Pi^+$,

(iv) $C\oplus D\in\Pi \Rightarrow \begin{pmatrix} C & 0 \\ C' & D\end{pmatrix}$ and $\begin{pmatrix} C & D' \\ 0 & D\end{pmatrix}\in\Pi$ for any matrices C' and D' of appropriate size,

(v) $C\oplus D\in\Pi^+ \Rightarrow \begin{pmatrix} C & 0 \\ C' & D\end{pmatrix}$ and $\begin{pmatrix} C & D' \\ 0 & D\end{pmatrix}\in\Pi^+$ for any matrices C' and D' of appropriate size.

Proof. (i). Let $A, B\in\Pi^+$; then $A\oplus B\in\Pi$ by M1 so it remains to show that $A\oplus B\notin-\Pi$. Assume on the contrary that $-1\oplus(A\oplus B)=(-1\oplus A)\oplus(-1\oplus B)\in\Pi$; thus $A\oplus B\in\mathcal{P}$ and hence $A\in\mathcal{P}$ or $B\in\mathcal{P}$ by M7. This contradicts the assumption $A, B\in\Pi^+$.

(ii). Let $A\in\Pi^+\cap R_n$ and $E, E'\in E_n(R)$. We have $EAE'\in\Pi$ by M3; it remains to verify that $EAE'\notin-\Pi$. Assume $EAE'\in-\Pi$; then $-1\oplus EAE'\in\Pi$ by definition. By M3 again it follows that

$$(1\dotplus E^{-1})(-1\dotplus EAE')(1\dotplus E'^{-1})=-1\dotplus A\in \Pi,$$

contradicting the assumption $A\in \Pi^+$.

(iii). Let $A\in\Pi^+$ and $B\in\Pi$ and assume that the determinantal sum $A\nabla B$ is defined. Without loss of generality we may assume that the determinantal sum is taken with respect to the first column. By M2 we have $A\nabla B\in\Pi$ so we only have to show that $A\nabla B\in -\Pi$. Let $A=(A_o,A_*)$, $B=(B_o,A_*)$ and assume that $A\nabla B=(A_o+B_o,A_*)\in -\Pi$. Now $1\dotplus B\in\Pi$ by M1 and M5, hence, by M5 again, $-1\dotplus(-B_o,A_*)$ is also in Π. Using M2 we find that

$$-1\dotplus A=(-1\dotplus(A_o+B_o,A_*))\ \nabla\ (-1\dotplus(-B_o,A_*))\in\Pi$$

which implies $A\in-\Pi$, a contradiction.

(iv). Let $C\in R_n$, $D\in R_m$ and denote the columns of C by $C_1,\dots,C_n$. Suppose that $C\dotplus D\in\Pi$. We show that

$$B' = \begin{pmatrix} C_1 & C_2 & \dots & C_n & 0 \\ C_o & 0 & \dots & 0 & D \end{pmatrix}$$

is in Π for any $C_o\in {}^mR$; for arbitrary $C'\in {}^mR^n$ the assertion can be proved by repeated application of a similar argument and $\begin{pmatrix} C & D' \\ 0 & D\end{pmatrix}\in\Pi$ follows by symmetry. We can write B' as the following determinantal sum:

$$B' = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \nabla \begin{pmatrix} 0 & C_2 & \dots & C_n & 0 \\ C_o & 0 & \dots & 0 & D \end{pmatrix}.$$

Now $C\dotplus D\in\Pi$ by hypothesis and the second matrix in the determinantal sum is clearly non-full, hence it is also in Π. By M2 we deduce that $B'\in\Pi$.

(v). The verificatin of the statement is similar to that of (iv), except that in the last step we use (iii) above instead of M2.

We have seen that if Π is a matrix cone over a ring R then $\mathcal{P} = \Pi \cap -\Pi$ is a prime matrix ideal; in consequence: there exists an epic R-field, say K, with singular kernel $\mathcal{P}$. We define a subset $P(\Pi)$ of K as follows:

$$P(\Pi)=\{p\in K \mid \exists \text{ an admissible matrix } (A_o, A_*, A_n) \text{ for p such that } (A_*, A_n) \nabla (A_*, -A_o) \in \Pi\}.$$

When Π is square-positive we have

<u>Theorem 5.3</u>. Let Π be a square-positive matrix cone over a ring R with associated singular kernel $\mathcal{P}$ and let K be the epic R-field with prime matrix ideal $\mathcal{P}$. Then $P(\Pi)$ is the positive cone of a square-positive partial order on K.

<u>Proof</u>. First we check P1-P3. To verify P1 and P2 it will clearly suffice to show that

$$p,\ q \in P(\Pi)\setminus\{0\} \implies p+q \in P(\Pi)\setminus\{0\}. \qquad (15)$$

We note that a non-zero element p of K is in $P(\Pi)$ if and only if there exists an admissible matrix (A_o, A_*, A_n) for p such that $(A_*, A_n)\nabla(A_*, -A_o) \in \Pi^+$. Thus given $p,\ q \in P(\Pi)\setminus\{0\}$ we can find admissible systems

$$(A_o, A_*, A_n)(1, u, p)^T=0, \qquad (16)$$

$$(B_o, B_*, B_m)(1, v, q)^T=0$$

for p anq respectively, so that $(A_*, A_n)\nabla(A_*, -A_o)$ and $(B_*, B_m)\nabla(B_*, -B_o)$ are in Π^+. Then

$$\begin{pmatrix} A_o & A_* & A_n & 0 & 0 \\ B_o & 0 & -B_m & B_* & B_m \end{pmatrix} (1,u,p,v,p+q)^T = 0$$

is a system admissible for p+q. We show that

$$D = \begin{pmatrix} A_* & A_n & 0 & 0 \\ 0 & -B_m & B_* & B_m \end{pmatrix} \dotplus \begin{pmatrix} A_* & A_n & 0 & -A_o \\ 0 & -B_m & B_* & -B_o \end{pmatrix}$$

is in $\mathfrak{M}^+$ which will prove that $p+q \in \mathcal{P}(\mathfrak{M}) \setminus \{0\}$. Write D as the determinantal sum with respect to the last column:

$$\begin{pmatrix} \begin{matrix} A_* & A_n & 0 & 0 \\ 0 & -B_m & B_* & B_m \end{matrix} & 0 \\ 0 & \begin{matrix} A_* & A_n & 0 & 0 \\ 0 & -B_m & B_* & -B_o \end{matrix} \end{pmatrix} \nabla \begin{pmatrix} \begin{matrix} A_* & A_n & 0 & 0 \\ 0 & -B_m & B_* & B_m \end{matrix} & 0 \\ 0 & \begin{matrix} A_* & A_n & 0 & -A_o \\ 0 & -B_m & B_* & 0 \end{matrix} \end{pmatrix}.$$

Put D_1 for the first and D_2 for the second matrix in the above determinantal sum. We claim that both D_1 and D_2 are E-associated to a matrix in $\mathfrak{M}^+$; by Lemma 5.2 (ii) it follows then that $D_1, D_2 \in \mathfrak{M}^+$ and hence, by Lemma 5.2 (iii), $D = D_1 \nabla D_2 \in \mathfrak{M}^+$. To prove the claim consider the matrix

$$D_1' = \begin{pmatrix} \begin{matrix} A_* & A_n & 0 & 0 \\ 0 & 0 & A_* & A_n \end{matrix} & 0 \\ \begin{matrix} 0 & -B_m & 0 & 0 \\ 0 & 0 & 0 & -B_m \end{matrix} & \begin{matrix} B_* & B_m & 0 & 0 \\ 0 & 0 & B_* & -B_o \end{matrix} \end{pmatrix}$$

which is E-associated to D_1. Now (A_*, A_n) is the denominator of p in (16) hence $(A_*, A_n) \notin \mathcal{P}$ and so $(A_*, A_n) \dotplus (A_*, A_n)$ is in $\mathfrak{M}^+$ since $\mathfrak{M}$ is square-positive. Furthermore

$$\begin{pmatrix} B_* & B_m & 0 & 0 \\ 0 & 0 & B_* & -B_o \end{pmatrix} \in \Pi^+$$

since $q \in P(\Pi) \setminus \{0\}$ by assumption. By Lemma 5.2 (i) and (v) we deduce that $D_1' \in \Pi^+$. Similarly, D_2 is E-associated to

$$D_2' = \begin{pmatrix} A_* & A_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_* & -A_o & 0 & 0 & 0 & -A_o \\ & & & & B_* & B_m & 0 & 0 \\ & & 0 & & 0 & 0 & B_* & B_m \end{pmatrix}$$

and using an argument, analogous to the one employed to verify that D_1' is in Π^+, we deduce that $D_2 \in \Pi^+$. This completes the proof of (15).

P3. Let p, q, (A_o, A_*, A_n), (B_o, B_*, B_m) be as before and set

$$C = \begin{pmatrix} A_o & A_* & A_n & 0 & 0 \\ 0 & 0 & B_o & B_* & B_m \end{pmatrix} .$$

Then

$$C(1, u, p, vp, qp)^T = 0$$

is a system admissible for qp. We have to show that

$$C' = \begin{pmatrix} A_* & A_n & 0 & 0 \\ 0 & B_o & B_* & B_m \end{pmatrix} + \begin{pmatrix} A_* & A_n & 0 & -A_o \\ 0 & B_o & B_* & 0 \end{pmatrix}$$

is in Π. Now C' is E-associated to

$$C'' = \begin{pmatrix} A_* & A_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_*-A_o & 0 & 0 & 0 & -A_n \\ 0 & B_o & 0 & 0 & B_* & B_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_*-B_o \end{pmatrix}$$

and C'' can be written as the determinantal sum with respect to the last column:

$$\begin{pmatrix} A_* & A_n & 0 & 0 & & \mathbf{0} & & \\ 0 & 0 & A_*-A_o & & & & & \\ 0 & B_o & 0 & 0 & B_* & B_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_*-B_o \end{pmatrix} \nabla \begin{pmatrix} A_* & A_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_*-A_o & 0 & 0 & 0 & -A_n \\ 0 & B_o & 0 & 0 & B_* & B_m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_* & 0 \end{pmatrix}$$

The first matrix in the determinantal sum is in Υ by M1 and Lemma 5.2 (iv) while the second one is non-full over R and hence it is also in Υ by M4. Thus $C'' \in \Upsilon$ by M2 and, consequently, $C' \in \Upsilon$ since C' and C'' are E-associated. It remains to verify that $\mathbf{P}(\Upsilon)$ is square-positive. In view of P2 and P3 it will suffice to show that $\mathbf{P}(\Upsilon)$ contains all squares. Let $p \in K$ and let (A_o, A_*, A_n) be any matrix admissible for p. Then

$$\begin{pmatrix} A_o & A_* & A_n & 0 & 0 \\ 0 & 0 & A_o & A_* & A_n \end{pmatrix}$$

is admissible for p^2. We show that

$$A = \begin{pmatrix} A_* & A_n & 0 & 0 \\ 0 & A_o & A_* & A_n \end{pmatrix} + \begin{pmatrix} A_* & A_n & 0 & -A_o \\ 0 & A_o & A_* & 0 \end{pmatrix}$$

is in Υ, this will prove that $p^2 \in \mathbf{P}(\Upsilon)$. The matrix A is E-associated to

$$A'=\begin{pmatrix} A_* & A_n & 0 & 0 \\ 0 & A_o & A_* & A_n \end{pmatrix} + \begin{pmatrix} A_*-A_o & 0 & -A_n \\ 0 & 0 & A_*-A_o \end{pmatrix}$$

and A' is in Π by Lemma 5.2 (iv), M1 and since Π is square-positive. Hence $A\in\Pi$ and so $p^2\in \mathcal{P}(\Pi)$.

We remark that if Π is any matrix cone over R with associated singular kernel $\mathcal{P}$ and K is the epic R-field with prime matrix ideal $\mathcal{P}$ then Π induces a partial order on K with positive cone

$$P=\Big\{p\in K \;\Big|\; \exists \text{ an admissible matrix } (A_o,A_*,A_n) \text{ for p such that } (A_*,A_n),\ (A_*,-A_o)\in\Pi\Big\}.$$

Further, if Π is square-positive then P coincides with $\mathcal{P}(\Pi)$. The verification of these facts is similar to that of the above theorem and will be omitted because we are mainly interested in square-positive matrix cones.

We shall want to obtain square-positive matrix cones from square-positive partial orders on epic R-fields. A method to do this is given in

<u>Theorem 5.4</u>. Let R be a ring and assume that $\alpha : R \rightarrow K$ is an epic R-field with singular kernel $\mathcal{P}$. Suppose that $\leq$ is a square-positive partial order on K with positive cone P. Then the set

$$\Pi(P)=\Big\{A\in M(R)\;\Big|\; \det A^{\alpha} \geq 0\Big\}$$

of matrices over R is a square-positive matrix cone with associated singular kernel $\mathcal{P}$.

<u>Proof</u>. M1, M2 and M3 follow by D2, D4 and D1 respectively while M4 is the consequence of having defined $\det T=0$ whenever T is a singular square matrix over R. Notice that

$\Pi(P) \cap -\Pi(P)$ is the subset of $M(R)$ consisting of all matrices which become singular over K; hence $\Pi(P) \cap -\Pi(P) = \mathcal{P}$ and now M4-M7 easily follow from the fact that $\mathcal{P}$ is a prime matrix ideal. Finally, $\Pi(P)$ is square-positive because P is so.

Proposition 5.5. Let R be a ring and assume that $\alpha : R \to K$ is an epic R-field. Assume further that P is the positive cone associated with a square-positive partial order on K. Then $P = \mathcal{P}(\Pi(P))$.

