SOME TOPICS IN SET THEORY

by

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Abstract

This thesis is divided into two parts. In the first of these we consider Ackermann-type set theories and many of our results concern natural models.

We prove a number of results about the existence of natural models of Ackermann's set theory, A, and applications of this work are shown to answer several questions raised by Reinhardt in [56]. A^+ (introduced in [56]) is another Ackermann-type set theory and we show that its set theoretic part is precisely ZF. Then we introduce the notion of natural models of A^+ and show how our results on natural models of A extend to these models. There are a number of results about other Ackermann-type set theories and some of the work which was already known for ZF is extended to A. This includes permutation models, which are shown to answer another of Reinhardt's questions.

In the second part we consider the different approaches to set theory; dealing mainly with the more philosophical aspects. We reconsider Cantor's work, suggest that it has frequently been misunderstood and indicate how quasi-constructive set theories seem to use a definite part of Cantor's earlier ideas. Other approaches to set theory are also considered and criticised. The section on NF includes some more technical observations on ordered pairs.

There is also an appendix, in which we outline some results on extended ordinal arithmetic.

Contents

Chapter 1

Introduction

1.1 Outline

Part 1 of this thesis concerns Ackermann-type set theories. In chapter 2 we introduce Ackermann's set theory, discuss its motivation, and show how it is related to other set theories.

Then, in chapter 3, we prove some results about the existence of natural models of Ackermann's set theory, A, and applications of this work are shown to answer several questions raised by Reinhardt in [56]. The subject of chapter 4 is A^+, an Ackermann-type set theory which was introduced in [56]. We show that its set theoretic part is precisely ZF, answering another question of Reinhardt's. Then we give several alternative axiomatisations of A^+, introduce the notion of its natural models and extend the results of chapter 3 to these models.

The first part of chapter 5 introduces two new Ackermann-type set theories and we investigate some of their properties. Some other Ackermann-type set theories are also discussed in that chapter. Chapter 6 contains several isolated results, including proofs that an Ackermann-type theory suggested by Wang is inconsistent and that extending permutation models to A enables us to answer another question of Reinhardt's.

In chapter 7 we consider some problems concerning natural models of ZF. One of them arises as a generalisation of the natural models of A and A^+. The others concern the structure of natural models under the relation of elementary extension: they were motivated by some of the results in chapter 3.

In part 2 of this thesis we consider and criticise the different approaches to set theory which have been made. This part mostly concerns the philosophical aspects and we often seem to disagree with the accepted views.

We describe Cantor's work in chapter 8. Then we emphasise its second order nature, indicate how it seems to have been misunderstood and suggest that a lot of later work was motivated by such misunderstandings. Part of chapter 9 gives a justification of ZF in Cantorian terms and in the remainder of that chapter we consider related problems and quasi-constructive approaches.

Most of chapter 10 concerns the theory NF. One section shows that it is very important to note which definition of ordered pair is used in this theory. In the remainder of chapter 10 we consider approaches to set theories with a universal set via theories of properties.

The appendix contains some results on extended ordinal arithmetic and a result on the number of ordinals obtained by permuting a given sequence of ordinals and taking their sums.

The main topics which we have considered are rather disconnected, but we hope that they do not form a discordant mixture. Also, it seems a shame that this thesis has to be linearly ordered as several of the topics intertwine. We apologise for the number of cross references.

1.2 Notation

Most of our notation is that which is becoming standard in set theory (for instance, $\alpha, \beta, \gamma, \delta, \kappa, \lambda, \mu$ are variables which range over ordinals) and, in general, we follow the notation of [56]. However, we wish to emphasise the following abbreviations which are not completely standard.

$\mathcal{P}$ is the power set operation, $R\lambda = \bigcup_{\beta<\lambda}(\mathcal{P}(R\beta))$ and the natural models of ZF are those of the form $\langle R\lambda, \in \upharpoonright R\lambda \rangle$.

We write $R\alpha+\beta$ for $R(\alpha+\beta)$ and we often drop the $\restriction R\lambda$, or even all mention of $\in$, from our natural model notation. Thus, for example, $R\alpha \prec R\beta$ means that $\langle R\alpha, \in\restriction R\alpha\rangle$ is a proper elementary substructure of $\langle R\beta, \in\restriction R\beta\rangle$. When $\gamma\in\alpha$, $R\alpha \prec_\gamma R\beta$ stands for $R\alpha+\gamma \prec R\beta+\gamma$.

If a and b are sets with a $\supseteq$b, then Df(a,b) is the set of those elements of a which are definable in $\langle a, \in\restriction a\rangle$ using a first order $\in$-formula and parameters from b. The letters ϕ, ψ, χ are always assumed to stand for $\in$-formulae and Φ, Ψ are allowed to be any formulae.

We write $\overline{\overline{x}}$ for the cardinality of x and $\overline{x}$ for the order type of x, where the ordering is assumed to be $\in\restriction x$ if no other ordering is mentioned. For convenience we always suppose that if κ is an inaccessible cardinal (written Inac(κ)) then $\kappa > \omega$.

Our notation for Ackermann-type set theories is explained in chapter 2 and the following abbreviations are used for other set theories.

ZF - Zermelo-Fraenkel set theory with the axiom of foundation, see [16].

Z - ZF without the axiom of replacement.

ZF° - ZF without the axiom of foundation.

ZM - ZF together with an axiom schema stating that every normal function has an inaccessible fixed point, see [39].

NBG - von Neumann-Bernays-Godel set theory, see [43]. For convenience we assume that NBG is axiomatised with an axiom of foundation for sets only: this is clearly equivalent to the formulation of [43].

MK - Morse-Kelly set theory, see [45]. This theory is NBG modified by allowing class quantifiers to appear in the class existence axiom and it is sometimes called impredicative NBG.

NF - Quine's system in [53], which is now called New Foundations.

1.3 Acknowledgements

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Chapter 2

Review of Ackermann's set theory

2.1 Ackermann's paper

Ackermann's set theory was introduced in [1]. In section one of that paper Ackermann describes his fundamental ideas in a heuristic form, and he starts from Cantor's 1895 definition of a set which says

" A set is any collection into a whole of definite, distinct objects of our perception or our thought. These objects are called the elements of the set. "

Whether or not Cantor intended this to be a definition is a problem which we shall consider later. Before describing Ackermann's ideas we note that many of his remarks seem more in keeping with Cantor's 1882 paper, rather than the later one. In the 1882 paper sets were thought of as " well defined " collections and Cantor says that

" A collection of elements belonging to any sphere of thought is said to be well defined when, in consequence of its definition and the logical principle of the excluded middle, it must be considered as intrinsically determined whether any object belonging to this sphere belongs to the collection or not and, secondly, whether two objects belonging to the collection are equal, or not, in spite of formal differences in the manner in which they are given. "

As is usual Ackermann disregards objects of our perception and he considers a general way of formalising Cantor's definition.

Although it is not made explicit in [1], Ackermann seems to be imagining a universe in which there are many objects which are not sets and, in general, these objects are called classes. Firstly he argues for the axioms

1. All classes are extensional,
2. For every property P(x), there is a class whose members are those sets which satisfy P(x),

and then he turns to his main axiom, and says

" The distinction between classes and sets can only be a matter of a satisfactorily fine definition of what belongs to the class and what does not. But the concept of set is completely open. "

Consequently, he argues, one will not be able to think of a class as sufficiently precisely distinguished if it can only be defined with reference to the concept of a set. Thus he is led to suppose that if the property P(x) (of a class which, it seems, is assumed to consist entirely of sets) is such that its definition does not refer to the property of being a set then the extension of this property will be a set. Other sets are allowed as parameters.

The fourth axiom which is justified states that all members and subclasses of sets are also sets: the argument for the latter is that such a class can be defined without reference to the general set concept. Ackermann then explains that the basic concepts of such a theory are identity, membership and sethood, and he indicates why such a system is not immediately inconsistent. He also suggests that choice is a logical axiom, but it can be added in the usual way if so desired.

In section two of [1] a formal system for Ackermann's set theory is set up and we explain the current formulation, which is easily seen to be equivalent to the original one. The system is set up in the first order predicate calculus with identity. $\in$ is used for membership and a constant, V, is included for the class of all sets. Remembering our convention that small Greek letters stand for $(\in, =)$-formulae and large Greek letters for any formulae, the counterparts of 1, 2 and 4 are

A1 $\forall t\ (t \in x \leftrightarrow t \in y) \rightarrow x = y,$

A2 $\exists z \forall t\ (t \in z \leftrightarrow t \in V \wedge \Phi)$, where Φ does not involve z,

A3 $x \in V \wedge (y \in x \vee y \subseteq x) \rightarrow y \in V.$

To formalise the main axiom we then take the following schema

A4 If ϕ has exactly three free variables, then

$$x, y \in V \wedge \forall t\ (\phi(x,y,t) \rightarrow t \in V) \rightarrow \exists z \in V \forall t\ (t \in z \leftrightarrow \phi(x,y,t)).$$

Ackermann did not include an axiom of foundation in his theory but it is convenient to introduce one here.

A5 $x \in V \wedge \exists u\ u \in x \rightarrow \exists u \in x \forall t \in x\ t \notin u.$

Next we give some abbreviations which we shall use throughout this work. WB(X) is intended to be read " X is well behaved ", and WB(V) is often the basis of a theory which generalises Ackermann's set theory (A).

Definition 2.1 A is the theory with axioms A1, A2, A3 and A4. A^* is A augmented by A5. WB(X) [$WB^*(X)$] is the collection of axioms A1, A2, A3 [and A5] in which V has been replaced by X.

Ackermann showed that the relativisations to V of all the axioms of Z, except foundation, are provable in A. These proofs are particularly elegant. For instance, to prove the power set axiom let θ be $t \subseteq x$ so that A3 gives $t \in V$ when $x \in V$. Then the power set of x is in V by A4. He also claimed to prove that this was true for replacement but, as Levy pointed out in [38], there is a mistake in his proof.

The only other notion from [1] which we shall refer to is a theory mentioned in the last part of that paper. This incorporates different orders of sets and, in an equivalent formulation, it has a constant V_ω and infinitely many constants V_n for $n \in \omega$. Ackermann did not feel that this theory was particularly important as he said

" Such a theory is of no great interest as all important sets are already contained in sets of the first order. "

Definition 2.2 A^*_∞ is a theory with $\in$ as a predicate and constants V_∞ and V_n for $n \in \omega$. Its axioms are all sentences of the form $\phi(V_\infty)$ and $\phi(V_n)$, where $\phi(V)$ is an axiom of A^* together with $V_n \in V_{n+1} \in V_\infty$ for $n \in \omega$.

2.2 The development of A

A, and related systems, have been studied in [38], [40], [22] and [56] and rather than attribute all results individually we shall just give the main known results for A. Further details are contained in [56] and we refer the reader to that work for our omissions.

It is possible to develop a theory of ordinals in A which is similar to that of ZF and in A^* it can be shown that $\exists \alpha\, V = R\alpha$. From this and A4 the following reflection principles can be obtained. These are extremely useful for proving results in A^*.

Theorem 2.3 (i) Downward reflection principle for V (DR).

If ϕ has exactly three free variables, then

$A^* \vdash x,y \in V \wedge \phi(x,y,V) \rightarrow \exists z \in V\ \phi(x,y,z)$.

(ii) Upward reflection principle for V (UR).

If ϕ has exactly three free variables, then

$A^* \vdash x,y \in V \wedge \phi(x,y,V) \rightarrow \exists z\ (V \in z \wedge \phi(x,y,z))$.

From UR we can see that in A^*, unlike NBG, there are proper classes (i.e. classes which are not sets) which have proper classes as members, and that constructions of such classes can be continued for a long way.

In [38] Levy proved the next theorem and this shows that A^* is not stronger than ZF.

Theorem 2.4 If ϕ is a sentence and $A^* \vdash \phi$ or $A^* \vdash \phi^V$, then $ZF \vdash \phi$.

After the discovery of the mistake in Ackermann's proof of replacement relativised to V, the main open question for A^* (and A, where it is still open) was whether or not it is provable. This has been answered affirmatively by Reinhardt in [56] and, consequently, the following result is now known.

Theorem 2.5 If ϕ is a sentence, then $A^* \vdash \phi^V$ iff $ZF \vdash \phi$.

It might be argued that on the basis of this theorem there is little point in continuing the study of A^*, but we have several arguments against this. Firstly, there is the fact that the reflection principles cannot be expressed in ZF. They are particularly interesting as in [40] it is shown that A^* can be axiomatised using $WB^*(V)$ and DR. The second argument is more important and it is that there are many natural generalisations of Ackermann's approach to set theory while this is not the case for ZF. In the motivation for A4 it was said that if a property can be used to form a set then this property must not depend on the set concept. Then, to insist that the set concept does not appear in the definition of the property is the crudest way of satisfying this condition. Consequently, further refinement of these ideas seems quite likely to give theories which are stronger than ZF. Such theories are called Ackermann-type set theories. Other reasons why we think that such theories are important are that the use of more basic notions might make proofs clearer and that their natural models often turn out to be of independent interest.

In [38] Levy also considered adding a strong replacement axiom (i.e. the replacement axiom of NBG) to A^* , and combining one of his results with one from [56] gives theorem 2.7.

Definition 2.6 A^*_0 is the theory A^* augmented by the axiom
$x \in V \wedge \forall u \in x \, \exists v \in V \, \langle u,v \rangle \in r \rightarrow \exists y \in V \, \forall u \in x \, \exists v \in y \, \langle u,v \rangle \in r.$

Theorem 2.7 If ϕ is a sentence, then $A^*_0 \vdash \phi^V$ iff $ZM \vdash \phi$.

Theorem 2.8 is proved in [56] using a straightforward, proof-theoretic argument and UR. It shows that A^*_∞ gives no new information about V_0 .

Theorem 2.8 If ϕ has exactly one free variable, then

$$A^*_\infty \vdash \phi(V_0) \text{ iff } A^* \vdash \phi(V).$$

We shall refer to other work which has been done on Ackermann-type set theories as we require it.

2.3 Comparison with other set theories

We shall summarise the main relationships between A^* , A^*_0 , ZF, NBG and MK, and, for this section, we assume that all of these theories are consistent. For convenience, we use V for both the constant of A^* and the defined constant of NBG.

Definition 2.9 $\Phi^{\vee}$ is the formula Φ with all its quantifiers relativised to '$\in V$' and $\Phi^{\circledcirc}$ is the formula Φ with all its quantifiers relativised to '$\subseteq V$'. If T is an appropriate set theory, then we put

$$T \mid V = \{ \phi \mid \phi \text{ is a sentence and } T \vdash \phi^{\vee} \},$$
$$T \mid (V) = \{ \Phi \mid \Phi \text{ is a sentence and } T \vdash \Phi^{\circledcirc} \}.$$

$\phi^{\vee}$ and $\Phi^{\circledcirc}$ are the natural ways of interpreting formulae of ZF and NBG, respectively, in A^* . The following results follow from theorems which we quoted in the last section, well known results or easy checks.

$$A^* \mid V = ZF = NBG \mid V \subset MK \mid V \subset A^*_0 \mid V,$$
$$NBG \subset MK \subset A^*_0 \mid (V) \supset A^* \mid (V) .$$

Now the only questions about such inclusions which are not answered are those concerning $A^* \mid (V)$ and NBG or MK. Theorem 2.10 shows that there are no further strict inclusions here, so that although $A^* \mid V = ZF$, the situation changes completely when we consider $A^* \mid (V)$.

Theorem 2.10 (i) $NBG \not\subseteq A^* \mid (V)$,

(ii) $A^* \mid (V) \not\subseteq MK$.

Proof (i) Suppose that $NBG \subseteq A^* \mid (V)$. Then strong replacement would be provable in A^* so that $A^* = A^*_0$. This is false.

(ii) Suppose that $A^* \mid \text{Ⓥ} \subseteq MK$. Then as $MK \vdash \exists\alpha(V = R\alpha \wedge Inac(\alpha))$ (see [43] , for instance) and downward reflection holds in A^* we would have $MK \vdash \exists\alpha \in V\ Inac(\alpha)$. This is a contradiction as the consistency of MK can be proved in $ZF + \exists\alpha\, Inac(\alpha)$. □

Next, we shall indicate how A^* is very useful for accomodating category theory. In [42] , MacLane said that NBG is sufficient to describe all of present day category theory with the exception that it cannot accomodate categories above large categories. Large categories are proper classes in the sense of NBG, superlarge categories contain functions which are themselves proper classes etc. NBG has only finitely many axioms and let ϕ be their conjunction. If $V = R\alpha$ then, from above, we know that $\phi^{R\alpha+1}$ holds in A^*_0. Hence we can derive in A^*_0, using the upward reflection principle, the existence of a least β for which $V \in R\beta$ and $\phi^{R\beta+1}$ holds. Now this $R\beta$ has the very pleasant attribute that if any $\in$-property can be proved to hold for all members of V (which corresponds to all small categories) then all members of $R\beta$ (which corresponds to all small, large, superlarge etc. categories) also have this property. To see this one need only take the conjunction of the property and $\phi^{R\alpha+1}$ and use the upward reflection scheme. Thus $R\beta$ is a suitable universe for category theorists as they need only worry about small categories.

In actual practice only finitely many axioms of ZF, rather than all of NBG, would be required for proofs (say, those axioms with less than 10^{12} symbols) so that the above procedure could be carried out with A^* in place of A^*_0. Thus we suggest that until a consistent axiomatisation of a (the?) category of all categories is given, category theory can be neatly handled in A^* without any artificialities having to be introduced.

2.4 Some reconsiderations

We do not find Ackermann's heuristic description of his theory in [1], or the arguments by which he obtains A from the basic idea, totally convincing. However this might, in some sense, be inevitable. Further, the ideas which led to A are not necessarily those which Ackermann published, and this suggestion is supported by the fact that A has been rediscovered by at least two other people who were not working from Cantor's definition.

In section one of [1] the fundamental point at which Ackermann diverges from Cantor is when he allows proper classes in his domain of individuals. It is clear that he thought of them as well defined entities. Of course this is alien to Cantor's work and even Ackermann insists that the set concept is throughly open despite the fact that his theory proves the existence of a unique class which is the class of all sets. The ability to prove this comes from A2 and in chapter 5 we show that the strength of Ackermann-type systems strongly depends upon this axiom. Now, although A cannot be viewed as an axiomatisation of Cantor's work, there are many other systems which assume the existence of a class of all sets. A is such a theory in which it is suggested that collections of sets which can only be defined by reference to V are of a different order of existence to those which can be defined without such reference. There are still problems about sets which are not definable and we consider some of these in chapter 8.

When Ackermann argues from his heuristic description to A he seems to ignore his earlier idea that the only difference between a set and a proper class is " a matter of a satisfactorily fine definition of what belongs to a class and what does not ". Such a statement surely leads us to question the use of the excluded middle for formulae which involve V. An intuitionistic version of A has been worked out in [51], but the motivation for this was completely different.

Despite these criticisms of the presentation of A, it can be viewed as just being based on the downward reflection principle for V so that it is not necessary to consider the original ideas behind it at all.

It is very interesting to consider other ways of formalising Ackermann's notion of the class of all sets not being sharply delimited. Reinhardt suggested (in [56]) that one way of doing this would be to suppose that there are alternative candidates for such a class: this idea is considered further in chapters 4 and 5.

Chapter 3

Natural models of A

3.1 Existence of natural models

Natural models of A are models of this theory which take the form $\langle R\alpha, R\beta, \in \restriction R\rangle$, where $R\alpha$ is the domain of the model and $R\beta$ is the class of all sets in the model. We shall usually drop the suffix from $\in$ and from now on we adopt the convention that $\mathcal{R}$ is $\langle R\alpha, R\beta, \in\rangle$ where $\alpha > \beta$.

The natural models of A were first studied in [22] and the main results which Grewe gives in that paper are the next three theorems.

Theorem 3.1 If $\langle B, U, E\rangle \models A$ then U is not definable in $\langle B, U, E\rangle$ using an $\in$-formula and parameters from U.

Theorem 3.2 If $\alpha > \beta$ and $R\beta \notin Df(R\alpha, R\beta)$, then $\mathcal{R} \models A$.

Theorem 3.3 If $\mathcal{R} \models A$ then either $R\beta \prec R\alpha$ or if ξ is the least ordinal such that $\beta < \xi < \alpha$ and $R\xi \in Df(R\alpha, R\beta)$, then $R\beta \prec R\xi$.

Actually Grewe only proves theorem 3.2 for the case when α is a limit ordinal, but it is straightforward to extend this proof to the case when $\alpha = \omega\xi + n$ (where $n \in \omega$) by relativising appropriate definitions to $R\omega\xi$.

Some further results on natural models of A are included in [56] and the main one of these is

Theorem 3.4 If $\gamma \in \beta$ and $R\beta \prec_\gamma R\alpha$ then $\langle R\alpha + \gamma, R\beta, \in\rangle \models A$.

The above theorems provide some knowledge about the structure of natural models of A, but only theorem 3.4 gives us examples of their existence. Our first main result, theorem 3.8, shows that assuming the existence of inaccessible cardinals there are a large number of natural models of A. We give some applications of this in the remainder of this chapter. Theorem 3.7 is a more general result which we shall use later, and theorem 3.6 is a straightforward modification of the main result of [47]: we include a proof for completeness.

Throughout this chapter we always assume that there are arbitarily large inaccessible cardinals, although the existence of one or two inaccessible cardinals suffices for most of our results. Theorem 3.9, however, seems to require a stronger hypothesis.

Definition 3.5 A function $f:\alpha \to \alpha$ is said to be regressive if $f(0) = 0$ and for $0 < \beta < \alpha$ $f(\beta) < \beta$.

Theorem 3.6 If κ is a regular cardinal greater than ω and f is a regressive function on κ, then there exists an $\alpha < \kappa$ such that for κ many $\beta < \kappa$, $f(\beta) = \alpha$.

Proof Suppose that the hypothesis of the theorem holds while the conclusion is false. Then for any $\lambda_0 < \kappa$ there is a $\lambda < \kappa$ such that for every $\xi \geq \lambda$ $f(\xi) > \lambda_0$, as κ is regular. Hence, for arbitary $\lambda_0 < \kappa$ we can obtain an ω-sequence of ordinals $< \kappa$

$$f(\lambda_0) < \lambda_0 < f(\lambda_1) < \lambda_1 < \ldots . \quad (*)$$

where for every $\xi \geq \lambda_{n+1}$ $f(\xi) > \lambda_n$. Let the supremum of $(*)$ be γ, which is $< \kappa$ by regularity, and then as $f(\gamma) < \gamma$ we have $f(\gamma) < \lambda_m$, for some $m \in \omega$. But then $\gamma > \lambda_{m+1}$ so that from the definition of $(*)$ $f(\gamma) > \lambda_m$, which is a contradiction. □

Theorem 3.7 If κ is an inaccessible cardinal, $\alpha \geqslant \kappa$ and x is a set of cardinality less than κ which is contained in $R\alpha$, then there are κ many $\beta < \kappa$ for which $R\beta \notin Df(R\alpha, R\beta \cup x)$.

Proof Let κ, α, x be as in the hypothesis of the theorem and suppose that there are less than κ β's with the required properties. If β has the required properties then put $f(\beta) = 0$, and we complete the definition of $f: \kappa \to \kappa$ as follows

If $f(\beta)$ has not already been defined then
$R\beta \in Df(R\alpha, R\beta \cup x)$ and put (**)
$f(\beta) =$ the least γ for which $R\beta \in Df(R\alpha, R\gamma \cup x)$.

It is clear that f is a regressive function on κ so that by theorem 3.6 there is a $\delta < \kappa$ such that for κ many $\beta < \kappa$ $f(\beta) = \delta$. Further, κ many of these β's must have had their f value defined by (**) so that there are κ many ordinals less than κ which are in $Df(R\alpha, R\delta \cup x)$. This is impossible as there are only countably many formulae and $\max(\overline{\overline{R\delta}}, \overline{\overline{x}})$ ($< \kappa$ as κ is inaccessible) many parameters available. □

Theorem 3.8 If κ is an inaccessible cardinal and $\alpha \geqslant \kappa$ then there are κ many natural models of A of the form $\langle R\alpha, R\beta, \in \rangle$, with $\beta < \kappa$.

Proof This follows directly from theorem 3.2 and theorem 3.7 □

Remark We have recently shown that V = L implies that theorem 3.8 is best possible, in the sense that the first inaccessible cardinal becomes the smallest cardinal for which the conclusion holds. V = L also decides some other questions about natural models of A and details of these results will appear elsewhere.

In [56] Reinhardt asked if there is a second order vesion of Grewe's theorem (i.e. theorem 3.3). He made this precise in Question 4.13 of that paper which is

" Suppose that for every $x \in R\beta$ β is not definable in $\langle R\alpha+1, \in, x\rangle$. Is there a $\gamma \leq \alpha$ such that $\beta \in \gamma$ and $R\beta \prec_1 R\gamma$? ".

We can use theorem 3.7 to show that the answer to this question is no, in general, as follows.

