

# Orientations of digraphs almost preserving diameter

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## Abstract

An orientation of a digraph  $D$  is a spanning subdigraph of  $D$  obtained from  $D$  by deleting exactly one arc between  $x$  and  $y$  for every pair  $x \neq y$  of vertices such that both  $xy$  and  $yx$  are in  $D$ . In this paper, we consider certain well-known classes of strong digraphs, each member  $D$  of which has an orientation with diameter not exceeding the diameter of  $D$  by more than a small constant.

## 1 Introduction, terminology and notation

An *orientation* of a digraph  $D$  is a spanning subdigraph of  $D$  obtained from  $D$  by deleting exactly one arc between  $x$  and  $y$  for every pair  $x \neq y$  of vertices such that both  $xy$  and  $yx$  are in  $D$ . In this paper, we consider certain well-known classes of generalizations of tournaments, each strongly connected member  $D$  of which has an orientation with diameter not exceeding the diameter of  $D$  by more than a small constant. While there is a large number of publications considering minimum diameter orientations of undirected graphs (see Sections 2.6-2.9 in [2] for results and references), the present paper is the first study of minimum diameter orientations of digraphs. It is shown in Section 2.11 of [2] that orientations  $H$  of digraphs  $D$  such that the diameter of  $H$  does not exceed the diameter of  $D$  by more than a small constant are of interest in a version of the gossip problem, see, e.g., [9, 10].

It is worth noting that there are a few papers [6, 8, 11] considering finite diameter orientations of mixed graphs (or, equivalently, of directed graphs), but none of these papers has been devoted to minimizing the diameter of an orientation of a given digraph. We restrict our attention to special classes of digraphs since even the problem of checking

whether a given undirected graph has an orientation of diameter 2 is proved to be  $\mathcal{NP}$ -complete by Chvátal and Thomassen [7] and the upper bound on the diameter of an orientation of an undirected graph obtained in [7] is far from best possible for many classes of undirected graphs. Notice that the minimum diameter orientation problem for undirected graphs is a special case of that for directed graphs since every undirected graph can be considered as the corresponding symmetric digraph.

This paper is organized as follows. In the rest of this section we give some terminology and notation. In Section 2 we prove a somewhat surprising upper bound for the minimum diameter of orientations of quasi-transitive digraphs and semicomplete bipartite digraphs. In particular, we show that if  $D$  is a strong quasi-transitive digraph on at least 3 vertices, then  $D$  has an orientation  $H$  such that  $\text{diam}(H) \leq \max\{3, \text{diam}(D)\}$ . The same bound, with 3 replaced by 5, holds for all semicomplete bipartite digraphs except for those in which one partite set consists of a unique vertex. While such a bound is not valid for the whole class of locally semicomplete digraphs, in Section 3 we prove that the bound  $\text{diam}(H) \leq \max\{4, \text{diam}(D)\}$  holds for locally semicomplete digraphs  $D$  without so-called similar vertices and  $\text{diam}(H) \leq \max\{4, \text{diam}(D)\} + 1$  is true for every locally semicomplete digraph  $D$  on at least three vertices.

We use the standard terminology and notation on digraphs as described in [2]. We still provide most of the necessary definitions for the convenience of the reader. A digraph  $D$  is *symmetric* if for every pair  $x \neq y$  of vertices in  $D$  either there is no arc between  $x$  and  $y$  or both  $xy$  and  $yx$  are in  $D$ . Symmetric digraphs are in natural correspondence to undirected graphs: for an undirected graph  $G$ , the symmetric digraph  $\overleftrightarrow{G}$  is obtained from  $G$  by replacing every edge  $xy$  with the pair  $xy, yx$  of arcs. Let  $D = (V, A)$  be a digraph and let  $x, y$  be a pair of vertices in  $D$ . If  $xy \in A$ , we say that  $y$  is an *out-neighbour* of  $x$ ,  $x$  is an *in-neighbour* of  $y$ , and  $x$  *dominates*  $y$  denoted by  $x \rightarrow y$ . For sets  $X, Y \subset V$ ,  $X \rightarrow Y$  means that  $x \rightarrow y$  for every  $x \in X, y \in Y$ . The set of in-neighbours (out-neighbours) of a vertex  $x$  is denoted by  $N^-(x)$  ( $N^+(x)$ ).