Proof. Let $p \in P$ and let (A_o, A_*, A_n) be a matrix over R, admissible for p. Then, as we have seen

$$\bar{p} = \det((A_*, A_n)^\alpha)^{-1}\det(A_*, -A_o)$$
$$= \det((A_*, A_n) \dotplus (A_*, -A_o))^\alpha (\det(A_*, A_n)^\alpha)^{-2}$$

and since P is square-positive we deduce that $(A_*, A_n) \dotplus (A_*, -A_o) \in \Pi(P)$. Thus $p \in \mathcal{P}(\Pi(P))$, by definition. Conversely, suppose that $p \in \mathcal{P}(\Pi(P))$; then there is an admissible matrix (A_o, A_*, A_n) over R such that $(A_*, A_n) \dotplus (A_*, -A_o) \in \Pi(P)$. It follows that

$$\det(A_*, A_n)^\alpha \det(A_*, -A_o)^\alpha \subseteq P$$

and since P is square-positive this implies that

$$\bar{p} = (\det(A_*, A_n)^\alpha)^{-1} \det(A_*, -A_o) \subseteq P$$

and so $p \in P$.

This proposition demonstrates that there is a one-to-one correspondence between square-positive partial orders on K and square-positive matrix cones over R of form $\Pi(P)$, where P is a square-positive partial order on K. Our next objective is to prove the matrix cone analogue of Theorem 3.3, Corollary. Let S be a subset of M(R); recall that the matrix semicone Π, generated by S, consists of all the matrices which can be built up from elements of S and non-full square matrices over R by the operations $\dotplus$, ∇ and multiplication by elements of E(R). Let $A \in \Pi$; by the length of A (in Π) we shall understand the minimum number of operations $\dotplus$, ∇ and left or right multiplication by an element of E(R) needed to obtain A as an element of Π. The length of A will be denoted by $l(A)$. Thus, for instance, the length of A is zero if and only if $A \in S$ or A is non-full.

Lemma 5.6. Let R be a ring and assume that $\alpha : R \to K$ is an epic R-field with singular kernel $\mathcal{P}$. Assume further that $\leq$ is a square-positive partial order on K and denote by Π the matrix semicone generated by $\mathcal{P}$, 1 and $S=\{A \dotplus A \in M(R) \mid A \in M(R)\}$. Then $\det A^{\alpha} \geq 0$ for all $A \in \Pi$.

Proof. Notice that all non-full square matrices are contained in $\mathcal{P}$. We use induction on the length. Let $A \in \Pi$: if $l(A)=0$ then $A \in \mathcal{P}$ or $A=1$ or $A \in S$, in each case we have $\det A^{\alpha} \geq 0$. Now let $l(A)>0$; then

$$A=B \dotplus C \quad \text{or} \quad A=B\nabla C \quad \text{or} \quad A=EB \quad \text{or} \quad A=BE,$$

where $B, C \in \Pi$, $E \in E(R)$ and further $l(A) > l(B), l(C)$. By the induction hypothesis we have $\det B^{\alpha} \geq 0$ and $\det C^{\alpha} \geq 0$. Suppose that $A = B \dotplus C$. By D2 we have

$$\det A = \det B \det C$$

and hence $\det A \geq 0$. The other cases can be treated similarly; when $A = B\nabla C$ we use Lemma 4.1 while if $A = BE$ or $A = EB$ then D1 is employed.

Corollary. Let R, K, $\mathcal{P}$, Π and $\leq$ be as in the lemma. Then $\Pi \cap -\Pi = \mathcal{P}$.

Proof. The inclusion $\Pi \cap -\Pi \supseteq \mathcal{P}$ is obvious. To see the converse let $A \in \Pi \cap -\Pi$; then by the lemma we have $0 \leq \det A^{\alpha}$ and $0 \leq \det(-1 \dotplus A)^{\alpha} = -\det A^{\alpha}$. Hence $\det A^{\alpha} = 0$ and it follows that $A \in \mathcal{P}$.

We can now prove the analogue of Theorem 3.3 Corollary.

Theorem 5.7. Let R be a ring and suppose that $\alpha : R \to K$ is an epic R-field with singular kernel $\mathcal{P}$. Then K can be fully ordered if and only if the matrix semicone generated by $\mathcal{P}$, 1 and $S = \{A \dotplus A \in M(R) \mid A \in M(R)\}$ is a matrix cone.

Proof. Put Π for the matrix semicone in the statement. Assume first that $\leq$ is a full order on K. Then Π satisfies the matrix cone axioms M1-M4 and M5-M7 also hold since $\Pi \cap -\Pi = \mathcal{P}$ is a prime matrix ideal by the above corollary. Hence Π is a matrix cone.

Conversely, assume that Π is a matrix cone; we have to verify that K can be fully ordered. Put $\mathcal{P}' = \Pi \cap -\Pi$ and let K' be the epic R-field with singular kernel $\mathcal{P}'$. Clearly Π is square-positive so $P(\Pi)$ is the positive cone of a square-positive partial order on K', by Theorem 5.3, which can be extended to a full order on K' by Theorem 3.3. It remains to verify that $\mathcal{P}' = \mathcal{P}$ for this implies that $K \cong K'$. The inclusion $\mathcal{P} \subseteq \mathcal{P}'$ is obvious. Let $A \in R_n \cap \mathcal{P}'$; we use double induction, first on n

and then on $l(A)$, to show that $A \in \mathcal{P}$.
If $n=1$ then $A \in R$ and by the construction of Π this can only be the case if $A \in \mathcal{P}$. Let $n > 1$ and assume that $l(A)=0$. Then

$$A=B\dotplus B \quad \text{or} \quad A \in \mathcal{P}\,.$$

If $A \in \mathcal{P}$ there is nothing to prove. Let $A=B\dotplus B$; then $B \in \mathcal{P}'$ since $A \in \mathcal{P}'$ and, further, $B \in R_m$ where $m < n$. Thus $B \in \mathcal{P}$ by induction hypothesis and hence $A \in \mathcal{P}$. Suppose now that $l(A) > 0$. Then

$$A=B\dotplus C \quad \text{or} \quad A=B\nabla C \quad \text{or} \quad A=EB \quad \text{or} \quad A=BE,$$

where $B, C \in \Pi$, $E \in E(R)$ and $l(A) > l(B), l(C)$. We consider the above four cases separately. If $A=B\dotplus C$ then $B \in \mathcal{P}'$ or $C \in \mathcal{P}'$ since $\mathcal{P}'$ is a prime matrix ideal. But both B and C are of smaller size than A and so, by the induction hypothesis, $B \in \mathcal{P}$ or $C \in \mathcal{P}$ whence $A \in \mathcal{P}$. Assume next that $A=B\nabla C$. Then we have $\det A \subseteq \det B+\det C$ by D4 and further $\det A=0$ since $A \in \mathcal{P}'$. From Lemma 5.6 we can now deduce that

$$\det B=\det C=0.$$

It follows that $B, C \in \mathcal{P}'$ and so, by the induction hypothesis, both B and C belong to $\mathcal{P}$. In consequence: $A=B\nabla C \in \mathcal{P}$. Finally, let $A=EB$. Now $B \in \mathcal{P}'$ since $A \in \mathcal{P}'$ and by the induction hypothesis this implies that $B \in \mathcal{P}$ whence $A \in \mathcal{P}$. The fourth case is treated similarly. This completes the proof.

The next theorem treats the case when K is a field of fractions of R.

Theorem 5.8. Let R be a partially ordered ring with positive cone P and assume that K is a field of fractions of R with singular kernel $\mathcal{P}$. Then P can be extended to a full order on K if and only if there exists a square-positive matrix cone over R, say Π, such that

(i) $\Pi \cap -\Pi = \mathcal{P}$,

(ii) $\Pi \supset P$,

where elements of P are considered as 1x1 matrices. Moreover, P extends to a unique full order on K if and only if $\mathbf{P}(\Pi)$ is a total cone for every square-positive matrix cone which satisfies (i) and (ii).

Proof. Let S_1 denote the set of total cones over K containing P and S_2 the set of square-positive matrix cones over R which satisfy (i) and (ii). With this notation the first assertion of the theorem states that S_1 is non-empty precisely if S_2 is non-empty. Let P' be an element of S_1; we claim that $\Pi(P') \in S_2$. By Theorem 5.4 $\Pi(P')$ is square-positive and satisfies (i). Let $a \in P$; then $\det(a)=\bar{a} \subset P \subseteq P'$ whence $a \in \Pi(P')$ and hence (ii) also holds. Conversely, assume $\Pi \in S_2$; then $\mathbf{P}(\Pi)$ is a square-positive partial order on K by Theorem 5.3. For any $a \in R$ the matrix $(-a,1)$ is admissible for a. Now if $a \in P$ then a is also in Π by (ii) and consequently $1 \dotplus a \in \Pi$. This proves that $P \subset \mathbf{P}(\Pi)$ and by Theorem 3.3 we deduce that $\mathbf{P}(\Pi)$, hence also P, has a full extension. We now turn to the second assertion of the theorem. Suppose first that that P_1 and P_2 are distinct elements of S_1. Then $P'=P_1 \cap P_2$ is not a total cone over R, but it is square-positive since both P_1 and P_2 are so. Furthermore P' contains P and it is easy to see now that $\Pi(P')\in S_2$. Moreover, by Proposition 5.5, $P'=\mathcal{P}(\Pi(P'))$; thus $\Pi(P')$ is a square-positive matrix cone satisfying (i) and (ii), such that $\mathcal{P}(\Pi(P'))$ is not total. Conversely, suppose that P can be extended to a unique full order on K; we have to show that $\mathcal{P}(\Pi)$ is a total cone for each $\Pi\in S_2$. Assume the contrary, namely: $\Pi'\in S_2$ but $P'=\mathcal{P}(\Pi')$ is not total. Clearly, P' is square-positive and contains P. Let $p\in K\setminus(P'\cup -P')$. Then by Theorem 3.3 there exist total cones P_1 and P_2, both containing P, such that $p\in P_1$ and $-p\in P_2$. Thus P_1 and P_2 are distinct elements of S_1, a contradiction.

As an application we prove the following

<u>Proposition 5.9</u>. Let K be an ordered field with centre k and let X be a set. Then the ordering of K can be extended to $K_k\langle X\rangle$.

<u>Proof</u>. Put $R=K_k\langle X\rangle$, let $\leq$ be the given ordering of K and write P for the positive cone associated with $\leq$. Assume first that $[K:k]=\infty$. We know from the above theorem that to prove the claim it will suffice to find a square-positive matrix cone, with associated singular kernel $\mathcal{P}_0$ (=set of all non-full matrices over K), which contains P. Let us verify that

$$\Pi=\left\{A\in M(R)\ \middle|\ \det A^s\geq 0 \text{ for every evaluation } s: R\to K\right\}$$

is such a matrix cone. Let s be an arbitrary evaluation; then s is the identity map on K and hence for each $a\in P$ we have $\det(a^s)=\bar{a}\geq 0$. This shows that

$P \subset \Pi$. Let A be a square matrix over R; then

$$\det(A \dotplus A)^s = \det(A^s)^2 \geq 0$$

and so Π is square-positive, provided it is a matrix cone. Thus it remains to show that Π is a matrix cone with the prescribed associated singular kernel. The matrix cone axioms M1-M3 follow by properties D1-D4 of det. We check M2, M1 and M3 can be proved similarly. Let A and B be in Π and assume that $A \nabla B$ is defined. For every evaluation s we have $\det A^s \geq 0$ and $\det B^s \geq 0$. Further $(A \nabla B)^s = A^s \nabla B^s$ and so by Lemma 4.1 we have

$$\det(A \nabla B)^s = \det(A^s \nabla B^s) \geq 0$$

which shows that $A \nabla B \in \Pi$. To verify M4-M7 and and prove that $\mathcal{P}_0$ is the singular kernel associated with Π it will suffice to show that $\Pi \cap -\Pi = \mathcal{P}_0$. Suppose that A is a non-full square matrix over R; then for any evaluation s, A^s is singular over K and hence so is $-1 \dotplus A$. It follows that

$$\det A^s = \det(-1 \dotplus A)^s = 0$$

which implies that $A \in \Pi \cap -\Pi$. In consequence: $\mathcal{P}_0 \subseteq \Pi \cap -\Pi$. Now let $A \in M(R)$ be full over R; by Theorem 1.3.1 we can choose an evaluation, say s', so that $A^{s'}$ is non-singular over K. Then

$$\det(-1 \dotplus A)^{s'} = -\det A^{s'} \neq 0$$

so either $\det(A^{s'}) < 0$ or $\det(-1 \dotplus A) < 0$. It follows that $A \notin \Pi \cap -\Pi$ and this shows that $\Pi \cap -\Pi \subseteq \mathcal{P}_0$. Thus $\mathcal{P}_0 = \Pi \cap -\Pi$, as claimed.

When $[K:k] < \infty$ embed K in $D=K(t)(x;\alpha)$, where α is the endomorphism of $K(t)$ induced by $t \mapsto t^2$. Then $\leq$ can be extended to a full order as described at the beginning of Example 1.10. Now the centre of D is clearly k and further $[D:k]=\infty$. It follows that the ordering of D, and hence also of K, can be extended to a full order on $D_k\langle\!\langle X\rangle\!\rangle$. Thus $\leq$ can be extended to $K_k\langle\!\langle X\rangle\!\rangle$ since, by Proposition 5.4.2, Corollary of [5], $K_k\langle\!\langle X\rangle\!\rangle \subseteq D_k\langle\!\langle X\rangle\!\rangle$.