Let κ be an inaccessible cardinal and $\alpha \geq \kappa$. Then, by theorem 3.7, there is a $\beta < \kappa$ such that for every $x \in R\beta$, β is not definable in $\langle R\alpha+1, \in, x\rangle$. Now suppose that the answer to Reinhardt's question is yes so that there is a $\gamma \leq \alpha$ such that $\beta \in \gamma$ and $R\beta \prec_1 R\gamma$. Theorem 4.12 of [56] shows that if $V = L$ holds then we can derive the existence of arbitarily large inaccessibles in $R\beta$ from $R\beta \prec_1 R\gamma$. Hence the usual consistency proof of $ZF + \exists\alpha\, Inac(\alpha) + V = L$ relative to $ZF + \exists\alpha\, Inac(\alpha)$ shows that we can derive the consistency of $ZF + \forall\alpha\, \exists\beta > \alpha\; Inac(\beta)$ from the consistency of $ZF + \exists\alpha\, Inac(\alpha)$. This is well known to be false.

Reinhardt's question is a straightforward generalisation of Grewe's theorem and it still might be true that there is a less obvious generalisation. The reason why the original proof does not generalise to higher orders is that the most natural generalisations of the following statement fail.

For limit λ, $R\alpha \prec R\lambda$ iff $\forall\xi(\alpha \leq \xi < \lambda \rightarrow R\xi \notin Df(R\lambda, R\alpha))$.

It might be possible to get a higher order version of this result by adding further conditions, which are vacuous in the first order case, to the right hand side.

3.2 Bounded upward reflection

In the theory A^*_∞ the different constants were intended to represent different orders of sets. As an upward reflection principle is provable in A, if $x \in V_i \wedge \phi(x, V_i)$ holds in A^*_∞ then it seems natural to insist that there is a y of the same order as V_i for which $V_i \in y \wedge \phi(x,y)$ holds. Thus we are led to consider the following principle of bounded upward reflection (BUR) in A^*_∞.

BUR. If ϕ has exactly two free variables, then

$$x \in V_i \wedge \phi(x, V_i) \rightarrow \exists y\, (V_i \in y \in V_{i+1} \wedge \phi(x,y)).$$

Our next theorem shows that BUR is not derivable in A^*_∞ by constructing a natural model. This suggests that there might be some intuitively reasonable generalisations of A^*_∞ and we return to this in chapter 5. Theorem 3.10 constructs a natural model of A^*_∞ + BUR using only one inaccessible cardinal.

Theorem 3.9 If there is a 1-indescribable cardinal (see [56] for a definition), then there is a natural model of A^*_∞ in which BUR is false.

Proof Let λ be a 1-indescribable cardinal and we firstly show that there is an inaccessible $\beta < \lambda$ such that $R\beta \prec R\lambda$.

Theorem 2.2 of [45] shows that there is a normal function $f : \lambda \rightarrow \lambda$ such that if β is a fixed point of f then $R\beta \prec R\lambda$. Hence we can suppose that

$\langle R\lambda + 1, \in \rangle \models$ " f is a normal function on λ with the above property, and λ is inaccessible ".

Then, by indescribability, there is a $\beta < \lambda$ for which

$\langle R\beta + 1, \in \rangle \models$ " $f \cap R\beta$ is a normal function on β with the above property, and β is inaccessible ".

It is clear that β is a fixed point of f so that β is an inaccessible cardinal satisfying $R\beta \prec R\lambda$, as required.

Theorem 3.4 shows that $\langle R\lambda, R\beta, \in\rangle \models A$. Let β' be the least inaccessible cardinal greater than β and then theorem 3.8 shows that there is an $(\omega+1)$-sequence of ordinals

$$\delta_0 < \delta_1 < \delta_2 < \ldots < \delta_\omega,$$

all between β and β', satisfying $\langle R\lambda, R\delta_i, \in\rangle \models A$ for all $i \in \omega+1$. Then $\langle R\lambda, R\beta, R\delta_1, \ldots R\delta_\omega, \in\rangle$ is a natural model of A^*_∞ in which $V_0 = R\alpha \wedge \mathrm{Inac}(\alpha)$ holds but $\exists\gamma(V_0 \in R\gamma \in V_1 \wedge \mathrm{Inac}(\gamma))$ fails. Thus BUR is false. □

Theorem 3.10 There is a natural model of A^*_∞ + BUR.

Proof Let κ be an inaccessible cardinal so theorem 3.8 shows $\exists\beta < \kappa\ \langle R\kappa, R\beta, \in\rangle \models A$. UR holds in this model so that for any formula ϕ with exactly two free variables

$$R\kappa \models x \in R\beta \wedge \phi(x, R\beta) \rightarrow \exists z\ (R\beta \in z \wedge \phi(x,z)).$$

Let the supremum of the least ranks of such z's (over all $x \in R\beta$ and all suitable formulae ϕ) be α_1 and then $\alpha_1 < \kappa$ as κ is inaccessible. Using theorem 3.8 again, let β_1 be the least ordinal greater than α_1 for which $\langle R\kappa, R\beta_1, \in\rangle \models A$. Then use $R\beta$ as V_0, $R\beta_1$ as V_1 and iterate the above construction to obtain a natural model of A^*_∞ with domain $R\kappa$. Clearly BUR holds in this model. □

3.3 Some questions of Reinhardt's

Theorem 3.11 answers question 4.14 of [56] negatively. This, in turn, shows that theorem 3.9 of [56] cannot be improved to a version without a parameter, answering another question of Reinhardt's. It is noted in [56] that question 4.14 is equivalent to asking whether or not the schema

$$x \in V_0 \to (\theta^{V_0}(x) \leftrightarrow \theta^{V_1}(x))$$

is provable in A^*_∞, where θ is a formula with exactly one free variable.

<u>Theorem 3.11</u> Let ψ be the sentence $\exists\beta\, \text{Inac}(\beta)$. Then there is a natural model in which $(\psi^{V_0} \leftrightarrow \psi^{V_1})$ is false.

<u>Proof</u> We construct a natural model of A^*_∞ in which ψ^{V_1} and $\neg\psi^{V_0}$ hold. Let κ_0, κ_1 be the first two inacessible cardinals. Then, by theorem 3.8, there is a model of A of the form $\langle R\kappa_1, R\alpha_0, \in\rangle$ with $\alpha_0 < \kappa_0$ As in the proof of theorem 3.9 we can then find an $(\omega+1)$-sequence of ordinals α_{i+1} all between κ_0 and κ_1, such that $\langle R\kappa_1, R\alpha_0, R\alpha_1, \ldots R\alpha_\omega, \in\rangle \models A^*_\infty$. In this model $\exists\beta \in V_1\, \text{Inac}(\beta)$ and $\neg\exists\beta \in V_0\, \text{Inac}(\beta)$ so that as V_0 and V_1 are supertransitive, ψ^{V_1} and $\neg\psi^{V_0}$ hold. □

Our next result proves Conjecture 4.16(b) of [56], again by using our natural model methods.

<u>Theorem 3.12</u> A + ZF is not finitely axiomatisable over A.

<u>Proof</u> Suppose that $\theta_1, \ldots \theta_n$ are axioms of ZF. We construct a natural model $\langle R\beta, R\delta, \in\rangle$ of $A + \theta_1 + \ldots + \theta_n$ where $\langle R\beta, \in\rangle$ is not a model of ZF, and the theorem will then follow.

Let κ be the least inaccessible cardinal and let α be the least ordinal greater than κ for which $\langle R\alpha, \in\rangle \models$ ZF. Using the reflection principle for ZF we then see that in $R\alpha$

$$\exists\beta\ (\kappa < \beta \wedge \theta_1^{R\beta} \wedge \ldots \wedge \theta_n^{R\beta}). \qquad (*)$$

Let β be the least ordinal which satisfies the bracketed part of $(*)$ in $R\alpha$. Then, by theorem 3.8, there is a $\delta < \kappa$ satisfying $\langle R\beta, R\delta, \in\rangle \models A$. Now $\langle R\beta, \in\rangle \models \theta_1 \wedge \ldots \theta_n$ from $(*)$, but

$\langle R\beta, \in\rangle \nvDash$ ZF from the definition of β so that this model is as required. $\square$

In proving theorem 3.12 one of the facts which we used is that if θ is a sentence and ZF $\vdash \theta$ then $\forall x \exists y\ (x \subsetneq y \wedge \langle y, \in\rangle \models \theta \wedge \langle y, \in\rangle \nvDash \text{ZF})$: we constructed y using a reduction of the length of the universe. An analogous width reducing principle for Z would be

If θ is a sentence and Z $\vdash \theta$ then
$\forall x \exists y\ (x \subsetneq y \wedge \langle y, \in\rangle \models \theta \wedge \langle y, \in\rangle \nvDash \text{Z})$. (**)

We cannot yet prove (**), but we end this section by showing that if its true then it answers another of Reinhardt's questions. It also seems possible that further results about one theory not being finitely axiomatisable over another can be obtained by these methods as problems of equiconsistency are avoided by constructing models in a stronger theory. Question 4.22 of [56] is

" T will be a theory formulated in a language with $\in$ and individual constants V_n $(n \in \omega)$. The axioms of T include (for each n) the pairing, union and power set axioms relativised to V_n, a comprehension axiom for each V_n and an axiom $V_n \in V_{n+1}$. Is T+Z finitely axiomatisable over T ? ".

We answer this question negatively, subject to (**), as follows. Let $\theta_1, \ldots \theta_n$ be axioms of Z and then, using (**), let y be such that $R\omega^2 \subseteq y$, $\langle y, \in\rangle \models \theta_1 \wedge \ldots \wedge \theta_n$ and $\langle y, \in\rangle \nvDash$ Z. Then by taking $R\omega + i$ as V_i we see that $\langle y, \in\rangle$ gives a model for T, $\theta_1, \ldots \theta_n$, but not for Z, as required.

Remark It is known that Z is not finitely axiomatisable (see Montague's paper Semantic closure and non-finite axiomatisability I, in Infinitistic Methods, Pergamon Press, 1961), but this result does not seem to give (**) directly.

3.4 Existence of more natural models

Theorem 3.8 shows that given a large α , there are many β s for which $R_\beta \models A$ and in this section we consider the possibility of finding results of the form ' given a certain β , there are many αs for which $R_\alpha \models A$ '. Most of the proofs in this section are just outlines as otherwise we would have to give a full treatment of absoluteness conditions, as is done in [22] .

Definition 3.13 β is said to be suitable if $\exists \alpha > \beta \;\; R_\alpha \models A$.

$$U(\beta) = \{ \alpha \mid R_\alpha \models A \} ,$$

$$E(\beta) = \{ \alpha \mid R\beta \prec R\alpha \} .$$

Our use of $U(\beta)$ and $E(\beta)$ will be quite loose as these abbreviations stand for sets, virtual classes (in the sense of Quine, see [55]) and proper classes at different times. We hope that the reader can see which use is intended from the context.

Theorem 3.14 β is suitable iff $\exists \alpha > \beta \;\; R\beta \prec R\alpha$.

Proof If β is suitable then theorem 3.3 shows that $\exists \alpha > \beta \; R\beta \prec R\alpha$. Theorem 3.4 implies the other half of this theorem. □

Theorem 3.15 If κ is an inaccessible cardinal, then there are κ many $\beta < \kappa$ for which $U(\beta)$ is unbounded.

Proof Let κ be an inaccessible cardinal and we suppose that there are less than κ many $\beta < \kappa$ for which $U(\beta)$ is unbounded. Let λ be the supremum of

$$\{\alpha \mid \exists \beta < \kappa (\mathfrak{R} \models A \wedge U(\beta) \text{ is bounded })\} .$$

Then by theorem 3.8 there are κ many $\beta < \kappa$ for which $\langle R\lambda + \omega, R\beta, \in \rangle \models A$ so that there are κ many $\beta < \kappa$ for which $U(\beta)$ is unbounded. This contradicts our assumption so that the theorem holds. $\square$

Corollary 3.16 If κ is an inaccessible cardinal, then there are κ many $\beta < \kappa$ for which $\{\alpha \mid \alpha \in U(\beta) \wedge \alpha \text{ is a limit ordinal}\}$ is unbounded.

Proof If $\langle R\omega\zeta + n, R\beta, \in \rangle \models A$ then, by theorem 3.1, $R\beta \notin Df(R\omega\zeta + n, R\beta)$ so that $R\beta \notin Df(R\omega\zeta, R\beta)$. By theorem 3.2 $\langle R\omega\zeta, R\beta, \in \rangle \models A$ so that the corollary follows from the theorem. $\square$

Theorem 3.15 shows that for many suitable β s $U(\beta)$ will be unbounded, but the next result shows that this will not be true of all suitable β s.

Theorem 3.17 $\forall \delta, \gamma \; \exists \beta > \delta (\overline{\overline{U(\beta)}} = \gamma)$.

Proof Choose δ, γ and let κ be the least inaccessible cardinal which is greater than $\max(\delta, \gamma)$. Then, by theorem 3.15, there is a β' with $\max(\delta, \gamma) \in \beta' \in \kappa$ and $U(\beta')$ unbounded. Thus there is a β which satisfies

$$\gamma \in \hat{\beta} \wedge \delta \in \beta \wedge \overline{\overline{U(\beta)}} \geqslant \gamma \qquad (*)$$

where we assume that $(*)$ is written in a way which does not assume $U(\beta)$ to be a set.

Now let β be the least ordinal which satisfies $(*)$, and if $\overline{\overline{U(\beta)}} > \gamma$, let α be the γth. member of $U(\beta)$ under the natural ordering. If γ is a successor ordinal then β will be definable in $R\alpha$ using $(*)$, which uses δ, γ as parameters. By theorem 3.1 this contradicts $\mathcal{R} \models A$. If γ is a limit ordinal then it is straightforward to see that β is definable in $R\alpha$, in terms of δ, γ, using

$$\gamma \in \beta \wedge \delta \in \beta \wedge \forall \mu < \gamma (\text{there is a set of ordinals } \alpha \text{ for which } \mathcal{R} \models A \text{ and the order type of the set, under } \in \text{, is } \mu).$$

Again, this contradicts theorem 3.1 □

Theorem 3.14 characterised suitable ordinals and it might suggest that $U(\beta)$ " looks like " $E(\beta)$, but our next theorem shows that this is not true from the point of view of order types.

<u>Theorem 3.18</u> If κ is the first inaccessible cardinal and $\gamma < \kappa$, then $\exists \beta (\overline{\overline{U(\beta)}} = \kappa + \overline{\overline{E(\beta)}} + \gamma)$.

<u>Proof</u> Choose $\gamma < \kappa$ and then, by theorem 3.15, there is a β' with $\gamma < \beta' < \kappa$ and $U(\beta')$ unbounded. $E(\beta') \subsetneq \kappa$ as $\neg \exists \alpha \, Inac(\alpha)$ is true in $R\beta'$, so that there is a β' which satisfies

$$\gamma < \beta < \kappa \wedge \overline{\overline{U(\beta)}} \geqslant \kappa + \overline{\overline{E(\beta)}} + \gamma \qquad (*)$$

where $(*)$ is written in a way which does not assume $U(\beta)$ is a set.

Let β be the least ordinal which satisfies $(*)$ and put $\eta = \kappa + \overline{\overline{E(\beta)}} + \gamma$. Now suppose that $\overline{\overline{U(\beta)}} > \eta$ and let α be the η th. member of $U(\beta)$, under the natural ordering. Then we can get a contradiction as in the proof of theorem 3.17. □

Theorem 3.18 admits some generalisations, but this method does not seem to give results of the form $\forall \gamma \exists \beta (\overline{\overline{U(\beta)}} = \overline{\overline{E(\beta)}} + \gamma)$. The main trouble seems to be our lack of knowledge about the structure of $E(\beta)$ and we return to this problem in chapter 7.

Although $U(\beta)$ and $E(\beta)$ can be quite different, theorem 3.4 shows that $E(\beta) \subsetneq U(\beta)$. If there are ordinals β, γ, δ for which $R\beta \prec_1 R\gamma \prec R\delta$ then theorem 3.4 shows that $E(\beta)$ need not be an initial segment of $U(\beta)$, but this leads to our next question.

Question 3.19 Is $\{\alpha+\gamma \mid \gamma \in \beta \wedge R\beta \prec_\gamma R\alpha\}$ an initial segment of $U(\beta)$?

3.5 The smallest natural model of A

In this section we just note that there is a reasonable definition of a smallest natural model of A.

Definition 3.20 $\mathcal{R}_{M1}$ $[\mathcal{R}_{M2}]$ is that natural model of A determined by letting α_{M1} $[\beta_{M2}]$ be the least ordinal for which $\exists\beta \langle R\alpha_{M1}, R\beta, \in\rangle \models A$ $[\exists\alpha \langle R\alpha, R\beta_{M2}, \in\rangle \models A]$ and letting β_{M1} $[\alpha_{M2}]$ be the least ordinal for which $\langle R\alpha_{M1}, R\beta_{M1}, \in\rangle \models A$ $[\langle R\alpha_{M2}, R\beta_{M2}, \in\rangle \models A]$.

Theorem 3.21 $\mathcal{R}_{M1} = \mathcal{R}_{M2}$.

Proof From the definitions $\beta_{M2} \leqslant \beta_{M1}$ and $\alpha_{M1} \leqslant \alpha_{M2}$. Suppose that $\alpha_{M1} < \alpha_{M2}$ and then α_{M1} is definable in $R\alpha_{M2}$ so that from theorem 3.3 $\langle R\alpha_{M1}, R\beta_{M2}, \in\rangle \models A$, which contradicts the definition of α_{M2}. Thus $\alpha_{M1} = \alpha_{M2}$ and hence $\mathcal{R}_{M1} = \mathcal{R}_{M2}$. □

We now put $\mathcal{R}_M = \mathcal{R}_{M1} = \mathcal{R}_{M2}$ and we call $\mathcal{R}_M$ the smallest natural model of A. It is also straightforward to see that α_M, β_M are the least ordinals (again, it makes no difference which order these are taken in) for which $R\beta_M \prec R\alpha_M$.

$R\beta_M$ is appreciably larger than the smallest natural model of ZF and we can see this as follows. As $R\beta_M \prec R\alpha_M$, $R\beta_M$ is larger than the γ th. natural model of ZF for any $\gamma \in \beta_M$. Hence there are β_M natural models of ZF smaller than $R\beta_M$. But β_M is not the first natural model of ZF with this property as $\beta_M \notin Df(R\alpha_M, R\beta_M)$. Consequently, there are β_M smaller natural models of ZF each having this property, etc. etc.

Chapter 4

The set theory A^+

4.1 Background

The set theory A^+ was introduced in [56] as an alternative way of formalising Ackermann's principle that " the collection of all sets is not sharply delimited ". Reinhardt interpreted this by suggesting that there are different classes V_0 , V_1 , . . . all of which are possible candidates for " the class of all sets ". Then, to interpret the principle that " sharply delimited collections of sets are sets ", he suggested that if the extension of $\phi(V) \wedge t \in V$ is independent of which candidate V is, then $\phi(V) \wedge t \in V$ can be used as an abstraction term. Parameters in this expression are assumed to be sets, as in A.

In the formal theory A^+ only two possible classes of all sets are considered, and we have constants V and V' for them. The only predicate is $\in$. The axioms of A^+ are $WB^*(V)$ together with

$A4^+$ If ϕ has exactly four free variables, then

$$x,y \in V \wedge \forall t\, (\phi(V,x,y,t) \wedge t \in V \leftrightarrow \phi(V',x,y,t) \wedge t \in V') \rightarrow$$
$$\exists z \in V\, \forall t\, (t \in z \leftrightarrow t \in V \wedge \phi(V,x,y,t)),$$

A6 $V \subseteq V'$.

In [56] Reinhardt indicates that

(i) $A^* \subseteq A^+$,

(ii) $A^+ \vdash V \in V'$,

(iii) A^+ is consistent if A^* + the following schema of indescribability is consistent.

$$x,y \subseteq V \wedge \phi(V,x,y) \rightarrow \exists v \in V\, \phi(v, x \cap v, y \cap v) \qquad (1)$$

We shall use (i) and (ii) without explicitly mentioning them. (iii), together with the result of [58], shows that if ZF + ' there is a Ramsey cardinal ' is consistent, then so is A^+. However, it did not seem very likely that A^+ was much stronger than A and we confirm this in the next section.

For A, we can see that V cannot be definable in terms of $\in$, but this proof does not work for V' in A^+: we shall exploit this fact in the next section.

Theorem 4.1 shows that a bounded upward reflection principle (see section 3.2) is provable in A^+. This shows that V' cannot be defined using a certain type of expression and that V' must be ' quite a bit ' larger than V. The proof of the theorem is an extension of Reinhardt's proof of (ii).

<u>Theorem 4.1</u> $A^+ \vdash x,y \in V \wedge \phi(x,y,V) \rightarrow \exists z\ (V \in z \in V' \wedge z \subseteq V' \wedge \phi(x,y,z))$.

<u>Proof</u> We work in A^+. Suppose that $x,y \in V$ and $\phi(x,y,V)$ holds. Then, by considering the downward reflection scheme, we know that

$$V = \{t \mid \exists \beta \in V\ (\phi(x,y,R\beta) \wedge t \in R\beta \in V \wedge R\beta \subseteq V)\} \quad . \qquad (2)$$

If we also have

$$V = \{t \mid \exists \beta \in V'\ (\phi(x,y,R\beta) \wedge t \in R\beta \in V' \wedge R\beta \subseteq V')\} \quad , \qquad (3)$$

then from (2), (3) and $A4^+$ we obtain $V \in V$. This is a contradiction as it implies the existence of the Russell set. Hence there is a β for which $R\beta \not\subseteq V$ and $R\beta \in V' \wedge \phi(x,y,R\beta) \wedge R\beta \subseteq V'$. From the development of ordinal theory which is given in [56] we know that if $R\beta \not\subseteq V$, then $V \in R\beta$, so that $R\beta$ can be used as z in the conclusion of the theorem. $\Box$

<u>4.2</u> <u>The strength of A^+</u>

The main result of this section is corollary 4.7, which shows that the set theoretic part of A^+ (i.e. $A^+ \restriction V$) is precisely ZF.

On the way to this result we also show that if a theory is an extension of A^* with a definable class (i.e. definable by an $\in$ formula) which contains V, then the ($\in$,V)-theorems of this theory include those of A^+ (this is theorem 4.2); and that it is relatively consistent with A^* that there is a class x satisfying $\forall y\ y \in x$ (this is theorem 4.4, essentially). All Ackermann-type set theories are assumed to have the appropriate language in this section.

Theorem 4.2 If ψ is a formula with exactly one free variable, then $A^* + \forall x\ (x \in V' \leftrightarrow \psi(x)) + \forall x \in V\ \psi(x) \vdash A^+$.

Proof We need only show that $A4^+$ is derivable in the given theory and we do this as follows.

In the given theory we can replace $x \in V'$ by $\psi(x)$ and $V' \subseteq x$ by $\exists t\ (\forall x\ (x \in t \leftrightarrow \psi(x)) \wedge t \subseteq x)$. We suppose that the $\in$-formula obtained from $\phi(V',V)$ by such replacements is $\phi_\psi(V)$. Then an instance of the hypothesis of $A4^+$ becomes

$$x,y \in V \wedge \forall t\ (\phi(V,x,y,t) \wedge t \in V \leftrightarrow \phi_\psi(x,y,t) \wedge \psi(t)).$$

$\phi_\psi(x,y,t) \wedge \psi(t)$ is an $\in$-formula, $\eta(x,y,t)$ say, so that we have

$$x,y \in V \wedge \forall t\ (\eta(x,y,t) \rightarrow t \in V).$$

The conclusion of $A4^+$ then follows by applying A4 to the formula η, as required. □

Lemma 4.3 If ϕ is a sentence, T_0 is the theory with axioms A1-A4, and $\exists u\ u \in x \rightarrow x \notin x \wedge \exists u \in x \forall t \in u\ t \notin x$, and $T_0 \vdash \phi^V$ then $ZF \vdash \phi$.

Proof This result follows directly from the proof of theorem 1 of [38]. □

<u>Theorem 4.4</u> If T is the theory $A^* + \exists x \forall y\ y \in x$ and T_0 is consistent, then T is also consistent.

<u>Proof</u> We interpret T in T_0 and we use $\in$, V as the basic symbols of T and $\in_0$, V_0 as those of T_0. ϕ is the empty set of T_0, $\iota\phi$ is $\{\phi\}$ and $\iota^{n+1}\phi$ is $\{\iota^n\phi\}$: these sets being defined using $\in_0$. The membership relation is defined by

$$y \in \phi \text{ iff } y \notin_0 \phi,$$
$$y \in \iota^{n+1}\phi \text{ iff } y \in_0 \iota^n \phi,$$
$$y \in x \text{ iff } y \in_0 x, \text{ in all other cases.}$$

We also put $V = V_0 - (\{\phi\} \cup \{x \in_0 V_0 | \phi \in TC(x)\})$, where TC(x) is the transitive closure of x in T. The membership part of this interpretation is similar to an idea used in [11] .

It now remains to show that the interpretations of the axioms of T hold in T_0. The axiom of foundation guarentees this for A1, and we obviously have the interpretation of $\exists x \forall y\ y \in x$ holding. It is also clear that for every $\in$-formula ψ, there is an equivalent $\in_0$-formula, ψ_0 say, and that for every $(\in, V)$-formula Φ, there is an equivalent $(\in_0, V_0)$-formula, Φ_0 say.

