GENERALIZATIONS OF CONVEXITY

Ph.D THESIS

DAVID IAN CALVERT

ROYAL HOLLOWAY COLLEGE

ABSTRACT

Generalizations of Convexity

by

DAVID IAN CALVERT

I consider four generalizations of the concept of a convex set in R^d.

A subset X of R belongs to the family $T(\underline{a})$ if for all $x, y \in X$ $\underline{a}x + (1-\underline{a})y \in X$ where $\underline{a} \in R$. Properties of elements of $T(\underline{a})$ are considered in Chapter 1.

Also in Chapter 1 a planar generalization of the family $T(\underline{a})$ is considered.

In Chapter 2 a study of m-convex sets is made and the extensive literature is constructively reviewed.

In Chapter 3 some properties of locally starshaped sets are obtained.

CONTENTS

PREFACE

I would like to take this opportunity to thank my supervisor H.G. Eggleston for the many helpful comments he has made on both the presentation and contents of this thesis.

Specific results and comments of his are acknowledged at the point in the text where they appear.

Thanks to Bev for typing the thesis.

SYMBOLS

Uncommon symbols used in the text are described when they first appear. Below more common ones are described.

The real number parameter $\underline{a}$ is underlined in Chapter 1 as are vectors which in this thesis are points of R^d, $d \geq 2$. In Chapter 2 and Chapter 3 however, when no ambiguity results, the underlining is usually dispensed with.

Symbol	Description
R	Set of real numbers.
$\|X\|$	Cardinality of the set X.
Z	Set of integers.
$\|x\|$	Absolute value of the real number x.
$\|x\|$ or $\|\underline{x}\|$	Length of the vector x
aff X or aff $\{X\}$	Affine hull of the set X.
$S(x,\delta)$	Set of vectors y such that $\|x-y\| < \delta$ and $y \in R^d$ (d obvious from context).
conv X or conv $\{X\}$	Convex hull of the set X.
$[x_1,\ldots,x_n]$	Convex hull of $\{x_1,\ldots,x_n\}$.
(x_1,x_2)	Convex hull of $\{x_1,x_2\}$ without x_1 and x_2.
Fr X	Frontier of X.
Int X or int$\{X\}$	Interior of X in some R^d (d obvious from context).
rel int X	Interior of X in aff X.
Ker X	Set of points y such that for all $x \in X$ $[x,y] \subset X$.

CHAPTER 1

The first part of this chapter appears in my paper [1.1].

Definition 1.1: A subset, X, of R belongs to the family $T(a)$ if $|X| \geq 2$ and for all x and y belonging to X $ax + (1 - a)y \in X$ where $a \in R$.

Firstly, I consider the problem of determining for which values of $a > 1$ all elements of $T(a)$ are dense in R.

Notation. Denote the closure of a set X by $cl(X)$. The brackets will be omitted where no ambiguity results. Denote the set of non-negative integers by Z^+, the set of non-negative reals by R^+ and the set of non-positive reals by R^-. Write q for $a/(a - 1)$.

Note:

1. If $X \in T(a)$ then $cl\ X \in T(a)$ and $\mu X + x_o \in T(a)$ where $x_o \in R$ and $\mu \in R$, $\mu \neq 0$.
2. The intersection of a family of elements of $T(a)$ containing two fixed points belongs to $T(a)$. So given 0 and 1, since $R \in T(a)$, there exists a smallest element of $T(a)$ containing them written $t\{a; 0, 1\}$.
3. Define $X_o = \{0,1\}$ and, for $n \geq 1$, define $X_n = \{z : z = ax + (1 - a)y$ where $x, y \in X_{n-1}\}$ then $\bigcup_{n=o}^{\infty} X_n = t\{a; 0, 1\}$.
4. If $0 < a < 1$ an element of $T(a)$ is dense in its convex cover.
5. Henceforth assume $a > 1$. The theory for $a < 0$ is essentially the same.

THEOREM 1.1. An uncountable element of $T(a)$ is dense in R. Moreover if a is not an algebraic integer all elements of $T(a)$ are dense in R.

Proof. It is sufficient to prove that, if $X \in T(a)$ and $cl\ X \neq R$ then $cl\ X$ is countable and a is an algebraic integer.

If $cl\ X \neq R$, since $R \setminus cl\ X$ is open and hence a countable union of intervals, there exist $u, v \in cl\ X$ with $(u, v) \subset R \setminus cl\ X$. By note (1), assume $u = 0$ and $v = 1$. Now there does not exist

$x \in \mathrm{cl}(X) \cap (1, q)$ for

$$a\,1 + (1 - a)\left(1 + \frac{\lambda}{a - 1}\right) = 1 - \lambda$$

and $(0, 1) \subset R \setminus \mathrm{cl}\, X$. I shall prove by induction, that the only points of $\mathrm{cl}(X) \cap (q^r, q^{r+1})$ are finite sums of the form $\Sigma\, c_i q^i$, $c_i \in Z^+$. Note that $c_i \leqq q^{r+1-i}$. Assume that the result has been proved for all $r \leqq n - 1$ and let $r = n$. Let $x \in (q^n, q^{n+1}) \cap \mathrm{cl}(X)$ then

$$a\, q^n + (1 - a)x > 0 \quad \text{and} \quad ax + (1 - a)q^{n+1} > 0.$$

Also $x = (aq^n + (1 - a)x) + (ax + (1 - a)q^{n+1})$. Since $aq^n + (1 - a)x < q^n$ the inductive hypothesis gives the required expression for x if $ax + (1 - a)q^{n+1} \leqq q^n$ which is equivalent to $x \leqq q^n(1 + 1/a)$. So if $x \leqq q^n(1 + 1/a)$ the required expression for x is obtained. Similarly if $ax + (1 - a)q^{n+1} \leqq q^n\,(1 + 1/a)$ that is if

$$x \leqq q^n \sum_{i=0}^{2} \frac{1}{a^i}$$

the result follows. Since

$$\sum_{i=0}^{\infty} \frac{1}{a^i} = q$$

the result concerning points of $\mathrm{cl}(X) \cap R^+$ follows. Further, since $-\mathrm{cl}(X) + 1 \in T(a)$, $\mathrm{cl}(X) \cap R^-$ is countable. Finally since $a = \Sigma\, c_i q^i$, $c_i \in Z^+$, a satisfies a monic polynomial equation and hence is an algebraic integer.

I now present two lemmas for

THEOREM 1.2. If $a < \frac{1}{2}(3 + \sqrt{5})$, $a \neq 2$ and $X \in T(a)$ then $\mathrm{cl}\, X = R$.

Lemma 1. If $X \in T(a)$ then $X \in T(a^2 - 2a + 1)$. Hence if $a < 2$, $\mathrm{cl}\, X = R$.

Proof. Let x and $y \in X$ then $ax + (1 - a)y \in X$ and so $ax + (1 - a)(ax + (1 - a)y) = (1 - (a - 1)^2)x + (a - 1)^2 y \in X$. Similarly $(a - 1)^2 x + (1 - (a - 1)^2)y \in X$. The second part follows from note (4).

Lemma 2. If $X \varepsilon T(1 + \sqrt{2})$ then $X \varepsilon T(\sqrt{2})$ and hence $\text{cl } X = R$.

Proof. Let x and $y \varepsilon X$ then

$$(1 + \sqrt{2})x - \sqrt{2}y = x + \sqrt{2}(x - y)$$

and

$$(1 + \sqrt{2})x - \sqrt{2}(x + \sqrt{2}(x - y)) = 2y - x$$

so

$$y(1 + \sqrt{2}) - \sqrt{2}(2y - x) = x\sqrt{2} + y(1 - \sqrt{2}) \ \varepsilon\ X.$$

Similarly

$$(1 - \sqrt{2})x + y\sqrt{2} \ \varepsilon\ X.$$

I now prove Theorem 1.2. From Lemma 1, $X \varepsilon T(a_n)$ where $a_o = \underline{a}$ and $a_n = (a_{n-1} - 1)^2$. If $a_m = 0$ for some smallest m, then $a_{m-1} = 1$ and $a_{m-2} = 2$, by definition of m, hence $a_{m-3} = 1 + \sqrt{2}$ since $a_n \geqq 0$ for all n. Hence suppose $a_n \neq 0$ for all n. Since $a_o > 1$ and $a_n < a_{n-1}$ if and only if $(a_{n-1} - \frac{1}{2}(3 + \sqrt{5}))(a_{n-1} - \frac{1}{2}(3 - \sqrt{5})) < 0$ either for some n $0 < a_n < \frac{1}{2}(3 - \sqrt{5})$ and hence $\text{cl}(X) = R$ by note (4) or $\{a_n\}$ is decreasing and hence convergent to $\frac{1}{2}(3 - \sqrt{5})$ in which case X is dense by note (4).

Using a computer I have obtained the elements of the X_i, $0 \leqq i \leqq 5$, of note (3) for $\underline{a} = \frac{1}{2}(3 + \sqrt{5})$ in the form $m + n\frac{1}{2}(1 + \sqrt{5})$ where $m, n \ \varepsilon\ Z$. The output convinces me of the correctness of the following

CONJECTURE. $t\{\frac{1}{2}(3 + \sqrt{5}); 0, 1\}$ is not dense in R.

I shall now prove some results on uncountable sets belonging to $T(\underline{a})$.

The following theorem suitably modifies an argument of Theorem 3.2 of [1.2]. It extends the obvious result that if $X \varepsilon T(\underline{a})$ and X contains an interval then $X = R$.

RESULT 1.3: Let $X \varepsilon T(\underline{a})$ and let X have positive inner Lebesgue measure then $X = R$.

Proof. Let $X \varepsilon T(\underline{a})$ and let X have positive inner Lebesgue measure, that is X contains a Lebesgue measurable set M of positive measure. Choose $\kappa = \max \{2, q\}$ and $\tau = \dfrac{(2\underline{a}-1)\kappa+(1-\underline{a})}{\kappa(2\underline{a}-1)}$ then $0 < \tau < 1$ as $\underline{a} > 1$. There exists an open interval, I, centre q of length 2δ such that $m(I \cap M) > \tau\, m(I)$.

Let I_q be the interval centre q of length δ then $I_q \subset X$. Suppose, on the contrary, that some point p of I_q (which I may assume is the origin) does not lie in X. Let I_p be the interval centre p, length $\frac{2\delta}{\kappa}$ then $I_p \subset I$. If $M_p = M \cap I_p$, then

$$m(M_p) = m(M \cap I_p) = m(I \cap M) - m(M \cap I \setminus I_p).$$

Now,

$$m(M \cap I \setminus I_p) \leqq m(I \setminus I_p) = m(I) - m(I_p)$$

$$= (\kappa - 1)\, m\,(I_p)$$

whence

$$m(M_p) > \tau\, \kappa\, m(I_p) - (\kappa - 1)\, m\,(I_p)$$

$$= (\kappa(\tau - 1) + 1)\, m\,(I_p)$$

$$= \left(\frac{\underline{a}}{2\underline{a}-1}\right) m\,(I_p) \ldots \quad (1)$$

Now define a function f on X by,

$$p = \underline{a}\, f(x) + (1 - \underline{a})x.$$

Since $p = 0$ and $p \notin X$,

$$f(M_p) \subset f(X) \subset R \setminus X \subset R \setminus M$$

moreover

$$|f(x)| = \frac{|x|}{q}$$

Consequently $f(M_p)$ and M_p are disjoint measurable sets of I_p so,

$$m(I_p) \geqq m(M_p) + m(f(M_p)) = (1 + 1 - \tfrac{1}{\underline{a}})m(M_p)$$

$$= \frac{2\underline{a}-1}{\underline{a}}\, m(M_p)$$

in contradiction to (1).

Thus $I_q \subset X$ and hence $X = R$ since $X \in T(\underline{a})$.

In view of Result 1.3 it is natural to ask whether there exist uncountable, elements of $T(\underline{a})$ with measure zero and whether there exist Lebesgue non-measurable sets belonging to $T(\underline{a})$. The answer to both questions is affirmative as Examples 1.1 and 1.2 show.

Example 1.1: To construct an uncountable element of $T(\underline{a})$ of measure zero, the method is to construct a perfect and hence uncountable subset X_o of $[0, 1]$ such that $m(X_n) \leqq 1$ for all $n \geqq 1$ where $X_n = \{z : z = \underline{a}x + (1 - \underline{a})y$ where $x, y \in X_{n-1}\}$. It follows that $\bigcup_{n=0}^{\infty} X_n$ is an F_σ set with $m(\bigcup_{n=0}^{\infty} X_n) \leqq 1$. Hence $\bigcup_{n=0}^{\infty} X_n$ is a measurable set of $T(\underline{a})$ with $\bigcup_{n=0}^{\infty} X_n \neq R$ and hence, by Theorem 1.3, $\bigcup_{n=0}^{\infty} X_n$ is an uncountable, F_σ set of measure zero.

Construct, for each n, a collection of closed intervals $I(a_1,\ldots,a_n)$ of $[0, 1]$ corresponding to each of the 2^n sequences of 0's and 1's, of length n, such that,

$$I(a_1,\ldots,a_{n-1}) \supset I(a_1,\ldots,a_{n-1},a_n)$$

and

$$I(a_1,\ldots,a_{n-1}0) \cap I(a_1,\ldots,a_{n-1},1) = \emptyset.$$

Define $x \in X_o$ if there exists an infinite sequence $a_1\ldots a_n\ldots$ such that, for all n, $x \in I(a_1\ldots a_n)$.

Note that if $I_i = [a_i,b_i]$ $i = 1, 2$ then $m(\underline{a}I_1 + (1 - \underline{a})I_2) = \underline{a}\, m(I_1) + (\underline{a} - 1)m(I_2)$, so if $m(I_1) = m(I_2)$ then $m(\underline{a}I_1 + (1 - \underline{a})I_2) = (2\underline{a} - 1)m(I_1)$.

Returning to the construction, for $n = 1$, choose from $[0,1]$ two sub-intervals $I(0)$ and $I(1)$ of equal length with $I(0)$ having its left hand end point at 0 and $I(1)$ having its right hand end point at 1 with $I(0) \cap I(1) = \emptyset$ and $m(I(0)) \leqq \frac{1}{8\underline{a}-4}$.

Inductively define the 2^n intervals of the n^{th} stage, $n \geqq 2$, as follows, $I(a_1\ldots a_n)$ is a subinterval of $I(a_1\ldots a_{n-1})$; if $a_n = 0$, $I(a_1\ldots a_n)$ has its left end point coincident with the left end point of $I(a_1\ldots a_{n-1})$; if $a_n = 1$, $I(a_1\ldots a_n)$ has its right end point coincident with the right end point of $I(a_1,\ldots,a_{n-1})$. Take $m(I(a_1\ldots a_n)) = d_n$ for all the finite sequences $a_1\ldots a_n$. By the note above, it is clear that d_n may be chosen sufficiently small that $m(Y_{n,n}) \leqq 1$ where $Y_{o,n} = \bigcup_{a_1\ldots a_n} I(a_1,\ldots,a_{n-1},a_n)$ and for $m \geqq 1$

$Y_{m,n} = \{z : z = \underline{a}x + (1 - \underline{a})y$ where $x, y \in Y_{m-1,n}\}$. d_n is easily chosen so that $I(a_1,\ldots,a_{n-1}, 0)$ is disjoint from $I(a_1,\ldots,a_{n-1}, 1)$.

Now $X_n \subset Y_{n,n}$ so $m(\bigcup_{n=1}^{\infty} X_n) \leqq 1$. Since X_o is closed, without isolated points it is uncountable and hence $\bigcup_{n=1}^{\infty} X_n$ is the required set.

The second example, of the non-measurable set which belongs to $T(\underline{a})$, was made known to me by H.G. Eggleston.

Example 1.2. Sierpinski's non-measurable set constructed from a Hamel Basis for R [1.3], belongs to $T(\underline{a})$ for all rational $\underline{a}$.

I have no idea whether the complex number generalisation of the family $T(\underline{a})$ to planar sets yields interesting results. However, I have studied a planar, two parameter generalisation of the family $T(\underline{a})$.

Definition 1.2: A subset, X, of R^2 belongs to the family $H(\lambda,K)$ if $|X| \geqq 2$ and for all $\underline{x}$ and $\underline{y}$ belonging to X $\lambda\underline{x} + (1 - \lambda)y + K\underline{u}|\underline{x} - y| \in X$ and $\lambda\underline{x} + (1 - \lambda)y - Ku|\underline{x} - \underline{y}| \in X$ where $\lambda, K \in R$ $K \geqq 0$ and if $\underline{x} = (x_1,x_2)$ and $\underline{y} = (y_1,y_2)$, $\underline{u} = \dfrac{(y_2-x_2,x_1-y_1)}{((y_2-x_2)^2+(y_1-x_1)^2)^{\frac{1}{2}}}$.

Note:

1. If $X \in H(\lambda,K)$ then $\text{cl}\, X \in H(\lambda,K)$.
2. If $X \in H(\lambda,K)$, A is an orthogonal linear transformation, $\mu \in R$ $\mu \neq 0$ and $\underset{\sim}{x}_o \in R^2$ then $\mu AX + \underset{\sim}{x}_o = Y \in H(\lambda,K)$.
3. The intersection of a family of elements of $H(\lambda,K)$ containing two fixed points belongs to $H(\lambda,K)$. So given (0, 0) and (1, 0) there exists a smallest element of $H(\lambda,K)$ containing them, written $h\{(\lambda,K); (0, 0), (1, 0)\}$. Moreover there exists a smallest closed element of $H(\lambda,K)$ containing (0, 0) and (1, 0) written $\text{clh}\{(\lambda,K); (0, 0), (1, 0)\}$.
4. $\text{cl}(h\{(\lambda,K); (0, 0), (1, 0)\}) = \text{clh}\{(\lambda,K); (0, 0), (1, 0)\}$

5. The smallest element of $H(\lambda, K)$ containing $(0, 0)$ and $(1, 0)$ can be constructed as follows. Define $X_o = \{(0, 0), (1, 0)\}$ and for $n \geq 1$, $X_n = \{\underline{z} : \underline{z} = \lambda\underline{x} + (1 - \lambda)\underline{y} + K\,\underline{u}\,|\underline{x} - \underline{y}|$ or $z = \lambda\underline{x} + (1 - \lambda)\underline{y} - K\,\underline{u}\,|\underline{x} - \underline{y}|$ where $\underline{x}, y \in X_{n-1}$ then

$$\bigcup_{n=0}^{\infty} X_n = h\{(\lambda, K); (0, 0), (1, 0)\}$$

6. If $X \in H(\lambda, 0)$ then for all x and y belonging to X, $X \cap \text{aff}\,\{x, y\} \in T(\underline{\lambda})$.

7. Henceforth, assume $K \neq 0$ and $\lambda \geq 0$ for elements of $H(\lambda, K)$.

Geometrically, Definition 1.2 means that given $\underline{x}, \underline{y} \in X$ the points $\underline{z}_i$ on the half lines obtained by a clockwise or anticlockwise rotation of the ray containing $\underline{y}$ and terminating at $\underline{x}$ or the ray containing $\underline{x}$ and terminating at $\underline{y}$ through $\alpha = \text{arc tan}\,K/\lambda$, distant $(\lambda^2 + K^2)^{\frac{1}{2}}|\underline{x} - \underline{y}|$ from $\underline{x}$ or $\underline{y}$ respectively belong to X. See Fig. 1.

Consider $\underline{x}, \underline{y} \in X \in H(\lambda, K)$ and note that by considering $\underline{x}$ and $\underline{z}_1$ (Fig. 1) and $\underline{y}$ and $\underline{z}_2$ one has $\underline{z}_5$ and $\underline{z}_6 \in X$ as shown in Fig. 2. In other words, on identifying $\text{aff}\,\{\underline{x}, \underline{y}\}$ with R, $X \cap \text{aff}\,\{\underline{x}, \underline{y}\} \in T(\underline{\lambda^2 + K^2})$.