2.6. Examples

In this section we discuss three examples. In the first oné we use the ordered monoid with no groups of fractions, constructed in §2 to obtain an ordered ring with no fields of fractions. The second example demonstrates that a full order on a free algebra need not be extendible to a field of fractions while in the third example we define a full order on a free algebra whose extension to the universal field of fractions is not unique.

Example 6.1. Let S' and $\preceq$ be as in §2. Thus S' is a cancellation monoid with no groups of fraction and $\preceq$ is a full order on S'. Further let k be a field with a full order $\leq$ and consider the semigroup algebra R=kS'. Then every non-zero element f of R can be written uniquely as follows:

$$\sum_{i=1}^{n} a_i s_i \qquad (0 \neq a_i,\ s_i \in S',\ s_1 \prec s_2 \prec \dots \prec s_n).$$

We define a full order $\leq'$ on R by putting

$$f \;'\!> 0 \quad \text{if} \quad a_n > 0.$$

Hence R is an ordered ring which cannot be embedded in a field since any field containing R would also contain a group of fractions of S'.

Example 6.2. Let k be a commutative field with a full order $\leq$ and let M be the free monoid on a, b and c. Then the free algebra $k\langle a,b,c\rangle$ is just the semigroup algebra kM. Order M lexicographically so that

$$a < b < c$$

and define an ordering $\leq'$ of $k\langle a,b,c\rangle$ by putting

$$\sum_{i=1}^{n} a_i m_i \;'\!> 0 \quad (0 \neq a_i \in k,\ m_i \in M,\ m_1 < \ldots < m_n) \text{ if } a_n > 0.$$

Consider now the free algebra $R=k\langle x,y\rangle$. As is well-known the subalgebra S of R generated by x, xy and xy^2 is free on these elements. Thus, using the ordering of $k\langle a,b,c\rangle$ defined above, S can be fully ordered so that

$$xy^2 < x < xy.$$

The natural embedding $S \to k\not\langle x,y\not\rangle$ is easily seen to be epic so $k\not\langle x,y\not\rangle$ is a field of fractions of S (clearly not the universal one). We claim that $\leq$ cannot be extended to $k\not\langle x,y\not\rangle$. In fact, $\leq$ cannot even be extended to R for $xy^2 < x$ implies that $y^2 < 0$, hence $y < 0$, while $x < xy$ implies $y > 0$.

Before giving the last example we note that if K is a field with a full order $\leq$ and G is a group with a full order $\preceq$ then the Malcev-Neumann field, $K((G,\preceq))$, has a natural ordering induced by $\leq$ and $\preceq$:

$$f = \sum_{g \in G} a_g g \succ 0 \quad \text{if} \quad a_{g_0} \succ 0,$$

where $g_0 = \min(\mathrm{supp}(f))$ (cf. [11; Corollary 11, p.137] and the end of §1.3).

Example 6.3. We have shown in Example 1.7 that the free monoid S on x and y has a full ordering, say $\preccurlyeq^0$, which extends to the free group F on x and y in several distinct ways. Let $\preccurlyeq^1$ and $\preccurlyeq^2$ be distinct full orders on F which agree with $\preccurlyeq^0$ on S. Further let k be a commutative field with a full order $\leq$. Order the free algebra $k\langle x,y\rangle$ as follows:

$$\sum_{i=1}^{n} a_i s_i \overset{0}{\succ} 0 \; (0 \neq a_i \in k,\; s_i \in S,\; s_1 \preccurlyeq^0 \ldots \preccurlyeq^0 s_n) \text{ if } a_1 \succ 0.$$

As we have mentioned at the end of §1.3, Lewin has shown in [12] that for any full order $\preccurlyeq$ on F the subfield of $k((F,\preccurlyeq))$, generated by $k\langle x,y\rangle$ is $k\Lsh x,y\Rsh$. It follows that the natural ordering of $k((F,\preccurlyeq))$ also fully orders $k\Lsh x,y\Rsh$. Now let $\leq^i$ denote the natural order on $k((F,\preccurlyeq^i))$, (i=1,2). Then $\leq^1$ and $\leq^2$ differ on $k\Lsh x,y\Rsh$ since they already differ on F. However $\leq^1$ and $\leq^2$ agree with $\leq^0$ on $k\langle x,y\rangle$, as is easily checked.

3

The abelianized multiplicative group of universal fields of fractions

Let R be a commutative integral domain; the field of quotients Q of R can be obtained by first constructing the group of quotients of the abelian monoid R^{x} and then defining addition on it. Let us denote the universal abelian group of a given monoid M by $\mathfrak{a}(M)$, then the above assertion can be expressed as follows: $Q^{x} = \mathfrak{a}(R^{x})$. Write Σ_n for the submonoid of R_n consisting of those matrices which become invertible over Q. Then Σ_n is an Ore monoid, because R is commutative, and $GL_n(Q)$ is its group of quotients. Recall that, except when $Q=F_2$ and $n=2$, the determinant map can be considered as an isomorphism $GL_n(Q)^{ab} \rightarrow Q^{x}$. Putting these facts together we find

$$\mathfrak{a}(\Sigma_n) = GL_n(Q)^{ab} \cong Q^{x} = \mathfrak{a}(R^{x}).$$

Assume now that R is also a UFD and write $\mathcal{P}$ for the set of equialence classes of associated primes of R; it follows by unique factorization that

$$Q^{x} \cong G(R) \times D,$$

where G(R) is the group of units of R and D is the free abelian group on $\mathcal{P}$. In this chapter we obtain analogues of these results in the non-commutative case.

Let R be a Sylvester domain, write U for its universal field of fractions and let Σ denote the set of full matrices over R. Then Σ can be viewed as a submonoid of

M(R) and we have the inclusion

$$\Sigma \subseteq GL(U). \tag{1}$$

Our concern is the universal abelian group, $\mathfrak{a}(\Sigma)$, of Σ. The above inclusion induces a natural homomorphism of abelian groups

$$\mathfrak{a}(\Sigma) \to GL(U)^{ab}$$

and we show in section 1 that this is an isomorphism (Theorem 1.4). Recall that the Dieudonné determinant establishes an isomorphism $GL(U)^{ab} \cong U^{\times ab}$. Clearly, the image of Σ in $\mathfrak{a}(\Sigma)$ under the natural map is the universal cancellation monoid of Σ and so the above result allows us to compute the Dieudonné determinant over R, analogously to the way ordinary determinants can be calculated over a commutative integral domain.

The inclusion (1), restricted to GL(R), induces an abelian group map

$$GL(R)^{ab} \longrightarrow GL(U)^{ab}$$

whose cokernel will be denoted by D(R), it is called the divisor group of R. We prove in §2 that when R is a fully atomic semifir (e.g. if R is a fir) then D(R) is free abelian on the set of equivalence classes of stably associated matrix atoms and moreover:

$$GL(U)^{ab} = \overline{GL(R)^{ab}} \times D(R),$$

where $\overline{GL(R)^{ab}}$ denotes the image of $GL(R)^{ab}$ in $GL(U)^{ab}$

(Theorem 2.4). It follows that if, in addition, the canonical map

$$G(R) \to GL(R)^{ab}$$

is surjective (i.e. if R is a GE-ring), computing $GL(U)^{ab}$, and hence U^{xab}, reduces to determining the image of $G(R)$ in U^{xab}.

In section 3 we collect certain facts about field extensions which will be needed in §4 where the structure of U^{xab} of specific firs are described (e.g. skew polynomial rings, free rings, coproducts of commutative fields amalgamating a common subfield). A typical result is the following: let k be a commutative field and X a set; then

$$k\langle X\rangle^{xab} \cong k^{x} \times D(k\langle X\rangle).$$

3.1 $GL(U(R))^{ab}$ of a Sylvester domain

Let R be a ring, Σ a set of square matrices over R and consider R_Σ, the universal Σ-inverting ring. Denote by $\hat{\Sigma}$ the inverse image of $GL(R_\Sigma)$ in $M(R)$; then $\hat{\Sigma}$ is a submonoid of $M(R)$. We shall be investigating the universal abelian group of $\hat{\Sigma}$. We outline the construction of $\mathfrak{a}(\hat{\Sigma})$; note that the same procedure yields $\mathfrak{a}(M)$ for any monoid M.

Let us call A and A' in $\hat{\Sigma}$ <u>1-related</u> if

$$A=C_1C_2\ldots C_n \quad \text{and} \quad A'=C_{\pi(1)}C_{\pi(2)}\ldots C_{\pi(n)},$$

where $n \in \mathbb{N}^+$, $C_i \in \hat{\Sigma}$ and π is a permutation on $\{1,\ldots,n\}$. When A and A' are 1-related we write $A \underset{1}{\sim} A'$. We define A and A' to be <u>k-related</u>, $k > 1$, if there exists a sequence $A_0, A_1, \ldots, A_k$ of elements of $\hat{\Sigma}$, such that $A_0 = A$, $A_k = A'$ and $A_i \underset{1}{\sim} A_{i+1}$ for all $0 \leq i \leq k-1$. Finally, define the relation $\sim$ on $\hat{\Sigma}$ as follows:

$$A \sim A' \quad \text{if} \quad A \text{ is k-related to } A' \text{ for some } k \geq 1.$$

It is easy to see that $\sim$ is a semigroup congruence; moreover for every pair A, B of elements of $\hat{\Sigma}$ we have $AB \underset{1}{\sim} BA$ and hence $AB \sim BA$. In fact $\sim$ is the semigroup congruence generated by $AB \sim BA$. Consequently: $\hat{\Sigma}/\sim$ is the universal abelian monoid of $\hat{\Sigma}$. Write $\overline{A}$ for the image of $A \in \hat{\Sigma}$ in $\hat{\Sigma}/\sim$ and define the congruence $\approx$ on $\hat{\Sigma}/\sim$ by putting

$$\overline{A} \approx \overline{B} \quad \text{if} \quad \overline{A}\,\overline{C} = \overline{B}\,\overline{C} \quad \text{for some} \quad \overline{C} \in \hat{\Sigma}/\sim .$$

In terms of $\hat{\Sigma}$ we have

$$\overline{A} \approx \overline{B} \quad \text{if and only if} \quad AC \sim BC \quad \text{for some } C \in \hat{\Sigma}.$$

Factor out $\approx$ and write $\hat{\Sigma}^{abc}$ for the resulting abelian cancellation monoid, it can be checked directly that, as such, $\hat{\Sigma}^{abc}$ is universal for $\hat{\Sigma}$. It follows that the group of quotients of $\hat{\Sigma}^{abc}$ is the universal abelian group of $\hat{\Sigma}$. Write $[A]$ for the image of $A \in \hat{\Sigma}$ in $\hat{\Sigma}^{abc}$; then every element of $\mathfrak{a}(\hat{\Sigma})$ is of form

$$\frac{[A]}{[B]} \qquad (A,\ B \in \hat{\Sigma}).$$

We note that $E(R) \subseteq \hat{\Sigma}$ and further each $E \in E(R)$ is a product of commutators of $E(R)$ which implies that $[E]=[1]$. In consequence: stably E-associated elements of $\hat{\Sigma}$ have the same $[\,]$-value.

Clearly, $\mathfrak{a}(\hat{\Sigma})$ is characterized by the universal property that for any semigroup homomorphism h of $\hat{\Sigma}$ into an abelian group A, the diagram

$$\begin{tikzcd} & \mathfrak{a}(\hat{\Sigma}) \\ \hat{\Sigma} \arrow[ur, "{[\,]}"] \arrow[dr, "h"'] & \\ & A \end{tikzcd}$$

can be completed by a unique homomorphism of abelian groups $h' : \mathfrak{a}(\hat{\Sigma}) \to A$, so that the resulting diagram commutes. Under the natural map $\lambda : R \to R_{\Sigma}$ elements of $\hat{\Sigma}$ become invertible; thus we obtain a natural homomorphism

$$\bar{\lambda} : \hat{\Sigma} \to GL(R_{\Sigma})^{ab}.$$

Now by the universal property of $\mathfrak{a}(\hat{\Sigma})$ we get the commutative diagram

$$\begin{tikzcd} & \mathfrak{a}(\hat{\Sigma}) \arrow[d, "\bar{\lambda}'"] \\ \hat{\Sigma} \arrow[ur, "{[\,]}"] \arrow[r, "\bar{\lambda}"'] & GL(R_{\Sigma})^{ab} \end{tikzcd} .$$

Our aim is to show that in certain cases $\bar{\lambda}'$ is an isomorphism. We need a couple of lemmas first.

Lemma 1.1. Let R be a ring and let Σ be a set of square matrices over R. Then the canonical homomorphism $\bar{\lambda}' : \mathfrak{a}(\hat{\Sigma}) \to GL(R_{\Sigma})^{ab}$ is surjective.

Proof. Let P be an element of $GL(R_\Sigma)$, say $P \in GL_N(R_\Sigma)$, and denote the image of P in $GL(R_\Sigma)^{ab}$ by $\overline{P}$. From Proposition 1.1.3 we know that we can choose an admissible system

$$(A_o, A_*, A_n)^\lambda (I_N, X_*, P)^T = 0$$

for P, then

$$(A_*, A_n)^\lambda \begin{pmatrix} I_{N(n-1)} & X_* \\ 0 & P \end{pmatrix} = (A_*, -A_o)^\lambda,$$

where $(A_*, A_n),\ (A_*, -A_o) \in \hat{\Sigma}$. It follows now that

$$((A_*, A_n)^{\overline{\lambda}})^{-1} (A_*, -A_o)^{\overline{\lambda}} = \overline{P}$$

and consequently:

$$\left(\frac{[(A_*, -A_o)]}{[(A_*, A_n)]} \right)^{\overline{\lambda}'} = \overline{P}.$$

Hence $\overline{\lambda}'$ is surjective.