To show that the interpretation of an instance of A2 holds for a formula Φ, we just need to use $(x \in V \wedge \Phi)_0$ in A2. Now suppose that $x \in V$ and we prove the interpretation of A3. If $y \in x$, then from the definition of V and as $TC(y) \subseteq TC(x)$ we get $y \in V$. If $y \subseteq x$, then $y \in_0 V_0$ by A3, $y \neq \phi$ (ϕ being the empty set defined using $\in_0$) as V_0 is not a member of any of its members and $TC(y) \subseteq TC(x)$. Hence we have $y \in V$ from the definition of V.

Now suppose that the hypothesis of A4 holds for a formula ψ,

i.e. $x, y \in V \wedge \forall t\ (\psi(x,y,t) \rightarrow t \in V)$. (4)

Then we also have $x, y \in_0 V_0 \wedge \forall t\ (\psi_0(x,y,t) \rightarrow t \in_0 V_0)$, so that by A4 $\exists z \in_0 V_0 \forall t\ (t \in_0 z \leftrightarrow \psi_0(x,y,t))$. If $z = \iota^n\phi$, then $\forall t\ (t \in_0 z \leftrightarrow t \in \iota^{n+1}\phi)$ so that the interpretation of the conclusion of A4 holds in this instance. Thus we can assume that $\forall t\,(t \in_0 z \leftrightarrow t \in z)$ and $z \neq \iota``\phi$. If $\phi \in TC(z)$, then for some $y \in_0 z$, $\phi \in TC(y)$ which contradicts (4). Hence $z \in V$ and the interpretation of A4 holds.

Similar, straightforward arguments show that the interpretation of A5 also holds. □

<u>Corollary 4.5</u> If ϕ is a sentence and $T \vdash \phi^{V}$, then $T_0 \vdash \phi$.

<u>Proof</u> By inspection of the construction used in the proof of the theorem, it is straightforward to check that there is a natural isomorphism between V and V_0, so that the corollary holds. □

<u>Theorem 4.6</u> If ZF is consistent, then A^{+} is also consistent.

<u>Proof</u> Suppose that ZF is consistent. Lemma 4.3 and theorem 4.4 then show that T is also consistent. We can use T in theorem 4.2 by taking ψ as $x = x$, so that, by that theorem, A^{+} is also consistent. □

<u>Corollary 4.7</u> If ϕ is a sentence, then $A^{+} \vdash \phi^{V}$ iff $ZF \vdash \phi$.

<u>Proof</u> By theorem 4.2, corollary 4.5 and lemma 4.3 we see that if $A^{+} \vdash \phi^{V}$, then $ZF \vdash \phi$. The converse follows from the main result of [56]. □

In [56], Reinhardt asked if $A^{+} + (1)$ is stronger than A^{+}. Theorem 4.6 answers this question positively, provided that ZF is consistent, as the existence of inaccessible cardinals, for instance, is derivable in $A^{+} + (1)$.

4.3 Reflection principles

In the last section we showed that it is relatively consistent with A^+ for V' to be definable in terms of $\in$, but our next theorem shows that V cannot be defined in terms of $\in$ and V'. This leads us to show (in theorems 4.9 and 4.10) that extended reflection principles are provable in A^+, in which V' can be used as a parameter. Then we show that A^+ can be axiomatised using the extended downward reflection principle in place of $A4^+$. This is analogous to the situation in A, where the corresponding result was proved in [40] .

Theorem 4.8 If ψ is a formula with exactly four free variables, then $A^+ \vdash x,y \in V \rightarrow \neg \forall t \, (t \in V \leftrightarrow \psi(x,y,V',t))$.

Proof Suppose that $x,y \in V$ and that for a suitable ψ

$$\forall t \, (t \in V \leftrightarrow \psi(x,y,V',t)). \qquad (5)$$

Firstly we will show that

$$\forall t \in V \; \psi(x,y,V,t). \qquad (6)$$

Suppose that (6) does not hold so that for some $t' \in V$, we have $\neg \psi(x,y,V,t')$. Now consider the formula

$$\chi(x,y,X,t',t) = (\neg \psi(x,y,X,t') \wedge t \in X) \vee (\psi(x,y,X,t') \wedge \psi(x,y,X,t)).$$

Then

$\chi(x,y,V,t',t) \leftrightarrow t \in V$ follows from (6),

$\chi(x,y,V',t',t) \leftrightarrow t \in V$ follows from (5).

Using $A4^+$ with $\chi(x,y,V,t',t) \wedge t \in V$ gives $V \in V$, a contradictiion, so that we know (6) holds. Then

$\psi(x,y,V,t) \wedge t \in V \leftrightarrow t \in V$ follows from (6),

$\psi(x,y,V',t) \wedge t \in V' \leftrightarrow t \in V$ follows from (5).

Using $A4^+$ again we get a contradiction. Hence (5) is false and the theorem holds. $\square$

Theorem 4.9 Extended downward reflection.

If ϕ is a formula with exactly four free variables, then

$A^{+} \vdash x,y \in V \wedge \phi(x,y,V,V') \rightarrow \exists z \in V\ \phi(x,y,z,V')$.

Proof Suppose that $x,y \in V$, $\phi(x,y,V,V')$ and $\neg \exists z \in V\ \phi(x,y,z,V')$. Then, by the usual theory of odrinals in A^{*}, we have

$t \in V \leftrightarrow$ ' $t \in R\alpha$, where α is the least ordinal for which

$x,y \in R\alpha$ and $\phi(x,y,R\alpha,V')$ '

$\leftrightarrow \psi(x,y,V',t)$, say.

This contradicts theorem 4.8, so that the result holds. □

Theorem 4.10 Extended upward reflection.

If ϕ is a formula with exactly four free variables, then

$A^{+} \vdash x,y \in V \wedge \phi(x,y\ V,V') \rightarrow \exists z\ (\ V \in z \in V' \wedge z \subseteq V' \wedge \phi(x,y,z,V')\ )$.

Proof Suppose that $x,y \in V$, $\phi(x,y,V,V')$ and $\neg \exists z\ (\ V \in z \in V' \wedge$ $z \subseteq V' \wedge \phi(x,y,z,V')$. Then,as in the proof of the last theorem, we have

$t \in V \leftrightarrow$ ' $t \in R\alpha$, where α is the supremum of those ordinals

for which $x,y \in R\alpha$ and $R\alpha \in V'$ $R\alpha \subsetneq V' \wedge \phi(x,y,R\alpha,V')$.

The result then follows from theorem 4.8. □

Theorem 4.11 A^{+} can be axiomatised using the extended downward reflection principle in place of $A4^{+}$.

Proof We need only show that $A4^{+}$ is provable from the other axioms of A^{+} and the extended downward reflection scheme. Suppose that an instance of the hypothesis of A4 holds, i.e. $x,y \in V \wedge \forall t\ (\phi(x,y,V,t) \wedge t \in V \leftrightarrow \phi(x,y,V',t) \wedge t \in V'\ )$. Then, by A2, $\exists z \subseteq V \forall t\ (\ t \in z \leftrightarrow t \in V' \wedge \phi(x,y,V',t)\ )$. Applying extended downward reflection to this formula gives

$$\exists w \in V \exists z \subseteq w \forall t\ (\ t \in z \leftrightarrow t \in V' \wedge \phi(x,y,V',t)\ ).$$

Hence, by A3,

$$\exists z \in V \forall t\ (\ t \in z \leftrightarrow t \in V' \wedge \phi(x,y,V',t)\ ),$$

and we immediately get the conclusion of $A4^{+}$ from this. □

The last theorem in this section gives another alternative axiomatisation of A^{+}. This one is more akin to the original system A. It shows that the only additional assumption in A^{+} is that there is a class containing V which can be used as a parameter in A4.

Theorem 4.12 A^{+} can be axiomatised using the following schema in place of $A4^{+}$.

If ϕ is a formula with exactly four free variables, then

$$x,y \in V \wedge \forall t\ (\ \phi(x,y,V',t) \rightarrow t \in V\ ) \rightarrow \exists z \in V \forall t\ (\ t \in z \leftrightarrow \phi(x,y,V',t)).$$

Proof Firstly, we show that the above schema is provable in A^{+}. Suppose that $x,y \in V \wedge \forall t\ (\ \phi(x,y,V',t) \rightarrow t \in V\ )$, and then by A2 $\exists z \subseteq V \forall t\ (t \in z \leftrightarrow \phi(x,y,V',t)$. The result then follows as in the proof of theorem 4.11.

Now suppose that this schema holds, and we prove $A4^{+}$. Suppose that the hypothesis of $A4^{+}$ holds for a suitable formula ψ. Then we just need to apply this schema to the formula $\psi(x,y,V',t) \wedge t \in V'$. □

4.4 Natural models

Natural models of A^{+} are models of this theory which are of the form $\langle R\alpha, R\beta, R\gamma, \in \restriction R\alpha \rangle$, where $R\alpha$ is the domain of the model, $R\beta$ is the interpretation of V' and $R\gamma$ is the interpretation of V. We extend our conventions about natural models of A to those of A^{+} and we use δ for the structure $\langle R\alpha, R\beta, R\gamma, \in \rangle$, in which we assume $\alpha > \beta > \gamma$.

Our first theorem is directly analogous to theorem 3.1, and theorem 4.14 gives a precise characterisation of the natural models of A^+. We shall not include full details of absoluteness considerations in the proofs of these results.

<u>Theorem 4.13</u> If $\mathfrak{A} = \langle B, U', U, E \rangle \models A^+$, then U is not definable in $\langle B, U', U, E \rangle$ by an $\in$-formula with parameters from $U \cup \{U'\}$.

<u>Proof</u> Suppose that the hypothesis of the theorem holds, but that the conclusion is false. Then, for some elements x,y of U and some formula ϕ, we have

$$t \in U \quad \text{iff} \quad \mathfrak{A} \models \phi(x,y,t,U'), \qquad (7)$$

where all free variables are shown and we confuse objects with their names. Then $\mathfrak{A} \models x,y \in V \wedge \forall t\ (\ \phi(x,y,V',t) \rightarrow t \in V\ )$, so that, by theorem 4.12, we get

$$\mathfrak{A} \models x,y \in V \wedge \exists z \in V \forall t\ (\ t \in z \leftrightarrow \phi(x,y,V',t)\ ).$$

From (7) we then see that $\mathfrak{A} \models V \in V$, which is impossible. $\Box$

<u>Theorem 4.14</u> If $R\gamma \notin Df(R\alpha, R\gamma \cup \{R\beta\}\ )$ and $\alpha > \beta > \gamma$, then $\mathcal{S} \models A$.

<u>Proof</u> Suppose that $\alpha > \beta > \gamma$ and $R\gamma \notin Df(R\alpha, R\gamma \cup \{R\beta\}\ )$. It is clear that we need only show that $A4^+$ holds in $\mathcal{S}$. Hence, by theorem 4.11, it suffices to show that the extended downward reflection principle holds in $\mathcal{S}$.

Suppose that α is a limit ordinal and that for some $x,y \in R\gamma$, $\mathcal{S} \models \phi(x,y,V,V')$. Then if $\mathcal{S} \not\models \exists z \in V \phi(x,y,z,V')$ we would have $\mathcal{S} \models$ ' $V = R\delta$, where δ is the least ordinal for which $x,y \in R\delta$ and $\phi(x,y,R\delta,V')$ '.

This contradicts our assumption so the result holds when α is a limit ordinal. If $\alpha = \omega\xi + n$ then it is straightforward to modify the above proof by relativising $V = R\gamma$ to $R\omega\xi$. □

Corollary 4.15 If $\langle R\alpha, R\delta, \in\rangle \models A$, $\alpha > \beta > \delta$ and $\beta \in Df(R\alpha, R\delta)$, then $\delta \vdash A^+$.

Proof This follows directly from theorem 3.2 and the theorem. □

Our next result is analogous to part of theorem 3.3. Theorem 4.17 extends the method of obtaining natural models of A, which we introduced in the last chapter, to those of A^+.

Theorem 4.16 If $\delta \models A^+$, $\beta < \alpha' < \alpha$ and $\alpha' \in Df(R\alpha, R\gamma)$, then $\langle R\alpha', R\beta, R\gamma, \in\rangle \models A^+$.

Proof Suppose that the hypothesis of the theorem holds while the conclusion fails. Then $R\gamma \in Df(R\alpha', R\gamma \cup \{R\beta\})$ so that $R\delta \in Df(R\alpha, R\gamma \cup \{R\beta\})$, as $\alpha' \in Df(R\alpha, R\delta)$. This contradicts theorem 4.14. □

Theorem 4.17 If κ is an inaccessible cardinal and $\alpha > \beta \geq \kappa$, then there are κ many ordinals $\gamma < \kappa$ for which $\delta \models A^+$.

Proof This follows from theorems 3.7 and 4.14. □

The original idea behind A^+ was that V' would be an alternative candidate for the class of all sets. However, theorem 4.17 enables us to construct natural models of A^+ in which V' can be any $R\beta$, where β is greater than the first inaccessible cardinal. We shall consider other ways of formalising the notion of a class being an alternative candidate for the class of all sets in chapter 5.

Next we shall compare the natural models of A^+ with those of A.

Theorem 4.18 If $\mathcal{S} \models A^+$, then $\langle R\beta, R\gamma, \in \rangle \models A$.

Proof Suppose that $\mathcal{S} \models A^+$. Then $R\gamma \notin Df(R\alpha, R\gamma \cup \{R\beta\})$, from theorem 4.12, and hence $R\gamma \notin Df(R\beta, R\gamma)$, as required. □

If $\langle R\alpha, R\beta, \in \rangle \models A$ and $\langle R\alpha, R\beta', \in \rangle \models A$ where $\beta' > \beta$ then it might seem plausible that $\langle R\alpha, R\beta', R\beta, \in \rangle \models A^+$, but we next give a counterexample to this.

Let $R\alpha, R\beta, R\gamma$ be the first three natural models which are elementary substructures of $R\kappa$, where κ is the first inaccessible cardinal. Then $\langle R\gamma, R\beta, \in \rangle \models A$ and $\langle R\gamma, R\alpha, \in \rangle \models A$ as $R\alpha \prec R\beta \prec R\gamma$. However, $R\alpha \in Df(R\gamma, R\alpha \cup \{R\beta\})$ as α is " the largest ordinal δ for which $R\delta \prec R\beta$ ", so that by theorem 4.13 $\langle R\gamma, R\beta, R\alpha, \in \rangle \not\models A^+$.

Consideration of the smallest natural model of A shows that there are set universes which occur in natural models of A but not in those of A^+. In theorem 4.20 we note that the smallest natural set universes of various theories form a strictly increase sequence. It is amusing to note that the provable set theoretic statements of all the theories mentioned in that theorem are the same, so that from a natural model point of view what constitutes a smallest natural set universe depends heavily on the formalism used.

Definition 4.19 The smallest natural set universe of an appropriate set theory is $R\alpha$, where α is the least ordinal for which there is a natural model of the theory in which $R\alpha$ is the class of all sets (or domain, if the theory has no proper classes).

<u>Theorem 4.20</u> The smallest natural set universes of the theories ZF, A, A^+, and NBG form a strictly increasing sequence of sets.

<u>Proof</u> In section 3.5 we noted this result for ZF and A. Let $R\gamma_M$ be the smallest natural set universe of A^+. Then if $\langle R\alpha, R\beta, R\gamma_M, \in\rangle \models A^+$, theorem 4.18 shows that $\langle R\beta, R\gamma_M, \in\rangle \models A$. Hence $\mathcal{R}_M$, the smallest natural model of A, will be definable in any natural model of A^+ so that $\beta_M < \gamma_M$. Now the second part of the theorem holds. Theorem 4.17 shows that γ_M is an accessible ordinal. It is well known that the smallest natural set universe of NBG is $R\kappa$, where κ is the first inaccessible cardinal, so that the last part also holds. $\square$

Chapter 5

Some Ackermann-type set theories

5.1 AT1

In section 4.1 we discussed the motivation behind the theory A^+. Some of the later results of chapter 4 suggest that the objectives of A^+ have not been reached in the formal theory. AT1 is an Ackermann-type set theory which is based on only part of the intuition which led to A^+. We again suppose that V and V' are alternative candidates for the class of all sets, both of them being models of a Zermelo-type theory so that they are much " larger " than all of their members. Consequently, we include $WB^*(V)$ among the axioms of AT1.

To express the idea that V and V' are equally good choices from the point of view of $\in$-formulae we assume that they have the same $\in$-properties. As it is natural to allow parameters from V in this schema, it becomes

$$x,y \in V \rightarrow (\phi(x,y,V) \leftrightarrow \phi(x,y,V')).$$

Finally, we suppose that $V \in V'$ as V' is the larger candidate. We have not considered formalising Ackermann's idea that " well determined collections of sets are sets " within this system.

Definition 5.1 AT1 is a theory with language $\in$, V, V'. Its axioms are $WB^*(V)$, $V \in V'$ and the schema

(AT1) If ϕ is a formula with exactly three free variables, then

$$x,y \in V \rightarrow (\phi(x,y,V) \leftrightarrow \phi(x,y,V')).$$

Next we note that $A^+ \subsetneq$ AT1. This result had previously been observed by Reinhardt in a different context.

Theorem 5.2 If $A^+ \vdash \phi(V,V')$ then AT1 $\vdash \phi(V,V')$.

Proof We need only show that $A4^+$ is provable in AT1, so we suppose that $x,y \in V \wedge \forall t\ (\phi(x,y,V,t) \wedge t \in V \leftrightarrow \phi(x,y,V',t) \wedge t \in V')$.
Then, by A2, $\exists z \subseteq V \forall t\ (t \in z \leftrightarrow \phi(x,y,V',t) \wedge t \in V')$. By A3 and (AT1) we have $z \subseteq V \in V' \rightarrow z \in V'$, and hence we have

$$\exists z \in V' \forall t\ (t \in z \leftrightarrow \phi(x,y,V',t) \wedge t \in V').$$

Applying (AT1) to this sentence gives the required result. □

From theorem 5.2 we know, very indirectly, that ZF^V can be derived in AT1. Theorem 5.3 shows that this can be proved straightforwardly, without using the axiom of foundation, and that the existence of arbitarily large natural models of ZF is derivable in AT1. We find it interesting that the motivation behind AT1 leads to the latter result.

Theorem 5.3 (i) If ϕ is a sentence and $ZF \vdash \phi$, then $AT1 \vdash \phi^V$.
(ii) $AT1 \vdash (\forall \alpha\ \exists \beta > \alpha R\beta \models ZF)$, and further $AT1 \vdash \forall \alpha \in V\ V \prec_\alpha V'$.

Proof (i) If ϕ is a formula with exactly one free variable and $A \vdash \phi(V)$, then from theorem 5.2 $AT1 \vdash \phi(V)$. Hence, by the easy proofs given by Ackermann in [1], it only remains to show that replacement holds when relativised to V. Suppose that $x,y \in V \wedge \forall u \in V \exists! t \in V\ \phi^V(x,y,u,t)$ and then, as in the proof of theorem 5.2, $\exists z \in V' \forall t\ (t \in z \leftrightarrow \exists u \in x\ \phi^V)$. From (AT1) we have $\phi^V \leftrightarrow \phi^{V'}$ so that replacing ϕ^V by $\phi^{V'}$ and using (AT1) again we get the required result.

(ii) We can work in classes " above " V' in exactly the same way that we worked in classes above V in A^*. Formalising the proof of (i) in such a class shows that $V \models ZF$ can be derived in AT1. Hence the first result follows by the downward reflection principle. To obtain the second result, we just need to work an appropriate distance above V' and relativise in the obvious way. □

As $V \prec_{\alpha} V'$, for all $\alpha \in V$, it seems quite likely that it will be difficult to give an extension of ZF which is the set theoretic part of AT1, for we do not know how strong the condition $R\alpha \prec_1 R\beta$ is. Another question is

Question 5.4 Is strong replacement provable in AT1 ?

Theorems 5.2 and 4.1 show that bounded upward reflection is provable in AT1. The next theorem shows that, analogously to theorem 2.8, there is no proof-theoretic weakening of AT1 by our considering only two possible set universes. Consequently, we suggest that AT1 is an improvement on A^{*}_{∞} , as well as on A^{+} .

Theorem 5.5 If V_i are constants (for $i \in \alpha$) and we add to $WB^{*}(V)$ the axioms $i < j < \alpha \rightarrow V \in V_i \in V_j$ and $x,y \in V \rightarrow (\phi(x,y,V) \leftrightarrow \phi(x,y,V_i))$ (for $i \in \alpha$ and ϕ any formula with 3 free variables) then this theory $\vdash \psi(V,V_0)$ iff $AT1 \vdash \psi(V,V')$.

Proof (Outline) Suppose that the hypothesis of the theorem holds for a formula ψ. Let $i_0 < i_1 \ldots i_n$ be the indices of the V s which occur in the axioms used in a proof of $\psi(V,V_0)$. The upward reflection principle holds for $V_{i_{n-1}}$ so the instances of those axioms akin to (AT1) involving V_{i_n} can be replaced by those which involve only $V_{i_{n-1}}$. Iterating this process as in the proof of theorem 4.5 in [56] gives the result. □

Definition 5.6 AIR is the theory A^* augmented by the following schema of indescribable replacement.

(IR) If θ is a formula with exactly three free variables, then

$$x,y \subseteq V \wedge \theta(V,x,y) \rightarrow \exists v \in V\, \theta(v, x \cap v, y \cap v).$$

Theorem 5.7 shows that the consistency of AIR implies the consistency of AT1, so that by the result of [58], if ZF + there is a Ramsey cardinal is consistent, then so is AT1. The proof which we give is due to Reinhardt and we include it as it has not been published.

Theorem 5.7 If AIR is consistent, then so is AT1.

Proof By compactness and consideration of the axioms of A^* it suffices to show that if ϕ is a formula with exactly two free variables, then AIR $\vdash \exists v \in V \forall t \in v\, (\, \phi(t,v) \leftrightarrow \phi(t,V)\,)$. To show this, note that $\exists x \subseteq V \forall t \in V\, (\, \phi(V,t) \leftrightarrow t \in x\,)$, and applying (IR) with x as a parameter we get $\exists v \in V \forall t \in v\, (\, \phi(v,t) \leftrightarrow t \in x \cap v\,)$, as required. □

5.2 AT2

In A4 the parameters must be sets although it seems natural to allow certain classes as well. Such classes cannot be too " close " to V (e.g. not V-3) or have a structure from which V can be extracted (e.g. not $\alpha \cup \{V\}$, for any ordinal α). In this section we suggest one way of approaching this idea and we restrict our attention to classes $x \subseteq V$. Then if $\phi(y,t) \rightarrow t \in V$, for all $y \supseteq x$, we allow $\phi(x,t)$ to be used as an abstraction term. This was partially inspired by Poincaré's notion of predicativity, although it certainly does not follow from it.

The formal system AT2 is set up as an extension of $WB^*(V)$.

Its only other axiom schema is as follows. We include parameters from V for convenience.

CA. If ϕ is a formula with exactly four free variables, then

$$x,y\in V \wedge p \subseteq V \wedge \forall q\ (p\subseteq q \rightarrow \forall t(\phi(x,y,q,t) \rightarrow t\in V\ )\ ) \rightarrow$$
$$\exists z\in V\forall t\ (\ t\in z \leftrightarrow \phi(x,y,p,t)\ ).$$

The next result shows that strong replacement is straightforwardly provable in AT2 and theorem 5.9 strengthens this.

<u>Theorem 5.8</u> Strong replacement for V is provable in AT2.

<u>Proof</u> Suppose that for some $p \subseteq V$ we have $x\in V \wedge \forall u\in x\, \exists! v\, \langle u,v\rangle \in p$. Let $\phi(x,p,t)$ be the formula

$$\forall u\in x\ \exists! v\ \langle u,v\rangle \in p \wedge \exists u\in x\ \langle u,t\rangle \in p,$$

and now suppose that $p \subseteq q$ and $\phi(x,q,t)$. Then $\forall u\in x\forall v(\langle u,v\rangle \in p \leftrightarrow \langle u,v\rangle \in q)$ so that $t\in V$ and the hypothesis of CA is satisfied. The conclusion of CA shows that strong replacement for V holds. $\square$

<u>Theorem 5.9</u> AIR $\subseteq$AT2.

<u>Proof</u> We need only show that IR is provable in AT2 and it clearly suffices to prove a version of IR which has only one parameter. Suppose that $x \subseteq V \wedge \theta(V,x)$ holds. We may then also suppose that $x \notin V$, as otherwise the result follows from the downward reflection scheme. Now suppose that the conclusion of IR is false, so that

$$\neg \exists \alpha \in V\ \ (R\alpha ,x \cap R\alpha).$$

Put $y = \{ \langle \alpha , \{\{z\}\}\rangle \mid \alpha \in V \wedge z = R\alpha \cap x\}$ and let $\psi(x,y)$ be a formula which asserts that

$$\exists \delta (y : \delta \rightarrow \cdot \wedge \forall \beta \in \delta \; \exists z \subseteq R\beta \;\; y(\beta) = \{\{z\}\} \wedge$$
$$x = \cup \{ z \mid \exists \beta \in \delta \;\; y(\beta) = \{\{z\}\} \; \} \;).$$

Then we have $\psi(x,y) \wedge \phi(V,x)$. Let $\theta(q,t)$ be the formula $\exists x' \; (\psi(x',q) \wedge \forall \gamma (\phi(R\gamma , x' \cap R\gamma) \rightarrow t \in R\gamma) \;)$, and we next prove

$$\forall q \; (\; y \subseteq q \rightarrow \forall t \; (\; \theta(q,t) \rightarrow t \in V \;) \;). \qquad (*)$$

Suppose that $y \subseteq q$ and $\theta(q,t)$ hold. Then $\exists ! x' \psi(x',t)$, and let x' be this set. q is a function from an ordinal δ and $y \subseteq q$ so that if $V = R\alpha$, we have $\alpha \leqslant \delta$. From the definition of ψ it is also clear that $x' \cap R\alpha = y$ so that $\phi(R\alpha , x' \cap R\alpha)$ holds. Hence $t \in V$ and $(*)$ holds.