It is clear from the geometric interpretation that if $X \in H(\lambda, K)$ contains a line then $X = R^2$. Further, it is clear that if $\lambda^2 + K^2 < 1$ and X is a closed set with $X \in H(\lambda, K)$ then X is convex.

<u>THEOREM 1.3</u>: If $\lambda^2 + K^2 < 1$ and $X \in H(\lambda, K)$ then $\text{cl}\,X = R^2$.

<u>Proof.</u> It is sufficient to prove that the only closed set X with $X \in H(\lambda, K)$ and $\lambda^2 + K^2 < 1$ is R^2. If $X \neq R^2$ since X is convex there exists $\underline{x} \in \text{Fr}\,X$ and a support line, L, to X through $\underline{x}$. Since $X \neq \{\underline{x}\}$ there exists $\underline{y}_o \in X$ with $\underline{y}_o \neq \underline{x}$. Since $X \in H(\lambda, K)$, there exists $\underline{y}_1$ at a distance $(\lambda^2 + K^2)^{\frac{1}{2}}|\underline{x}_o - \underline{y}_o|$ on the half-line terminating at $\underline{x}$ obtained by an anticlockwise rotation through $\alpha = \text{arc tan}\,K/\lambda < \pi$ of the half-line containing $\underline{y}_o$ and terminating at $\underline{x}$. It is clear that a set of $\underline{y}_i$ may be generated with $\underline{y}_i \in X$ and some $\underline{y}_i$ on both sides of L. Hence there does not exist a support line at $\underline{x}$ and so $X = R^2$ since $\text{Fr}\,X$ is empty.

Figure 1.

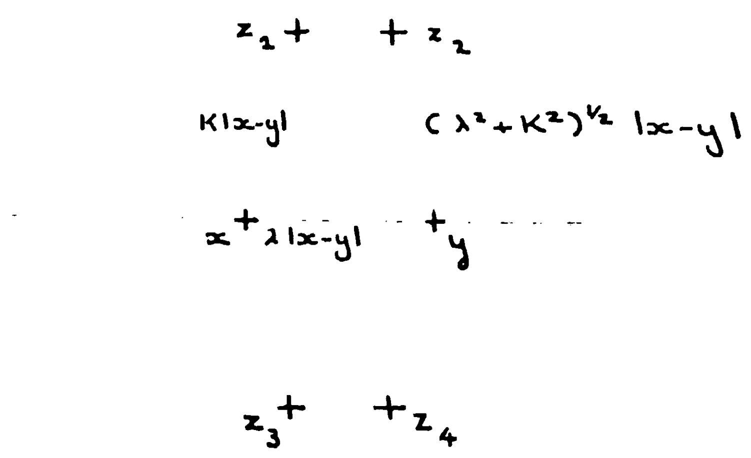
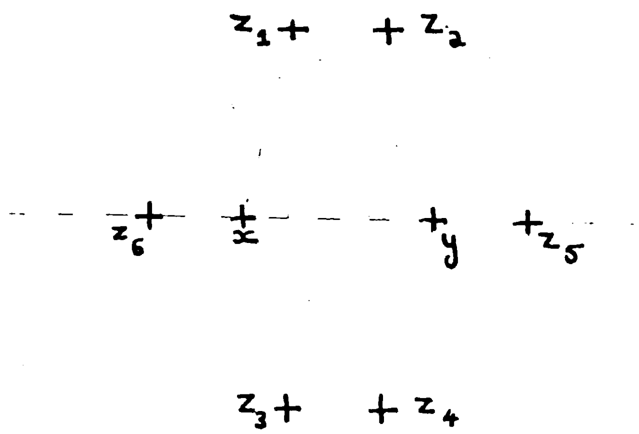


Figure 2.



THEOREM 1.4: If $X \epsilon H(\lambda, K)$ with $\lambda^2 + K^2 = 1$ then $\text{cl } X = R^2$ unless $(\lambda, K) = (0, 1), (\frac{1}{2}, \frac{\sqrt{3}}{2})$ or $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Proof. The method is to show that if $X \epsilon H(\lambda, K)$ with (λ, K) not one of the stated pairs, but $\lambda^2 + K^2 = 1$, then $X \epsilon H(\lambda, K)$ with $\lambda^2 + K^2 < 1$. To do this I consider two points $\underline{x}$ and $\underline{y}$ and show that it is possible to generate the four other points required. Many subcases of Case 2 below are reduced to Case 1. It is more convenient to work with angles, α and β as shown in Fig. 3 than with λ's and K's.

Case 1. Acute α and β, $\alpha \neq \beta$, where α and β are as shown in Fig. 3. (The circles and lines are just aids to perception). Consider two of the generated points as shown by crosses in Fig. 3. Generate a point $\underline{w}$ from them as shown. It is clearly possible to generate the three other points necessary to show $X \epsilon H(\lambda, K)$ with $\lambda^2 + K^2 < 1$. (Note that there will indeed be three other points since $\lambda + (1 - \lambda)(1 - 2\lambda) = 2\lambda^2 - 2\lambda + 1$ and $2\lambda^2 - 2\lambda + 1 = \frac{1}{2}$ iff $\lambda = \frac{1}{2}$)

Case 2. Obtuse $\alpha = 90 + \gamma$. From $\underline{x}$ and $\underline{z}$ generate $\underline{w}$ as shown in Fig. 4. Clearly for $60 < 180 - 2\gamma < 90$ or $0 < 180 - 2\gamma < 60$ that is $90 > \gamma > 60$ or $60 > \gamma > 45$ the result follows by Case 1 as it is clear that the three other points required can be obtained. For the subcase $\gamma = 60$ see Fig. 5. Generate $\underline{W}(\underline{x}, \underline{z}_1)$ from $\underline{x}$ and $\underline{z}_1$ and $\underline{v}$ from $\underline{W}(\underline{x}, \underline{z}_2)$ and $\underline{x}$. Clearly $\underline{v}$ and $\underline{W}(\underline{x}, \underline{z}_1)$ will generate a point within both circles. It is clear that three similar points may be generated. For $0 < \gamma \leq 45$, $\gamma \neq 30$, see Fig. 6. Consider two cases $0 < \gamma < 30$ and $30 < \gamma \leq 45$. For $30 < \gamma \leq 45$, generate $\underline{z}_1$ from $\underline{x}$ and $\underline{y}$, $\underline{z}_2$ from $\underline{z}_1$ and $\underline{x}$ and $\underline{z}_3$ from $\underline{z}_2$ and $\underline{x}$ as shown. Since $0 < F \leq 45$ the result follows. For $0 < \gamma < 30$ consider two subcases (1) $\gamma \neq 10$ (2) $\gamma = 10$. In the first case proceed as in $30 < \gamma \leq 45$ case in the second note that one can generate Case 1 with $\alpha = 40$.

Figure 3.

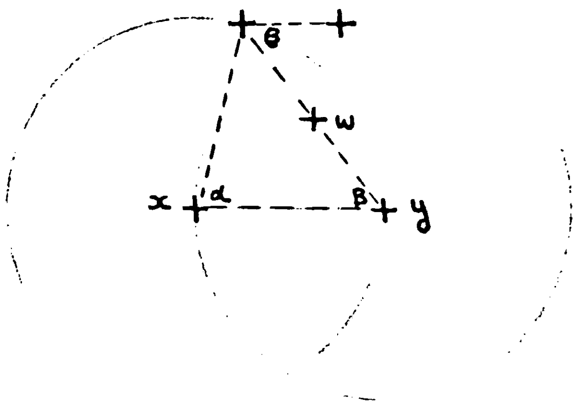


Figure 4.

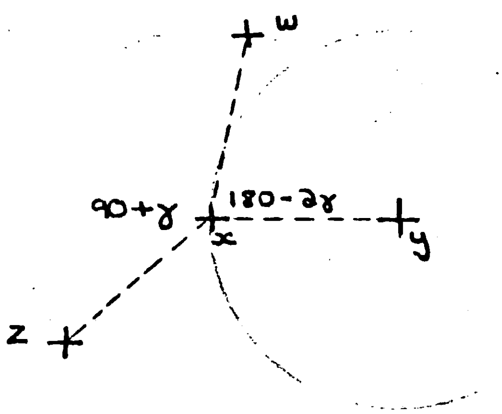


Figure 5.

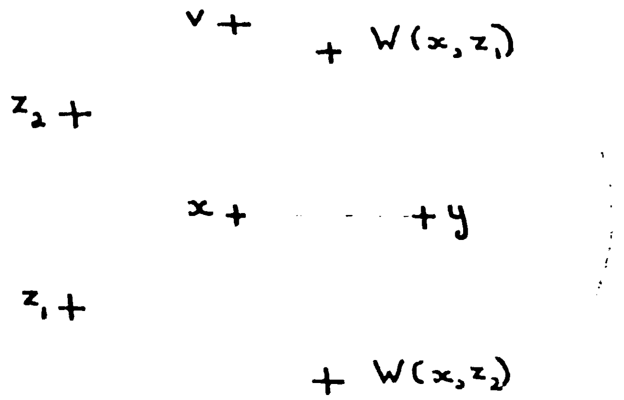
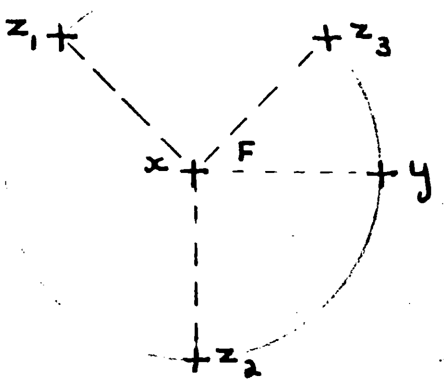


Figure 6.



THEOREM 1.5: $h\{(\frac{1}{2}, \frac{\sqrt{3}}{2}); (0, 0), (1, 0)\} = \text{clh}\{(\frac{1}{2}, \frac{\sqrt{3}}{2}); (0, 0), (1, 0)\}$ $= \{\underline{z} : \underline{z} = m(1, 0) + n(\frac{1}{2}, \frac{\sqrt{3}}{2})\ m, n \in Z\}$.

Proof. Firstly, if $\underline{x} = a(1, 0) + b(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $\underline{y} = c(1, 0) + d(\frac{1}{2}, \frac{\sqrt{3}}{2})$ then $\underline{x} = (a + \frac{b}{2}, b\frac{\sqrt{3}}{2})$ and $\underline{y} = (c + \frac{d}{2}, d\frac{\sqrt{3}}{2})$. Now $\underline{x} - \underline{y} = (a - c + \frac{1}{2}(b - d), \frac{\sqrt{3}}{2}(b - d))$ and $|\underline{x} - \underline{y}|\underline{u} = (-\frac{\sqrt{3}}{2}(b - d), a - c + \frac{1}{2}(b - d))$ so $\frac{\sqrt{3}}{2}|\underline{x} - \underline{y}|\underline{u} = (-\frac{3}{4}(b - d), \frac{\sqrt{3}}{2}(a - c) + \frac{\sqrt{3}}{4}(b - d))$ $= \frac{1}{4}(-3(b - d), \sqrt{3}(2a - 2c + b - d))$. Further $\frac{1}{2}(\underline{x} + \underline{y}) = \frac{1}{4}(2a + 2c + b + d, (b + d)\sqrt{3})$ so that $\frac{1}{2}(\underline{x} + \underline{y}) + \frac{\sqrt{3}}{2}\underline{u}|\underline{x} - \underline{y}| = (c - b + d)(1, 0) + (a - c + b)(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Similarly, $\frac{1}{2}(\underline{x} + \underline{y}) - \frac{\sqrt{3}}{2}\underline{u}|\underline{x} - \underline{y}| = (a - d + b)(1, 0) + (c - a + d)(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Hence it is clear that $\{\underline{z}:\underline{z} = m(1, 0) + n(\frac{1}{2}, \frac{\sqrt{3}}{2})\ m, n \in Z\} \in H(\frac{1}{2}, \frac{\sqrt{3}}{2})$. To prove the theorem it is sufficient to show that any element of $H(\frac{1}{2}, \frac{\sqrt{3}}{2})$ containing (0, 0) and (1, 0) contains $\{\underline{z}:\underline{z} = m(1,0) + n(\frac{1}{2}, \frac{\sqrt{3}}{2})\ m, n \in Z\}$. By note (2), following Definition 1.2 it is clearly sufficient to show any element of $H(\frac{1}{2}, \frac{\sqrt{3}}{2})$ containing (0, 0) and (1, 0) contains $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(\frac{3}{2}, \frac{\sqrt{3}}{2})$ $(\frac{5}{2}, \frac{\sqrt{3}}{2})$, $(-1, 0)$, $(2, 0)$, $(-\frac{3}{2}, -\frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ and $(\frac{3}{2}, -\frac{\sqrt{3}}{2})$ which is geometrically obvious.

THEOREM 1.6: $h\{(0, 1); (0, 0), (1, 0)\} = \text{clh}\{(0, 1); (0, 0), (1, 0)\} = \{\underline{z}:\underline{z} = m(1, 0) + n(0, 1)\ m, n \in Z\}$.

Proof. Clearly $h\{(0, 1); (0, 0), (1, 0)\}$ $\{z:z = m(1, 0) + n(0, 1)\ m, n \in Z\}$. Hence it is sufficient to show that $\{\underline{z}:\underline{z} = m(1, 0) + n(0, 1)\ m, n \in Z\} \in H(0, 1)$. Let $\underline{x} = a(1, 0) + b(0, 1)$ and $\underline{y} = c(1, 0) + d(0, 1)$ then $|\underline{x} - \underline{y}|\underline{u} = (d - b, a - c)$. Hence that result is clear.

THEOREM 1.7: $h\{(-\frac{1}{2}, \frac{\sqrt{3}}{2}); (0, 0), (1, 0)\} = \text{clh}\{(-\frac{1}{2}, \frac{\sqrt{3}}{2}); (0, 0), (1, 0)\} = \{\underline{z}:\underline{z} = m(1, 0) + n(\frac{1}{2}, \frac{\sqrt{3}}{2})\ n \not\equiv m + 1(3)\ m, n \in Z\}$.

Proof. Firstly if $\underline{x} = a(1, 0) + b(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $\underline{y} = c(1, 0) + d(\frac{1}{2}, \frac{\sqrt{3}}{2})$ then as in Theorem 1.5 $\frac{\sqrt{3}}{2}|\underline{x} - \underline{y}|\underline{u} = \frac{1}{4}(-3(b - d), \sqrt{3}(2a - 2c + b - d))$. Further $-\frac{1}{2}\underline{x} + \frac{3}{2}\underline{y} = \frac{1}{4}(-2a - b + 6c + 3d, \sqrt{3}(3d - b))$. So

$-\frac{1}{2}\underline{x} + \frac{3}{2}\underline{y} + \frac{\sqrt{3}}{2}|\underline{x} - \underline{y}|\underline{u} = (a - c + d)(\frac{1}{2}, \frac{\sqrt{3}}{2}) + (-a - b + 2c + d)(1, 0)$,

$-\frac{1}{2}\underline{x} + \frac{3}{2}\underline{y} - \frac{\sqrt{3}}{2}|\underline{x} - \underline{y}|\underline{u} = (-a - b + c + 2d)(\frac{1}{2}, \frac{\sqrt{3}}{2}) + (b + c - d)(1, 0)$,

$\frac{3}{2}\underline{x} - \frac{1}{2}\underline{y} + \frac{\sqrt{3}}{2}\,|\underline{x} - \underline{y}|\underline{u} = (a + 2b - c - d)(\frac{1}{2}, \frac{\sqrt{3}}{2}) + (a - b + d)(1, 0)$ and
$\frac{3}{2}\underline{x} - \frac{1}{2}\underline{y} - \frac{\sqrt{3}}{2}\,|\underline{x} - \underline{y}|\underline{u} = (-a + b + c)(\frac{1}{2}, \frac{\sqrt{3}}{2}) + (2a + b - c - d)(1, 0)$.
On the assumption $b \not\equiv a + 1\ (3)\ d \not\equiv c + 1\ a, b, c, d \in Z$ it is easy to verify that these four new points belong to $\{\underline{z}:\underline{z} = m(1, 0) + n(\frac{1}{2}. \frac{\sqrt{3}}{2})\ n \not\equiv m + 1\ (3)\ m, n \in Z\}$ which thus belongs to $H(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. To prove the theorem it is sufficient to prove that any element of $H(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ containing $(0, 0)$ and $(1, 0)$ contains $\{\underline{z}:\underline{z} = m(1, 0) + n(\frac{1}{2}, \frac{\sqrt{3}}{2})\ n \not\equiv m + 1\ (3)\}$. It is clearly sufficient to show that any element of $H(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ containing $(0, 0)$ and $(1, 0)$ contains $(0, \sqrt{3})$, $(1, \sqrt{3})$, $(\frac{5}{2}, \frac{\sqrt{3}}{2})$, $(-\frac{3}{2}, \frac{\sqrt{3}}{2})$, $(\frac{3}{2}, \frac{\sqrt{3}}{2})$, $(-2, 0)$, $(3, 0)$, $(-\frac{3}{2}, -\frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $(\frac{3}{2}, -\frac{\sqrt{3}}{2})$, $(\frac{5}{2}, -\frac{\sqrt{3}}{2})$, $(0, \sqrt{3})$ and $(1, -\sqrt{3})$ which is geometrically obvious.

The situation for $\lambda^2 + K^2 > 1$ appears to be more complicated. However for $1 < \lambda^2 + K^2 < \frac{1}{2}(3 + \sqrt{5})$, $\lambda^2 + K^2 \neq 2$ or $\lambda^2 + K^2$ not an algebraic integer Theorems 1.1 and 1.2 and an earlier remark imply that if $X \in H(\lambda, K)$ then $\mathrm{cl}X = R^2$.

The final result I present for $\lambda^2 + K^2 > 1$ is the following theorem.

THEOREM 1.8. If $X \in H(\lambda, K)$ with $0 < \lambda < 1$, $\lambda \neq \frac{1}{2}$, then $\mathrm{cl}X = R^2$.

Proof. Without loss of generality since $H(\lambda, K) = H(1 - \lambda, K)$, I may suppose $0 < \lambda < \frac{1}{2}$. I consider two cases

Case 1: $0 < \lambda \leq \frac{1}{4}$

Case 2: $\frac{1}{4} < \lambda < \frac{1}{2}$.

I begin by presenting a lemma which will be useful in both Case 1 and Case 2.

LEMMA 1. If $X \in H(\lambda, K)$ with $0 < \lambda < \frac{1}{2}$ then $X \in H(\frac{1}{2}, \frac{K(4\lambda-1)}{2\lambda})$

Proof. Let $\underline{x}_o, \underline{y}_o \in X$ and let L_o^+ and L_o^- denote the open half bounded by aff $\{\underline{x}_o, \underline{y}_o\}$.

Let $\underline{x}_1 = \lambda\underline{y}_o + (1 - \lambda)\underline{x}_o + K\underline{u}\,|\underline{x}_o - \underline{y}_o|$ and $\underline{y}_1 = \lambda\underline{x}_o + (1 - \lambda)\underline{y}_o + K\,\underline{u}\,|\underline{x}_o - \underline{y}_o|$ belong to L_o^+ and let L_1^+ and L_1^- denote the corresponding open half planes. Note that $|\underline{x}_1 - \underline{y}_1| = (1 - 2\lambda)\,|\underline{x}_o - \underline{y}_o|$ and $0 < 1 - 2\lambda < 1$.

Since X is closed and $(1-2\lambda)^n \to 0$ as $n \to 0$ it is clear that $X \varepsilon H(\frac{1}{2}, S)$ where $S = K(1 - \sum_{n=1}^{\infty} (1-2\lambda)^n) = \frac{K(4\lambda-1)}{2\lambda}$: consider x_n, $y_n = \bar{L}_{n-1}$ for $n \geqq 2$.

The second lemma will be used for Case 1 only.

