The Linear Arrangement Problem Parameterized Above Guaranteed Value

Gregory Gutin* Arash Rafiey† Stefan Szeider‡ Anders Yeo§

Abstract

A linear arrangement (LA) is an assignment of distinct integers to the vertices of a graph. The cost of an LA is the sum of lengths of the edges of the graph, where the length of an edge is defined as the absolute value of the difference of the integers assigned to its ends. For many application one hopes to find an LA with small cost. However, it is a classical NP-complete problem to decide whether a given graph G admits an LA of cost bounded by a given integer. Since every edge of G contributes at least one to the cost of any LA, the problem becomes trivially fixed-parameter tractable (FPT) if parameterized by the upper bound of the cost. Fernau asked whether the problem remains FPT if parameterized by the upper bound of the cost minus the number of edges of the given graph; thus whether the problem is FPT "parameterized above guaranteed value." We answer this question positively by deriving an algorithm which decides in time $O(m + n + 5.88^k)$ whether a given graph with m edges and n vertices admits an LA of cost at most $m + k$ (the algorithm computes such an LA if it exists). Our algorithm is based on a procedure which generates a problem kernel of linear size in linear time for a connected graph G. We also prove that more general parameterized LA problems stated by Serna and Thilikos are not FPT, unless P=NP.

Key words: linear arrangement, fixed-parameter tractability, parametrization above guaranteed value, para-NP-complete.

*Corresponding author. Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 OEX, UK, gutin@cs.rhul.ac.uk and Department of Computer Science, University of Haifa, Israel

†Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 OEX, UK, arash@cs.rhul.ac.uk

‡Department of Computer Science, Durham University Science Labs, South Road, Durham DH1 3LE, UK, stefan.szeider@durham.ac.uk

§Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 OEX, UK, anders@cs.rhul.ac.uk

1 Introduction

All graphs considered in this paper do not have loops or parallel edges. A *linear arrangement* of a graph $G = (V, E)$ is a one-to-one mapping $\alpha : V \to \{1, \dots, |V|\}$. The *length* of an edge $uv \in E$ relative to α is defined as

$$\lambda_\alpha(uv) = |\alpha(u) - \alpha(v)|.$$

The *cost* $\mathrm{c}(\alpha, G)$ of a linear arrangement α is the sum of lengths of all edges of G relative to α, i.e.,

$$\mathrm{c}(\alpha, G) = \sum_{e \in E} \lambda_\alpha(e).$$

Linear arrangements of minimal cost are *optimal*; $\mathrm{ola}(G)$ denotes the cost of an optimal linear arrangement of G.

The **Linear Arrangement Problem (LAP)** is the problem of deciding whether, given a graph G and an integer k, G admits a linear arrangement of cost at most k, i.e., whether $\mathrm{ola}(G) \leq k$. The problem has numerous application; in particular, the first published work on the subject appears to be the 1964 paper of Harper [14], where a polynomial-time algorithm for finding optimal linear arrangement for n-cubes is developed, which has applications in error-correcting codes. Goldberg and Klipker [13] were first to obtain a polynomial-time algorithm for computing optimal linear arrangements of trees. Faster algorithms for trees were obtained by Shiloach [17] and Chung [2]. However, we cannot hope to find optimal linear arrangements for the class of all graphs in polynomial time since LAP is a classical NP-complete problem [11, 12].

Recently, LAP was studied under the framework of parameterized complexity [6, 18]. We recall some basic notions of parameterized complexity here, for a more in-depth treatment of the topic we refer the reader to [4, 5, 6, 10, 16]. A parameterized problem Π can be considered as a set of pairs (I, k) where I is the *problem instance* and k (usually an integer) is the *parameter*. Π is called *fixed-parameter tractable (FPT)* if membership of (I, k) in Π can be decided in time $O(f(k)|I|^c)$, where $|I|$ is the size of I, $f(k)$ is a computable function, and c is a constant independent from k and I. Let Π and Π' be parameterized problems with parameters k and k', respectively. An *fpt-reduction R from Π to Π'* is a many-to-one transformation from Π to Π', such that (i) $(I, k) \in \Pi$ if and only if $(I', k') \in \Pi'$ with $k' \leq g(k)$ for a fixed computable function g and (ii) R is of complexity $O(f(k)|I|^c)$. A *reduction to problem kernel* (or *kernelization*) is an fpt-reduction R from a parameterized problem Π to itself. In kernelization, an instance (I, k) is reduced to another instance (I', k'), which is called the *problem kernel*. It is easy to see that a decidable parameterized problem is FPT if and only if it admits a kernelization (see, e.g., [5, 16]); however, the problem kernels obtained by this general result have impractically large size. Therefore, one tries to develop kernelizations that yield problem kernels of smaller size, if possible of size linear in the parameter.

The following is a straightforward way to parameterize LAP [6, 18]:

Parameterized LAP
Instance: A graph G.
Parameter: A positive integer k.
Question: Does G have a linear arrangement of cost at most k?

An edge has length at least 1 in any linear arrangement. Thus, for a graph G with m edges always $\mathrm{ola}(G) \geq m$ prevails; in other words, m is a guaranteed value for $\mathrm{ola}(G)$. Consequently, parameterized LAP is FPT by trivial reasons (we reject a graph with more than k edges and solve LAP by brute force if the graph has at most k edges). Hence it makes sense to consider the *net cost* $\mathrm{nc}(\alpha, G)$ of a linear arrangement α defined as follows:

$$\mathrm{nc}(\alpha, G) = \sum_{e \in E} (\lambda_\alpha(e) - 1) = \mathrm{c}(\alpha, G) - m.$$

We denote the net cost of an optimal linear arrangement of G by $\mathrm{ola}^+(G)$. Indeed, the following non-trivial parameterization of LAP is considered by Fernau [6, 7]:

LA parameterized above guaranteed value (LAPAGV)
Instance: A graph G.
Parameter: A positive integer k.
Question: Does G have a linear arrangement of net cost at most k?

Parameterizations above a guaranteed value were first considered by Mahajan and Raman [15] for the problems Max-SAT and Max-Cut; such parameterizations have lately gained much attention [6, 16]. However, apparently only a few nontrivial problems parameterized above guaranteed value are known to be FPT.

Fernau [6, 7, 8] raises the question of whether LAPAGV is FPT (the status of this problem was reported open in Cesati's compendium [1]). We answer this question positively by deriving a kernelization procedure for LAPAGV that yields problem kernels of linear size in linear time for connected graphs G. Moreover, using the method of bounded search trees, we develop an algorithm that solves LAPAGV for the obtained kernel more efficiently than by brute force. In summary, we obtain an algorithm that decides in time $O(m+n+5.88^k)$ whether a given graph with m edges and n vertices admits an LA of cost at most $m+k$. Our algorithm also produces an optimal linear arrangement if $\mathrm{ola}^+(G) \leq k$. A key concept of our kernelization is the suppression of vertices of degree 2, a standard technique used in the design of parameterized algorithms (e.g., for finding small feedback vertex sets in graphs [4]). For LAPAGV, however, we need a more sophisticated approach where we suppress only vertices of degree 2 that satisfy a certain condition depending on the parameter k.

Fernau [8] proposes a bounded search tree approach to prove that LAPAGV is FPT. The description of the approach is incomplete (for example, it is unclear how to deal with vertices of degree 2 without rejecting any yes-instances) and an inequality, which is required by Fernau's approach to show that LAPAGV is FPT, is not proved. These conclusions are confirmed in our private communication with Fernau (February, 2006) and it remains to be seen whether a bounded search tree approach can be used to prove that LAPAGV is FPT.

Serna and Thilikos [18] formulate more general parameterized LA problems (see Section 4) and ask whether their problems are FPT. We prove that the problems are not FPT (unless P=NP) by demonstrating that for almost all fixed values of the parameter, the corresponding decision problems are NP-complete. This implies that the problems are para-NP-complete [10]. We conclude the paper by Theorem 4.3, which indicates that our FPT result cannot be extended much further, in a sense.