Using CA and $(*)$ we obtain

$$\exists z \in V \forall t \; (\; t \in z \leftrightarrow \theta(x,t) \;). \qquad (**)$$

From our assumption that $\forall \gamma \in V \neg \phi(R\gamma , x \cap R\gamma)$ and the definition of θ we get $\theta(x,t) \leftrightarrow t \in V$. Hence, from $(**)$, $V \in V$ which is a contradiction. □

Theorem 5.11 gives a weak relative consistency result for AT2, but we leave open the next question.

<u>Question 5.10</u> How strong is AT2 ?

<u>Theorem 5.11</u> Suppose that α and β are ordinals with $\alpha > \beta$, and that there is an elementary embedding $j : R\beta + 1 \rightarrow R\alpha + 1$ which is fixed on $R\beta$ and has $j(\beta) = \alpha$. Then $\langle R\alpha + 1, R\beta , \in \rangle \models$ AT2.

<u>Proof</u> Let α, β, j be as in the statement of the theorem. From theorem 3.4 we need only show that CA holds in $\langle R\alpha + 1, R\beta , \in \rangle$. Suppose that, in this model, $x \subseteq V$ and $\forall p \; (\; x \subseteq p \rightarrow \forall t \; (\; \phi(q,t) \rightarrow t \in V \;) \;)$ (we are ignoring parameters

from V for convenience). Then $x \subseteq j(x)$ and, in the model, we have $\forall t\ (\phi(j(x),t) \rightarrow t \in V)$. Comprehension then gives $\{t \mid \phi(j(x),t)\} \in R\beta+1 \subseteq R\alpha = j(V)$, and as j is an elementary embedding, we obtain $\{t \mid \phi(x,t)\} \in V$, as required. □

5.3 Weak forms of Ackermann-type set theories

In the original motivation for A it was asserted that the notion of a set is not " sharply delimited " and one way of interpreting this would be to ensure that there are collections which may or may not be sets. Consequently, there may be no class of all sets and we shall therefore return to Ackermann's original formulation of A with the predicate M(x) for " x is a set " instead of the constant V, in this section. To formalise the above ideas, or doubts, we would have to drop the law of the excluded middle for formulae involving M and to alter A2 so that it only applied to subclasses of sets. However, theorem 5.13 shows that such a theory would be very weak as A^-, a theory in which it would be contained, cannot even prove one version of the axiom of infinity.

Definition 5.12 A^- is a theory set up in the predicate calculus using $\in$ for membership and M(x) for " x is a set ". Its axioms are A1, A3 and A4, where V is replaced by M in the obvious way, together with the following weakened form of A2.

$A2^-$ $\quad M(x) \rightarrow \exists z \forall t\ (t \in z \leftrightarrow t \in x \wedge \Sigma)$.

Theorem 5.13 If ZF is consistent then $A^- \nvdash (\mathrm{Inf})^M$, where Inf is the sentence $\exists x\ (\phi \in x \wedge \forall y \in x\ y \cup \{y\} \in x)$.

Proof Let $\langle B,E \rangle$ be a model of $Z + \forall x \exists \alpha\, x \in R\alpha$, in which the natural numbers are non-standard. We extend this to a model of A^- by taking B as the domain, E as membership and M(x) holding iff x has rank n in the model where n is in the isomorphic copy of the true natural numbers which is an initial segment of the ordinals in $\langle B,E\rangle$. A1 and A3 are clearly true in this model and A2$^-$ holds as all subsets of the (externally) finite sets will be in the model. To show that A4 holds suppose that

$$M(x) \wedge M(y) \wedge \forall t \; (\phi(x,y,t) \rightarrow M(t)). \qquad (*)$$

If there are only finitely many t's which satisfy $\phi(x,y,t)$ then then there is obviously a set in the model which satisfies the conclusion of A4. Otherwise, the overspill lemma shows that there must be an infinite natural number in the model which is the rank of a t which satisfies ϕ. This contradicts (*) so that A4 holds.

As all the sets for which M(x) holds are finite, it is clear that $(\mathrm{Inf})^M$ is false in the model, as required. □

Next, we briefly consider alternative developments of AT1 in which V and V' are not assumed to be " much larger than all of their elements ". Here, not all members of V could be used as parameters in (AT1), but theorem 5.15 shows that if none are allowed we again get a very weak theory. It still might be possible to find a natural way of distinguishing suitable sets, however.

Definition 5.14 AT1$^-$ is the theory with axioms A1, A2, A3, $V \in V'$ and $\phi(V) \leftrightarrow \phi(V')$, for all formulae ϕ with exactly one free variable.

Theorem 5.15 If ZF is consistent, then AT1$^- \nvdash$ Inf.

Proof We produce a model of A1, A2, A3 and a finite number of instances of $\phi(V) \leftrightarrow \phi(V')$ in which V is Rn, V' is Rm for some $n \in m \in \omega$, and the domain is $R\omega$. The result will then follow from the compactness theorem.

Clearly A1, A2, A3 and $V \in V'$ will hold in such a model. Suppose that the given instances of the schema contain the formulae $\phi_0(x), \ldots \phi_{n-1}(x)$. Then at least one of $\{m \in \omega \mid R\omega \models \phi_0(Rm)\}$ and $\{m \in \omega \mid R\omega \models \neg \phi_0(Rm)\}$ is infinite and let A_0 be one of them which is infinite. Then at least one of $\{m \in A_0 \mid R\omega \models \phi_1(Rm)\}$ and $\{m \in A_0 \mid R\omega \models \neg \phi_1(Rm)$ is infinite and let A_1 be one of them which is infinite. Continuing this process we get an infinite set A_{n-1} with the property that $\forall i < n \forall m, m' \in A_{n-1}(R\omega \models \phi_i(Rm) \leftrightarrow R\omega \models \phi_i(Rm'))$. Choosing two numbers from A_{n-1} then gives us a model with the required property. □

Chapter 6

Some isolated results

6.1 Permutation models of Λ

In this section we extend some of the work on permutation models of ZF (see [14] , for instance) to models of Λ. This suggests that it should be possible to extend much of the Frankel-Mostowski machinery to models of Λ but, at the moment, we do not think that this would give sufficiently interesting results to be worthwhile. The main result of this section is theorem 6.2 and we use it to answer a question raised in [56] .

Definition 6.1 A functional formula, $y = F(x)$, is said to be a permutation if it represents a bijection of the universe onto itself. If $y = F(x)$ is a permutation then we write $x \in_F y$ for $F(x) \in y$ and Φ_F for the formula Φ with all instances of $\in$ replaced by $\in_F$.

Theorem 6.2 If $y = F(x)$ is a functional $\in$-formula such that

(i) F is a permutation,

(ii) $x \in V$ iff $F(x) \in V$,

then we can interpret Λ in Λ using $\in_F$ for the membership relation and V as V.

Proof Firstly note that $x \in_F V \leftrightarrow F(x) \in V \leftrightarrow x \in V$, and we often use this in showing that the interpretations of the axioms hold.

$(A1)_F$ Extensionality holds as F is a bijection.

$(A2)_F$ We show that $\exists z \forall t\, (t \in_F z \leftrightarrow t \in V \wedge \Phi_F)$. By A2 and (ii) $\exists z \forall p\, (p \in z \leftrightarrow \exists t \in V (p = F(t) \wedge \Phi_F))$. Then $t \in_F z \leftrightarrow F(t) \in z \leftrightarrow t \in V \wedge \Phi_F$, as required.

$(A3)_F$ Firstly we show that $x \in_F y \in V \to x \in V$.

$x \in_F y \to F(x) \in y \to F(x) \in V \to x \in V$, by A3 and (ii). Now it remains to show that $\forall t\ (\ t \in_F x \to t \in_F y\ ) \wedge y \in V \to x \in V$.

$$\forall t\ (\ t \in_F x \to t \in_F y\ ) \to \forall t\ (\ F(t) \in x \to F(t) \in y\ ) \quad \text{by definition,}$$

$$\to x \subseteq y \quad \text{by (ii),}$$

$$\to x \in V \quad \text{by A3.}$$

$(A4)_F$ Suppose that $x,y \in V$ and $\forall t\ (\ \phi_F(x,y,t) \to t \in V\ )$, and we show that $\exists z \in V \forall t\ (\ t \in_F z \leftrightarrow \phi_F(x,y,t)\ )$. Let $\psi(x,y,z)$ be the formula $\exists t\ (\ p = F(t) \wedge \phi_F(x,y,t)\ )$ and then from (ii) we get $\forall p\ (\ \psi(x,y,p) \to p \in V\ )$. Then by A4 $\exists z \in V \forall p\ (\ p \in z \leftrightarrow \psi(x,y,p)\ )$. Now $t \in_F z \leftrightarrow F(t) \in z \leftrightarrow \phi_F(x,y,t)$ so that $(A4)_F$ holds. □

Corollary 6.3 If $y = F(x)$ satisfies the conditions of the theorem then we can interpret $A + ZF^{\circ}$ in $A + ZF^{\circ}$ using $\in_F$ for membership and V for V.

Proof This follows directly from the theorem and the usual result for ZF° which is proved in [14] , for instance. □

Question 4.24(c) of [56] asks

" If we add the following schema of downward reflection to A, then do we get A^* ?

DR If ϕ has exactly two free variables, then

$$y \in V \wedge \phi(V,y) \to \exists x \in V \phi(x,y). "$$

We shall answer this question negatively, provided that ZF is consistent, by interpreting A + DR in A + DR in such a way that the interpretation of A5 fails.

Let $y = F(x)$ be a functional $\in$-formula which says that

$$F(72) = \{72\},$$
$$F(\{72\}) = 72,$$
$$F(x) = x, \text{ otherwise.}$$

F obviously satisfies the hypothesis of theorem 6.2 so that result shows we can interpret A in A+DR using $\in_F$ for membership. An instance of $(DR)_F$ becomes

$$y \in V \wedge \phi_F(V,y) \rightarrow \exists x \in V\ \phi_F(x,y).$$

This is just another instance of DR so that we can interpret A+DR in A+DR using $\in_F$ for membership. The interpretation of A5 does not hold as $x \in_F \{72\} \leftrightarrow F(x) \in \{72\} \leftrightarrow F(x) = 72 \leftrightarrow x = \{72\}$, as required.

6.2 Real classes

In a first order theory in which all the objects are sets questions about arbitary subcollections of sets do not really arise. However, when we add classes to the theory it becomes conceivable, from some points of view, that there are subclasses of sets which are not sets. This possibility has been considered for NBG in [70] and we indicate, in this section, how it is possible to set up an analogous system for A^*.

It is natural to consider real classes in such a system, where a real class is one which intersects all sets in a set. This is a problem for producing an Ackermann-type theory as the notion of a real class is defined in terms of V and so cannot be used in A4. Our formal theory gets round this by introducing a new predicate for real classes and a further predicate for hereditary real classes. The latter notion cannot be defined by recursion as we are not guarenteed any structure above V.

The axioms of the theory are obvious modifications of those of A and the condition which replaces $x \subseteq y \in V \rightarrow x \in V$ is intended to say that subclasses of sets which are defined by reference to nicely behaved classes are themselves sets.

Definition 6.4 The theory R has the language of A extended by two unary predicates: Rc(x) for " x is a real class " and HRc(x) for " x is a hereditary real class ". Its axioms are

R0 $Rc(x) \leftrightarrow \forall y \in V \exists z \in V \forall t\ (t \in z \leftrightarrow t \in x \wedge t \in y)$

$HRc(x) \leftrightarrow Rc(x) \wedge \forall y \in x\ HRc(x)$,

R1 $\forall t\ (t \in x \leftrightarrow t \in y) \rightarrow x = y$,

R2 If Φ is any formula not involving z, then $\exists z \forall t\ (t \in z \leftrightarrow t \in V \wedge \Phi)$,

R3 $x \in y \in V \rightarrow x \in V$

If Φ has exactly three free variables, then

$Rc(x) \wedge Rc(y) \wedge z \in V \wedge \{t \in z \mid \Phi^{Rc}(x,y,t)\} \in V$,

R4 If Φ is any formula with exactly three free variables which does not involve V, then

$x,y \in V \wedge \forall t\ (\Phi(x,y,t) \rightarrow t \in V) \rightarrow \exists z \in V \forall t\ (t \in z \leftrightarrow \Phi(x,y,t))$,

R5 $x \in V \wedge \exists y\ y \in x \rightarrow \exists y \in x \forall z \in y\ z \notin x$.

Theorem 6.5 If Φ is an $\in$,V-sentence, then $R + \forall x Rc(x) \vdash \Phi$ iff $A^* \vdash \Phi$.

Proof This follows directly from considering the axioms. □

Theorem 6.5 shows that R can be viewed as a refinement of A^* as one would expect. It is straightforward to prove a number of of elementary results in R (similar to theorem 1.6 in [56]) and, using these, we can prove the next theorem which shows that the hereditary real classes form an inner model for A^* . The only surprising fact about this is that R0 gives us sufficient information about HRc.

Theorem 6.6 If Φ is an $\in$,V-sentence, then $R \vdash \Phi^{HRc}$ iff $A^* \vdash \Phi$.

Proof Omitted, due to length and lack of originality. □

6.3 Modified abstraction principles

On page 428 of [71] Wang briefly considers Ackermann's set theory and he suggests that it might be possible to allow any formula to occur in A4 if we modify the axiom to

$$A4_W \quad x,y \in V \wedge \forall t\ (\Phi(x,y,t) \rightarrow t \in V) \wedge \exists t \in V\ \neg\Phi(x,y,t) \rightarrow$$
$$\exists z \in V\ \forall t\ (t \in z \leftrightarrow \Phi(x,y,t)),$$

where all free variables are shown.

He also mentions that it might be necessary to add the existence of the empty set as an axiom, but it is straightforward to check that there is a model of such axioms in which $V = \{\phi\}$. However, our next result shows that if we add an axiom asserting the existence of two sets then the theory becomes inconsistent.

Theorem 6.7 The theory with axioms A1, A2, A3, $A4_W$, and $\exists x \in V\ \exists y \in V\ x \neq y$ is inconsistent.

Proof In this theory we firstly prove

$$\forall x \in V\ x \notin x. \qquad (*)$$

Suppose that $\exists x \in V\ x \in x$ and let $\Phi(x)$ be the formula $x \in V \wedge x \notin x$. Then by $A4_W$ $z = \{t \mid t \in V \wedge t \notin t\} \in V$, but $z \in z \leftrightarrow z \notin z$, so that $(*)$ holds. From $(*)$ we know that

$$V \notin V. \qquad (**)$$

Let $x \in V$ and we can suppose that $x \neq \phi$. Using $t \neq t$ in $A4_W$ we see that $\phi \in V$. Let $\Phi_1(t)$ be $t \in V \wedge t \neq \phi$ so that by $A4_W$ $z' = \{x \mid x \in V \wedge x \neq \phi\} \in V$. Let $\Phi_2(t)$ be $t \in z' \vee t = \phi$. Then $\forall t\ (\Phi_2(t) \rightarrow t \in V)$, $z' \notin z'$ by $(*)$ and $z' \neq \phi$ by our assumption that there are at least two sets. Hence we can use Φ_2 in $A4_W$ and this gives $\{x \mid x \in z' \vee x = \phi\} \in V$, i.e. $V \in V$, which contradicts $(**)$. □

Wang's idea was to allow V to appear in the formula of A4 and to alter this axiom to give an extension of A. If Φ is a formula with three variables which could be used then we would have

$$\text{(i)} \quad \forall z\ (\ \Phi(x,y,z) \to z \in V\ ).$$

The other conditions which we put on Φ must prevent it from being equivalent to $z \in V$, as this implies $V \in V$. In fact, noticing the downfall of $A4_w$, we must prevent the possibility of Φ being converted to a definition of V when the process of conversion does not mention V. More precisely, if ϕ is a formula with exactly four free variables, then Φ would satisfy

$$\text{(ii)} \quad \neg \forall t\ (t \in V \leftrightarrow \exists z (\ \Phi(x,y,z) \wedge \phi(x,y,z,t)\ )\ ).$$

Our next result shows that if Φ satisfies the above conditions then $\{z \mid \Phi(x,y,z)\} \in V$ is already provable in A^*. Consequently, A^* is maximal in this sense.

<u>Theorem 6.8</u> If Φ is a formula with exactly three free variables, $x,y \in V$ and Φ satisfies (i) and (ii), then $A^* \vdash \{z \mid \Phi(x,y,z)\} \in V$.

<u>Proof</u> Suppose that the hypothesis of the theorem holds and let ϕ be the formula $\exists \alpha, \beta\ (\ \alpha = \text{the rank of } z \wedge \beta \leqslant \alpha \wedge t \in R\beta\ )$. Then (i) and (ii) show that $s = \{t \mid \exists z\ (\ \Phi(x,y,z) \wedge \phi(z,t)\} \subset V$. The ordinals in s form an initial segment of those in V, and let γ be their supremum. Then $\forall z\ (\ \Phi\ (x,y,z) \to z \in R\gamma + 1\ )$ so that the result follows by A2. □

There are some similarities between Wang's idea and a modification of the general abstraction principle (i.e. $\exists x \forall y\ (y \in x \leftrightarrow K)$, for any $\in$-formula K) which Hintikka described in [26] . We shall show that Hintikka's axiom together with certain other axioms is inconsistent. This indicates another direction in which A4 cannot be generalised, Hintikka proposed two modified versions of the abstraction principle. The first of these is

$$\exists x \forall y \; (y \neq x \rightarrow (y \in x \leftrightarrow \phi^{+})), \qquad (1)$$

where ϕ is any formula which does not involve x and ϕ^{+} is obtained from ϕ by replacing subformulae of the form $\exists z \psi$ by $\exists z \; (z \neq x \wedge \psi)$ and those of the form $\forall z \psi$ by $\forall z (z \neq x \rightarrow \psi)$ etc., in such a way that all the variables become distinct. The second version is

$$\exists x \forall y \; (y \neq x \wedge y \neq z_1 \; . \; . \; . \wedge y \neq z_k \rightarrow (y \in x \leftrightarrow \phi^{+})), \qquad (2)$$

where ϕ^{+} is a formula of the type described above and $z_1, \; . \; . \; z_k$ are all its free variables.

In [27], Hintikka derived a contradiction from (1), but not from (2), and he argued that this was a disproof of Russell's vicious circle principle. Shiman has suggested (in [65]) that this is not the case as the contradiction requires instances of (1) which contain free variables. He has produced a more complicated theory which ensures that the set being defined by an instance of (1) cannot occur as a value of a bound variable in the specification of a parameter used in this instance.

Theorem 6.9 shows that (2) is inconsistent with some very reasonable set theoretic principles. We will also indicate how this result extends to a number of weaker theories. The reasonable principles are extensionality (this is a basic assumption for set theory), the nonexistence of two cycles of sets (this seems essential in formalising the vicious circle principle) and the existence of three sets.

<u>Theorem 6.9</u> The axiom of extensionality and (2) are inconsistent with the following axioms

(i) $\forall x \forall y \in x \; x \notin y$,

(ii) $\exists x,y,z \; (x \neq y \wedge y \neq z \wedge z \neq x)$.

<u>Proof</u> We assume that extensionality, (2), (i) and (ii) all hold and we derive a contradiction. Obviously we have

(iii) $\forall x \; x \notin x$.

By (2) $\exists x \forall y \; (y \neq x \rightarrow (y \in x \leftrightarrow y = y))$ and then by (iii) $\exists x \forall t \; (t \in x \leftrightarrow t = t \wedge t \neq x)$. Then (i) shows there is a unique x satisfying this condition and we call it a. We next show that

$$\neg \exists x \; a \in x. \qquad (*)$$

Suppose that $\exists x \; a \in x$ and then $a \neq x$ by (iii), so that $a \in x \in a$ by the definition of a, which contradicts (i). Thus (*) holds.

By (2) $\exists x \forall y \; (y \neq x \rightarrow (y \in x \leftrightarrow \neg \exists z (z \neq x \wedge z \neq y \wedge y \in z)))$, and let b be an x satisfying this expression. If $b \neq a$, then from (*), $a \in b \in a$, which contradicts (i). Hence $b = a$ and we get

$$\forall y \in a \, \neg \exists z \; (z \neq a \wedge y \in z). \qquad (**)$$

By (2) $\exists x \forall y \; (y \neq x \rightarrow (y \in x \leftrightarrow y \neq y))$ and we then get $\exists ! x \forall y \; (y \in x \leftrightarrow y \neq y)$ and we call this x ϕ, as usual. From (ii) $\exists z \; (\phi \neq z \neq a)$ and $z \in a$ follows from the definition of a. Also, $\exists y \; y \in z$ by extensionality, but this $y \neq a$ by (i) so that $y \in a$ as well. This contradicts (**). □

The only instances of (2) which have been used in the proof of theorem 6.9 have no parameters so that this proof will go through for any version of (2), no matter what conditions are put on the free variables. In particular, it applies to Shiman's system of [65].

Just for completeness theorem 6.10 shows that assumption (ii) was necessary in the last theorem.

Theorem 6.10 The theory with axioms extensionality, (2), (i) and $\exists x,y\ x \neq y$ is consistent.

Proof Consider the model $\{\phi, \{\phi\}\}$ where the membership relation is used for $\in$. Clearly we only need to show that (2) holds in this model and let ψ be the formula used in an instance of (2).

If ψ has two or more free variables then (2) will be vacuously satisfied so we can suppose that ψ has only one free variable. Firstly we consider the case when ψ contains no quantifiers. Then $\psi = \psi^{+}$ and it will always be true or always false so that letting x be $\{\phi\}$ or ϕ, respectively, shows that (2) is satisfied. If ψ contains at least one quantifier, on a variable z say, then after this quantifier we have $z \neq x \wedge z \neq y$. Again, ψ^{+} becomes always true or always false and (2) is satisfied as before. $\square$

Chapter 7

Some natural model problems

7.1 Introduction

In this chapter we consider some natural model problems which were motivated by earlier results in this thesis. A few results are given but really we do no more than to point out some possible directions for future research.

Sections 2, 3 and 4 are concerned with $E(\beta)$ and $U(\beta)$, where, as in chapter 3, $E(\beta) = \{\alpha \mid R\beta \prec R\alpha\}$ and $U(\beta) = \{\alpha \mid \langle R\alpha, R\beta, \in\rangle \models A\}$. It is clear that $E(\beta)$ forms a tree under the ordering $\alpha' \ll \beta'$ iff $R\alpha' \prec R\beta'$, and we shall often refer to this tree structure on $E(\beta)$ without explicitly mentioning it. There are a large number of questions concerning the structure of $E(\beta)$ which seem interesting in their own right, but we shall only consider some basic structual properties.

In the last section we introduce a problem concerning the definability of ordinals in natural models which is a generalisation of the natural models of A and A^+.

7.2 Is $E(\beta)$ always bounded ?

Theorem 3.15 shows that $U(\beta)$ is not always bounded and it is natural to ask if this is also true of $E(\beta)$. We shall consider the following three ways of expressing the idea that $E(\beta)$ cannot be bounded: only the first two of them can be written as statements of ZF.

(i) $E(\beta)$, considered as a virtual class, is unbounded,

(ii) the lengths of the branches of $E(\beta)$ are unbounded,

(iii) $E(\beta)$ has a branch which is unbounded.

The existence of a β satisfying all of these statements is provable in MK (see [45]). Clearly (iii) $\Rightarrow$(ii) $\Rightarrow$(i), and we next show that neither of these implications can be reversed, with respect to natural models of ZM.

(a) To show that (ii) $\not\Rightarrow$ (iii).
Let κ be an inaccessible cardinal such that $R\kappa \models ZM$ and let κ' be the least ordinal for which $R\kappa' \prec R\kappa$. Then $R\kappa' \models ZM + $(ii), for some $\beta < \kappa'$, but (iii) cannot hold for this β as otherwise taking the union of the unbounded chain would contradict the definition of κ'.

(b) To show that (i) $\not\Rightarrow$ (ii).
We assume that $\exists \alpha, \beta, \gamma \;\; R\alpha \prec R\beta \prec R\gamma \models ZM$, and then let γ' be the least ordinal which satisfies

$$\exists \alpha, \beta \;\; R\alpha \prec R\beta \prec R\gamma \wedge R\gamma \models ZM. \qquad (*)$$

Then let α', β' be the corresponding α, β. From theorem 7.5, below, $E(\alpha')$ is unbounded in $R\gamma'$ so that (i) is true of α' in $R\gamma'$. Further, $E(\alpha')$ cannot have a branch of length 3 in $R\gamma'$ from the definition of γ', so that (ii) fails for α' in $R\gamma'$.