All paths and cycles we consider in this paper are directed. A path from  $x$  to  $y$  is an  $(x, y)$ -*path*. A digraph  $D$  is *strongly connected* (or, *strong*) if there exist an  $(x, y)$ -path and a  $(y, x)$ -path for every pair  $x, y$  of distinct vertices in  $D$ . The *distance*,  $\text{dist}_D(x, y)$ , *from  $x$  to  $y$*  in  $D$  is the least length of an  $(x, y)$ -path if  $y$  is reachable from  $x$ , and is equal to  $\infty$ , otherwise. We assume that  $\text{dist}_D(x, x) = 0$  for every vertex  $x \in V$ . The *diameter* of  $D$ ,  $\text{diam}(D)$ , is the maximum distance between an ordered pair of vertices in  $D$ . Observe that a digraph  $D$  is strong if and only if  $\text{diam}(D) < \infty$ . A digraph  $D$  is *connected* if the underlying undirected graph of  $D$  is connected. For a digraph  $D$ , let  $\text{diam}_{\min}(D)$  denote the minimum diameter of an orientation of  $D$ . The *converse* of a digraph  $D$  is the digraph obtained from  $D$  by replacing every arc  $xy$  of  $D$  by the arc  $yx$ .

A digraph  $D$  is *semicomplete* if there is at least one arc between any pair of distinct vertices of  $D$ . A *tournament* is a semicomplete digraph with no cycle of length 2. A digraph  $D$  is *quasi-transitive* if the existence of a pair  $xy, yz$  of arcs in  $D$  implies the existence of

$xz$  or  $zx$  (or both). By definition, every semicomplete digraph is quasi-transitive. To see that there are quasi-transitive digraphs, which are not semicomplete (and not transitive), replace every vertex of a tournament  $T$  by a set of independent (i.e. with no arc between them) vertices. The resulting digraph  $D$  is quasi-transitive: if  $xy, yz$  are in  $D$ , then  $x$  and  $y$  belong to different sets of independent vertices (as  $T$  has no 2-cycle) and, thus, are joint by an arc. A recursive characterization of quasi-transitive digraphs is given by Bang-Jensen and Huang [5].

A digraph  $D$  is *locally semicomplete* if, for every vertex  $x$ , the subdigraphs of  $D$  induced by  $N^+(x)$  and  $N^-(x)$  are semicomplete. One of the simplest examples of a locally semicomplete digraph is a cycle. A digraph  $D$  is semicomplete  $k$ -partite,  $k \geq 2$ , if the vertices of  $D$  can be partitioned into  $k$  partite sets  $V_1, V_2, \dots, V_k$  such that every partite set is independent, but, for every pair  $x, y$  of vertices from distinct partite sets,  $xy$  or  $yx$  (or both) is in  $D$ . When  $k = 2$ , we speak of semicomplete bipartite digraphs. By definition, every semicomplete digraph with  $n$  vertices is a semicomplete  $n$ -partite digraph. A characterization of locally semicomplete digraphs is obtained in [1].

Quasi-transitive digraphs, locally semicomplete digraphs and semicomplete  $k$ -partite digraphs are well-known generalizations of tournaments, they share several nice structural properties with tournaments and have been extensively studied in the literature (cf. [2, 3] and the bibliography therein). In particular, we know now that the hamiltonian cycle is polynomial time solvable when restricted to any of these classes. (A highly non-trivial proof that the hamiltonian cycle problem is polynomial time solvable for semicomplete  $k$ -partite digraphs can be found in [4].)

We conclude this section with the following useful result by Boesch and Tindell [6], whose short proof is given by Volkmann [11]:

**Theorem 1.1** *A strong digraph  $D$  has no strong orientation if and only if there is a pair  $x, y$  of vertices in  $D$  such that the deletion of the arcs  $xy, yx$  leaves  $D$  disconnected.*

## 2 Orientations of quasi-transitive digraphs and semicomplete bipartite digraphs

Applying Theorem 1.1 it is easy to see that every strong quasi-transitive digraph of order  $n \geq 3$  has a strong orientation. Volkmann [11] observed that a strong semicomplete  $k$ -partite digraph  $D$ ,  $k \geq 2$ , has a strong orientation unless  $D$  is a semicomplete bipartite digraph with a partite set consisting of a single vertex. (By Theorem 1.1, a semicomplete bipartite digraph with a partite set consisting of a single vertex does not have a strong orientation.) This justifies the consideration of the following two classes of digraphs. Let  $\mathcal{D}_0$  be the set of strong quasi-transitive digraphs of order  $n \geq 3$ . Let  $\mathcal{D}_1$  be the set of strong semicomplete bipartite digraphs with at least two vertices in each partite set.

In this section, we shall use the following basic result:

**Proposition 2.1** [5] *Let  $D$  be a quasi-transitive digraph. Suppose that  $P = x_0x_1x_2\dots x_k$  is a minimal  $(x_0, x_k)$ -path. Then the subdigraph induced by  $V(P)$  is semicomplete and  $x_j \rightarrow x_i$  for every  $2 \leq i+1 < j \leq k$ , unless  $k = 3$ , in which case the arc between  $x_0$  and  $x_k$  may be absent.*

For digraphs from the class  $\mathcal{D}_0 \cup \mathcal{D}_1$  the following somewhat surprising bound on the minimum diameter of an orientation holds.