<u>LEMMA 2</u>: If $a_n = r^n$ with $\frac{1}{2} \leqq r < 1$ and $\sum_{n=1}^{\infty} a_n = \alpha$ then there exists a sequence b_n with $b_n = 1$ or $b_n = -1$ such that for t with $a_1 \leqq t \lneqq \alpha$ $\sum_{n=1}^{\infty} a_n b_n = t$.

<u>Proof.</u> Write $s[n]$ for $\sum_{m=1}^{n} a_m$. Define $P(j)$ and $N(j)$ inductively as follows $P(1) = 1$, $P(2) =$ the first integer n such that $t < \sum_{m=1}^{n} a_m$ then $\sum_{m=1}^{P(2)-1} a_m \leqq t < \sum_{m=1}^{P(2)} a_m = S[P(2)]$. Define $N(1) = P(2) + 1$ and $N(2) =$ the first integer n such that $\sum_{j=N(1)}^{n-1} a_j < S[P(2)] - t \leqq \sum_{j=N(1)}^{n} a_j$.

Note that $S[P(2)] - t < a_{P(2)}$ and so $N(2)$ exists if $r > \frac{1}{2}$ since $r^{n-1} < \frac{r^n}{1-r}$. If $r = \frac{1}{2}$ the result is clear from the $r > \frac{1}{2}$ argument. For $r > \frac{1}{2}$ define $b_n = 1$ for $n \varepsilon \bigcup_{L=1}^{\infty} (P(2L-1), P(2L))$ and $b_n = -1$ for $n \varepsilon \bigcup_{L=1}^{\infty} (N(2L-1), N(2L))^{L=1}$ where $P(j)$ and $N(j)$ are defined in the obvious manner then b_n is the required sequence.

I now complete the proof of Case 1. Choose t such that $1 - 2\lambda < t < \sum_{n=1}^{\infty} (1-2\lambda)^n$ and such that $K^2(1-t)^2 + \frac{1}{4}$ is not an algebraic integer. By Lemmas 1 and 2 $X \varepsilon H(\frac{1}{2}, K(1-t))$ and hence $\text{cl } X = R^2$.

<u>Case 2.</u>: Note that since the algebraic integers are countable to prove the result it is sufficient to prove that $X \varepsilon H(\frac{1}{2}, S_\lambda)$ for uncountably many S_λ. That result follows from the result that the 2^W numbers obtained by letting $\{b_n\}$ range over all possible sequences of -1 and 1 are distinct where $a_n = r^n$ with $0 < r = 1 - 2\lambda < \frac{1}{2}$.

Let $\{b_{j,1}\}$ and $\{b_{j,2}\}$ be two sequences of $+1$ and -1 and let n be the smallest integer m such that $b_{m,1} \neq b_{m,2}$ and suppose without loss of generality, $b_{m,1} = 1$ and $b_{m,2} = -1$ then,

$$\sum_{j=1}^{\infty} b_{j,1} a_j \geq \sum_{j=1}^{n} b_{j1} a_j - \sum_{j=n+1}^{\infty} a_j$$

$$= \tau + r^n - \frac{r^{n+1}}{1-r} > \tau$$

and,

$$\sum_{j=1}^{\infty} b_{j,2}\, a_j \leqq \sum_{j=1}^{n} b_{j,2}\, a_j + \sum_{j=n+1}^{\infty} a_j$$

$$= \tau - r^n + \frac{r^{n+1}}{1-r} < \tau$$

Hence

$$\sum_{j=1}^{\infty} b_{j,1}\, a_j \neq \sum_{j=1}^{\infty} b_{j,2}\, a_j .$$

Appendix to Chapter 1

The study of the denseness of sets belonging to $H(\lambda,K)$ with $\lambda^2 + K^2 < \frac{1}{2}(3 + \sqrt{5})$ is completed by Theorem 1.9.

Theorem 1.9. If $X \in H(\lambda,K)$ with $\lambda^2 + K^2 = 2$ then $\mathrm{cl}\, X = R^2$ unless,

$$(\lambda,K) = (0,\sqrt{2}),\ (-1,1),\ (-\tfrac{1}{2}, \tfrac{\sqrt{7}}{2}) \text{ or } (\tfrac{1}{2}, \tfrac{\sqrt{7}}{2}).$$

Proof. The method is to show that if $X \in H(\lambda,K)$ with (λ,K) not one of the four exceptional values given in the statement of Theorem 1.9 then an argument based on Theorem 1.8 may be used to deduce $c\ell\ X = R^2$.

It is assumed $-\sqrt{2} < \lambda < 0$, without loss of generality by virtue of Theorem 1.8 and that fact that $X \in H(\lambda,K)$ is equivalent to $X \in H(1-\lambda,K)$.

Recall that if $X \in H(\lambda,K)$ then for all $\underline{x},\underline{y} \in X$ aff $\{\underline{x},\underline{y}\}$ $X \in T(\underline{\lambda^2 + K^2})$. Since

$$\lambda\{\underline{y} + (\lambda^2 + K^2)(\underline{x} - \underline{y})\} + (1 - \lambda)\underline{x}$$

$$= \lambda\{1 - (\lambda^2 + K^2)\}\underline{y} + \{1 - \lambda + \lambda(\lambda^2 + K^2)\}\underline{x},$$

it is clear that if $-1 < \lambda < 0$, $\lambda \neq -\frac{1}{2}$, Theorem 1.8 may be applied to X to deduce $c\ell\ X = R^2$.

Finally if $-\sqrt{2} < \lambda < -1$ with $\lambda^2 + K^2 = 2$ $(\lambda + 1)^2 + K^2 < 1$ and Theorem 1.3 may be applied to X to deduce $c\ell\ X = R^2$.

It is easily verified that

$h\{(0,\sqrt{2});\ (0,0),\ (1,0)\} = \{z : z = m\,(1,0) + n(0,\sqrt{2})\ m,n \in Z\}$

$h\{(-1,1);\ (0,0),\ (1,0)\} = \{z : z = m(1,0) + n(0,1)\ m,n \in Z\}$

$h\{(\frac{1}{2}, \frac{\sqrt{7}}{2});\ (0,0),\ (1,0)\} = h\{(-\frac{1}{2}, \frac{\sqrt{7}}{2});\ (0,0),\ (1,0)\}$

$= \{z : z = m\,(1,0) + n(\frac{1}{2}, \frac{\sqrt{7}}{2})\ m,n \in Z\}.$

Example 1.1 was an easy modification of a construction of Souslin [1.4]. A similar construction yields an uncountable set of measure zero such that for all x and y belonging to X xy, $x + y$, $x - y$ and $\frac{1}{x}$, $x \neq 0$ belong to X. For all $x \in X$, $X \in T(\underline{x})$ and clearly a construction similar to Example 1.2 gives a non-measurable set belonging to $T(\underline{x})$ for all $x \in X$.

The construction of uncountable sets belonging to $H(\lambda,K)$ of zero planar measure follows from the proposition:

If X and Y are convex and if $Z = \{\underline{z}:\underline{z} = \lambda\underline{x} + (1 - \lambda)\underline{y} + K\underline{u}|\underline{x}-\underline{y}|\ \underline{x} \in X, \underline{y} \in Y\}$ then Z is convex. Moreover if the diameter of $X =$ diameter of $Y = \varepsilon$ then Z has diameter $\leq (T + K\sqrt{2})\varepsilon$ where $T = |\lambda| + |1 - \lambda|$.

Proof $s\{\lambda\underline{x}_1 + (1 - \lambda\underline{y}_1) + K\underline{u}_1|\underline{x}_1 - \underline{y}_1|\} + (1 - s)\{\lambda\underline{x}_2 + (1 - \lambda)\underline{y}_2$
$+ K\underline{u}_2\ |\underline{x}_2 - \underline{y}_2|\} = \lambda\{s\underline{x}_1 + (1 - s)\underline{x}_2\} + (1 - \lambda)\{s\underline{y}_1 + (1 - s)\underline{y}_2\} + sK\underline{u}_1|\underline{x}_1 - \underline{y}_1|$
$+ (1 - s)\ K\underline{u}_2|\underline{x}_2 - \underline{y}_2| = \lambda\underline{x}_s + (1 - \lambda)\underline{y}_s + K\underline{u}_s|\underline{x}_s - \underline{y}_s|$ where if
$\underline{x}_t = (a_{t_1}, a_{t_2})$ and $\underline{y}_t = (b_{t_1}, b_{t_2})$ then $\underline{u}_t|\underline{x}_t - \underline{y}_t| = (a_{t_2} - b_{t_2}, b_{t_1} - a_{t_1})$
$t = 1, 2$ or s.

Moreover it is clear that $|\lambda\underline{x}_1 + (1 - \lambda)\underline{y}_1 + K\underline{u}_1|\underline{x}_1 - \underline{y}_1| - \lambda\underline{x}_2 - (1 - \lambda)\underline{y}_2$
$- K\underline{u}_2|\underline{x}_2 - \underline{y}_2|\ | \leq |\lambda|\ \varepsilon + |1 - \lambda|\ \varepsilon + K\ \varepsilon\ \sqrt{2} = (T + K\sqrt{2})\ \varepsilon$.

Uncountable sets belonging to $H(\lambda,K)$ can be constructed using the Souslin argument and the fact that a planar set with diameter $\leq 2\varepsilon$ has area $\leq \pi\varepsilon^2$.

Finally, a natural generalisation of the family $H(\lambda,K)$ is the following:

Let aff $X = R^d$ then $X \in H_d(\lambda,K)$ if for each $\underline{x}$ and $\underline{y}$ belonging to X:

$$\lambda\underline{x} + (1 - \lambda)\underline{y} + K\ \mathrm{Fr}\ S_{d-1} \subset X$$

where $S_{d-1} = \{\underline{z} : |\underline{z}| \leq 1, \underline{Z} \in R^{d-1}\}$, $K > 0$ and S_{d-1} lies in the hyperplane with normal $\underline{x} - \underline{y}$ centre $\lambda\underline{x} + (1 - \lambda)\underline{y}$.

However for $d > 2$ the concept is not as fruitful as the case $d = 2$. The following theorem illustrates that point. Note that for $d > 2$ each non-empty planar section belongs to the family $H(\lambda,K) \equiv H_2(\lambda,K)$.

Theorem 1.10. Let $X \in H_d(\lambda,K)$ then $X = R^d$ if $d > 2$.

Proof. By the note above and the fact that if $X \in H(\lambda,K)$ and X contains a line then $X = R^2$, it is sufficient to show that if $X \in H(\lambda,K)$ and X contains $\mathrm{Fr}\, S_2$ then $X = R^2$.

Firstly I shall show that each point $\underline{z}_o$ outside $\mathrm{Fr}\, S_2$ must belong to X. Consider rays from $\underline{z}_o$ meeting $\mathrm{Fr}\, S_2$ in two points. Let the nearer one to $\underline{z}_o$ be $\underline{z}_1$ and let the further one be $\underline{z}_2$. As $\underline{z}_1$ varies over the frontier of the semi-circle which $\underline{z}_o$ sees via the complement of S_2 the function $\dfrac{|\underline{z}_1 - \underline{z}_o|}{|\underline{z}_1 - \underline{z}_2|}$ is a continuous real valued function which is not bounded above. Since each line meets X in a set which can be identified with an $X \in \underline{T(\lambda^2 + K^2)}$ with $\lambda^2 + K^2 > 1$ it is possible to choose $\underline{z}_1$ so that it is clear that $\underline{z}_o \in X$.

Now similarly take $\underline{z}_o \in \mathrm{int}\, S_2$ and consider any $\underline{z}_1 \in \mathrm{Fr}\, S_2$ and a $\underline{z}_2$ sufficiently close to $\underline{z}_1$ outside S_2 on $\mathrm{aff}\,\{\underline{z}_o, \underline{z}_1\}$ so that again $\underline{z}_o \in X$.

REFERENCES

[1.1] I. Calvert, On the closure of a class of subsets of the real line, Math. Proc. Cam. Phil. Soc., (1978).,

[1.2] J.W. Green and W. Gustin, Quasiconvex Sets, Can. J. Math. 2, (1950), 489-507.

[1.3] W. Sierpinski, Sur la question de la mesurabilite de la base de M. Hamel, Fund. Math. 1, (1920), 105-111.

[1.4] M. Souslin, Fund. Math. 4,(1923),311-315.

CHAPTER 2

Introduction

For brevity, I shall write that a set X is in R^d if and only if $\operatorname{aff} X = R^d$.

Definition 2.1: A set, X, in R^d is said to be m-convex $m \geq 2$ if for every m distinct points of X at least one of the line segments determined by those points belongs to X.

Definition 2.2: An m-convex set, X, is said to be exactly m-convex if it is not $(m - 1)$ - convex.

Definition 2.3: A point x of a set X in R^d is a point of local convexity of X if there is some neighbourhood S of x such that if y and z belong to $X \cap S$ then $[y, z] \subset X \cap S$.

Definition 2.4: If X fails to be locally convex at some point q then q is a point of local non-convexity (ℓnc point) of X.

I denote the set of points of local non-convexity of X by $Q(X)$ or more usually Q when the set X is obvious. I point out the important result that Q is a closed set and I note Tietze's Theorem, Valentine [2.1] pp.48-50, that a closed, connected, locally convex set in R^d is convex.

Following the literature terminology, I shall use the phrase "decomposition theorem" to describe results where a set X can be written as a union of a, not necessarily least, number of convex sets.

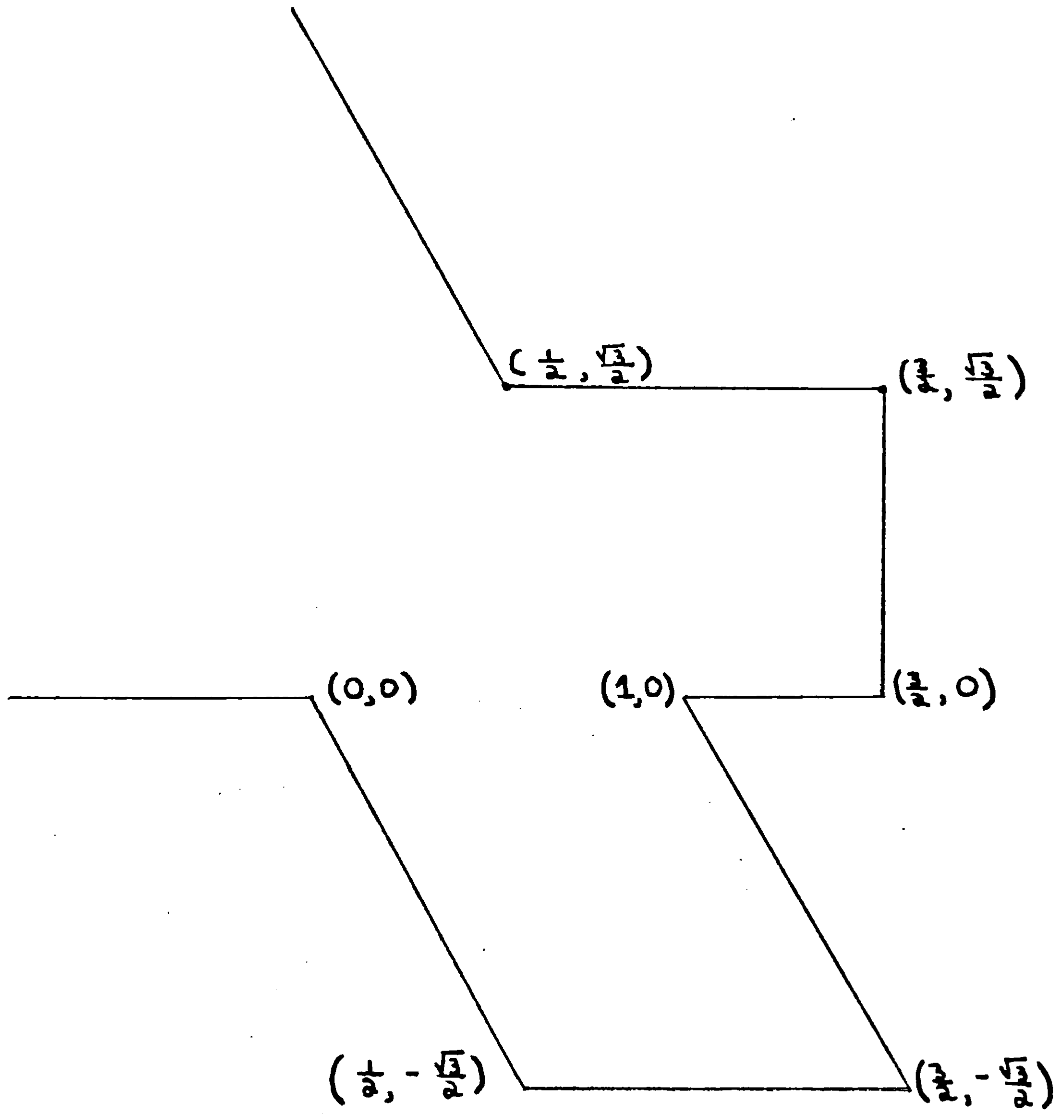
Valentine introduced the concept of 3-convexity in [2.2] where he proved that a closed, 3-convex set in R^2 could be decomposed into a union of three convex sets, in Theorem 2. Further he showed that if $|Q|$ was one, even or infinite then X was the union of two convex sets in Theorem 3. He also proved that for a closed, connected 3-convex set in R^d, $Q \subset \operatorname{Ker} X$. Breen notes in [2.3] that it follows from Lemma 5 of [2.2] that for a closed, planar set, X, with $|Q| \geq 4$, $Q \subset \operatorname{Ker} X$ implies X is 3-convex. Note that the result is false if $|Q| = 3$

as Example 2.1, which will be used again later shows. Example 2.1 is shown in Fig. 2.1.

Example 2.1: Let $X_1 = \text{conv}\{(0, 0), (1, 0), (\frac{1}{2}, -\frac{\sqrt{3}}{2}), (\frac{3}{2}, -\frac{\sqrt{3}}{2})\}$, $X_2 = \text{conv}\{(\frac{1}{2}, \frac{\sqrt{3}}{2}), (\frac{3}{2}, \frac{\sqrt{3}}{2}), (1, 0), (\frac{3}{2}, 0)\}$ and let $X_3 = \{z : z = \lambda(1, 0) + (1 - \lambda)(\mu(0, 0) + (1 - \mu)(\frac{1}{2}, \frac{\sqrt{3}}{2}))\ \lambda \leq 1\ \ 0 \leq \mu \leq 1\}$. Then if $X = X_1 \cup X_2 \cup X_3$, $Q(X) = \{(0, 0), (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2})\} \subset \text{Ker } X$ but X is not 3-convex. Note that X may be modified to ensure X is compact while maintaining $Q(X) \subset \text{Ker } X$ and X not 3-convex.

In his concluding remarks of [2.2] Valentine pointed out that the theory in R^3 needed to be settled. That is still true twenty yerrs later. Eggleston [2.4] has given an example of a compact, 3-convex set, X, in R^4 such that X is not the union of finitely many convex sets. In [2.4] he also proved that if Ker X is of lower dimension than X and X is a compact, 3-convex set in R^d, then X is the union of two convex sets. Exactly the same method can be used to show that the consequence follows if the hypothesis is replaced by conv Q is of lower dimension than X where X is a closed 3-convex set in R^d, a result also proved in Breen [2.5] with a generalisation to m-convex sets in the unpublished Breen [2.6]. Another greater than two dimensional decomposition theorem for 3-convex sets also restricts Q. Buchmann [2.7] has proved that if X is a compact 3-convex set in R^d, $d \geq 3$, such that $Q \subset \text{int}(\text{conv } X)$ and $\text{int}(\text{Ker } X) \neq \phi$ then X is the union of two convex sets. He gives one example to show the result is false for $d = 2$ and another to show that compact may not be replaced by closed. Buchmann uses a result of Valentine [2.8] that, if X is a compact 3-convex set in R^d with $\text{int}(\text{Ker } X) \neq \phi$ and $Q \subset \text{int}(\text{conv } X)$ then Q can be expressed as a finite union of disjoint $(d-2)$-dimensional manifolds. For completeness, I mention Breen [2.9] and Breen [2.10] which give decomposition theorems for sets whose ℓnc points satisfy very restrictive conditions.