Part (a) shows that the existence of a β satisfying (iii) is not always true in natural models of ZM. We next suggest that, from a Cantorian viewpoint, there will be no such β. The motivation behind this will be explained in the next chapter. If, in reality, there is such a β , then taking the " union " of the unbounded branch of $E(\beta)$ would give the Absolute. Hence there would be an an ordinal such that first order truth in $R\beta$ is the same as first order truth in the Absolute: we consider this very unlikely. There is a large factor of analogy in this argument as the union of elementary chains argument is only proved for the case when the chain is a set, but we still think that it is suggestive.

We conclude this section by showing that the existence of a β satisfying (ii) is provable in ZM.

<u>Theorem 7.1</u> ZM $\vdash \exists \beta$ ' the lengths of the branches of $E(\beta)$ are unbounded '.

<u>Proof</u> We work in ZM and suppose that for every β , the lengths of the branches of $E(\beta)$ are bounded. Then put

$F(\alpha) = \bigcup \{ \gamma \mid \exists \beta < \alpha \;\; \gamma = \bigcup \{\text{the lengths of the branches of } E(\beta)\}\}$

and as there are arbitarily large inaccessible cardinals we know that there are arbitarily large $F(\alpha)$s. Hence there is a functional formula G which satisfies

$$G(0) = 0,$$
$$G(\alpha+1) = F(\text{' the least } \mu \text{ satisfying } F(\mu) > G(\alpha) \text{ '}),$$
$$G(\lambda) = \bigcup_{\mu<\lambda} G(\mu), \text{ for } \lambda \text{ a limit ordinal.}$$

We clearly have $G(\alpha) \geqslant F(\alpha)$ for all ordinals α . G is a normal function so that, in ZM, it has an inaccessible fixed point. Let η be such a fixed point. As η is inaccessible $R_\eta \vdash \exists \beta$ ' the lengths of the branches of $E(\beta)$ are unbounded '. Let β be an ordinal which satisfies this condition in R_η , and then $F(\beta) \geqslant \eta$. Hence we have $G(\eta) > G(\beta) \geqslant F(\beta) \geqslant \eta = G(\eta)$ as η is a fixed point of G. This is a contradiction. $\square$

<u>7.3</u> <u>The structure of $E(\beta)$</u>

We work in ZM and our first three results concern possible lengths of $E(\beta)$.

<u>Theorem 7.2</u> $\forall \alpha, \gamma \; \exists \beta > \gamma \;\; \overline{E(\beta)} = \alpha$.

Proof We know that there are arbitarily large β s with $\overline{E(\beta)} \geqslant \alpha$ and let β' be the least ordinal which satisfies

$$\alpha \in \beta \wedge \gamma \in \beta \wedge \overline{E(\beta)} \geqslant \alpha.$$

Suppose that $\overline{E(\beta')} > \alpha$ and let δ be the αth member of $E(\beta')$. Then ' the least β for which $\alpha \in \beta$ and $\gamma \in \beta$ and $\forall \mu < \alpha$ there is a set of ordinals λ, for which $R\beta' \prec R\lambda$, which has order type $\geqslant \mu + 1$ ' shows that $\beta' \in Df(R\delta, R\beta')$. This is a contradiction so that $\overline{E(\beta')} = \alpha$. □

Theorem 7.3 $\forall \alpha, \gamma \ \exists \beta > \gamma$ ' the length of $E(\beta)$, considered as a tree, is equal to α'.

Proof Similar to the proof of theorem 7.2 □

Theorem 7.4 $\forall \alpha, \gamma$ ($\exists \beta > \gamma$ $E(\beta)$ has a branch of length α iff α is a successor ordinal)

Proof If α is a successor ordinal then a proof similar to that of theorem 7.2 shows that there is a β with the desired property. If α is a limit ordinal then no $E(\beta)$ can have a branch of length α by the union of elementary chains result. □

The next two results give some indication of the width of the trees which occur as $E(\beta)$s, and theorem 7.7 shows that all trees of length 2 occur as $E(\beta)$s. The method of proof used for theorem 7.6 can be extended to give a number of similar results.

Theorem 7.5 If $R\beta \prec \ldots \prec R\alpha_i \prec R\alpha_{i+1}$, then $\overline{E(\beta) \cap \alpha_{i+1}} = \alpha_{i+1}$

Proof Suppose that $R\beta \prec R\alpha_i \prec R\alpha_{i+1}$ and that $\overline{E(\beta) \cap \alpha_{i+1}} < \alpha_{i+1}$. Then in $R\alpha_{i+1}$ we know that $\sup \{ \gamma \mid R\beta \prec R\gamma \}$ exists and is

$\geqslant \alpha_i$. This contradicts $Df(R\alpha_{i+1}, \{\beta\}) = Df(R\alpha_i, \{\beta\})$, so that the result holds. $\square$

<u>Theorem 7.6</u> If $R\beta \prec \ldots \prec R\alpha_i \prec R\alpha_{i+1} \prec R\alpha_{i+2} \ldots$ occurs in a branch of $E(\beta)$, then $E(\beta)$ has at least α_{i+1} branches all of which have a different member less than α_{i+1}.

<u>Proof</u> Suppose that the hypothesis of the theorem holds and we show that there are at least α_{i+1} ' splittings ' of $E(\beta)$ at α_i. Suppose that there are only $\xi < \alpha_{i+1}$ such splittings and then for some $\xi' \leqslant \xi$, α_{i+1} is the ξ'th ordinal for which $R\alpha_i \prec R\lambda$ and $\neg\exists\gamma\ R\alpha_i \prec R\gamma \prec R\lambda$ in $R\alpha_{i+2}$. Thus $\alpha_{i+1} \in Df(R\alpha_{i+2}, R\alpha_{i+1})$, which is a contradiction. $\square$

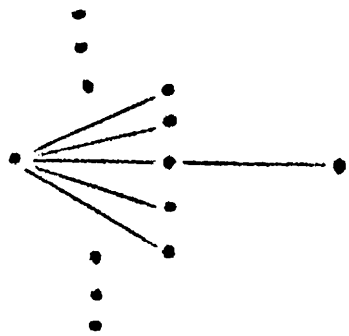
<u>Theorem 7.7</u> For every tree of length 2, there is a β such that $E(\beta)$ has that tree structure.

<u>Proof</u> Given a tree of length 2 with κ many branches, theorem 7.2 shows that there is a $\beta > \kappa$ with $\overline{\overline{E(\beta)}} = \kappa$. Theorem 7.5 shows that this $E(\beta)$ is of length 2, as required. $\square$

We finish this section by indicating how one might consider the possible structures of short $E(\beta)$s.

<u>Definition 7.8</u> Let γ_1 be the least ordinal for which $\exists\alpha, \beta\ R\alpha \prec R\beta \prec R\gamma_1$ and let α_1 and β_1 be the corresponding α and β .

By the usual arguments of this section $E(\alpha_1)$ has only one branch of length 3 and $E(\alpha_1) \subseteq \gamma_1 + 1$. Theorem 7.5 shows that $\overline{\overline{E(\alpha_1)}} \geqslant \gamma_1$ so that $E(\alpha_1)$ has precisely γ_1 branches so that its structure is



This method of constructing β s for which $E(\beta)$ has a certain shape can easily be extended to other short trees, but we leave open the next problem.

<u>Question 7.9</u> Can those trees which occur as $E(\beta)$s be characterised in any nice way ?

<u>7.4 E(β) and U(β)</u>

In chapter 3 we indicated that we often have $E(\beta) \neq U(\beta)$, but we do not know when equality holds. The two theorems of this section give conditions which imply equality.

<u>Theorem 7.10</u> If the length of $E(\beta)$ is $\leqslant \omega$, then $E(\beta) = U(\beta)$.

<u>Proof</u> Suppose that the length of $E(\beta)$ is $\leqslant \omega$ and that $\alpha \in U(\beta) - E(\beta)$. Then, by theorem 3.3, $\exists \gamma \in \alpha (\gamma \in Df(R\alpha, R\beta) \wedge R\beta < R\gamma)$. Choose such a γ and then by our assumption on $E(\beta)$, γ will be the nth element of a branch of $E(\beta)$, for some $n \in \omega$. Then consider the greatest β such that there are n-1 ordinals α_i satisfying $R\beta < R\alpha_1 < . . . R\alpha_{n-1} < R\gamma$ and ' the definition of γ '. This shows that $\beta \in Df(R\alpha, R\beta)$, contradicting theorem 3.1. $\square$

Definition 7.11 β is said to be low if $\neg\exists\gamma\ R\gamma \prec R\beta$.

Theorem 7.12 If β is low, then $E(\beta) = U(\beta)$.

Proof Suppose that β is low and that $\alpha \in U(\beta) - E(\beta)$. Then, by theorem 3.3, $\exists\gamma\in\alpha(\gamma\in Df(R\alpha, R\beta) \wedge R\beta \prec R\gamma)$, and let γ be such an ordinal. Then the least β for which $R\beta \prec R\gamma$ and ' the definition of γ ' shows that $\beta \in Df(R\alpha, R\beta)$, contradicting theorem 3.1. □

Corollary 7.13 $E(\beta) \subseteq E(\beta') \not\Rightarrow U(\beta) \subseteq U(\beta')$.

Proof Let β be an ordinal less than the first inaccessible cardinal for which $U(\beta)$ is unbounded (such an ordinal exists by theorem 3.15) and let β' be the least ordinal for which $R\beta' \prec R\beta$. Then $E(\beta) \subseteq E(\beta')$, but as β' is low, the theorem shows that $U(\beta') = E(\beta')$, which is bounded so that $U(\beta) \not\subseteq U(\beta')$. □

Theorem 7.12 admits some generalisation, but we leave open the next problem.

Question 7.14 When does $E(\beta) = U(\beta)$?

7.5 Definability of ordinals using parameters

Theorems 3.2 and 4.14 show that natural models of A and A^+ are equivalent to the nondefinability of ordinals in natural models, using certain parameters. This suggests that more general results might be obtainable and we briefly consider the problem of when there is an ordinal $\beta \notin Df(R\alpha, R\beta \cup x)$, for $x \subseteq R\alpha$ and $\beta < \alpha$. This is clearly a generalisation of the notions of natural models of A and A^+. A partial solution was given in theorem 3.7 which says

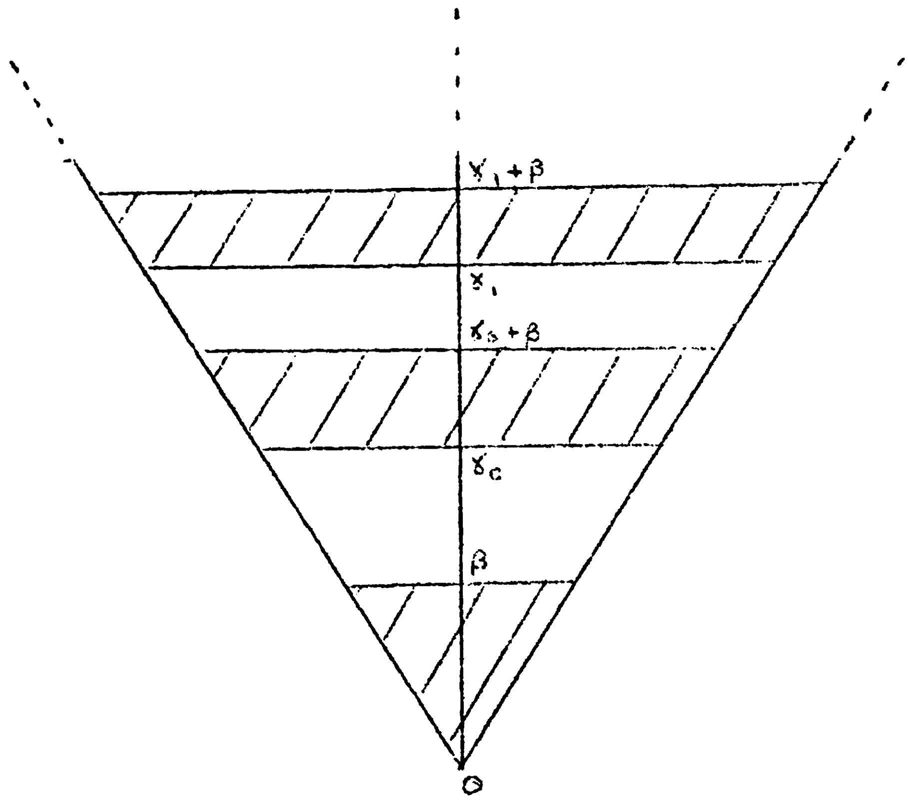
If κ is an inaccessible cardinal, $\alpha \geqslant \kappa$ and x is a set of cardinality $< \kappa$ such that $x \subseteq R\alpha$, then there are κ many $\beta < \kappa$ for which $\beta \notin Df(R\alpha, R\beta \cup x)$.

Thus the remaining problem is to extend this theorem to sets x of larger cardinality. It is clear that the members of x must fall into " β bands " (shaded portions in the diagram below) and that x cannot contain any complete β band, but we leave open the next problem.

<u>Question 7.15</u> How large, relative to $R\alpha$, can x be when

$$\exists \beta < \alpha \quad \beta \notin Df(R\alpha, R\beta \cup x) \ ?$$

In this question ' large ' can firstly be interpreted as cardinality, but it might also be possible to interpret it as the inclusion relation when results concerning scales might even hold.



Chapter 8

Cantor's work

8.1 Introduction

Although it seems possible to trace the notion of a set back for an indefinite period, it is indisputable that Cantor's work made the greatest step, by far, in the development of the idea. This is one of the reasons why we think it important to consider his work here. The other is that its nature is often misrepresented in textbooks and mythology today.

Basically we shall give an account of Cantor's work on the notion of a set and, from his publications, we can discern three stages in the development of his ideas. It is quite possible that Cantor's views remained constant and that we are really only considering different stages of presentation, but we shall always write as if his papers correspond to his ideas. The main references which we shall use are [8], [9], [10] and [30] and we shall usually refer to Cantor's (or C's for the rest of this chapter) papers just by the year in which they were first published.

As well as describing C's ideas we shall often comment on points at which various problems arise and sometimes we shall investigate them further. Also, we shall try to show how, in the development of set theory, some people have gone astray (knowingly, or otherwise) from the original ideas. Frequently, we shall impose certain ways of thinking on the published work so that we cannot be sure that we are faithfully presenting C's work, but we leave others to argue over such problems.

Actually, C has written relatively little on the notion of a set (or aggregate, as it was called at the end of the ninteenth centuary; we shall always update such terminology without further mention) and most of his work concerns infinite ordinals and cardinals. He did not view these in the current way, but firstly as newly postulated entities and later as abstractions from ordered sets. During this chapter the terms ordinal and cardinal have a variable status (among the three meanings) and we hope that the intended usage will be clear from the context.

A reasonable introduction to C's earlier work and some indications of his motivation are given in [30]. This also describes his first work on powers of sets (two sets were said to have the same power if there is a bijection between them so this corresponds to cardinality) and we shall not discuss this. For a discussion of the prior opinions and uses of the notion of infinity in mathematics and philosophy C's 1883 paper is very good.

8.2 Early work on ordinals

In the last part of [1883] C explains certain principles by which, he argues, we can form new infinite ordinals. His language is very suggestive of one's creating new objects in time and we shall discuss this interpretation in section 9.4.

C's considerations start with the sequence of natural numbers

(I) $1, 2, 3, \ldots, \nu, \ldots$

In this sequence each element is obtained from the previous element by adding a unit to it, and this process is called the first principle of generation. C then argues that we can posit a new number, ω, which is the least number greater than all of the elements of (I). Then, applying the first principle of generation repeatedly, we obtain the new sequence

$$\omega+1,\ \omega+2,\ \omega+3,\ \ldots,\ \omega+\nu,\ \ldots$$

On the basis of this, and other, examples C defined the second principle of generation as follows.

" If any definite succession of ordinals is given, for which there is no greatest, a new number can be created on the basis of the second principle, which is defined to be the least number greater than all of the elements of the sequence. "

Using this principle C then introduced $\omega.n$ and ω^n in the obvious way and he proceeded to illustrate the dazzling array of small countable ordinals. C then defined the totality of all numbers of the same power as (I) as the second number class, (II) ((I) was called the first number class). From the existence of (II) and the second principle C then obtained a least member of the third number class, and so on. In making these definitions C has used the third principle which takes the form of a restricting, or limiting, principle on the second one. This states that the numbers to be next formed using the second principle are all to be of the power of a smaller number. To be precise, the 1883 paper does not actually state the third principle, but it is said that (II) has the required property and hence it is said to satisfy the third principle. From the introductory part of [1883] (see page 547) it seems that C might have wanted the third principle to give the number classes rather than to restrict all uses of the second principle in this way.

Some theorems on ordinal arithmetic and a proof that the power of (II) is the next greater cardinal to that of (I) form the remaining technical results of [1883] . These proofs are always of a higher order nature (i.e. they consider sets of ordinals etc.) but we shall consider this point again with respect to the later work.

We learn from [30] that in 1883 the above approach to ordinals had already been replaced by C (probably for reasons which we shall outline in the next section) and the notion of an order type was introduced as an abstraction from an ordered set. Further details of C's work between 1883 and 1890 are given in [30] and we only note that some of the work which was published in 1895 (which we call later work) had been completed ten years earlier.

8.3 Some comments on the early work

The main criticisms of C's earlier work on ordinals seem to concern certain uses of the second principle and we find it convenient to split the uses of this principle into the following cases.

(2a) When we apply it to a countable, increasing sequence of ordinals which have already been introduced and for which we have a notation. Such sequences are called fundamental ones.

(2b) When we are producing a least ordinal of the next higher cardinality.

(2a) leaves no doubts that we have a definite succession of ordinals, but this does not seem to be true of (2b). The third principle might have been intended just as an assertion that (2b) is dealing with a definite succession of ordinals, but this still gives no reason for believing it. It seems intuitely reasonable that however we describe any procedure which only uses fundamental sequences of ordinals we shall never be able to generate the first uncountable ordinal. The work on ordinal notations in ZF (see page 215 of [59] , for instance) also suggests this. Thus, if the second number class is to be thought of as a completed totality we seem to require a more detailed description of the process by which it is to be generated. In particular, what is a definite, uncountable process ?

It is hard to imagine an answer to this question which does not use an uncountable set to index the process, and the only way to get such a set, at the moment, seems to be by using the power set axiom. We cannot assume that C had such a scheme in mind as no indication of it is given and it would hardly have been obvious to his readers. An alternative solution to this question would be to allow (2b) without the power set axiom by adding the proviso that the class of all ordinals less than the new one is essentially incomplete. However, we do not think that such an approach is intuitively very plausible.

On the basis of the above arguments we suggest that C's justification of the existence of the second number class is not completely convincing. It is equally possible to advance analogous criticisms of the notion of a set which was given in C's 1882 paper, which, in chapter 2, we suggested was the basic idea behind Ackermann's set theory. In that paper the concept of power was considered as an attribute of " well defined collections ", where

" A collection of elements belonging to any well defined sphere of thought is said to be well defined when, in consequence of its definition and the logical principle of the excluded middle, it must be considered as intrinsically determined whether any object belonging to this sphere of thought belongs to the collection, or not, and, secondly, whether two objects belonging to the collection are equal or not, in spite of formal differences in the manner in which they are given. "

C went on to emphasise that " intrinsically determined " does not mean that we can actually find the answer. With this notion of a set it is hard not to jump to the conclusion that all sets are definable, in some sense, so that there cannot be a first uncountable ordinal, all of whose members are sets. It might be worthwhile to consider how far one could go in formalising a system of sets and objects where all sets are definable, and we mention this again in section 8.8. This notion is also slightly evident in the following definition of a set which C gives in a note to the 1883 paper. It is also possible to see the later ideas developing here.

" By a set I understand, generally, any multiplicity which can be thought of as one, that is to say, any totality of definite elements which can be bound up into a whole by means of a law. "

8.4 Cantor's later work

By the later work we mean the papers of 1895 and 1897. Here, the main aims are to establish a rigourous basis for the ordinals and cardinals, and to start the development of their theories. Throughout these papers set theory is not treated in general although C says that he intended to formulate this theory later. The 1895 paper starts with the oft quoted " definition " of a set,

" By a set we are to understand any collection into a whole of definite and seperate objects of our intuition or thought. "

It seems highly unlikely that C intended this to be anything more than a heuristic guideline as he frequently explains why certain sets can be said to exist. Consequently, we shall not treat this statement as a definition. We take it to mean that any collection which can be consistently " visualised ", in some sense, can be thought of as a set.

Next in [1895] C explained his basic ideas about cardinality and the relationships between cardinals. He also defined arithmetic operations on the cardinals, proved some results about $\aleph_0$ and indicated some results concerning increasing sequences of cardinals. The most important point, from our point of view, is that C no longer based these ideas on direct intuition, but says that for a rigourous foundation of these matters we must turn to the theory of order types, which he considered next.

C starts from the notion of a linearly ordered set. He considered this as a set with a seperate ordering relation rather than the current view which includes the ordering as a set. Order types are considered as abstractions from these ordered sets where the abstraction is thought of as a set, all of whose elements are " unity ", which has the same order precedence as the given set. C then discussed similarity of order types and finite order types. Finally, in [1895] , addition and multiplication of order types are considered and the order types of the rationals and the reals are discussed. The results include the well known characterisations of the latter two order types.

This work continues in the 1897 paper where C defines well ordered sets as linearly ordered ones for which

(i) there is a least element,

(ii) if a part, f, of the set has one or more elements of the set above it, then there is an element of the set which follows immediately after f.

It is clear that this is equivalent to the usual definition of a well ordering. C then proved the results on well orderings which now form a well known part of courses on set theory. Ordinals are defined as the order types of well ordered sets and the law of trichotomy for ordinals is proved rigourously. Then, at the beginning of section 15 of [1897] , there comes what, from our point of view, is the most important definition in the paper. This is

" The second number class, $Z(\aleph_0)$, is the set of all order types of well ordered sets of cardinality $\aleph_0$. "

In effect, this is allowing us to gather into a whole all the different well orderings of ω and, as such, it is a new principle which has not been previously used in these two papers. It is quite clear when an ordering of ω is a well ordering and, although we cannot give a process which enumerates the well orderings of ω, we are allowed to gather them all together at one sweep. Thus $Z(\aleph_0)$ is defined in a single second order way (we take all well orderings of ω - these can obviously be obtained from all subsets of ω), rather than by a vague belief that the building up processes for obtaining ordinals can be continued through all countable ordinals.

C then proceeded to analyse $Z(\aleph_0)$ and he proved that its cardinality is the next greater one to $\aleph_0$. He also proved his normal form theorem and this illustrates C's approach to set theory: he studied the structure of $Z(\aleph_0)$ in some detail, rather than getting involved in vaguer macro problems.

8.5 The second order nature of the later work

We think that, at the moment, the second order nature of C's work cannot be overemphasised. If we were to begin to formalise his work on ordinals, then the principles akin to (2a) could easily be handled within a first order system, but this does not seem to be true when it comes to the existence of $Z(\aleph_0)$ and the power set axiom. We do not think that C would have assented to founding set theory on full second order logic, where the variables X, Y, . . . range over subcollections of the ' universe of all sets ', for reasons which we shall discuss in the next section. We suggest that a suitable form of second order logic (we call it a mild second order theory) would be one where X, Y, . . . range over <u>all</u> those collections of sets which are equipotent to a set, and x, y, . . . range over sets, as usual. Then the power set axiom (the existence of $Z(\aleph_0)$ can be derived from this) would take the form

$$\forall x \exists y \ (\forall t \in y \ t \subseteq x \wedge \forall X \subseteq x \ \exists t \in y \ t = X).$$

This essential viewpoint gets lost in first order axiomatisations of set theory, such as ZF.

It is also important to notice how, on the basis of the above ideas, we can justify the comprehension axiom of ZF without any reference to truth considerations, as follows. We consider a set y and, for convenience, a formula ϕ with exactly one free variable. If $x \in y$, C would argue that by the logical principle of the excluded middle, we would have $\phi(x)$ or $\neg\phi(x)$. Then, as the power set of y contains <u>all</u> subcollections of y, there must be one, z say, for which $\forall x \ (x \in z \leftrightarrow x \in y \wedge \phi(x) \ )$. Hence the comprehension axiom holds. This reduces the truth of comprehension to a question of logic and although people can, and do, work in non classical logics, classical logic is presupposed in all of C's work.

This justification of the comprehension axiom runs counter to what some people have recently suggested and we think that model theory is partly to blame for this shift of emphasis. Here, one frequently considers first order ZF (a quaint theory, as it only ensures that certain definable subsets exist although it is not at all clear what the variables range over so that we do not know in what sense these subsets are definable anyway) and then from Skolem's work we know that there are countable models of ZF so that people get very worried about which subsets of ω, for instance, " really exist ". They also begin to think that comprehension is true because, for a given formula ϕ, they can check the truth definition of ϕ in the model, whereas questions of truth in set theory cannot use Tarski's truth definition for it assumes that the universe is a single consistent totality.