**Theorem 2.2** *If  $D \in \mathcal{D}_i$  for  $i \in \{0, 1\}$ , then*

$$\text{diam}_{\min}(D) \leq \max\{3 + 2i, \text{diam}(D)\}.$$

**Proof:** Assume that this theorem is false and that  $D$  is a counter-example to the theorem with as few 2-cycles as possible. Let  $D \in \mathcal{D}_i$  for  $i \in \{0, 1\}$  and let  $\gamma = 3 + 2i$ . Let  $xyx$  be a 2-cycle in  $D$ . Clearly, the diameter of  $D$  increases by at least one when we delete either of the arcs  $xy$  or  $yx$  from  $D$ . Therefore, there exist vertices  $s_{xy}, t_{xy}, s_{yx}, t_{yx}$  in  $D$ , such that  $\text{dist}_{D-xy}(s_{xy}, t_{xy}) > \max\{\gamma, \text{diam}(D)\}$  and  $\text{dist}_{D-yx}(s_{yx}, t_{yx}) > \max\{\gamma, \text{diam}(D)\}$ . Let  $P = p_0p_1\dots p_l$  be an  $(s_{xy}, t_{xy})$ -path in  $D$  of minimum length (in particular,  $l \leq \text{diam}(D)$ ) and let  $Q = q_0q_1\dots q_m$  be an  $(s_{yx}, t_{yx})$ -path in  $D$  of minimum length (in particular,  $m \leq \text{diam}(D)$ ). Let  $\rho$  and  $\eta$  be defined such that  $xy = p_\rho p_{\rho+1}$  and  $yx = q_\eta q_{\eta+1}$ .

We now consider the following cases, which exhaust all possibilities:

**Case 1:**  $\rho + 1 < l$ ,  $\eta + 1 < m$  and  $D \in \mathcal{D}_0 \cup \mathcal{D}_1$ . We first show that  $p_{\rho+2}$  and  $q_{\eta+2}$  are adjacent. This is clearly true if  $D$  is semicomplete bipartite as these two vertices belong to different partite sets of  $D$ . If  $D$  is quasi-transitive, then  $p_\rho$  and  $p_{\rho+2}$  are adjacent. Therefore,  $p_{\rho+2} \rightarrow p_\rho$  by the minimality of  $l$ . However, this implies that  $p_{\rho+2}$  and  $q_{\eta+2}$  are adjacent, as  $p_{\rho+2} \rightarrow (p_\rho = q_{\eta+1}) \rightarrow q_{\eta+2}$ .

If  $p_{\rho+2} \rightarrow q_{\eta+2}$ , then by  $q_\eta = p_{\rho+1}$ ,

$$q_0q_1\dots q_\eta p_{\rho+2} q_{\eta+2} \dots q_m$$

is a  $(q_0, q_m)$ -path of length  $m \leq \text{diam}(D)$  in  $D - yx$ , a contradiction. The case when  $q_{\eta+2} \rightarrow p_{\rho+2}$  can be considered analogously.

**Case 2:**  $\rho > 0$ ,  $\eta > 0$  and  $D \in \mathcal{D}_0 \cup \mathcal{D}_1$ . This case can be transformed into Case 1 by considering the converse of  $D$ .

**Case 3:**  $\rho = 0$ ,  $\eta + 1 = m$  and  $D \in \mathcal{D}_0$ . We first prove that  $l + m \geq 3$ . Suppose that  $l = m = 1$ , i.e.  $x = p_0 = q_1, y = p_1 = q_0$ . Let  $z_0z_1\dots z_k$  be a shortest  $(y, x)$ -path in  $D - yx$ .

By the choice of  $x, y, k \geq 4$ . By Proposition 2.1,  $z_k \rightarrow z_1$  and  $z_2 \rightarrow z_0$ . Hence,  $z_k z_1 z_2 z_0$  is an  $(x, y)$ -path in  $D - xy$  of length three, contradiction. Therefore, we may assume, without loss of generality, that  $l \geq 2$ .

Let  $R = r_0 r_1 \dots r_t$  be a shortest path from  $q_0$  to  $p_l$  in  $D$ . The path  $R$  can be chosen such that it does not contain  $yx$ . Indeed, if  $y = r_j, x = r_{j+1}$  for some  $j$ , then  $r_0 r_1 \dots r_j p_2 p_3 \dots p_l$  is not longer than  $R$  (as  $p_1 p_2 \dots p_l$  is a shortest  $(p_1, p_l)$ -path in  $D$ ). So, we may assume that  $R$  does not contain  $yx$ . Similarly, it is not difficult to see that we may assume that  $R$  does not contain  $xy$ .