Figure 2.1


$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
$(\frac{3}{2}, \frac{\sqrt{3}}{2})$
$(0,0)$
$(1,0)$
$(\frac{3}{2}, 0)$
$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$
$(\frac{3}{2}, -\frac{\sqrt{3}}{2})$

Theorem 3 of [2.2] suggests the problem of characterising those closed, planar 3-convex sets with $|Q| = 2n + 1$ $n \geq 1$ which are the unions of two convex sets. Stamey and Marr [2.11] have proved that if X is a compact, planar 3-convex set with $|Q| = 2n + 1$ $n \geq 1$ then X is the union of two convex sets if and only if $(X \setminus Q) \cap \text{Ker } X \cap \text{Fr } X \neq \phi$. They give an example to show that the only if result is false if compact is replaced by closed in the hypotheses.

Breen [2.12] generalised Stamey and Marr's result to: Let X be a closed m-convex subset of the plane with $\text{conv } Q \subset X$. If there is some point $p \in (X \setminus Q) \cap \text{Ker } X \cap \text{Fr } X$ then X is a union of $m - 1$ closed, convex sets. Thus the "if" part of the Stamey and Marr result holds if compact is replaced by closed in the hypotheses.

Breen's result in [2.12] was strengthened in Breen and Kay [2.13], Theorem 1, where it was shown that the hypothesis $\text{conv } Q \subset X$ was superfluous and that it was sufficient for X to be supported at $p \in \text{Ker } X$. In [2.13] Breen and Kay found a bound for $\sigma(m)$, the number such that every closed, planar, m-convex set is decomposable into $\sigma(m)$ convex sets. The existence of $\sigma(m)$, Eggleston's Theorem, was established in [2.14]. The closed case follows immediately from the compact by Lawrence, Hare and Kenelly [2.15], Theorem 2 which is:

Let S be a subset of a linear space such that each finite subset $F \subset S$ has a K-partition $\{F_1, \ldots F_K\}$ where $\text{conv } F_i \subset S$ $1 \leq i \leq K$. Then S is the union of K convex sets.

Eggleston's methods give a worse bound than Breen and Kay's $(m - 1)^3\, 2^{m-3}$. Breen and Kay obtain much better bounds by considering the effect of further restrictions on X; for closed, starshaped, m-convex planar sets $\left[\frac{3(m-1)}{2}\right] \leq \sigma(m) \leq 2(m - 1)$ as an example in [2.16] and Corollary 3 of [2.13] show. However a beautiful example, due to Perles, is given in [2.13] of a class of m-convex sets for m taking a sequence of values approaching infinity which are not the union of $\frac{1}{4} m^{3/2}$ convex sets. Note that while Theorem 4 of [2.13] is true the proof is false.

I prove the result and state their error in Theorem 2.7 below.

Two unsolved problems suggested by [2.13] are:

Conjecture 2.1. Let X be a closed, planar, 4-convex set then X is the union of five convex sets. See Guay [2.17], Tattersall [2.18], Kay and Guay [2.16] and Breen and Kay [2.13] for partial results.

Unsolved Problem 2.1. Given a closed, m-convex set, X, how many bounded components of the complement of X may there be?

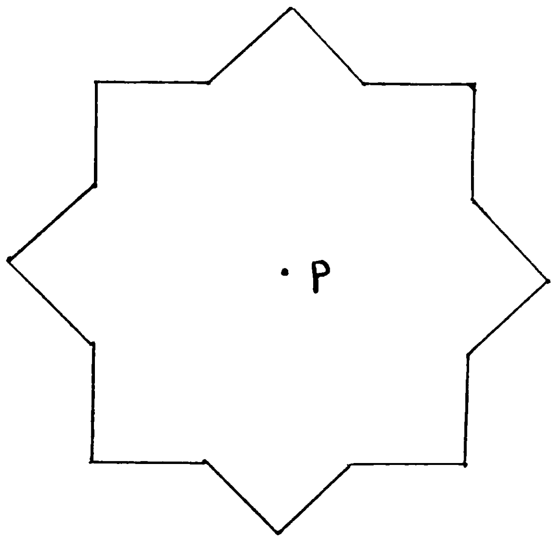
Decomposition Theorems I now turn to the decomposition theorems due to Breen [2.19] for planar, 3-convex sets, which are not closed. Breen [2.19], Theorem 7, showed that a planar 3-convex set is a union of six convex sets which she claimed was best possible. Her, otherwise admirable, paper containes three serious errors. Example 1 and Example 3 are both unions of three convex sets and Example 4 is a union of four convex sets contrary to her claims. Her Example 1 may be replaced by my Example 2.2. However her Theorem 6 is not best possible as she claims. I shall prove a bound of three in Theorem 2.3 below. Whether Theorem 7 is best possible is an open question.

Example 2.2: Let X be the compact set bounded by the Jordan curve in Fig. 2.2. Then $X \setminus \{p\}$ is 3-convex but it is not the union of three convex sets for if it were then one of them must contain three of the extreme points of X which is clearly impossible.

I note that [2.16] includes an example, announced in Kay [2.20], of a planar 4-convex set, X, which is not the union of finitely many convex sets. That example stimulated Breen [2.21] to prove: If X is an m-convex set in the plane, $m \geq 3$, having the property that $(\text{int cl } X) \setminus X$ contains no isolated points then X is expressible as a union of $(m-1)^4\, 2^{m-3}\, \{1 + (2^{m-2} - 1)2^{m-3}\}$ convex sets. In fact her proof requires that $(\text{int cl } X) \setminus X$ has no single point components.

In fact, with the hypothesis, that $(\text{int cl } X) \setminus X$ has no single point components Breen's bound in [2.21] may be easily improved to $(m-1)^4\, 2^{m-3}$.

Figure 2.2



By Theorem 7 of [2.19] assume $m \geq 4$. By the argument of Lemma 4 of [2.21] which stated: Let X be an m-convex set in the plane if $x \in \text{int}(\text{cl } X) \setminus X$ and x is not an isolated point then x lies in a segment of $(\text{int cl } X) \setminus X$; either $(\text{int cl } X) \setminus X$ contains a segment (r, s) or $(\text{int cl } X) \setminus X = \phi$. In the former case let $L = \text{aff}(r, s)$ and let L^+ and L^- denote the open half spaces, bounded by L. Then $L \cap X$ is a union of at most $m - 1$ convex sets and if $X_1 = L^+ \cap X$ and $X_2 = L^- \cap X$ then X_1 and X_2 are $(m - 1)$-convex and $(\text{int cl } X_i) \setminus X_i$ $i = 1, 2$ contains no single point components. Hence by induction X is the union of $(m - 1)^4 \, 2^{m-3}$ convex sets since $(m - 1)^4 \, 2^{m-3} \geq (m - 2)^4 \, 2^{m-3} + m - 1$.

Consider now the second case, since $\text{cl } X$ is m-convex, $\text{cl } X$ may be decomposed into $2^{m-3} (m - 1)^3$ convex sets, [2.13] .

If C is one of these sets let $T = C \cap X$. Then T is m-convex. There are two subcases to consider.

Case 1: If C is 1-dimensional, T contains at most $m - 1$ convex components.

Case 2: If C is not 1-dimensional $C = \text{cl } T$ as Breen shows in the corollary to Theorem 1. Note that $(\text{cl } T) \setminus T \subset \text{Fr cl } T = \text{Fr } C$ since $x \in (\text{int cl } T) \setminus T$ implies $x \in \text{int cl } X$ implies $x \in X$ implies $x \notin C$ a contradiction. Hence the result follows by Breen's Lemma which is :

Let T be an m-convex set in the plane $m \geq 3$ such that $\text{cl } T$ is convex. If all points of $(\text{cl } T) \setminus T$ are in $\text{Fr}(\text{cl } T)$ then T is a union of $\max(m - 1, 3)$ or fewer convex sets.

Given an $(m + 1)$-convex set X, without isolated points and p points in X, it is natural to ask how many, q, of the associated segments must belong to X. As Breen shows in [2.22] the answer is $q(p, m) = \sum_{i=1}^{m} \binom{\left[\frac{p+m-i}{m}\right]}{2}$ where $p = Km + r$ $0 \leq r \leq m - 1$. Note that as she points out, at the end of the paper, the result follows from a remark in [2.16] p 42 as do her results on minimal p subsets. Note that the result is best possible for closed sets X without isolated points.

The problem suggests the study of (p, q)-convexity a concept due to Kay [2.20].

Definition 2.5: A set X in R^d is said to be (p, q)-convex if given any p points of X at least q of the associated segments belong to X for $p \geqq 2$ and $1 \leqq q \leqq \binom{p}{2}$.

Definition 2.6: A (p, q)-convex set is said to be exactly (p, q)-convex if it is not $(p, q + 1)$-convex.

In [2.16] p40 examples are given of exactly (p, q)-convex sets. The examples can be slightly modified to achieve connected, exactly (p, q)-convex sets. For closed sets the situation is different. Kaapke [2.23] has shown that if $K(p, q)$ denotes the class of closed (p,q)-convex sets without isolated points, $K(p,q) = K(p-1, q-K-1)$ where K $K = \left[\frac{2(q-1)}{p+1}\right]$. Moreover a closed (p,q)-convex set without isolated points is exactly $(m,1)$-convex for some m a result false for non-closed sets as Kaapke notes, and "m" can be calculated using Kaapke's result.

Conversely given p points in a closed, m-convex set X without isolated points the number of segments q in X can be calculated by a second method using $K(p, q) = K(p + 1, q + K_1 + 1)$ where $K_1 = \left[\frac{2q}{p}\right]$ which can be deduced from Kaapke's results as follows. Kaapke proves $K(p, q) \subset K(p + 1, q + K_1 + 1)$ and that $K(p, q) = K(p - 1, q - K_2 - 1)$ where $K_2 = \left[\frac{2(q-1)}{p+1}\right]$ so it is sufficient to prove $K_1 - K_2 \leqq 0$ which follows from $u - \frac{up+r+2u}{p+2} \leqq 0$ where $2q = up + r$.

I shall consider (3, 2)-convexity briefly below. Before I do, I think it is valuable to make some remarks on the literature. In [2.16], Kay and Guay prove that if X is a closed (p, q)-convex set with $q > \frac{1}{4}(p - 1)^2$ then X is either convex or the union of a closed convex set X_1 and s isolated points where $s \leqq p - \frac{1}{2}(1 + \sqrt{8q + 1})$. In [2.18], Tattersall proves that a $(p, \binom{p}{2} - 1)$ set in R^d is a union of two convex sets for $p > 3$ and that a planar, bounded (p, q)-convex set with $q > \binom{p-1}{2}$ is a union of $\frac{1}{2}(1 + \sqrt{(8p - 15)})$ convex sets, a best possible result.

The first theorem of this chapter is simple but pleasing.

THEOREM 2.1: A (3, 2)-convex set, X, in R^d is a union of two convex sets.

Proof. Note that each point of X fails to see at most one point of X and that by the (3, 2)-convexity of X if $x_\alpha, x_\beta \in X$ and $[x_\alpha, x_\beta] \not\subset X$ then $X \cap \text{aff}\{x_\alpha, x_\beta\} = \{x_\alpha, x_\beta\}$. Now consider the collection C of segments $[x_\alpha, x_\beta]$ such that $x_\alpha, x_\beta \in X$ and $[x_\alpha, x_\beta] \not\subset X$ where $x_\alpha < x_\beta$ in the sense that $(x_\alpha)_j < (x_\beta)_j$ where $j = \min\{i:(x_\alpha)_i \neq (x_\beta)_i\}$. Note that if $[x_\alpha, x_\beta] \in C$ then $[x_\alpha, x_\gamma] \notin C$ for any γ. Define $X_1 = \text{conv}\{x_\alpha: [x_\alpha x_\beta] \in C\} = \text{conv } A$ and $X_2 = X \setminus A$. Clearly X_2 is a convex subset of X. To prove $X_1 \subset X$ it is sufficient to show that every simplex with n vertices, $x_1 \ldots x_n$, belonging to A is a subset of X. For $n = 2$, the result follows from a remark above. By an induction hypothesis concerning S_{n-1}, I may assume the frontier of S_n belongs to X. Now suppose some interior point, t, of S_n does not belong to X; $\text{aff}\{x_1,t\}$ meets $\text{conv}\{x_2 \ldots x_n\}$ in $u \in X$. Now u sees every point of $\text{Fr } S_n$ except x_1 via X. Hence $S_n \setminus (x_1,u) = X \cap S_n$ but then $X \cap S_n$ is not (3,2) - convex, a contradiction, so $S_n \subset X$.

Note that a planar, exactly (3,2)-convex set is a convex set with, an at most countable, collection of segments removed from its frontier.

Together, the next two theorems prove that a simply connected, planar 3-convex set is the union of three convex sets.

THEOREM 2.2: Let X be a simply connected, planar, 3-convex set. If $(\text{int cl } X) \setminus X \neq \phi$ then X is a union of three convex sets but it may not be the union of two convex sets.

Proof. By Theorem 1 of [2.19] which states: If X is a planar 3-convex set and $\text{cl } X \neq \text{cl (int } X)$ then X is the union of two convex sets; suppose $\text{cl (int } X) \supset X$.

Furthermore, since X is simply connected, by Lemma 3 of [2.19], $(\mathrm{cl}\ X) \setminus X$ contains an interval (r,s) disjoint from $\mathrm{Fr}(\mathrm{cl}\ X)$. Lemma 3 of [2.19] states that: If X is a planar, 3-convex set and if T is $(\mathrm{int\ cl}\ X) \setminus X$. Then every connected component of T is either an isolated point of $(\mathrm{cl}\ X) \setminus X$ or an interval.

Let $\mathrm{aff}\{r,s\} = L$ and let L_1 and L_2 denote the open half spaces bounded by L. $X = (X \cap \mathrm{cl}(L_1 \cap X)) \cup (X \cap \mathrm{cl}(L_2 \cap X))$.

Consider $u \in (r,s)$ then for δ sufficiently small $S(u,\delta)$ meets $L_i \cap X$ $i = 1,2$, is an open half disc that is $S(u,\delta) \cap (R^2 \setminus X) = (r,s) \cap S(u,\delta)$. Hence $L_i \cap X$ $i = 1,2$, is convex by the 3-convexity of X.

By the 3-convexity of X, $L \cap X$ has at most two components.

If $L \cap X$ has exactly one component C, then X is a union of three convex sets C, $L_1 \cap X$ and $L_2 \cap X$. That X need not be the union of two convex sets may be seen from Example 2.3 below.

Paradoxically if $L \cap X$ has two components, C_1 and C_2, X is the union of two convex sets.

If $C_2 \not\subset \mathrm{cl}(L_1 \cap X)$ then $C_1 \subset \mathrm{cl}\ (L_1 \cap X)$ by the 3-convexity of X. By the simply connectedness of X not both $\mathrm{cl}(L_1 \cap X)$ and $\mathrm{cl}(L_2 \cap X)$ meet both C_1 and C_2. Suppose without loss of generality $C_1 \subset \mathrm{cl}(L_1 \cap X)$ and $C_2 \cap \mathrm{cl}(L_1 \cap X) = \phi$. Then $C_1 \cup (L_1 \cap X)$ is convex by the 3-convexity of X. Further $C_2 \cup (L_2 \cap X)$ is convex for suppose there exists an $x \in C_2$ $y \in L_2 \cap X$ with $[xy] \not\subset C_2 \cup (L_2 \cap X)$ which implies $[xy] \not\subset X$, then for all $z \in L_1 \cap X$ $[yz] \subset X$. That is easily seen to be impossible by considering a point z in $L_1 \cap X$ sufficiently close to u.

<u>Example 2.3</u>. Let $Y_1 = \mathrm{conv}\ \{(0,\ 0),\ (\frac{1}{2},\ \frac{\sqrt{3}}{2}),\ (1,\ 0)\}$ and let $Y_2 =$ $\mathrm{conv}\ \{(0,\ 0),\ (2,\ 0),\ (2,\ -\frac{\sqrt{3}}{2}),\ (\frac{1}{2},\ -\frac{\sqrt{3}}{2})\}$. From $Y_1 \cup Y_2$ remove the open segments joining $(\frac{1}{2},\ \frac{\sqrt{3}}{2})$ to $(0,\ 0)$, $(0,\ 0)$ to $(\frac{1}{2},\ -\frac{\sqrt{3}}{2})$ and $(2,\ 0)$ to $(\frac{3}{4},\ 0)$ and remove the point $(2,\ 0)$. The resulting set satisfies the conditions of Theorem 2.2 but it is not the union of two convex sets. Consider the five points $x_1 = (\frac{1}{2},\ \frac{\sqrt{3}}{2})$, $x_2 = (0,\ 0)$,

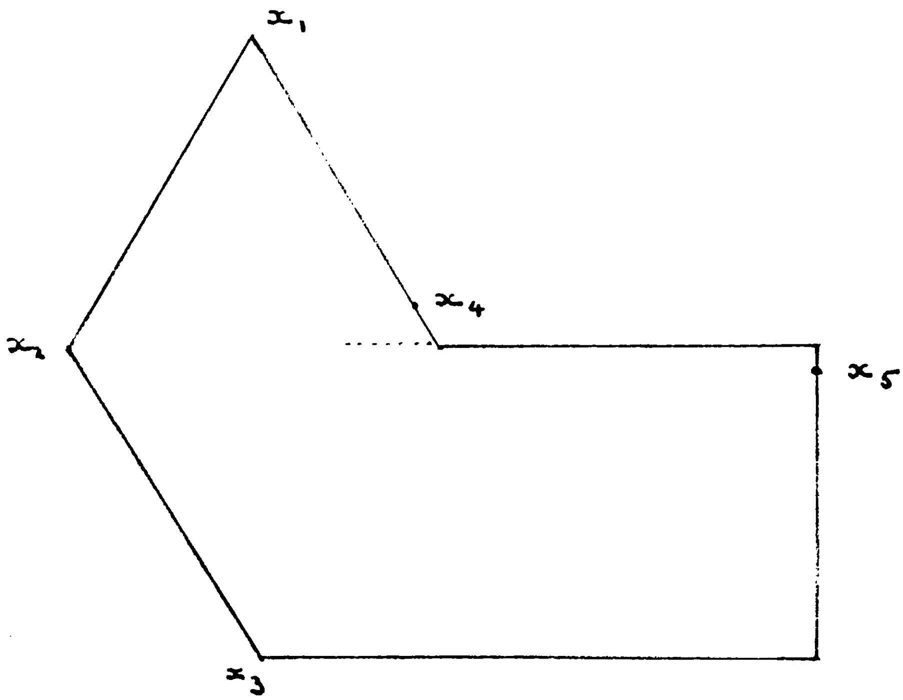


Fig. 2.3

$x_3 = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $x_4 = (\frac{15}{16}, \frac{\sqrt{3}}{16})$ and $x_5 = (2, -\frac{1}{16})$ shown in Fig. 2.3.

The next theorem improves Theorem 6 of [2.19].

THEOREM 2.3: Let X be a simply-connected, planar 3-convex set with $(\text{cl } X) \setminus X \subset \text{Fr}(\text{cl } X)$ then X is the union of three convex sets.

Proof: Firstly consider the graph G on $3n$ vertices $n \geq 3$ described below. On the unit circle consider n points $x_1 \ldots x_n$, forming an n-cycle in clockwise order. With each x_r associate two points $y_{1,r}$ and $y_{2,r}$ on $S((0, 0), 2)$ with $y_{1,r} < y_{2,r}$ that is $y_{1,r}, y_{2,r}$ in clockwise order. Let $y_{\kappa,i} < y_{m,j}$ $\kappa, m = 1,2$ identifying $y_{\kappa,j}$ and $y_{\kappa,L}$ if $L \equiv j(n)$, if $i < j$. Let $y_{\kappa,i}$ $\kappa = 1,2$ form a $2n$-cycle with $y_{1,i}$ joined to $y_{2,i}$ and $y_{2,i-1}$ and $y_{2,i}$ joined to $y_{1,i}$ and $y_{1,i+1}$. Finally join x_i to $y_{2,i-1}$ and $y_{1,i+1}$. Diagram 2.3.1 shows the case $n = 5$.

