Transformation of Generalised Ordinary and Basic Hypergeometric
Series and Identities of the Rogers-Ramanujan Type

M. Jackson

Masters

Royal Holloway and Bedford New College

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Transformations of generalised ordinary and basic hypergeometric series and identities of the Rogers – Ramanujan type.

The name hypergeometric series was given by Gauss to the series $_2F_1\left[a, b; c; x\right] = 1 + \dfrac{abx}{1c} + \cdots + \dfrac{a(a+1)\ldots(a+n-1)b(b+1)\ldots(b+n-1)x^n}{1 \cdot 2 \ldots n \cdot c(c+1) \ldots (c+n-1)} + \cdots$ This series had occurred previously in analysis, in particular in the works of Euler, but Gauss was the first mathematician to make any systematic study of its properties. Gauss showed that certain well known expansions, e.g. the binomial and exponential functions, were particular cases of the hypergeometric series and then enumerated certain obvious properties of the series before proceeding to make a more detailed study of some less elementary properties. The first section of his paper is concerned with linear relations between series of the type $F\ (\alpha+\lambda, \beta+\mu', \gamma+\nu)$ and $F(\alpha+\lambda', \beta+\mu'; \gamma+\nu')$ where $\lambda, \mu, \nu, \lambda', \mu', \nu'$ take the values $0, \pm 1$; these relationships included, of course, those between contiguous hypergeometric functions. In the second section, the series $G\ (\alpha, \beta, \gamma, x) = F(\alpha, \beta+1; \gamma+1; x)/F(\alpha, \beta; \gamma; x)$ is expressed as a continued fraction. Section three gives the sum of the series when $x = 1$, a result used in the analysis of this thesis. In a further paper (1A), Gauss studied the differential equation satisfied by the series and by various changes of variable, he obtained several transformations of hypergeometric series.

Since the time of Gauss, the ordinary hypergeometric series, the generalised hypergeometric series, the basic hypergeometric series introduced into analysis by Heine and Apell's hypergeometric series of two variables have been studied by various mathematicians. Until 1923, however, the literature dealing with the subject was very scattered, consisting of isolated papers in the transactions of the various learned societies. Then, in 1923, the paper by Hardy "A Chapter from Ramanujan's Notebook" was published in the Proceedings of the Cambridge Philosophical Society. This paper which was concerned with the generalised hypergeometric series of the ordinary type gave a summary, with proofs, of most of the results which were then known, together with some formulae stated, in practically all cases without proof, by the Indian Ramanujan. In 1926, Appell and Kampé de Fériet had their book, "Fonctions hypergéometriques et hypersphériques," published. This gave a summary of the more important properties of the ordinary hypergeometric series and a very detailed account of their researches on the hypergeometric functions of two variables.

Hardy's paper occasioned a considerable number of papers, principally by Professors Bailey and Watson and Mr.Whipple, in the Proceedings of the London Mathematical Society.

(a)

In 1935, Professor Bailey's Tract was published. This gives a detailed account of the more recent researches on the subject. Since 1935, a few more papers have appeared.

This thesis is concerned primarily with transformations of generalised hypergeometric series both of the ordinary and the basic type and with indentitied of the Rogers – Ramanujan type which are essentially limiting cases of transformations of basic series. When the Tract was published, various methods of obtaining transformations of generalised series of the ordinary type, had been given. Gauss' method of changing the variable in the differential equation becomes very tedious in the case of generalised hypergeometric series, but transformations had also been obtained by summing series of lower order, by Dougall's method and Carlson's Theorem, and by using Barne's Contour Integrals. Details of the last three methods are to be found in the Tract. The most powerful method was the one given by Professor Bailey and which will be used in the first part of this thesis, but the only method which was applicable to basic series was Dougall's. Since publication of the Tract, Professor Bailey has published two further papers. In the first, entitled, "Some Identities in Combinatory Analysis" (3) he gives a method of obtaining transformations of basic series and in the second, "Identities of the Rogers Ramanujan type," (4) he gives a simple fundamental result, which is the basic idea under- lying the methods of finding transformations of hyper- geometric series, and shows that the two methods which he had previously given, were just particular cases of this more general result.

Professor Bailey obtained many of the transformations of generalised hypergeometric series which had been proved previously, together with many new ones, including, as limiting cases of the transformations of basic series, identities of the Rogers-Ramanujan type.

Two sums of ordinary generalised hypergeometric series to which this method is applicable are those given by Saalschütz and Dougall. In the first part of this thesis, these two formulae have been taken in all possible ways and the method of finding transformations has been applied. The second part is a detailed study of three formulae given by Professor Bailey — (6.2) (6.4) (6.5) of "Some Identities of the Rogers-Ramanujan type," from which various identities have been deduced. The third part gives results involving bilateral series of basic hypergeometric type, which have been obtained from a formula due to D.B.Sears and are generalisations of known results for ordinary basic series. Finally, some miscellaneous results are given. These have been proved during the course of the work, but cannot logically be included in any of the three main sections.

(1)   For transformations of ordinary generalised series, we use the method as it was first given. Starting from a known sum, we deduce a transformation. In fact, we know that, if the formula

$$\sum_{r=0}^{m} \frac{(\alpha_1)_r (\alpha_2)_r (\alpha_3)_r (-n)_r (K+n)_r (K_1+n)_{r(\mu-1)} (K_2+n)_{r(\mu-1)} (\ell_1)_{nr} (-1)^r c^r}{r! (\beta_1)_r (\beta_2)_r (\ell_1+n)_{r(\nu-1)} (K_1)_{\mu r} (K_2)_{\mu r}} = \frac{(\rho_1)_n (\rho_2)_n (\rho_3)_n (\ell_1)_n b^\nu}{(\sigma_1)_n (\sigma_2)_n (X_1)_n (X_2)_n}$$

(α)

gives the sum of a certain hypergeometric series, then the formula

$$F\left[\begin{array}{c} \rho_1, \rho_2, \rho_3, a_1, a_2, -m \;;\; b \\ \sigma_1, \sigma_2, p_1, p_2, p_3 \end{array}\right] = \sum_{r=0}^{m} \frac{(a_1)_r (a_2)_r (-m)_r (\alpha_1)_r (\alpha_2)_r (\alpha_3)_r c^r}{r! (p_1)_r (p_2)_r (p_3)_r (\beta_1)_r (\beta_2)_r}$$

$$\times F\left[\begin{array}{c} K_1+\mu_1 r, K_2+\mu_2 r, a_1+r, a_2+r, -m+r \;; \\ \ell_1+\nu_1 r, p_1+r, p_2+r, p_3+r \end{array}\right] - (\beta)$$

is true. In these formulae, there may be any number of the quantities α, β, ρ, σ, K, ℓ, a, p. The numbers μ & ν may be positive or negative integers provided that $(a)_m$ is replaced by $(-1)^m / (1-a)_m$.

The analysis which led to the above result is as follows.

We start by supposing that we have a relationship of the type:-

$$_3F_2\left[\begin{array}{c} \rho_1, \rho_2, \rho_3 \;; bX \\ \sigma_1, \sigma_2 \end{array}\right] = (1-X)^{-K} \;_3F_2\left[\begin{array}{c} \alpha_1, \alpha_2, \alpha_3 \;; \frac{cX}{(1-X)^\mu} \\ \beta_1, \beta_2 \end{array}\right] - (1.1),$$

where μ is an integer.[+] By means of the substitution $X = xy/(1-x)(1-y)$ and multiplying by $(1-x)^{-p}(1-y)^{-q}$ we shew that the relationship

$$_5F_4\left[\begin{array}{c} \rho_1, \rho_2, \rho_3, -m, -n \;; b \\ \sigma_1, \sigma_2, p, q \end{array}\right] = \frac{(p-k)_m}{(p)_m} \sum_{r=0}^{} \frac{(-1)^r c^r (\alpha_1)_r (\alpha_2)_r (-m)_r (-n)_r (1-p+k)_r (\mu+r)(\alpha_3)_r}{r! (\beta_1)_r (\beta_2)_r (1-p-m+k)_{\mu r} (q)_r}$$

$$\times \;_3F_2\left[\begin{array}{c} K+\mu r, q+n, -m+r \\ 1-p+K-m+\mu r, q+r \end{array}\right], \qquad \qquad (1.2)$$

is obtained by equating coefficients of $x^m y^n$ on both sides of (1.1).

This, by means of a known relationship between series of the type $_3F_2$ with unit argument, is seen to give the relationship:-

$$_5F_4\left[\begin{array}{c} \rho_1, \rho_2, \rho_3, -n, -m \;; b \\ \sigma_1, \sigma_2, p, q \end{array}\right] = \sum_{r=0}^{m} \frac{c^r (\alpha_1)_r (\alpha_2)_r (\alpha_3)_r (-n)_r (-m)_r}{r! (\beta_1)_r (\beta_2)_r (p)_r (q)_r} \times \;_3F_2\left[\begin{array}{c} K+\mu r, -n+\lambda r, -m+\lambda r \;; \\ p+\lambda r, q+\lambda r \end{array}\right] - (1.3)$$

If we equate coefficients of x^n on both sides of (1.1) we see that if (1.1) is true then, so is

$$\sum_{r=0}^{m} \frac{(\alpha_1)_r (\alpha_2)_r (\alpha_3)_r (-n)_r (K+n)_r (\mu-1)(-1)^r c^r}{r! (\beta_1)_r (\beta_2)_r (K)_{\mu r}} = \frac{(\rho_1)_n (\rho_2)_n (\rho_3)_n b^n}{(\sigma_1)_n (\sigma_2)_n (K)_n} . - (1.4).$$

Footnote. + We have taken the λ of the paper (5) equal to 1, the case which gives the most interesting results.

Thus, if (1.4) is true, so are (1.2) & (1.3). The form (1.3) however is seen to lend itself much more easily to generalisation. Thus, generalising and replacing the positive integer n by any real number a, Professor Bailey proved, as in the Tract, that, if (α) is true, then (β) is true.

It is, however, possible to obtain a slightly more general form of the results (α) & (β), with the a's negative integers, directly from the equation (1.1), by making the substitution $X = \overset{\sim}{\underset{n=1}{\prod}} k x_n/(1-x_n)$ and equating coefficients of $x_1^{n_1} x_2^{n_2} \ldots \ldots x_n^{n_n}$. We give the details of the working for the case in which n = 3 and, for convenience, change the notation and make the substitution $X = kxyz/(1-x)(1-y)(1-z)$ and equate coefficients of $x^m y^n z^N$ on both sides, i.e. in (1.1) put $X = \dfrac{kxyz}{(1-x)(1-y)(1-z)}$ and multiply by $(1-x)^{-P}(1-y)^{-q}(1-z)^{-R}$.

Then,

$$\text{L.H.S.} = \sum_{r=0}^{\infty} \frac{(\rho_1)_r (\rho_2)_r (\rho_3)_n b^r k^r x^r y^r z^r (1-x)^{-P-r}(1-y)^{-q-r}(1-z)^{-R-r}}{r!\,(\sigma_1)_r (\sigma_2)_r}$$

and the coefficient of $x^m y^n z^N$ in this is:-

$$\sum_{r=0}^{\infty} \frac{(\rho_1)_r (\rho_2)_r (\rho_3)_n (p+r)_{m-r}(q+r)_{n-r}(R+r)_{N-r} b^r k^r}{r!\,(\sigma_1)_r (\sigma_2)_r (m-r)!\,(n-r)!\,(N-r)!}$$

$$= \frac{(p)_m (q)_n (R)_N}{m!\,n!\,N!} \times F\left[\begin{array}{c} \rho_1,\ \rho_2,\ \rho_3,\ -m,\ -n,\ -N\ ;\ -bk \\ \sigma_1,\ \sigma_2,\ p,\ q,\ R \end{array}\right]$$

R.H.S. is

$$\sum_{r=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r (\alpha_3)_r c^r X^r (1-X)^{-K-\mu r}(1-x)^{-P}(1-y)^{-q}(1-z)^{-R}}{r!\,(\beta_1)_r (\beta_2)_r}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r (\alpha_3)_r c^r (K+\mu r)_s X^{r+s}(1-x)^{-P}(1-y)^{-q}(1-z)^{-R}}{r!\,s!\,(\beta_1)_r (\beta_2)_r}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r (\alpha_3)_r c^r (K+\mu r)_s k^{r+s}(1-x)^{-P-r-s}(1-y)^{-q-r-s}(1-z)^{-R-r-s}}{r!\,s!\,(\beta_1)_r (\beta_2)_r}$$

and the coefficient of $x^m y^n z^N$ in this is

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r (\alpha_3)_r c^r (K+\mu r)_s k^{r+s}}{r!\,(\beta_1)_r (\beta_2)_r} \frac{(p+r+s)_{m-r-s}}{s!} \frac{(q+r+s)_{n-r-s}}{(m-r-s)!} \frac{(R+r+s)_{N-r-s}}{(n-r-s)!}\frac{}{(N-r-s)!}$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r (\alpha_3)_r (-m)_r (-n)_r (-N)_r (-ck)(p)_m (q)_n (R)_N}{r!\,(\beta_1)_r (\beta_2)_r (p)_r (q)_r (R)_r\,m!\,n!\,N!} \times F\left[\begin{array}{c} -m+r,\ -n+r,\ -N+r,\ K+\mu r\ ;\ -k \\ p+r,\ q+r,\ R+r \end{array}\right]$$

Therefore we have that if

$$\sum_{r=0}^{n} \frac{(\alpha_1)_r (\alpha_2)_r (\alpha_3)_r (-n)_r (K+n)_{r(\mu-1)}(-1)^r c^r}{r!\,(\beta_1)_r (\beta_2)_r (K)_{\mu r}} = \frac{(\rho_1)_n (\rho_2)_n (\rho_3)_n b^n}{(\sigma_1)_n (\sigma_2)_n (K)_n}$$

then

$$F\left[\begin{array}{c} \rho_1, \ \rho_2, \ \rho_3, \ -m, \ -n, \ -N \ ; \ (-bk) \\ \sigma_1, \ \sigma_2, \ p, \ q, \ R \end{array}\right]$$

$$=\sum_{r=0}^{\infty}\frac{(\alpha_1)_r(\alpha_2)_r(\alpha_3)_r(-m)_r(-n)_r(-N)_r(-ck)^r}{r!\ (\beta_1)_r(\beta_2)_r(p)_r(q)_r(R)_r}\times F\left[\begin{array}{c} K+\mu r, \ -m+r, \ -n+r, \ -N+r \ ; \ -k \\ p+r, \ q+r, \ R+r \end{array}\right]. \quad (1-5)$$

The case $k = -1$, gives the formula given by Professor Bailey (with the a's negative integers).

Unfortunately, this k is of little use, except when it equals -1, because very little is known about the sums of series with argument x, except when x = 1, and $-$ 1 and the cases when x = $-$ 1, are limiting forms of those where x = 1. Thus although we might be able to sum the series on the right of (1 $-$ 5) by Kummer's formula, we could obtain the same transformation by summing by Saalschütz's formula and letting one of the parameters tend to infinity. The only other possibility would be to use the formulae for the sum of a $_2F_1(\tfrac{1}{2})$ given in the Tract Page 11. In this case, however, the parameters involved are of such a specialised form, that it has not been possible to find a case where these sums could be used.

(2).   We now proceed to consideration of transformations obtained from a formula given by Saalschütz (6), viz:—

$$_3F_2\left[a, \ b, \ -n \ ; \ c, \ 1+a+b-c-n\right] = (c-a)_n(c-b)_n/(c)_n(c-a-b)_n \ .$$

More recently, a very simple proof of this formula has been given by Professor Watson, (7), who pointed out that it could be obtained by equating coefficients of x^n on both sides of the identity

$$(1-x)^{a+b-c}\,_2F_1\left[a, b; c; x\right] = \,_2F_1\left[c-a, c-b; c; x\right].$$

The cases in which we write Saalschütz's formula in the following forms:-

$$_3F_2\left[1+a-b-c, \ a+n, \ -n \ ; \ 1+a-b, 1+a-c\right] = (b)_n(c)_n/(1+a-b)_n(1+a-c)_n \ ,$$

$$_3F_2\left[a, b+n, -n \ ; \ c+n, 1+a+b-c-n\right] = (c)_n(c-a)_{2n}(c-b)_n/(c)_{2n}(c-a)_n(c-a-b)_n \ ,$$

have been fully investigated by Professor Bailey. (5) In both these cases, he was able to sum the hypergeometric series on the right of (/3) by Saalschütz and Dougall's Theorem: he thus obtained the most general possible transformations.