By Proposition 2.1, we obtain immediately that  $p_l \rightarrow p_0$  if  $l \neq 3$  and  $p_l \rightarrow p_1$  if  $l = 3$ . If  $l = 3$ , then we have  $p_3 \rightarrow p_1$  and  $p_0 \rightarrow p_1$ . Therefore, by the minimality of  $l$ ,  $p_3 \rightarrow p_0$ . Hence, for every  $l \geq 2$ ,  $p_l \rightarrow p_0$ .

We have  $t > 2$ , for otherwise  $r_0 r_1 \dots r_t p_0$  would be a path from  $q_0$  to  $q_m$  of length  $t + 1 \leq 3$  in  $D - yx$ . Since  $p_l \rightarrow p_0$  and  $r_{t-1} \rightarrow r_t = p_l$ , we conclude that  $r_{t-1}$  and  $p_0$  are adjacent. If  $r_{t-1} \rightarrow p_0$ , then  $r_0 r_1 \dots r_{t-1} p_0$  is a path from  $q_0$  to  $q_m$  of length  $t \leq \text{diam}(D)$  in  $D - yx$ , a contradiction. If  $p_0 \rightarrow r_{t-1}$ , then  $p_0 r_{t-1} p_l$  is a path of length two from  $p_0$  to  $p_l$  in  $D - xy$ , a contradiction.

**Case 4:**  $\eta = 0, \rho + 1 = l$  and  $D \in \mathcal{D}_0$ . This case can be transformed into Case 3 by considering the converse of  $D$ .

**Case 5:**  $\rho = 0, \eta + 1 = m$  and  $D \in \mathcal{D}_1$ . Suppose that  $l = m = 1$ . Let  $z_0 z_1 \dots z_k$  be a shortest  $(y, x)$ -path in  $D - yx$ . By the choice of  $x, y, k \geq 6$ . By the minimality of  $k$ ,  $z_3 \rightarrow z_0$  ( $z_0$  and  $z_3$  belong to different partite sets of  $D$ ) and  $z_k \rightarrow z_j$ , where  $j = 2$  or  $3$  ( $z_k$  and  $z_j$  belong to different partite sets of  $D$ ). Hence, either  $z_k z_3 z_0$  or  $z_k z_2 z_3 z_0$  is an  $(x, y)$ -path in  $D - xy$ , a contradiction. So, we may assume, without loss of generality, that  $m \geq 2$ .

Let  $R = r_0 r_1 \dots r_t$  be a shortest path from  $q_0$  to  $p_l$  in  $D$ . As in Case 3, we may assume that  $R$  contains neither  $xy$  nor  $yx$ .

Suppose that  $t = 0$ , implying that  $q_0 = p_l$  and  $l, m \geq 2$ . Assume that  $l \geq 3$ . If  $p_0$  and  $p_l$  belong to different partite sets of  $D$ , then, by the minimality of  $l$  and the assumption that  $D$  is semicomplete bipartite,  $p_l \rightarrow p_0$ , which is impossible as  $p_l p_0$  is a  $(q_0, q_m)$ -path of length one in  $D - yx$ , a contradiction. If  $p_0$  and  $p_l$  belong to the same partite set of  $D$ , then  $p_l \rightarrow p_1$  (by the minimality of  $l$ ) and  $p_l p_1 p_2 p_3 p_0$  is a  $(q_0, q_m)$ -path of length four in  $D - yx$ , a contradiction. So,  $l = 2$ . Analogously, we can prove that  $m = 2$ . Since  $D - xy$  has a  $(p_0, p_2)$ -path and  $p_2 = q_0 \rightarrow q_1 = p_1$ , there is a  $(p_0, p_1)$ -path  $S = s_0 s_1 \dots s_a$  in  $D - xy$ . Assume that  $S$  has minimum length and observe that  $a \geq 5$ , as  $s_0 s_1 \dots s_a p_l$  is a  $(p_0, p_l)$ -path in  $D - xy$ . Furthermore,  $s_3 \rightarrow s_0$  as  $s_0$  and  $s_3$  lie in different partite sets of  $D$  and  $S$  is of minimum length. Observe that if  $p_2 \rightarrow s_3$ , then  $p_2 s_3 s_0$  is a  $(q_0, q_m)$ -path in  $D - yx$  of length 2, and if  $s_3 \rightarrow p_2$  then  $s_0 s_1 s_2 s_3 p_2$  is a  $(p_0, p_l)$ -path in  $D - xy$  of length 4. In both cases we obtain a contradiction. Hence,  $t > 0$ .