For a graph G and a set X of its vertices, $V(G)$, $E(G)$ and $G[X]$ denote the vertex set of G, the edge set of G, and the subgraph of G induced by X, respectively. An edge e in a graph G is a *bridge* if $G-e$ has more components than G has. A connected graph with at least two vertices and without bridges is called *2-edge-connected*. A *bridgeless component* of a graph G is a maximal induced subgraph of G with no bridges. Observe that the bridgeless components of G are the connected components that we get after removing all bridges from G. A bridgeless component is either a 2-edge-connected graph or is isomorphic to K_1; in the latter case we call it *trivial*. Further graph-theoretic terminology can be found in Diestel's book [3].

2 Kernelization

In the next section, we use the following simple lemma to solve LAPAGV for the general case of an arbitrary graph input G. The lemma allows us to confine our attention to connected graphs in the rest of this section.

Lemma 2.1 *Let $G_1,\dots,G_p$ be the connected components of a graph G. Then* $\mathrm{ola}^+(G)=\sum_{i=1}^p \mathrm{ola}^+(G_i)$.

Proof: Follows directly from the definitions. □

Let α be a linear arrangement of a graph G. It is convenient to use for subgraphs G' of G the notation $\mathrm{nc}(\alpha,G')=\sum_{uv\in E(G')}(\lambda_\alpha(uv)-1)$.

Lemma 2.2 *Let G be a graph, let $X\subseteq V(G)$, and let u,v be two distinct vertices of G that belong to the same connected component of $G-X$. Let α be a linear arrangement of G with $\alpha(u)<\alpha(x)<\alpha(v)$ for every $x\in X$. Then* $\mathrm{nc}(\alpha,G-X)\geq|X|$.

Proof: We proceed by induction on $|X|$. If $|X| = 0$ then the lemma holds vacuously. Hence we assume $|X| \geq 1$ and pick $x \in X$. We define $G' = G - x$, $X' = X \setminus \{x\}$, and we let α' be the linear arrangement of G' obtained from α by setting, for $y \in V(G')$, $\alpha'(y) = \alpha(y)$ if $\alpha(y) < \alpha(x)$, and $\alpha'(y) = \alpha(y) - 1$ otherwise. By induction hypothesis, $\mathrm{nc}(\alpha', G' - X') \geq |X'|$. By assumption, $G - X$ contains a path P from u to v; hence P contains at least one edge $w_1 w_2$ with $\alpha(w_1) < \alpha(x) < \alpha(w_2)$ (and $w_1, w_2 \notin X$). By definition of α', we have $\lambda_\alpha(w_1 w_2) = \lambda_{\alpha'}(w_1 w_2) + 1$. Since for all other edges $e \in E(G' - X')$ we have $\lambda_\alpha(e) \geq \lambda_{\alpha'}(e)$, $\mathrm{nc}(\alpha, G - X) \geq \mathrm{nc}(\alpha', G' - X') + 1$ follows. □

Let G be a connected graph and let α be a linear arrangement of G. We say that two disjoint subgraphs A, B of G are *α-comparable* if either $\alpha(a) < \alpha(b)$ holds for all $a \in V(A), b \in V(B)$, or $\alpha(a) > \alpha(b)$ holds for all $a \in V(A), b \in V(B)$. Moreover, let e be a bridge of G and let G_1, G_2 be the two connected components of $G - e$. For a positive integer k, we say that e is *k-separating* if both $|V(G_1)|, |V(G_2)| > k$.

Lemma 2.3 *Let G be a connected graph and let k be a positive integer such that $k \geq$ $\mathrm{ola}^+(G)$. Then for every optimal linear arrangement α of G and every k-separating bridge e of G, the two connected components of $G - e$ are α-comparable.*

Proof: Let α be an optimal linear arrangement. Let e be a k-separating bridge of G and let G_1, G_2 be the two connected components of $G - e$. Since e is a k-separating bridge, $|V(G_1)|, |V(G_2)| > k$ holds by definition. We denote the extremal values of the vertices of G_1 and G_2 with respect to α by $l_i = \min_{v \in V(G_i)} \alpha(v)$ and $r_i = \max_{v \in V(G_i)} \alpha(v)$, $i = 1, 2$. We may assume, w.l.o.g., that $l_1 < l_2$.

First we show that $r_1 < r_2$. Assume to the contrary that $r_1 > r_2$. Now $\alpha^{-1}(l_1)$ and $\alpha^{-1}(r_1)$ belong to the same connected component of $G - V(G_2)$, and Lemma 2.2 implies $\mathrm{nc}(\alpha, G) \geq |V(G_2)| > k$, contradicting the assumption $\mathrm{nc}(\alpha, G) \leq k$. Hence indeed $l_1 < l_2$ and $r_1 < r_2$.

Next we show that $r_1 < l_2$. Assume to the contrary that $l_2 < r_1$. From α we obtain a new linear arrangement α' of G, changing the order of vertices in $X = \{\, x \in V(G) : l_2 \leq \alpha(x) \leq r_1 \,\}$ such that G_1 and G_2 become α'-comparable, without changing the relative order of vertices within G_1 or changing the relative order of vertices within G_2. That is, for $X \cap V(G_i) = \{v_1^{(i)}, \ldots, v_{j_i}^{(i)}\}$ and $\alpha(v_1^{(i)}) < \ldots < \alpha(v_{j_i}^{(i)})$, $i = 1, 2$, we have $\alpha'(v_1^{(1)}) < \ldots < \alpha'(v_{j_1}^{(1)}) < \alpha'(v_1^{(2)}) < \ldots < \alpha'(v_{j_2}^{(2)})$.

Since e is a bridge, we have

$$\mathrm{nc}(\alpha, G) = \mathrm{nc}(\alpha, G - e) + \lambda_\alpha(e) - 1 \text{ and } \mathrm{nc}(\alpha', G) = \mathrm{nc}(\alpha', G - e) + \lambda_{\alpha'}(e) - 1 \qquad (1)$$

Although $\lambda_{\alpha'}(e)$ can be greater than $\lambda_\alpha(e)$, we will show that an increase of the length of e is more than compensated by the reduced cost of $G - e$ under α'. Again using Lemma 2.2 we conclude that $\mathrm{nc}(\alpha', G_i) \leq \mathrm{nc}(\alpha, G_i) - |X \cap V(G_{3-i})|$ holds for $i = 1, 2$ (observe that the vertices $\alpha^{-1}(l_i), \alpha^{-1}(r_i)$ are in the same component of $G - V(G_{3-i})$, and for each vertex x in $X \cap V(G_i)$ we have $\alpha(l_i) < \alpha(x) < \alpha(r_i)$). In summary, we have

$$\mathrm{nc}(\alpha', G - e) \leq \mathrm{nc}(\alpha, G - e) - |X|. \tag{2}$$

Using the fact that $|\alpha(x) - \alpha'(x)| \leq |X| - 1$ holds for all vertices $x \in V(G)$, it is easy to see that

$$\lambda_{\alpha'}(e) \leq \lambda_\alpha(e) + |X| - 1. \tag{3}$$

Indeed, if at least one of the ends of e is in $V(G) \setminus X$, then clearly $\lambda_{\alpha'}(e) \leq \lambda_\alpha(e) + |X| - 1$; otherwise, if both ends of e are in X, then $\lambda'_\alpha(e) \leq |X| - 1$, and since $\lambda_\alpha(e) \geq 1$, we have even $\lambda_{\alpha'}(e) \leq \lambda_\alpha(e) + |X| - 2$.

By (1),(2) and (3), we obtain $\mathrm{nc}(\alpha', G) \leq \mathrm{nc}(\alpha, G) - 1$. This contradicts the assumption that α is an optimal linear arrangement. Hence $l_1 < r_1 < l_2 < r_2$, and so G_1 and G_2 are α-comparable as claimed. □

Lemma 2.4 *If G is a connected bridgeless graph of order $n \geq 1$, then* $\mathrm{ola}^+(G) \geq (n-1)/2$.