The forms of Saalschütz's theorem which I have taken and which yield results are

$$_3F_2\left[a, b, -n; c, 1+a+b-c-n\right] = (c-a)_n(c-b)_n/(c)_n(c-a-b)_n \quad , \qquad - (2.1)$$

$$_3F_2\left[a, b-n, -n; c-n, 1+a+b-c-n\right] = (c-a-n)_n(c-b)_n/(c-n)_n(c-a-b)_n , \qquad - (2.2)$$

$$_3F_2\left[a+n, b, -n; c+2n, 1+a+b-c-2n\right] = (c-a+n)_n(c-b+2n)_n/(c+2n)_n(c-a-b+n)_n , \qquad - (2.3)$$

$$_3F_2\left[a+n, b, -n; c+3n, 1+a+b-c-3n\right] = (c-a+2n)_n(c-b+3n)_n/(c+3n)_n(c-a-b+2n)_n . \qquad - (2.4)$$

(2.1)
→Taking Saal. in the form $_3F_2\left[a, b, -n; c, 1+a+b-c-n\right] = (c-a)_n(c-b)_n/(c)_n(c-a-b)_n$

we have, using
$(a)_{-r} = (-1)^r/(1-a)_r$, $\sum_{r=0}^{n} \frac{(a)_r(b)_r(c-a-b+n)_r(-n)_r(-1)^r}{r! \, (c)_r} = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}$

This is formula (α) with

$\alpha_1 = a, \quad \alpha_2 = b ; \quad K_1 = (c-a-b), \quad \mu_1 = 0 ; \quad c = 1 ;$

$\beta_1 = c ;$

$\rho_1 = c-a, \quad \rho_2 = c-b; \quad \sigma_1 = c ; \quad b = 1$

Therefore using formula (β) we obtain the transformation

$$_{s+3}F_{s+2}\left[\begin{matrix} c-a, c-b, a_1, a_2, \ldots, a_s, -m ; \\ c, p_1, p_2, \ldots, p_s, p_{s+1} \end{matrix}\right]$$

$$= \sum_{r=0}^{m} \frac{(a_1)_r(a_2)_r \ldots (a_s)_r(-m)_r(a)_r(b)_r}{r! \, (p_1)_r \ldots (p_{s+1})_r (c)_r} \times {}_{s+2}F_{s+1}\left[\begin{matrix} c-a-b, a_1+r, a_2+r, \ldots, a_s+r, -m+r ; \\ p_1+r, p_2+r, \ldots, p_s+r, p_{s+1}+r \end{matrix}\right]. \quad - (2.1A)$$

If in (2.1A) we take s = 0, no a's, and one p = p(say), and (2.1A) becomes

$$_3F_2\left[\begin{matrix} c-a, c-b, -m ; \\ c, p \end{matrix}\right] = \sum_{r=0}^{m} \frac{(a)_r(b)_r(-m)_r}{r! \, (c)_r(p)_r} \times {}_2F_1\left[\begin{matrix} c-a-b, -m+r ; \\ p+r \end{matrix}\right] .$$

We can now sum the $_2F_1$ on the right, using Vandermonde's Theorem viz :—
$$_2F_1\left[-n, b; c; 1\right] = (c-b)_n/(c)_n$$

$$\therefore \quad _3F_2\left[\begin{matrix} c-a, c-b, -m ; \\ c, p \end{matrix}\right]$$

$$= \sum_{r=0}^{m} \frac{(a)_r(b)_r(-m)_r}{r! \, (c)_r(p)_r} \times \frac{(p+r-c+a+b)_{m-r}}{(p+r)_{m-r}}$$

$$= \frac{(p-c+a+b)_m}{(p)_m} \times \sum_{r=0}^{m} \frac{(a)_r(b)_r(-m)_r}{r! \, (c)_r(p-c+a+b)_r}$$

since $(p+r)_{m-r} = (p)_m/(p)_r$

We thus obtain the transformation

$$_3F_2\left[\begin{matrix}c-a,\,c-b,\,-m\,;\\c,\,p\end{matrix}\right]=\frac{(p-c+a+b)_m}{(p)_m}\times\,_3F_2\left[\begin{matrix}a,\,b,\,-m\,;\\c,\,p-c+a+b\end{matrix}\right]$$

This, with a change of notation, can be written,

$$_3F_2\left[\begin{matrix}a,\,b,\,-m\,;\\e,\,f\end{matrix}\right]=\frac{(f+e-a-b)_m}{(f)_m}\times\,_3F_2\left[\begin{matrix}e-a,\,e-b,\,-m\,;\\e,\,e+f-a-b\end{matrix}\right].\qquad\text{---}\;(2.1B)$$

Using the notation introduced by Whipple (see Page 17 Tract) and taking $c=-m$, (2.1B) can be written

$$\Gamma(\alpha_{1,3})\Gamma(\beta_{4,0})\Gamma(\beta_{5,0})\,F_p(0\,;\,4,5)$$

$$=(s+c)_m/(f)_m\,F_p(0;\,3,4)\,\Gamma(\alpha_{1,2,5})\Gamma(\beta_{4,0})\Gamma(\beta_{3,0}).$$

$$=\Gamma(s)\Gamma(f)/\Gamma(s+c)\Gamma(f-c)\,\Gamma(\alpha_{1,2,5})\Gamma(\beta_{4,0})\Gamma(\beta_{3,0})\,F_p(0\,;\,3,4).$$

$$=\Gamma(\alpha_{1,3})\Gamma(\beta_{5,0})\Gamma(\beta_{4,0})\,F_p(0\,;\,3,4).$$

$$\therefore\;F_p(0\,;\,4,5)=F_p(0\,;\,3,4),\qquad\text{---}\;(2.1B)$$

a relationship which was first given by Barnes and subsequently by Whipple and Hardy. The relationship is true, in the more general case, where c replaces the $-m$, i.e. when the $_3F_2$'s do not terminate.

Now in (2.1A) take $a_1=d$ (say) and two p's equal to p & q then

$$_4F_3\left[\begin{matrix}c-a,\,c-b,\,d,\,-m\,;\\c,\,p,\,q\end{matrix}\right]=\sum_{r=0}^{m}\frac{(a)_r(b)_r(d)_r(-m)_r}{r!\,(c)_r(p)_r(q)_r}\times\,_3F_2\left[\begin{matrix}c-a-b,\,d+r,\,-m+r\,;\\p+r,\,q+r\end{matrix}\right].$$

Now choose q so that the $_3F_2$ on the right is Saalschützian i.e. take $q=c-a-b+d-p-m+1$.

Then, summing the $_3F_2$ on the right by Saalschütz,

$$_4F_3\left[\begin{matrix}c-a,\,c-b,\,d,\,-m\\c,\,p,\,q\end{matrix}\right]$$

$$=\sum_{r=0}^{m}\frac{(a)_r(b)_r(d)_r(-m)_r}{r!\,(c)_r(p)_r(q)_r}\times\frac{(p+a+b-c+r)_{m-r}(p-d)_{m-r}}{(p+r)_{m-r}(p+a+b-c-d)_{m-r}}$$

$$=\frac{(p+a-b-c)_m(p-d)_m}{(p)_m(p+a+b-c-d)_m}\times\sum_{r=0}^{m}\frac{(d)_r(a)_r(b)_r(c+d-p-a-b-m+1)_r(-m)_r}{r!\,(q)_r(c)_r(d+1-p-m)_r(p+a+b-c)_r},$$

since

$$(p)_{m-r}=(p)_m/(-1)^r(1-p-m)_r,$$

therefore we are led to the transformation

$$_4F_3\left[\begin{matrix}c-a,\,c-b,\,d,\,-m\,;\\c,\,p,\,c+d-a-b-m+1\end{matrix}\right]=\frac{(p+a+b-c)_m(p-d)_m}{(p)_m(p+a+b-c-d)_m}\times\,_4F_3\left[\begin{matrix}d,\,a,\,b,\,-m\,;\\c,\,1+d-p-m,\,p+a+b-c\end{matrix}\right]$$

this may be written,

$$_4F_3\left[\begin{matrix}a,\,b,\,c,\,-m\,;\\e,\,f,\,1+a+b+c-e-f-m\end{matrix}\right]$$

$$=\frac{(f-a-b+e)_m(f-c)_m}{(f)_m(f-a-b+e-c)_m}\times\,_4F_3\left[\begin{matrix}e-a,\,e-b,\,c,\,-m\,;\\e,\,e+f-a-b,\,c-f-m+1\end{matrix}\right],\qquad\text{---}\;(2.1C)$$

a transformation of two Saalschützian $_4F_3$'s .

Using the relation $(a)_n = (-1)^n (1-a-n)_n$, (2.1C) may be written

$$_4F_3 \left[\begin{array}{c} a, b, c, -m \\ e, f, 1+a+b+c-e-f-m \end{array} \right]$$

$$= \frac{(f-c)_m (1+a+b-e-f-m)_m}{(f)_m (1+a+b+c-e-f-m)_m} \times {}_4F_3 \left[\begin{array}{c} e-a, e-b, c, -m \; ; \\ e, 1-f+c-m, e+f-a-b \end{array} \right] .$$

This is the formula of the Tract Page 56 §7.2 (1) with a,b,c, e & f replacing x,y,z,u,v, a formula which can be obtained directly as shewn and first given by Whipple.

(2.2)

We now take Saalschütz's Theorem in the form

$$_3F_2 \left[a, b-n, -n \; ; c-n, 1+a+b-c-n \right] = \frac{(c-a-n)_n (c-b)_n}{(c-n)_n (c-a-b)_n}$$ which can be written

$$\sum_{r=0}^{n} \frac{(a)_r (1-c+n)_r (c-a-b+n)_r (-n)_r (-1)^r}{r! \, (1-b+n)_r} = \frac{(a-c+1)_n (c-b)_n}{(1-c)_n (c-a-b)_n}$$

This is the formula (α) with

$\alpha_1 = a$, $\kappa_1 = 1-c$, $\mu_1 = 0$; $\kappa_2 = c-a-b$, $\mu_2 = 0$; $c = 1$;

$\ell_1 = 1-b$, $v_1 = 0$;

$\rho_1 = a-c+1$, $\rho_2 = c-b$; $\sigma_1 = 1-b$; $b = 1$;

therefore using (β) we obtain the transformation

$$_{s+3}F_{s+2} \left[\begin{array}{c} a-c+1, c-b, a_1, a_2, \ldots, a_s, -m \; ; \\ 1-b, \rho_1, \rho_2, \ldots, \rho_s, \rho_{s+1} \end{array} \right] \qquad - (2.2A)$$

$$= \frac{(a)_r (a_1)_r \ldots (a_s)_r (-m)_r (a)_r}{r! \, (\rho_1)_r \ldots (\rho_{s+1})_r} \times {}_{s+3}F_{s+2} \left[\begin{array}{c} 1-c, c-a-b, a_1+r, a_2+r, \ldots, a_s+r, -m+r \; ; \\ 1-b, \rho_1+r, \rho_2+r, \ldots, \rho_s+r, \rho_{s+1}+r \end{array} \right]$$

Now sum on the right of (2.2A) by Saalschütz's theorem taking s = 0 and one p = - a - m + 1 and (2.2A) becomes

$$_3F_2 \left[\begin{array}{c} a-c+1, c-b, -m \; ; \\ 1-b, p \end{array} \right] = \frac{(a)_r (-m)_r}{r! \, (p)_r} \times {}_3F_2 \left[\begin{array}{c} 1-c, c-a-b, -m+r \; ; \\ 1-b, 1-a-m+r \end{array} \right]$$

$$= \frac{(c-b)_m (1+a-c)_m}{(1-b)_m (a)_m} \times {}_3F_2 \left[\begin{array}{c} a, b-m, -m \; ; \\ 1+b-c-m, c-a-m \end{array} \right]$$

therefore we have the transformation

$$_3F_2 \left[\begin{array}{c} a-c+1, c-b, -m \; ; \\ 1-b, 1-a-m \end{array} \right] = \frac{(c-b)_m (1+a-c)_m}{(1-b)_m (a)_m} \times {}_3F_2 \left[\begin{array}{c} a, b-m, -m \; ; \\ b-c-m+1, c-a-m \end{array} \right] .$$

This may be written

$$_3F_2\left[\begin{matrix}a,\ b,\ -m\ ;\\ c,\ c-a-b-m+1\end{matrix}\right] = \frac{(b)_m (a)_m}{(c)_m (a+b-c)_m} \times {}_3F_2\left[\begin{matrix}a+b-c,\ 1-c-m,\ -m\ ;\\ 1-b-m,\ 1-a-m\end{matrix}\right]. - (2.2B)$$

If in relation (2.2B) we put $c = e$ and $f = c-a-b-m+1$
i.e. $f = e-a-b-m+1$, then (2.2B) can be written

$$_3F_2\left[\begin{matrix}a,\ b,\ -m\ ;\\ e,\ f\end{matrix}\right] = \frac{(-1)^m (1-b-m)_m (1-a-m)_m}{(e)_m (1-a-b+e-m)_m} \times {}_3F_2\left[\begin{matrix}a+b-e,\ 1-e-m,\ -m\ ;\\ 1-b-m,\ 1-a-m\end{matrix}\right].$$

Now, when in Whipple's notation we put $\alpha_{345} = c = -m$ this
becomes

$$\Gamma(\alpha_{123})\Gamma(\beta_{40})\Gamma(\beta_{50})F_p(0;\ 4,5)$$

$$= \frac{(-1)^m \Gamma(\alpha_{011})\Gamma(\alpha_{023})\Gamma(\beta_{40})\Gamma(\beta_{50})\Gamma(\beta_{13})\Gamma(\beta_{22})F_n(3;\ 1,2)\Gamma(\alpha_{123})}{\Gamma(\beta_{32})\Gamma(\beta_{31})\Gamma(\alpha_{124})\Gamma(\alpha_{125})}$$

i.e. $\Gamma(\alpha_{124})\Gamma(\alpha_{125})F_p(0;\ 4,5) = \Gamma(\alpha_{013})\Gamma(\alpha_{023})(-1)^m F_n(3;\ 1,2)$.

This result was proved by Whipple in the case of general f
and can be proved directly by reversing the series on the
left of (2.2B). It is a degenerate form of Tract (3.7)
(1) with $f = e - b - a - m + 1$.

Before we can proceed to the consideration of cases
(2.3) and (2.4), it is necessary to prove a slight modification
of formulae (α) and (β).
We prove that if we have a formula of the type

$$\sum_{r=0}^{n} \frac{(\alpha_1)_r(\alpha_2)_r(\alpha_3)_r(-n)_r(K_1+\lambda_1 n)_{\lambda_1 r(\mu_1-1)}(K_2+\lambda_2 n)_{\lambda_2 r(\mu_2-1)}(\ell_1)_{\phi_1 v_1 r}(-1)^r c^r}{r!\ (\beta_1)_r(\beta_2)_r(\ell_1+\phi_1 n)_{\phi_1 r(v_1-1)}(K_1)_{\lambda_1\mu_1 r}(K_2)_{\lambda_2\mu_2 r}} = \frac{(\rho_1)_n(\rho_2)_n(\rho_3)_n(\ell_1)_{\phi_1 n}b^n}{(\sigma_1)_n(\sigma_2)_n(K_1)_{\lambda_1 n}(K_2)_{\lambda_2 n}} \quad - (A)$$

giving the sum of a certain hypergeometric series, then the
transformation

$$F\left[\begin{matrix}\rho_1,\ \rho_2,\ \rho_3,\ a_1,\ a_2,\ -m\ ;\ b\\ \sigma_1,\ \sigma_2,\ \rho_1,\ \rho_2,\ \rho_3\end{matrix}\right]$$

$$= \sum_{r=0}^{m} \frac{(a_1)_r(a_2)_r(-m)_r(\alpha_1)_r(\alpha_2)_r(\alpha_3)_r c^r}{r!\ (\rho_1)_r(\rho_2)_r(\rho_3)_r(\beta_1)_r(\beta_2)_r} \times \frac{(K_1+\lambda_1\mu_1 r)_{\lambda_1 t}(K_2+\lambda_2\mu_2 r)_{\lambda_2 t}(a_1+r)_t(a_2+r)_t(-m+r)_t}{t!\ (\ell_1+\phi_1 v_1 r)_{\phi_1 t}(\rho_1+r)_t(\rho_2+r)_t(\rho_3+r)_t} \quad - (B)$$

is true. As before there may be any number of the quantities
$\alpha,\ \beta,\ \rho,\ \sigma,\ \ell,\ a,\ \rho,$ but μ & v may now have fractional
values, provided that $\lambda\mu$ & ϕv are integral.
For,

$$F\left[\begin{matrix}\rho_1,\ \rho_2,\ \rho_3,\ a_1,\ a_2,\ -m\ ;\ b\\ \sigma_1,\ \sigma_2,\ \rho_1,\ \rho_2,\ \rho_3\end{matrix}\right] = \sum_{n=0}^{m} \frac{(a_1)_n(a_2)_n(-m)_n(K_1)_{\lambda_1 n}(K_2)_{\lambda_2 n}}{n!\ (\rho_1)_n(\rho_2)_n(\rho_3)_n(\ell_1)_{\phi_1 n}}$$

$$\times \sum_{r=0}^{n} \frac{(\alpha_1)_r(\alpha_2)_r(\alpha_3)_r(-n)_r(K_1+\lambda_1 n)_{\lambda_1 r(\mu_1-1)}(K_2+\lambda_2 n)_{\lambda_2 r(\mu_2-1)}(\ell_1)_{\phi_1 v_1 r}(-1)^r c^r}{r!\ (\beta_1)_r(\beta_2)_r(\ell_1+\phi_1 n)_{\phi_1 r(v_1-1)}(K_1)_{\lambda_1\mu_1 r}(K_2)_{\lambda_2\mu_2 r}} \quad \text{using (A)}$$

$$= \sum_{r=0}^{m}\sum_{n=r}^{m} \frac{(a_1)_n(a_2)_n(-m)_n(K_1+\lambda_1\mu_1 r)_{\lambda_1(n-r)}(K_2+\lambda_2\mu_2 r)_{\lambda_2(n-r)}(\alpha_1)_r(\alpha_2)_r(\alpha_3)_r c^r}{r!\ (n-r)!\ (\rho_1)_n(\rho_2)_n(\rho_3)_n(\ell_1+\phi_1 v_1 r)_{\phi_1(n-r)}(\beta_1)_r(\beta_2)_r}$$

(and putting n = r+t we get)

$$= \sum_{r=0}^{m}\sum_{t=0}^{m-r} \frac{(a_1)_{r+t}(a_2)_{r+t}(-m)_{r+t}(K_1+\lambda_1\mu_1 r)_{\lambda_1 t}(K_2+\lambda_2\mu_2 r)_{\lambda_2 t}(\alpha_1)_r(\alpha_2)_r(\alpha_3)_r c^r}{r!\ t!\ (\rho_1)_{r+t}(\rho_2)_{r+t}(\rho_3)_{r+t}(\ell_1+\phi_1 v_1 r)_{\phi_1 t}(\beta_1)_r(\beta_2)_r}.$$

$$= \sum_{r=0}^{\infty} \frac{(a_1)_r (a_2)_r (-m)_r (\alpha_1)_r (\alpha_2)_r (\alpha_3)_r c^r}{r! (p_1)_r (p_2)_r (p_3)_r (\beta_1)_r (\beta_2)_r} \times \frac{(K_1 + \lambda_1 \mu_1 r)_{\lambda_1 t} (K_2 + \lambda_2 \mu_2 r)_{\lambda_2 t} (a_1 + r)_t (a_2 + r)_t (-m + r)_t}{t! (\ell_1 + \phi_1 \nu_1 r)_{\phi_1 t} (p_1 + r)_t (p_2 + r)_t (p_3 + r)_t}$$

which is the required result (B).