Suppose that  $1 \leq t \leq 2$ . Clearly  $r_0$  and  $r_1$  lie in different partite sets, so we may assume, without loss of generality, that  $r_0$  and  $p_0$  are adjacent (the case when  $r_1$  and  $p_0$  are adjacent can be considered analogously). Clearly  $p_0$  dominates  $r_0$  by the minimality of  $m$ . However,  $p_0 r_0 \dots r_t$  is a  $(p_0, p_l)$ -path in  $D - xy$  of length of  $t + 1 \leq 3$ , a contradiction. Hence,  $t \geq 3$ .

Clearly  $r_1$  and  $r_2$  lie in different partite sets, so we may assume, without loss of generality, that  $r_1$  and  $p_0$  are adjacent (the case when  $r_2$  and  $p_0$  are adjacent can be considered analogously). Clearly  $p_0$  dominates  $r_1$  by the minimality of  $m$ . However the path  $p_0 r_1 \dots r_t$  in  $D - xy$  is of length  $t \leq \text{diam}(D)$ .

**Case 6:**  $i_\eta = 0, i_\rho + 1 = l$  and  $D \in \mathcal{D}_1$ . This case can be transformed into Case 5 by considering the converse of  $D$ .  $\square$

The upper bound of this theorem is sharp as one can see from the following examples. Let  $T_k, k \geq 3$ , be a (transitive) tournament with vertices  $x_1, x_2, \dots, x_k$  and arcs  $x_i x_j$  for every  $1 \leq i < j \leq k$ . Let  $y$  be a vertex not in  $T_k$ , which dominates all vertices of  $T_k$  but  $x_k$  and is dominated by all vertices of  $T_k$  but  $x_1$ . The resulting semicomplete digraph  $D_{k+1}$  has diameter 2. However, the deletion of any arc of  $D_{k+1}$  between  $y$  and the set  $\{x_2, x_3, \dots, x_{k-1}\}$  leaves a digraph with diameter 3. Indeed, if we delete  $yx_i, 2 \leq i \leq k-1$ , then a shortest  $(x_k, x_i)$ -path becomes of length 3.

Let  $H$  be a strong semicomplete bipartite digraph with the following partite sets  $V_1$  and  $V_2$  and arc set  $A: V_1 = \{x_1, x_2, x_3\}, V_2 = \{y_1, y_2, y_3\}$ , and

$$A = \{x_1 y_1, y_1 x_1, x_1 y_2, y_3 x_1, x_2 y_1, y_2 x_2, y_3 x_2, y_1 x_3, x_3 y_3, x_3 y_2\}.$$

Let  $H' = H - x_1 y_1$  and  $H'' = H - y_1 x_1$ . It is easy to verify that  $\text{diam}(H) = 4$  (in particular,  $\text{dist}(y_2, y_3) = 4$ ) and that  $\text{diam}(H') = \text{diam}(H'') = 5$  (a shortest  $(x_1, y_3)$ -path in  $H'$  and a shortest  $(y_2, x_1)$ -path in  $H''$  are of length 5). The digraph  $H$  can be used to generate an infinite family of semicomplete bipartite digraphs with the above property: replace, say,  $x_3$  by a set of independent vertices.

### 3 Orientations of locally semicomplete digraphs

Unfortunately, the bound of the type

$$\text{diam}_{\min}(D) \leq \max\{c, \text{diam}(D)\}, \quad (1)$$

where  $c$  is a constant, is not valid for the whole class of strong locally semicomplete digraphs. Consider the following digraph  $D_k = (V, A)$ :

$$V = \{x_1, x_2, \dots, x_k\}, A = \{x_i x_{i+1} : i = 1, 2, \dots, k-1\} \cup \{x_k x_1, x_k x_2, x_1 x_3, x_2 x_1\}.$$

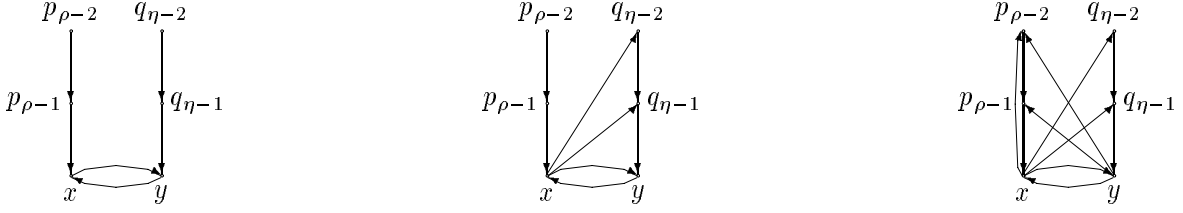


Figure 1: The leftmost picture contains the given arcs. These arcs imply  $x \rightarrow q_{\eta-1}$ , and thus  $x \rightarrow q_{\eta-2}$ , as seen in the middle picture. Analogously we obtain  $y \rightarrow \{p_{\rho-2}, p_{\rho-1}\}$ , which implies that  $x \rightarrow p_{\rho-2}$ , as seen in the last picture.