Proof: If $n \leq 2$, then the inequality trivially holds. Thus, we may assume that $n \geq 3$ and G is 2-edge-connected. Let α be an optimal linear arrangement of G and put $u = \alpha^{-1}(1)$ and $w = \alpha^{-1}(n)$. Since G is 2-edge-connected, Menger's Theorem (see, e.g., [3]) implies that there are two paths P, P' between u to w such that $E(P) \cap E(P') = \{u, w\}$. Observe that the subgraph G' of G induced by $E(P) \cup E(P')$ is a collection of $t \geq 1$ edge-disjoint cycles. Let n' be the number of vertices in G'. Since G' has $t - 1$ vertices of degree 4 and $n' - t + 1$ vertices of degree 2, $|E(G')| = (n' - t + 1) + 2(t - 1) = n' + t - 1$. Since $n' \leq n$ and $t \leq \frac{n-1}{2}$, we conclude that $|E(G')| \leq \frac{3}{2}(n-1)$. Observe that $\mathrm{nc}(\alpha, P) \geq n - 1 - |E(P)|$ and $\mathrm{nc}(\alpha, P') \geq n - 1 - |E(P')|$. Hence,

$$\mathrm{ola}^+(G) = \mathrm{nc}(\alpha, G) \geq \mathrm{nc}(\alpha, G') \geq 2(n-1) - |E(G')| \geq (n-1)/2.$$

□

Lemma 2.5 *A connected graph G on at least two vertices has a pair u, v of distinct vertices such that both $G - u$ and $G - v$ are connected.*

Proof: Let T be a spanning tree in G and let u, v be leaves in T. Then $T - x$ is a spanning tree in $G - x$ for $x \in \{u, v\}$. □

Let α be an optimal linear arrangement of G. We call a vertex $u \in V(G)$ *α-special* if $G - u$ is connected and $\alpha(u) \notin \{1, n\}$.

Lemma 2.6 *Let G be a connected graph. Let X be a vertex set of G such that $G[X]$ is connected and let $G-X$ have connected components $G_1, G_2, \ldots, G_r$ with $n_1, n_2, \ldots, n_r$ vertices, respectively, such that $n_1 \le n_2 \le \ldots \le n_r$. Then* $\mathrm{ola}^+(G) \ge \mathrm{ola}^+(G[X]) + \sum_{i=1}^{r-2} n_i$.

Proof: Let α be an optimal linear arrangement of G. If $r \le 2$, then $\sum_{i=1}^{r-2} n_i = 0$ and, thus, this lemma holds. Now assume that $r \ge 3$. By Lemma 2.5, each nontrivial G_i has a pair u_i, v_i of distinct vertices such that $G_i - u_i$ and $G_i - v_i$ are connected. If G_i is trivial, i.e., it has just one vertex x, then set $u_i = v_i = x$. Since $r \ge 3$, for some $j \in \{1, 2, \ldots, r\}$, we have $\alpha(u_j) \notin \{1, n\}$ and $\alpha(v_j) \notin \{1, n\}$. Now we claim that there is a vertex $u \in V(G_j)$ such that $G-u$ is connected. Indeed, we set $u = u_j$ if there are edges between v_j and $G[X]$, we set $u = v_j$, otherwise.

We have proved that G has an α-special vertex u not in X. Let α_u be a linear arrangement of $G-u$ defined as follows: $\alpha_u(x) = \alpha(x)$ for all $x \in V(G)$ with $\alpha(x) < \alpha(u)$, and $\alpha_u(x) = \alpha(x) - 1$ for all $x \in V(G)$ with $\alpha(x) > \alpha(u)$. Since G is connected, it has an edge yz such that $\alpha(y) < \alpha(u) < \alpha(z)$. Observe that $\lambda_\alpha(yz) = \lambda_{\alpha_u}(yz) + 1$. Hence, we have

$$\mathrm{ola}^+(G) = \mathrm{nc}(\alpha, G) \ge \mathrm{nc}(\alpha_u, G-u) + 1 \ge \mathrm{ola}^+(G-u) + 1.$$

Thus,

$$\mathrm{ola}^+(G) \ge \mathrm{ola}^+(G-u) + 1 \text{ for an } \alpha\text{-special vertex } u \text{ of } G \tag{4}$$

Run the following procedure: while $G-X$ has a least three components, choose a β-special vertex $u \notin X$ of G for an optimal linear arrangement β of G and replace G with $G-u$. By the end of this procedure, we have deleted some t vertices from G obtaining a subgraph H of G. By (4), we have $\mathrm{ola}^+(G) \ge \mathrm{ola}^+(G[X]) + t$. Observe that $H-X$ has at most two components, if all vertices of at least $r-2$ components $G_1, G_2, \ldots, G_r$ are deleted from G during the procedure. Thus, $t \le \sum_{i=1}^{r-2} n_i$ and $\mathrm{ola}^+(G) \ge \mathrm{ola}^+(G[X]) + \sum_{i=1}^{r-2} n_i$.□

The proof of the next lemma is illustrated in Figure 1.

Lemma 2.7 *Let k be a positive integer and let G be a connected graph with n vertices with* $\mathrm{ola}^+(G) \le k$*. Then either G has a k-separating bridge or $n \le 4k+1$.*

Proof: Assume that G does not have a k-separating bridge. If G is a bridgeless graph, then by Lemma 2.4 we know that $n \le 2k+1$. So, we may assume that G has a bridge. Choose a bridge e_1 with maximal $\min\{|V(F_1)|, |V(F_0)|\}$, where F_1, F_0 are the components of $G - e_1$. Assume, w.l.o.g., that $|V(F_1)| \le |V(F_0)|$. Since e_1 is not a k-separating bridge, $|V(F_1)| \le k$ follows of necessity. Let F_0^* denote the bridgeless component of F_0 that contains a vertex incident to e_1. If $F_0 = F_0^*$ then $|V(F_0)| \le 2k+1$ follows by Lemma 2.4 and we are done; hence we assume that $F_0 \ne F_0^*$.

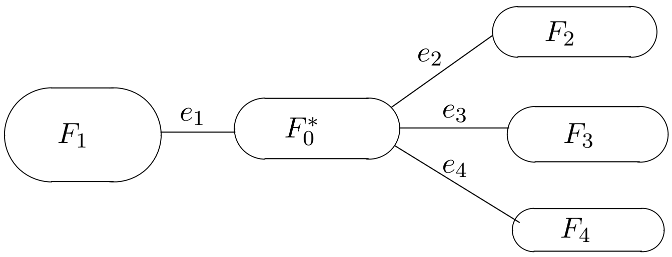

Figure 1: Illustration for Lemma 2.7.