(2.3)   We now take Saalschütz's formula in the form

$$_3F_2 \left[\begin{array}{c} a+n, \; b, \; -n \; ; \\ c+2n, \; 1+a+b-c-2n \end{array} \right] = \frac{(c-a+n)_n (c-b+2n)_n}{(c+2n)_n (c-a-b+n)_n}$$

This can be written

$$\sum_{r=0}^{n} \frac{(a+n)_r (b)_r (c-a-b+2n)_r (-1)^r (-n)_r}{r! \; (c+2n)_r} = \frac{(c-a)_{2n} (c-b)_{3n} (c)_{2n} (c-a-b)_n}{(c-a)_n (c-b)_{2n} (c)_{3n} (c-a-b)_{2n}}$$

This is formula (A) with

$$\alpha_1 = b, \; \alpha_2 = \frac{a}{2}, \; \alpha_3 = \frac{a+1}{2}, \; \alpha_4 = c-a-b; \; \beta_1 = \frac{c}{3}, \; \beta_2 = \frac{c+1}{3}, \; \beta_3 = \frac{c+2}{3} \; ; \; b = 1;$$

$$K_1 = a, \; \lambda_1 = 1, \; \mu_1 = 2, \; K_2 = c-a-b, \; \lambda_2 = 2, \; \mu_2 = \frac{1}{2}; \; \ell_1 = c, \; \phi_1 = 2, \; \nu_1 = \frac{3}{2}; \; c = \frac{4}{27};$$

$$\rho_1 = \frac{c-a}{2}, \; \rho_2 = \frac{c-a+1}{2}, \; \rho_3 = c-a-b, \; \rho_4 = \frac{c-b}{3}, \; \rho_5 = \frac{c-b+1}{3}, \; \rho_6 = \frac{c-b+2}{3}, \; \rho_7 = a \; ;$$

$$\sigma_1 = c-a, \; \sigma_2 = \frac{c-b}{2}, \; \sigma_3 = \frac{c-b+1}{2}, \; \sigma_4 = \frac{c}{3}, \; \sigma_5 = \frac{c+1}{3}, \; \sigma_6 = \frac{c+2}{3} \; ;$$

therefore using (B) we obtain the transformation

$$_{s+8}F_{s+7} \left[\begin{array}{c} \frac{c-a}{2}, \frac{c-a+1}{2}, c-a-b, \frac{c-b}{3}, \frac{c-b+1}{3}, \frac{c-b+2}{3}, a, a_1, a_2, \cdots, a_s, -m \; ; \\ c-a, \frac{c-b}{2}, \frac{c-b+1}{2}, \frac{c}{3}, \frac{c+1}{3}, \frac{c+2}{3}, p_1, p_2, \cdots, p_s, p_{s+1} \end{array} \right]$$

$$= \sum_{r=0}^{\infty} \frac{(a_1)_r (a_2)_r \cdots (a_s)_r (-m)_r (b)_r \left(\frac{a}{2}\right)_r \left(\frac{a+1}{2}\right)_r (c-a-b)_r \left(\frac{4}{27}\right)^r}{r! \; (p_1)_r (p_2)_r \cdots (p_{s+1})_r \left(\frac{c}{3}\right)_r \left(\frac{c+1}{3}\right)_r \left(\frac{c+2}{3}\right)_r}$$

$$\times \;_{s+4}F_{s+3} \left[\begin{array}{c} a+2r, \frac{c-a-b+r}{2}, \frac{c-a-b+1+r}{2}, a_1+r, a_2+r, \ldots, a_s+r, -m+r \; ; \\ \frac{c+3r+1}{2}, \frac{c+3r}{2}, p_1+r, p_2+r, \cdots, p_s+r, p_{s+1}+r \end{array} \right] . \; \text{—— (2.3A)}$$

We can sum the series on the right of (2.3A) by Dougall's Theorem if the parameters satisfy certain conditions viz,

$$b = 2c - 3a - 1$$

and we take $a_3 = 1 + \frac{1}{2}a$, $p_3 = \frac{1}{2}a$; then we take two more a's a_1 and a_2 (say) and p_1 and p_2 equal to $1+a-a_1$, $1+a-a_2$, respectively. Also since the sum of the denominator parameters must exceed the sum of the numerator by 2 we require $a_2 = c - \frac{1}{2} + m - a_4$.

The hypergeometric series on the right of (2.3A) then becomes

$$_7F_6\left[\begin{array}{c} a+2r,\ 1+\frac{a}{2}+r,\ \frac{c-a-b+r}{2},\ \frac{c-a-b+1+r}{2},\ a_1+r,\ a_2+r,\ -m+r\ ; \\ \frac{a}{2}+r,\ \frac{c+1+3r}{2},\ \frac{c+3r}{2},\ 1+a-a_1+r,\ 1+a-a_2+r,\ 1+a+m+r \end{array}\right]$$

and the sum of this (by Dougall's Theorem) is

$$\frac{(1+a+2r)_{m-r}\ (\frac{1}{2}+2a-c+b+r)_{m-r}\ (1+3a+b-c-2a_1+r)_{2(m-r)}}{(1+a-a_1+r)_{m-r}\ (1+3a+b-c+3r)_{2(m-r)}\ (\frac{1}{2}+2a-c+b-a_1)_{m-r}}$$

But $(p+r)_{m-r} = \dfrac{(p)_m}{(p)_r}$, $(p+r)_{2(m-r)} = \dfrac{(p)_{2m}}{(-1)^r(p)_r(1-p-2m)_r}$, $(p+2r)_{m-r} = \dfrac{(p)_m(p+m)_r}{(p)_{2r}}$,

$(p+3r)_{2(m-r)} = \dfrac{(p)_{2m}(p+2m)_r}{(p)_{3r}}$, $(p)_{m-r} = \dfrac{(p)_m}{(-1)^r(1-p-m)_r}$

and substituting $2c - 3a -1$ for b, we obtain as the sum of the $_7F_6$.

$$\frac{(1+a)_m(1+a+m)_r(c-a-\frac{1}{2})_m(c-2a_1)_{2m}(1+a-a_1)_r(-1)^r(a+a_1+\frac{1}{2}-m-c)_r\,3^{3r}(\frac{c}{3})_r(\frac{c+1}{3})_r(\frac{c+2}{3})_r}{2^{2r}(\frac{1+a}{2})_r(1+\frac{a}{2})_r(c-a-\frac{1}{2})_r(c-2a_1)_r(-1)^r(1-c+2a_1-2m)_r(c-a-a_1-\frac{1}{2})_m(c)_{2m}(c+2m)_r}$$

$$\therefore\ _{11}F_{10}\left[\begin{array}{c} a,\ 1+\frac{1}{2}a,\ \frac{c-a}{2},\ \frac{c-a+1}{2},\ 2a-c+1,\ \frac{3a+1-c}{3},\ \frac{3a+2-c}{3},\ \frac{3a+3-c}{3},\ a_1,\ c-\frac{1}{2}+m-a_1,\ -m\ ; \\ \frac{1}{2}a,\ \frac{3a+2-c}{2},\ \frac{3a+1-c}{2},\ c-a,\ \frac{c+2}{3},\ \frac{c+1}{3},\ \frac{c}{3},\ 1+a-a_1,\ \frac{3}{2}+a+a_1-c-m,\ 1+a+m \end{array}\right]$$

$$= \frac{(1+a)_m(c-2a_1)_{2m}(c-a-\frac{1}{2})_m}{(1+a-a_1)_m(\frac{1}{2}+2a+b-c-a_1)_m(c)_{2m}}\ .$$

$$\times\ \frac{(1+a+m)_r(1+a-a_1)_r(\frac{3}{2}-c+a+a_1+m)_r(\frac{c}{3})_r(\frac{c+1}{3})_r(\frac{c+2}{3})_r\,3^{3r}(1+\frac{a}{2})_r(a_1)_r(c-\frac{1}{2}+m-a_1)_r(-m)_r}{(\frac{1+a}{2})_r(1+\frac{a}{2})_r(c-a-\frac{1}{2})_r(c-2a_1)_r(1-c+2a_1-2m)_r(c+2m)_r\,r!\,(\frac{c}{3})_r(1+a-a_1)_r(\frac{3}{2}+a+a_1-c-m)_r}$$

$$\times\ \frac{(2c-3a-1)_r(\frac{a}{2})_r(\frac{a+1}{2})_r(c-a-b)_r(\frac{4}{3}a)_r^{}}{(1+a+m)_r(\frac{c}{3})_r(\frac{c+1}{3})_r(\frac{c+2}{3})_r}$$

Therefore we have the transformation

$$_{11}F_{10}\left[\begin{array}{c} a,\ 1+\frac{1}{2}a,\ \frac{c-a}{2},\ \frac{c-a+1}{2},\ 2a-c+1,\ \frac{3a+1-c}{3},\ \frac{3a+2-c}{3},\ \frac{3a+3-c}{3},\ a_1,\ c-\frac{1}{2}+m-a_1,\ -m\ ; \\ \frac{1}{2}a,\ \frac{3a+2-c}{2},\ \frac{3a+1-c}{2},\ c-a,\ \frac{c+2}{3},\ \frac{c+1}{3},\ \frac{c}{3},\ 1+a-a_1,\ \frac{3}{2}+a+a_1-c-m,\ 1+a+m \end{array}\right]$$

$$= \frac{(1+a)_m(c-2a_1)_{2m}(c-a-\frac{1}{2})_m}{(1+a-a_1)_m(c-a-\frac{1}{2}-a_1)_m(c)_{2m}}\ \times\ _5F_4\left[\begin{array}{c} a_1,\ c-\frac{1}{2}-a_1+m,\ 2c-3a-1,\ 2a-c+1,\ -m\ ; \\ c-a-\frac{1}{2},\ c-2a_1,\ c+2m,\ 1-c+2a_1-2m \end{array}\right]\ \ -\ (2.3B)\ .$$

This formula (2.3B) transforms a particular well-poised $_{11}F_{10}$ with the special form of second parameter into a Saalschützian $_6F_4$ and is thought not to have been proved previously.

(2.4)   Now take Saalschütz's theorem in the form

$$_3F_2\left[\begin{array}{c} a+n, b, -n \; ; \\ c+3n, 1+a+b-c-3n \end{array}\right] = \frac{(c-a+2n)_n (c-b+3n)_n}{(c+3n)_n (c-a-b+2n)_n} \; .$$

This can be written

$$\sum_{r=0}^{n} \frac{(a+n)_r (b)_r (c-a-b+3n)_r (-1)^r (-n)_r}{r! \, (c+3n)_r} = \frac{(c-a)_{3n} (c-b)_{4n} (c)_{3n} (c-a-b)_{3n}}{(c-a)_{2n} (c-b)_{3n} (c)_{4n} (c-a-b)_{3n}} \; .$$

This is formula (A) with

$$\alpha_1 = b, \; \alpha_2 = \frac{a}{2}, \; \alpha_3 = \frac{a+1}{2}, \; \alpha_4 = \frac{c-a-b}{2}, \alpha_5 = \frac{c-a-b+1}{2} \; ; \; \beta_1 = \frac{c}{4}, \; \beta_2 = \frac{c+1}{4}, \; \beta_3 = \frac{c+2}{4}, \; \beta_4 = \frac{c+3}{4} \; ;$$

$$K_1 = a, \lambda_1 = 1, \mu_1 = 2 \; ; \; K_2 = c-a-b, \lambda_2 = 3, \; \mu_2 = \frac{2}{3} \; ; \; l_1 = c, \; \phi_1 = 3, \nu_1 = \frac{4}{3} \; ;$$

$$\rho_1 = \frac{c-a}{3}, \; \rho_2 = \frac{c-a+1}{3}, \rho_3 = \frac{c-a+2}{3}, \; \rho_4 = \frac{c-b}{4}, \; \rho_5 = \frac{c-b+1}{4}, \; \rho_6 = \frac{c-b+2}{4}, \; \rho_7 = \frac{c-b+3}{4},$$

$$\rho_8 = \frac{c-a-b}{2}, \; \rho_9 = \frac{c-a-b+1}{2}, \; \rho_{10} = a \; ;$$

$$\sigma_1 = \frac{c-a}{2}, \; \sigma_2 = \frac{c-a+1}{2}, \; \sigma_3 = \frac{c-b}{3}, \; \sigma_4 = \frac{c-b+1}{3}, \; \sigma_5 = \frac{c-b+2}{3}, \; \sigma_6 = \frac{c}{4}, \; \sigma_7 = \frac{c+1}{4},$$

$$\sigma_8 = \frac{c+2}{4}, \; \sigma_9 = \frac{c+3}{4} \; ;$$

therefore using formula (B) we obtain the transformation

$$_{s+11}F_{s+10}\left[\begin{array}{c} a, \frac{c-a}{3}, \frac{c-a+1}{3}, \frac{c-a+2}{3}, \frac{c-b}{4}, \frac{c-b+1}{4}, \frac{c-b+2}{4}, \frac{c-b+3}{4}, \frac{c-a-b}{2}, \frac{c-a-b+1}{2}, a_1, a_2, \ldots, a_s, -m \; ; \\ \frac{c-a}{2}, \frac{c-a+1}{2}, \frac{c-b}{3}, \frac{c-b+1}{3}, \frac{c-b+2}{3}, \frac{c}{4}, \frac{c+1}{4}, \frac{c+2}{4}, \frac{c+3}{4}, p_1, p_2, \ldots, p_s, p_{s+1} \end{array}\right]$$

$$= \sum_{r=0}^{m} \frac{(a_1)_r (a_2)_r \ldots (a_s)_r (-m)_r (b)_r \left(\frac{a}{2}\right)_r \left(\frac{a+1}{2}\right)_r \left(\frac{c-a-b}{2}\right)_r \left(\frac{c-a-b+1}{2}\right)_r}{(p_1)_r (p_2)_r \ldots (p_{s+1})_r \left(\frac{c}{4}\right)_r \left(\frac{c+1}{4}\right)_r \left(\frac{c+2}{4}\right)_r \left(\frac{c+3}{4}\right)_r}$$

$$\times \; _{s+5}F_{s+4}\left[\begin{array}{c} a+2r, \frac{c-a-b+2r}{3}, \frac{c-a-b+2r+1}{3}, \frac{c-a-b+2r+2}{3}, a_1+r, a_2+r, \ldots, a_s+r, -m+r \; ; \\ \frac{c+4r+2}{3}, \frac{c+4r+1}{3}, \frac{c+4r}{3}, p_1+r, p_2+r, \ldots, p_s+r, p_{s+1}+r \end{array}\right]. \; - \; (2.4A)$$

Now we choose the parameters so that we have on the right of (2.4A) a $_7F_6$ which we can sum by Dougall's Theorem. That is we take a_1 and $a_2 = 1 + \frac{1}{2}a$, $p_1 = 1 + a - a_1$ and $p_2 = \frac{1}{2}a$. Then, for the $_7F_6$ to be well-poised we have $b = 2c - 4a - 1$, and for the sum of the denominator parameters to exceed the sum of the numerator parameters by 2, we require the further condition

$$a_1 = c - a - 1 + m$$

The series on the right of (2.4A) then becomes

$$_7F_6\left[\begin{array}{c} a+2r,\ 1+\frac{1}{2}a+r,\ \frac{3a-c+1+2r}{3},\ \frac{3a-c+2+2r}{3},\ \frac{3a-c+3+2r}{3},\ c-a-1+m+r,\ -m+r\ ; \\ \frac{1}{2}a+r,\ \frac{c+4r+2}{3},\ \frac{c+4r+1}{3},\ \frac{c+4r}{3},\ 2+2a-c-m+r,\ 1+a+m+r \end{array}\right]$$

$$= \frac{(1+a+2r)_{m-r}\left(\frac{2r}{3}-a+\frac{2c}{3}\right)_{m-r}\left(\frac{2r}{3}-a+\frac{2c}{3}-\frac{1}{3}\right)_{m-r}\left(\frac{2r}{3}-a+\frac{2c}{3}-\frac{2}{3}\right)_{m-r}}{\left(\frac{c}{3}+\frac{4r}{3}+\frac{2}{3}\right)_{m-r}\left(\frac{c}{3}+\frac{1}{3}+\frac{4r}{3}\right)_{m-r}\left(\frac{c}{3}+\frac{4r}{3}\right)_{m-r}(c-2a-1)_{m-r}} \quad\text{by Dougall}$$

$$= \frac{(1+a+2r)_{m-r}\ (2r-3a+2c-2)_{3(m-r)}}{(c+4r)_{3(m-r)}\ (c-2a-1)_{m-r}}$$

but

$$(p+2r)_{m-r} = \frac{(p)_m(p+m)_r}{(p)_{2r}}\ ,\quad (p+2r)_{3(m-r)} = \frac{(p)_{3m}}{(p)_{2r}(-1)^r(1-p-3m)_r}\ ,$$

$$(p+4r)_{3(m-r)} = \frac{(p)_{3m}(p+3m)_r}{(p)_{4r}}\ ,\quad (p)_{m-r} = \frac{(p)_m}{(-1)^r(1-p-m)_r}\ ,$$

$\therefore$ the sum of the $_7F_6$ is

$$\frac{(1+a)_m(1+a+m)_r(2c-2-3a)_{3m}\left(\frac{c}{4}\right)_r\left(\frac{c+1}{4}\right)_r\left(\frac{c+2}{4}\right)_r\left(\frac{c+3}{4}\right)_r(2-c+2a-m)_r}{\left(\frac{1+a}{2}\right)_r\left(1+\frac{a}{2}\right)_r(3-2c+3a-3m)_r\left(c-1-\frac{3a}{2}\right)_r\left(c-\frac{1}{2}-\frac{3a}{2}\right)_r(c)_{3m}(c+3m)_r(c-2a-1)_m}\ .$$

Therefore, substituting in (2.3A), we have after a little reduction the transformation

$$_{13}F_{12}\left[\begin{array}{c} a,\ 1+\frac{1}{2}a,\ \frac{c-a}{3},\ \frac{c-a+1}{3},\ \frac{c-a+2}{3},\ a-\frac{c-1}{4},\ a-\frac{c-2}{4},\ a-\frac{c-3}{4},\ a-\frac{c-4}{4},\ \frac{3a-c+1}{2},\ \frac{3a-c+2}{2},\ c-a-1+m,\ -m\ ; \\ \frac{1}{2}a,\ \frac{4a-c}{3}+1,\ \frac{4a-c+1}{3},\ \frac{4a-c+1}{3},\ \frac{c+3}{4},\ \frac{c+2}{4},\ \frac{c+1}{4},\ \frac{c}{4},\ \frac{c-a+1}{2},\ \frac{c-a}{2},\ 2+2a-c-m,\ 1+a+m \end{array}\right]$$

$$= \frac{(1+a)_m(2c-2-3a)_{3m}}{(c)_{3m}(c-2a-1)_m} \times {}_5F_4\left[\begin{array}{c} \frac{3a-c+1}{2},\ \frac{3a-c+2}{2},\ 2c-4a-1,\ c-a-1+m,\ -m\ ; \\ c-1-\frac{3a}{2},\ c-\frac{1}{2}-\frac{3a}{2},\ 3-2c+3a-3m,\ c+3m \end{array}\right]$$

a transformation which expresses a particular well-poised $_{13}F_{12}$ in terms of a $_5F_4$.

(2.5)

A result could be obtained by taking Saalschütz's formula in the form

$$_3F_2\left[\begin{array}{c} a+n,\ b,\ -n\ ; \\ c+4n,\ 1+a+b-c-4n \end{array}\right] = \frac{(c-a+3n)_n\,(c-b+4n)_n}{(c+4n)_n\,(c-a-b+3n)_n}\ ,$$

but in order to sum the series on the right of (B) by Dougall's formula, we should have to reduce the number of parameters involved to two, namely a & —m. Thus in view of the highly specialised form of the result which would be obtained, it does not seem worth while carrying out the analysis. The same remark applies of course to values of $\lambda > 4$ when we take the form $c+\lambda n$.