A good example of bad motivation which follows from such misunderstandings is Barwise's paper [3]. In the concluding remarks of that work he says that to allow all first order formulae to occur in the comprehension axiom (a suggestion due to Skolem which is obviously inadequate for giving all subsets) assumes that we can form a true universe of all sets. Why this should be true, unless Barwise is worried about truth definitions, remains a mystery. Barwise considers restricting the comprehension axiom to $\Delta_0(\mathcal{P})$ formulae (i.e. those formulae which are Δ_0 when we allow $\mathcal{P}$, the power set operator, as a new basic symbol) and he seems quite willing to believe these instances. But now if one is willing to believe the power set axiom in its mild second order form then all instances of comprehension follow, and if one believes it in some other form it seems to be a harder problem to say which subsets exist than to accept the comprehension axiom.

It seems that [72] is the origin of such heresies and the presupposition of this paper is that set theory is a first order theory rather than a mild second order one. This fallacious belief seems to be held largely by people who publish in logic journals: mathematicians, in general, seem quite happy to believe in a genuine power set operation which cannot be first order. In [72] Zermelo talks of comprehension holding for " definite properties " and this notion is an open ended extension of Skolem's restriction to first order formulae. Although all instances of Zermelo's comprehension axiom will be true from a Cantorian viewpoint, there does not seem to be any reason for supposing that these ideas suffice for describing the true power set operation.

Finally, in this section, we note that, in [7], Borel criticised C's work on ordinal numbers and he was probably referring to the earlier work so that his reasons might have been similar to those of section 3. Borel acknowledged C's proof that $\mathcal{P}\omega$ was larger than $\aleph_0$, but he did not believe in the existence of ω_1. This was the motivation for his later (famous) work. In a footnote Borel asks why there should be a least cardinal greater than $\aleph_0$, although from C's later work and the Schroder-Bernstein theorem (both of which had been published before [7]) there seems to be a convincing proof of this fact. Of course, we do not know that Borel was aquainted with these results and, as he offered no criticisms of them, we assume that he was not. Thus his work was motivated by doubts about the principle (2b) and we shall later suggest that other work also arose in this way.

8.6 Inconsistent multiplicities

A letter which C wrote in 1899 (see [10]) contains what we consider to be his final conclusions about the notions of set, ordinal and cardinal. The discussion in the letter assumes that there are multiplicities (we hope that this word does not have any connotations of oneness) which are not sets. The main point of the letter is to show that all cardinal numbers are alephs, or, in effect, that every set can be well ordered. However, C firstly outlines his general ideas.

C says that it is necessary to distinguish between two sorts of multiplicities (he always assumes that we are considering only definite multiplicities) and he says that for some multiplicities the assumption that " all of its elements are together " leads to a contradiction, so that it cannot be conceived of as " one finished thing ". On the other hand, if the elements of a multiplicity can be thought of as " being together ", then it is called a consistent multiplicity, or a set. Thus all notions of processes and building up are eliminated and the whole of set theory is given in one psychological (though not obvious) swoop.

Then C gives informal versions of the axioms of ZF as ways of getting from one set to another. Hence it would seem more reasonable for this theory to be called CZF than ZF. Two of the statements which are of interest to us are

(a) Two equivalent multiplicities are either both sets or both inconsistent.

(b) Every submultiplicity of a set is a set.

(a) obviously implies the replacement axiom and (b) suggests that our mild second order theory is a reasonable formalisation of part of C's ideas. C probably believed these axioms because of considerations of the Absolute, although he does not explicitly say this.

As examples of inconsistent multiplicities C gives " the totality of all things thinkable " and Ω, which is the system of all ordinals under their natural ordering. The proofs that these multiplicities are inconsistent are, of course, the usual paradoxes. C then reiterates his work on ordinals and gives the following proof that if v is a definite multiplicity and no aleph corresponds to it as its cardinal number, then v must be inconsistent.

Suppose that v is a definite multiplicity and that no aleph corresponds to it as its cardinal number. Then " we readily see that, on the assumption made, the whole system Ω is projectible into the multiplicity v, that is, there must exist a submultiplicity v' of v that is equipotent to the system Ω. v' is inconsistent because Ω is and the same must therefore be asserted of v. "

From this C proved the law of trichotomy for cardinals.

The quoted proof was objected to by Zermelo as it used inconsistent multiplicities: we consider this further in section 8.

C's considerations of inconsistent multiplicities can be argued to follow logically from his earlier work as, in [1883], he says that considering the infinite in the sense of finite increasing without bound implies the existence of the truely infinite as the domain for the variables. In this way, the use of variables over sets necessitates the existence of inconsistent multiplicities as their domains.

In the introduction to C's letter in [24], van Heijenoort says that C's inconsistent multiplicities prefigure the distinction between sets and classes which was introduced by von Neumann. This seems to be untrue as the nature of proper classes assumes that they are definite, fixed totalities which are not inconsistent by their very existence. The idea of a proper class seems far more likely to have originated with Zermelo's definite properties.

8.7 Cantor's notions and set theoretic developments

Before we consider some of the interelations between C's notions and set theoretic developments, we shall return to the so called definition in [1895], which says

" By a set we are to understand any collection into a whole of definite and seperate objects of our intuition or thought. "

It is often claimed that this leads to an inconsistent theory and, as an example of this, we quote from pages 285-6 of [31]. We do not think that the sense is altered by the omissions.

" Cantor's definition has not been retained in quite its original form by later authors, but was replaced at an early stage by a more abstractly conceived principle, or axiom, that has become known as the principle of comprehension [We refer to it as the abstraction principle so as not to confuse it with the axiom of comprehension] . . . . [This] can be expressed in the following form

$$\exists z \forall x \ (\ x \in z \leftrightarrow H(x) \ )$$

. . . . The formal system which we have obtained in this way [the abstraction principle and extensionality formulated in the first order predicate calculus with $\in$] . . . . may indeed be regarded as a reasonable formalisation of Cantor's naive theory of sets. "

This argument simply does not seem to be valid. Presumably the variables of the formal system are ranging over sets, but then the abstraction principle shows certain objects to be sets whilst C showed that they were not sets. The formal system has more in sympathy with Frege than with C as it ignores C's insistence on our being able to visualise all the members of a set being together.

Also, on page 262 of [20] , Godel suggests that " a satisfactory foundation of Cantor's theory in its whole original extent and meaning " can be given on the basis of iterations of the notion of " set of ", and this contrasts sharply with the suggestion that a reasonable formalisation of C's theory is inconsistent.

Next we point out three areas where people have extended set theory using new principles which run contry to C's ideas. Their justifications do not seem to be as well motivated as C's work.

The first example is Ackermann's set theory which we discussed in chapter 2. The second is the notion of building up sets " in time "; [51] and [52] being examples of this. On page 573 of [1883] C says that, in his opinion, it is wrong to use the concept of time to explain the much more basic concept of a continuum and hence it is reasonable to suggest that this is also true for the notion of a set. Thirdly, there is the topic of reflection principles and their connections with the Absolute. In [5] and [50], for instance, axioms are asserted which suggest that there exist sets (or at least consistent multiplicities for the notion of set in such theories is often weaker than C's notion) which resemble (e.g. are elementary substructures of) the Absolute. It is quite clear that C believed we could not have any good approximation to the Absolute and on page 587 of [1883] he says

" There is no doubt in my mind that in this way [producing new number classes] we may mount even higher, never arriving at any approximate comprehension of the Absolute. The Absolute can only be recognised, never known, not even approximately. "

Thus if we are to have any strong reflection principles and to maintain a Cantorian viewpoint then we must believe that the expressive power of the language under consideration is hopelesly inadequate for truth in the Absolute. However, such ideas do not seem to be considered at all in the works on reflection principles. One way of making reflection principles and Ackermann's set theory more reasonable is to consider them as ways of picking out certain ordinals which occur in their natural models, but this was not the original motivation for these ideas.

Comparing the kind of results which C proved with those which are proved today we get another contrast, this time in methodology. He concentrated on structual problems for small sets rather than larger cardinals, for instance. Although C was investigating problems which occur in nature (specifically the continuum hypothesis, of course) perhaps we could still gain much guidance from small, structual considerations.

Sierpinski is one of the very few mathematicians who have continued to work in C's original spirit. Some further topics for structual considerations are countable order types (although there is quite a bit in the literature on this topic) and other countable partial orderings. Another topic which seems to have been neglected is n dimensional order types (for $n \in \omega$ see page 80 of [9]) and higher dimensional ones. It might be possible to show that all interesting questions concerning these objects can be reduced, in some uniform way, to questions about ordinary order types, but we know of no such results.

8.8 Formalising parts of Cantor's work

Here, we shall briefly outline three problems connected with formalising parts of C's work. Firstly, there is the " constructive " notion of building up sets by a definite process, which we shall again refer to in the next chapter. These ideas have been considered by Lorenzen, [41] , Wang, [71] , Borel, [7] , and many others. We consider all this work to be motivated by C's ideas which lead to the first principle and the principle (2a). Is it possible to isolate a definite part of set theory which results from just these principles (when (2a) is modified to deal with sets as well as ordinals) ?

Our next considerations concern the interpretation of C's earlier work, mentioned in section 3, which suggests that all sets are definable. Although we cannot easily formalise such statements in a first order system we indicate how a first order system, analogous to ZF, could be set up , the axioms of which would be true under this interpretation. It would not be assumed that all members of sets are sets so that an additional predicate, M(x), would be introduced for " x is a (definable) set ". We then let $\Phi_i(x)$ stand for $\exists ! y\, \phi_i(y) \wedge \phi_i(x)$, where $\phi(x)$ is an $\in$-formula with one free variable, and we would have the schema

$$\Phi(x) \rightarrow M(x).$$

The other axioms would be obvious variants of those of ZF and, for instance, the comprehension axiom would take the form

$$\Phi_1(x) \wedge \Phi_2(y) \rightarrow \exists z\ (\ M(z) \wedge \forall t\ (\ t \in z \leftrightarrow t \in x \wedge \phi(t,y)\ )\ ).$$

This system would be quite similar to one which Friedman introduced in [17] and if we add $\forall x\, M(x)$ (which is false under our intended interpretation) to our system it becomes Friedman's. Obvious questions which one could ask for this system are its relative consistency and the structure of its models, but we shall not pursue these questions.

Our final considerations in this chapter concern C's notions of inconsistent multiplicities and the Absolute. We hope to consider, elsewhere, the general problems of formalising these notions and here we only consider the conversion of C's proof that every set has a cardinality which is an aleph (see section 6) into a proof which would be acceptable in a ZF like system.

We assume that all variables range over sets and then the hypothesis of the proof is

$$\neg \exists \alpha\ v \approx \aleph_\alpha \qquad (*)$$

C then considered it obvious that we could project the whole of Ω into v. If we interpret this as meaning that there is an injection from Ω into v, then this leads to a contradiction in ZF. Hence the question reduces to showing that Ω can be projected into v.

C seems to have used the axiom of choice as a logical principle so that we feel it is reasonable to assume the existence of a choice function $F : \mathcal{P}(v) - \{\phi\} \rightarrow v$ with $F(x) \in x$. Now the argument that Ω can be projected into v can be represented by defining the following function by recursion

$$g(0) = F(v)$$

$$g(\gamma) = F(v - g[\gamma]),$$

and then we know that g must be defined on all ordinals as, otherwise, consideration of the least ordinal for which g is not defined contradicts (∗).

Thus it is possible to get a proof of the well ordering theorem from C's proof (by eliminating one of the reductio ad absudrums) so that there are grounds for believing his proof. However, it remains true that Zermelo was the first person to rigourously prove the well ordering theorem without using inconsistent multiplicities.

Chapter 9

ZF and quasi-constructive approaches to set theory

9.1 Historical developments of ZF and NBG

Briefly, Zermelo first axiomatised part of Cantor's work (see [72]) and then Frankel noted the omission of the replacement axiom (see [15]). However, Zermelo's axiomatisation included the notion of a " definite property ", or definite assertion, so that his comprehension axiom took the form

For every definite propositional function F(x),

$\forall y \exists z \forall t \; (\; t \in z \leftrightarrow F(t) \wedge t \in y \;).$

It is not completely clear what Zermelo meant by a definite property, but Skolem suggested that it could be taken as any first order expression (see [69]), giving us the theory which is now known as ZF. We believe that Skolem's suggestion is, essentially, a correct interpretation of Zermelo's ideas, except that Zermelo wanted to allow all (definite) predicates to appear in the comprehension axiom rather than just $\in$, so that his notion is open ended.

Another line of development from Zermelo's axioms is that which considers definite properties as objects in themselves. This started with von Neumann (see [46]) and his justification of this step seems to be somewhat formalistic as he talks of how far the abstraction principle can be extended without generating the paradoxes. We shall ignore the fact that von Neumann's work is couched in terms of functions, but just note that the theory was put nearer modern NBG by Bernays in [4] : his theory explicitly considers two types of individuals, sets and classes, adopting an extensional view of both. For the rest of this chapter we shall use the term class for proper classes (i.e. those classes for which there is no set which has the same members). The obvious question which we must now consider is what these classes are supposed to be.

From the Cantorian viewpoint it would seem natural to think of classes as inconsistent multiplicities, but this is alien to their appearing as definite collections in a formal system. The next alternatives is to think of classes as genuine properties (rather than collections of sets) or as the extensions of properties, possibly over some given collection. One criticism of both these approaches is that the notion of a property seems to be at least as complex as that of a set so that it is just as much in need of clarification: one need only consider the property of " not holding of itself ". Also, if we think of classes as genuine properties, then NBG does not seem to be reasonable for

(i) why should properties be extensional ?

(ii) presumably there is a property U with $x \in U$ corresponding to " x is identical with x ", so that $U \in U$ would have to hold.

There have been attempts to modify NBG to meet the second of these criticisms and we shall consider these in chapter 10.

The second of the alternative programmes was to consider classes as the extensions of properties, possibly over some given collection. Without the added condition, this view is still open to an obvious modification of (ii). Further, it is not at all obvious that the amended scheme could be carried out as the following situation might well arise. Suppose that we are taking classes as the extensions of properties over V, where, as in Ackermann's set theory, V is thought of as the collection of all sets. Then there should be a property P meaning " is a set " and a property Q meaning " is identical to itself " so that although these properties have the same extensions on V we would obviously want $\exists x\ (x \in Q \wedge \neg x \in P)$ to be true.

Thus none of the above explanations of the intended meaning of classes seem to be convincing. This leads us to consider two weaker alternatives. Firstly, classes could be thought of as virtual objects, in the sense of Quine in [55], so that they are identified with first order definable predicates. On this view they become a convenient aid and , although they add nothing to our understanding of the nature of sets, they might make proofs easier to follow. Finally, one could adopt a formalist position and maintain that one is only interested in the usual models of first order ZF. Then classes are thought of as (certain) subcollections of the domain of the relevant model. This view, possibly that which is held by a number of people who work with NBG, has the disadvantage that it becomes meaningless when applied to the intended Cantorian interpretation of sets. It could still be useful though, if one thinks of formal set theories as picking out certain sets via their natural models etc.

9.2 Shoenfield's principle

When introducing ZF in set theory courses now it is very popular to use the idea of building up a cumulative type structure as the heuristic guide. A typical treatment of this is given in [66], where we find

" We then form sets in successive stages. At each stage we have already the urelements and the sets formed at earlier stages; and we form into sets all collections of these objects. A collection is said to be a set only if it is formed at some stage in this construction . . . . Since we wish to allow a set to be as arbitary a collection as possible, we agree that there shall be such a stage [i.e. one following a given collection of stages] whenever possible, i.e. whenever we can visualise a situation in which all the stages of the collection are completed. . . . If a collection consists of an infinite sequence S_1, S_2, . . . of stages, then we can visualise a situation in which all of these stages are completed, so there is to be a stage after all of the S_n. . . . . Suppose that we have a set A and that we have assigned a stage S_a to each element a of A. Since we can visualise the collection A as a single object (viz. the set A), we can also visualise a situation in which all of these stages are completed. This result is called the principle of cofinality. "

There are certain problems connected with a literal interpretation of these ideas, such as what indexes the stages and what " assigned " means, but these do not affect what is the intended meaning. Shoenfield goes on to justify all the axioms of ZF using this principle. We consider this principle, which is sometimes known as Shoenfield's principle, to be a variant of Cantor's second principle (from the 1883 paper) combined with the power set axiom. Later, we shall show that it follows from considerations of the Absolute so that, in an imprecise sense, it is half way between ZF and the Absolute.

A significant problem for Shoenfield's principle is that it is phrased in terms of the notions of building up stages and visualising situations so that the usual first order semantics do not give an intended model. Thus it only justifies ZF if we can jump to the conclusion that the process of visualising and completing has itself been completed as otherwise it is not obvious that the law of the excluded middle would hold. This is suggested by Kripke's constructive semantics (see [35]) where the law of the excluded middle can fail although, as Kreisel mentions in [33] , this only holds for models which are themselves sets. Also, this slightly dubious point (if the building up and visualising is completed, then why can we not start again ?) makes the set concept seem more complex than is necessary (see the next section). This makes some people worry about such building up processes.

The problem of formalising Shoenfield's principle is considered in [57] . Reinhardt slightly modifies it to

(S) " If P is a property of stages and if we can imagine a situation in which all the stages having P have been built up, then there exists a stage s beyond all of the stages which have P. "

He introduces a new constant V such that $x \in V$ is to be thought of as " x is a set ", and then he produces a set theory S^+ which has some similarities with Ackermann's system. S^+ has variables for properties and an axiom corresponding to (S). Reinhardt shows that S^+ is very much stronger than ZF and, although this is very interesting, there are still problems about what V and the properties are intended to be. It is suggested in [57] that the usual semantics are not really adequate for these ideas and it is a significant open problem to introduce a suitable semantics. Perhaps this is where one should start in formalising classes. In the philosophical remarks at the end of [57] , Reinhardt states that

" I have tried to introduce the axioms for properties in such a way that the naive reader will find them natural for naive (or Cantor's) set theory ",

but, again at the risk of overemphasing a point, we do not think that it is reasonable to introduce properties as consistent collections whilst maintaining a Cantorian viewpoint.

Finally, we note that Shoenfield's principle could be argued to give answers to some questions which are independent of ZF. For instance, it seems much easier to visualise a situation in which there is a scale for $^{\omega}\omega$ than one where the is no such scale. Are we then justified in asserting the existence of such a scale ?

9.3 ZF from the Absolute

In this section we hope to show that ZF can be justified by considerations of the Absolute. The viewpoint which we adopt is an extrapolation from that of [10] , but we do not claim that this is an exposition of Cantor's views.

We are thinking in terms of collections of objects where a collection is thought of as a ' bringing together ' of the objects under consideration. However, we must firstly ask what the Absolute is. Basically, we think of it in terms of everything which has ultimate existence: we shall not consider its metaphysical overtones. With Cantor, we believe that the Absolute can be recognised (which implies that it is a meaningful notion, of course) but that it can never be known. The latter point means that it is not good enough to imagine some very large set playing the part of the Absolute because the inherent nature of the Absolute ensures that it cannot be thought of as a unity in itself. Our usage of consistent and inconsistent multiplicities will be as in the last chapter and we identify sets and consistent multiplicities. It does not seem to be immediately true that all inconsistent multiplicities have the same " size " as the Absolute, but we shall often assume that they share much of the nature of the Absolute. If we add a new principle saying that all inconsistent multiplicities are of the same " size " (this would be analogous to von Neumann's maximal principle), then many of our arguments would flow more smoothly. We shall not do this as we do not find such a principle completely convincing.

Extensionality is basic for the view of sets which we have adopted and we next indicate how a version of Shoenfield's principle can be justified. The axiom of infinity follows from this by considering the natural numbers. Consider the version of (S) with ' property ' replaced by ' collection ', and then if we imagine a situation in which all the stages in P have been completed, we can imagine the collection of those stages as a consistent totality. The nature of this collection is not that of the Absolute (or any other inconsistent multiplicity) so that we have a consistent multiplicity and there is a stage beyond all those in the collection P. We shall not use Shoenfield's principle to justify the remaining axioms of ZF as we believe it overcomplicates matters, but we indicate how they can be got directly from considerations of the Absolute.

The replacement axiom follows from Cantor's statement that " two equipotent multiplicities are both consistent or both inconsistent " . This is the same as saying that there cannot be two equipotent collections, one of which is an inconsistent multiplicity and the other of which is a set: this seems a transparent fact from the nature of the Absolute. The comprehension axiom, in the form that every subcollection of a set is a set, similarly follows from the nature of inconsistent multiplicities.

The sum and power set axioms follow as it is inconceivable that an inconsistent multiplicity could be obtained from a set by one of these visualisable operations. This even clearer if we assume that all inconsistent multiplicities are the same size, for then the power set axiom, for instance, says that there is no set for which the collection of all its subcollections is the same size as the Absolute.

The axiom of foundation does not seem to be evident on this interpretation, although there is no reason why one should not restrict one's attention to well founded sets if it is desired. Of course, the non existence of cycles of sets follows from our basic viewpoint of forming collections by bringing together certain objects. We consider the axiom of choice to be a logical principle for sets so that it is not in need of justification.

Now we consider two other kinds of axioms from this point of view.

(i) Let Ω be the inconsistent multiplicity consisting of all ordinals, ordered by their natural ordering. We consider certain axioms about " stopping points " in Ω. It is convenient to think in terms of processes for going up Ω and then the nature of the Absolute shows that there cannot be any definite process, the completion of which is Ω. Thus if $\forall\alpha\,\exists\beta\;\phi(\alpha,\beta)$ there must be a cardinal κ such that from below κ this process (i.e. going from α to the least β satisfying $\phi(\alpha,\beta)$) does not get beyond κ. Further, it is reasonable to insist that κ is regular as otherwise the process can be continued by taking the union of a shorter cofinal sequence. Consequently, we have the schema

$\forall\alpha\,\exists\beta\;\phi(\alpha,\beta)\rightarrow\exists\kappa(\,\mathrm{Reg}(\kappa)\wedge\forall\alpha\in\kappa\,\exists\beta\in\kappa\;\;\phi(\alpha,\beta)\,)$,

which, together with ZF, gives the theory ZM (we showed that in [37])

(ii) The existence of a measurable cardinal does not seem to be justified, at the moment, by arguments similar to those which we have already encountered.

(i) shows that ZM can be justified from the Absolute and (ii) suggests that one should investigate other ways of justifying axioms from the Absolute. Whether or not measurable cardinals turn out to be reasonable, the latter programme should be very useful. For instance, does it give any new structual information ?

9.4 Intuitionistic ZF

Intuitionistic ZF is ZF set theory based on intuitionistic logic. Myhill, in a seminar, suggested that such a theory, without the axiom of choice, corresponds to that part of ZF which gives effective results, using this word in the sense of [68]. This is a thoroughly reasonable attitude and, like Church's thesis in recursion theory, the conjecture is open to empirical testing.

However, intuitionistic ZF is also the end product of a paper of Pozsgay's, [52], and for the remainder of this section we shall be considering this paper. Pozsgay claims to be formalising a certain intuitive approach to set theory which he thinks represents the basic insights underlying the ZF axioms. He thinks of sets as mental constructions and he gives the following principle for set construction.

" Any well defined mental process for constructing sets which has been clearly envisioned without ambiguities or contradictions may be regarded as already completed, regardless of any merely practical difficulties which may prevent one from actually carrying it out. "

On the basis of this principle Pozsgay argues that we can justify the axioms of ZF and, in particular, the power set axiom. But what mental process is available for constructing the power set of ω? Certainly we cannot give any step by step procedure for doing this as any countable number of countable processes will remain countable. Somehow we need to jump to the uncountable set. Consequently, we feel that this principle does not justify the power set axiom, but that it must be added as a further principle. Then we seem to get Shoenfield's principle, though.

Pozsgay's paper splits into two sections and in the second he turns to the problem of formalising his principle, where he says

" As far as set theoretic axioms go, the best available seem to be the ZF axioms, and the main question is whether the underlying logic should be intuitionistic or classical. "

The procedure now seems to have very little to do with the original principle. For example, a first order theory is assumed without any explanation of how this affects the power set operation, although, in justifying the comprehension axiom Pozsgay circumvented the problem of impredicativity by saying that he took all possible subcollections of a set in the power set. Consequently we feel that the reasons for using ZF to formalise this work are a little obscure, but the reason for using intuitionistic logic seems even less clear.

Pozsgay states that he wants $\exists x\ B(x)$ only to be provable if there is " at hand a definite construction for producing a set x with the property B(x) ". Two pages previously he justified the axiom of choice and it remains a complete mystery how we are to give a definite construction for a choice function on infinitely many pairs of socks.

Basically, [52] belongs to those approaches to set theory which can be thought of as " building up in time " and hence we do not see how ω_1 can be thought to exist (unless one adds the power set as an additional basic operation). Hence Powell's approach to such a theory in [51] seems more reasonable, if one is not going to allow time to be completed.

In [37] we gave a possible axiomatisation of Pozsgay's building up ideas, but we now think that Wang's system of predicative set theory (Σ , see [71]) is probably a better candidate for such a theory. To really axiomatise Σ we should make explicit the principles by which one indexes the types: perhaps we could just allow completions of fundamental sequences for some given system of notations. Section 4 of [37] contains some considerations of the power set axiom and we now believe that the ideas of that section are superseded by that of a mild second order logic, which we introduced in the last chapter.

Chapter 10

Set theories with a universal set

10.1 Introduction

In this chapter we shall consider some aspects of set theories in which there is a universal set (i.e. a set x such that for all sets y $y \in x$). Such a set cannot exist from a Cantorian viewpoint so there must be some other motivation for such theories. One possible approach is via properties and such theories are discussed in sections 5 and 6. The remaining theories all seem to result from formalist inspiration and the main one of these theories is NF: sections 2-4 are devoted to questions related to this theory.