It is easy to check that  $\text{diam}(D_k) = k - 2$  and  $\text{diam}(D_k - x_1x_2) = \text{diam}(D_k - x_2x_1) = k - 1$ . The digraph  $D_k$  does not satisfy (1) due to the existence of so-called similar vertices  $x_1$  and  $x_2$ . Two vertices  $x$  and  $y$  of a digraph  $D$  are *similar* if  $N^+(x) \cup \{x\} = N^+(y) \cup \{y\}$  and  $N^-(x) \cup \{x\} = N^-(y) \cup \{y\}$ . Observe that if  $x$  and  $y$  are similar, then the 2-cycle  $xyx$  is in  $D$ .

The main result of this section, Theorem 3.2, can be proved using the classification of locally semicomplete digraphs obtained in [1] and Theorem 2.2 for the case of quasi-transitive digraphs (actually, for just semicomplete digraphs). Even though such a ‘classification-based’ proof is slightly shorter than the one we provide below, the ‘classification-based’ proof relies heavily on the classification and related results in [1]. The presented proof is direct and does not require any previous knowledge. Provided with enough detail, the ‘classification-based’ proof along with the classification itself and additional results and definitions would require more space than our proof below. We start from the following result.

**Theorem 3.1** *If  $D$  is a strong locally semicomplete digraph with no similar vertices then  $\text{diam}_{\min}(D) \leq \max\{4, \text{diam}(D)\}$ .*

**Proof:** Assume that this theorem is false and that  $D$  is a counter-example, with as few 2-cycles as possible. Let  $xyx$  be a 2-cycle in  $D$ . Since  $x$  and  $y$  are not similar, we may without loss of generality find a vertex  $u$ , such that  $xu \in A(D)$ , but  $yu \notin A(D)$ . However this implies that  $uy \in A(D)$ , as  $x \rightarrow \{u, y\}$ . Since  $\text{diam}(D - xy) > \max\{4, \text{diam}(D)\}$ , there are vertices  $s_{xy}$  and  $t_{xy}$  such that  $\text{dist}_{D-xy}(s_{xy}, t_{xy}) > \max\{4, \text{diam}(D)\}$ . Let  $P = p_0p_1 \dots p_l$  be a shortest  $(s_{xy}, t_{xy})$ -path in  $D$ . Since  $\text{dist}_{D-xy}(s_{xy}, t_{xy}) > \text{diam}(D)$  the arc  $xy$  must be used in the path  $P$ , so let  $\rho$  be defined such that  $xy = p_\rho p_{\rho+1}$ . The path  $P' = p_0p_1 \dots p_\rho u p_{\rho+1} \dots p_l$  is a path in  $D - xy$ , implying that  $l = \text{diam}(D) \geq 4$ . If  $\rho \geq 1$

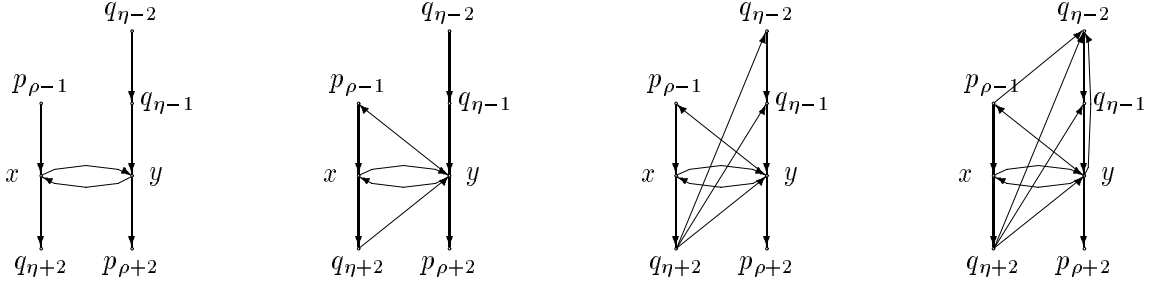


Figure 2: The leftmost picture contains the given arcs. This implies that the arcs  $q_{\eta+2} \rightarrow y$  and  $y \rightarrow p_{\rho-1}$  must be present, as seen in the next picture. This implies that  $q_{\eta+2} \rightarrow q_{\eta-1}$ , which implies that  $q_{\eta+2} \rightarrow q_{\eta-2}$ , as seen in picture 3. Finally we must therefore have arc  $y \rightarrow q_{\eta-2}$ , which implies that  $p_{\rho-1} \rightarrow q_{\eta-2}$ , as seen in the last picture.

then we observe that  $p_{\rho+1}p_{\rho-1} \in A(D)$  (as  $\{p_{\rho+1}, p_{\rho-1}\} \rightarrow p_{\rho}$  and  $l$  is minimum). If  $\rho = 0$  then  $p_2 \rightarrow p_0$  by a similar argument. So there is a  $(y, x)$ -path of length 2 in  $D - yx$ .