(2.6)

As Vandermonde's Theorem is really a limiting form of Saalschütz's Theorem, there is not much point in applying the method of obtaining transformations to Vandermonde's Theorem. When however, we take Saalschütz's formula in the form

$$_3F_2\left[\begin{array}{c} a+n,\ b-n,\ -n\ ; \\ c,\ 1+a+b-c-n \end{array}\right] = \frac{(c-a-n)_n\,(c-b+n)_n}{(c)_n\,(c-a-b)_n}\ ,\qquad \text{we obtain}$$

on the right of (B) a hypergeometric series which is not well-poised, and contains too many parameters to be summed by Saalschütz's Theorem. In this case therefore, instead of reducing the number of terms by substituting for one parameter in terms of the others, or letting a parameter tend to infinity it seems more sensible to take the limiting form of Saalschütz's Theorem to begin with. We therefore take Vandermonde's Theorem in the form

$$_2F_1\left[\begin{array}{c} b-n,\ -n\ ;\ 1 \\ c \end{array}\right] = \frac{(c-b+n)_n}{(c)_n}\ ,\qquad - (2.6).$$

and apply the method of obtaining transformations.

The formula (2.6) may be written

$$\sum_{r=0}^{n} \frac{(-1)^r\,(-n)_r}{r!\,(c)_r\,(1-b+n)_{-r}} = \frac{\left(\frac{c-b}{2}\right)_n\left(\frac{1+c-b}{2}\right)_n 4^n}{(c)_n\,(c-b)_n}$$

This is the formula (α) with

$$\beta_1 = c\ ;\quad \ell_1 = 1-b,\quad v_1 = 0\ ;\quad c = 1\ ;$$

$$\rho_1 = \frac{c-b}{2}\ ,\quad \rho_2 = \frac{1+c-b}{2}\ ;\quad b = 4\ ;\quad \sigma_1 = c,\quad \sigma_2 = c-b,\quad \sigma_3 = 1-b\ ;$$

$\therefore$ using formula (β) we have the transformation

$$_5F_4 \left[\begin{array}{c} \frac{c-b}{2}, \frac{1+c-b}{2}, a_1, a_2, -m \; ; \; 4 \\ c, c-b, 1-b, p \end{array} \right]$$

$$= \sum_{r=0}^{m} \frac{(a_1)_r (a_2)_r (-m)_r}{r! \, (p)_r (c)_r} \times \; _3F_2 \left[\begin{array}{c} a_1+r, a_2+r, -m+r \; ; \\ 1-b, p+r \end{array} \right]. \quad — (2.6A)$$

Now the only case in which we can sum the series on the right of (2.6A) is when it is a $_2F_1$ and can be summed by Vandermonde's Theorem.

$\therefore$ In (2.6A) we take 1 a = a & no p's and we obtain the transformation

$$_4F_3 \left[\begin{array}{c} \frac{c-b}{2}, \frac{1+c-b}{2}, a, -m \; ; \; 4 \\ c, c-b, 1-b \end{array} \right] = \sum_{r=0}^{m} \frac{(a)_r (-m)_r}{r! \, (c)_r} \times \; _2F_1 \left[\begin{array}{c} a+r, -m+r \; ; \\ 1-b \end{array} \right]$$

and summing by Vandermonde's Theorem, the R.H.S. is

$$\sum_{r=0}^{m} \frac{(a)_r (-m)_r}{r! \, (c)_r} \times \frac{(1-a-b-r)_{m-r}}{(1-b)_{m-r}}$$

and since

$$(p-r)_{m-r} = (-1)^r (p)_m (1-p)_r / (1-p-2m)_{2r}$$

the R.H.S. is

$$\sum_{r=0}^{m} \frac{(a)_r (-m)_r}{r! \, (c)_r} \times \frac{(1-a-b)_m (-1)^r (a+b)_r (b-m)_r}{(a+b-m)_{2r} (-1)^r (1-b)_m}$$

$$= \frac{(1-a-b)_m}{(1-b)_m} \times \; _4F_3 \left[\begin{array}{c} a, a+b, b-m, -m \; ; \; \frac{1}{4} \\ c, \frac{a+b-m}{2}, \frac{1+a+b-m}{2} \end{array} \right]$$

$\therefore$ we have the transformation

$$_4F_3 \left[\begin{array}{c} \frac{c-b}{2}, \frac{1+c-b}{2}, a, -m \; ; \; 4 \\ c, c-b, 1-b \end{array} \right] = \frac{(1-a-b)_m}{(1-b)_m} \times \; _4F_3 \left[\begin{array}{c} a, a+b, b-m, -m \; ; \; \frac{1}{4} \\ c, \frac{a+b-m}{2}, \frac{1+a+b-m}{2} \end{array} \right]$$

This may be written

$$_4F_3 \left[\begin{array}{c} a, b, b-a-m, -m \; ; \; \frac{1}{4} \\ c, \frac{b-m}{2}, \frac{1+b-m}{2} \end{array} \right]$$

$$= \frac{(1+a-b)_m}{(1-b)_m} \times \; _4F_3 \left[\begin{array}{c} a, \frac{c+a-b}{2}, \frac{1+c+a-b}{2}, -m \; ; \; 4 \\ c, c+a-b, 1+a-b \end{array} \right]. \quad — (2.6B)$$

---

(3). We now turn to a formula given by Dougall viz:—

$${}_7F_6\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c,\ d,\ e,\ -m\ ; \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a+m \end{matrix}\right] = \frac{(1+a)_m(1+a-b-c)_m(1+a-b-d)_m(1+a-c-d)_m}{(1+a-b-c-d)_m(1+a-b)_m(1+a-c)_m(1+a-d)_m}$$

provided that

$$1 + 2a = b+c+d+e-m\ .$$

Dougall's method of proof is an ingenious application of the algebraical theorem that if a polynomial of degree n vanishes for more than n different values of the variable, it vanishes identically. More recently, Preece[9] has given a direct proof which builds up the series from the factorial product, but his proof is very much more complicated than that given by Dougall.

Professor Bailey[5] has worked out the two following cases in which Dougall's Theorem is written in the forms

$${}_7F_6\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ c,\ d,\ e,\ k+n,\ -n\ ; \\ \tfrac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-k-n,\ 1+a+n \end{matrix}\right] = \frac{(1+a)_n(1+a-c-d)_n(1+a-c-e)_n(1+a-d-e)_n}{(1+a-c)_n(1+a-d)_n(1+a-e)_n(1+a-c-d-e)_n},$$

$${}_7F_6\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ c,\ d+n,\ e-n,\ k+n,\ -n\ ; \\ \tfrac{1}{2}a,\ 1+a-c,\ 1+a-d-n,\ 1+a-e+n,\ 1+a-k-n,\ 1+a+n \end{matrix}\right] = \frac{(1+a)_n(c+d-a)_n(1+a-e)_n(1+a-c-e)_{2n}(1+a-d-e)_n}{(1+a-c)_n(1+a-e)_{2n}(d-a)_n(1+a-c-e)_n(1+a-c-d-e)_n}$$

where $k = 1 + 2a - c - d - e$.

The only other form from which has been obtained a result is the following.

$${}_7F_6\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ b,\ c+n,\ d-2n,\ k+2n,\ -n\ ; \\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c-n,\ 1+a-d+2n,\ 1+a-k-2n,\ 1+a+n \end{matrix}\right]$$

$$= \frac{(1+a)_n(1+a-b-c-n)_n(1+a-b-d+2n)_n(1+a-c-d+n)_n}{(1+a-b)_n(1+a-d+2n)_n(1+a-c-n)_n(1+a-b-c-d+n)_n} \quad \text{where}\quad k = 1+2a-b-c-d.$$

Equation (3.1) can be written

$$\sum_{r=0}^{n} \frac{(a)_r(1+\tfrac{1}{2}a)_r(b)_r(c+n)_r(k+2n)_r(-n)_r(-1)^r(c-a-n)_{-r}(k-a+2n)_{-r}}{r!\,(\tfrac{1}{2}a)_r(1+a-b)_r(1+a-d+2n)_r(1+a+n)_r(1-d+2n)_{-r}}$$

$$= \frac{(1+a)_n(b+c-a)_n\left(\tfrac{1+a-b-d}{3}\right)_n\left(\tfrac{2+a-b-d}{3}\right)_n\left(\tfrac{3+a-b-d}{3}\right)_n\left(\tfrac{1+a-c-d}{2}\right)_n\left(\tfrac{2+a-c-d}{2}\right)_n\left(\tfrac{1+a-d}{2}\right)_n\left(\tfrac{2+a-d}{2}\right)_n(1+a-b-c-d)_n}{(1+a-b)_n\left(\tfrac{1+a-d}{3}\right)_n\left(\tfrac{2+a-d}{3}\right)_n\left(\tfrac{3+a-d}{3}\right)_n(c-a)_n\left(\tfrac{1+a-b-d}{2}\right)_n\left(\tfrac{2+a-b-d}{2}\right)_n(1+a-c-d)_n\left(\tfrac{1+a-b-c-d}{2}\right)_n\left(\tfrac{2+a-b-c-d}{2}\right)_n}\ .$$

This is formula (A) with the following particular values:—

$$\alpha_1 = a,\ \alpha_2 = b,\ \alpha_3 = \tfrac{c}{2},\ \alpha_4 = \tfrac{c+1}{2},\ \alpha_5 = \tfrac{k}{3},\ \alpha_6 = \tfrac{k+1}{3},\ \alpha_7 = \tfrac{k+2}{3},\ \alpha_8 = k-a\ ;\quad b = 1\ ;\ c = 1\ ;$$

$$\beta_1 = \tfrac{1}{2}a,\ \beta_2 = 1+a-b,\ \beta_3 = \tfrac{1+a-d}{3},\ \beta_4 = \tfrac{2+a-d}{3},\ \beta_5 = \tfrac{3+a-d}{3},\ \beta_6 = \tfrac{1+a}{2},\ \beta_7 = 1-a\ ;$$

$$K_1 = c,\ \lambda_1 = 1,\ \mu_1 = 2\ ;\quad K_2 = k,\ \lambda_2 = 2,\ \mu_2 = \tfrac{1}{2}\ ;\quad K_3 = c-a,\ \lambda_3 = 1,\ \mu_3 = 0\ ;$$

$$K_4 = k-a,\ \lambda_4 = 2,\ \mu_4 = \tfrac{1}{2}\ ;$$

$$L_1 = 1+a-d,\ \phi_1 = 2,\ \nu_1 = \tfrac{1}{2}\ ;\quad L_2 = 1+a,\ \phi_2 = 1,\ \nu_2 = 2\ ;\quad L_3 = 1-d,\ \phi_3 = 2,\ \nu_3 = \tfrac{1}{2}\ ;$$

$$\rho_1 = b+c-a,\ \rho_2 = \tfrac{1+a-b-d}{3},\ \rho_3 = \tfrac{2+a-b-d}{3},\ \rho_4 = \tfrac{3+a-b-d}{3},\ \rho_5 = \tfrac{1+a-c-d}{2},\ \rho_6 = \tfrac{2+a-c-d}{2},$$

$\rho_7 = \frac{1+a-d}{2}, \rho_8 = \frac{2+a-d}{2}, \rho_9 = c, \rho_{10} = \frac{k}{2}, \rho_{11} = \frac{k+1}{2}, \rho_{12} = \frac{k-a}{2}, \rho_{13} = \frac{k-a+1}{2}, \rho_{14} = 1+a-b-c-d;$

$\sigma_1 = 1+a-b, \sigma_2 = \frac{1+a-d}{3}, \sigma_3 = \frac{2+a-d}{3}, \sigma_4 = \frac{3+a-d}{3}, \sigma_5 = \frac{1+a-b-c-d}{2}, \sigma_6 = \frac{2+a-b-c-d}{2}, \sigma_7 = \frac{1+a-b-d}{2},$

$\sigma_8 = \frac{2+a-b-d}{2}, \sigma_9 = 1+a-c-d, \sigma_{10} = \frac{1+a-d}{2}, \sigma_{11} = \frac{2+a-d}{2}, \sigma_{12} = \frac{1-d}{2}, \sigma_{13} = \frac{2-d}{2};$

∴ using (B) we have the transformation

$${}_{s+5}F_{s+4}\left[\begin{array}{c} b+c-a, \frac{1+a-b-d}{3}, \frac{2+a-b-d}{3}, \frac{3+a-b-d}{3}, \frac{1+a-c-d}{2}, \frac{2+a-c-d}{2}, \frac{1+a-d}{2}, \frac{2+a-d}{2}, c, \frac{k}{2}, \frac{k+1}{2}, \frac{k-a}{2}, \frac{1+k-a}{2}, 1+a-b-c-d, a_1, \ldots, a_{s}, -m; \\ 1+a-b, \frac{1+a-d}{3}, \frac{2+a-d}{3}, \frac{3+a-d}{3}, \frac{1+a-b-c-d}{2}, \frac{2+a-b-c-d}{2}, \frac{1+a-b-d}{2}, \frac{2+a-b-d}{2}, 1+a-c-d, \frac{1+a-d}{2}, \frac{2+a-d}{2}, \frac{1-d}{2}, \frac{2-d}{2}, \rho_1 \ldots \rho_{s+1} \end{array}\right]$$

$$= \sum_{r=0}^{m} \frac{(a_1)_r (a_2)_r (a_3)_r (-m)_r (a)_r (b)_r \left(\frac{c}{2}\right)_r \left(\frac{c+1}{2}\right)_r \left(\frac{k}{3}\right)_r \left(\frac{k+1}{3}\right)_r \left(\frac{k+2}{3}\right)_r (k-a)_r}{r! (\rho_1)_r \ldots (\rho_{s+1})_r \left(\frac{1}{2}a\right)_r (1+a-b)_r \left(\frac{1+a-d}{2}\right)_r \left(\frac{2+a-d}{3}\right)_r \left(\frac{3+a-d}{3}\right)_r \left(\frac{1+a}{2}\right)_r (1-a)_r}$$

$$\times {}_{s+1}F_{s+4}\left[\begin{array}{c} c+2r, \frac{k+3r}{2}, \frac{1+k+3r}{2}, c-a, \frac{k-a+r}{2}, \frac{1+k-a+r}{2}, a_1+r, a_2+r, \ldots, a_s+r, -m+r; \\ \frac{1-a+r}{2}, \frac{2-a+r}{2}, 1+a+2r, \frac{1+a-d+3r}{2}, \frac{2+a-d+3r}{2}, \rho_1+r, \rho_2+r, \ldots, \rho_s+r, \rho_{s+1}+r \end{array}\right] \qquad - (3.1A)$$

Now if in (3.1A) we take $k = 1$, $d = a = b + c$, the series
on the left has a zero parameter in the numerator ∴ its sum
is 1, and if we take no a's in the series on the right and
one $p = -2b-m$, the series on the right of (3.1A) becomes

${}_3F_2\left[c+2r, -b, -m+r; 1+b+c+2r, -2b-m+r\right]$ which by Saalschütz's Theorem

$= (1+b)_{m-r}(1+2b+c+2r)_{m-r}/(1+b+c+2r)_{m-r}(1+2b)_{m-r}$.

∴ with these values, (3.1A) becomes

$$\frac{(1+b+c)_m (1+2b)_m}{(1+b)_m (1+2b+c)_m} = \sum_{r=0}^{m} \frac{(b+c)_r (b)_r \left(\frac{c}{2}\right)_r \left(\frac{c+1}{2}\right)_r (1+2b+c+m)_r \left(\frac{2+b+c}{2}\right)_r (-m)_r}{r! \left(\frac{b+c}{2}\right)_r (1+c)_r (-b-m)_r \left(\frac{1+2b+c}{2}\right)_r \left(\frac{2+2b+c}{2}\right)_r (1+b+c+m)_r}$$

∴ we have the sum of a well-poised ${}_7F_6$ viz :-

$${}_7F_6\left[\begin{array}{c} b+c, \frac{1+b+c}{2}, \frac{c}{2}, \frac{1}{2}+\frac{c}{2}, 1+2b+c+m, b, -m ; \\ \frac{b+c}{2}, \frac{2+2b+c}{2}, \frac{1+2b+c}{2}, -b-m, 1+c, 1+b+c+m \end{array}\right] = \frac{(1+b+c)_m (1+2b)_m}{(1+b)_m (1+2b+c)_m} \qquad - (3.1B)$$

This is not a particular case of Dougall's Theorem, but the
formula has been previously obtained by Professor Bailey[15] with
$b+c = a$ and $c = d$.

If in (3.1A) we take $k = 1$ and $d = a = b + c$, as before,
we can also sum the hypergeometric series on the right using
Dougall's Theorem if we take

$a_1 = 1+\frac{c}{2}, a_2, a_3, a_4, \rho_1 = \frac{c}{2}, \rho_2 = 1+c+m, \rho_3 = 1+c-a_2, \rho_4 = 1+c-a_3, \rho_5 = 1+c-a_4$
where

$a_4 = 1+b+2c+m-a_2-a_3.$

In this case, the sum of the $_7F_6$ on the right is

$$\frac{(1+c+2r)_{m-r}(1+b+c-a_2+r)_{m-r}(1+b+c-a_3+r)_{m-r}(1+c-a_2-a_3)_{m-r}}{(1+b+c+2r)_{m-r}(1+c-a_2+r)_{m-r}(1+c-a_3+r)_{m-r}(1+b+c-a_2-a_3)_{m-r}}$$

and we are led to Dougall's Theorem itself in the following form viz,

$$_7F_6\left[\begin{array}{c} b+c, \ 1+\tfrac{b+c}{2}, \ b, \ a_2, \ a_3, \ 1+b+2c+m-a_2-a_3, -m \ ; \\ \tfrac{b+c}{2}, 1+c, 1+b+c-a_2, 1+b+c-a_3, a_2+a_3-c-m, 1+b+c+m \end{array}\right]$$

$$= \frac{(1+b+c)_m(1+c-a_2)_m(1+c-a_3)_m(1+b+c-a_2-a_3)_m}{(1+c)_m(1+b+c-a_2)_m(1+b+c-a_3)_m(1+c-a_2-a_3)_m} \ .$$

We cannot obtain any transformations from (3.1A).

Professor Bailey[3] has used the formula (3.1B) to obtain transformations.