Another approach to set theories with a universal set has been made by Church in [11] . Here the motivation is that the abstraction principle is desirable but (unfortunately ?) it turns out to be inconsistent so that we must investigate all (formalistic) ways of approximating to it whilst remaining within the realms of consistency or, at least, relative consistency. This view also seems to be an assumption for the book by Frankel, Bar-Hillel and Levy ([16]). We have little sympathy with such ideas as there does not seem to be any clear reason why we should have believed the abstraction principle in the first place.

10.2 Quine's NF

The theory NF was introduced in [53] and is formulated with $\in$ as the only predicate. Equality is introduced by definition and there is an axiom of extensionality. The only other axiom is the abstraction principle for those formulae ϕ which are stratified (i.e. one can attach numerals to the variables in such a way that whenever $x \in y$ occurs in ϕ with n attached to x, then n+1 is attached to y). The motivation behind this is that stratified formulae correspond, in an obvious way, to those of type theory and that the paradoxes (at least, the old familiar favourites) do not seem to be derivable in the theory. Thus NF is a formalist's theory, but it still could be a reasonable set theory as well.

In [16] it is suggested that the unprovability of all instances of induction in NF, if this theory is consistent, shows that it is not a reasonable theory, but it would be nicer to have a stronger condemnation. The next section contains some arguments which show that NF is not, as it stands, a good set theory, in the sense that it is not adequate to describe certain mathematical notions.

Section 2 of Rosser and Wang's paper [62] claims to show that if NF is consistent (we always assume this when discussing its models) then it does not have a standard model. Briefly, the argument is as follows. NF is assumed to have a model in which the natural numbers are standard and then, using Rosser's paper [60] , one shows that transfinite induction cannot hold for all the formulae of NF. Consequently, the order relation of the ordinals in the model is not really well founded and NF cannot have a standard model.

The actual arguments which are used in the proof are correct but it is implicit throughout that the definition of ordinal which is used (equivalence classes of ' well ordered classes ', in the sense of NF - ordinal(NF), say) corresponds to the intuitive notion of ordinal (ordinal(I), say). There is no attempt in [62] to show that ordinal(NF) is a good approximation to ordinal(I). Usually, the definition of an ordinal occurs within an enviroment where we may suppose that all instances of the comprehension axiom hold and when this is not the case the definition of an ordinal is suitably modified (see, for instance, [17] or [40]). From page 474 of [61] we know that NF does not ensure that the order type of the class of ordinals(NF) less than an ordinal(NF) α is α, so that it is natural to strengthen the definition ordinal(NF) to

$$\text{ordinal}'(\text{NF})(x) \equiv \text{ordinal}(\text{NF})(x) \wedge \text{' the order type of the ordinals(NF) less than } x \text{ is } x \text{ '.}$$

However, we still would not know that ordinal'(NF) is a good approximation to ordinal(I) in NF. Indeed, there might be no formula of NF which satisfies this requirement.

On this basis we suggest that Rosser and Wang's result shows that if NF has a standard model, then ordinal(NF) does not represent the notion ordinal(I) in NF. This suggests that one should look at the adequacy of the representations of the usual mathematical notions in NF, rather than assuming that a formal definition gets its intended meaning; we start this in the next section.

10.3 Ordered pairs in NF

In any set theory, two sets are said to have the same cardinality if there is a bijection between them. Thus the notion of having the same cardinality (which we call being equipollent) is dependent on that of function and hence on that of ordered pair. We shall show that in NF, the definition of ordered pair which is used affects whether, or not, two sets are equipollent, and we make some further considerations based on this fact. The following definitions will aid our discussion; we hope that it is obvious how they could be made precise.

Definition 10.1 A formula $\psi(x,y,z)$, with exactly three free variables, is said to represent an ordered pair relation in a set theory T if

(i) $T \vdash \forall x,y \exists ! z\ \psi(x,y,z)$, and

(ii) $T \vdash \forall x,x',y,y',z\ (\ \psi(x,y,z) \wedge \psi(x',y',z) \rightarrow x = x' \wedge y = y'\ )$.

Definition 10.2 If ψ represents an ordered pair relation in a set theory T, then $x \approx_{\psi} y$ is a formula which, in a natural way, says that there is a function, represented as a set of ordered pairs which are defined using ψ, which is a bijection from x to y.

We shall always assume that $z = \langle x,y \rangle$ is a formula which says that z is the Kuratowski ordered pair (i.e. $\{\{x\},\{x,y\}\}$) and this represents an ordered pair relation in both ZF and NF. Also, $x \approx y$ means $x \approx_{\phi} y$, where ϕ is the formula $z = \langle x,y \rangle$.

The next theorem shows that, in a certain sense, the notion of being equipollent is independent of the representation of ordered pairs in ZF set theory.

Theorem 10.3 If ψ represents an ordered pair relation in ZF, then $ZF \vdash \forall u,v \; (u \approx v \leftrightarrow u \approx_{\psi} v)$.

The proof of this result is completely straightforward. For instance, if $u \approx v$ then let f be a bijection from u to v, put $f' = \{ z \mid \exists x,y \; (\psi(x,y,z) \wedge \langle x,y \rangle \in f \}$ and verify that $u \approx_{\psi} v$ using f'.

From a mathematical point of view theorem 10.3 is highly desirable as the actual structure of the ordered pair does not seem to be important for two sets being equipollent. However, provided that NF is consistent, the analogous form of theorem 10.3 for NF is false, even if we restrict ψ to being a stratified formula. This can be seen as follows. If $\psi'(x,y,z)$ is the formula $z = \{\{x\},\{x,\{y\}\}\}$,then ψ' represents an ordered pair relation in NF. By considering Cantor's theorem for NF in [54] Quine showed that if $V = \{x \mid x = x\}$ and $S = \{x \mid \exists y \; x = \{y\}\}$, then $\neg(V \approx S)$, but it is straightforward to show that $S \approx_{\psi'} V$, in NF, as required.

The key point in this counterexample is that we have represented an ordered pair relation using a formula which can only be shown to be stratified by attaching different numerals to x and y. It might be argued that this is not desirable in NF, but then one must explain the process of stratification in such a way that this becomes highly unreasonable as, from a mathematical point of view, there is no significance in the representation of ordered pairs. The following weak form of theorem 10.3 does hold for NF.

Theorem 10.4 If $\psi(x,y,z)$ and $\psi'(x',y',z')$ are formulae which represent ordered pair relations in NF and can be shown to be stratified in such a way that one numeral can be attached to both x and x' and another to both y and y', then
$NF \vdash \forall u,v \; (u \approx_{\psi} v \leftrightarrow u \approx_{\psi'} v)$.

Theorem 10.4 shows that when considering sets being equipollent in NF, it is only the way in which the ordered pair relation can be shown to be stratified (we restrict our attention to stratified definitions from now on) which is important. Hence the following definition of $\approx_i$ is independent of which ψ we choose.

Definition 10.5 If $\psi(x,y,z)$ represents an ordered pair relation in NF and can be shown to be stratified by attaching a numeral n to x and a numeral m to y, and $i = m-n$, then we write $x \approx_i y$ for $x \approx_{\psi} y$. For definiteness, we could take $z = \langle x, \underbrace{\{ \ldots \{}_{i \text{ brackets}} y\} \ldots \} \rangle$ for ψ when $i \geqslant 0$, and $z = \langle \underbrace{\{ \ldots \{}_{-i \text{ brackets}} x\} \ldots \}, y \rangle$ for ψ when $i < 0$.

We can now reformulate the results which we quoted earlier as $\neg(V \approx_0 S)$ and $S \approx_1 V$. Another result of [54] shows that $\neg(V \approx_1 V)$ although, of course, $V \approx_0 V$. Our next theorem notes some properties of being i-equipollent (i.e. $\approx_i$) and it is obvious how these are generalisations of being O-equipollent.

<u>Definition 10.6</u> $x^{(m)} = \{y \mid \exists t \in x\ y = \underbrace{\{\ldots\{t\}\ldots\}}_{m \text{ brackets}}\}$.

<u>Theorem 10.7</u> The universal closures of the following statements are provable in NF

(i) $x \approx_0 x$ (i.e. $x \approx x$),

(ii) $x \approx_i y \rightarrow y \approx_{-i} x$,

(iii) $x \approx_i y \wedge y \approx_j z \rightarrow x \approx_{i+j} z$,

(iv) $x^{(m)} \approx_m x$.

The proof of theorem 10.7 is straightforward. It might be interesting to investigate further properties of i-equipollence, but we shall next consider a method of extending NF.

It seems emminently reasonable to suggest that if $u \approx_i v$, for any integer i, then u and v are equipollent in an intuitive sense. Consequently, we let ENF be NF extended by adding a new symbol $\approxeq$, together with the axiom

$$u \approxeq v \leftrightarrow \text{for some integer } i,\ u \approx_i v. \qquad (*)$$

We shall not consider methods of formalising (*) in first order terms but will continue to treat it in an intuitive sense. Theorem 10.7 then shows that $\approxeq$ has the properties

(i) $x \approxeq x$,

(ii) $x \approxeq y \rightarrow y \approxeq x$,

(iii) $x \approxeq y \wedge y \approxeq z \rightarrow x \approxeq z$,

(iv) $x^{(m)} \approxeq x$,

so it seems that $\approxeq$ is a more reasonable formulation of being equipollent than $\approx$ in NF as $\approxeq$ also posesses the intuitively true property (iv). To actually work in ENF we would probably have to add axioms asserting the existence of cardinals, as equivalence classes under $\approxeq$, and other comprehension principles, but we shall leave these problems. We shall next consider the interpretations, when $\approx$ is replaced by $\approxeq$, of two results which have been proved for NF.

In [49] it is shown that if NF is consistent, then the axiom of counting is not provable in NF. This axiom is the intuitively true statement

$\forall n(Nn(n) \rightarrow \{m \mid Nn(m) \wedge m < n\} \in n)$,

where Nn(n) is a formula saying ' n is a natural number ' (using $\approx$ for equipollence). Hence, the axiom of counting says that if Nn(n), then for some $t \in n$,

$$\{m \mid Nn(m) \wedge m < n\} \approx t. \qquad (**)$$

(To consider (**) in ENF we should really consider natural numbers as equivalence classes under $\approxeq$, rather than $\approx$, but, for convenience, we continue to use Nn defined using $\approx$.)

Intuitively, the reason why (**) is not derivable in NF is that the objects on the left and the right are of different " types ", although it is straightforward to show that the following version of (**) is provable for some $t \in n$,

$\{m \mid Nn(m) \wedge m < n\} \approx_1 t.$

Thus in ENF we have $\{m \mid Nn(m) \wedge m < n\} \approxeq t$, again suggesting that $\approxeq$ is a better notion of being equipollent than $\approx$.

Henson showed in [25] that if Nc(x) is the cardinal of x, then it is relatively consistent with NF that for finite sets x we have $Nc(\mathcal{P}(x)) < Nc(x)$ or $Nc(\mathcal{P}(x)) > Nc(x)$. He also showed that we can have $Nc(x^{(1)}) <$, = or > $Nc(x)$. We have already noted that the latter pathologies are eliminated in ENF as $x^{(1)} \approxeq x$, In ENF $x \approx x^{(1)} \subseteq \mathcal{P}(x)$ so that $\mathcal{P}(x)$ will probably be at least as big as x, but we do not seem to get an immediate answer to this problem.

There are a number of similar problems which could be investigated in ENF and one could also consider other properties which depend on ordered pairs. For instance, in NF ordinals are equivalence classes under similarity where this is defined using a 0-bijection (an obvious extension of our notation), but it seems more natural to allow all i-bijections. To formulate such a theory in detail seems to require an inordinate amount of work. It would be nice to show that any such extension of ENF is inconsistent, but proofs using the idea of Cantor's theorem do not seem to yield such a result. The Burali-Forti paradox, perhaps ?

On the above basis we think it reasonable to claim that NF is not a nice set theory as various natural notions, such as equipollence, depend on the way in which ordered pairs are represented. Further, if the theory is extended to take care of these problems, then the resulting system would be extremely complicated and completely unusable.

10.4 Consistency of a fragment of NF

In [23] Halperin showed that NF can be finitely axiomatised using extensionality and P1-P9, which are all instances of NF's comprehension axiom. By constructing a model in number theory in [6] , Benes showed that extensionality and P1-P8 are consistent. Starting from a Benes-like construction and iterating transfinitely we proved theorem 10.8. The proof which we now indicate starts from a model of NFU (this is NF with extensionality replaced by $\exists z \in x \wedge \forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y$), which is proved consistent in [29] , as this is more straightforward. We only outline the proof as we will refer to this method again later. Modifications of it yield the relative consistency of other fragments of NF, but none of these methods seem to give a result for full NF.

Theorem 10.8 In ZF we can prove the consistency of the theory whose axioms are extensionality, P1-P5 and P7-P9.

Proof (Outline) Let $\langle N_0, E_0 \rangle \models$ NFU in which N_0 is the set of natural numbers. We define a sequence of models $\langle N_\lambda, E_\lambda \rangle$ for $\lambda \in \omega_1 + 1$. E_λ will always be $N_\lambda^2 \cap E_0$ and we define N_λ by induction:

(i) Suppose that $\lambda = \gamma + 1$ and we are given N_γ. Put $A_\gamma = \{ t \mid t \subseteq N_\gamma$ and $\forall y, z \in t \, \forall n \in N_\gamma (nE_0 y \leftrightarrow nE_0 z) \}$ and $N_{\gamma+1} = \{ m \mid$ for some $t \in A_\gamma$, m is the least member of $t \}$.

(ii) For limit λ, put $N_\lambda = \bigcap_{\gamma < \lambda} N_\gamma$.

We have $N_0 \supseteq N_1 \supseteq N_2 \supseteq \ldots \supseteq N_{\omega_1}$ and we next show that N_{ω_1} is of cardinality ω. Let k be the least natural number which represents an urelement in $\langle N_0, E_0 \rangle$ and then consider the sets k, $\{k\}$, $\{\{k\}\}$, ..., where $\forall n (nE_0 \{k\} \leftrightarrow n = k)$ etc. Each of these numbers can be ' replaced ' only finitely many times in the production of the N_λs and let $\breve{x}$ be that number which x ' ends up ' as. Clearly, $\breve{k}$, $\breve{\{k\}}$, ... will all be different numbers so that N_{ω_1} is infinite.

As N_λ is always countable there must be some $\gamma < \omega_1$ for which $N_\gamma = N_{\gamma+1}$ and let η be the least such ordinal. Extensionality clearly holds in $\langle N_\eta, E_\eta \rangle$ and we indicate why P1 holds in this structure: the verifications of the other axioms are similar. P1 is $\forall u, v \exists y \forall x (x \in y \leftrightarrow \neg x \in u \wedge \neg x \in v)$, and suppose that $u, v \in N_\lambda$. Then as $\langle N_0, E_0 \rangle \models$ P1, there is a $y \in N_0$ which satisfies P1 there. $\breve{y}$ has the required property in $\langle N_\eta, E_\lambda \rangle$. □

10.5 Properties as properties

Sets can be considered as collections of objects which satisfy a given property, or in other words, as the extensions of properties. This is the usual view from which people argue that the abstraction principle is intuitively plausible, but there seems to be no agreement as to whether the variables are ranging over properties, objects, extensions over some collection, or anything else.

The property of " not satisfying itself " might show that if properties are allowed to apply to properties, then we cannot expect them to be everywhere defined: this is probably the motivation behind Kreisel's following remarks on properties in [34].

" For this notion, with $y \in x$ being interpreted as: the property y has the property x, $\exists x \forall y \, (y \in x \leftrightarrow P)$ [i.e. the abstraction principle] is indeed evident, provided that the most general kind of property is considered, including properties which are not everywhere defined. "

He goes on to say that we cannot expect the usual logical laws to hold in such a system but we find it unlikely that the logical laws must be altered before we can talk about properties: we consider another way of approaching this problem below. Kreisel also suggests that no property can be defined for itself as argument whilst consideration of the property of " being a property " suggests that sometimes this might be quite harmless.

An earlier suggestion regarding an approach to properties (or concepts - we make no distinction between these notions) was given by Godel in [19] , where he says

" It is not impossible that the idea of limited ranges of significance could be carried out without the above restrictive principle [referring to type theory]. It might even turn out that it is possible to assume every concept to be significant everywhere except for certain " singular points " or " limiting points ", so that the paradoxes would appear as something analogous to dividing by zero. "

We next outline a framework, based on the first order predicate calculus with identity, within which such ideas can be formalised. There are two predicates:

$M(x,y)$ for " it is meaningful to ask if the property x has the property y ", and

$x \eta y$ for $\begin{cases} \text{" the property x has the property y ", if } M(x,y) \\ \text{no intended interpretation, if } \neg M(x,y). \end{cases}$

If K is any η-formula, then we define a translation giving a formula K^{+}, as follows: every instance of $\forall x\ x \eta y$ is replaced by $\forall x\ (M(x,y) \rightarrow x \eta y)$, of $\exists x\ x \eta y$ by $\exists x\ (M(x,y) \wedge x \eta y)$ etc., in such a way that $x \eta y$ only occurs when we have $M(x,y)$. (This is an obvious generalisation of the translation described in section 6.3.) Then if K is an η-formula, the abstraction principle takes the form

$$\exists y \forall x\ (M(x,y) \rightarrow (x \eta y \leftrightarrow K^{+})). \qquad (*)$$

Thus we have formalised a framework for talking about properties which are not meaningfully defined everywhere, without altering the underlying logic. The paradoxes give us examples of properties for which $\neg M(x,y)$ holds and the main open problem is to say for which properties we have $M(x,y)$. [27] shows that if we have

$\forall x \not\exists y\, M(x,y)$, then (*) is still inconsistent, and if we take $M(x,y)$ as $\exists z\, x \eta z$, then (*) turns into the class existence axiom of NBG.

Question 10.9 Is there any natural way (syntatic, or otherwise) of saying when $M(x,y)$ holds in the above system ?

During the above considerations the variables were assumed to be ranging over properties. Given that a system of properties could be produced, it is often suggested that extensional collections can be obtained just by " taking the extensions of the properties ". Two possible interpretations of this view are

(i) the extensions are taken over all possible objects, and

(ii) the extensions are taken over some given collection of individuals,

and we suppose that x, y, . . . range over the resulting extensions. If (i) is assumed and we suppose that the extensions are already objects, then it seems quite possible for two extensions to have the same extensions as members, but to differ over some property. Thus, such a system would only be extensional if there are urelements in the theory: this seems a little surprising. If (ii) is adopted, then it is not at all clear what the membership relation is intended to mean and it certainly cannot be the original η. Consequently, we suggest that the notion of taking the extensions of properties to get an extensional system is still in need of clarification.

10.6 Other views of properties

The approach to properties with which most people are familiar is that of Zermelo's in [72] ; which was refined in [4] and [21]. Basically, this view assumes the existence of a totality of all sets and works with it exactly as if it were a set: we criticised this in chapter 9.

Zermelo's original motivation seems to be similar to Russell's notion of a propositional function and, although it is not completely clear, one way of viewing this is as a variable ranging over the first order formulae of a given language (cf. a weak second order logic). However, during his later work (see [73]) Zermelo has extended his ideas to arbitary propositional functions and it might be possible to make some sense of this idea without using proper classes.

One method of extending NBG is considered by Powell in [50] . Here, properties are identified with their extensions on V and a different predicate is used for " has the property ". This is shown to lead to quite a strong theory with other interesting features, but a point which does not seem to have been considered is why two different properties should not have the same extension over V. Also, this approach does not allow quantifiers over properties to occur in the main comprehension axiom.

Another extension of Zermelo's approach is [57] , where Reinhardt includes an axiom corresponding to Shoenfield's principle (see section 9.2). The intended semantics of this system has modal overtones and there are some similarities between the systems of [57] and [50] .

Despite our doubts about the ontological overtones of systems such as NBG, it is still possible to view these theories as ways of delimiting various levels in the cumulative hierarchy by means of their natural models. There seems to be an implicit belief that any reasonable set theory will have such a natural model, but next we attempt to give a counterexample to this. In chapter 7 we suggested that the following is a reasonable axiom of set theory

(C) If X is a class of ordinals such that for some β, X is a branch of $E(\beta)$, then X is a set.

We suggest that NBG + (C) is a suitable theory as it clearly has no natural models (i.e. models of the form $\langle R_{\alpha+1}, \in \rangle$). The consistency of NBG + (C) can be proved in MK as follows. Let κ be the least cardinal for which $R_\kappa \prec V$ and then $R_\kappa \models ZF$ with the property that (C) is true for X being any subclass of R_κ. The usual relative consistency proof for NBG and ZF (see [48]) then gives a model $\langle R_\kappa \cup A, \in \rangle$ of NBG + (C), for some $A \subseteq R_{\kappa+1}$.

Of course, NBG + (C) is not a reasonable set theory from our point of view because of the existence of proper classes, but it might be possible to include the essence of its axioms in a modified version of ZF (strong replacement is catered for in a mild second order logic so (C) is the only remaining problem). Also, the fact that MK + (C) is inconsistent can be taken as a condemnation of the naive approach to proper classes.

Appendix

Some results on extended ordinal arithmetic

A.1 Introduction

Extended ordinal arithmetic was introduced by Doner and Tarski (in [12]) as a continuation of the recursive definition of ordinal multiplication in terms of ordinal addition. The extended operations, O_γ, are defined by

$$\alpha O_\gamma \beta = \alpha + \beta \quad \text{, when } \gamma = 0,$$
$$\alpha O_\gamma \beta = \bigcup_{\eta<\beta,\, \zeta<\gamma} ((\alpha O_\gamma \eta) O_\zeta \alpha), \text{ when } \gamma > 0,$$

and it is straightforward to check that this is a natural generalisation with O_1 corresponding (essentially) to multiplication and O_2 to exponentiation. Theorem 3 of [12] shows that $\alpha O_2 (1+\beta) = \alpha^{\alpha^\beta}$, but a few calculations show that the higher operations increase much faster. Some basic properties of the extended ordinal operations, some identities and some results concerning main numbers (i.e. those ordinals δ such that for a given $\alpha, \beta < \delta \quad \alpha O_\gamma \beta < \delta$) are also proved in [12]. For convenience, we shall refer to [12] as [D-T] in this appendix: similarly, we refer to [63] and [64] as [RR1] and [RR2], respectively.

Part of [RR1] gives necessary and sufficient conditions for the associativity and commutativity of O_γ when γ is a limit ordinal. The corresponding results for O_0, O_1 and O_2 are classical. In section 2 we prove analogous theorems for O_3 and indicate some other results. Section 3 gives some inequalities for the extended ordinal operations and we show how these prevent one from giving a straightforward answer to one of the problems which was raised in [D-T].

It was suggested by J. Rubin (in a letter) that it should be possible to extend some of the classical results about permuting the elements of infinite sums to the extended operations and in section 5 we indicate how some of these results can be directly transferred. However, it seems that many of the classical results for infinite sums are far from best possible and we improve one of them to a best possible result in section 4. Anderson has also tackled some problems in this field (see [2]).

A.2 Some properties of O_3

Our first result gives necessary and sufficient conditions for the associativity of ordinals with respect to O_3.

Theorem A.1 $(\alpha O_3 \beta) O_3 \delta = \alpha O_3 (\beta O_3 \delta)$ iff one of the following conditions holds

(1) any one of α, β, δ is 0 or 1,

(2) δ is an ε-number and $\alpha, \beta < \delta$,

(3) $\delta = \omega$ and $\alpha, \beta < \delta$,

(4) $\delta = 2, \alpha < \omega$ and β is any ordinal for which $\beta = \omega\beta = \beta_1 \beta$ where $\beta = \omega^{\beta_1} b_1 + \dots$ is the normal form of β,

(5) $\delta = 2$, $\alpha, \beta \geqslant \omega$ and β is a limit ordinal, where the normal form of β is as above and that of α is $\omega^{\alpha_1} a_1 + \dots$, & $\beta_1 \cdot \beta = \alpha_1 \beta$,

(6) $\delta = 2$, $\alpha = \beta$, $\alpha \geqslant \omega$ and α is a successor ordinal,

(7) $\delta \in \omega - 3$, β is an infinite successor ordinal such that $(\delta - 1) \mid b_1$ where the normal form of β is as above, and a process for obtaining a unique α from β and δ can be described,

(8) $\alpha, \beta, \delta \in \omega - 2$ and $\beta + \alpha^{\beta - 1}(\delta - 1) = \beta^{\beta^{\delta - 1}}$

Proof We omit the proof as the number of subcases which have to be considered would make it about ten pages long: the result does not seem to justify this. □

Cases (1) - (7) of theorem A.1 give complete answers to the associativity of O_3. Although it is easy to solve case (8) when one of the variables takes a small value, we have not found a general solution. From page 363 of [68] we see that the ordinals which are associative with respect to O_3 do not coincide with those which are associative with respect to O_2. Theorem A.4 shows that this situation is unlike that with respect to commutativity.

We cannot extend the methods used in proving theorem A1.1 to higher γ as we do not have a suitable representation of the first term of the normal form of $\alpha O_\gamma \beta$ in terms of the normal forms of α and β, but the the next theorem shows that a partial generalisation of theorem A.1 gives a much nicer result.