Lemma 1.2. Let R be a ring and let Σ be a set of square matrices over R. Assume that R_Σ is weakly finite and $\lambda : R \to R_\Sigma$ is rank-preserving. Then there is a commutative diagram

$$\begin{tikzcd} & \mathfrak{a}(\hat{\Sigma}) \\ \hat{\Sigma} \arrow[ur, "{[\ ]}"] \arrow[dr, "\lambda"'] & \\ & GL(R_\Sigma) \arrow[uu, "\mu"'] \end{tikzcd}$$

Proof. The homomorphism μ will be defined as the direct limit of homomorphisms $\mu_N : GL_N(R_\Sigma) \to \mathfrak{a}(\hat{\Sigma})$.

Let $N > 1$ and let P be an element of $GL_N(R_\Sigma)$. Further let

$$(A_o, A_*, A_n)^{\lambda} (I_N, X_*, P)^T = 0 \tag{2}$$

be an admissible system for P; such a system exists by Proposition 1.1.3. We define μ_N as follows:

$$\mu_N : P \longmapsto \frac{[(A_*, -A_o)]}{[(A_*, A_n)]} .$$

To see that μ_N is a homomorphism let Q be another element of $GL_N(R_\Sigma)$ and let

$$(B_o, B_*, B_m)^{\lambda} (I_N, Y_*, Q)^T = 0$$

be an admissible system for Q. Then

$$\begin{pmatrix} A_o & A_* & A_n & 0 & 0 \\ 0 & 0 & B_o & B_* & B_m \end{pmatrix}^{\lambda} (I_N, X_*, P, Y_*P, QP)^T = 0 \tag{3}$$

is an admissible system for $QP \in GL_N(R_\Sigma)$. The $[\;]$-value of the numerator of QP in (3) remains unchanged if we interchange the n^{th} and the $(n+m)^{th}$ columns and then multiply the last column of the resulting matrix by -1; for this transformation can be performed by multiplication by elements of E(R). We thus obtain

$$\begin{pmatrix} A_* & -A_o & 0 & -A_n \\ 0 & 0 & B_* & -B_o \end{pmatrix}$$

and further

$$\begin{pmatrix} A_* & -A_o & 0 & -A_n \\ 0 & 0 & B_* & -B_o \end{pmatrix} = \begin{pmatrix} I_{Nn} & 0 & 0 \\ 0 & B_* & -B_o \end{pmatrix} \begin{pmatrix} A_* & -A_o & 0 & -A_n \\ 0 & 0 & I_{Nm} \end{pmatrix} . \tag{4}$$

Similarly, the denominator of QP in (3) can be factorized as follows:

$$\begin{pmatrix} A_* & A_n & 0 & 0 \\ 0 & B_o & B_* & B_m \end{pmatrix} = \begin{pmatrix} A_* & A_n & 0 \\ 0 & B_o & I_{Nm} \end{pmatrix} \begin{pmatrix} I_{Nn} & 0 & 0 \\ 0 & B_* & B_m \end{pmatrix}. \tag{5}$$

On the right-hand side of (4) and (5) the factors are stably E-associated to $(B_*,-B_o)$, $(A_*,-A_o)$, (A_*,A_n) and (B_*,B_m) respectively. Hence

$$(QP)\mu_N = \frac{[(A_*,-A_o)][(B_*,-B_o)]}{[(A_*,A_n)][(B_*,B_m)]} = Q\mu_N\, P\mu_N\,.$$

Furthermore the system

$$(A_n,A_*,A_o)^{\lambda}(I_N,X_*P^{-1},P^{-1})^T = 0$$

is admissible for $P^{-1} \in GL_N(R_\Sigma)$ and we have

$$(P^{-1})\mu_N = \frac{[(A_*,-A_n)]}{[(A_*,A_o)]} = \frac{[-1][(A_*,A_n)]}{[-1][(A_*,-A_o)]} = (P\mu_N)^{-1}\,.$$

Now to prove that μ_N is a well-defined homomorphism it will suffice to verify that if the matrix (G_o,G_*,G_n) over R is admissible for I_N then

$$[(G_*,G_n)] = [(G_*,-G_o)]\,.$$

Accordingly, let us suppose that

$$(G_o,G_*,G_n)^{\lambda}(I_N,Z_*,I_N)^T = 0 \tag{6}$$

is an admissible system for I_N of order n. It follows that

$$(G_*,G_n+G_o)^{\lambda} = G_*^{\lambda}(I_{N(n-1)},-Z_*^T)$$

and since λ is rankpreserving by hypothesis, we can find $B \in {}^{Nn}R^{N(n-1)}$, $C_* \in R_{N(n-1)}$ and $C_o \in {}^{N(n-1)}R$, such that

$$(G_*, G_n + G_o) = B(C_*, C_o). \tag{7}$$

Now we have

$$(G_*, G_n) = (BC_*, G_n) = (B, G_n)\begin{pmatrix} C_* & 0 \\ 0 & I_N \end{pmatrix}$$

and

$$(G_*, -G_o) = (BC_*, -G_o) = (BC_*, G_n - BC_o) = (B, G_n)\begin{pmatrix} C_* & -C_o \\ 0 & I_N \end{pmatrix};$$

furthermore (B, G_n) and C_* belong to $\hat{\Sigma}$ because R_Σ is weakly finite. Thus in $\mathfrak{a}(\hat{\Sigma})$ we find

$$[(G_*, G_n)] = [(B, G_n)][C_*] = [(G_*, -G_o)]$$

and this completes the proof of the fact that μ_N is a homomorphism for each $N \geq 1$. To obtain a homomorphism $\mu : GL(R_\Sigma) \rightarrow \mathfrak{a}(\hat{\Sigma})$, induced by the maps μ_N, by the direct limit property of $GL(R_\Sigma)$, we verify that for each $N \geq 1$ and $P \in GL_N(R_\Sigma)$

$$P\mu_N = (P \dotplus 1)\mu_{N+1}.$$

Consider (2), an admissible system for $P \in GL_N(R_\Sigma)$ of order n. It is easily checked that

$$\left(\begin{pmatrix} 0 & 0 \\ A_o & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} I_{n-1} & 0 \\ 0 & A_* \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ A_n & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} I_{N+1} & \\ 0 & 0 \\ X_*^T & 0 \\ P & 0 \\ 0 & 1 \end{pmatrix} = 0$$

is an admissible system for $P\dagger 1 \in GL_{N+1}(R_\Sigma)$ of order n. Further:

$$(P\dagger 1)\mu_{N+1} = \frac{\left[\begin{pmatrix} I_{n-1} & 0 & 0 & 0 \\ 0 & A_* & -A_o & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]}{\left[\begin{pmatrix} I_{n-1} & 0 & 0 & 0 \\ 0 & A_* & A_n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]} = \frac{[(A_*,-A_o)]}{[(A_*,A_n)]} = P\mu_N,$$

as required.

It remains to show that the diagram in the statement of the theorem commutes. Let $A \in \hat{\Sigma} \cap R_N$, then

$$(A,-I_N)^\lambda (I_N,A^\lambda)^T = 0$$

is an admissible system for A^λ of order 1 and clearly

$$(A^\lambda)\mu = (A^\lambda)\mu_N = \frac{[A]}{[I_N]} = [A].$$

This completes the proof.

<u>Theorem 1.3</u>. Let R be a ring and let Σ be a set of square matrices over R. If the universal Σ-inverting ring R_Σ is weakly finite and $\lambda : R \rightarrow R_\Sigma$ is rank-preserving then the canonical map

$$\bar{\lambda}' : \mathfrak{a}(\hat{\Sigma}) \longrightarrow GL(R_\Sigma)^{ab}$$

is an isomorphism.

<u>Proof</u>. That $\bar{\lambda}'$ is surjective has been proved in Lemma 1.1. Consider the diagram

$$\begin{tikzcd}
& \mathfrak{a}(\hat{\Sigma}) \arrow[dr, "\bar{\lambda}'"] & \\
\hat{\Sigma} \arrow[ur, "{[\ ]}"] \arrow[dr, "\lambda"'] & & GL(R_{\Sigma})^{ab}, \\
& GL(R_{\Sigma}) \arrow[uu, "\mu"'] \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where π denotes the natural surjection. By Lemma 1.2 and the definition of μ the above diagram is commutative. To see that $\bar{\lambda}'$ is injective assume that $[A]/[B]$ is in the kernel of $\bar{\lambda}'$. Then $A^{\lambda}(B^{\lambda})^{-1}$ is a product of commutators in $GL(R_{\Sigma})$ and hence

$$\frac{[A]}{[B]} = (A^{\lambda}(B^{\lambda})^{-1})\mu = [1].$$

We can now prove the main result of this section.

Theorem 1.4. Let R be a Sylvester domain and let Σ denote the set of full matrices over R. Write U for the universal field of fractions of R. Then

$$GL(U)^{ab} = \mathfrak{a}(\Sigma).$$

Proof. Recall that the natural embedding $R \to U$ is rank-preserving (cf. §1.2.). Further, by Theorem 7.6.5, Corollary of [4], $U=R_{\Sigma}$, so R_{Σ} is clearly weakly finite. The claim now follows from Theorem 1.3 observing that $\Sigma = \hat{\Sigma}$.

Remark. Lemma 1.2 and Theorem 1.3 can be strengthened so as to cover left and right Ore domains as follows. Let us call an embedding of rings $R \to S$ weakly rank-preserving if for each matrix A over R the following condition holds: $\rho_S(A) = t$ implies that there exist $D_1, D_2 \in M(R) \cap GL(S)$ such that $\rho_R(D_1AD_2) = t$. For instance, assume that R is a right Ore domain and write Q for its field of quotiens, then the inclusion $R \subseteq Q$ is weakly rank-preserving. To see this assume that $A \in {}^mR^n$ is of rank t over Q. Then for some matrices $B \in {}^mQ^t$ and $C \in {}^tQ^n$ we have $A=BC$. Choosing a common denominator for the entries of B and C we can write

$$A=B'(b^{-1}I_t)C'(c^{-1}I_n),$$

where $B' \in {}^mR^t$, $C' \in {}^tR^n$ and $b, c \in R^x$. Further b^{-1} can be pulled to the right and so we obtain

$$A=B'C''(d^{-1}I_n),$$

where $C'' \in {}^tR^n$ and $d \in R^x$, whence

$$A(dI_n)=B'C''.$$

Thus the rank of $A(dI_n)$ over R is at most t and it is easy to see that in fact $\rho_R(A(dI_n)) = t$. A similar argument applies to left Ore domains.

The only place in Lemma 1.2 and Theorem 1.3 where we exploit the fact that $R \to R_\Sigma$ is rank-preserving is to obtain relation (7) from (6). If $R \to R_\Sigma$ is only weakly rank-preserving, instead of (7) we obtain

$$D_1(G_*, G_o+G_n)D_2 = B(C_*, C_o),$$

where B, C_*, C_o are of the same size as in (7) and $D_1, D_2 \in \hat{\Sigma} \cap R_{Nn}$. As in the proof of Lemma 1.2, we can easily verify that

$$[D_1(G_*, G_n)D_2] = [D_1(G_*, -G_o)D_2]$$

whence

$$[(G_*,G_n)] = [(G_*,-G_o)].$$

Thus Lemma 1.2 and Theorem 1.3 hold with the hypothesis '$R \rightarrow R_\Sigma$ is rank-preserving' replaced by '$R \rightarrow R_\Sigma$ is weakly rank-preserving'. Now let R be a right or left Ore domain with field of quotients Q. Put $\Sigma = R^x$; then R_Σ is just Q and so $GL(Q)^{ab} = \mathfrak{a}(\hat{\Sigma})$.

3.2 $GL(U(R))^{ab}$ of a fully atomic semifir

A ring R is said to be fully atomic if every full matrix over R can be written as a product of matrix atoms. For instance every fir is fully atomic by Theorem 5.6.4 of [4]. In this section we specialize the results of §1 to fully atomic semifirs. Most of what follows is based on two notions: the factorization theorem for full matrices over a fully atomic semifir of [4] and the divisor group defined in [8]. We begin by recalling a few results from [4].

Let R be a semifir and M a finitely ~~generated~~ presented right R-module with presentation

$$0 \rightarrow {}^nR \xrightarrow{\lambda} {}^mR \rightarrow M \rightarrow 0.$$

Then λ is realized as left multiplication by an mxn matrix over R, say A, and then $M \cong {}^mR/A\,{}^nR$. The characteristic of M is defined as follows:

$$\chi(M) = m-n;$$

by Schanuel's lemma $\chi(M)$ is independent of the choice of the presentation. M is called a <u>torsion module</u> if

(i) $\chi(M) = 0$ and

(ii) $\chi(N) \geq 0$ for every submodule N of M.

Theorem 5.6.2 of [4] states that M is a torsion module if and only if it is presented by a full matrix; further two torsion modules over R are isomorphic if and only if the full matrices presenting them are stably associated (cf. [6; Thm.2.1, Corollary 1]). Torsion modules over R and homomorphisms between them form a full subcategory $\mathcal{T}_R$ of $\mathcal{M}_R$ (cf. [4; Theorem 5.3.3]); if R is fully atomic then torsion modules over R satisfy both chain conditions on torsion submodules and so $\mathcal{T}_R$ satisfies the hypotheses of the Jordan-Hölder theorem. The simple objects of $\mathcal{T}_R$ are precisely the torsion modules presented by a matrix atom and hence the Jordan-Hölder theorem in $\mathcal{T}_R$ can be interpreted as follows:

<u>Theorem 2.1</u>. ([4; Thm.5.6.4]) Let R be a fully atomic semifir and denote by Σ the monoid of full matrices over R. Then Σ has unique factorization in the sense that every full matrix can be written as a product of matrix atoms and the atomic factors are unique up to stable association and order.