(4)

We now obtain transformations of basic series. To obtain such transformations, we use the fact that if

$$\beta_n = \sum_{r=0}^{n} \frac{(q^{-n})_r(kq^n)_r d_r}{(aq^{n+1})_r(aq^{1-n}/k)_r} \quad — (c)$$

where d_r is independent of n then

$$\sum_{n=0}^{\infty} \frac{(k)_n(kb)_n(\mu_1)_n(\mu_2)_n t^n \beta_n}{(x)_n(ax)_n(\sigma_1)_n(\sigma_2)_n} = \sum_{r=0}^{\infty} \frac{(k)_{2r}(\mu_1)_r(\mu_2)_r}{(ax)_{2r}(\sigma_1)_r(\sigma_2)_r}\left(\frac{kt}{a}\right)^r d_r \times \mathcal{\Phi}\left[\begin{array}{c} kq^{2r}, k/a, \mu_1 q^r, \mu_2 q^r \ ; \ t \\ aq^{2r+1}, \sigma_1 q^r, \sigma_2 q^r \end{array}\right]; — (D)$$

a result which has been established by Professor Bailey. (3)

Using this result, we proceed to obtain, wherever possible, the analogues of the results which we obtained for ordinary series.

(5)

We start therefore with the basic analogue of Saalschütz's formula viz,

$$_3\Phi_2\left[\begin{array}{c} b, \ c, \ q^{-n} \ ; \ q \\ d, \ bcq^{1-n}/d \end{array}\right] = \frac{(d/b)_n(d/c)_n}{(d)_n(d/bc)_n} \quad$$

which was first proved by Professor Watson, and consider it in various forms from which, using formulae (c) and (D) we obtain transformations The cases which give results are

$$_3\Phi_2\left[\begin{array}{c} a,\ b,\ q^{-n};\ q \\ c,\ abq^{-n+1}/c \end{array}\right] = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n}, \quad\text{—— (5.1)}$$

$$_3\Phi_2\left[\begin{array}{c} a,\ bq^{-n},\ q^{-n};\ q \\ cq^{-n},\ abq^{-n+1}/c \end{array}\right] = \frac{(cq^{-n}/a)_n (c/b)_n}{(cq^{-n})_n (c/ab)_n}, \quad\text{—— (5.2)}$$

$$_3\Phi_2\left[\begin{array}{c} a,\ bq^{n},\ q^{-n};\ q \\ cq^{n},\ abq^{-n+1}/c \end{array}\right] = \frac{(cq^{n}/a)_n (c/b)_n}{(cq^{n})_n (c/ab)_n}. \quad\text{—— (5.3)}$$

(5.1) We start with the formula

$$_3\Phi_2\left[\begin{array}{c} a,\ b,\ q^{-n};\ q \\ c,\ abq^{-n+1}/c \end{array}\right] = \frac{(c/a)_n (c/b)_n}{(c)_n (c/ab)_n}$$

Then

$$\sum_{n=0}^{\infty} \frac{(c/ab)_n (\mu_1)_n (\mu_2)_n t^n \beta_n}{(q)_n (\sigma_1)_n (\sigma_2)_n} = \sum_{n=0}^{\infty} \frac{(c/ab)_n (\mu_1)_n (\mu_2)_n t^n}{(q)_n (\sigma_1)_n (\sigma_2)_n} \sum_{r=0}^{n} \frac{(q^{-n})_r \alpha_r}{(abq^{1-n}/c)_r}$$

where $\beta_n = \sum\limits_{r=0}^{n} \dfrac{(q^{-n})_r \alpha_r}{(abq^{1-n}/c)_r}$ and $\alpha_r = \dfrac{(a)_r (b)_r q^r}{(q)_r (c)_r}$

Putting $n = r+s$,

$$\sum_{n=0}^{\infty} \frac{(c/ab)_n (\mu_1)_n (\mu_2)_n t^n}{(q)_n (\sigma_1)_n (\sigma_2)_n} \sum_{r=0}^{n} \frac{(q^{-n})_r \alpha_r}{(abq^{1-n}/c)_r}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\mu_1)_{r+s} (\mu_2)_{r+s} (c/ab)_s \alpha_r t^{r+s}}{(\sigma_1)_{r+s} (\sigma_2)_{r+s} (q)_s} \left(\frac{c}{ab}\right)^r$$

$$= \sum_{r=0}^{\infty} \frac{(\mu_1)_r (\mu_2)_r}{(\sigma_1)_r (\sigma_2)_r (q)_r} \left(\frac{ct}{abq}\right)^r \alpha_r \times \Phi\left[\begin{array}{c} c/ab,\ \mu_1 q^r,\ \mu_2 q^r;\ t \\ \sigma_1 q^r,\ \sigma_2 q^r \end{array}\right]$$

∴ we have the transformation

$$\sum_{n=0}^{\infty} \frac{(c/ab)_n (\mu_1)_n (\mu_2)_n t^n \beta_n}{(q)_n (\sigma_1)_n (\sigma_2)_n} = \sum_{r=0}^{\infty} \frac{(\mu_1)_r (\mu_2)_r \alpha_r}{(\sigma_1)_r (\sigma_2)_r (q)_r} \left(\frac{ct}{abq}\right)^r \times \Phi\left[\begin{array}{c} c/ab,\ \mu_1 q^r,\ \mu_2 q^r;\ t \\ \sigma_1 q^r,\ \sigma_2 q^r \end{array}\right]$$

i.e. we have, using (5.1)

$$\sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n (\mu_1)_n (\mu_2)_n t^n}{(q)_n (c)_n (\sigma_1)_n (\sigma_2)_n} = \sum_{r=0}^{\infty} \frac{(\mu_1)_r (\mu_2)_r (a)_r (b)_r}{(q)_r (c)_r (\sigma_1)_r (\sigma_2)_r} \left(\frac{ct}{ab}\right)^r$$

$$\times \Phi\left[\begin{array}{c} c/ab,\ \mu_1 q^r,\ \mu_2 q^r;\ t \\ \sigma_1 q^r,\ \sigma_2 q^r \end{array}\right]. \quad\text{—— (5.1A)}$$

If in (5.1A) we take one $\mu = q^{-n}$ and one $\sigma = \sigma$ and $t = q$, we obtain

$$\sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n (q^{-n})_n q^{n}}{(q)_n (c)_n (\sigma)_n} \overset{\text{?}}{=} \sum_{r=0}^{N} \frac{(q^{-n})_r (a)_r (b)_r}{(q)_r (c)_r (\sigma)_r} \left(\frac{cq}{ab}\right)^r \times {}_2\Phi_1\left[\begin{array}{c} c/ab, \; q^{-n+r} \; ; \; q \\ \sigma q^r \end{array}\right]$$

Now, we sum the ${}_2\Phi_1$ on the right using the analogue of Vandermonde's Theorem

$${}_2\Phi_1\left[\begin{array}{c} b, \; q^{-N} \; ; \; q \\ d \end{array}\right] = \frac{(d/b)_N \, b^N}{(d)_N}$$

and replacing $(pq^r)_{n-r}$ by $(p)_n/(p)_r$ we obtain the transformation

$${}_3\Phi_2\left[\begin{array}{c} c/a, c/b, \; q^{-N} \; ; \; q \\ c, \; \sigma \end{array}\right] = \frac{(ab\sigma/c)(c/ab)^N}{(\sigma)_n} \times {}_3\Phi_2\left[\begin{array}{c} a, \; b, \; q^{-N} \; ; \; q \\ c, \; ab\sigma/c \end{array}\right].$$

This result is the analogue of (2.1B) as may be seen if we change the notation and write the above as

$${}_3\Phi_2\left[\begin{array}{c} a, \; b, \; q^{-N} \; ; \; q \\ e, \; f \end{array}\right] = \frac{(ef/ab)_n (e/ab)^N}{(f)_n} \times {}_3\Phi_2\left[\begin{array}{c} c/a, \; c/b, \; q^{-N} \; ; \; q \\ e, \; ef/ab \end{array}\right]. \quad - (5.1B)$$

Now we use the analogue of Saalchutz's Theorem to sum on the right of (5.1A) i.e. we take

$$\mu_1 = q^{-N}, \quad \mu_2 = \mu, \quad t = q, \quad \sigma_1 = c\mu q^{1-n}/ab\sigma \quad \text{and} \quad \sigma_2 = \sigma.$$

Then

$$\sum_{r=0}^{N} \frac{(c/a)_n (c/b)_n (\mu)_n (q^{-n})_n q^N}{(q)_n (c)_n (\sigma)_n (c\mu q^{1-n}/ab\sigma)_n}$$

$$= \sum_{r=0}^{\infty} \frac{(\mu)_r (q^{-n})_r (a)_r (b)_r}{(c)_r (q)_r (\sigma)_r (c\mu q^{1-n}/ab\sigma)_r} \left(\frac{cq}{ab}\right)^r \times {}_3\Phi_2\left[\begin{array}{c} c/ab, \; \mu q^r, \; q^{-n+r} \; ; \; q \\ \sigma q^r, \; c\mu q^{1-n+r}/ab\sigma \end{array}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(\mu)_r (q^{-n})_r}{(\sigma)_r (c\mu q^{1-n}/ab\sigma)_r} \left(\frac{c}{ab}\right)^r \frac{(a)_r (b)_n (\sigma/\mu)_n (\sigma ab/c)_n (\sigma)_r (\sigma ab/\mu c)^r (\mu c q^{1-n}/ab\sigma)_r}{(c)_r (q)_r (\sigma/\mu)_r (\mu q^{-n+1}/\sigma)_n (\sigma ab/c)_r (\sigma)_n (\sigma ab/\mu c)_N}$$

(since $(A)_{n-r} = (A)_n / (A)^r (-1)^r q^{(\frac{r}{2})} \cdot q^{(n-r)}(q^{-n+1}/A)_r$)

$$= \frac{(\sigma ab/c)_n (\sigma/\mu)_n}{(\sigma)_n (\sigma ab/\mu c)_n} \times {}_4\Phi_3\left[\begin{array}{c} a, \; b, \; \mu, \; q^{-n} \; ; \; q \\ c, \; \sigma ab/c, \; \mu q^{1-n}/\sigma \end{array}\right]$$

∴ we obtain the transformation

$${}_4\Phi_3\left[\begin{array}{c} c/a, c/b, \mu, \; q^{-n} \; ; \; q \\ \sigma, c, \; c\mu q^{1-n}/ab\sigma \end{array}\right] = \frac{(\sigma ab/c)_n (\sigma/\mu)_n}{(\sigma)_n (\sigma ab/\mu c)_n} \times {}_4\Phi_3\left[\begin{array}{c} a, \; b, \; \mu, \; q^{-n} \; ; \; q \\ c, \sigma ab/c, \; \mu q^{1-n}/\sigma \end{array}\right].$$

i.e. with a change of notation

$$_4\Phi_3\left[\begin{matrix} a, b, c, q^{-n}; \\ e, f, abcq^{-n+1}/ef \end{matrix}\ q\right] = \frac{(ef/ab)_n (f/c)_n}{(f)_n (ef/abc)_n} \times {}_4\Phi_3\left[\begin{matrix} e/a, e/b, c, q^{-n}; \\ ef/ab, e, cq^{1-n}/f \end{matrix}\ q\right], -(5.1C)$$

— the analogue of (2.1C)

(5.2)   Now take the analogue of Saalschütz's formula in the form

$$_3\Phi_2\left[\begin{matrix} a, bq^{-N}, q^{-N}; \\ cq^{-n}, abq^{-n+1}/c \end{matrix}\ q\right] = \frac{(cq^{-n}/a)_n (c/b)_n}{(cq^{-n})_n (c/ab)_n} = \frac{(aq/c)_n (c/b)_n}{(q/c)_n (c/ab)_n\, a^N}$$

Then

$$\sum_{n=0}^{\infty} \frac{(c/ab)_n (q/c)_n (\mu_1)_n (\mu_2)_n t^n \beta_n}{(q)_n (\sigma_1)_n (\sigma_2)_n (q/b)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(c/ab)_n (q/c)_n (\mu_1)_n (\mu_2)_n t^n}{(q)_n (q/b)_n (\sigma_1)_n (\sigma_2)_n} \sum_{r=0}^{N} \frac{(q^{-n})_r (bq^{-n})_r \alpha_r}{(abq^{1-n}/c)_n (cq^{-n})_r} \quad \text{where } \alpha_r = \frac{(a)_r q^r}{(a)_r}$$

and putting n = r + s   and summing β_n by (5.2) we have

$$\sum_{n=0}^{\infty} \frac{(aq/c)_n (c/b)_n (\mu_1)_n (\mu_2)_n}{(q)_n (q/b)_n (\sigma_1)_n (\sigma_2)_n} \left(\frac{t}{a}\right)^N$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\mu_1)_{r+s} (\mu_2)_{r+s} (c/ab)_s (q/c)_s \alpha_r t^{r+s}}{(\sigma_1)_{r+s} (\sigma_2)_{r+s} (q)_s (q/b)_s} \left(\frac{1}{aq}\right)^r$$

$$= \sum_{r=0}^{\infty} \frac{(\mu_1)_r (\mu_2)_r (a)_r}{(\sigma_1)_r (\sigma_2)_r (q)_r} \left(\frac{t}{a}\right)^r \times \Phi\left[\begin{matrix} c/ab, q/c, \mu_1 q^r, \mu_2 q^r; t \\ q/b, \sigma_1 q^r, \sigma_2 q^r \end{matrix}\right]$$

and if we take t = q we are led to the transformation (5.2A) viz

$$\sum_{n=0}^{\infty} \frac{(aq/c)_n (c/b)_n (\mu_1)_n (\mu_2)_n}{(q)_n (q/b)_n (\sigma_1)_n (\sigma_2)_n} \left(\frac{q}{a}\right)^n = \sum_{r=0}^{\infty} \frac{(\mu_1)_r (\mu_2)_r (a)_r}{(q)_r (\sigma_1)_r (\sigma_2)_r} \left(\frac{q}{a}\right)^r \times \Phi\left[\begin{matrix} c/ab, q/c, \mu_1 q^r, \mu_2 q^r; q \\ q/b, \sigma_1 q^r, \sigma_2 q^r \end{matrix}\right]. -(5.2A)$$

We can now sum on the right using the analogue of Saalschütz's
Theorem, if we take

$$\mu_1 = q^{-N} \quad \text{and} \quad \sigma_1 = q^{1-n}/a$$

Then

$$\sum_{n=0}^{\infty} \frac{(aq/c)_n (c/b)_n (q^{1-n})_n (q/a)^n}{(q)_n (q/b)_n (q^{1-n}/a)_n}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (q^{-n})_r}{(q)_r (q^{1-n}/a)_r} \left(\frac{q}{a}\right)^r \times \frac{(aq/c)_{n-r} (c/b)_{n-r}}{(q/b)_{n-r} (a)_{n-r}}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (q^{-n})_r (q/a)^r (aq/c)_n (c/b)_n (1/b)^r (bq^{-n})_r (a)^r (q^{-n+1}/a)_r}{(q)_r (q^{1-n}/a)_r (a/c)^r (cq^{-n}/a)_r (c/b)^r (bq^{-n+1}/c)_r (q/b)_n (a)_n}$$

$\therefore$ we are led to the transformation

$$_3\Phi_2\left[\begin{matrix} aq/c,\ c/b,\ q^{-\nu};\ q/a \\ q/b,\ q^{-n+1}/a \end{matrix}\right] = \frac{(aq/c)_n\,(c/b)_n}{(q/b)_n\,(a)_n} \times\ _3\Phi_2\left[\begin{matrix} a,\ bq^{-\nu},\ q^{-\nu};\ q/a \\ cq^{-n}/a,\ bq^{-n+1}/a \end{matrix}\right]$$

which may be written

$$_3\Phi_2\left[\begin{matrix} a,\ b,\ q^{-\nu};\ cq/ab \\ c,\ bcq^{-n+1}/a \end{matrix}\right] = \frac{(a)_n\,(b)_n}{(c)_n\,(ab/c)_n} \times\ _3\Phi_2\left[\begin{matrix} ab/c,\ q^{-n+1}/c,\ q^{-n};\ cq/ab \\ q^{-n+1}/b,\ q^{-n+1}/a \end{matrix}\right] - (5.2B).$$

This formula (5.2B) is analogue to (2.2B).

(5.3)

We now take the analogue of Saalschütz in the form

$$_3\Phi_2\left[\begin{matrix} a,\ bq^{-\nu},\ q^{-\nu};\ q \\ cq^{\nu},\ abq^{-n+1}/c \end{matrix}\right] = \frac{(cq^n/a)_n\,(c/b)_n}{(cq^n)_n\,(c/ab)_n} = \frac{(\sqrt{c/a})_n\,(-\sqrt{c/a})_n\,(\sqrt{cq/a})_n\,(-\sqrt{cq/a})_n\,(c/b)_n\,(c)_n}{(c/a)_n\,(\sqrt{c})_n\,(-\sqrt{c})_n\,(\sqrt{cq})_n\,(-\sqrt{cq})_n\,(c/ab)_\nu}.$$

In this case, we take

$$\beta_n = \sum_{r=0}^{\nu} \frac{(bq^n)_r\,(q^{-n})_r\,d_r}{(cq^n)_r\,(abq^{-n+1}/c)_r} \qquad \text{where}\ \ d_r = \frac{(a)_r\,q^r}{(q)_r}$$

Then

$$\sum_{n=0}^{\infty} \frac{(c/ab)_n\,(b)_n\,(\mu_1)_n\,(\mu_2)_n\,t^n\,\beta_n}{(q)_n\,(c)_n\,(\sigma_1)_n\,(\sigma_2)_n} = \sum_{n=0}^{\infty} \frac{(c/ab)_n\,(b)_n\,(\mu_1)_n\,(\mu_2)_n\,t^n}{(q)_n\,(c)_n\,(\sigma_1)_n\,(\sigma_2)_n} \sum_{r=0}^{\nu} \frac{(bq^n)_r\,(q^{-n})_r\,d_r}{(cq^n)_r\,(abq^{-n+1}/c)_r}$$

$$= \sum_{r=0}^{\infty} \frac{(\mu_1)_r\,(\mu_2)_r\,(b)_{2r}}{(\sigma_1)_r\,(\sigma_2)_r\,(c)_{2r}} \left(\frac{ct}{abq}\right)^r \times\ \Phi\left[\begin{matrix} c/ab,\ bq^{2r},\ \mu_1 q^r,\ \mu_2 q^r;\ t \\ cq^{2r},\ \sigma_1 q^r,\ \sigma_2 q^r \end{matrix}\right]$$

$\therefore$ Putting $t = q$ and using (5.3) on the L.H.S. we obtain the transformation

$$\Phi\left[\begin{matrix} b,\ \sqrt{c/a},\ -\sqrt{c/a},\ \sqrt{cq/a},\ -\sqrt{cq/a},\ c/b,\ \mu_1,\ \mu_2;\ q \\ \sqrt{c},\ -\sqrt{c},\ \sqrt{cq},\ -\sqrt{cq},\ c/a,\ \sigma_1,\ \sigma_2 \end{matrix}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(\mu_1)_r\,(\mu_2)_r\,(b)_{2r}\,(a)_r}{(q)_r\,(\sigma_1)_r\,(\sigma_2)_r\,(c)_r} \left(\frac{cq}{ab}\right)^r \times\ \Phi\left[\begin{matrix} c/ab,\ bq^{2r},\ \mu_1 q^r,\ \mu_2 q^r;\ q \\ cq^{2r},\ \sigma_1 q^r,\ \sigma_2 q^r \end{matrix}\right], - (5.3A).$$