There exist vertices  $s_{yx}$  and  $t_{yx}$  in  $D$ , such that  $\text{dist}_{D-yx}(s_{yx}, t_{yx}) > \max\{4, \text{diam}(D)\}$ . Analogously to the above we let  $Q = q_0q_1 \dots q_m$  be a shortest  $(s_{xy}, t_{xy})$ -path in  $D$ , and observe that  $yx \in A(Q)$ , which implies that there is some  $\eta$ , such that  $yx = q_{\eta}q_{\eta+1}$ . Furthermore  $m = \text{diam}(D) \geq 4$ , as there is a path from  $y$  to  $x$  of length 2 in  $D - yx$ .

Assume without loss of generality that  $\eta \geq 2$ , as otherwise we can reverse all arcs and swap the names  $x$  and  $y$ , in order to get  $\eta \geq 2$  (this is true since  $m \geq 4$ ). We now consider the following cases, which exhaust all possibilities:

**Case 1:**  $\rho > 1$ . Using the minimality of  $l$  and  $m$  we observe that the arguments in Figure 1 imply that  $q_{\eta-2}$  and  $p_{\rho-2}$  are adjacent, as  $x \rightarrow \{q_{\eta-2}, p_{\rho-2}\}$  in the last picture. If  $q_{\eta-2} \rightarrow p_{\rho-2}$  then the path  $q_0q_1 \dots q_{\eta-2}p_{\rho-2}p_{\rho-1}q_{\eta+1} \dots q_m$  is a path of length  $m$  in  $D - yx$ , a contradiction. If  $p_{\rho-2} \rightarrow q_{\eta-2}$ , then we analogously arrive to a contradiction.

**Case 2:**  $\rho = 1$  and  $\eta + 2 \leq m$ . Then, by the minimality of  $l$  and  $m$ , we obtain the arcs seen in the last picture of Figure 2. Since  $\{p_{\rho-1}, q_{\eta+2}\} \rightarrow q_{\eta-2}$ , the vertices  $p_{\rho-1}$  and  $q_{\eta+2}$  are adjacent. We cannot have  $p_{\rho-1} \rightarrow q_{\eta+2}$  as then the path  $(p_0 = p_{\rho-1})q_{\eta+2}p_{\rho+1} \dots p_l$  is a  $(p_0, p_l)$ -path of length  $l$  in  $D - xy$ . Therefore  $q_{\eta+2} \rightarrow p_{\rho-1}$ . However this implies that  $p_{\rho-1}$  and  $q_{\eta-1}$  are adjacent. We can now get a contradiction analogously to Case 1.

**Case 3:**  $\rho = 0$ . We see from Figure 3 that  $x \rightarrow \{q_0, q_1, \dots, q_{\eta-1}\}$ . Let  $R = r_0r_1 \dots r_t$  be a shortest path from  $q_0$  to  $p_l$  in  $D$  (see Figure 3). We have  $t \geq 3$  as  $(p_0 = x) \rightarrow q_0$  and there is no  $(p_0, p_l)$ -path of length at most four in  $D - xy$ . Observe that if  $x$  and



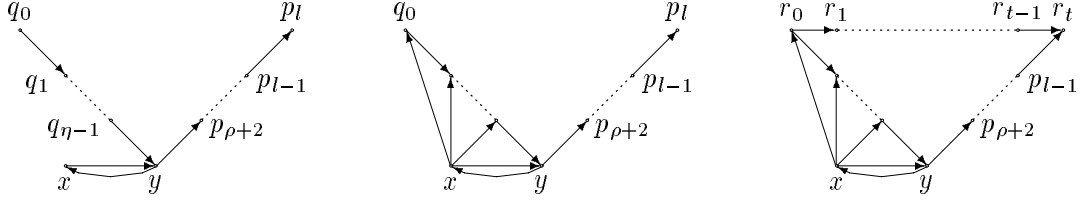


Figure 3: The first picture contains the given arcs. This implies that the arc  $x \rightarrow q_{\eta-1}$ , which implies that  $x \rightarrow q_{\eta-2}$ . Continuing this process we see that  $x \rightarrow \{q_0, q_1, \dots, q_{\eta-1}\}$ , as seen in the middle picture. In the last picture we have added a shortest  $(q_0, p_l)$ -path.

$r_1$  are adjacent then either  $q_0 r_1 q_{\eta+1} \dots q_m$  or  $p_0 r_1 r_2 \dots r_{t-1} p_l$  are paths of length at most  $\text{diam}(D)$  in  $D - \{xy, yx\}$  a contradiction. Therefore  $x$  and  $r_1$  are not adjacent in  $D$ .