Let R be a fully atomic semifir, let Σ be the monoid of full matrices over R and write U for the universal field of fractions of R. We shall use the above theorem to describe the structure of $GL(U)^{ab}$. It is easy to see that stable association is an equivalence relation on Σ; we write $\mathfrak{A}$ for the set of similarity classes of matrix atoms and the elements of $\mathfrak{A}$ are called <u>prime divisors</u>. Let $\mathfrak{A}^*$ be an arbitrary set of representatives of $\mathfrak{A}$. By what has been said it is clear that any element A of Σ can be decomposed as follows:

$$A = U_1P_1V_1U_2P_2V_2\ldots U_nP_nV_n, \tag{8}$$

where U_i, $V_i \in GL(R)$, $P_i \in \mathfrak{A}^*$ and further n and the P_i are unique. On applying $[\;]$ to (8) and setting $W = U_1V_1\ldots U_nV_n$ we find

$$[A] = [W][P_1][P_2]\ldots[P_n]. \tag{9}$$

Thus we can write

$$[A] = [W] \prod_{P_i \in \mathfrak{A}^*} [P_i]^{n_i}, \qquad (W \in GL(R),\ n_i \in \mathbb{N})$$

where only a finite number of the n_i are non-zero. The $[P_i]^{n_i}$ for which $n_i \neq 0$ are called the <u>primary divisors</u> of A. (Notice that according to this definition $[P_i]$ and $[P_i]^2$ are counted as distinct primary divisors.)

<u>Lemma 2.2</u>. Let R be a fully atomic semifir and let Σ denote the monoid of full matrices over R. Let A, B $\in \Sigma$; if $[A] = [B]$ then A and B have the same primary divisors.

<u>Proof</u>. Suppose $[A] = [B]$; by the construction of $\mathfrak{a}(\Sigma)$ it follows that AC and BC are k-related for some $k \geq 1$ and $C \in \Sigma$. We claim that AC and BC have the same primary divisors, this is clearly equivalent to A and B having the same primary divisors. By the definition of k-relatedness (see the beginning of §1) it suffices to verify the claim when k=1 and this is straightforward.

By the above lemma the $[P_i]$ in (9) are unique and hence, by cancellation, so is $[W]$. We get

Theorem 2.3. Let R be a fully atomic semifir and let Σ denote the monoid of full matrices over R. Write N for the image of GL(R) in Σ^{abc} and D'(R) for the submonoid of Σ^{abc} generated by the prime divisors. Then D'(R) is the free abelian monoid on the set of prime divisors of R and

$$\Sigma^{abc} = N \times D'(R). \tag{10}$$

Proof. From Lemma 2.2 it follows that

$$[P_1][P_2]\dots[P_n] = [Q_1][Q_2]\dots[Q_m] \quad (P_i,\ Q_j \in \mathfrak{R}^*)$$

if and only if $m=n$ and a suitable rearrangement of the indeces gives $P_i = Q_i$, $i=1,\dots,n$. This proves that D'(R) is free abelian on $\mathfrak{R}$. Further (9) shows that $\Sigma^{abc} = N.D'(R)$ and the remark preceding the theorem completes the proof.

Write D(R) for the group of fractions of D'(R) in the setup of the above theorem, clearly D(R) is the free abelian group on $\mathfrak{R}$; it is called the divisor group of R. Recall that we have defined $\mathfrak{a}(\Sigma)$ as the group of fractions of Σ^{abc}, hence from Theorem 2.3 it follows that $\mathfrak{a}(\Sigma) = N \times D(R)$. Moreover $\bar{\lambda}' : \mathfrak{a}(\Sigma) \to GL(U(R))^{ab}$ is an isomorphism by Theorem 1.4 and, under $\bar{\lambda}'$, N is mapped onto

$$\frac{GL(R)\,GL(U(R))'}{GL(U(R))'}$$

which is also the image of of $GL(R)^{ab}$ under the natural homomorphism $GL(R)^{ab} \rightarrow GL(U(R))^{ab}$. This proves

Theorem 2.4. Let R be a fully atomic semifir and put $U=U(R)$. Then

$$GL(U)^{ab} = \overline{GL(R)^{ab}} \times D(R),$$

where $\overline{GL(R)^{ab}}$ denotes the image of $GL(R)^{ab}$ in $GL(U)^{ab}$ and $D(R)$ is the free abelian group on the set of prime divisors of R.

By a GE-ring we understand a ring over which every invertible matrix can be written as a product DE, where $E \in E(R)$ and D is a diagonal matrix with units on its diagonal. A GE-ring which is also a semifir is said to be a strong GE-ring (see [4; pp.52-53] for an alternative definition). Euclidean rings, more generally rings with weak algorithm (see [4] for the definition) are examples of strong GE-rings. Furthermore the coproduct of a family of strong GE-rings, amalgamating a common subfield, is again a strong GE-ring (cf. [3; Thm.3.4]).

Assume now that R is a fully atomic strong GE-ring and write U for its universal field of fractions. The Dieudonné determinant establishes an isomorphism $GL(U)^{ab} \rightarrow U^{\times ab}$ and so the above theorem implies that

$$U^{\times ab} = G \times D(R),$$

where G denotes the isomorphic image of $\overline{GL(R)^{ab}}$ under det. Let $T \in GL(R)$; since R is a GE-ring we can write $T=DE$, where $D=d_1 \dotplus d_2 \dotplus \ldots \dotplus d_n$, $d_i \in G(R)$, and $E \in E_n(R)$. Set $d=d_1 d_2 \ldots d_n$, then clearly $\det T = \det D = dU^{\times\prime}$. It follows that

$$G = \frac{G(R)\,U^{x\prime}}{U^{x\prime}}$$

and the second isomorphism theorem completes the proof of

Corollary 1. Let R be a fully atomic strong GE-ring with universal field of fractions U. Then

$$U^{xab} \cong \frac{G(R)}{G(R) \cap U^{x\prime}} \times D(R).$$

This corollary shows that to compute U^{xab} of a fully atomic strong GE-ring we have to determine $G(R) \cap U^{x\prime}$; next we consider this problem in a more general context.

An embedding of rings $f : R \rightarrow S$ is said to be commutator-pure if the induced abelian group homomorphism $G(R)^{ab} \rightarrow G(S)^{ab}$ is an embedding; alternatively we say that R is commutator-pure in S with respect to f. Observe that f is commutator-pure if and only if

$$G(R)^f \cap G(S)' = (G(R)')^f.$$

Clearly, $(G(R)')^f$ is always contained in $G(S)'$, so to prove that f is commutator-pure it suffices to establish the reverse inclusion. For instance, an embedding of commutative rings is commutator-pure; we shall soon see examples of embeddings which are not commutator-pure. Let R be a Sylvester domain; if the inclusion $R \subset U(R)$ is commutator-pure we shall say that R is commutator-pure. An immediate consequence of this terminology and Corollary 1 is

Corollary 2. Let R be a fully atomic strong GE-semifir which is commutator-pure and put $U=U(R)$. Then

$$U^{\times ab} = G(R)^{ab} \times D(R).$$

In section 4 we shall prove that free algebras and certain related rings are commutator-pure. An example of a fully atomic strong GE-ring which is not commutator-pure is the skew polynomial ring $S=k[x;\alpha]$ where k is a commutative field and $1\neq\alpha\in \mathrm{Aut}(k)$. S is a Euclidean, hence an Ore, ring and $K=k(x;\alpha)$ is its universal field of fractions. The group of units of S is just $k^{\times}$, which is abelian, hence $G(S)'=1$. From the commutation formula $ax=xa^{\alpha}$, $a\in k$, we obtain the relation

$$a^{-1}x^{-1}ax=a^{-1}a^{\alpha}$$

over K which shows that $a^{-1}a^{\alpha}\in K^{\times\prime}$. Picking a so that $a^{\alpha}\neq a$ we have $a^{-1}a^{\alpha}\in (G(S)\cap K^{\times\prime})\setminus G(S)'$.

Another interesting example of a non-commutator-pure fully atomic strong GE-ring is the following. Let k be a commutative field and let k(a), k(b) be isomorphic simple algebraic extensions of k. Adjoin commuting indeterminates x, y to k(a) and k(b), respectively, and put

$$R = k(x)(a)\underset{k}{\sqcup}k(y)(b)$$

and $U=U(R)$. Every field is a strong GE-ring, hence R, as a coproduct of strong GE-rings, is itself a strong GE-ring. Moreover R is a fir, by Theorem 5.3.2 of [5], and so it is fully atomic. The group of units of R can be determined using Proposition 5.3.4 of [5]; we have:

$$G(R) = k(x)(a)^x \underset{k^x}{*} k(y)(b)^x,$$

viz. the group theoretical coproduct of $k(x)(a)^x$ and $k(y)(b)^x$, amalgamating k^x. We claim that

$$b^{-1}a \in (G(R) \cap U^{x\prime}) \setminus G(R)'. \qquad (11)$$

The groups k^x, $k(x)(a)^x$, $k(y)(b)^x$ are abelian; in the category of abelian groups the pushout of $k^x \subset k(x)(a)$ and $k^x \subset k(y)(b)$ is

$$P = \frac{k(x)(a)^x \times k(y)(b)^x}{H},$$

where $H = \{(c, c^{-1}) \in k(x)(a)^x \times k(y)(b)^x \mid c \in k^x\}$. By the coproduct property of $G(R)$ we obtain a homomorphism $h : G(R) \to P$, then $G(R)'$ is contained in the kernel of h since P is abelian. However, $b^{-1}a \notin \ker h$ and therefore $b^{-1}a \notin G(R)'$. Now the centre of U is k by Theorem 4.7 of [7]; we have adjoined the indeterminates so as to be able to apply this theorem. Thus a and b are zeros of the same irreducible polynomial over $\operatorname{ctr} U$; it follows by the Skolem-Noether theorem that a and b are conjugates in U. Consequently $a = t^{-1}bt$, for some $t \in U^x$, and hence $b^{-1}a = b^{-1}t^{-1}bt$ which proves (11). Hence R is not commutator-pure.

Remark 1. As we have mentioned in the previous section, for any monoid M the universal abelian monoid, M^{ab}, can be obtained by factoring out the congruence relation generated by $a \sim b$, $a, b \in M$. Observe that if M' is a submonoid of M, closed under factorizations, then $M'^{ab} \to M^{ab}$ is an embedding. It follows that in the setting of Theorem 2.3 $GL(R)^{ab} \to \Sigma^{ab}$ is an embedding. Consider the natural map $\Sigma \to D'(R)$ (cf. (10)). This induces a surjection

$$\frac{\Sigma^{ab}}{GL(R)^{ab}} \to D'(R)$$

and an argument similar to the proof of Theorem 2.3 shows that it is one-to-one. Hence $\Sigma^{ab}/GL(R)^{ab} \cong D'(R)$. In particular $\Sigma^{ab} \to GL(U)^{ab}$ is an embedding if and only if $GL(R)^{ab} \to GL(U)^{ab}$ is an embedding and the former is clearly equivalent to Σ^{ab} having cancellation. Thus the examples preceding this remark show that Σ^{ab} need not have cancellation.

Remark 2. For any Sylvester domain the divisor group of R can be defined as the cokernel of the natural map

$$GL(R)^{ab} \to GL(U)^{ab}$$

where $U = U(R)$. From Theorem 2.4 it follows that this definition coincides with the one given for fully atomic semifirs. When R is a semifir the divisor group has the following interpretation. We have indicated at the beginning of this section that torsion modules over R and homomorphism between them form an abelian category $\mathcal{T}_R$. In this category $\oplus$ is a product so we can form $K_o \mathcal{T}_R$, the Grothendieck group of $\mathcal{T}_R$ (cf. [2]). We claim that

$$K_o \mathcal{T}_R = D(R).$$

Put Σ for the monoid of full matrices over R and denote by $[[A]]$ the image of $A \in \Sigma$ under the canonical map

$$\Sigma \to \frac{GL(U)^{ab}}{GL(R)^{ab}} = D(R).$$

Define a map

$$\gamma : \mathrm{Ob}\,\mathcal{T}_R \to D(R), \qquad M \mapsto [[A]],$$

where A is any full matrix presenting the torsion module M. Let M_1 and M_2 be torsion modules over R presented by full matrices A_1 and A_2, respectively: $M_1 \cong M_2$ if and only if A_1 and A_2 are stably associated over R ([6; Thm.2.1, Corollary]) and, further, $M_1 \oplus M_2$ is presented by $A_1 \dotplus A_2$, as is easily verified. Using these facts it is not hard to show that γ satisfies the required universal property.

3.3 Commutator-pure embeddings of fields

In this section we establish some results which will be needed later. Let D be a field with a central subfield k; our aim is to find a commutator-pure embedding of D into a field E, such that ctrE=k and, further, E and k satisfy the hypotheses of the specialization lemma (Thm. 1.3.1).