Let $e_2, \ldots, e_r$ denote the bridges of F_0 that are incident to vertices in F_0^*. Moreover, let $F_2, \ldots, F_r$ denote the connected components of $F_0 - V(F_0^*)$ such that each e_i is incident with a vertex of F_i, $i = 2, \ldots, r$. Assume that $|V(F_2)| \geq |V(F_3)| \geq \ldots \geq |V(F_r)|$. Suppose that $|V(F_2)| > |V(F_1)|$. Then the component of $G - e_2$ different from F_2 has more vertices than F_1, which is impossible by the choice of e_1 and the assumption that G has no k-separating bridges. We conclude that $|V(F_1)| \geq |V(F_2)|$. By Lemma 2.6, $\mathrm{ola}^+(G) \geq \mathrm{ola}^+(F_0^*) + \sum_{i=3}^{r} |V(F_i)|$. Thus, $\sum_{i=3}^{r} |V(F_i)| \leq k - \mathrm{ola}^+(F_0^*)$. Since $|V(F_2)| \leq |V(F_1)| \leq k$ and, by Lemma 2.4, $|V(F_0^*)| \leq 2 \cdot \mathrm{ola}^+(F_0^*) + 1$, we obtain that

$$n = |V(F_0^*)| + \sum_{i=1}^{r} |V(F_i)| \leq (2 \cdot \mathrm{ola}^+(F_0^*) + 1) + (3k - \mathrm{ola}^+(F_0^*)) = 3k + \mathrm{ola}^+(F_0^*) + 1 \leq 4k + 1.$$

□

Lemma 2.8 *Let k be a positive integer and let G be a connected graph with the following structure:*

1) *G has bridgeless components $C_1, C_2, \ldots, C_t$, $t \geq 2$, such that every two consecutive components C_i and C_{i+1} are linked by a single edge e_i, which is a k-separating bridge in G, $i = 1, 2, \ldots, t-1$.*

2) *Let $L = G[\bigcup_{i=1}^{t} V(C_i)]$. The graph $G' = G - V(L)$ has connected components $G_1, G_2, \ldots, G_r$ such that each G_j has edges only to one subgraph $C_{\pi(j)}$, $\pi(j) \in \{1, 2, \ldots, t\}$.*

Let J_p be the indices of all G_j such that $\pi(j) = p$, $p = 1, 2, \ldots, t$. Let $n_i = \max\{|V(G_j)| : j \in J_i\}$, $i = 1, 2, \ldots, t$. Then $\mathrm{ola}^+(G) \geq \mathrm{ola}^+(L) + |V(G')| - n_1 - n_t$.

Proof: Let α be an optimal linear arrangement of G. Let

$$A_p = \left(\bigcup_{j \in J_1 \cup J_2 \cup \cdots \cup J_p} V(G_j) \right) \cup \left(\bigcup_{j=1}^{p} V(C_j) \right)$$

for $p = 1, 2, \ldots, t$. By Lemma 2.3, the two components of $G - e_1$ are α-comparable. We may assume, w.l.o.g., that $\alpha(x) < \alpha(y)$ for each $x \in A_1$, $y \notin A_1$. Because of the assumption and since the two components of $G - e_2$ are α-comparable, we have $\alpha(x) < \alpha(y) < \alpha(z)$ for each $x \in A_1$, $y \in A_2 - A_1$ and $z \notin A_2$. Continuing this argument, we can prove that $\alpha(x_i) < \alpha(x_{i+1})$ for each $x_i \in A_i$ and $x_{i+1} \in A_{i+1} \setminus \bigcup_{j=1}^{i} A_j$.

By the above conclusion and the arguments similar to those used in the proof of Lemma 2.6, we can prove that each G_j, apart from at most one graph G_p with $p \in J_1$ and at most one graph G_q with $q \in J_t$, has an α-special vertex u. As in Lemma 2.6, it follows that $\text{ola}^+(G - u) \leq \text{ola}^+(G) - 1$. Now we apply a procedure similar to that used in the proof of Lemma 2.6: until $|J_1| \leq 1$, $|J_t| \leq 1$ and $J_2 = \cdots = J_{t-1} = \emptyset$, choose a β-special vertex $u \in V(G')$ for an optimal linear arrangement β of G and replace G with $G - u$ and G' with $G' - u$. The procedure will have at most $|V(G')| - n_1 - n_t$ steps each decreasing $\text{ola}^+(G)$ by at least 1. Hence $\text{ola}^+(G) \geq \text{ola}^+(L) + |V(G')| - n_1 - n_t$. $\square$

Let G be a graph and let v be a vertex of degree 2 of G. Let vu_1, vu_2 denote be the edges incident with v. Assume that $u_1u_2 \notin E(G)$. We obtain a graph G' from G by removing v (and the edges vu_1, vu_2) from G and adding instead the edge u_1u_2. We say that G' is obtained from G by *suppressing* vertex v. Furthermore, if the two edges incident with v are k-separating bridges for some positive integer k, then we say that v is *k-suppressible*. The last definition is justified by the following lemma.

Lemma 2.9 *Let G be a connected graph and let v be an* $\text{ola}^+(G)$*-suppressible vertex of G. Then* $\text{ola}^+(G) = \text{ola}^+(G')$ *holds for the graph G' obtained from G by suppressing v.*

Proof: Let u_1, u_2 denote the neighbors of v and let G_1, G_2 denote the connected components of $G - v$ with $u_i \in V(G_i)$, $i = 1, 2$. Consider an optimal linear arrangement α of G. As above we use the notation $l_i = \min_{w \in V(G_i)} \alpha(w)$ and $r_i = \max_{w \in V(G_i)} \alpha(w)$, $i = 1, 2$, and we assume, w.l.o.g., that $l_1 < l_2$. Since vu_1, vu_2 are $\text{ola}^+(G)$-separating bridges, Lemma 2.3 implies that α assigns to the vertices of G_i an interval of consecutive integers. Thus, we conclude that $l_1 < r_1 < \alpha(v) < l_2 < r_2$. We define a linear arrangement α' of G' by setting $\alpha'(w) = \alpha(w)$ for $w \in V(G_1)$ and $\alpha'(w) = \alpha(w) - 1$ for $w \in V(G_2)$. Evidently $\text{ola}^+(G') \leq \text{nc}(\alpha', G') = \text{nc}(\alpha, G) = \text{ola}^+(G)$.

Conversely, assume that α' is an optimal linear arrangement of G'. We proceed symmetrically to the first part of this proof. Let $l_i = \min_{w \in V(G_i)} \alpha'(w)$ and $r_i = \max_{w \in V(G_i)} \alpha'(w)$, $i = 1, 2$, and assume, w.l.o.g., that $l_1 < l_2$. Observe that u_1u_2 is an

$\mathrm{ola}^+(G')$-separating bridge of G', hence Lemma 2.3 applies. Thus $l_1 < r_1 < l_2 < r_2$. We define a linear arrangement α of G by setting $\alpha(w) = \alpha'(w)$ for $w \in V(G_1)$, $\alpha(v) = r_1+1$, and $\alpha'(w) = \alpha(w)+1$ for $w \in V(G_2)$. Evidently $\mathrm{ola}^+(G) \leq \mathrm{nc}(\alpha', G) = \mathrm{nc}(\alpha', G') = \mathrm{ola}^+(G')$.

Hence $\mathrm{ola}^+(G) = \mathrm{ola}^+(G')$ as claimed. □

Theorem 2.10 *Let k be a positive integer, and let G be a connected graph without k-suppressible vertices. If $\mathrm{ola}^+(G) \leq k$, then G has at most $5k+2$ vertices and at most $6k+1$ edges.*

Proof: Let $n = |V(G)| > 1$, and let $\mathrm{ola}^+(G) \leq k$.

Any linear arrangement of G can have at most $n-1$ edges of length 1, and each additional edge contributes at least 1 to the net cost. Thus, $m \leq n-1+k$ and it suffices to show that $n \leq 5k+3$.

If G does not have a k-separating bridge, then by Lemma 2.7 we have $n \leq 5k+1$. Assume now that G has a k-separating bridge. Let $e = uv$ be such a bridge, and let H_1, H_2 be two connected component of $G-e$, where H_1 contains u. Let C^u (C^v) be the bridgeless components containing u (v). Let $C_1^u, C_2^u, ..., C_p^u$ ($C_1^v, C_2^v, ..., C_q^v$) be all connected components of $H_1 - V(C^u)$ ($H_2 - V(C^v)$). Observe that each of the components C_i^u (C_i^v) is linked to C^u (C^v) by a bridge. Assume that $|V(C_i^x)| \leq |V(C_j^x)|$ for $i < j$, where $x \in \{u, v\}$. By Lemma 2.6, we have $\sum_{i=1}^{i=p-1} |V(C_i^u)| \leq k$ and $\sum_{i=1}^{i=q-1} |V(C_i^v)| \leq k$. If the bridge between C_p^u and C^u (C_q^v and C^v) is k-separating, we consider the bridgeless component of C_p^u (C_q^v) containing an endvertex of the bridge and the connected components obtained from C_p^u (C_q^v) by deleting the vertices of the bridgeless component. Continuation of the procedure above as long as possible will bring us the following decomposition of G:

1) G has bridgeless components $C_1, C_2, \ldots, C_t$, $t \geq 2$, such that every two consecutive components C_i and C_{i+1} are linked by a single edge e_i, which is a k-separating bridge in G, $i = 1, 2, \ldots, t-1$.