First we sum on the series on the right of (5.3A) by taking one $\mu = q^{-\nu}$ and one $\sigma = q^{-n+1}/a$ then we have when $t = q$,

$$_7\Phi_6\left[\begin{matrix} b,\ \sqrt{c/a},\ -\sqrt{c/a},\ \sqrt{cq/a},\ -\sqrt{cq/a},\ c/b,\ q^{-\nu};\ q \\ \sqrt{c},\ -\sqrt{c},\ \sqrt{cq},\ -\sqrt{cq},\ c/a,\ q^{-n+1}/a \end{matrix}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(q^{-n})_r\,(\sqrt{b})_r\,(-\sqrt{b})_r\,(\sqrt{bq})_r\,(-\sqrt{bq})_r\,(a)_r}{(q)_r\,(q^{-n+1}/a)_r\,(\sqrt{c})_r\,(-\sqrt{c})_r\,(\sqrt{cq})_r\,(-\sqrt{cq})_r} \left(\frac{cq}{ab}\right)^r \times\ _3\Phi_2\left[\begin{matrix} c/ab,\ bq^{2r},\ q^{-n+r};\ q \\ cq^{2r},\ q^{-n+r+1}/a \end{matrix}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\,(\sqrt{b})_r\,(-\sqrt{b})_r\,(\sqrt{bq})_r\,(-\sqrt{bq})_r\,(q^{-n})_r\,(cq/ab)^r\,(ab)_n\,(abq^n)_r\,(c)_{2r}\,(c/b)_n\,(q^{-n}/a)_r\,(a)^r}{(q)_r\,(q^{-n+1}/a)_r\,(\sqrt{c})_r\,(-\sqrt{c})_r\,(\sqrt{cq})_r\,(-\sqrt{cq})_r\,(ab)_{2r}\,(c)_n\,(cq^n)_r\,(c/b)^r\,(bq^{-n}/c)_r\,(a)_\nu}.$$

(6)

We now consider the analogue of Dougall's formula which was first proved by Jackson (11) viz:—

$$_8\Phi_7\left[\begin{array}{c} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ q^{-n}\ ;\ q \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq^{n+1} \end{array}\right]$$

$$= \frac{(aq)_n\,(aq/cd)_n\,(aq/bd)_n\,(aq/bc)_n}{(aq/b)_n\,(aq/c)_n\,(aq/d)_n\,(aq/bcd)_n} \quad \text{where}\ e = a^2 q^{n+1}/bcd.$$

Jackson's proof followed the same general lines as Dougall's proof of his theorem. The case in which we take Jackson's formula in the above form has been investigated and leads to the formula given in Tract —§(8.5) (1) — a transformation between two well-poised $_{10}\Phi_9$'s.

(6.1)

We now proceed to investigate the other form of Jackson's theorem, which yields a result, by taking the analogue of Dougall's theorem in the form

$$_8\Phi_7\left[\begin{array}{c} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ cq^n,\ dq^{-n},\ kq^n,\ q^{-n}\ ;\ q \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq^{-n+1}/c,\ aq^{n+1}/d,\ aq^{-n+1}/k,\ aq^{n+1} \end{array}\right]$$

$$= \frac{(aq)_n\,(aq/cd)_n\,(aq^{n+1}/bd)_n\,(aq^{-n+1}/bc)_n}{(aq/b)_n\,(aq^{-n+1}/c)_n\,(aq^{n+1}/d)_n\,(aq/bcd)_n} \quad \text{where}\ k = a^2 q/bcd. \quad -(6.1).$$

If we write

$$(Aq^{n+1})_n = (Aq)_{2n}/(Aq)_n = (\sqrt{Aq})_n(-\sqrt{Aq})_n(q\sqrt{A})_n(-q\sqrt{A})_n/(Aq)_n$$

and

$$(Aq^{-n+1})_n = (A)^n(-1)^n q^{-\frac{n}{2}(n+1)}(1/A)_n$$

then (6.1) may be written as

$$_8\Phi_7\left[\begin{array}{c} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ cq^n,\ dq^{-n},\ kq^n,\ q^{-n}\ ;\ q \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq^{-n+1}/c,\ aq^{n+1}/d,\ aq^{-n+1}/k,\ aq^{n+1} \end{array}\right] \quad \text{where}\ k = a^2 q/bcd$$

$$= \frac{(aq)_n\,(aq/cd)_n\,(\sqrt{aq/bd})_n(-\sqrt{aq/bd})_n(q\sqrt{a/bd})_n(-q\sqrt{a/bd})_n(bc/a)_n(aq/d)_n}{(aq/b)_n\,(c/a)_n\,(\sqrt{aq/d})_n(-\sqrt{aq/d})_n(q\sqrt{a/d})_n(-q\sqrt{a/d})_n(aq/bcd)_n(aq/bd)_n} \times \frac{1}{b^n}.$$

If

$$\beta_n = \sum_{r=0}^{n} \frac{(q^{-n})_r\,(kq^n)_r\,(cq^n)_r\,(dq^{-n})_r\,d_r}{(aq^{n+1})_r\,(aq^{1-n}/k)_r\,(aq^{1-n}/c)_r\,(aq^{n+1}/d)_r}$$

Then

$$\sum_{n=0}^{\infty} \frac{(k)_n\,(k/a)_n\,(c)_n\,(c/a)_n\,(\mu_1)_n\,(\mu_2)_n\,t^n\,\beta_n}{(q)_n\,(aq)_n\,(\sigma_1)_n\,(\sigma_2)_n\,(q/d)_n\,(aq/d)_n}$$

$$= \sum_{n=0}^{\infty} \frac{(k)_n\,(k/a)_n\,(c)_n\,(c/a)_n\,(\mu_1)_n\,(\mu_2)_n\,t^n}{(q)_n\,(aq)_n\,(\sigma_1)_n\,(\sigma_2)_n\,(q/d)_n\,(aq/d)_n} \times \sum_{r=0}^{\infty} \frac{(kq^n)_r\,(cq^n)_r\,(dq^{-n})_r\,(q^{-n})_r\,d_r}{(aq^{n+1})_r\,(aq^{1-n}/k)_r\,(aq^{1-n}/c)_r\,(aq^{n+1}/d)_r}$$

Putting $n = r+s$ and using $(q^{-n})_r = q^{-n} q^{-(n+1)} \cdots (q^{-n+r-1})(-1)^r (q^s)_r$ we obtain

$$(q^{-n})_r / (q)_r = q^{-n} q^{-n+1} \cdots q^{-n+r-1} / (q)_s,$$

$$(aq^{1-n}/k)_r = (-1)^r (aq/k)^r (kq^2/a)_r q^{1-n} q^{2-n} \cdots q^{-n+r-1} q^{-n},$$

$$(kq^s/a)_r / (k/a)_{r+s} = 1 / (k/a)_s.$$

Thus, $\displaystyle \sum_{n=0}^{\infty} \frac{(k)_n (k/a)_n (c)_n (c/a)_n (\mu_1)_n (\mu_2)_n t^n \beta_n}{(q)_n (aq)_n (\sigma_1)_n (\sigma_2)_n (q/d)_n (aq/d)_n}$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(k)_{2r+s}(k/a)_s (\mu_1)_{r+s}(\mu_2)_{r+s}(c)_{2r+s} d_r t^{r+s} (c/a)_s (cdk/a^2 q^2)^r}{(q)_s (aq)_{2r+s}(\sigma_1)_{r+s}(\sigma_2)_{r+s}(q/d)_s (aq/d)_{2r+s}}$$

$$= \sum_{r=0}^{\infty} \frac{(k)_{2r}(c)_{2r}(\mu_1)_r (\mu_2)_r}{(aq)_{2r}(\sigma_1)_r (\sigma_2)_r (aq/d)_{2r}} \left(\frac{cdtk}{a^2 q^2}\right)^r d_r \times {}_?\Phi \left[\begin{array}{c} kq^{2r}, cq^{2r}, k/a, c/a, \mu_1 q^r, \mu_2 q^r ; t \\ aq^{2r+1}, aq^{2r+1}/d, q/d, \sigma_1 q^r, \sigma_2 q^r \end{array} \right]$$

Using Jackson's theorem on the L.H.S. and taking

$$d_r = (a)_r (q\sqrt{a})_r (-q\sqrt{a})_r (b)_r q^r / (q)_r (\sqrt{a})_r (-\sqrt{a})_r (aq/b)_r$$

we obtain the transformation

$${}_{11}\Phi_{10}\left[\begin{array}{c} bc/a, \, aq/cd, \, \sqrt{aq/bd}, -\sqrt{aq/bd}, \, q\sqrt{a/bd}, -q\sqrt{a/bd}, \, c, \, k, \, \mu_1, \mu_2, \mu_3 ; \, t/b \\ q/d, \, aq/bd, \, aq/b, \, \sqrt{aq/d}, -\sqrt{aq/d}, \, q\sqrt{a/d}, -q\sqrt{a/d}, \, \sigma_1, \sigma_2, \sigma_2 \end{array} \right].$$

$$= \sum_{r=0}^{\infty} \frac{(k)_{2r}(c)_{2r}(\mu_1)_r (\mu_2)_r (\mu_3)_r (a)_r (q\sqrt{a})_r (-q\sqrt{a})_r (b)_r q^r}{(aq)_{2r}(\sigma_1)_r (\sigma_2)_r (\sigma_3)_r (aq/d)_{2r}(q)_r (\sqrt{a})_r (-\sqrt{a})_r (aq/b)_r} \times \left(\frac{cdkt}{a^2 q^2}\right)^r$$

$$\times \, {}_7\Phi_6 \left[\begin{array}{c} kq^{2r}, \, cq^{2r}, \, k/a, \, c/a, \, \mu_1 q^r, \, \mu_2 q^r, \, \mu_3 q^r ; \, t \\ aq^{2r+1}, \, aq^{2r+1}/d, \, q/d, \, \sigma_1 q^r, \, \sigma_2 q^r, \, \sigma_3 q^r \end{array} \right]. \quad —— (6.1A)$$

If in the basic series on the R.H.S. of (6.1A) we take

$\mu_1 = q\sqrt{c}, \; \mu_2 = -q\sqrt{c}, \; \mu_3 = \mu, \; \mu_4 = q^{-n}; \; t = q;$

$\sigma_1 = \sqrt{c}, \; \sigma_2 = -\sqrt{c}, \; \sigma_3 = cq/\mu, \; \sigma_4 = cq^{n+1};$

we obtain on the R.H.S. of (6.1A) the series

$${}_8\Phi_7 \left[\begin{array}{c} cq^{2r}, \, kq^{2r}, \, k/a, \, c/a, \, -q^{r+1}\sqrt{c}, \, +q^{r+1}\sqrt{c}, \, \mu q^r, \, q^{-n+r} ; \, q \\ aq^{2r+1}, \, aq^{2r+1}/d, \, q/d, \, -q^r\sqrt{c}, \, q^r\sqrt{c}, \, cq^{n+1}/\mu, \, cq^{n+r+1} \end{array} \right]$$

for this ${}_8\Phi_7$ to be well-poised we require $d = k/c$ and if we are to sum it by Jackson's theorem we require the further condition that $\mu = a^2 cq^{n+1}/k^2$.

If the parameters have these values, we have as the sum of the $_8\Phi_7$

$$\frac{(cq^{2r+1})_{n-r}\,(acq/k^2)_{n-r}\,(aq/k)_{n-r}\,(a^2q^{2r+1}/k)_{n-r}}{(cq/k)_{n-r}\,(acq^{2r+1}/k)_{n-r}\,(aq^{2r+1})_{n-r}\,(a^2q/k^2)_{n-r}}$$

and using

$$(aq^{2r+n})_{n-r} = (aq)_n\,(aq^{n+1})_r\,/(aq)_{2r}, \quad\text{and}\quad (aq)_{n-r} = (aq)_n\, q^{\frac{r(r-1)}{2}}/(aq)^r\,(-1)^r\,(q^{-n}/a)_r$$

the sum

$$= \frac{(cq)_n(cq^{n+1})_r\,(acq/k^2)_n(aq/k)_n(a^2q/k)_n(a^2q^{n+1}/k)_r(cq^n/k)^r(kq^{-n}/c)_r(acq/k)_{2r}\,(aq)_{2r}\,(a^2q^n/k^2)^r(k^2q^{-n}/a^2)_r}{(cq)_{2r}\,(acq^n/k^2)^r(k^2q^{-n}/ac)_r\,(aq/k)^r(kq^{-n}/a)_r(a^2q/k)_{2r}(cq/k)_n\,(ac/k)_n(acq^{n+1}/k)_r(aq)_n(aq^{n+1})_r(a^2q/k^2)_n}$$

$\therefore$ substituting in (6.1A) we have

$$\sum_{n=0}^{\infty}\frac{(k)_n(c)_n(q\sqrt{c})_n(-q\sqrt{c})_n(\mu)_n(q^n)_n(aq/k)_n(\sqrt{kcqa})_n(-\sqrt{kcqa})_n(\sqrt{kcqa})_n(-\sqrt{kcqa})_n(aqc/k^2)_n(q/b)^n}{(q)_n\,(\sqrt{c})_n(-\sqrt{c})_n(cq/\mu)_n(cq^{n+1})_n(cq/k)_n(ck/a)_n(k^2/a)_n(\sqrt{aqc/k})_n(-\sqrt{aqc/k})_n(q\sqrt{ac/k})_n(-q\sqrt{ac/k})_n}$$

$$= \frac{(cq)_n(acq/k^2)_n(aq/k)_n(a^2q/k)_n}{(cq/k)_n\,(acq/k)_n(aq)_n\,(a^2q/k^2)_n}$$

$$\times\sum_{r=0}^{\infty}\frac{(k)_{2r}(c)_{2r}(q\sqrt{c})_r(-q\sqrt{c})_r(\mu)_r(q^n)_r(a)_r(q\sqrt{a})_r(-q\sqrt{a})_r(\mu q^{-n}/k)_r(a^2q/k^2)_r(cq^{n+1})_r(a^2q^{n+1}/k)_r(kq^{-n}/c)_r(acq/k)_{2r}(aq)_{2r}(k^2q^{-n}/a^2)_r(q/b)^r}{(aq)_{2r}(aqc/k)_{2r}(\sqrt{c})_r(-\sqrt{c})_r(cq/\mu)_r(cq^{n+1})_r(q)_r(\sqrt{a})_r(-\sqrt{a})_r(k^2/a)_r(cq)_{2r}(k^2q^{-n}/ac)_r(kq^{-n}/a)_r(a^2q/k)_{2r}(acq^{n}/k)_n(aq^{n+1})_r}$$

$\therefore$ we obtain the transformation

$$_{12}\Phi_{11}\left[\begin{array}{c} a,\ q\sqrt{a},\ -q\sqrt{a},\ \sqrt{k},\ -\sqrt{k},\ \sqrt{kq},\ -\sqrt{kq},\ \mu,\ \mu q^{-n}/c,\ kq^{-n}/c,\ a^2q^{n+1}/k,\ q^{-n}\ ;\quad q/b \\ \sqrt{a},\ -\sqrt{a},\ aq/\sqrt{k},\ -aq/\sqrt{k},\ a\sqrt{q/k},\ -a\sqrt{q/k},\ aq/\mu,\ acq^{n+1}/\mu,\ acq^{n+1}/k,\ kq^{-n}/a,\ aq^{n+1} \end{array}\right]$$

$$= \frac{(aq)_n\,(cq/k)_n(acq/k)_n(ac/\mu)_n}{(cq)_n\,(aq/k)_n(a^2q/k)_n(aq/\mu)_n}$$

$$\times\ _{12}\Phi_{11}\left[\begin{array}{c} c,\ q\sqrt{c},\ -q\sqrt{c},\ k,\ \mu,\ acq/k^2,\ \sqrt{kcqa},\ -\sqrt{kcqa},\ \sqrt{kcqa},\ -\sqrt{kcqa},\ aq/k,\ q^{-n}\ ;\quad q/b \\ \sqrt{c},\ -\sqrt{c},\ cq/k,\ cq/\mu,\ k^2/a,\ \sqrt{aqc/k},\ -\sqrt{aqc/k},\ q\sqrt{ac/k},\ -q\sqrt{ac/k},\ ck/a,\ cq^{n+1} \end{array}\right],\quad -\ (6.1B)$$

where

$$k^2 = a^2cq^{n+1}/\mu\ .$$

Formula (6.1B) is a transformation between two well-poised $_{12}\Phi_{11}$'s, though not of general type, and is the analogue of (7.41) of (S).

where the parameters satisfy the further relation that

$\mu_1\mu_2\mu_3 = a^3 q^{n+2}/c$ thus the sum of the $_8\Sigma_7$ on the right is

$$\frac{(a^2 q^{2r+2}/c)_{n-r} \, (a^2 q^2/c\mu_1\mu_2)_{n-r} \, (aq^{n+1}/\mu_1)_{n-r} \, (aq^{n+1}/\mu_2)_{n-r}}{(aq^{2r+1})_{n-r} \, (a^2 q^{r+2}/c\mu_1)_{n-r} \, (a^2 q^{r+2}/c\mu_2)_{n-r} \, (aq/\mu_1\mu_2)_{n-r}}$$

$$= \frac{(a^2 q^2/c)_n (a^2 q^2/c\mu_1\mu_2)_n (aq/\mu_1)_n (aq/\mu_2)_n}{(aq)_n (aq/\mu_1\mu_2)_n (a^2 q^2/c\mu_1)_n (a^2 q^2/c\mu_2)_n} \times \frac{(a^2 q^{n+2}/c)_r (aq)_{2r} (\mu_1\mu_2 q^{-n}/a)_r (a^2 q^2/c\mu_1)_r (a^2 q^2/c\mu_2)_r (c/aq)^r}{(a^2 q^2/c)_{2r} (aq^{n+1})_r (\mu_1\mu_2 c q^{-n+1}/a^2)_r (a/\mu_1)_r (a/\mu_2)_r}$$

and $\therefore$ from (7.1A) we obtain the transformation

$$_7\Phi_6\left[\begin{array}{c} a^2 q/c^2, \; aq^{\frac{3}{2}}/\sqrt{c}, \; -aq^{\frac{3}{2}}/\sqrt{c}, \; \mu_1, \; \mu_2, \; \mu_3, \; q^{-n} \; ; \\ aq^{\frac{1}{2}}/\sqrt{c}, \; -aq^{\frac{1}{2}}/\sqrt{c}, \; a^2 q^2/c\mu_1, \; a^2 q^2/c\mu_2, \; a^2 q^2/c\mu_3, \; a^2 q^{n+2}/c \end{array} \; q \;\right]$$

$$= \frac{(a^2 q^2/c)_n (a^2 q^2/c\mu_1\mu_2)_n (aq/\mu_1)_n (aq/\mu_2)_n}{(aq)_n (aq/\mu_1\mu_2)_n (a^2 q^2/c\mu_1)_n (a^2 q^2/c\mu_2)_n}$$

$$\times \; _{12}\Phi_{11}\left[\begin{array}{c} a, \; q\sqrt{a}, \; -q\sqrt{a}, \; \mu_1, \mu_2, \mu_3, \; a/c, \; \sqrt{c}, \; -\sqrt{c}, \; \sqrt{cq}, \; -\sqrt{cq}, \; q^{-n} \; ; \\ \sqrt{a}, \; -\sqrt{a}, \; aq/\mu_1, \; aq/\mu_2, \; aq/\mu_3, \; cq, \; aq/\sqrt{c}, \; -aq/\sqrt{c}, \; a\sqrt{q/c}, \; -a\sqrt{q/c}, \; aq^{n+1} \end{array} \; q \;\right] \quad - (7.1B)$$

where $\mu_1\mu_2\mu_3 = a^3 q^{n+2}/c$.