We begin by recalling the construction of a certain type of fields of formal power series. Let K be a field and let δ be a derivation on K; the set of all formal power series in y, of form

$$\sum_{i=o}^{\infty} a_i y_i \,, \qquad (a_i \in K)$$

can be made into a ring with the usual addition and multiplication defined by the commutation formula

$$ya = (\sum_{i=o}^{\infty} a^{\delta^i} y^i)\, y \qquad \text{for all } a \in K.$$

The ring so obtained is denoted by $K[[y;1,\delta]]$. Then $K[[y;1,\delta]]$ is a PID, its field of quotients is the field formal Laurent series in y which we denote by $K((y;1,\delta))$. We note that in $K((y;1,\delta))$ the commutation formula for y^{-1} is

$$y^{-1}a = ay^{-1} - a^{\delta} \qquad \text{for all } a \in K.$$

Put $S=K[[y;1,\delta]]$ and $D=K((y;1,\delta))$; every non-zero element f of D can be written in normal form as follows:

$$f = f_1 y^n,$$

where $n \in \mathbb{Z}$, $f_1 \in S \setminus Sy$ and both are unique. Let $n \in \mathbb{Z}$ and $f_1 \in S \setminus Sy$; it is easy to see that

$$y^n f_1 = f_1' y^n$$

for some $f_1' \in S \setminus Sy$ and the commutation formulae show that the constant terms of f_1 and f_1' agree. Furthermore $f_1^{-1} \in S \setminus Sy$ and the constant term of f_1^{-1} is the inverse of the constant term of f_1. These facts will be used in the proof of

Lemma 3.1. Let K be a field with centre k and let δ be a derivation on K. Put $D = K((y;1,\delta))$. Then the inclusion $K \subset D$ is commutator-pure. Furthermore $\mathrm{ctr} D = \mathrm{ctr} K \cap \ker \delta$ and if δ is non-zero then D is infinite dimensional over its centre.

Proof. Set $S = K[[y;1,\delta]]$. Consider first any commutator in D^x. Let

$$f = f_1 y^n \qquad \text{and} \qquad g = g_1 y^m$$

be elements of D in normal form and suppose that a and b are the constant terms of f and g, respectively. Then

$$(f,g) = y^{-n} f_1^{-1} y^{-m} g_1^{-1} f_1 y^n g_1 y^m$$

and we can pull powers of y to one side which, as we have seen, does not change the constant terms of f_1, g_1, f_1^{-1} and g_1^{-1}. It follows that

$$(f,g) = (a,b)+h,$$

$a, b \in K^x$ and $h \in Sy$. Assume now that $e \in K^x$ is a product of commutators in D^x:

$$e = \prod (f_i,g_i), \qquad f_i,\ g_i \in D^x.$$

By what has been said it is clear that

$$e = \prod((a_i,b_i)+h_i)$$

where a_i, $b_i \in K^x$ and $h_i \in Sy$. On comparing constant terms we find that

$$e = \prod (a_i,b_i)$$

and hence $e \in K^{x\prime}$. This proves that K is commutator-pure in D. As to the centre of D, it follows from the commutation rules that $\mathrm{ctr}D = \mathrm{ctr}K \cap \ker\delta$. Further if $\delta \neq 0$ then $y \notin \mathrm{ctr}D$ and hence $[D:\mathrm{ctr}D] = \infty$.

Proposition 3.2. Let E be a field with centre k and put $K = E(t)$. Write δ for the derivation d/dt on K and let D be as in Lemma 3.1. Then E is commutator-pure in D. Further, in characteristic 0 the centre of D is k while in characteristic p: $\mathrm{ctr}D = k(t^p)$. In either case $[D:\mathrm{ctr}D] = \infty$.

Proof. We know from the above lemma that E is commutator-pure in E((t)) (put $\delta=0$) and clearly this implies that E is commutator-pure in E(t). Applying Lemma 3.1 again we find that E(t) is commutator-pure in D; in consequence E is commutator-pure in D. Observe that in characteristic 0 the kernel of δ is E while in characteristic p $\ker\delta = E(t^p)$. Now the assertions of the proposition follow by Lemma 3.1.

Recall that a commutative field K is said to be a regular extension of its subfield k if $K \otimes \bar{k}$ is an integral domain, where $\bar{k}$ is the algebraic closure of k. In characteristic 0 this is equivalent to k being algebraically closed in K. Let K be a finitely generated regular extension of a field k of characteristic 0. It is well-known that K is either a rational function field $k(t_1,\dots,t_n)$ or is of form $k(t_1,\dots,t_n)(a)$ where a is algebraic over $k(t_1,\dots,t_n)$, but not over k. We shall want to know that K has a commutator-pure embedding into a field D with centre k, such that $[D:k] = \infty$. First we need a lemma.

Lemma 3.3. Let k be a commutative field of characteristic 0 and let $T=\{t_1,t_2,\dots\}$ be a set of commuting indeterminates indexed by the positive natural numbers. Set $F=k(T)$ and define a derivation d on F by putting $t_i^d=t_{i+1}$ for all i. Then

(i) $f\in k[t_1,\dots,t_{n+1}]\setminus k[t_1,\dots,t_n]$ implies that $f^d\in k[t_1,\dots,t_{n+2}]\setminus k[t_1,\dots,t_{n+1}]$,

(ii) $\ker d\cap k[T]=k$,

(iii) $f\in k[T]\Rightarrow \deg f=\deg f^d$,

(iv) $\ker d=k$.

Proof. The verification of (i) is straightforward and (ii) follows from (i). When f is a monomial (iii) can be proved by induction on the degree of f, the general case follows by linearity and (i).

(iv). It is clear that $k\subseteq\ker d$. Let $q\in F\setminus k$, say $q\in k(t_1,\dots,t_n)$; then $q=f/g$ where f and g are non-zero elements of $k[t_1,\dots,t_n]$. Furthermore we may assume f and g to be coprime because $k[t_1,\dots,t_n]$ is a UFD. Then

$$q^d=\frac{f^d g-f g^d}{g^2}$$

and thus $q^d=0$ if and only if $f^d g=f g^d$. Suppose $q^d=0$; then $f\mid f^d$ and $g\mid g^d$ since f and g are coprime. If $f^d=0$ then $g^d=0$ and by (ii) this would imply that $q\in k$. Hence $f^d\neq 0\neq g^d$ and so

$$f^d=f_1 f \quad\text{and}\quad g^d=g_1 g$$

for some $f_1, g_1 \in k[t_1,\ldots,t_{n+1}]$. Now (iii) implies that, in fact, f_1 and g_1 belong to k; but this is impossible since in f^d and g^d new indeterminates are introduced. Hence $q^d \in \ker d$ and this completes the proof of (iv).

Proposition 3.4. Let k be a commutative field of characteristic 0 and let K be a finitely generated regular extension of k. Then k has a regular extension L, containing K, with a derivation D such that $\ker D = k$.

Proof. The above lemma provides the required field extension of K and derivation if K is form $k(t_1,\ldots,t_n)$. Assume $K = k(t_1,\ldots,t_n)(a)$ where a is algebraic over $k(t_1,\ldots,t_n)$, but not over k. Let F and d be as in the above lemma and embed K in $L = F(a)$; then L is a regular extension of k and, by Lemma 3.3 (iv), $\ker d = k$. Now d extends uniquely to a derivation, say D, on L; writing p for the minimal polynomial of a over F we have

$$a^D = \frac{-p^d(a)}{p'(a)},$$

where p' is the usual derivative of p and p^d is obtained from p by applying d to its coefficients (cf. [16; Thm.4.3.5]). Not all the coefficients of p are in k since a is not algebraic over k; hence $p^d \neq 0$ and so $a^D \neq 0$. Let $b \in L \setminus k$, we claim that $b^D \neq 0$; this would clearly imply that $\ker D = k$. If b is an element of F then $b^D = b^d \neq 0$, by Lemma 3.3 (iv). Assume $b \notin F$, then b is algebraic over F, but not over k. By the above argument, putting b instead of a, we see that d can be extended to a unique derivation, say D_1, on $F(b)$ and further $b^{D_1} \neq 0$. Now $F(a)$ is a simple algebraic extension of $F(b)$ so D_1 can be uniquely extended to a derivation, say D_2, on $F(a)$. Thus both D and D_2 extend d; it follows that $D_2 = D$ and hence

$$b^D = b^{D_2} = b^{D_1} \neq 0,$$

as claimed.

Putting Lemma 3.1 and Proposition 3.4 together we obtain

Proposition 3.5. Let k be a commutative field of characteristic 0 and let K be a finitely generated regular extension of k. Then there is a commutator-pure embedding of K into a field E, such that $\mathrm{ctr}E = k$ and $[E:k] = \infty$.

Proof. By Proposition 3.4 we have a regular extension L of k, containing K, and a derivation D on L whose kernel is k. K is clearly commutator-pure in L. Set $E = L((y;1,D))$; by Lemma 3.1 L, hence also K, is commutator-pure in E. From the same lemma we deduce that $\mathrm{ctr}E = k$ and that E is infinite dimensional over k.

To see that not every commutative extension of k can be embedded in a field with properties as in the above proposition, assume that K contains two distinct elements, α and β, algebraic over k, with the same minimal polynomial. In any extension E of K, with centre k, α and β are conjugates by the Skolem-Noether theorem. Hence for the embedding $K \subset E$ to be commutator-pure α and β would have to coincide because K is commutative.

3.4 Applications

From Theorem 2.4, Corollary we know that if R is a fully atomic strong GE-ring with universal field of fractions U, in order to describe the structure of U^{xab} we have to find $G(R) \cap U^{x1}$. Now we shall consider two classes of rings, namely: skew polynomial rings $K[x;\alpha]$, where K is a field and $\alpha \in \mathrm{End}D$, and free E-rings $E_k\langle X\rangle$ where E is a field with centre k. All these are filtered rings with weak algorithm and hence fully atomic strong GE-rings (cf. [4; Thm.2.2.4, Thm2.2.5 and Exercises 2.4.6 and 2.4.7]). In particular, we shall prove that if E and k are fields which satisfy the hypotheses of the specialization lemma then $E_k\langle X\rangle$ is commutator-pure. This result will be used to show that certain related rings are also commutator-pure.

Let K be a field and let α be an endomorphism of K, put $R = K[x;\alpha]$. The commutation rule for x is

$$ax = xa^{\alpha} \qquad \text{for all } a \in K.$$

R is a right Ore domain so U(R) is its usual field of quotients, $K(x;\alpha)$, which will be denoted by U. Further $G(R) = K^{x}$, as is easily checked. Using the same commutation formula we can form the ring of formal power series $S = K[[x;\alpha]]$ which is a local right PID. Write F for its field of quotients $K((x;\alpha))$; we note that every element of F can be written in the following form:

$$x^{t}(a+h)x^{-s},$$

where $t,\ s \in \mathbb{N}$, $a \in K^{\times}$ and $h \in xS$. The commuting square of embeddings

$$\begin{array}{ccc} R & \longrightarrow & U \\ \downarrow & & \downarrow \\ S & \longrightarrow & F \end{array}$$

will be used to determine $G(R) \cap U^{\times\prime} = K^{\times} \cap U^{\times\prime}$. Retaining the above notation we first prove

Lemma 5.1. Let $f,\ g \in U^{\times}$ and, considered as elements of F, put

$$f = x^{t_1}(a+f_1)\,x^{-s_1} \quad \text{and} \quad g = x^{t_2}(b+g_1)\,x^{-s_2},$$

where $t_1,\ t_2,\ g_1,\ g_2 \in \mathbb{N}$, $a,\ b \in K^{\times}$ and $f_1,\ g_1 \in xS$. Then

$$(f,g) = x^{m}(c+h)x^{-m},$$

where $m = s_1+s_2$ and

$$c = (a^{-1})^{\alpha^{s_2}}(b^{-1})^{\alpha^{t_1}}a^{\alpha^{t_2}}b^{\alpha^{s_1}}.$$

Proof. Using elementary facts about F we have

$$(f,g) = x^{s_1}(a^{-1}+f_1')x^{-t_1}x^{s_2}(b^{-1}+g_1')x^{-t_2}.$$
$$.x^{t_1}(a+f_1)x^{-s_1}x^{t_2}(b+g_1)x^{-s_2}$$

for suitable $f_1',\ g_1' \in xS$. By the commutation rule positive powers of x can be moved to the left while negative powers of x can be moved to the right and thus we obtain

$$(f,g) = x^{s_1+s_2}((a^{-1})^{\alpha^{s_2}} + f_1^{(2)})((b^{-1})^{\alpha^{t_1}} + g_1^{(2)}).$$
$$.(a^{\alpha^{t_2}} + f_1^{(3)})(b^{\alpha^{s_1}} + g_1^{(3)})x^{-(s_1+s_2)}$$

for suitable $f_1^{(2)}$, $f_1^{(3)}$, $g_1^{(2)}$, $g_1^{(3)} \in xS$. The assertion now easily follows.