2) Let $L = G[\bigcup_{i=1}^{t} V(C_i)]$. The graph $G' = G - V(L)$ has connected components $G_1, G_2, \ldots, G_r$ such that each G_j has edges only to one subgraph $C_{\pi(j)}$, $\pi(j) \in \{1, 2, \ldots, t\}$.

Since we have carried out the above procedure as long as possible, all bridges between G' and L are not k-separating. Thus, $|V(G_j)| \leq k$ for each $j = 1, 2, \ldots, t$. Recall that J_p is the set of indices of all G_j such that $\pi(j) = p$, $p = 1, 2, \ldots, t$, and $n_i = \max\{|V(G_j)| : j \in J_i\}$, $p = 1, 2, \ldots, t$. By Lemma 2.8, $\mathrm{ola}^+(G) \geq \mathrm{ola}^+(L) + |V(G')| - n_1 - n_t$. Since $n_1 \leq k$, $n_t \leq k$ and $\mathrm{ola}^+(G) \leq k$, we obtain

$$|V(G')| \leq 3k - \mathrm{ola}^+(L). \tag{5}$$

Since G has no k-suppressible vertices, the bridgeless components $C_2, C_3, \ldots, C_{t-1}$ are not trivial. Observe that $\sum_{i=2}^{t-1} \mathrm{ola}^+(C_i) \leq \mathrm{ola}^+(L)$. By Lemma 2.4, every component $\mathrm{ola}^+(C_i) \geq 1$, $2 \leq i \leq t-1$, and thus $t-2 \leq \mathrm{ola}^+(L)$. By Lemma 2.4, $|V(C_i)| \leq 2 \cdot \mathrm{ola}^+(C_i) + 1$ for each $i = 1, 2, \ldots, t$. Hence,

$$|V(L)| = \sum_{i=1}^{t} |V(C_i)| \leq 2(\sum_{i=1}^{t} \mathrm{ola}^+(C_i)) + t \leq 3 \cdot \mathrm{ola}^+(L) + 2. \tag{6}$$

Combining (5) and (6), we obtain

$$|V(G)| = |V(G')| + |V(L)| \leq (3k - \mathrm{ola}^+(L)) + (3 \cdot \mathrm{ola}^+(L) + 2) \leq 3k + 2 \cdot \mathrm{ola}^+(L) + 2 \leq 5k + 2.$$

□

Theorem 2.11 *Let $f(n, m)$ be the time sufficient for checking whether* $\mathrm{ola}^+(G) \leq k$ *for a connected graph G with n vertices and m edges. Then*

$$f(n, m) = O(m + n + f(5k + 2, 6k + 1)).$$

Proof: We assume that G is represented by adjacency lists. Using a depth-first-search (DFS) algorithm, we can determine the cut vertices of G in time $O(n+m)$ (see Tarjan [19]). Let T be a spanning rooted tree of G (say, as obtained by the DFS algorithm). For each vertex $v \in V(G)$, let T_v denote the subtree of T rooted at v. That is, T_v contains v and all descendants of v in T. We assign to each vertex v the integer $t_v = |V(T_v)|$. This can be done in time $O(n + m)$ by a single bottom-up traversal of T where we assign 1 to leaves, and to non-leaves we assign the sum of the integers assigned to their immediate descendants plus one.

Consider now a cut vertex v of G of degree 2. Let u, w be the neighbors of v. Since the edges vu and vw are bridges of G, they are edges of T. It follows now directly from the definition that v is k-suppressible if and only if one of the following conditions holds.

1. v is the root of T and $t_u, t_w > k$.
2. v is not the root of T and $k + 1 < t_v < n - k$.

Since these conditions can be checked in constant time for each cut vertex v of G, we can find the set S of all k-suppressible vertices of G in time $O(n + m)$. Note that if H is the graph obtained by suppressing some $v \in S$, some vertices of $S \setminus \{v\}$ may not be k-suppressible in H; however, any k-suppressible vertex of H belongs to $S \setminus \{v\}$.

We compute a set $S' \subseteq S$ starting with the empty set and successively adding some of the vertices of S to S'. We visit the vertices of G according to a bottom-up traversal of T (i.e., if v is a descendant of v' then we visit v before v'). During this traversal we assign to each vertex v an integer t'_v which is the number of vertices in $S' \cap V(T_v)$.

Assume we visit a vertex $v \in V(G) \setminus S$. If v is a leaf of T we put $t'_v = 0$; otherwise we let t'_v be the sum of the values $t'_{v'}$ for the direct descendants v' of v in T. Assume we visit a vertex $v \in S$. Let u and w be the neighbors of v such that u is a direct descendant of v. Let H denote the graph obtained from G by suppressing all vertices in the current set S'. It follows from the considerations above that v is a k-suppressible vertex of H if and only if one of the following conditions holds.

1. v is the root of T and $t_u - t'_u, t_w - t'_w > k$.
2. v is not the root of T and $k + 1 < t_v - t'_u < n - k - |S'|$.

If v is a k-suppressible vertex of H we add v to S', put $t'_v = t'_u + 1$, and continue; otherwise we leave S' unchanged, put $t'_v = t'_u$, and continue.

Performing a further bottom-up traversal of T we suppress the vertices in S' one after the other, and we are left with a graph G' which has no k-suppressible vertices. If $|V(G')| > 5k+2$ or $|E(G')| > 6k+1$, then we know from Theorem 2.10 that $\mathrm{ola}^+(G') > k$. It follows from Lemma 2.9 that $\mathrm{ola}^+(G) > k$ as well, and we can reject G. On the other hand, if $|V(G')| \leq 5k + 2$ and $|E(G')| \leq 6k + 1$, then we can find an optimal linear arrangement α' for G' in time $f(5k+2, 6k+1)$. By means of the construction in the proof of Lemma 2.9 we can transform in time $O(n + m)$ the arrangement α' into an optimal linear arrangement α of G. □

The proof of Theorem 2.11 implies the following:

Corollary 2.12 *The problem LAPAGV, with a connected graph G as an input, has a linear problem kernel, which can be found in linear time. If $\mathrm{ola}^+(G) \leq k$, then the reduced graph (i.e., kernel) has at most $5k + 2$ vertices and $6k + 1$ edges.*

In the next section, we give an upper bound for the function $g(k) = f(5k + 2, 6k + 1)$ in Theorem 2.11.

3 Computing Optimal Linear Arrangements

Lemma 3.1 *Let G be a 2-vertex-connected graph on n vertices and let α be a linear arrangement of G. Then $\mathrm{nc}(\alpha, G) \geq n - 2$.*

Proof: Define x and y such that $\alpha(x) = 1$ and $\alpha(y) = n$. For an edge $e = uv$ in G in which $\alpha(u) < \alpha(v)$, let $Q(e) = \{w : \alpha(u) < \alpha(w) < \alpha(v)\}$. For every vertex $w \in V(G) - \{x, y\}$, there is a path between x and y in $G - w$, and therefore there is an edge uv such that $\alpha(u) < \alpha(w) < \alpha(v)$. This implies that $\bigcup_{e \in E(G)} Q(e) = V(G) - \{x, y\}$.