     The $_7\Phi_6$ is nearly-poised and the product of the denominator parameters is q times the product of the numerator parameters; the $_{12}\Phi_{11}$ is well-poised. It is the analogue of

$$_6F_5\left[\begin{array}{c} 2K-1-2a, \; 1+\tfrac{1}{2}K, \; b, \; c, \; d, \; -m \; ; \\ \tfrac{1}{2}K, \; 1+K-b, \; 1+K-c, \; 1+K-d, \; 1+K+m \end{array}\right] = \frac{(1+K)_m (1+a-b)_m (1+a-c)_m (1+a-d)_m}{(1+a)_m (1+K-b)_m (1+K-c)_m (1+K-d)_m}$$

$$\times \; _9F_8\left[\begin{array}{c} a, \; 1+\tfrac{1}{2}a, \; b, \; c, \; d, \; \tfrac{1}{2}+a-\tfrac{1}{2}K, \; 1+a-\tfrac{1}{2}K, \; K-a-1, \; -m \; ; \\ \tfrac{1}{2}a, \; 1+a-b, \; 1+a-c, \; 1+a-d, \; \tfrac{1}{2}+\tfrac{1}{2}K, \; \tfrac{1}{2}K, \; 2+2a-K, \; 1+a+m \end{array}\right] \qquad \text{a formula given}$$

by Professor Bailey[+] with $K = 1 + 2a - d$, $b = a_1$, $c = a_2$ and $d = a_3$

Footnote. [+] (8.41) of (5).

(8)   We now consider identities of the Rogers —
Ramanujan type. The two identities discovered by Rogers
and then rediscovered by Ramanujan are

$$1 + q^{(1-q)} + q^{4}/(1-q)(1-q^2) + q^{9}/(1-q)(1-q^2)(1-q^3) + \cdots$$

$$= 1/(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})(1-q^{14}) \cdots \qquad ,$$

$$1 + q^{2}/(1-q) + q^{6}/(1-q)(1-q^2) + q^{12}/(1-q)(1-q^2)(1-q^3) + \cdots$$

$$= 1/(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})(1-q^{13}) \cdots \qquad .$$

In these formulae the indices of the powers of q in the
numerator on the left are n^2 and $n(n+1)$ while, in the products
on the right. the indices of the powers of q form two
arithmetic progressions with difference 5. These identities
were obtained in a paper published by Rogers in 1894, (12), wh
where they appeared as corollaries of a series of general
theorems. They seemed to escape notice in spite of
their particular elegance until Ramanujan, who had then no
proof of the formula, but has arrived at them by a process
of induction, communicated them to Hardy in a letter sent
from India in Feb. 1913. Further proofs were published by
Rogers, Ramanujan and Schur, (12), Watson, (10), Rogers
third proof and Ramanujan's proof were the same in
principle, though the details differed; they consisted in
establishing the preliminary lemma

$$1 + \sum_{n=1}^{\infty} (-1)^n a^{2n} q^{\frac{1}{2}n(5n-1)} \frac{(1-aq^{2n})(1-aq)(1-aq^2) \cdots (1-aq^{n-1})}{(1-q)(1-q^2) \cdots \cdots (1-q^n)}$$

$$= \prod_{n=1}^{\infty}(1-aq^n)\left[1 + \sum_{n=1}^{\infty} \frac{a^{2} q^{n^2}}{(1-q) \cdots (1-q^n)}\right] \qquad - (8.1)$$

and then using Jacobi's
well known formula

$$\prod_{n=1}^{\infty}(1-q^{2n-1}z)(1-q^{2n-1}/z)(1-q^{2n}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} z^n$$

and putting a = 1 and $a = q$,   we have the
required results.

    In 1929, Watson published his proof of these identities.
He pointed out, that equation (8.1) was the limiting
form of a much more general identity viz:—

$$_8\Phi_7 \left[\begin{array}{c} a, q\sqrt{a}, -q\sqrt{a}, c, d, e, f, q \\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, aq/g \end{array} ; a^2q^2/cdefg \right]$$

$$= \prod_{n=1}^{\infty} \left[\frac{(1-aq^n)(1-aq^n/fg)(1-aq^n/ge)(1-aq^n/ef)}{(1-aq^n/e)(1-aq^n/f)(1-aq^n/g)(1-aq^n/efg)} \right] \times {}_4\Phi_3 \left[\begin{array}{c} aq/cd, e, f, q \\ efg/a, aq/c, aq/d \end{array} ; q \right] \qquad - (8.2)$$

where e, f, g is of the form q^{-N} (N a +ve integer),
which Watson proved by induction, after establishing as a
preliminary lemma the analogue of Saalschütz's formula viz:—

$$_3\Phi_2 \left[\begin{array}{c} aq/cd, aq^{-N}, q^{-N} ; q \\ aq/c, aq/d \end{array} \right] = \prod_{n=0}^{N-1} \left[\frac{aq(1-cq^n)(1-dq^n)}{(c-aq^{n+1})(d-aq^{n+1})} \right] .$$

This he proved by equating coefficients of x^n in the expansions of both sides of Heine's equation,

$$_2\Phi_1\left[\alpha, \beta; \gamma; x\right] = {_1\Phi_0}\left[\alpha\beta/\gamma; x\right] \times {_2\Phi_1}\left[\gamma/\alpha, \gamma/\beta; \gamma; \frac{\alpha\beta x}{\gamma}\right]$$

It is however, easier to obtain (8.2) as in the Tract. If in (8.2) we make c, d, e, f, g $\rightarrow \infty$ then

$$\lim_{c \to \infty} (c)_n / c^n (aq/c)_n = (-1)^r q^{\frac{1}{2}n(n-1)},$$

and we arrive at (8.1). Watson then obtained the Rogers-Ramanujan identities as in their proofs.

Watson's proof was extremely interesting in that it shewed the Rogers-Ramanujan identities to be essentially limiting cases of transformations of basic hypergeometric series. Since Watson's proof was published, further identities of the Rogers-Ramanujan type have been obtained from basic hypergeometric series in particular by Professor Bailey and Mr. Dyson. We now proceed to consider identities of this type.

(8.1)   As the formula (6.1C) giving the sum of a well-poised $_{10}\Phi_9$ has not been obtained before, we obtain its limiting form, to see if it leads to any new identities of the Rogers-Ramanujan type.

If in (6.1C) we let c $\rightarrow \infty$ it becomes

$$1 + \sum_{r=1}^{n} \frac{(a)_r (1-aq^{2r}) a^r \left(q^{\frac{1}{2}r(r-1)}\right)^2 (q^{\frac{1}{2}rn})^2 (q^{-n})_r q^r}{(q)_r (1-a) q^{\frac{1}{2}r(r+1)} q^{-nr+\frac{1}{2}r(r-1)} (aq^{n+1})_r}$$

Now let n $\rightarrow \infty$ in this and we obtain

$$1 + \sum_{r=1}^{\infty} \frac{(a)_r (1-aq^{2r}) q^{r^2} (-1)^r q^{\frac{1}{2}r(r-1)}}{(q)_r (1-a)} = \prod_{n=1}^{\infty} (1-aq^n)$$

$$\therefore 1 + \sum_{r=1}^{\infty} \frac{(aq)_{r-1} (1-aq^{2r})(-1)^r q^{\frac{1}{2}r(3r-1)}}{(q)_r} = \prod_{n=1}^{\infty} (1-aq^n) .$$

Putting a = 1 in this becomes

$$1 + \sum_{n=1}^{\infty} (-1)^n \left\{ q^{\frac{1}{2}n(3n-1)} + q^{\frac{1}{2}n(3n+1)} \right\} = \prod_{n=1}^{\infty} (1-q^n),$$

which is a classical result due to Euler.

31.

(9)   We now study identities of the Rogers-Ramanujan type which can be obtained from 3 formulae due to Professor Bailey.(4). The following notation is used throughout this section.

$$(a)_n = (1-a)(1-ax) \dots (1-ax^{n-1}), \quad [a]_{n+1} = (1-a)(1-ax^2)(1-ax^4) \dots (1-ax^{2n}),$$

$$\{a\}_n = (1-a)(1-ax^3) \dots (1-ax^{3n-3}),$$

$$x_n = 1-x^n, \quad \bar{x}_n = 1+x^n, \quad x_n! = (x)_n, \quad x_n!! = x_1 x_3 x_5 \dots x_{2n-1},$$

$$x_n^{r} = x_{rn}, \quad x_n^{r}! = x_1^r x_2^r \dots x_n^r, \quad x_n^{r}!! = x_1^r x_3^r \dots x_{2n-1}^r,$$

$$x! = x_\infty!, \quad x^{\lambda}! = x_\infty^{\lambda}!$$

and similarly for abbreviations with bars E.g. $\bar{x} = \bar{x}_\infty!$.   The three formulae are:—

$$\sum_{n=0}^{\infty} \frac{(ax)_{3n} \{\rho_1\}_n \{\rho_2\}_n}{x_n^3! \{a^3x^3\}_{3n}} \left(\frac{a^3x^3}{\rho_1\rho_2}\right)^n = \prod_{m=1}^{\infty} \left[\frac{(1-a^3x^{3m}/\rho_1)(1-a^3x^{3m}/\rho_2)}{(1-a^3x^{3m})(1-a^3x^{3m}/\rho_1\rho_2)} \right]$$

$$\times \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n (ax)_{n-1} \{\rho_1\}_n \{\rho_2\}_n (1-ax^{2n}) a^n x^{\frac{1}{2}(3n^2-n)}}{x_n! \{a^3x^3/\rho_1\}_n \{a^3x^3/\rho_2\}_n} \left(\frac{a^3x^3}{\rho_1\rho_2}\right)^n \right], \qquad —\ (9.1)$$

$$1 + \sum_{n=1}^{\infty} \frac{(\rho_1)_n (\rho_2)_n \{ax^3\}_{n-1}}{x_n! (ax)_{2n-1}} \left(\frac{ax}{\rho_1\rho_2}\right)^n = \prod_{m=1}^{\infty} \left[\frac{(1-ax^m/\rho_1)(1-ax^m/\rho_2)}{(1-ax^m)(1-ax^m/\rho_1\rho_2)} \right]$$

$$\times \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n \{ax^3\}_{n-1} (1-ax^{6n})(\rho_1)_{3n} (\rho_2)_{3n} a^{4n} x^{\frac{3n}{2}(3n+1)}}{x_n^3! (ax/\rho_1)_{3n} (ax/\rho_2)_{3n} \rho_1^{3n} \rho_2^{3n}} \right], \qquad —\ (9.2)$$

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n \{ax^3\}_{n-1} (\rho_1)_n a^n x^{\frac{1}{2}n(n-1)}}{x_n! (ax)_{2n-1} \rho_1^n} = \prod_{m=1}^{\infty} \left[\frac{(1-ax^m/\rho_1)}{(1-ax^m)} \right]$$

$$\times \left[1 + a - a/\rho_1 + \sum_{n=1}^{\infty} \frac{\{ax^3\}_{n-1}(1-ax^{6n})(\rho_1)_{3n}(1-ax^{6n}-ax^{3n}/\rho_1) a^{4n} x^{\frac{3n}{2}(3n-1)}}{x_n^3! (ax/\rho_1)_{3n} \rho_1^{3n}} \right]. \qquad —\ (9.3)$$

These formulae were obtained from the following result of which the methods of obtaining transformations which I have previously used are particular cases.

If $\quad \beta_n = \sum_{r=0}^{n} d_r u_{n-r} v_{n+r} \quad$ and $\quad \gamma_n = \sum_{r=n}^{\infty} d_r u_{r-n} v_{r+n}$,

then $\quad \sum_{n=0}^{\infty} d_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n$. —— (9.4)

The proof is simple, for

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \sum_{r=n}^{\infty} \alpha_n \delta_r u_{r-n} v_{n+r}$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{r} \delta_r \alpha_n u_{r-n} v_{n+r}$$

$$= \sum_{r=0}^{\infty} \delta_r \beta_r \quad , \qquad \text{assuming that the series converge}$$

and that the change in the order of summation is permissible.

If we take $\gamma_n = \sum_{r=n}^{\infty} \dfrac{\delta_r}{(x)_{r-n}(ax)_{r+n}} = \sum_{s=0}^{\infty} \dfrac{\delta_{n+s}}{(x)_s (ax)_{2n+s}}$

$\&\ \delta_r = (\rho_1)_r (\rho_2)_r (ax/\rho_1\rho_2)^r$ we obtain using the basic analogue of Gauss's theorem,

$$\gamma_n = \frac{(\rho_1)_n (\rho_2)_n}{(ax/\rho_1)_n (ax/\rho_2)_n} \left(\frac{ax}{\rho_1\rho_2}\right)^n \prod_{n=1}^{\infty} \left[\frac{(1-ax^m/\rho_1)(1-ax^m/\rho_2)}{(1-ax^m)(1-ax^m/\rho_1\rho_2)}\right].$$

Hence, using (9.4),

$$\sum_{n=0}^{\infty} (\rho_1)_n (\rho_2)_n \left(\frac{ax}{\rho_1\rho_2}\right)^n \beta_n = \prod_{m=1}^{\infty}\left[\frac{(1-ax^m/\rho_1)(1-ax^m/\rho_2)}{(1-ax^m)(1-ax^m/\rho_1\rho_2)}\right]\sum_{n=0}^{\infty}\frac{(\rho_1)_n (\rho_2)_n}{(ax/\rho_1)_n (ax/\rho_2)_n}\left(\frac{ax}{\rho_1\rho_2}\right)^n \alpha_n \quad .\,-(9.5)$$

Taking

$$\alpha_n = (-1)^n (1-ax^{2n})(ax)_{n-1}\, a^n x^{\frac{1}{2}n(3n-1)}/x_n !$$

β_n is obtained from the formula:—

$${}_6\Phi_5\left[\begin{array}{c} a,\ x\sqrt{a},\ -x\sqrt{a},\ b,\ c,\ x^{-n}\,;\\ \sqrt{a},\ -\sqrt{a},\ ax/b,\ ax/c,\ ax^{n+1} \end{array}\ \frac{ax^{n+1}}{bc}\right]=\frac{(ax)_n (ax/bc)_n}{(ax/b)_n (ax/c)_n}$$

and we find that

$$\beta_n = (ax)_{3n}/x_n^3 !\ \{a^3 x^3\}_{2n}$$

Changing a into a^3, x into x^3, in (9.5), we obtain (9.1).

If we take $\alpha_{3n\pm1}=0,\ \alpha_{3n}=\dfrac{(-1)^n\{ax^3\}_{n-1}(1-ax^{6n})a^n x^{\frac{9}{2}n(3n-1)}}{x_n^3 !}$ we sum β_n using Jackson's analogue of Dougall's theorem and we obtain $\beta_n = \dfrac{\{ax^3\}_{n-1}}{x_n^3 !\,(ax)_{n-1}}$;

substituting these values in (9.5) we have (9.2).

The formula (9.3) was obtained using different α's & γ's We use the formula

$${}_2\Phi_1\left[\begin{array}{c} f,\ g\,;\\ ax \end{array}\ \frac{a}{fg}\right]=\prod_{m=1}^{\infty}\frac{(1-ax^m/f)(1-ax^m/g)}{(1-ax^m)(1-ax^m/fg)}\left[\frac{fg(1+a)-(f+g)a}{fg-a}\right]\quad\text{obtained from}$$

§8.5 (2) of the Tract by letting $c,\ e\to\infty$ and putting $d=\frac{1}{x}$

Then if $\delta_r = (\rho_1)_r (\rho_2)_r (a/\rho_1\rho_2)^r$

$$\gamma_n = \frac{(\rho_1)_n (\rho_2)_n}{(ax/\rho_1)_n (ax/\rho_2)_n} \left(\frac{a}{\rho_1\rho_2}\right)^n \left[\frac{\rho_1\rho_2(1+ax^{2n})-ax^n(\rho_1+\rho_2)}{\rho_1\rho_2-a}\right] \times \prod_{m=1}^{\infty} \left[\frac{(1-ax^m/\rho_1)(1-ax^m/\rho_2)}{(1-ax^m)(1-ax^m/\rho_1\rho_2)}\right]$$

When $\rho_2 \to \infty$, we obtain

$$\delta_r = (-1)^r (\rho_1)_r \, a^r x^{\frac{1}{2}r(r-1)} / \rho_1^r$$

$$\gamma_n = \frac{(-1)^n (\rho_1)_n a^n x^{\frac{1}{2}n(n-1)} (1+ax^{2n}-ax^n/\rho_1)}{(ax/\rho_1)_n \rho_1^n} \prod_{m=1}^{\infty} \left[\frac{(1-ax^m/\rho_1)}{(1-ax^m)}\right]$$
$$\qquad\qquad (9.6)$$

The formula (9.6) combined with the values of used to give (9.2) gives (9.3).

The results obtained may be divided into three groups, viz:— identities of the Rogers-Ramanujan type, identities in which we have the difference of two false ϑ series expressed as a single sum, and identities in which we have the sum of two false ϑ series expressed as a single sum. In the last case, the identities obtained are rather complicated and their only interest appears to be in their existence. The formula (9.3) gives linear combinations of results in practically all cases.

For the sake of completeness, I shall include identities which have been obtained previously. (3) & (4). As the method used is the same for the various identities of any one type, details of the working will only be given for typical cases.

9.1

The following formula have already been obtained from the formula (9.1)

Identities of the Rogers-Ramanujan type.