I shall show that G is three colourable. The subgraph generated by $\{x_1 \ldots x_n\}$ is three colourable. Colour $y_{1,i}$ with the colour of x_i and $y_{2,i}$ with a colour different from x_i and x_{i+1} giving a three colouring of G.

Note that only two colours are used on each triple $\{x_i, y_{1,i}, y_{2,i}\}$.

By Lawrence, Hare and Kenelly [2.15], Theorem 2, assume without loss of generality that $\text{cl } X$ has finitely many leaves, [2.19] p.43 and p.51. Hence $|Q(\text{cl } X)| = n$ is finite and each point of $Q(\text{cl } X)$ is isolated.

If $|Q(\text{cl } X)| = 0$, X is the union of three convex sets which is Theorem 5 of [2.19].

Some care is needed with the case $|Q(\text{cl } X)| = 1$ and $X \setminus q$ not connected where $Q = \{q\}$.

Let $X \setminus q$, have components X_i where clearly $2 \leq i \leq 6$ by Ramsey's Theorem and the 3-convexity of X. If X_i has interior points for some i then for each $j \neq i$ X_j is convex again by the 3-convexity of X. If $X \setminus q$ had two full dimensional components X_1 and X_2 then $X_\kappa = \phi$ for $\kappa \neq 1, 2$ as X is 3-convex. So X is a union of two

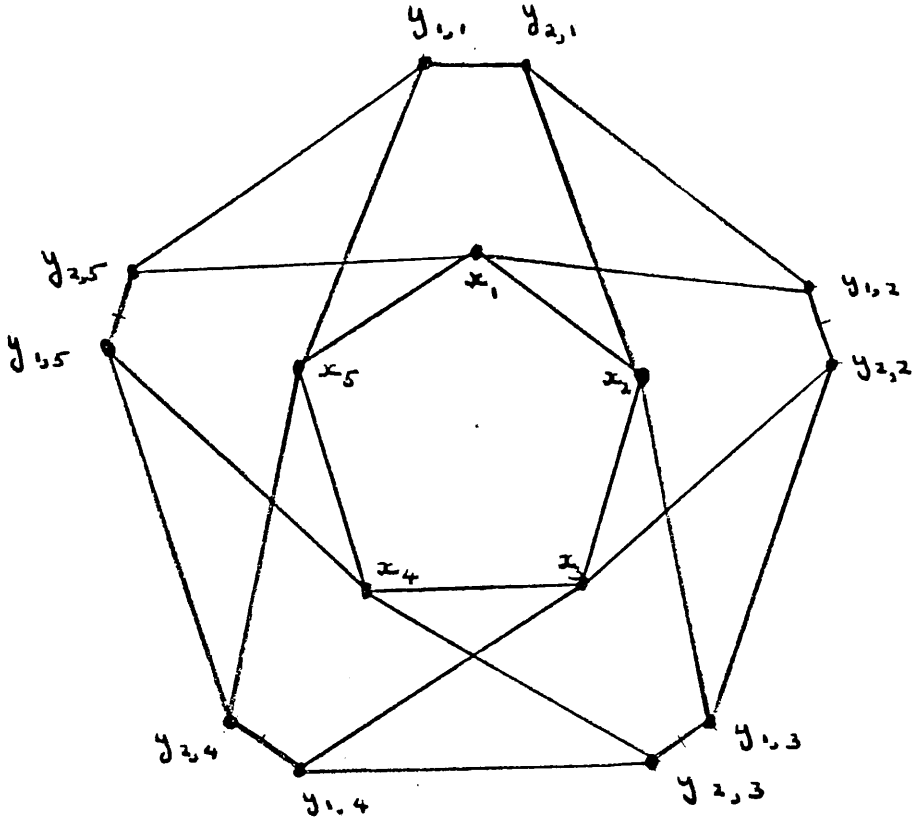


Diagram 2.3.1

convex sets since q fails to see points in at most one of X_i or X_j via X as X is 3-convex.

Suppose $X \setminus q$ has a component X_i with $\text{int}(X_i) = \phi$. Since $X \setminus q$ has finitely many components $X \not\subset \text{cl}(\text{int } X)$ in that case and hence by Theorem 1 of [2.19] X is a union of two convex sets.

In 2.19 p.55 Breen proves that if $|Q(\text{cl } X)| = 1$ and $X \setminus Q$ is connected or $|Q(\text{cl } X)| = 2$ then X is a union of three convex sets.

So assume $|Q(\text{cl } X)| \geqq 3$. Using the terminology of Eggleston [2.24] Theorem 3, $Q = \bigcup_{i=1}^{n} c_i$. Let $c_i c_{i+1}$ bound a leaf W_i of X. L_i is still a support line to J_i' at c_i and Y_i is convex by the 3-convexity of X. Similarly for Z_i and m_i.

Consider W_i, classify W_i according as e_i occurs after or before d_{i+1}. W_i is of Type 1 if e_i occurs after d_{i+1}, see Diagram 2.3.2. W_i is of Type 2 if e_i occurs before d_{i+1} or $e_i = d_{i+1}$, see Diagram 2.3.3.

I shall show that whether W_i is of Type 1 or Type 2 it is possible to express W_i as a union of three convex sets $x_i, y_{1,i}, y_{2,i}$ such that if $\text{conv}\{a,b\} \not\subset X$ then $\{a,b\}$ is an edge of G where $a, b \in \bigcup_{i=1}^{n} \{x_i, y_{1,i}, y_{2,i}\}$.

Consider Type 1 leaves shown in Diagram 2.3.2. e_i occurs after d_{i+1}. Let $Y_{1,i} = Y_i \subset X$ and $Y_{1,i} = Z_{i+1} \subset X$. If aff m_{i+1} meets X in a component not containing (c_{i+1}, d_{i+1}) let x_i be the closed (in X) set bounded by L_i and m_{i+1} containing e_i and d_{i+1} in its closure in R^2. Similarly if aff L_i meets X in a component not containing $(c_i e_i)$ define x_i in the same way. In either of the above cases define $y_{1,i} = (X \cap \text{cl } Y_{1,i}) \setminus x_i$ and $y_{2,i} = (X \cap \text{cl } Y_{2,i}) \setminus x_i$. Note that $x_i, y_{1,i}$ and $y_{2,i}$ are convex sets. If neither of the above cases holds define: $x_i = (\text{cl } Y_{1,i}) \cap Y_{2,i} \setminus \{e_i, d_{i+1}\}$, $y_{1,i} = (\text{cl}(Y_{1,i}) \cap X) \setminus \{x_i, e_i\}$ and $y_{2,i} = e_i \cup \text{cl}(Y_{2,i}) \cap X \setminus \{x_i \cup y_{1,i}\}$. Note that x_i, $y_{1,i}, y_{2,i}$ are convex and that $W_i \setminus \{x_i \cup y_{1,i} \cup y_{2,i}\} \subset \text{Ker } X$.

Now consider Type 2 leaves shown in Diagram 2.3.3 e_i occurs

Diagram 2.3.2

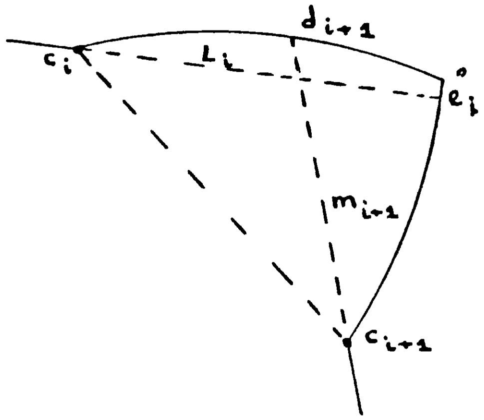
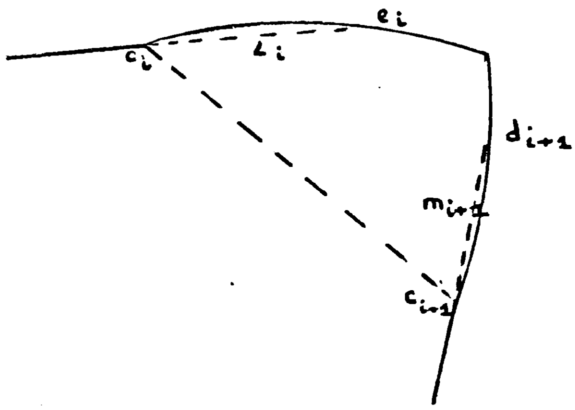


Diagram 2.3.3



before d_{i+1} or $e_i = d_{i+1}$. Let $X_i = W_i \quad (Y_{1,i} \quad Y_{2,i})$ then each point of X_i except possibly e_i and d_{i+1} sees every point of $X \setminus W_i$ by Lemma 5 of [2.2] and each interior point of $X_i \in$ Ker X.

The result now follows from the three colouring of G described above, Caratheodory's Theorem the simply connectedness of X and Lemma5 of [2.2]. That is clear if all leaves are of Type 1. If W_i is of Type 2 one has a two colouring of $Y_{1,i} \quad Y_{2,i}$ from G and if one colours $(c_i e_i) \cap X$ with the colour of $Y_{1,i}$ and $(d_{i+1} \; c_{i+1})$ with the colour of $Y_{2,i}$ one may three colour the points of $X_i \cap$ Fr (cl X) so that x and y are coloured differently if $[x,y] \not\subset X$ for all x and y belonging to X.

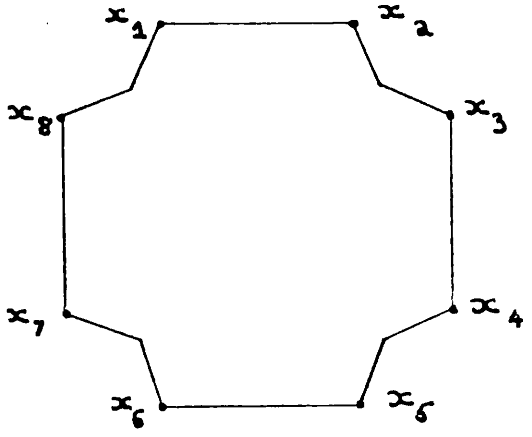
Note that if $|Q(\text{cl } X)|$ is even ≥ 4 or infinite then X need not be the union of two convex sets as Example 2.4 shows. The case $n = 4$ is given but the example may be modified for the other cases.

Example 2.4: Let X be the compact set bounded by the Jordan curve in Fig. 2.4. with the four open segments $(x_{2i+1} \; x_{2i+2})$ $0 \leq i \leq 3$ removed from its frontier. Then X satisfies the conditions of Theorem 3.3. with $|Q(\text{cl } X)| = 4$ but X is not the union of two convex sets.

Intersection of s-convex and t-convex sets: I shall now consider the intersection of an s-convex and a t-convex set. It follows from Ramsey's Theorem, Behzad and Chartrand [2.25] p240-244 that the intersection of an s-convex and a t-convex set is R(s,t)-convex. The first non-trivial and the only case I shall consider is $s = t = 3$.

Recall that $R(3, 3) = 6$. I shall show that there exists two planar 3-convex sets with one closed and hence simply connected with intersection which is not 5-convex, Example 2.5. However if both sets are planar ans simply-connected then I prove, Theorem 2.4, that the intersection is 5-convex which is best possible, Example 2.6. Finally I shall give an example of two compact 3-convex sets in R^4 with intersection which is not 5-convex, Example 2.7.

Figure 2.4



Example 2.5: Let X_1 be the compact set bounded by the Jordan Curve ABCDEFG shown in Fig. 2.5 top and let P be the point of intersection of AD and BG. Let S, F, Q and R be as shown. Let X_2 be the compact set shown in Fig. 2.5 below bounded by the Jordan curve AKBDLEGM where K,M lie on BG, M lies DL produced and K on EL produced. Then $X_1 \setminus \{p\}$ and X_2 are 3-convex but $X_2 \cap (X_1 \setminus \{p\})$ has five visually independent points A, B, D, E and G.

The last assertion follows from the fact that each of the associated segments fails to lie in X_1 or X_2.

Consider three supposedly visually independent points of $X_1 \setminus \{p\}$. None of them belongs to Ker X_1, since, as the other two points fail to see each other, one of them must see the point in Ker X_1 via $X_1 \setminus \{p\}$. So the three points lie in the union of the regions bounded by the curves FGAR, SBC, CDQ and QEF. That $X_1 \setminus \{p\}$ is 3-convex is now clear.

I shall now prove a sequence of Lemmas leading to Theorem 2.4. For the graph theory notation and terminology used and undefined below see [2.25].

Lemma 2.4.1: The only graph, G, of order five such that neither G nor its complement contains a triangle is C_5.

Proof: If some vertex of G has valency ≥ 3, since the complement of G contains no triangle, G contains a triangle. Thus each vertex of G and similarly of the complement on G has valency two. Hence $G = C_5$.

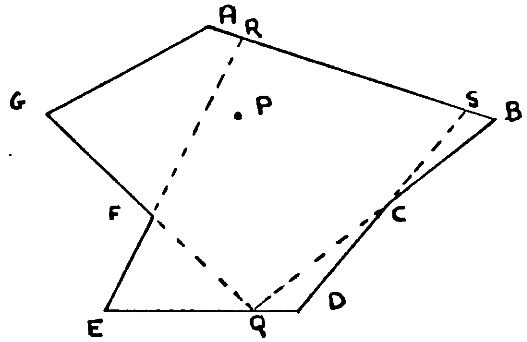
Definition 2.7. The non-visibility graph G(S, X) of a subset S, of a set X relative to X is the graph whose vertices are points of S and whose edges are defined by: if x and y belong to S then x and y are joined by an edge in G(S, X) if $[x,y] \not\subset X$. If S = X write G(X) for G(X, X) and call G(X) the non-visibility graph of X.

G(X) was introduced in Hare and Kenelly [2.26].

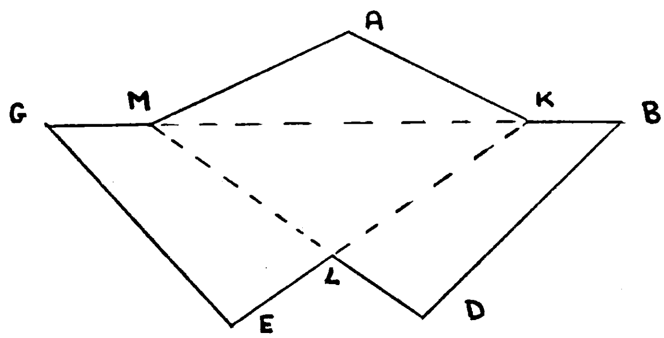
Lemma 2.4.2: Let X_1 and X_2 be 3-convex sets. If $x_1 \ldots x_5 \in X_1 \cap X_2$ are such that $[x_i, x_j] \not\subset X_1 \cap X_2$ for all $i \neq j$ $1 \leq i, j \leq 5$ then

Figure 2.5

X_1:



X_2:



$[x_ix_j] \subset X_1$ or $[x_ix_j] \subset X_2$.

Proof: Let $S = \{x_1,\ldots,x_5\}$ then $G(S, X_1)$ has no triangles and since X_2 is 3-convex and S has five visually independent points of $X_1 \cap X_2$, $G(S, X_1) = C_5$ by Lemma 2.4.1. Similarly $G(S, X_2) = C_5$ and the result follows.

Note that it follows from Lemma 2.4.2 that if either X_1 or X_2 is a union of two convex sets then $X_1 \cap X_2$ is 5-convex and that with the hypotheses of Lemma 2.4.2 $G(S, X_1)$ and $G(S, X_2)$ are complementary 5 cycles.

Lemma 2.4.3 If X_1, X_2 and $\{x_1,\ldots,x_5\}$ are as in Lemma 2.4.2 then no three members of $\{x_1,\ldots x_5\}$ are collinear.

Proof. The result follows trivially from Lemma 2.4.2.

A proof of the next Lemma appears in Erdös and Szekeres [2.27].

Lemma 2.4.4: "From five points in the plane of which no three lie on the same straight line it is always possible, to select four points determining a convex quadrilateral".

I now come to the proof of

THEOREM 2.4. If X_1 and X_2 are simply connected, planar, 3-convex sets then $X_1 \cap X_2$ is 5-oonvex.

Proof. The proof is by contradiction. The method is to select four points $x_1,\ldots,x_4$ determining a convex quadrilateral, P, from the five supposedly visually independent points of $X_1 \cap X_2$ and then to show that wherever x_5 is placed one gets some segment x_i, x_j in both X_1 and X_2.

Suppose P has vertices A, B, C, D in clockwise order. If $[A,B]$ and $[B,C]$ belong to X_1 consider two cases. Firstly, $[AB]$,$[BC]$, $[CD]$, $[DA]$ all belong to X_1 when, since X_1 is simply connected $[BD]$ also belongs to X_1 and by the 3-convexity of X_2 one of the segments $[AB]$, $[BD]$, $[DA]$ belongs to X_2 when by the 3-convexity of X_2 $[AC] \subset X_2$ and by the 3-convexity of X_1 $[AC] \subset X_1$. Hence it is to consider two cases:

Case 1: three edges of P belong to X_1.

Case 2: two non-adjacent edges of P belong to X_1.

Case 1: Note that no x_j may see more than two other x_K via X_i for each i. Without loss of generality one has the configuration in Diagram 2.4.1 though possibly A_3 may be unbounded if $\text{aff}\{x_1,x_2\}$ is parallel to $\text{aff}\{x_3,x_4\}$. In Diagram 2.4.1 Ⓚ indicates the X_K which contains $[x_i,x_j]$, $K = 1,2$. It was assumed that $[x_1x_2], [x_2,x_3]$ and $[x_1,x_4]$ belonged to X_1 and it was deduced from the 3-convexity of X_2 that $[x_1,x_3]$ and $[x_2,x_4]$ belonged to X_2.

Remembering that no three x_i are collinear I define regions A_i as follows:

Let A_1 be the open half-plane bounded by $\text{aff}\{x_1,x_2\}$ not containing P.

Let A_2 be the open half-plane bounded by $\text{aff}\{x_3,x_4\}$ not containing P.

Let A_3 be the open half-plane bounded by $\text{aff}\{x_2,x_3\}$ not containing P without $\text{cl}\, A_1 \cup \text{cl}\, A_2$.

Let A_4 be the open half-plane bounded by $\text{aff}\{x_1,x_4\}$ not containing P without $\text{cl}\, A_1 \cup \text{cl}\, A_2$.

Let A_5 be the interior of P.

Then as can easily be seen from Diagram 2.4.1, using the simply connectedness of X_1 and X_2 and the result in the proof of Lemma 2.4.2 that $G(S, X_1)$ and $G(S, X_2)$ are complementary 5 cycles if:

	1.	$x_5 \in A_1$	then $[x_1,x_2] \subset X_1 \cap X_2$,
	2.	$x_5 \in A_2$	then $[x_3,x_4] \subset X_1 \cap X_2$,
	3.	$x_5 \in A_3$	then $[x_2,x_4] \subset X_1 \cap X_2$,
	4.	$x_5 \in A_4$	then $[x_1,x_3] \subset X_1 \cap X_2$,
or	5.	$x_5 \in A_5$	then $[x_5,x_1] \subset X_1 \cap X_2$.

Thus if the theorem is false Case 2 must occur.

Case 2: Without loss of generality one has the configuration shown in Diagram 2.4.2 though possibly B_5 and/or B_6 may be unbounded. In Digram 2.4.2, Ⓚ indicates the X_K which contains $[x_ix_j]$. It was assumed that $[x_1x_2], [x_3,x_4]$ lay in X_2 and $[x_1x_4]$ $[x_2,x_3]$ lay in X_1 from which, without loss of generality, it was deduced that $[x_1,x_3]$ lay in X_1 and $[x_2,x_4]$ lay in X_2.