<u>Definition A.2</u> δ is a main number of O_γ if for all $\alpha, \beta < \delta$ $\alpha O_\gamma \beta < \delta$. $M(O_\gamma)$ is the collection of main numbers of O_γ.

<u>Theorem A.3</u> If δ is a limit ordinal and $\gamma \geqslant 2$, then $(\alpha O_{2\gamma} \beta) O_{2\gamma} \delta = \alpha O_{2\gamma} (\beta O_{2\gamma} \delta)$ iff one of the following conditions is satisfied

(1) α or β is 0 or 1,

(2) $\beta < \delta \in M(O_{2\gamma})$.

<u>Proof</u> Suppose that δ is a limit ordinal and $\gamma \geqslant 2$. Then the theorem clearly holds if α or β is 0 or 1, so that from now on suppose that $\alpha, \beta \geqslant 2$. Theorem 32 of [D-T] then shows that $(\alpha O_{2\gamma} \beta) O_{2\gamma} \delta = \alpha O_{2\gamma} (\beta + \delta)$ so we have $(\alpha O_{2\gamma} \beta) O_{2\gamma} \delta = \alpha O_{2\gamma} (\beta O_{2\gamma} \delta)$ iff $\beta + \delta = \beta O_{2\gamma} \delta$.

Now suppose that

$$\beta + \delta = \beta O_{2\gamma} \delta \qquad (*)$$

If $\beta \geqslant \delta$ then $\beta O_{2\gamma} \delta \geqslant \beta \cdot \delta \geqslant \beta + \beta + \beta > \beta + \delta$, so that $\beta < \delta$. Theorem 47 of [D-T] shows that (2) will follow from $\beta O_{2\gamma} \delta = \delta$

and we now prove this. Let $\omega^{\delta_1} d_1 + \ldots$ be the normal form of δ and then the normal form of $\beta + \delta$ is $\omega^{\delta} e_1 + \ldots$, for some $e_1 \in \omega - 1$. $\beta\, O_{2\gamma}\, \delta \geqslant \beta O_2\, \delta = \beta^{\delta}$ so that as the first term of the normal form of β^{δ} is $\omega^{\beta_1 \cdot \delta}$, (*) shows that $\delta_1 \geqslant \beta_1 . \delta$. Hence $\delta_1 = \delta$ and δ is an ε-number. Then, from (*), $\beta\, O_{2\gamma}\, \delta = \delta$, as required.

If we have $\beta < \delta \in M(O_{2\gamma})$, then $\delta \leqslant \beta + \delta \leqslant \beta\, O_{2\gamma}\, \delta = \delta$ so that (*) holds and the theorem is proved. □

<u>Theorem A.4</u> $\alpha\, O_3\, \beta = \beta\, O_3\, \alpha$ iff $\alpha\, O_2\, \beta = \beta\, O_2\, \alpha$.

<u>Proof</u> Necessary and sufficient conditions for the commutativity of O_2 were given by Jacobsthal and his theorem is proved in [76]. Our proof uses modifications of his method and we omit it because of its length. □

As in the case of associativity, the method of proof used in theorem A.4 does not extend to higher γ. Our next result shows that for certain γ, the ordinals which are commutative with respect to O_γ are not commutative with respect to O_2.

<u>Theorem A.5</u> If $\alpha, \beta \geqslant \omega$, $\alpha < \beta$, $\gamma \geqslant 2$ and $\alpha\, O_{2\gamma+1}\, \beta = \beta\, O_{2\gamma+1}\, \alpha$, then $\alpha\, O_2\, \beta \neq \beta\, O_2\, \alpha$.

<u>Proof</u> Suppose that all the hypotheses of the theorem hold and that we also have $\alpha\, O_2\, \beta = \beta\, O_2\, \alpha$. Then α is a limit ordinal and $\beta = \tau\alpha$, where τ is an ε-number greater than α. Then, by theorem 33 of [D-T]

$\alpha\, O_{2\gamma+1}\, \beta = \alpha\, O_{2\gamma}(\alpha . \beta) = \alpha\, O_{2\gamma}(\alpha . \tau . \alpha) = \alpha\, O_{2\gamma}(\tau . \alpha)$.

Then, using that theorem again, $\alpha\, O_{2\gamma}\, \beta = \beta\, O_{2\gamma+1}\, \alpha = \beta\, O_{2\gamma}(\beta . \alpha)$.

However, $\alpha\, O_{2\gamma}\, \beta \leqslant \beta\, O_{2\gamma}\, \beta < \beta\, O_{2\gamma}(\beta . \alpha)$, contradicting this. □

Corollary A.6 If $\alpha < \beta$; α, β are limit ordinals, $\gamma \geqslant 2$ and $\alpha\, 0_{2\gamma+1}\, \beta = \beta\, 0_{2\gamma+1} \alpha$, then $\beta < \alpha^{\omega}.\omega$.

Proof From the proof of the theorem

$\alpha\, 0_{2\gamma+1}\, \beta = \alpha\, 0_{2\gamma}(\alpha.\beta) \leqslant \beta\, 0_{2\gamma}(\alpha.\beta)$ and $\beta\, 0_{2\gamma+1}\, \alpha = \beta\, 0_{2\gamma}(\beta.\alpha)$ so that we get

$$\beta.\alpha \leqslant \alpha.\beta. \qquad (**)$$

Assuming that the normal forms of α and β are as usual, considering the normal forms of the sides of (**) gives $\beta_1 + \alpha_1 \leqslant \alpha_1 + \beta_1$

Now let $\alpha_1 = \omega^{\eta_1} a' + \ldots$ and $\beta_1 = \omega^{\nu_1} b' + \ldots$

Then $\nu_1 = \eta_1$ as otherwise $\alpha_1 + \beta_1 = \beta_1$. Hence

$$\beta_1 = \omega^{\nu_1} b' + \ldots \leqslant \omega^{\nu_1+1} = (\omega^{\nu_1}).\omega \leqslant \alpha_1.\omega \ \&$$

$$\beta = \omega^{\beta_1} b_1 + \ldots \leqslant \omega^{\beta_1+1} \leqslant (\omega^{\alpha_1 \omega})\omega \leqslant \alpha^{\omega}.\omega. \ \square$$

The following relation was defined and studied in [RR1].

Definition A.7 $\alpha L_\gamma \beta$ iff $\exists \delta \neq 0\ (\delta\, 0_\gamma\, \alpha = \beta)$.

L_γ is transitive for $\gamma = 0,1,2$ or 3 and it is stated in [RR1] that L_4 is not transitive. It is left as an open question in [RR1] as to whether or not L_γ is transitive for limit γ. Our next result gives necessary and sufficient condition for the transitivity of L_γ, when γ is a limit ordinal, so that from theorem 28 of [RR1] it then follows that L_γ is not, in general, transitive for limit γ.

Theorem A.8 For limit γ; $\alpha L_\gamma \beta$, $\beta L_\gamma \delta$ and $\alpha L_\gamma \delta$ all hold iff one of the following conditions is satisfied

(1) $\alpha = \beta$ and $\alpha L_\gamma \delta$,

(2) $\beta = \delta$ and $\alpha L_\gamma \beta$,

(3) $\alpha = 1 < \beta$ and $\beta L_\gamma \delta$,

(4) $\alpha = 2$, $\beta = 4$, $\delta \in (\bigcap_{\eta < \gamma} M(O_\eta))$ and $4 L_\gamma \delta$.

Proof We omit the proof as it uses many of the results from [RR1]. The method is similar to, though more straightforward than, that used for theorem A.1. □

A.3 Some inequalities

It is noted in [D-T] that no identities involving the operations $O_0, \ldots O_4$ are known for the finite domain, except for those which are trivially implied by those which are already known for $O_0, \ldots O_3$. The straightforward method of proving such identities is to use induction together with an identity for $\alpha O_\gamma (\beta + \delta)$. For $\gamma = 1,2$ or 3 we have the following identities

$$\alpha O_1 (\beta O_0 \delta) = (\alpha O_1 \beta) O_0 (\alpha O_1 \delta),$$
$$\alpha O_2 (\beta O_0 \delta) = (\alpha O_2 \beta) O_1 (\alpha O_2 \delta),$$
$$\alpha O_3 (\beta O_0 \delta) = (\alpha O_3 \beta) O_2 (\alpha O_2 \delta).$$

This suggests that one might compare $\alpha O_\gamma (\beta O_0 \delta)$ with $(\alpha O_\gamma \beta) O_{\gamma-1} (\alpha O_i \delta)$ for $i \leq \gamma$, but theorem A.9 shows that there are always strict inequalities between these expressions in the finite domain for $\gamma \in \omega - 4$. Thus a new method is required to answer Doner and Tarski's question affirmatively.

Theorem A.9 Suppose that $\alpha, \beta, \delta \in \omega - 7$ and $\gamma \in \omega - 4$. Then, if γ is even

$$\alpha O_\gamma (\beta O_0 \delta) > (\alpha O_\gamma \beta) O_{\gamma-1} (\alpha O_\gamma \delta).$$

Then, if γ is odd

$$(\alpha O_\gamma \beta) O_{\gamma-1} (\alpha O_2 \beta) > \alpha O_\gamma (\beta O_0 \delta) > (\alpha O_\gamma \beta) O_{\gamma-1} (\alpha O_1 \delta).$$

Proof Suppose that all the hypotheses of the theorem hold and that γ is even. Then put $\eta = \max.(\beta, \delta)$ and we get

$$\alpha 0_\gamma(\beta 0_0 \delta) > \alpha 0_\gamma(\eta + 6)$$

$\geqslant (\alpha 0_\gamma \eta) 0_\gamma 2$ by lemma 31(i) of [D-T]

$= (\alpha 0_\gamma \eta) 0_{\gamma-1}(\alpha 0_\gamma \eta)$ from the definition of 0_γ

$\geqslant (\alpha 0_\gamma \beta) 0_{\gamma-1}(\alpha 0_\gamma \delta)$,

so that the first inequality of the theorem holds.

Next we prove that for $\alpha, \beta, \delta \in \omega - 2$, $\gamma \in \omega - 1$

$$\alpha 0_\gamma(\beta 0_0 \delta) > (\alpha 0_\gamma \beta) 0_{\gamma-1}(\alpha 0_1 \delta), \qquad (*)$$

and this implies one half of the second inequality. To prove (*) we firstly show that $(\alpha 0_\gamma \beta) 0_\gamma \delta > \alpha 0_\gamma (\beta 0_0 \delta)$ holds for the above range using induction on δ. Actually, we combine the induction and the basis steps by noticing that this is an equality when $\delta = 1$.

$(\alpha 0_\gamma \beta) 0_\gamma (\delta 0_0 1) \geqslant (\alpha 0_\gamma (\beta 0_0 \delta)) 0_{\gamma-1}(\alpha 0_\gamma \beta)$ by assumption

$> (\alpha 0_\gamma (\beta 0_0 \delta)) 0_{\gamma-1} \alpha$

$= \alpha 0_\gamma (\beta 0_0 (\delta 0_0 1))$, as required.

Now we prove (*) using the same method.

$\alpha 0_\gamma (\beta 0_0 \delta 0_0 1) \geqslant ((\alpha 0_\gamma \beta) 0_{\gamma-1}(\alpha 0_1 \delta)) 0_\gamma \alpha$ by assumption

$> (\alpha 0_\gamma \beta) 0_{\gamma-1}((\alpha 0_1 \delta) 0_0 \alpha)$ by above inequality

$= (\alpha 0_\gamma \beta) 0_{\gamma-1}(\alpha 0_1 (\delta 0_0 1))$, as required.

Finally we prove that for $\alpha, \beta \in \omega - 4$, $\delta, \gamma \in \omega - 2$

$$\alpha 0_{2\gamma+1}(\beta 0_0 \delta) < (\alpha 0_{2\gamma+1} \beta) 0_{2\gamma}(\alpha 0_2 \delta), \qquad (**)$$

and this implies the remaining half of the second inequality. The proof of (**) is similar to that of (*), using $(\alpha 0_{2\gamma} \beta) 0_{2\gamma} \delta < \alpha 0_{2\gamma}(\beta 0_1 \delta)$ as the first inequality. □

A.4 The sums of permutations of a sequence of ordinals

If we are given an α-sequence of ordinals and we permute the members of that sequence to give a new α-sequence, then the two sequences often have different infinite sums. This is the reason for considering the problem of this section.

<u>Definition A.10</u> If a is an α-sequence of ordinals, then $S_a(\alpha)$ is the number of different ordinals which can be obtained by permuting the members of a into an α-sequence and taking the sum of that sequence.

One natural question is to find a best possible upper bound for $S_a(\alpha)$ in terms of α. This has not been fully answered in the literature, although the following results appear.

(i) (Erdos, [13]) If α is finite, then
$S_a(\alpha) \leqslant \max_{k < \alpha,\, a'} (k.2^{k-1} + 1).S_{a'}(\alpha - k)$, and the proof shows that this result is best possible.

(ii) (Sierpinski, [67]) $S_a(\omega) < \omega$ and $S_a(\lambda) \leqslant \omega$ when λ is a countable ordinal. Clearly the first of these results is best possible and we shall show that the second one is also.

(iii) (Ginsberg, [18]) If ω_ρ is a regular cardinal, then
$S_a(\omega_\rho) \leqslant \aleph_\rho^{\breve{\aleph}_\rho}$.

Theorem A.11 improves on (iii) and completes the answer to the above question.

<u>Theorem A.11</u> Suppose that α is an infinite ordinal. Then
(a) if $\alpha = \omega$ or α is weakly inaccessible, $S_a(\alpha) < \alpha$, and
(b) otherwise, $S_a(\alpha) \leqslant \overline{\overline{\alpha}}$.
Further, these results are the best that can be obtained independently of a .

Proof Firstly, we shall show that $S_a(\alpha) \leqslant \overline{\overline{\alpha}}$ and this clearly follows from the following statement

If $\overline{\overline{\lambda}} \leqslant \aleph_\rho$, then the number of different sums of λ-sequences of ordinals, all of whose members are taken from a given set of ordinals of cardinality $\aleph_\rho$, is $\leqslant \aleph_\rho$. (1)

We prove (1) by induction on λ. Clearly its true for $\lambda = 1$. If its true for λ, then $\sum\limits_{i \leqslant \lambda+1} a_i = \sum\limits_{i \leqslant \lambda} a_i + a_\lambda$ so that its also true for $\lambda+1$ as $\aleph_\rho \aleph_\rho = \aleph_\rho$.

Now suppose that (1) holds for all $\beta < \lambda$ and λ is a limit ordinal. Then $\sum\limits_{i<\lambda} a_i = \bigcup\limits_{\beta<\lambda} (\sum\limits_{i<\beta} b_i)$ and as there are at most $\aleph_\rho$ different sums for $\beta < \lambda$, the given sum must be the supremum of a subset of a set, B say, of ordinals, which has cardinality $\aleph_\rho$. Either this subset has arbitarily large members in B or there is a least member of B which is not in it. Hence, the supremum of the subset is an initial segment of B (under the natural ordering) so that there are at most $\aleph_\rho$ different values for the λ sum. Thus (1) is proved.

If a is an α-sequence and $S_a(\alpha) = \kappa$, then for $\beta > \alpha$ we can obtain a β-sequence, b, with $S_b(\beta) \geqslant \kappa$ by letting its first α terms be the same as a's and the remainder C's. Let c be the $(\omega_\rho + 1)$ sum $\omega^1 + \omega^2 + \ldots + \omega^\alpha + \ldots + 1$, for $\alpha < \omega_\rho$. Then, by altering the last term, we get $S_c(\omega_\rho + 1) = \omega_\rho$, so that $S_a(\omega_\rho + 1) \leqslant \omega_\rho$ is the best possible inequality. The prior observation then shows that $S_a(\alpha) \leqslant \overline{\overline{\alpha}}$ is the best possible inequality when α is not a cardinal.

The following example of Ginsberg's shows that we can have $S_d(\omega_{\rho+1}) = \omega_{\rho+1}$: put $d_i = \omega_{\rho+1}$ for $i < \omega_\rho$ and $d_i = 1$ for $i \geqslant \omega_\rho$. Consequently, we can now restrict our attention to

α being a limit cardinal, ω_λ say, and we consider two cases.

Case 1 ω_λ is singular.

We show that $\exists$ a $S_a(\omega_\lambda) = \omega_\lambda$. This implies that all of (b) holds. Let $\kappa = cf(\omega_\lambda)$ and consider the sequence $b = \sum_{i<\omega_\lambda} i$. Clearly $\sum_{i<\omega_\lambda} i = \omega_\lambda$ and for any $\beta < \omega_\lambda$ we show that there is a permutation of b with ω_λ-sum $\omega_\lambda.(\beta+1)$, and the result will then follow.

As $\beta < \omega_\lambda$, $\beta < \omega_\rho$ for some $\rho < \lambda$ and we can choose β different cofinal κ sequences, each of which has sum ω_λ, as follows.
Let $f: \kappa \to \omega_\rho$ be such that $\omega_\lambda = \bigcup_{\delta\in\kappa} f(\delta)$,
$f(\delta+1) > f(\delta) + \beta$ and all $f(\delta) > \omega_\rho$. Then $f(\delta) + i$, for $i \leq \beta$ give the required sequences.

Now we form the required sum by letting the first $\kappa.\beta$ terms be the above cofinal sequences arranged one after another and then we " compress " the remaining elements of b, without altering their order, to take the remaining places. Clearly, the sum of this sequence is $\omega_\lambda.(\beta+1)$.

Case 2 ω_λ is regular

Let a be an ω_λ-sequence and we show that the number of different ω_λ sums which can be obtained by permuting the members of a to another ω_λ sequence and taking its sum is $< \omega_\lambda$. This completes the proof.

The method which we use for this case is an extension of that of [67] and [18] . We can clearly suppose that infinitely many of the members of a are non zero. Then, an element a_i is said to have the property P if
$\{a_j \mid a_j \geq a_i\} < \omega_\lambda$, and then [18] shows that $< \omega_\lambda$ elements of a have the property P.

Let $\sum_{i<\omega_\lambda} a_i = k_1 + \omega^\delta$ and then for some ϕ_0, $\sum_{i \geq \phi \geq \phi_0} a_i = \omega^\delta$ for

$\phi < \omega_\lambda$. Suppose that b is an ω_λ sequence which is a permutation of a, and then as ω_λ is regular, there is some index, q say, such that for $q \leq i < \omega_\lambda$, a_i does not have the property P. We show that

$$\sum_{q \leq i < \omega_\lambda} b_i = \omega^\delta \qquad (2)$$

By cardinality, there is some member of a, a_k say, such that $k \geq \phi$ and $a_k \geq b_q$. We can then continue this argument, as in [18], to get $\sum_{q \leq i < \omega_\lambda} b_i \leq \omega^\delta$. If we suppose that $\sum_{q \leq i < \omega_\lambda} b_i = \varrho'$ then, by reversing the argument, we get $\omega^\delta \leq \varrho'$, so that (2) holds.

Now we have shown that all sums take the form $\sum_{i<q} b_i + \omega^\delta$ for some $q < \omega_\lambda$, where all the terms with the property P occur before b_q. Next we show that all terms which do not have the property P may be replaced by 0 in the initial segment of b up to b_q, without altering the sum. We call the new terms b_i'. Suppose that this works up to $\beta < q$ and that b_β does not have the property P. Then $b_\beta < \omega^\delta$ as there are arbitarily large terms $\geq b_\beta$, so that $b_\beta + \sum_{\beta<i<q} b_i + \omega^\delta = \sum_{\beta<i<q} b_i + \omega^\delta$, by the lemma of [67]. Thus b_β can be replaced by 0 without altering the sum. Hence all sums are of the form $\sum_{i<q} b_i' + \omega^\delta$, where the only non zero terms are those having the property P. As there are less than ω_λ such terms the result now follows from (1). □

A.5 Infinite extended ordinal operations

We define the infinite extended ordinal operations as a natural extension of infinite sums and products in definition A.12. Theorem A.13 shows that for limit γ, we can reduce many questions about Ω_γ to questions about infinite sums, and corollary A.14 shows how theorem A.11 can be transferred to such operations.

Definition A.12 $\underset{\zeta<0}{\Omega_\gamma} a_\zeta = 1$ if $\gamma > 0$ or 0 if $\gamma = 0$.
When $\rho > 0$, $\underset{\zeta<\rho}{\Omega_\gamma} a_\zeta = \bigcup_{\beta<\rho} ((\underset{\zeta<\beta}{\Omega_\gamma} a_\zeta) 0_\gamma a_\beta)$.

Theorem A.13 If γ is a limit ordinal, $\rho \geqslant 2$, $a_0 \geqslant 2$ and all $a_\zeta \geqslant 1$, then $\underset{\zeta<\rho}{\Omega_\gamma} a_\zeta = a_0 \, 0_\gamma (\sum_{0<\zeta<\rho} (a_\zeta - 1))$.

Proof We prove this result by transfinite induction on ρ, and it clearly holds when $\rho = 2$. Suppose that it holds for all $\beta < \rho$, and then

$$\begin{aligned}
\underset{\zeta<\rho}{\Omega_\gamma} a_\zeta &= \bigcup_{\beta<\rho} ((\underset{\zeta<\beta}{\Omega_\gamma} a_\zeta) 0_\gamma a_\beta) && \text{by definition} \\
&= \bigcup_{2\leqslant\beta<\rho} ((a_0 \, 0_\gamma \sum_{0<\zeta<\beta} (a_\zeta - 1)) 0_\gamma a_\beta) && \text{by hypothesis} \\
&= \bigcup_{2\leqslant\beta<\rho} (a_0 \, 0_\gamma (\sum_{0<\zeta<\beta} (a_\zeta - 1) 0_0 (a_\beta - 1))) \\
& && \text{by theorem 27 of [D-T]} \\
& \bigcup_{2<\beta<\rho} (a_0 \, 0_\gamma \sum_{0<\zeta<\beta} (a_\zeta - 1)) && \text{by definition.}
\end{aligned}$$

If ρ is a successor ordinal then this immediately gives the result. If ρ is a limit ordinal then $\sum_{\zeta<\rho} a_\zeta$ is also a limit ordinal and the result follows by theorem 15(iii) of [D-T]. □

Corollary A.14 If $S_{\gamma;a}(\alpha)$ is the definition corresponding to $S_a(\alpha)$ with Ω_γ replacing infinite sum and γ is a limit ordinal, then theorem A.11 is also true when $S_a(\gamma)$ is replaced by $S_{\gamma,a}(\alpha)$.

Proof This follows from theorem A.11, the proof of theorem A.11 and the theorem. The only important point to note is that if b_0 does not have the property P in Case 2, then we may replace it by 2 rather than by 0. □

It is possible to prove other theorems, like A.13, which enable us to transfer questions about infinite extended ordinal operations to those concerning infinite sums. Theorem A.15 is an example of such a result.

<u>Theorem A.15</u> If $\gamma > 1$ and $\{a_\xi \mid \xi < \rho\}$ is a sequence of ordinals, all of which are $\geqslant \omega^2$, then

$$\Omega_{\xi<\rho}{}_{2\gamma}\, a_\xi = a_0 \; O_{2\gamma}\Big(\sum_{0<\xi<\rho} a_\xi\Big).$$

<u>Proof</u> Suppose that the hypothesis of the theorem holds and, initially, also suppose that all of the a_ξ are limit ordinals. We prove the result in this case using transfinite induction as follows.

$$\Omega_{\xi<\rho}{}_{2\gamma}\, a_\xi = \bigcup_{2\leq\beta<\rho} \Big(\big(a_0 \; O_{2\gamma}\big(\sum_{0<\xi<\beta} a_\xi\big)\big) O_{2\gamma} a_\beta \Big) \qquad \text{by hypothesis}$$

$$= \bigcup_{2\leq\beta<\rho} \Big(a_0 \, O_{2\gamma}\Big(\big(\sum_{0<\xi<\beta} a_\xi\big) O_0 \, a_\beta \Big)\Big)$$

by theorem 32(i) of [D-T]

$$= \bigcup_{2\leq\beta<\rho} \Big(a_0 \; O_{2\gamma}\big(\sum_{0<\xi<\beta} a_\xi\big)\Big) \qquad \text{by definition.}$$

The initial result then follows as in the proof of theorem A.13.

Now drop the assumption that all of the a_ξ 's are limit ordinals. It is clear that we can define a sequence of ordinals, a_ξ', with the property

$\omega \leqslant \omega . a_\xi' \leqslant a_\xi \leqslant \omega . (a_\xi' + 1)$ for $\xi > 0$, and $a_0' = a_0$.

Then, using the monoticity laws and the first result of this proof, we get

$$a_0 \; O_{2\gamma}\Big(\sum_{0<\xi<\rho} \omega . a_\xi'\Big) \leqslant \Omega_{0<\xi<\rho}{}_{2\gamma}\, a_\xi \leqslant a_0 \; O_{2\gamma}\Big(\sum_{0<\xi<\rho} \omega(a_\xi' + 1)\Big).$$

As all of the a_ξ are $\geqslant \omega^2$ we clearly have

$\sum_{0<\xi<\rho} \omega a_\xi' = \sum_{0<\xi<\rho} \omega(a_\xi' + 1)$, so the inequalities give the result. □

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