Let us call an element c of K an <u>α-commutator</u> if

$$c = (a^{-1})^{\alpha^{n_1}}(b^{-1})^{\alpha^{n_2}} a^{\alpha^{n_3}} b^{\alpha^{n_4}}$$

for some a, $b \in K^x$ and $n_i \geq 0$ (e.g. 1-commutators are just the usual commutators). Assume now that

$$d = \prod_{i=1}^{n} (f_i, g_i) \in G(R) \cap U^{x\prime} = K^x \cap U^{x\prime},$$

where f_i, $g_i \in U^x$, and let

$$f_i = x^{t_{i1}}(a_i + f_{i1})x^{-s_{i1}} \quad \text{and} \quad g_i = x^{t_{i2}}(b_i + g_{i1})x^{-s_{i2}},$$

a_i, $b_i \in K^x$, f_{i1}, $g_{i1} \in xS$, t_{i1}, t_{i2}, s_{i1}, $s_{i2} \in \mathbb{N}$, $i=1,\dots,n$. Put $m_i = s_{i1} + s_{i2}$; by the above lemma we have

$$d = \prod_{i=1}^{n} x^{m_i}((a_i^{-1})^{\alpha^{s_{i2}}}(b_i^{-1})^{\alpha^{t_{i1}}} a_i^{\alpha^{t_{i2}}} b_i^{\alpha^{s_{i1}}} + h_i)x^{-m_i}$$

where $h_i \in xS$. We may again pull through positive powers of x to the left and negative powers of x to the right; setting $m = \sum_i m_i$ and $m_i' = \sum_{j \neq i} m_j$ we find

$$x^{-m} d\, x^m = d^{\alpha^m} =$$
$$= \prod_{i=1}^{n} (a_i^{-1})^{\alpha^{(s_{i2}+m_i')}} (b_i^{-1})^{\alpha^{(t_{i1}+m_i')}} a_i^{\alpha^{(t_{i2}+m_i')}} b^{\alpha^{(s_{i1}+m_i')}} +$$
$$+ h',$$

where $h' \in xS$ and, in fact, $h' = 0$ since $d^{\alpha^m} \in K$. We have shown part of

<u>Proposition 4.2</u>. Let K be a field and let $\alpha \in \operatorname{End} K$. Put $R = K[x;\alpha]$ and $U = K(x;\alpha)$; then

$$G(R) \cap U^{\times\prime} = \left\{ d \in K^{\times} \;\middle|\; d^{\alpha^m} \text{ is a product of } \alpha\text{-commutators for some } m \geq 0 \right\}.$$

<u>Proof</u>. It remains to show that if d^{α^m} is a product of α-commutators then $d \in K^{\times} \cap U^{\times\prime}$. Let $a, b \in K^{\times}$ and let $n_i \in \mathbb{N}$, $i=1,2,3,4$. Then

$$f = x^{n_2} a\, x^{-n_4} \quad \text{and} \quad g = x^{n_3} b\, x^{-n_1}$$

are elements of $U^{\times}$ and further

$$(f,g) = x^{n_1+n_4}\left((a^{-1})^{\alpha^{n_1}} (b^{-1})^{\alpha^{n_2}} a^{\alpha^{n_3}} b^{\alpha^{n_4}}\right) x^{-(n_1+n_2)}.$$

It follows that α-commutators belong to $G(R) \cap U^{\times\prime}$. Now let $d \in K^{\times}$ be such that for some $m \geq 0$ d^{α^m} is a product of α-commutators. Then clearly $d^{\alpha^m} \in K^{\times} \cap U^{\times\prime}$ and, since $d^{\alpha^m} = x^{-m} d\, x^m$, d is also in $K^{\times} \cap U^{\times\prime}$, as claimed.

Assuming that α is an automorphism, the above proposition states that $G(R) \cap U^{\times\prime}$ is the subgroup of $K^{\times}$ generated by the α-commutators. If α is inner, it is easily checked that each α-commutator is a product of commutators. Hence, in this case, R is commutator-pure and so

$$U^{\times ab} \cong K^{\times ab} \times D(R),$$

by Theorem 2.4, Corollary 2. At the other extreme we find fields with an automorphism α, such that every element of the field is a product of α-commutators; then $U^{xab} \cong D(R)$. For instance, let K be the algebraic closure of F_p and consider the automorphism $\alpha : a \mapsto a^p$. For each $a \in K$

$$a^{-1} a^{\alpha} = a^{p-1}$$

is an α-commutator. Moreover every element of K^x is a $(p-1)^{th}$ root and hence an α-commutator.

We now turn to free E-rings. Let E be field with a central subfield k and let X be a set. We have seen that $E_k\langle X\rangle$ is a fully atomic strong GE-ring. We first show

Proposition 4.3. Let E be a field with centre k and let X be a set. Assume that (i) k is infinite and (ii) $[E:k] = \infty$. Then $E_k\langle X\rangle$ is commutator-pure.

Proof. The group of units of $E_k\langle X\rangle$ is E^x so we have to prove that

$$E^x \cap E_k\langle X\rangle^{x'} = E^{x'}.$$

Let $d \in E^x \cap E_k\langle X\rangle$ and assume that

$$d = \prod_{i=1}^{n} (a_i, b_i) \qquad (a_i,\ b_i \in E_k\langle X\rangle,\ n \geq 1).$$

By Theorem 1.3.2 we can choose a homomorphism of an E-subring of $E_k\langle X\rangle$, containing $\{a_i, a_i^{-1}, b_i, b_i^{-1}\}_{i=1,\ldots,n}$, onto E, which keeps d fixed. We find that d is a product of commutators in E^x, hence $E^x \cap E_k\langle X\rangle \subseteq E^x$. The reverse inclusion is obvious.

In order to prove similar results when (i) or (ii) of the hypotheses is not satisfied we prove

Lemma 4.4. Let $f : R \to S$ be an honest, commutator-pure homomorphism of Sylvester domains. If S is commutator-pure so is R.

Proof. By Theorem 1.2.1 we have the commuting diagram

$$\begin{array}{ccc} R & \longrightarrow & U(R) \\ f \downarrow & & \downarrow f' \\ S & \longrightarrow & U(S) \end{array} .$$

Let $a \in G(R) \cap U(R)^{\times\prime}$; to verify the assertion it will suffice to show that $a \in G(R)'$. Clearly $a^f \in G(S) \cap U(S)^{\times\prime}$ and so $a^f \in G(S)'$ because S is commutator-pure. But f is also commutator-pure which implies that $a \in G(R)'$.

Now we can strengthen Proposition 4.3. Let E be a field with a central subfield k. Suppose that D is a field extension of E with centre C, such that $k = C \cap D$ and further E and C satisfy the hypotheses of Proposition 4.3. The natural homomorphism $\Psi : E_k\langle X\rangle \to D_C\langle X\rangle$ need not be honest, but when Ψ is honest and commutator-pure the above lemma can be used to deduce that $E_k\langle X\rangle$ is commutator-pure. This is shown in

Theorem 4.5. Let D be a field with centre k, let E be a subfield of D and put $k = C \cap E$. Assume that (i) C is infinite, (ii) $[D:C] = \infty$, (iii) the inclusion $E \subset D$ is commutator-pure and (iv) the natural map $\Psi : E_k\langle X\rangle \longrightarrow D_C\langle X\rangle$ is honest. Then $E_k\langle X\rangle$ is commutator-pure.

Proof. Observe that if Ψ is an embedding then it is commutator-pure precisely when the inclusion $E \subset D$ is commutator-pure. Thus the assertion of the theorem follows by Proposition 4.3 and Lemma 4.4.

Corollary 1. Let E be a field with centre k and let X be a set. Then $E_k\langle X\rangle$ is commutator-pure.

Proof. Let D be as in Proposition 3.2 and assume first that k is of characteristic 0. Then the natural map $E_k\langle X\rangle \rightarrow D_k\langle X\rangle$ is honest, essentially by Proposition 5.4.2, Corollary of [5], and hence the claim follows by Proposition 3.2 and the theorem. Suppose now that $\mathrm{ch}\, k = p \neq 0$. Consider the maps

$$E_k\langle X\rangle \xrightarrow{\alpha} E(t^p)_{k(t^p)}\langle X\rangle \xrightarrow{\beta} D_{k(t^p)}\langle X\rangle .$$

From Proposition 3.2 we know that the inclusion $E \subset D$ is commutator-pure. Further α and β are honest, by Lemma 6.3.4 and Proposition 5.4.2, Corollary of [5], respectively, so $\alpha\beta$ is also honest. Thus D, $k(t^p)$, E and k satisfy the hypotheses of the theorem.

Corollary 2. Let k be a commutative field of characteistic 0 and let K be a regular extension of k. Further let X be a set. Then $K_k\langle X\rangle$ is commutator-pure.

Proof. Put $R = K_k\langle X\rangle$, then $G(R) = K^X$. We have to show that if $d \in K^X \cap U(R)^{X\prime}$ then $d \in K^{X\prime}$ (i.e. $d = 1$). Let $d \in K^X \cap U(R)^{X\prime}$ and suppose

$$d = \prod_{i=1}^{n} (a_i, b_i) \qquad\qquad (a_i,\ b_i \in U(R)^{X\prime}). \quad (12)$$

Every element of U(R) can be written as a (by no means unique) rational expression with parameters from K and X. Let S be a fixed set of rational expressions for d, a_i and b_i, i=1,...,n, further let A be the subset of K containing precisely those elements which are present in the expressions of S. Then A is a finite set. Put K_0 for the subfield of K (finitely) generated by A over k and write $R_0 = K_{0\,k}\langle X\rangle$. Then $U(R_0)$ is a subfield of U(R), by Proposition 5.4.2, Corollary, and hence (12) holds over $U(R_0)$. Thus if we could show that $d \in G(R_0)'$ it would follow that $d \in G(R)'$. Hence, without loss of generality we may assume that K is finitely generated over k. In Proposition 3.5 we have shown that K has a field extension D with centre k, such that D, K and k satisfy the hypotheses of the theorem. This completes the proof.

As an application consider the free algebra $k\langle X\rangle$. By Corollary 1 $k\langle X\rangle$ is commutator-pure and hence by Theorem 2.4, Corollary 2 we may deduce that

$$k\langle X\rangle \cong k^{x} \times D(k\langle X\rangle).$$

The above theorem and its corollaries enable us to show that certain related rings are also commutator-pure. Consider first the coproduct

$$R_2 = E \coprod_k kF,$$

where E is a field with a central subfield k, F is the free group on a set X and kF is the group algebra on F over k. Let $k[x]_x$ denote the ring obtained from the polynomial ring $k[x]$ by localization at $\{x^i \mid i \in \mathbb{N}\}$; it is easy to see that

$$kF = \coprod_k k[x]_x \qquad (x \in X)$$

and hence

$$R_2 = E \underset{k}{\sqcup}(\underset{k}{\sqcup} k[x]_x). \qquad (x \in X)$$

(R_2 and kF are sometimes denoted by $E_k\langle X, X^{-1}\rangle$ and $k\langle X, X^{-1}\rangle$.) For all $x \in X$, $k[x]_x$ is a PID, hence a fir and a GE-ring (cf. [4; Thm.8.1.1]). Thus R_2, as a coproduct of firs which are also GE-rings, is a fully atomic strong GE-ring. Set

$$R_1 = E \underset{k}{\sqcup} (\underset{k}{\sqcup} k[x]) = E_k\langle X\rangle \quad \text{and} \quad R_3 = E \underset{k}{\sqcup} (\underset{k}{\sqcup} k(x)) ;$$

then R_1 and R_3 are also fully atomic srong GE-rings. The inclusions

$$k[x] \subset k[x]_x \subset k(x)$$

are epic in the category of rings and hence, by the coproduct property, they induce epimorphisms

$$R_1 \xrightarrow{\alpha} R_2 \xrightarrow{\beta} R_3.$$

Moreover α and β are honest, essentially by Lemma 5.4.1 of [5], so we may take $R_1 \subset R_2 \subset R_3$ and then $U(R_1) = U(R_2) = U(R_3)$. We aim to prove that R_2 and R_3 are commutator-pure, provided R_1 is so. We need a lemma first.

<u>Lemma 4.6</u>. Let E be a field with a central subfield k and let x and y be elements of a set X. Then

(i) if $p \in k[x]$ is irreducible then it is an atom in $E_k\langle X\rangle$ and

(ii) if $p \in k[x]$ and $q \in k[y]$ are irreducible polynomials and p is stably associated to q over $E_k\langle X\rangle$ then $x = y$ and $p = cq$ for some $c \in k$.

<u>Proof</u>. (i). (P.M. Cohn) Assume p is not an atom in $E_k\langle X\rangle$,

we show this leads to a contradiction. Let $p = fg$, $f, g \in E_k\langle X\rangle$, $\deg f, \deg g \geq 1$. Clearly f and g belong to $E_k\langle x\rangle$; we put R for this ring. Then R is a fir, hence a UFD by Theorem 3.2.2, Corollary of [4] (see [4] for the definition); in particular R is atomic. Thus without loss of generality we may assume that f is an atom. Further p has non-zero constant term so we may also assume that $f = 1+f_1$ where the constant term of f_1 is 0. We have

$$xfg = xp = px = fgx$$

whence $fR \cap xfR \neq 0$. Thus $fR + xfR$ is a principal right ideal (cf. [4; Thm.1.1.1]) and since f is an atom it follows now that either $fR \supseteq xfR$ or $fR + xfR = R$. Assume the former, then $fh = xf$ for some $h \in R$ and clearly $\deg h = \deg x = 1$. Moreover the constant term of h is 0. Now we have

$$fh = (1+f_1)h = h + f_1 h = xf = x + x f_1 ;$$

equating coefficients we find $h=x$. Hence $fx=xf$ and consequently: $f \in k[x]$. But then g is also in $k[x]$ contradicting the assumption that p is irreducible.

Suppose now that $fR + xfR = R$; then $fu + xfv = 1$ for some $u, v \in R$. Let $\bar{a}$ denote the image of $a \in R$ under the canonical surjection $E_k\langle x\rangle \longrightarrow E[x]$. Passing over to $E[x]$ we find

$$\bar{f}(\bar{u} + \overline{xv}) = 1$$

whence $\bar{f} = 1$. Thus

$$p = \bar{p} = \bar{f}\,\bar{g} = \bar{g}$$

and since $\deg g \geq \deg \bar{g}$ we deduce that $\deg f = \deg \bar{f} = 0$, a contradiction.