Diagram 2.4.1

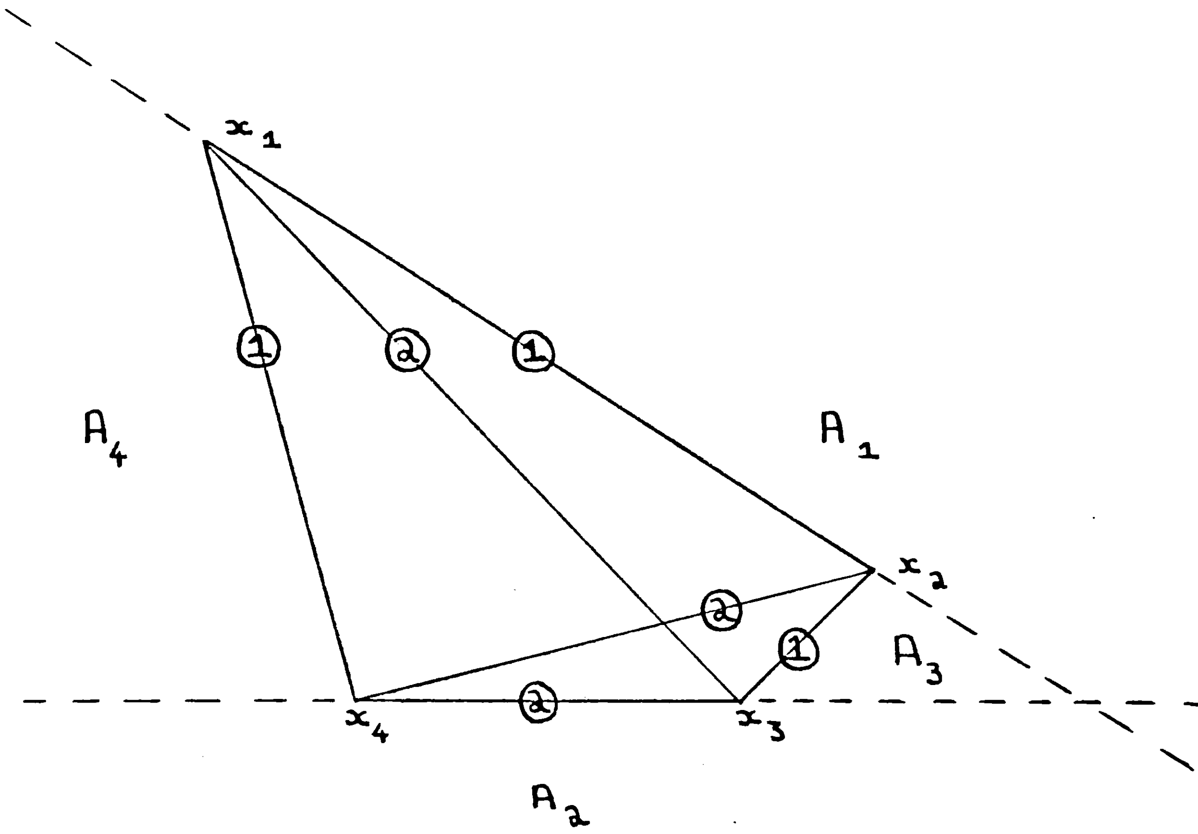
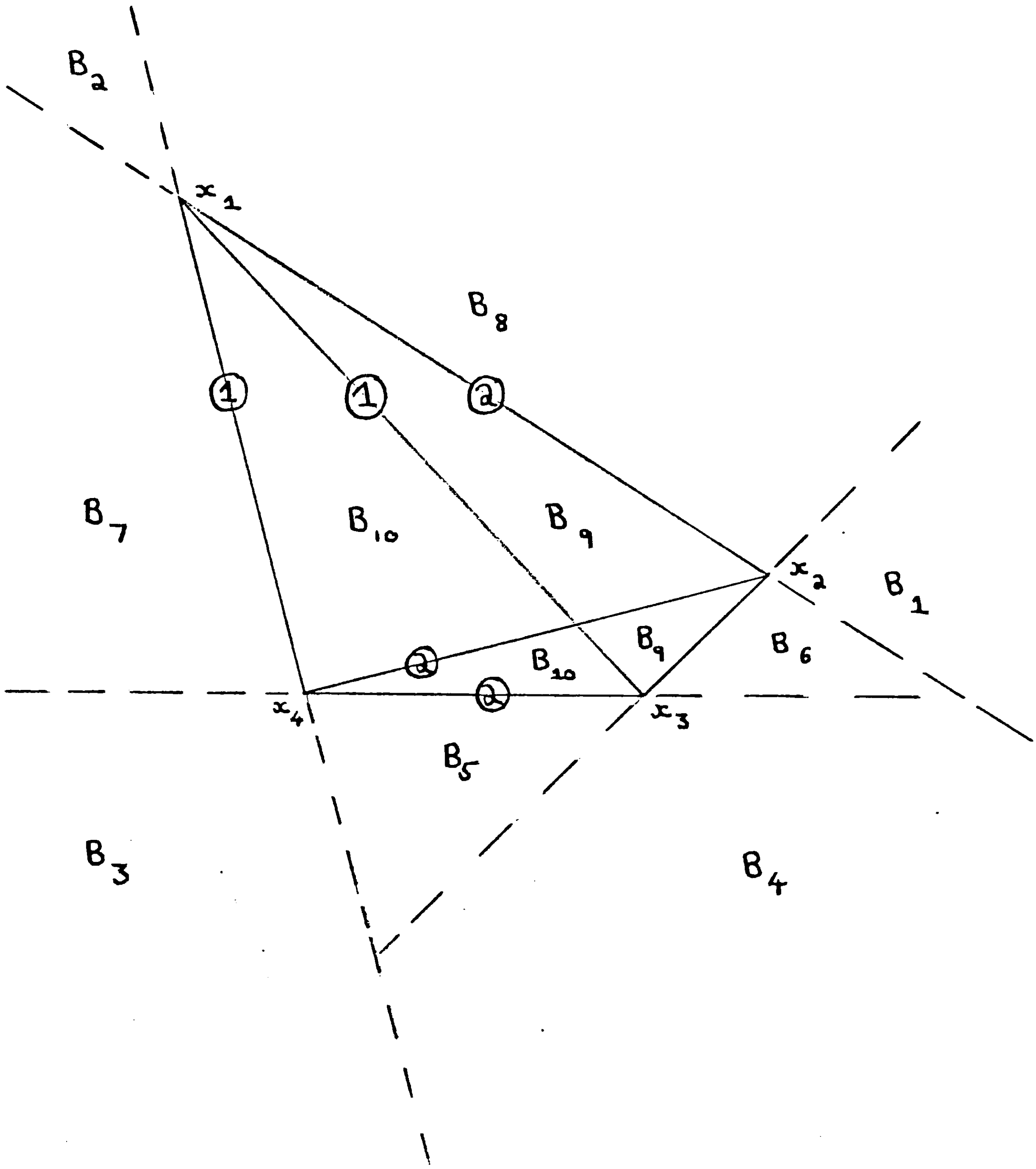


Diagram 2.4.2



As in Case 1 one may define the ten regions as shown in Diagram 2.4.2 in terms of open half-planes determined by the $\text{aff}\{x_i, x_j\}$. However the following results which complete the proof of the theorem can easily be seen from Diagram 2.4.2.

If	$x_5 \in B_1$	then	$[x_3, x_2] \subset X_1 \cap X_2$
If	$x_5 \in B_2$	then	$[x_2, x_1] \subset X_1 \cap X_2$
If	$x_5 \in B_3$	then	$[x_1, x_4] \subset X_1 \cap X_2$
If	$x_5 \in B_4$	then	$[x_4, x_3] \subset X_1 \cap X_2$
If	$x_5 \in B_5$	then	$[x_5, x_1] \subset X_1 \cap X_2$
If	$x_5 \in B_6$	then	$[x_5, x_4] \subset X_1 \cap X_2$
If	$x_5 \in B_7$	then	$[x_5, x_2] \subset X_1 \cap X_2$
If	$x_5 \in B_8$	then	$[x_5, x_3] \subset X_1 \cap X_2$
If	$x_5 \in B_9$	then	$[x_5, x_3] \subset X_1 \cap X_2$
If	$x_5 \in B_{10}$	then	$[x_5, x_1] \subset X_1 \cap X_2$

Thus Case 2 cannot occur and the assumption $X_1 \cap X_2$ is not 5-convex is false and Theorem 2.4. is proved.

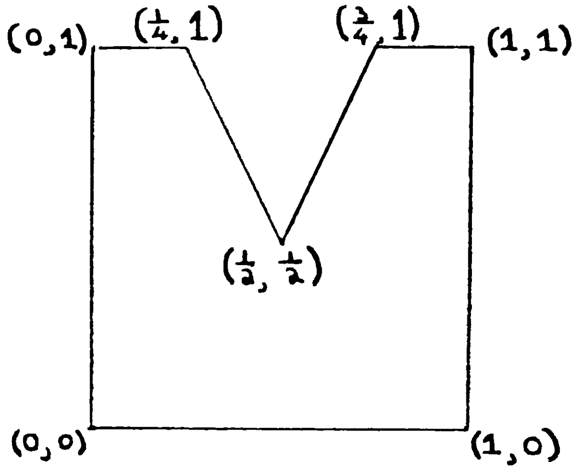
Corollary 2.4.1 If X_1 and X_2 are closed, planar 3-convex sets then $X_1 \cap X_2$ is 5-convex.

Proof It is sufficient to prove that a closed, planar, 3-convex set, X, is simply connected. That result follows from Tietze's Theorem and the fact that a closed 3-convex set is starlike from each of its lnc points [2.2], Theorem 1.

Corollary 2.4.1 is the best possible result for closed, planar, 3-convex sets even if X_1 and X_2 are connected and the union of two convex sets as Example 2.6 shown in Fig. 2.6 demonstrates.

Example 2.6: Let $S = \text{conv}\,\{(0, 0), (1, 0), (1, 1), (0, 1)\}$. Let $X_1 = \text{conv}\,\{(\frac{1}{4}, 1), (\frac{1}{2}, \frac{1}{2})\} \cup \text{conv}\,\{(\frac{1}{2}, \frac{1}{2}), (\frac{3}{4}, 1)\} \cup \{S \setminus \text{conv}\,\{(\frac{1}{2}, \frac{1}{2}). (\frac{1}{4}, 1), (\frac{3}{4}, 1)\}\}$ and let $X_2 = \text{conv}\,\{(\frac{1}{4}, \frac{5}{6}), (0, \frac{11}{12})\} \cup \text{conv}\,\{(\frac{1}{4}, \frac{5}{6}), (0, \frac{3}{4})\} \cup \text{conv}\,\{(1, \frac{11}{12}), (\frac{3}{4}, \frac{5}{6})\} \cup \text{conv}\,\{(1, \frac{3}{4}), (\frac{3}{4}, \frac{5}{6})\} \cup \{S \setminus (\text{conv}\,\{(\frac{1}{4}, \frac{5}{6}), (0, \frac{3}{4}), (0, \frac{11}{12})\} \cup \text{conv}\,\{(\frac{3}{4}, \frac{5}{6}), (1, \frac{11}{12}), (1, \frac{3}{4})\})\}$. $X_1 \cap X_2$ is not 4-convex

X_1 :



X_2 :

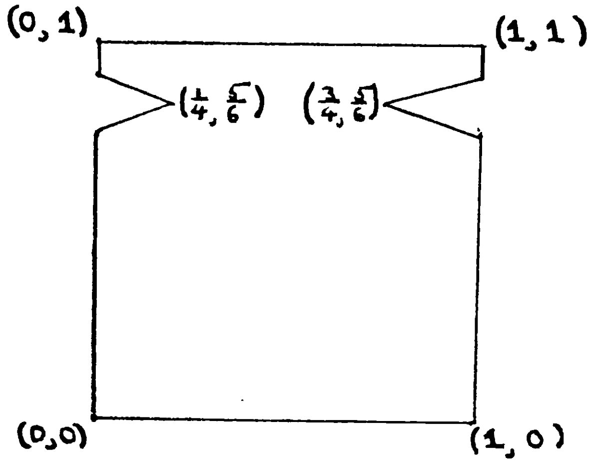


Fig 2.6

since $(1, \frac{11}{12})$, $(1, \frac{3}{4})$, $(0, \frac{11}{12})$ and $(0, \frac{3}{4})$ are visually independent.

The next example was inspired by Eggleston's Example in [2.4].

Example 2.7: Let $S = \{x_1, \ldots, x_5\}$ be a set of five points on the moment curve $p(\theta) = (\theta, \theta^2, \theta^3, \theta^4)$ with $x_i = p(\theta_i)$ with $0 < \theta_i \leqq 1$. As in Eggleston [2.4] it is possible to construct a compact 3-convex set $X_1 \supset S$ such that $G(S, X_1) = C_5$ and a compact 3-convex set $X_2 \supset S$ such that $G(S, X_2)$ is the complementary C_5. Hence $X_1 \cap X_2$ is not 5-convex.

Note that Eggleston's Example [2.4] shows that, given a countable graph G with no triangles there exists a compact 3-convex set in R^4 which contains G as a subgraph of its non-visibility graph.

Note that a most important step in the proof of Theorem 2.4 was to show that if $X_1 \cap X_2$ was not 5-convex then both X_1 and X_2 had to contain five cycles in their non-visibility graphs.

Consider the Minkowski or vector sum, $X_1 + X_2$, of two planar 3-convex sets. Clearly if one of them has no five cycles in its non-visibility graph then $X_1 + X_2$ is 5-convex. This suggests the following

Conjecture 2.2. If X_1 and X_2 are closed, planar 3-convex sets $X_1 + X_2$ is 5-convex.

Note that by Ramsey's Theorem if X_1 is s-convex and X_2 is t-convex $X_1 + X_2$ is $R(s,t)$-convex. Note also that if $|Q(X_i)|$ $i = 1, 2$ is not equal to 3 or 5 then Conjecture 2.2 is true by Lemma 2.4.2 and Lemma 5 of [2.2].

Miscellaneous Properties of M-convex Sets

Theorem 2.5 generalises a 3-convex result of Eggleston. His, informally presented, proof of that most important case did not generalise.

THEOREM 2.5 Let X be a closed, m-convex set in R^d and $p \notin X$ then there exists a $d - m + 1$ flat through p not meeting X if $d > m - 1$.

Proof. For $m = 2$ the result is known. Assume the result is true for $m \leqq s$ and for all $d > m - 1$. Let X be a closed $(s + 1)$-convex set in

R^d, $d > s$. Let q be a nearest point of X to p. Let H be a hyperplane through p with normal $p - q$. Then $X \cap H$ is s-convex and by the inductive hypothesis there exists a $(d - 1) - s + 1 = d - (s + 1) + 1$ flat through p in H not meeting $X \cap H$.

Corollary 2.5.1 If X is a closed m-convex set in R^d and $x \in \mathrm{Fr}\, X$ then there exists a $d - m + 1$ flat through x which does not meet int X if $d > m - 1$.

Proof Suppose firstly x is a nearest point of X to $p \notin X$. Let H be the hyperplane through x with normal $p - x$. Let H_1 denote the open half-space bounded by H containing p. Then $X \cap H_1$ is $(m - 1)$-convex for suppose there exist $m - 1$ visually independent points in $X \cap H_1$, since X is m-convex, one of them can see x via X but then x is not a nearest point of X to p. Let $S = \mathrm{cl}\,(X \cap H_1)$. Then S is a closed $(m - 1)$-convex subset of X. Since a closed, m-convex set is locally starshaped, Kay and Guay [2.16] Lemma 2, $x \notin S$ for otherwise x is not a nearest point of X to p. Now $S \cap H$ is $(m - 1)$-convex and by Theorem 2.5 there exists a $(d - 1)-(m - 1) + 1 = d - m + 1$ flat, in H, through x not meeting $S \cap H$. Now this $d - m + 1$ flat does not meet the interior of X for suppose it did at y then $y \in S \cap H$ a contradiction.

Now let $x \in \mathrm{Fr}\, X$ and let $\{x_i\}$ be a sequence of points of $R^d \setminus X$ with $x_i \to x$. Let y_i be the nearest point of X to x_i. Then $y_i \to x$. Now through each y_i there exists a $d - m + 1$ flat, F_i, through y_i which does not meet int X. Let x_i^K, $K = 1, \ldots d - m + 1$ be vectors in F_i with $x_i^K x_i^L = \delta_{K,L}$ where $\delta_{K,L}$ is the Kronecker delta. Now by taking, an appropriate subsequence $\{I_j\}$ of the integer sufficies "I" there exists an x^K such that $x_{I_j}^K \to x^K$ for $1 \leqq K \leqq d - m + 1$ and $x^K x^L = \delta_{K,L}$ $1 \leqq K,L \leqq d - m + 1$. Now the $d - m + 1$ flat through x spanned by the $d - m + 1$ vectors x^K does not meet the interior of X. For suppose it did at y with $S(y, \delta) \subset \mathrm{int}\, X$ and $y = x + \sum_{K=1}^{d-m+1} \lambda_K x^K$. Then for j sufficiently large $y' = y_{I_j} + \sum_{K=1}^{d-m+1} \lambda_K x_{I_j}$ is such that $y' \in S(y, \delta)$ which is a contradiction.

Definition 2.8: A subset S of a set X in R^d is relatively m-convex if for every set of m-points in S at least one of the associated segments lies in X.

Definition 2.8 given by Tattersall in [2.28] is stated because of my observation that if X is a compact set in R^d with Fr X relatively 3-convex then X is 3-convex. Note that if the frontier of X is relatively 3-convex $Q(X) \subset \text{Ker } X$ since X is closed. Since X is compact, if there exist three visually independent points x_1, x_2, x_3 of X one can obtain three relatively visually independent points of the frontier of X by considering its intersection with three rays one passing through each x_i having common end-point $q \in Q(X)$ if $Q(X) \neq \phi$. If $Q(X) = \phi$ X is the union of two convex sets by Tietze's Theorem. Note that for closed planar sets X, by virtue of Breen's note in [2.3] mentioned earlier, if $|Q| \geqq 4$ and the frontier of X is relatively 3-convex then X is 3-convex. Finally note that Example 2.1 shows that the hypothesis $|Q| \geqq 4$ is not superfluous

It has been noted by several writers that if X is m-convex then cl X is m-convex. The next theorem is an analogous result for the interior of a 3-convex set.

THEOREM 2.6. Let X be a 3-convex set in R^d then int X is 3-convex .

Proof Firstly I prove the following:

Lemma 2.6.1 Suppose $0 < \lambda, \mu, v < 1$ and three points $\underline{x}_i$, $i = 1,2,3$, in R^2 forming a non-degenerate triangle are given. Then there exists a triangle $\text{conv}\{\underline{y}_1, \underline{y}_2, \underline{y}_3\}$ with

1. $\underline{x}_1 = \mu\underline{y}_1 + (1 - \mu)\underline{y}_3$
2. $\underline{x}_2 = \lambda\underline{y}_1 + (1 - \lambda)\underline{y}_2$
3. $\underline{x}_3 = v\underline{y}_2 + (1 - v)\underline{y}_3$

Moreover if $\underline{x}_i = (x_{1i}, x_{2i})$ and $\underline{y}_i = (y_{1i}, y_{2i})$ them the $y_{j\ell}$ are linear combinations of x_{Ki} and hence continuous functions of the x_{Ki} for $0 \leqq i,\ell \leqq 3$ $j = 1, 2$ and $K = 1, 2$

Proof of Lemma. The statements (1), (2), and (3) yield six equations

$$\begin{pmatrix} \mu & 0 & 0 & 0 & 1-\mu & 0 \\ 0 & \mu & 0 & 0 & 0 & 1-\mu \\ 0 & 0 & V & 0 & 1-V & 0 \\ 0 & 0 & 0 & V & 0 & 1-V \\ \lambda & 0 & 1-\lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1-\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{21} \\ y_{12} \\ y_{22} \\ y_{13} \\ y_{23} \end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \\ x_{13} \\ x_{23} \end{pmatrix}$$

By inspection the six by six matrix A has linearly dependent rows if and only if

$$-\frac{1-\mu}{1-v} \cdot V + (1-\lambda)\left(-\frac{\mu}{\lambda}\right) = 0$$

that is $v\lambda(1-\mu) + \mu(1-\lambda)(1-v) = 0$ which is impossible since $0 < \lambda, \mu, v < 1$. Hence A is invertible and the result follows.