Since $\lambda_\alpha(e) - 1 = |Q(e)|$ we have

$$\mathrm{nc}(\alpha, G) = \sum_{e \in E(G)} |Q(e)| \geq |\bigcup_{e \in E(G)} Q(e)| = n - 2.$$

□

Let n and k be nonnegative integers. Let $P_n = p_1 p_2 \ldots p_n$ be a path of order n and let $OLA^+_{P_n}(k, j)$ be the set of linear arrangements α of P_n with net cost at most k and such that $\alpha(p_1) = j$ and $\alpha(p_n) = n$. We will first prove an upper bound for $|OLA^+_{P_n}(k, j)|$.

Theorem 3.2 *For all $n \geq 2$, $k \geq 0$ and $0 \leq j \leq n - 1$, we have*

$$|OLA^+_{P_n}(k, j)| \leq 2^{0.119n + 1.96k - 0.967095j + 2}.$$

Furthermore the following holds, when $d_2 = 0.497534$,

$$|OLA^+_{P_n}(k, 2)| \leq (1 - d_2) 2^{0.119n + 1.96k - 2 \cdot 0.967095 + 2}.$$

Proof: Let $j > k + 1$ and let G be P_n with the extra edge $p_1 p_n$. By Lemma 3.1, $\mathrm{nc}(\alpha, G) \geq n - 2$. Since $\lambda_\alpha(p_1 p_n) - 1 = n - j - 1$, we conclude that $\mathrm{nc}(\alpha, P_n) \geq n - 2 - (n - j - 1) = j - 1 > k$. Therefore $|OLA^+_{P_n}(k, j)| = 0$ when $j > k + 1$, and the theorem holds in this case. So assume that $j \leq k + 1$. We also note that the theorem holds when $k = 0$, as in this case $|OLA^+_{P_n}(k, j)| \leq 2$, so assume that $k \geq 1$.

We will prove the theorem by induction on n. Clearly the theorem is true when $n \leq 4$, as in this case $|OLA^+_{P_n}(k, j)| \leq (n - 2)! \leq 2$. So we may assume that $n > 4$.

Let $\alpha \in OLA^+_{P_n}(k, j)$ be arbitrary. Let α' be a linear arrangement of the path $P_n - p_1$ such that $\alpha'(z) = \alpha(z)$ when $\alpha(z) < \alpha(p_1)$ and $\alpha'(z) = \alpha(z) - 1$ when $\alpha(z) > \alpha(p_1)$. Furthermore, let $a = 0.119$, $b = 1.96$, $x = 0.967095$, $\Gamma = 2^{an + bk - xj + 2}$ and $\gamma = |OLA^+_{P_n}(k, j)|$ and consider the following two cases:

Case 1: $j = 1$. Let $\alpha(p_2) = q$. Observe that $\alpha' \in OLA^+_{P_{n-1}}(k - q + 2, q - 1)$ since $\alpha'(p_2) = q - 1$ and $\lambda_\alpha(p_1 p_2) - 1 = q - 2$. Since α' is uniquely determined by α we note that there are at most $|OLA^+_{P_{n-1}}(k - q + 2, q - 1)|$ linear arrangements in $OLA^+_{P_n}(k, j)$ with $\alpha(p_2) = q$. This implies the following:

$$\begin{aligned}
\gamma &\leq \sum_{q=2}^{k+2} |OLA^+_{P_{n-1}}(k - q + 2, q - 1)| \\
&\leq \left(\sum_{q=2}^{k+2} 2^{a(n-1) + b(k-q+2) - x(q-1) + 2} \right) - d_2 2^{a(n-1) + b(k-3+2) - x(3-1) + 2}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\left(2^{-a}\sum_{q=0}^{k}(2^{-b-x})^q\right) - d_2 2^{-a-b-x}\right)\Gamma \\
&\leq \left(\frac{2^{-a}}{1-2^{-b-x}} - d_2 2^{-a-b-x}\right)\Gamma \leq \Gamma.
\end{aligned}
$$

Case 2: $j \geq 2$. First assume that $q = j - \alpha(p_2) > 0$. Since $P_n - p_1$ is connected, there must be an edge e from the set of vertices with α-values in $\{1, 2, \ldots, j-1\}$ to the set of vertices with α-values in $\{j+1, j+2, \ldots, n\}$. Observe that $\lambda_\alpha(e) = \lambda_{\alpha'}(e) + 1$. Since $\lambda_\alpha(p_1p_2) - 1 = q - 1$, the net cost of α' is at most the net cost of α minus q. Since α' is uniquely determined by α we note that there are at most $|OLA^+_{P_{n-1}}(k-q, j-q)|$ linear arrangements in $OLA^+_{P_n}(k,j)$ with $\alpha(p_2) = j - q$.

Now assume that $q = \alpha(p_2) - j > 0$. Let p_i be the vertex with $\alpha(p_i) = 1$. Observe that the path $p_2p_3 \ldots p_i$ must contain some edge $e = uv$, where $\alpha(u) > j$ and $\alpha(v) < j$ (as $\alpha(p_2) > j$ and $\alpha(p_i) = 1 < j$). Furthermore, the path $p_ip_{i+1} \ldots p_n$ must contain some edge $e' = u'v'$, where $\alpha(v') > j$ and $\alpha(u') < j$ (as $\alpha(p_n) = n > j$ and $\alpha(p_i) = 1 < j$). As above we note that $\lambda_\alpha(e) = \lambda_{\alpha'}(e) + 1$ and $\lambda_\alpha(e') = \lambda_{\alpha'}(e') + 1$. Since $\lambda_\alpha(p_1p_2) - 1 = q - 1$, the net cost of α' is at most the net cost of α minus $q+1$. Since α' is uniquely determined by α, we note that there are at most $|OLA^+_{P_{n-1}}(k-q-1, j+q-1)|$ linear arrangements in $OLA^+_{P_n}(k,j)$ with $\alpha(p_2) = j + q$ (as $\alpha'(p_2) = j + q - 1$). This implies the following when $j \geq 3$:

$$
\begin{aligned}
\gamma &\leq \sum_{q=1}^{j-1}|OLA^+_{P_{n-1}}(k-q, j-q)| + \sum_{q=1}^{k-1}|OLA^+_{P_{n-1}}(k-q-1, j+q-1)| \\
&\leq \sum_{q=1}^{j-1} 2^{a(n-1)+b(k-q)-x(j-q)+2} + \sum_{q=1}^{k-1} 2^{a(n-1)+b(k-q-1)-x(j+q-1)+2} \\
&= \left(2^{-a-b+x}\sum_{q=0}^{j-2}(2^{-b+x})^q + 2^{-a-2b}\sum_{q=0}^{k-2}(2^{-b-x})^q\right)\Gamma \\
&\leq \left(\frac{2^{-a-b+x}}{1-2^{-b+x}} + \frac{2^{-a-2b}}{1-2^{-b-x}}\right)\Gamma \leq \Gamma.
\end{aligned}
$$

When $j = 2$ we get the following analogously to above.

$$
\begin{aligned}
\gamma &\leq \sum_{q=1}^{j-1}|OLA^+_{P_{n-1}}(k-q, j-q)| + \sum_{q=1}^{k-1}|OLA^+_{P_{n-1}}(k-q-1, j+q-1)| \\
&\leq \left(2^{-a-b+x}\sum_{q=0}^{2-2}(2^{-b+x})^q + 2^{-a-2b}\sum_{q=0}^{k-2}(2^{-b-x})^q\right)\Gamma - d_2 2^{a(n-1)+b(k-1-1)-2x}
\end{aligned}
$$

$$\leq \left(2^{-a-b+x} + \frac{2^{-a-2b}}{1-2^{-b-x}} - d_2 2^{-a-2b}\right)\Gamma$$
$$\leq (1-d_2)\Gamma$$

This completes the induction proof. $\square$

Remark 3.3 *Note that Theorem 3.2 implies that* $|OLA^+_{P_{5k}}(k,1)| = O(2^{2.555k}) = O(5.88^k)$*. It is possible to prove that* $|OLA^+_{P_{5k}}(k,1)| = \Omega(5.36^k)$*, which shows that our result cannot be significantly improved, in a sense. Due to space considerations we do not include the proof of* $|OLA^+_{P_{5k}}(k,1)| = \Omega(5.36^k)$*.*

Let n and k be nonnegative integers. Let $\mathcal{T}_n$ be the set of trees with n vertices. Let $T \in \mathcal{T}_n$ and let $X \subseteq V(T)$ be arbitrary. Let $OLA^+_T(n,k,X)$ be the set of linear arrangements α of T with net cost at most k and such that $\alpha(x) \in \{1,n\}$ for all $x \in X$. Note that $OLA^+_T(n,k,X) = \emptyset$ if $|X| \geq 3$. Now define $t(n,k,i)$ as follows:

$$t(n,k,i) = \max\{|OLA^+_T(n,k,X)| : \; T \in \mathcal{T}_n, |X| = i\}.$$

In other words, no tree T of order n has more than $t(n,k,i)$ linear arrangements such that the net cost is at most k and i prescribed vertices have to be mapped to either 1 or n (and $t(n,k,i)$ is the minimum such value).