(ii). Set $R = E_k\langle X\rangle$ and suppose

$$U\begin{pmatrix} p & 0 \\ 0 & I \end{pmatrix} V = \begin{pmatrix} q & 0 \\ 0 & I \end{pmatrix} \qquad (13)$$

where $U, V \in GL_n(R)$, $n \geq 2$. Putting $z = 0$ for all $x \neq z \in X$ we find that $x = y$ and p is stably associated to q over $E_k\langle x\rangle$. On applying the natural homomorphism $E_k\langle x\rangle \to E[x]$ we deduce that p is also stably associated to q over $E[x]$. Moreover, if E^* is any field extension of E then p and q are stably associated over $E^*[x]$. Choose E^* so that it contains a zero of p, say α (e.g. $E^* = U(E \underset{k}{\sqcup} \bar{k})$, $\bar{k}$ is the algebraic closure of k). Now (13) holds with $U, V \in GL_n(E^*[x])$. Put $x = \alpha$; it is easy to see that α is also a zero of q and hence $p = cq$ for some $c \in k$.

<u>Theorem 4.7</u>. Let E be a field with a central subfield k and let X be a set. Put

$$R_1 = E_k\langle X\rangle, \quad R_2 = E \underset{k}{\sqcup} (\underset{k}{\sqcup} k[x]_x) \text{ and } R_3 = E \underset{k}{\sqcup} (\underset{k}{\sqcup} k(x)). \quad (x \in X)$$

If R_1 is commutator-pure then so are R_2 and R_3.

<u>Proof</u>. We have seen that $U(R_1) = U(R_2) = U(R_3)$; we put U for this field. The groups of units of the R_i may be determined using Proposition 5.3.4 of [5]; we have

$$G(R_1) = E^x \subseteq G(R_2) = E^x \underset{k^x}{*} (k^x \times F) \subseteq G(R_3) = E^x \underset{k^x}{*} (\underset{k^x}{*} k(x)^x), \quad (x \in X)$$

where F denotes the free group on X. Assume that R_1 is commutator-pure, we show that R_3 is also commutator-pure;

the same argument works for R_2. Without loss of generality we may assume that $X=\{x_1,x_2,\dots,x_n\}$. Let $d\in G(R_3)\cap U^{X'}$, we have to show that $d\in G(R_3)'$. We know d is of the following form:

$$d = e_1\frac{f_1'}{g_1'}e_2\frac{f_2'}{g_2'}\cdots e_m\frac{f_m'}{g_m'}, \quad f_j',\ g_j'\in k[x_{i(j)}]^X,\ e_j\in E^X;$$

but of course the j(i) need not be distinct. However, we can modify d multiplying it by a suitable element of $G(R_3)'$ to obtain

$$d_1 = e\frac{f_1}{g_1}\frac{f_2}{g_2}\cdots\frac{f_n}{g_n}, \quad f_i,\ g_i\in k[x_i]^X,\ e\in E^X.$$

Clearly, $d_1\equiv d \pmod{G(R_3)'}$. Furthermore, setting $f=f_1f_2\dots f_n$ and $g=g_1g_2\dots g_n$ we have

$$d_2=efg^{-1}\equiv d_1\equiv d \pmod{G(R_3)'}. \tag{14}$$

Now $f,\ g\in R_1$; further f and g have the same prime divisors over R_1 because $d_2\equiv d_1\equiv d\equiv 1 \pmod{U^{X'}}$. Each f_i and g_i can be written as a product of irreducible polynomials in $k[x_i]$; by Lemma 4.6 (i) we thus obtain atomic factorizations of f and g in R_1. Furthermore the atomic factors of f and g are pairwise stably associated over R_1 (cf. Theorem 2.1 above). By Lemma 4.6 (ii) this means that, in fact, the atomic factors of f and g are pairwise k-associated, moreover factors of f_i can only be k-associated to factors of g_i. We deduce that for each i

$$f_i=c_i\,g_i \qquad (c_i\in k^X)$$

and consequently:

$$d_1 = ec_1 c_2 \dots c_n \in E^X \cap U^{X\prime} = G(R_1) \cap U^{X\prime}.$$

We have assumed that R_1 is commutator-pure, hence $d_1 \in E^X$ and from (14) we deduce that $d \in G(R_3)'$. This completes the proof.

Corollary. Let E be a field with a central subfield k and let X be a set. Let R_2 and R_3 be as in the theorem. If (a) $\operatorname{ctr} E = k$ or (b) $\operatorname{ch} k = 0$ and E is a regular extension of k, then R_2 and R_3 are commutator-pure.

Proof. The assertions follow from the theorem by Theorem 4.5, Corollaries 1 and 2.

We remark that the converse of the above theorem also holds as is easily verified.

In section 2 we have given an example which demonstrates that the coproduct of fields over a common subfield need not be commutator-pure; it also shows that elements algebraic over the ground field are relevant here. On the other hand, when $E = k$, Theorem 4.7, Corollary states that the coproduct over k of simple transcendental extensions of k is commutator-pure. Our last result generalizes this in characteristic 0. Let k be a commutative field of characteristic 0 and let $\{K_i\}$ be a family of regular extensions of k. We shall show that $R = \bigsqcup_k K_i$ is commutator-pure. Our plan is to construct a commutator-pure, fully atomic strong GE-ring S and a family $\{\alpha_i\}$ of honest embeddings $R \to S$ with the following property: for each $d \in G(R) \setminus G(R)'$ there exists $\alpha \in \{\alpha_i\}$ such that $d^{\alpha} \in G(S) \setminus G(S)'$. (One could say that R is then locally commutator-pure in S.) We prove a couple of lemmas first.

Lemma 4.8. Let F, G, H be fields with a common central subfield k. Let α be a k-algebra homomorphism of E into F. Then the natural homomorphism

$$\alpha' : E \underset{k}{\sqcup} G \longrightarrow F \underset{k}{\sqcup} G,$$

induced by α, is honest.

Proof. Write E^{α} for the image of E in F, then clearly $E \underset{k}{\sqcup} G \cong E^{\alpha} \underset{k}{\sqcup} G$ and further α' can be decomposed as follows:

$$E \underset{k}{\sqcup} G \cong E^{\alpha} \underset{k}{\sqcup} G \subseteq F \underset{k}{\sqcup} G. \tag{15}$$

Every isomorphism is honest, to see that the above inclusion is also honest consider the natural maps

$$E^{\alpha} \underset{k}{\sqcup} G \subseteq F \underset{k}{\sqcup} G \rightarrow F \underset{E^{\alpha}}{\sqcup} U(E^{\alpha} \underset{k}{\sqcup} G)$$

Let A be a full matrix over $E^{\alpha} \underset{k}{\sqcup} G$; then A becomes invertible over $F \underset{E^{\alpha}}{\sqcup} U(E^{\alpha} \underset{k}{\sqcup} G)$ and hence full over $F \underset{k}{\sqcup} G$. Thus the inclusion in (15) is honest and consequently α', as a composite of honest maps, is honest.

We note that the proof of the above lemma is based on the proof of Proposition 5.4.2, Corollary of [5].

Lemma 4.9. Let k be a commutative field of characteristic 0 and let L be a finitely generated regular extension of k. Then L has a commutative field extension L*, finitely generated and regular over k, such that there exists a homomorphism $L \rightarrow L^*$ which keeps no element of $L \setminus k$

fixed.

Proof. Put L^* for the field of quotients of $L \otimes_k L$; then L^* is finitely generated and regular over k (cf. [16; Thm. 4.5.2 and Thm. 4.5.3]). We can consider L as a subfield of L^* identifying $a \in L$ with $a \otimes 1 \in L^*$. Define an automorphism φ of L^* by the rule

$$a \otimes 1 \longmapsto 1 \otimes a.$$

Then φ, restricted to L, clearly has the required property.

We can now prove the promised generalization of Theorem 4.7, Corollary 2.

Theorem 4.10. Let k be a commutative field of characteristic 0 and let $\{K_i\}$ be a family of regular extensions of k. Then $\bigsqcup_k K_i$ is commutator-pure.

Proof. Using an argument similar to the one employed at the beginning of the proof of Theorem 4.5, Corollary 2 we may assume that the family $\{K_i\}$ is finite, say of cardinality n, and each K_i is finitely generated over k. We put $R = \bigsqcup_k K_i$. Let $T = \bigotimes_k K_i$, i=1,...,n; then T is an integral domain and its field of quotients, say L, is a finitely generated regular extension of k. When $n = 2$, this follows from Theorem 4.3.2 and Theorem 4.5.3 of [16], the general case can be easily proved by induction. For each i let α_i denote the embedding of K_i in L by the rule

$$a \longmapsto 1 \otimes .. 1 \otimes a \otimes 1 .. \otimes 1 \qquad \text{for all } a \in K_i;$$

then the α_i agree on k. Let L* be as in Lemma 4.9 and let φ be a homomorphism of L into L* which keeps only elements of k fixed. For each i, α_i can be viewed as a map $K_i \rightarrow L^*$ and then

$$a^{\alpha_i \varphi} = a^{\alpha_i}$$

precisely when $a \in k$. For any homomorphism λ of K_i into L^*, $i \in \{1,\ldots,n\}$, we define a homomorphism

$$\bar{\lambda} : K_i \longrightarrow L^* \coprod_k k(z)$$

as follows:

$$\bar{\lambda} : a \longmapsto z^{-i} a^{\lambda} z^{i} \qquad \text{for all } a \in K_i.$$

Hence for each i we have a map

$$\overline{\alpha_i} : K_i \longrightarrow L^* \coprod_k k(z);$$

and further, the family $\{\overline{\alpha_1}, \ldots, \overline{\alpha_n}\}$ induces a homomorphism

$$\bar{\alpha} : R = \coprod_k K_i \longrightarrow L^* \coprod_k k(z)$$

by the coproduct proprty of R. Similarly, for each $j \in \{1,\ldots,n\}$ let $\overline{\beta_j} : R \longrightarrow L^* \coprod_k k(z)$ be the homomorphism induced by the family of maps

$$\{\overline{\alpha_1}, \ldots, \overline{\alpha_{j-1}}, \overline{\alpha_j \varphi}, \overline{\alpha_{j+1}}, \ldots, \overline{\alpha_n}\}.$$

Then $\bar{\alpha}$ and the $\overline{\beta_j}$ are honest, essentially by Lemma 5.5.4 of [5] and hence the diagram

$$\begin{array}{ccc} R = \coprod_k K_i & \longrightarrow & U = U(R) \\ \downarrow & & \downarrow \\ L^* \coprod_k k(z) & \longrightarrow & V = U(L^* \coprod_k k(z)) = L^*_k\langle\langle z\rangle\rangle \end{array}$$

commutes using $\overline{\alpha}$ or any of the $\overline{\beta_j}$, and their extensions to embeddings $U \to V$, as vertical maps (cf. Thm.1.2.1). Let $d \in G(R) \cap U^{x\prime}$; we have to show that $d \in G(R)'$. By Proposition 5.3.4 of [5] we have

$$G(R) = \mathop{*}_{k^x} K_i^x$$

and thus

$$d = c_1' c_2' \dots c_m', \qquad c_j' \in K_{i(j)}.$$

We can multiply d by a suitable element of G(R)' to obtain

$$d_1 = c_1 c_2 \dots c_n, \qquad c_j \in K_j$$

and then clearly

$$d_1 \equiv d \pmod{G(R)'}.$$

We claim that $d_1 \in k$. Apply $\overline{\alpha}$ to d_1: we have

$$d_1^{\overline{\alpha}} = (z^{-1} c_1^{\alpha_1} z)(z^{-2} c_2^{\alpha_2} z^2)\dots(z^{-n} c_n^{\alpha_n} z^n) \in V^{x\prime}.$$

Moreover mod $V^{x\prime}$ we can pull all powers of z through to one side and then we find that

$$c = c_1^{\alpha_1} c_2^{\alpha_2} \dots c_n^{\alpha_n} \in V^{x\prime}.$$

Now each $c_j^{\alpha_j}$ is in L* so $c \in L^{*x} \cap V^{x\prime}$; but $L^{*x} \cap V^{x\prime} = 1$, by Theorem 4.5, Corollary 2 and hence $c = 1$. It follows that for each $j \in \{1,\ldots,n\}$

$$c_j^{\alpha_j} = (\prod_{i \neq j} c_i^{\alpha_i})^{-1} . \tag{16}$$

Similarly, applying $\overline{\beta_j}$ to d_1 we obtain

$$d_1^{\overline{\beta_j}} = (z^{-1} c_1^{\alpha_1} z) \ldots (z^{-j} c_j^{\alpha_j \varphi} z^j) \ldots (z^{-n} c_n^{\alpha_n} z^n) .$$

Further $d_1^{\overline{\beta_j}} \in V^{x\prime}$ and mod $V^{x\prime}$ we can pull powers of z through; thus we find that

$$c_1^{\alpha_1} \ldots c_{j-1}^{\alpha_{j-1}} c_j^{\alpha_j \varphi} c_{j+1}^{\alpha_{j+1}} \ldots c_n^{\alpha_n} \in L^{*x} \cap V^{x\prime} = 1 .$$

Hence

$$c_j^{\alpha_j \varphi} = (\prod_{i \neq j} c_i^{\alpha_i})^{-1}$$

and comparing this to (16) we have $c_j^{\alpha_j} = c_j^{\alpha_j \varphi}$. But φ was chosen so that it fixes only elements of k; consequently $c_j \in k$ for all j and so $d_1 \in k$. It follows that $d_1^{\overline{\alpha}} \in k \cap V^{x\prime} = 1$ and hence $d_1 = 1$. Thus we have shown that $d \equiv 1 \pmod{G(R)'}$ and this completes the proof.

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