Since  $q_0 \rightarrow \{q_1, r_1\}$  we observe that  $q_1 \rightarrow r_1$ , as if  $r_1 \rightarrow q_1$  then  $x$  and  $r_1$  would be adjacent (as  $q_{\eta+1} \rightarrow q_1$ ). Analogously  $q_2 \rightarrow r_1$ , as  $q_1 \rightarrow \{q_2, r_1\}$ . Continuing in this fashion we get that  $\{q_0, q_1, \dots, q_{\eta+1}\} \rightarrow r_1$ , which is a contradiction against  $q_{\eta+1}$  and  $r_1$  not being adjacent.

**Case 4:**  $\rho = 1$  and  $\eta + 2 > m$ . This clearly implies that  $\eta + 1 = m$ , as  $m \geq \eta + 1$ . By reversing all arcs we obtain the case when  $\rho = 0$  and  $\eta = l - 2 \geq 2$ , which we handled in Case 3.  $\square$

**Theorem 3.2** *If  $D$  is a strong locally semicomplete digraph of order  $n \geq 3$ , then*

$$\text{diam}_{\min}(D) \leq \max\{5, \text{diam}(D) + 1\}.$$

**Proof:** For a given vertex  $x \in V(D)$ , let  $(N^+(x) \cup \{x\}, N^-(x) \cup \{x\})$  be the neighbourhoods pair of  $x$ . Let  $V_1 = (N_1, M_1), V_2 = (N_2, M_2), \dots, V_k = (N_k, M_k)$  be the distinct neighbourhood pairs in  $D$ , and let  $v_i$  be some vertex in  $D$  with  $NT(v_i) = (N_i, M_i)$ , for  $i = 1, 2, \dots, k$ . Let  $D'$  be the subdigraph of  $D$  induced by  $\{v_1, v_2, \dots, v_k\}$ . If  $k = 1$ , then  $D = \overleftrightarrow{K}_n$ . In this case our result follows from Theorem 2.2. So, we may assume that  $k \geq 2$ .

We will now show that  $D'$  is a strong locally semicomplete digraph. Since  $D'$  is an induced subgraph of  $D$ , it is clearly a locally semicomplete digraph. Let  $v_j, v_t$  be a pair of distinct vertices in  $D'$  and let  $P = v_j p_0 p_1 \dots p_l v_t$  be a shortest  $(v_j, v_t)$ -path in  $D$ . Assume that  $p_i \in V_{a_i}$  for all  $i = 0, 1, \dots, l$ . Since  $P$  is shortest, all sets  $V_j, V_{a_1}, V_{a_2}, \dots, V_{a_l}, V_t$  are distinct. However this implies that  $v_j v_{a_1} v_{a_2} \dots v_{a_l} v_t$  is a path in  $D'$ . So  $D'$  is strong.

By Theorem 3.1 we can find an orientation  $D''$  of  $D'$  such that

$$\text{diam}(D'') \leq \max\{4, \text{diam}(D')\}.$$

We now let  $D'''$  be the digraph obtained from  $D''$  by replacing every vertex  $v_i$  with the set  $V_i$  and choosing arbitrary orientations for arcs between vertices in the same  $V_i$  ( $i = 1, 2, \dots, k$ ). As above we can easily see that the distance between vertices in distinct sets,  $V_j$  and  $V_i$ , remains the same in  $D'''$  as in  $D''$ . Let  $u \neq w \in V_i$ . Since  $D'''$  is strong, there is a vertex  $v \notin V_i$  such that  $v \rightarrow w$ . Clearly,  $\text{dist}_{D''}(u, w) \leq \text{dist}_{D''}(u, v) + 1$ . Thus, the distance between two vertices in the same set  $V_i$  in  $D'''$ , is at most  $\text{diam}(D'') + 1$  and  $D'''$  is an orientation of  $D$  with  $\text{diam}(D''') \leq \text{diam}(D'') + 1 \leq \max\{4, \text{diam}(D')\} + 1 \leq \max\{4, \text{diam}(D)\} + 1$ .  $\square$

## 4 Further research

We were not able to prove or disprove the following bound for strong semicomplete  $k$ -partite digraphs  $D$ :  $\text{diam}_{\min}(D) \leq \text{diam}(D) + c$ , where  $c$  is a constant.

Since every undirected graph can be considered as the corresponding symmetric digraph, it would be interesting to see what results on diameters of orientations of undirected graphs can be extended to digraphs. The results on minimum diameter orientations of undirected graphs form only a small part in the important area of orientations of undirected graphs (e.g., Chapter 8 in [2] is completely devoted to orientations of graphs). It would be interesting to investigate what results in the area can be (or cannot be) generalized to orientations of digraphs, see Section 7.14 in [2] for some examples of such results.

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