For a connected graph G, let T_G be a spanning tree of G. Since $\text{ola}^+(T_G) \leq \text{ola}^+(G)$ we only have to check all linear arrangements in $OLA^+_{T_G}(n,k,\emptyset)$ (but still considering all edges in G and not just T_G) to decide whether $\text{ola}^+(G) \leq k$. Since $|OLA^+_{T_G}(n,k,\emptyset)| \leq t(n,k,0)$ the values of $t(n,k,i)$ are of interest (especially when $i = 0$). We will prove an upper bound for $t(n,k,i)$ before indicating how to generate all linear arrangements in $OLA^+_{T_G}(n,k,\emptyset)$. Note that $t(n,k,3) = 0$.

Theorem 3.4 *For all* $n \geq 2$*,* $k \geq 0$ *and* $0 \leq i \leq 3$*, we have the following upper bound:*

$$t(n,k,i) \leq 2^{0.119n+1.96k-1.4625i+4}.$$

Proof: We will prove the theorem by induction on $n+k-i$. Clearly the theorem is true when $n = 2$ and $0 \leq i \leq 3$, as in this case $t(n,k,i) = 2$ if $i \in \{0,1,2\}$ and $t(n,k,3) = 0$. Furthermore when $i = 3$ the theorem also holds. So now let $i \leq 2$ and $n \geq 3$ (and $k \geq 0$) and assume that the theorem holds for all smaller values of $n+k-i$.

Let T be a tree of order n and let X be a set of i vertices in T. Let x be a leaf in the tree T and let y be the unique neighbor of x in T. Furthermore if some leaf in the tree T does not belong to X then let x be such a vertex (that is $x \notin X$). Let α be a linear arrangement of T with net cost at most k and with all vertices $q \in X$ having $\alpha(q) \in \{1, n\}$. Let α' be a linear arrangement of the tree $T - x$ such that $\alpha'(z) = \alpha(z)$ when $\alpha(z) < \alpha(x)$ and $\alpha'(z) = \alpha(z) - 1$ when $\alpha(z) > \alpha(x)$. Furthermore let $a = 0.119$, $b = 1.96$, $c = 1.4625$, $\Gamma = 2^{an+bk-ci+4}$, and $\gamma = |OLA_T(n, k, X)|$. Consider the following three cases:

Case 1: $x, y \notin X$**.** Observe that there are at most $t(n, k, i + 1)$ linear arrangements α in which $\alpha(x) \in \{1, n\}$, as we may add x to X and use our induction hypothesis. So now assume that $\alpha(x) \notin \{1, n\}$. Assume that $\alpha(x) - \alpha(y) = j$. This means that $\lambda_\alpha(xy) - 1 = j - 1$. However, since $\alpha(x) \notin \{1, n\}$ and $T - x$ is connected, there must be an edge e from the set of vertices with α-values in $\{1, 2, \ldots, \alpha(x) - 1\}$ to the set of vertices with α-values in $\{\alpha(x) + 1, \alpha(x) + 2, \ldots, n\}$. Observe that $\lambda_\alpha(e) = \lambda_{\alpha'}(e) + 1$. Therefore, the net cost of α' is at most the net cost of α minus j. Thus, there are at most $t(n-1, k-j, i)$ linear arrangements α'. Since α' is uniquely determined by α we note that there are at most $t(n-1, k-j, i)$ linear arrangements α with $\alpha(x) - \alpha(y) = j$. Analogously, there are at most $t(n - 1, k - j, i)$ linear arrangements α with $\alpha(x) - \alpha(y) = -j$. The above arguments imply the following:

$$
\begin{aligned}
\gamma &\leq t(n, k, i+1) + 2\sum_{j=1}^{k} t(n-1, k-j, i) \\
&\leq 2^{an+bk-c(i+1)+4} + 2\sum_{j=1}^{k} 2^{a(n-1)+b(k-j)-ci+4} \\
&= \left(\frac{1}{2^c} + \frac{2}{2^{a+b}}\sum_{j=0}^{k-1}(2^{-b})^j\right)\Gamma \\
&\leq \left(\frac{1}{2^c} + \frac{2}{2^{a+b}(1-2^{-b})}\right)\Gamma \leq \Gamma.
\end{aligned}
$$

Case 2: $x \notin X$ **and** $y \in X$**.** As in our first case there are at most $t(n, k, i + 1)$ linear arrangements α with $\alpha(x) \in \{1, n\}$. Now assume that $\alpha(x) \notin \{1, n\}$ and assume that $|\alpha(x) - \alpha(y)| = j$, which implies that $\lambda_\alpha(xy) - 1 = j - 1$. As in our first case we observe that there is an edge e in $T - x$ such that $\lambda_\alpha(e) = \lambda_{\alpha'}(e) + 1$. Therefore, there are at most $t(n - 1, k - j, i)$ linear arrangements α with the above property. Thus, $\gamma \leq t(n, k, i+1) + \sum_{j=1}^{k} t(n-1, k-j, i)$. By the computations in our first case, this implies that $\gamma \leq \Gamma$, so we have now proved the case when $x \notin X$ and $y \in X$.

Case 3: $x \in X$**.** Since $|X| \leq 2$ (and $n \geq 3$) we note that the tree T only has two leaves, by our definition of x. Furthermore X contains both leaves in T, which implies that $T = P_n = p_1 p_2 \ldots p_n$ is a path of order n and $X = \{p_1, p_n\}$. By Theorem 3.2 we now obtain the following:

$$\gamma \leq |OLA^+_{P_n}(k, 1)| \leq 2^{0.119n+1.96k-0.967095+2} \leq 2^{0.119n+1.96k-2\cdot1.4625+4} = \Gamma.$$

We have now bounded the value of $t(n, k, i)$ for all the values we needed. $\square$

Remark 3.5 *The values $a = 0.119$ and $b = 1.96$ in the above proofs could be changed in such a way that we decrease a but increase b (and change c and x accordingly) or we could decrease b but increase a (and change c and x accordingly). However the values we have chosen are the ones that minimize $5a + b$, as our final bound is basically $O((2^{5a+b})^k)$.*

It is not difficult to turn the computations in the proof of Theorem 3.2 and Theorem 3.4 into a recursive algorithm that generates $OLA^+_{T'}(n', k', X')$ for all the relevant n', k', X' and subtrees T' of T_G and $OLA^+_{P_{n'}}(k', j')$ for all relevant n', k' and n'. After computing $OLA^+_{T_G}(n, k, \emptyset)$ we only need to calculate the net cost of each linear arrangement in $OLA^+_{T_G}(n, k, \emptyset)$ with respect to G. This way we can find the value $\mathrm{ola}^+(G)$ if $\mathrm{ola}^+(G) \le k$.