Proof of Theorem. Let $\underline{z}_1, \underline{z}_2, \underline{z}_3$ be three, supposedly visually independent, points of int X. Then each segment $\underline{z}_i, \underline{z}_j$ with $1 \leq i < j \leq 3$ meets Fr X. Let $\underline{z}_i\underline{z}_j$ meet Fr X in $\underline{x(i,j)}$.

Let $\underline{x(1,2)} = \lambda\underline{z}_1 + (1-\lambda)\underline{z}_2$, $\underline{x(1,3)} = \mu\underline{z}_1 + (1-\mu)\underline{z}_3$ and let $\underline{x(2,3)} = v\underline{z}_1 + (1-v)\underline{z}_3$ where $0 < \lambda, \mu, v < 1$. Let $\underline{x(i,j;n)}$ for $n = 1, 2, \ldots$ be a sequence of points $R^d \setminus X$ converging to $\underline{x(i,j)}$. For n sufficiently large $\underline{x(1,2;n)}$ $\underline{x(1,3;n)}$ and $\underline{x(2,3;n)}$ determine F_n a flat of dimension at least equal to that of aff $\underline{z}_1, \underline{z}_2, z_3$.

In the case where aff $\underline{z}_1, \underline{z}_2, \underline{z}_3$ is one dimensional and without loss of generality $\underline{z}_2 \in (\underline{z}_1\underline{z}_3)$, consider the line L_n determined by $\underline{x(1,2;n)}$ and $\underline{x(2,3;n)}$. Note that for n sufficiently large, since $\underline{z}_i \in$ int X, L_n will contain three visually independent points of X.

In the case where aff $\underline{z}_1, \underline{z}_2, \underline{z}_3$ is two dimensional for n sufficiently large F_n is of dimension 2. It is then clear that three visually, independent of int X, one close to each $\underline{z}_i$ $1 \leq i \leq 3$, can be obtained by an application of Lemma 2.6.1 or its extension to R^d.

A false proof of Theorem 2.7 appeared in [2.13] as Theorem 4.

THOEREM 2.7. Let X be a closed planar m-convex set. If conv $Q \subset X$ and int (conv Q) = ϕ then X is a union of $m - 1$ convex sets.

Proof. The argument of Breen and Kay [2.13] is correct for the case $X \setminus Q$ disconnected. However it is not true that X has at most $m - 2$ lnc points when $X \setminus Q$ is connected.

I prove the result under the assumption $X \setminus Q$ is connected. For $m = 3$ $|Q| \leqq 2$ and the result was proved by Valentine in [2.2]. So assume $m \geqq 4$.

Now X = cl (int X) and int X is connected since X is m-convex and hence locally sitarshaped so that X = cl $(X \setminus Q)$ see Lemma 3.1.2. and Corollary 3.1.

Let aff $Q = L$ and let L_1 and L_2 denote the open half-planes bounded by L. It is clear that $L_1 \cap X$ may be at most $m - 1$ disjoint convex sets $C_1 \dots C_{m-1}$ containing interior points of R^2 whose closures meet L in one dimensional convex sets with disjoint relative interiors. Similarly for L_2 with $C_m \dots C_{2m-2}$. It is clear $|Q| \leqq 4\ (m - 1)$. Let $L \cap Q = \{q_1 \dots q_t\}$ in that order. It is clear that q_1 lies on the closure of precisely one C_i on each side of L. Let them be C_1 and C_s. Let $S(q_i) = \{x : x \in X, [xq_i] \subset X\}$ then $X \setminus S(q_1)$ is relatively (m - 2)-convex. Furthermore cl$(X \setminus S(q_1))$ is (m - 2)-convex. It is sufficient to show that if $x, y \in X \setminus S(q_1)$ and $[x,y] \subset X$ then $[x,y] \subset$ cl$(X \setminus S(q_1))$. If $x,y \in L_1 \cap X \setminus S(q_1)$ $[x,y] \subset X \setminus S(q_1)$ since both x and y belong to the same C_i $i \neq 1$. Similarly if $x,y \in L_2 \cap X \setminus S(q_1)$. If $x \in L_1$ and $y \in L_2$ and $[xy] \cap L = z$ then $[xz) \cup (zy] \subset X \setminus S(q_1)$. Hence cl$(X \setminus S(q_1))$ is (m - 2)-convex. Clearly cl $C_1 \cup$ cl $C_s \cup$ cl$(X \setminus S(q_1)) = X$. Thus, since the lnc points of $X \setminus S(q_1)$ belong to L, X is a union of $(m - 1)$ convex sets by induction.

I note that if X is a connected, closed, planar m-convex set and int (conv Q) = ϕ then conv $Q \subset X$. For if $(u,v) \subset$ (conv Q) $\setminus X$ then $(uv) \subset (q_s q_{s+1})$ for some s and by Lemmas 1 and 2 of Guay and Kay [2.2?] one can partition X into two disjoint non-empty closed sets $\bigcup_{i \leq s} S(q_i)$ and $\bigcup_{i \geq s+1} S(q_i)$, a contradiction. Thus by considering the components of X, the hypothesis $\text{conv } Q \subset X$ may be deleted.

Notice that while a closed m-convex set in R^2 with $\text{int}(\text{conv } Q) = \phi$ is a union of $m - 1$ convex sets and a closed, 3-convex set in R^d with $\text{int}(\text{conv } Q) = \phi$ is a union of two convex sets, the analogous result for closed m-convex sets in R^d is false as easily constructed examples show.

I sent a copy of the proof of Theorem 2.7 to Breen which she acknowledged in a letter of October 4th, 1976. Breen proved Theorem 2.6 again in [2.5] and extended the idea in [2.6].

The next theorem generalises a result of Valentine [2.8] Lemma 1.

<u>THEOREM 2.8.</u> Let X be an m-convex set in R^d then if $x \in \text{conv } X$ $x \in \text{conv}\{x_1 \ldots x_r\}$ where $x_i \in X$ and $r \leq m - 1$, $m \geq 3$.

<u>Proof.</u> The result is known if $d \leq m - 2$, see Eggleston [2.30] pages 35 and 36.

The proof is by induction on d. Suppose the result is true for all $d \leq n - 1$ and let X be an m-convex set, in R^n. Let $x \in \text{conv } X$ and suppose $x \notin X$. By Caratheodory's Theorem x belongs to a simplex S of dimension n with vertices in X. If x belongs to some facet F of S then the result is immediate by considering $(\text{aff } F) \cap X$. So assume x belongs to the interior of S. Let $S = \text{conv } x_1 \ldots x_{n+1}$ and let $\text{aff}\{x_{n+1}, x\}$ meet $\text{conv } x_1 \ldots x_n$ at y. Then by the inductive hypothesis there exist $y_1 \ldots y_r \in X$ with $r \leq m - 1$ such that $y \in \text{conv}\{y_1 \ldots y_r\}$. If $r < m - 1$ the result follows immediately. So suppose $r = m - 1$ with $y = \sum_{i=1}^{m-1} \lambda_i y_i$ with $\sum_{i=1}^{m-1} \lambda_i = 1$ $\lambda_i > 0$ $1 \leq i \leq m - 1$. By the m-convexity of X either for some $i \neq j$ $1 \leq i,j \leq m - 1$ $[y_i y_j] \subset X$ or $[y_i, x_{n+1}] \subset X$. In the first case suppose $[y_i y_j] \subset X$ then $\frac{\lambda_i}{\lambda_i+\lambda_j} y_i + \frac{\lambda_j}{\lambda_i+\lambda_j} y_j = y_{i,j} \in X$ and

$$y = \sum_{\substack{K=1 \\ K \neq i,j}}^{m-1} \lambda_K y_K + (\lambda_i+\lambda_j) y_{i,j}.$$

Hence y belongs to the convex cover of $m - 2$ points of X and hence x belongs to the convex cover of $m - 1$ points of X. In the second case suppose $[x_{n+1}, y_j] \subset X$ and $x = \lambda y + (1 - \lambda)x_{n+1}$ then since $\frac{1-\lambda}{1-\lambda+\lambda\lambda_j} x_{n+1} + \frac{\lambda\lambda_j}{1-\lambda+\lambda\lambda_j} y_j = y_{n+1,j} \in X$ and

$$x = \lambda(\sum_{K=1}^{m-1} \lambda_K y_K) + (1 - \lambda)x_{n+1} = \lambda(\sum_{\substack{K=1 \\ K \neq j}}^{m-1} \lambda_K x_K) + ((1 - \lambda) + \lambda\lambda_j)y_{n+1,j}$$

x belongs to the convex cover of $m - 1$ points of X.

I remark that $r \leqq m - 1$ is best possible for $d \geqq m - 1$. Consider $m - 1$ mutually perpendicular line segments through the origin.

Definition 2.9: The visibility graph of a subset S of a set X in R^d written $CG[S,X]$ is the graph with vertex set S and $\{x,y\}$ and edge of $CG[S,X]$ if and only if $[x,y] \subset X$.

For the other graph theoretic terminology and results used below see Harary [2.31] p 155-165.

THEOREM 2.9 Let X be a closed set in R^d and let $T(X) = \{[x,y] : x,y \in X$ and $[x,y] \subset X\}$. If for every three segments s_1, s_2, s_3 (possibly degenerate) of $T(X)$ at least one of the corresponding convex hulls conv $\{s_i \cup s_j\}$ $1 \leqq i < j \leqq 3$ lies in X then X is the union of two convex sets.

Proof. For every finite subset S of X $\alpha(CG[S,X]) = 2$. By considering three adjacent edges of a cycle in $CG[S,X]$ it follows that $CG[S,X]$ is triangulated and hence perfect. Hence $\theta(CG\ S,X\ ) = 2$.

Hence by Theorem 3 of [2.15] X is the union of two convex sets. Theorem 3 of [2.15] is: Let S be a closed subset of a topological linear space such that for every finite subset $F \subset S$ there is a 2-partition $\{F_1,F_2\}$ such that if $x,y \in F_i$ $(1 \leqq i \leqq 2)$ then $[xy] \subset S$. Then S is the union of two convex sets.

For completeness I mention Breen [2.32] and the two papers concerned with the intersection of maximal m-convex subsets of a closed set in R^d Breen [2.33] and Tattersall [2.28].

I note that Kay and Guay [2.16], Lemma 2, have shown that a closed m-convex set is locally starshaped. I shall consider closed, locally starshaped sets in Chapter 3.

Appendix to Chapter 2

The following theorem and examples may be of interest in view of Theorem 2.6.

Theorem 2.10. In R^d, the interior of a set X which is the union of two convex sets, may be expressed as the union of two convex subsets of int X.

Proof. Let $X = X_1 \cup X_2$ with $X_1, X_2 \subset X$ and X_1, X_2 convex. Assume $\dim X_1 = \dim X_2 = \dim(\text{aff } X)$ since if $\dim X_1$ and $\dim X_2 < \dim(\text{aff } X)$ then $\text{int } X = \phi$ and if $\dim X_1 < \dim X_2 = \dim(\text{aff } X)$ $\text{int } X = \text{int } X_2$.

To prove the theorem it is to prove that if $x \in \text{Fr } X_1 \cap \text{int } X$ or $x \in \text{Fr } X_2 \cap \text{int } X$ then x belongs to a convex subset of int X containing $\text{int } X_1$ or $\text{int } X_2$ respectively.

Let $x \in \text{int } X \setminus \text{int } X_1 \setminus \text{int } X_2$ then it follows that $x \in \text{Fr } X_1 \cap \text{Fr } X_2$ since $x \in \text{int } X$. Since there exists a support hyperplane H_1 to X_1 at x and a support hyperplane to X_2 at x and $x \in \text{int } X$, $H_1 = H_2 \equiv H$ and $(\text{Fr } X_1 \cap \text{Fr } X_2) \cap \text{int } X = H \cap \text{int } X$.

Hence $\text{int } X = \text{int } X_1 \cup \text{int } X_2 \cup (H \cap \text{int } X)$.

Now $H \cap \text{int } X = (H \cap \text{int } X \cap X_1) \cup (H \cap \text{int } X \cap X_2)$. I shall prove $H \cap \text{int } X \cap X_1$ is a convex subset of int X.

It is sufficient to prove that the line segment joining each pair of points x,y of $H \cap \text{int } X \cap X_1$ lies in int X which is clear since x and y belong to int X and H separates X_1 and X_2.

Hence $(H \cap \text{int } X \cap X_1) \cup \text{int } X_1$ is a convex subset of int X. Similarly for $H \cap \text{int } X \cap X_2 \cup \text{int } X_2$. Hence the theorem follows.

Note that it is not possible to insist that the convex subsets of int X in Theorem 2.10 are open even if X is compact as easily constructed examples show.

Example 2.8 shows that the interior of a union of three convex sets need not be 4-convex.

<u>Example 2.8</u>: Let X be the compact set bounded by the Jordan Curve ABCDEFA shown in Figure 2.8 without P_1 and P_2 where
ABCDEFA shown in Figure 2.8 without P_1 and P_2 where
$A = (0,0)$, $P_1 = (\frac{1}{2},0)$, $C = (1,0)$, $P_2 = (\frac{3}{2},0)$ and $E = (2,0)$.
Also for definiteness suppose

$B = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ $D = (\frac{3}{2}, \frac{\sqrt{3}}{2})$ and $F = (\frac{1}{2}, \frac{-\sqrt{3}}{2})$ as shown in Fig. 2.8.

Then X is the union of three convex sets but int X is not 4-convex.

For a compact set X if Fr X is relatively 4-convex then X is not necessarily 4-convex as the compact set bounded by the Jordan Curve ABCDEFGHJKLMN shown in Fig. 2.9 demonstrates.

Figure 2.8

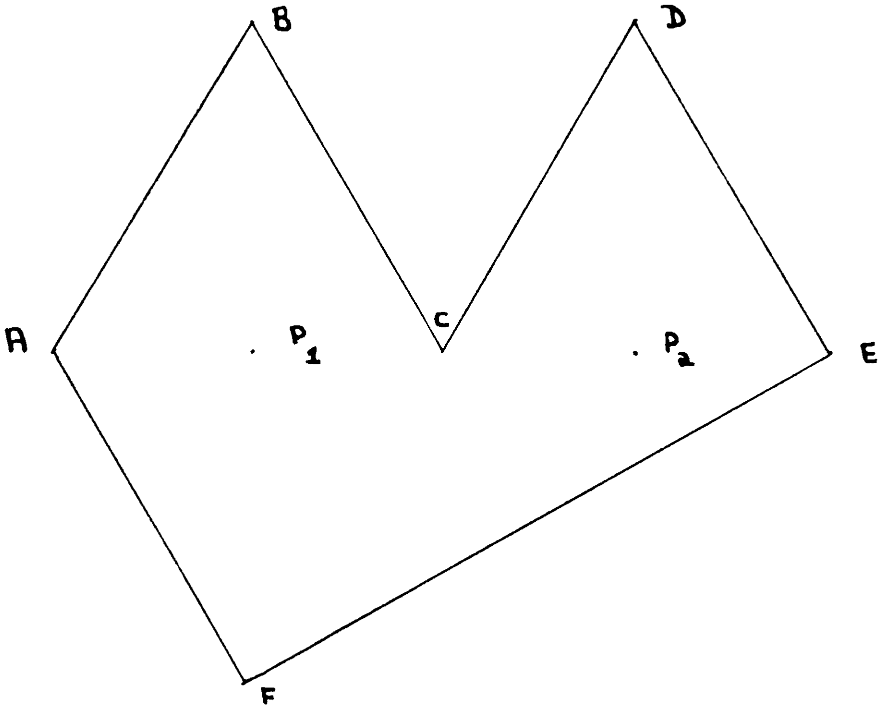
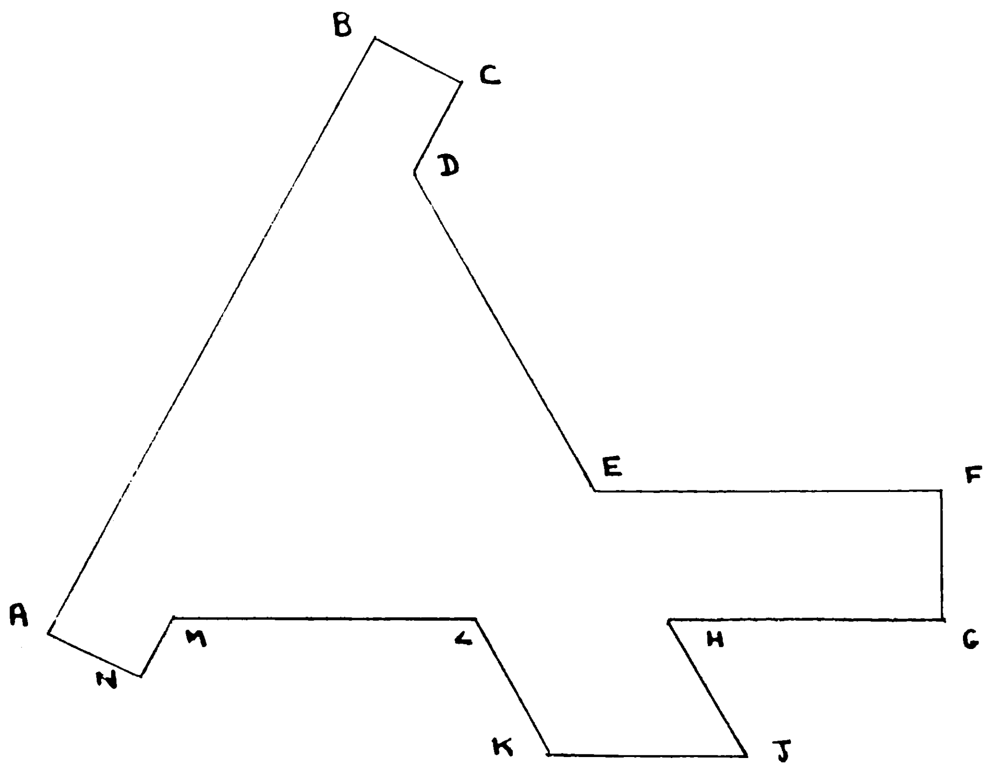


Figure 2.9



REFERENCES

[2.1] F.A. Valentine, Conxex Sets, McGraw-Hill (1964).

[2.2] F.A. Valentine, A three point convexity property, Pac. J. Math. 7, (1957), 1227-1235.

[2.3] M. Breen, An $n + 1$ member decomposition for sets whose lnc points form n convex sets, Can J. Math. 27, (1975), 1378-1383.

[2.4] H.G. Eggleston, Valentine convexity in n dimensions, Math. Proc. Cam. Phil. Soc. 80, (1976), 223-228.

[2.5] M. Breen, M-convex sets whose lnc points lie in a hyperplane, J.L.M.S. to appear.

[2.6] M. Breen, A bound for decompositions of m-convex sets whose lnc points lie in a hyperplane.

[2.7] E.O. Buchmann, Property P_3 and the union of two convex sets, Proc. Amer. Math. Soc. 25, (1970), 642-645.

[2.8] F.A. Valentine, The intersection of two convex surfaces and property P_3, Proc. Amer. Math. Soc. 9, (1958), 47-54.

[2.9] M. Breen, A decomposition theorem for m-convex sets in R^d, Can. J. Math. 28, (1976), 1051-1057

[2.10] M. Breen, An R^d analogue of Valentine's Theorem on 3-convex sets, Is. J. Math. 24, (1976), 206-210.