In order to do the above we need to generate at most $(n + 1)^2(n - 1)(k + 1)$ sets $OLA^+_{T'}(n', k', X')$, (there are at most $(n + 1)^2$ sets $|X'|$, $2 \le n' \le n$ and $0 \le k' \le k$). We also need to generate at most $n(k+1)(k+2)$ sets $OLA^+_{P_{n'}}(k', j')$ (as $1 \le n' \le n$, $0 \le k' \le k$ and $0 \le j \le k + 1$). Each of the above sets can be computed in at most $n \cdot t(n, k, \emptyset)$ time (as every set will be of size at most $t(n, k, \emptyset)$). Thus, we can obtain $OLA^+_{T_G}(n, k, \emptyset)$ in $O(n(n^3k + nk^2)t(n, k, \emptyset))$ time. We then need $O((n + m)t(n, k, \emptyset))$ time to consider each linear arrangement α in $OLA^+_{T_G}(n, k, \emptyset)$ and compute $\mathrm{nc}(\alpha, G)$, where $m = |E(G)|$. So the total time complexity, when $n \le 5k + 2$, is at most

$$O(k^5 t(n, k, \emptyset)) = O(k^5 2^{0.119(5k+2)+1.96k}) = O(2^{(5\cdot 0.119+1.96+0.0001)k}) = O(2^{2.5551k}).$$

We have proved the following:

Theorem 3.6 *Let n be the number of vertices in a connected graph G and let k be a nonnegative integer. If $n \le 5k + 2$, then we can check whether $\mathrm{ola}^+(G) \le k$ and compute $\mathrm{ola}^+(G)$, provided $\mathrm{ola}^+(G) \le k$, in time $O(2^{2.5551k})$.*

Now we are ready to prove the main result of this paper.

Theorem 3.7 *Let $G = (V, E)$ be a graph and let k be a nonnegative integer. We can check whether $\mathrm{ola}^+(G) \le k$ and compute $\mathrm{ola}^+(G)$ provided $\mathrm{ola}^+(G) \le k$ in time $O(|V| + |E| + 5.88^k)$.*

Proof: Let $G_1, G_2, \dots, G_p$ be the connected components of G. We can check, in time $O(|V(G_i)|)$, whether $\mathrm{ola}^+(G_i) = 0$ since $\mathrm{ola}^+(G_i) = 0$ if and only if G_i is a path. Thus, in time $O(|V|)$, we can detect all components of G of net cost zero. By Lemma 2.1, we do not need to take these components into consideration when computing $\mathrm{ola}^+(G)$. Thus, we may assume that for all components G_i, $i = 1, 2, \ldots, p$, we have $\mathrm{ola}^+(G_i) \geq 1$. Thus, if $\mathrm{ola}^+(G) \leq k$, then $\mathrm{ola}^+(G_i) \leq k-p+1$. By Lemma 2.1, Theorems 2.11 and 3.6, and the fact that $\mathrm{ola}^+(G_i) \leq k-p+1$ if $\mathrm{ola}^+(G) \leq k$, we can check whether $\mathrm{ola}^+(G) \leq k$ and compute $\mathrm{ola}^+(G)$ provided $\mathrm{ola}^+(G) \leq k$ in time $O(\sum_{i=1}^{p}(|V(G_i)| + |E(G_i)|) + p2^{2.5551(k-p+1)}) = O(|V| + |E| + 5.88^k)$. □

4 Stronger Parameterizations of LAP

Serna and Thilikos [18] introduce the following related problems. They ask whether either problem is FPT.

Vertex Average Min Linear Arrangement (VAMLA)
Instance: A graph G.
Parameter: A positive integer k.
Question: Does G have a linear arrangement of cost at most $k|V(G)|$?

Edge Average Min Linear Arrangement (EAMLA)
Instance: A graph G.
Parameter: A positive integer k.
Question: Does G have a linear arrangement of cost at most $k|E(G)|$?

Both problems are not FPT (unless P=NP), which follows from the next two theorems.

Theorem 4.1 *For any fixed integer $k \geq 2$, it is* NP*-complete to decide whether* $\mathrm{ola}(H) \leq k|V(H)|$ *for a given graph H.*

Proof: Let G be a graph and let r be an integer. We know that it is NP-complete to decide whether $\mathrm{ola}(G) \leq r$ (LAP). Let $n = |V(G)|$. Let k be a fixed integer, $k \geq 2$. Define G' as follows: G' contains k copies of G, j isolated vertices and a clique with i vertices (all of these subgraphs of G' are vertex disjoint). We have $n' = |V(G')| = kn + i + j$.

By the definition of G' and the fact that $\mathrm{ola}(K_i) = \binom{i+1}{3}$, we have

$$k \cdot \mathrm{ola}(G) = \mathrm{ola}(G') - \mathrm{ola}(K_i) = \mathrm{ola}(G') - \binom{i+1}{3}.$$

Therefore, $\mathrm{ola}(G) \leq r$ if and only if $\mathrm{ola}(G') \leq kr + \binom{i+1}{3}$. If there is a positive integer i such that $kr + \binom{i+1}{3} = kn'$ and the number of vertices in G' is bounded from above by a polynomial in n, then G' provides a reduction from LAP to VAMLA with the fixed k. Observe that $kr + \binom{i+1}{3} \geq k(kn + i)$ for $i = 6kn$. Thus, by setting $i = 6kn$ and

$j = r + \frac{1}{k}\binom{i+1}{3} - kn - i$, we ensure that G' exists and the number of vertices in G' is bounded from above by a polynomial in n. $\square$

The proof of the following theorem is similar, but G' is defined differently: G' contains k copies of G, a path with j edges and a clique with i vertices (all of these subgraphs of G' are vertex disjoint).

Theorem 4.2 *For any fixed integer $k \geq 2$, it is* NP*-complete to decide whether* $\mathrm{ola}(H) \leq k|E(H)|$ *for a given graph H.*

For a vertex v in a graph $G = (V, E)$, its *closed neighborhood* $N[v] = \{u \in V : uv \in E\} \cup \{v\}$. The *profile* of a linear arrangement α of G is

$$\mathrm{prf}(\alpha, G) = \sum_{z \in V} (\alpha(z) - \min\{\alpha(w) : w \in N[z]\}).$$

Serna and Thilikos [18] introduce also the following problem and ask whether it is FPT.

Vertex Average Profile (VAP)
Instance: A graph $G = (V, E)$.
Parameter: A positive integer k.
Question: Does G have a linear arrangement of profile $\leq k|V|$?

Similarly to Theorem 4.1 we can prove that VAP is NP-complete for every fixed $k \geq 2$.

Recently, Flum and Grohe [9, 10] introduced para-NP and other parameterized complexity classes. Recall that a parameterized problem Π can be considered as a set of pairs (I, k) where I is the problem instance and k is the parameter. Π is in *para-NP* if membership of (I, k) in Π can be decided in nondeterministic time $O(f(k)|I|^c)$, where $|I|$ is the size of I, $f(k)$ is a computable function, and c is a constant independent from k and I. Here, nondeterministic time means that we can use nondeterministic Turing machine. A parameterized problem Π' is *para-NP-complete* if it is in para-NP and for any parameterized problem Π in para-NP there is an fpt-reduction from Π to Π'. Observe that VAMLA, EAMLA and VAP are in para-NP. Moreover, it follows directly form our results that the three problems are para-NP-complete (see Corollary 2.16 in [10]).

Similarly to Theorem 4.2 we can prove the following:

Theorem 4.3 *For each fixed $0 < \epsilon \leq 1$, it is* NP*-complete to decide whether* $\mathrm{ola}^+(H) \leq |E(H)|^\epsilon$ *for a given graph H.*

Notice that Theorem 3.7 implies that we can decide, in polynomial time, whether $\mathrm{ola}(H) \leq |E(H)| + \log |E(H)|$ for a graph H. Theorem 4.3 indicates that the possibility to strengthen the last result is rather limited. It would be interesting to determine the complexity of the problem to verify whether $\text{ola}(H) \leq |E(H)| + \log^2 |E(H)|$ for a graph H.

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