[2.11] W.L. Stamey and J.M. Marr, Unions of two convex sets, Can. J. Math. 15, (1963), 152-156.

[2.12] M. Breen, A decomposition theorem for m-convex sets, Is. J. Math. 24, (1976), 211-216.

[2.13] M. Breen and D.C. Kay, General decomposition theorems for m-convex sets in the plane, Is. J. Math, 24, (1976), 217-233.

[2.14] H.G. Eggleston, A condition for a compact plane set to be the union of finitely many convex sets, Proc. Cam. Phil. Soc. 76, (1974), 61-66.

[2.15] J.F. Lawrence, W.R. Hare and J.W. Kenelly, Finite unions of convex sets, Proc. Amer. Math. Soc. 34, (1972), 225-228.

[2.16] D.C. Kay and M.D. Guay, Convexity and a certain property P_m, Is. J. Math. 8, (1970), 39-51.

[2.17] M.D. Guay, Planar sets having property P^n, Doctoral Dissertation, Michigan State University, East Lansing, (1967).

[2.18] J.J. Tattersall, A generalisation of convexity, Ph.D. Thesis, Oklahoma University, (1971).

[2.19]. M. Breen, Decomposition theorems for 3-convex subsets of the plane, Pac. J. Math. 53, (1974), 43-57.

[2.20] D.C. Kay, On a generalisation of convexity. Summer Research Project, University of Wyoming, Laramie, (1965).

[2.21] M. Breen, Decomposition theorems for nonclosed planar m-convex sets, Pac. J. Math. 69, (1977), 317-324.

[2.22] M. Breen, The combinatorial structure of (m,n)-convex sets, Is. J. Math. 15, (1973), 367-374.

[2.23] J. Kaapke, Uber ein verallgemeinerung der Valentineschen P_3-eigenschaft. Geometricae Dedicata, 2, (1973), 111-114.

[2.24] H.G. Eggleston, A proof of a theorem of Valentine, Math. Proc. Cam. Phil. Soc. 80, (1976), 223-228.

[2.25] M. Behzad and G. Chartrand, Introduction to the Theory of Graphs, Allyn and Bacon, (1971).

[2.26] W.R. Hare and J.W. Kenelly, Sets expressible as unions of two convex sets, Proc. Amer. Math. Soc. 25, (1970), 379-380.

[2.27] P Erdős and G. Szekeres, A combinatorial problem in geometry, Compos. Math. 2, (1935), 463-470.

[2.28] J.J. Tattersall, On the intersection of maximal m-convex subsets, Is. J. Math. 16, (1973), 300-305.

[2.29] M.D. Guay and D.C. Kay, On sets having finitely many points of local nonconvexity and property P_m, Is. J. Math. 10, (1971), 196-209.

[2.30] H.G. Eggleston, Convexity, C.U.P. (1969).

[2.31] F. Harary, Graph Theory and Theoretical Physics, Academic Press (1967).

[2.32] M. Breen, Sets which can be extended to m-convex sets, P.A.M.S. 62, (1977), 124-128.

[2.33] M. Breen, Intersections of m-convex sets, Can. J. Math. 27, (1975), 1384-1391.

CHAPTER 3

In this chapter I shall prove some results on locally starshaped sets.

Definition 3.1: A set, X, is locally starshaped if for each $x \in X$ there exists $\delta > 0$ such that $S(x,\delta) \cap X$ is starshaped from x that is for all $y \in S(x,\delta) \cap X$ $[x,y] \subset S(x,\delta) \cap X$.

Closed, locally starshaped sets seem to be a fruitful generalisation of finite unions of closed, convex sets.

THEOREM 3.1 Let X be a closed, locally starshaped set in R^d, $d \leq 3$, then $X = \bigcup_{i=1}^{\infty}$ cl C_i where cl C_i is the closure in R^d of an open (in aff C_i) connected set int(cl C_i). Moreover cl C_i is the closure of a component of $X \setminus Q$.

I prove the result using a sequence of lemmas. Note that X need not be a finite union of cl C_i as the following example shows

$X = \bigcup_{n=0}^{\infty} (0,0), (\frac{1}{n}, \frac{1}{2^{n-1}})$.

Lemma 3.1.1: For a closed set X in R^d, $X \setminus Q$, has at most countably many components.

Proof. Take a countable dense sequence $\{y_i\}$ in X. Suppose $X \setminus Q$ has uncountably many components. Let x belong to one of them so that no y_i belongs to the same component. Since Q is closed there exists $\delta > 0$ such that $S(x,\delta) \cap Q = \phi$. Now let $y_{i_j} \to x$. Since x does not belong to Q for i_j sufficiently large $[xy_{i_j}]$ $X \setminus Q$ so x and y_{i_j} belong to the same component of $X \setminus Q$, a contradiction.

Lemma 3.1.2: Let $X \subset R^d$ be locally starshaped then $X \subset$ cl$(X \setminus Q)$.

Proof. Let $x_o \in X$ and let $S(x_o,\delta_o)$ be starshaped from x_o. Since without loss of generality $x_o \in Q$ and for all $\delta > 0$ $S(x_o,\delta) \setminus \{x_o\} \neq \phi$ choose $y \in S(x_o, \delta_o)$ and let $x_1 \in (x_o,y) \subset X$. Now choose δ_1 so that $0 < \delta_1 < \min \{\delta_o - \rho(x_o,x_1), \frac{1}{2}\rho(x_o,x_1), \frac{1}{2}\rho(x_1,y)\}$ and $S(x_1,\delta_1)$ is starshaped from x_1. Let dim(aff $S(x_1,\delta_1) \cap X) = n \geq 1$. If dim (aff $S(x_1, \delta_1) \cap X) = 1$ it is clear that $S(x_1, \delta_1) \cap X \subset (x_o, y)$ and so $S(x_o, \delta_o)$ contains a locally convex point of X. Thus, assume

$S(x_1, \delta_1) \cap X$ contains a point x_2', $x_2' \notin$ aff $\{x_o, x_1\}$. Notice that $[x_2' x_1] \subset S(x_o, \delta_o)$ and $[x_o, x_1, x_2'] \subset X$. Take x_2 belonging to the relative interior of $[x_o, x_1, x_2']$ such that $x_2 \in S(x_1, \delta_1)$ (and hence $x_2 \in S(x_o, \delta_o)$). Now choose δ_2 such that $0 < \delta_2 < \delta_1 - \rho(x_1 x_2)$ and such that $S(x_2, \delta_2) \cap$ aff $\{x_o, x_1, x_2'\} \subset$ rel int $[x_o, x_2, x_2']$ with $S(x_2, \delta_2) \cap X$ starshaped from x_2. Clearly $\dim(\text{aff}\{S(x_2, \delta_2) \cap X\}) \geq 2$ and if $\dim(\text{aff}\{S(x_2, \delta_2) \cap X\}) = 2$ then x_2 is a locally convex point of X in $S(x_o, \delta_o)$. Proceeding in the obvious way I can find an r-simplex in $S(x_o, \delta_o)$ with vertices $x_o, x_1 \ldots x_r'$ such that either the simplex contains a locally convex point of X or it is possible to construct an $r + 1$ - simplex in $S(x_o, \delta_o) \cap X$. Clearly if $r = d$ a relative interior point of $x_o \ldots x_d$ is a locally convex point of X in $S(x_o, \delta_o)$.

<u>Lemma 3.1.3</u>: Let X be a closed, locally starshaped set in R^d, $d \leq 3$, then for all $q \in Q$ $q \in$ cl C_i where C_i is a component of $X \setminus Q$.

<u>Proof</u>. For $d = 1$, $Q = \phi$. Assume $d = 2$ or 3. By Lemma 3.1.2. $q \in$ cl$(X \setminus Q)$. Let $y \in S(q, \delta)$ where $y \in C_i$ and $S(q, \delta) \cap X$ is starshaped from q. If $(qy] \subset X \setminus Q$ $q \in$ cl C_i otherwise there exists $q_1 \in (qy]$, $q_1 \in Q$. Hence by the argument of Lemma 3.1.2. construct a non-degenerate triangle T contained in $S(q, \delta) \cap X$. For $d = 2$ the result is clear. For $d = 3$ if rel int $T \cap Q = \phi$ the result follows, otherwise note that for each point, x, of $Q \cap$ rel int T a disc, D, centre x starshaped from x meets $Q \cap$ rel int T in a subset H of aff $\{q, x\} \cap D$ since otherwise one clearly has a 3-dimensional tetrahedron with q as a vertex since $T \subset S(q, \delta) \cap X$. Thus $Q \cap$ rel int T is at most a countable union of segments each of whose affine hulls contains q. Hence the result follows since $T \subset S(q, \delta) \cap X$.

The case $d = 3$ in Lemma 3.1.3. is due to H.G. Eggleston.

Returning to Theorem 3.1, it is now clear that $X = \bigcup_{i=1}^{\infty}$ cl C_i where cl C_i is the closure of a component of $X \setminus Q$. Consider cl C_i and $Q(\text{cl } C_i)$, note that if $x \in C_i$ then $x \notin Q(\text{cl } C_i)$ for $x \in C_i$

implies there exists $\delta > 0$ such that $cl(S(x, \delta) \cap X) \subset X \setminus Q$ and $cl(S(x, \delta)) \cap X$ is starshaped from x so that by Tietze's Theorem $cl(S(x, \delta)) \cap X$ is convex hence $cl(S(x, \delta)) \cap X \subset C_i$ but $cl(S(x, \delta)) \cap X$ is convex and $cl(S(x, \delta)) \cap X = cl(S(x, \delta)) \cap C_i = cl(S(x, \delta)) \cap cl\ C_i$. Hence $C_i \subset cl\ C_i \setminus Q(cl\ C_i) \subset d\ C_i$ and so $cl\ C_i \setminus Q(cl\ C_i)$ is connected. Note that $cl\ C_i = cl\ (cl(C_i) \setminus Q\ (cl\ C_i))$.

I now prove Theorem 3.1 with two further lemmas. A proof of the first, in the case $d = 3$, using Zorn's Lemma, has been given by Stavrakas [3.1]

<u>Lemma 3.1.4:</u> Let $\dim(\text{aff } X) = d$ then if $X \subset cl(X \setminus Q)$ and $X \setminus Q$ is connected then $X \subset cl(\text{int } X)$ where the interior is taken in $R^d = \text{aff } X$.

<u>Proof</u> It is sufficient to prove that $X \setminus Q \subset cl(\text{int } X)$. Let $u \in X \setminus Q$ and let $S(u, \delta)$ be such that $S(u, \delta) \cap X$ is convex. If $u \notin cl(\text{int } X)$, $\dim (S(u, \delta) \cap X) < d$. Then there exists $v \in X \setminus \text{aff } (S(u, \delta) \cap X)$ and since $X \subset cl(X \setminus Q)$ suppose $v \in X \setminus Q$. Since $X \setminus Q$ is locally convex and connected it is polygonally connected. Let ℓ be a polygonal arc joining u and v, $\ell = \{x: x = f(t)\ t \in [0,1]\}$ with $f(0) = u$ and $f(1) = v$. Let C_α be the component of $X \setminus Q \cap \text{aff}(S(u, \delta) \cap X)$ containing u. Suppose ℓ meets some other component C_β of $X \setminus Q \cap \text{aff } \{S(u, \delta) \cap X\}$ at $w = f(s)$ then there exists r with $r < s$ such that $f(r) \notin \text{aff}(S(u, \delta) \cap X)$ since otherwise $C_\alpha = C_\beta$. Let $t_o = \sup \{t: f(t) \in \text{aff}(S(u, \delta) \cap X)$ and $f(s) \in \text{aff}(S(u, \delta) \cap X)$ for all $s \leqq t\}$ then $f(t_o) \in \text{aff } S(u, \delta) \cap X$ and for all $s \leqq t_o f(s) \in C_\alpha$. However $f(t_o) \in cl(X \setminus \text{aff}\{S(u, \delta) \cap X\})$ for let $t_n \downarrow t_o$ then there exists $s_n \downarrow t_o$ with $t_o < s_n \leqq t_n$ such that $f(s_n) \notin \text{aff } S(u, \delta) \cap X$. So one may partition C_α into two sets C_1 and C_2 both open and closed in C_α thus contradicting C_α connected

$$C_1 = \{x: x \in C_\alpha, \quad x \in cl(X \setminus \text{aff}(S(u, \delta) \cap X))\}$$

$$C_2 = \{x: x \in C_\alpha, \quad x \notin cl(X \setminus \text{aff}(S(u, \delta) \cap X)\}$$

Now $f(t_o) \varepsilon C_1$ and $u \varepsilon C_2$ moreover C_1 is clearly closed in C_α. It only remains to show C_1 is open in C_α. Let $x \varepsilon C_1$ then $x \varepsilon X \setminus Q$. There exists $\eta > 0$ such that $X \cap S(x, \eta)$ is convex, and contains a point $z \varepsilon X \setminus \text{aff}\,(S(u, \delta) \cap X)$ then for all $x' \varepsilon S(x, \eta) \cap C_\alpha$ $[zx'] \subset X$ which implies $x' \varepsilon C_1$

The final lemma completes the proof of Theorem 3.1.

Lemma 3.1.5: Let $X \subset R^d$ $X \subset \text{cl(int } X)$ then $X \setminus Q$ is connected if and only if int X is connected.

Proof. The "if" part is trivial. To prove the "only if" part suppose x and y belong to int X. Since $X \setminus Q$ is connected and locally convex it is polygonally connected. Let P be a polygonal arc joining x to y via $X \setminus Q$ consisting of finitely many segments. I shall prove that such an arc exists joining x to y via int X.

Suppose n the number of segments in P is one $P = [x,y]$. Since P is compact let $P \subset \bigcup_{i=1}^{t} S(x_i, \delta_i) \cap X$ where $S(x_i, \delta_i) \cap X$ is a full dimensional convex set since $P \subset X \setminus Q$ and $X \subset \text{cl(int } X)$ with $x_1 = x$, $x_t = y$ and if $i < j$ then $x_i \varepsilon [x_j\, x_1]$. Now $S(x_1, \delta_1) \cap P$ meets $S(x_j, \delta_j)$ for some $j \geq 2$, with highest suffix, κ, otherwise P is not connected. Let $x_2' \varepsilon S(x_1, \delta_1) \cap S(x_\kappa, \delta_\kappa) \cap P$. Take an interior point, α_{11}, of $S(x_1, \delta_1) \cap X$ then $[\alpha_{11}\, x_2') \cup [\alpha_{11}\, x) \subset \text{int } X$. Further since x_2' is a locally convex point there exists $\delta > 0$ such that $S(x_2', \delta) \cap X$ is a full dimensional convex set. Hence there clearly exists a polygonal arc from x_1 to x_κ (having five segments) in the interior of X apart possibly from x_κ.

Noting that the argument applied at x_2' may be applied at x_κ, it is clear by considering $[x_\kappa, y]$ that there exists a polygonal arc in the interior of X joining x and y. (It is assumed that for all $i \neq j$ $S(x_j, \delta_j) \cap P \not\subset S(x_i, \delta_i) \cap P$.)

For $n \geq 1$, the argument can be applied to each segment of P and the x_2 argument can be applied at the vertices of P completing the proof.

Corollary 3.1. X is the closure of an open connected set if and only if $X \setminus Q$ is connected and $X = cl(X \setminus Q)$.

Proof. The result follows from Lemma 3.1.4 and Lemma 3.1.5.

Definition 3.2 A point $x \in Fr\ X$, $X \subset R^d$, is a point of mild convexity if no segment $[uv]$, $u \neq v$, exists having x as midpoint and having $[uv] \setminus \{x\} \subset int\ X$.

Corollary 3.2. Let X be a locally starshaped set in R^d with $X \setminus Q$ connected and let each point of the frontier of X be a point of mild convexity then int X and cl X are convex.

Proof. The result follows from Lemmas 3.1.4. and 3.1.5. and Theorem 4.9, p.53, of Valentine [3.2] which states:

Let S be an open connected set in a topological linear space L, and suppose each point $x \in Fr\ S$ is a point of mild convexity of S. Then S is convex.

It is easy to construct examples to show that for $X \subset R^d$ none of the three hypotheses about X is superfluous if the same conclusion is to be obtained in Corollary 3.2.

The final result of this chapter is

THEOREM 3.2. Let X be a compact, locally starshaped set in R^2 then Q(X) has finite 1-dimensional measure.

Proof. Since X is compact and locally starshaped it may be written as a finite union of compact, starshaped,locally starshaped sets $\bigcup_{i=1}^{n} X_i$ with $X_i = cl\ (S(x_i, \delta_i) \cap X)$ starshaped from x_i. If $q \in Q(X)$ $q \in \bigcup_{i=1}^{n} Fr\ (S(x_i, \delta_i))$ which is of finite one dimensional measure or $q \in S(x_i, \delta_i)$. In the latter case $q \in Q(X_i)$ for otherwise there exists a neighbourhood, $S(q, \delta)$, with $S(q, \delta) \cap X_i$ convex and $S(q, \delta) \cap X \subset X_i$ so that $q \notin Q(X)$.

Hence it is sufficient to show that the result is true for compact, starshaped, locally starshaped sets. Let X be such a set starshaped from x with $X \subset S(x, \delta)$. Firstly I show that X may have at most countably many segments $[a_i, b_i] \subset \mathrm{Fr}\, X$ and $x \in \mathrm{aff}[a_i, b_i]$. Let $[a_\lambda, b_\lambda]$ $\lambda \in \Lambda$ be a collection of such segments. If there are uncountably many, for n sufficiently large, uncountably many contain an interval of length $> \frac{2}{n}$. It follows that uncountably many a_λ, b_λ contain $x + te_\alpha$ where $\frac{m}{n}\delta \leq t \leq \frac{m+1}{n}\delta$ for some $m \in \{1 \ldots n-1\}$ and e_α is a unit vector along $\mathrm{aff}[a_\alpha, b_\alpha]$. Now some sequence of e_α, $\{e_i\}$, containing infinitely many elements converges to e. Let $[y_i, z_i] \subset \mathrm{Fr}\, X$ with $[y_i, z_i] = x + te_i, \frac{m}{n}\delta \leq t \leq \frac{m+1}{n}\delta$. Let $v = x + se, \frac{m}{n}\delta < s < \frac{m+1}{n}\delta$. Note that $x + te, \frac{m}{n}\delta \leq t \leq \frac{m+1}{n}\delta$, is a segment in $\mathrm{Fr}\, X$ since $\mathrm{Fr}\, X$ is closed. It is clear that since X is starshaped from x and X is locally starshaped from v eventually some points of $[y_i\ z_i]$ lie in $\mathrm{int}\, X$, a contradiction.

So, since each segment in $\mathrm{Fr}\, X$ with x in its affine hull contains at most three points of $Q(X)$, by discarding a countable subset of $Q(X)$, one may assume that for each $q \in Q$ the ray from x through q meets $\mathrm{Fr}\, X$ in at most q and x. Since X is bounded $\mathrm{Fr}\, X \setminus \bigcup_{i=1}^{\infty} y_i\, z_i$ is the image of a subset of $\mathrm{Fr}\, S(x,\delta)$ under a Lipschitz map and hence Q has finite one dimensional measure.

For completeness, I note that H.G. Eggleston has constructed a compact, planar set, X, which is the union of two convex sets for which $Q(X)$ has positive 1-dimensional measure. He has also proved that a closed, locally starshaped set in R^d is a countable union of convex sets.

Appendix to Chapter 3

Note that the example given at the beginning of this chapter can be easily modified to produce a compact subset X, of R^2 which is a countable union of convex sets with Q(X) having infinite one dimensional measure.

REFERENCES

[3.1] N.M. Stavrakas, A generalization of Tietze's Theorem on convex sets in R^3, Proc. Amer. Math. Soc. 40 (1973) 565-567.

[3.2] F.A. Valentine, Convex Sets, McGraw-Hill (1964).