$\rho_1 \to \infty$, $\rho_2 \to \infty$, $a = 1$ gives

$$\sum_{n=0}^{\infty} \frac{x^{3n^2} x_{3n}!}{x_n^3! \, x_{2n}^3!} = \prod_{n=1}^{\infty} \frac{(1-x^{9n-5})(1-x^{9n-4})(1-x^{9n})}{(1-x^{3n})}, \qquad (9.1A)$$

$\rho_1 \to \infty$, $\rho_2 \to \infty$, $a = x$ gives

$$\sum_{n=0}^{\infty} \frac{x^{3n(n+1)} x_{3n+1}!}{x_n^3! \, x_{2n+1}^3!} = \prod_{n=1}^{\infty} \frac{(1-x^{9n-8})(1-x^{9n-1})(1-x^{9n})}{(1-x^{3n})},$$

$\rho_1 = -\sqrt{a^3 x}$, $\rho_2 \to \infty$, $a = 1$ gives

$$\sum_{n=0}^{\infty} \frac{x^{3n^2} x_{3n}^2!}{x_n^{12}! \, x_{3n}^3!} = \prod_{n=1}^{\infty} \frac{(1-x^{12n-5})(1-x^{12n-7})}{(1-x^{6n-3})},$$

$\rho_1 = -\sqrt{a^3 x^3}$, $\rho_2 \to \infty$, $a = x$ gives

$$\sum_{n=0}^{\infty} \frac{x_{3n+1}! \, x^{\frac{3}{2}(n^2+n)}}{(x_n^3!)^2 \, x_{3n+1}^3!!} = \prod_{n=1}^{\infty} \frac{(1-x^{6n-5})(1-x^{6n-1})}{(1-x^{6n-3})^2},$$

Obtained in linear combination with another result,.

$$\sum_{n=0}^{\infty} \frac{x_{3n+4}! \, x^{\frac{3}{2}(n^2+n)}}{x_n^3! \, x_{n+1}^3! \, x_{n+2}^3!} = (1-x) \prod_{n=1}^{\infty} \frac{(1-x^{3n})}{(1-x^{3n})(1-x^{6n-3})}.$$

Identities of the false ϑ series type

$\rho_1 = \sqrt{a^3 x^3}$, $\rho_2 \to \infty$, $a = x$ gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{3}{2}(n^2+n)} x_{3n+1}!}{x_{2n+1}^3!} = \sum_{n=0}^{\infty} x^{3n^2+2n} - \sum_{n=1}^{\infty} x^{3n^2-2n},$$

and given in a linear combination

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{3}{2}(n^2+n)} x_{3n}! (1-x^{3n+2})}{x_{2n+1}^3!} = \sum_{n=0}^{\infty} x^{3n^2+n} - \sum_{n=1}^{\infty} x^{3n^2-n}.$$

Further identities have been obtained as follows:—

Identities of the Rogers-Ramanujan type

$\rho_1 = -a^{\frac{3}{2}} x^3$, $\rho_2 \to \infty$, $a = 1$ gives

$$\sum_{n=0}^{\infty} \frac{(x)_{3n} x^{\frac{3n}{2}(n-1)}}{(x_n^3!)^2 \, x_{2n}^3!!} = \prod_{m=1}^{\infty} \frac{(1-x^{6m-5})(1-x^{6m-1})}{(1-x^{6m-3})^2} + \prod_{m=1}^{\infty} \frac{(1-x^{2m})}{(1-x^{3m})(1-x^{6m-3})},$$

$\rho_1 = -a^{\frac{3}{2}} x^3$, $\rho_2 \to \infty$, $a = x$ gives

$$\frac{1}{(1-x^2)} \sum_{n=0}^{\infty} \frac{x_{3n+1}^2! \, x^{3n^2}}{x_n^{12}! \, x_{2n+1}^3!} = \frac{(1+x^3)}{(1-x^2)} \prod_{m=1}^{\infty} \frac{(1-x^{12m-7})(1-x^{12m-5})}{(1-x^{6m-3})} - \frac{x^2(1+x^3)}{(1-x^2)} \prod_{m=1}^{\infty} \frac{(1-x^{12m-11})(1-x^{12m-1})}{(1-x^{6m-3})}.$$

Identities of the false ϑ series type

$$\rho_1 = -a^{\frac{3}{2}} x^{\frac{3}{2}}, \quad \rho_2 = x^3, \quad a = x, \text{ gives, on changing } x \text{ into } x^2$$

$$2 \sum_{n=0}^{\infty} \frac{(-1)^n x^2_{3n+1}!}{x_n^6! \, x_n^6!!} = \sum_{n=0}^{\infty} x^{3n^2+n} - \sum_{n=1}^{\infty} x^{3n^2-n} \qquad (9.1B)$$

We can also obtain a result of the third type by taking $\rho_1 = a^{\frac{1}{2}} x^3, \; \rho_2 \to \infty \;$ & $a = x$.

We give the proofs of (9.1A) and (9.1B)

In (9.1) let $\rho_1 + \rho_2 \to \infty$ and we obtain

$$\sum_{n=0}^{\infty} \frac{(ax)_{3n} (a^3 x^3)^n x^{3n(n-1)}}{x_n^3! \, \{a^3 x^3\}_{2n}} = \prod_{m=1}^{\infty} \frac{1}{(1-a x^{3m})} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n (ax)_{n-1} (1-a x^{2n}) a^{4n} x^{\frac{1}{2}(3n^2-n)} a^{3n} x^{3n^2}}{x_n!}\right].$$

Taking $a = 1$, we have

$$\sum_{n=0}^{\infty} \frac{(x)_{3n} x^{3n^2}}{x_n^3! \, \{x^3\}_{2n}} = \prod_{m=1}^{\infty} \frac{1}{(1-x^{3m})} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n (x)_{n-1} (1-x^{2n}) x^{\frac{1}{2}n(9n-1)}}{x_n!}\right]$$

$$\therefore \sum_{n=0}^{\infty} \frac{(x)_{3n} x^{3n^2}}{x_n^3! \, \{x^3\}_{2n}} = \prod_{m=1}^{\infty} \frac{1}{(1-x^{3m})} \left[1 + \sum_{n=1}^{\infty} (-1)^n (1+x^n) x^{\frac{1}{2}n(9n-1)}\right]$$

$$= \prod_{m=1}^{\infty} \frac{1}{(1-x^{3m})} \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(9n-1)} = \prod_{n=1}^{\infty} \frac{(1-x^{9n})(1-x^{9n-4})(1-x^{9n-5})}{(1-x^{3n})}$$

$$\therefore \sum_{n=0}^{\infty} \frac{x^{3n^2} x_{3n}!}{x_n^3! \, x_{2n}^3!} = \prod_{n=1}^{\infty} \frac{(1-x^{9n-4})(1-x^{9n-5})(1-x^{9n})}{(1-x^{3n})}$$

In (9.1) take $\rho_1 = -a^{\frac{3}{2}} x^{\frac{3}{2}} \;, \; \rho_2 = x^3, \; a = x$ and we have

$$\sum_{n=0}^{\infty} \frac{(x^2)_{3n} \{-x^3\}_n \{+x^3\}_n (-1)^n}{x_n^3! \, \{x^6\}_{2n}}$$

$$= \prod_{m=1}^{\infty} \left[\frac{(1+x^{3m})(1-x^{3m})}{(1-x^{3m+3})(1+x^{3m-3})}\right] \times \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n (x^2)_{n-1} \{-x^3\}_n \{x^3\}_n (1-x^{4n+1}) x^n x^{\frac{1}{2}(3n^2-n)} (-1)^n}{x_n! \, \{-x^3\}_n \{x^3\}_n}\right]$$

R.H.S.

$$= (1-x^3)/2 \left[1 + \sum_{n=1}^{\infty} (1-x^{2n+1}) x^{\frac{1}{2}(3n^2+n)} / (1-x)\right]$$

$$= (1-x^3) 2(1-x) \left[1 - x + \sum_{n=1}^{\infty} x^{\frac{1}{2}(3n^2+n)} - \sum_{n=1}^{\infty} x^{\frac{1}{2}(n+1)(3n+2)}\right]$$

and changing x into x^2 the R.H.S. becomes

$$(1-x^6)/2(1-x^2)\left[\sum_{n=0}^{\infty} x^{3n^2+n} - \sum_{n=1}^{\infty} x^{3n^2-n}\right].$$

Changing x into x^2.

$$\frac{(x^2)_{3n}\ \{-x^3\}_n}{\{x^6\}_{2n}} \qquad \text{becomes} \qquad \frac{(1-x^4)(1-x^6)\ldots(1-x^{6n+2})(1+x^6)(1+x^{12})\ldots(1+x^{6n})}{(1-x^{12})(1-x^{18})\ldots(1-x^{12n+6})}$$

$$\therefore \frac{2(1-x^2)}{(1-x^6)} \times L.H.S = 2\sum_{n=0}^{\infty} \frac{(-1)^n x^2_{3n+1}!}{x_n^6!\ x_n^6!!}$$

$\therefore$ we obtain the identity

$$2\sum_{n=0}^{\infty} \frac{(-1)^n x^2_{3n+1}!}{x_n^6!\ x_n^6!!} = \sum_{n=0}^{\infty} x^{3n^2+n} - \sum_{n=1}^{\infty} x^{3n^2-n}.$$

(9.2)

From the formula (9.2) the following identities have already been obtained.

Identities of the Rogers-Ramanujan type

$\rho_1, \rho_2 \to \infty, \quad a = 1$ gives

$$1 + \sum_{n=1}^{\infty} \frac{x^{n^2} x_{n-1}!}{x_n! \, x_{2n-1}!} = \prod_{n=1}^{\infty} \frac{(1-x^{27n-15})(1-x^{27n-12})(1-x^{27n})}{(1-x^n)}, \quad - (9.2A).$$

$\rho_1, \rho_2 \to \infty, \quad a = x$ gives

$$\sum_{n=0}^{\infty} \frac{x^{n^2+3n} x_n!}{x_n! \, x_{2n+2}!} = \prod_{n=1}^{\infty} \frac{(1-x^{27n-24})(1-x^{27n-3})(1-x^{27n})}{(1-x^n)},$$

$\rho_1 = -\sqrt{ax}, \quad \rho_2 \to \infty, \quad a = 1$ and finally changing x into x^2 gives

$$1 + \sum_{n=1}^{\infty} \frac{x^{n^2} x_{n-1}!}{x_{2n}! \, x_{n-1}^4!} = \prod_{n=1}^{\infty} \frac{(1-x^{36n-24})(1-x^{36n-15})(1-x^{36n})}{(1-x^{2n-1})(1-x^{4n})},$$

$\rho_1 = -\sqrt{ax}, \quad \rho_2 \to \infty, a = x^3$ gives

$$\sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}(n^2+3n)} x_n!}{x_n! \, x_{n+1}! \, x_{n+1}!!} = \prod_{n=1}^{\infty} \frac{(1-x^{6n})}{(1-x^n)(1-x^{2n-1})},$$

$\rho_1 = -\sqrt{a}, \quad \rho_2 \to \infty, \quad a = 1$ gives

$$1 + 2\sum_{n=1}^{\infty} \frac{x^{\frac{1}{2}(n^2+n)} x_{n-1}!}{x_{n-1}! \, x_n! \, x_n!!} = \prod_{n=1}^{\infty} \frac{(1-x^{9n})(1-x^{18n-9})}{(1-x^n)(1-x^{2n-1})},$$

$\rho_1 = -\sqrt{a}, \quad \rho_2 \to \infty, a = x^3$ and finally changing x into x^2 gives

$$\sum_{n=0}^{\infty} \frac{x^{n^2+4n} x_n!}{x_{2n+1}! \, x_{n+1}!} = \prod_{n=1}^{\infty} \frac{(1-x^{36n-33})(1-x^{36n-3})(1-x^{36n})}{(1-x^{2n-1})(1-x^{4n})},$$

$\rho_1 = -\sqrt{a}, \quad \rho_2 = -\sqrt{ax}, \quad a = 1$ and changing x into x^2 gives

$$1 + 2\sum_{n=1}^{\infty} \frac{x_{n-1}! \, x^n}{x_{2n+1}! \, x_n^2!} = \prod_{n=1}^{\infty} \frac{(1-x^{9n})(1-x^{18n-9})}{(1-x^n)(1-x^{2n-1})} = \prod_{n=1}^{\infty} \frac{(1+x^n)(1-x^{9n})}{(1-x^n)(1+x^{9n})},$$

$\rho_1 = -\sqrt{a}, \quad \rho_2 = -\sqrt{ax}, \quad a = x^3$ and changing x into x^2 gives

$$\sum_{n=0}^{\infty} \frac{x_n! \, x^n}{x_{n+2}! \, x_n^2!} = \prod_{n=1}^{\infty} \frac{(1-x^{18n})(1-x^{18n-3})(1-x^{18n-15})}{(1-x^n)(1-x^{2n-1})},$$

Identities involving false ϑ series

$\rho_1 = \sqrt{ax}, \quad \rho_2 \to \infty, \quad a = x^3$ gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n x_n! \, x^{\frac{1}{2}(n^2+3n)}}{(1+x^{n+1}) x_{2n+1}!} = \sum_{n=0}^{\infty} x^{9n^2+6n} - \sum_{n=1}^{\infty} x^{9n^2-6n}. \quad - (9.2B).$$

Further results have been obtained as follows:-

Identities of the Rogers-Ramanujan type

$p_1 = -\sqrt{a}\,x, \quad p_2 \to \infty, \quad a = 1$ gives

$$1 + \sum_{n=1}^{\infty} \frac{(1-x^{2n})\,x_{n-1}!\,x^{\frac{1}{2}(n^2-n)}}{(x_n!)^2\,x_n!!} = \prod_{n=1}^{\infty} \frac{(1-x^{6n})}{(1-x^n)(1-x^{2n-1})} + \prod_{n=1}^{\infty} \frac{(1-x^{9n})(1-x^{18n-9})}{(1-x^n)(1-x^{2n-1})} ,$$

This result may be written

$$\sum_{n=0}^{\infty} \frac{x_n!\,x^{\frac{1}{2}(n^2+n)}}{(x_n!)^2\,x_{n+1}!!} + 1 + 2\sum_{n=1}^{\infty} \frac{x_{n-1}!\,x^{\frac{1}{2}(n^2+n)}}{x_{n-1}!\,x_n!\,x_n!!} = \prod_{n=1}^{\infty} \frac{(1-x^{6n})}{(1-x^n)(1-x^{2n-1})} + \prod_{n=1}^{\infty} \frac{(1-x^{9n})(1-x^{18n-9})}{(1-x^n)(1-x^{2n-1})} ,$$

which shows that it is the sum of two known results.

$p_1 = -\sqrt{a}\,x, \quad p_2 \to \infty, \quad a = x^3$ gives

$$1 + \frac{(1-x)(1-x^4)}{(1+x^3)}\sum_{n=1}^{\infty} \frac{x_n^6!\,(1+x^{2n+3})x^{n^2+n}}{x_{2n+1}!\,x_{n+1}^4!}$$

$$= \frac{(1-x)(1-x^4)}{(1+x^3)}\left[\prod_{n=1}^{\infty} \frac{(1-x^{36n})(1-x^{18n-9})}{(1-x^{4n})(1-x^{2n-1})} + x^3\prod_{n=1}^{\infty} \frac{(1-x^{36n})(1-x^{36n-33})(1-x^{36n-3})}{(1-x^{4n})(1-x^{2n-1})}\right]$$

and this again is a linear combination of two known results

$p_2 = x^3, \quad p_2 = x^3 \quad a = x^6$ gives

$$1 + \frac{1}{(1+x^2)(x^2-1)}\sum_{n=1}^{\infty} \frac{[x^6]_n\,x_{n+1}^6!\,x^{2n+6}}{x_n^2!\,[x^8]_{n+1}\,]x^{10}[_{n+1}} = \prod_{n=1}^{\infty} (1-x^{6n})$$

where $]x^{10}[_{n+1}$ denotes $(1-x^{10})(1-x^{14}) \ldots (1-x^{4n+10})$.

Identities of the third type

In all three identities x has been replaced by x^2 to give the final result

$p_2 = \sqrt{a}\,x, \quad p_2 \to \infty, \quad a = 1$ gives

$$\sum_{n=0}^{\infty} x^{9n^2+3n} + \sum_{n=1}^{\infty} x^{9n^2-3n} = \frac{x_\infty^2}{x_\infty!!}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n\,x_{n-1}^6!\,x^{n^2}}{x_n^2!\,x_n!!\,x_{n-1}^4!}\right], \quad - \quad (9.20)$$

$p_1 = \sqrt{a}, \quad p_2 \to \infty, \quad a = x^3$ gives

$$\sum_{n=0}^{\infty} x^{18n^2+15n} + \sum_{n=1}^{\infty} x^{18n^2-15n} = \frac{(1+x^3)[x^8]_\infty}{[x^5]_\infty}\left[1 + \sum_{n=1}^{\infty} \frac{(1-x^4)(-1)^n\,x_{n-1}^6!\,x^{n^2+4n}}{x_n^2!\,[x^5]_n\,x_n^4!}\right],$$

$p_2 = -\sqrt{a}, \quad p_2 = +\sqrt{a}\,x, \quad a = 1$ gives

$$\sum_{n=-\infty}^{\infty} x^{9n^2} = \frac{x_\infty^2!\,x_\infty!!}{x_\infty^2!\,x_\infty!!}\left[1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n\,x_{n-1}^6!\,x^n}{x_n^2!\,x_{n+2}!}\right].$$

We now give the proof of (9.2A)

In (9.2) let $p_1, p_2 \to \infty$ then, since

$$(p_1)_n / p_1^n \to (-1)^n x^{\frac{n}{2}(n-1)} \quad \text{and} \quad (p_1)_{3n}/p_1^{3n} \to (-1)^{3n} x^{\frac{3n}{2}(3n-1)}$$

we obtain

$$1 + \sum_{n=1}^{\infty} \frac{\{ax^3\}_{n-1} a^n x^{n^2}}{x_n! (ax)_{2n-1}}$$

$$= \prod_{m=1}^{\infty} \frac{1}{(1-ax^m)} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n \{ax^3\}_{n-1} (1-ax^{6n}) a^{4n} x^{\frac{1}{2}(27n^2-3n)}}{x_n!} \right]$$

putting $a = 1$, we have

$$1 + \sum_{n=1}^{\infty} \frac{\{x^3\}_{n-1} x^{n^2}}{x_n! (x)_{2n-1}} = \prod_{m=1}^{\infty} \frac{1}{(1-x^m)} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n \{x^3\}_{n-1} (1-x^{6n}) x^{\frac{1}{2}(27n^2-3n)}}{x_n!} \right].$$

Now

$$\{x^3\}_{n-1} = (1-x^3)(1-x^6) \cdots (1-x^{3n-3})$$

$$x_n! = (1-x^3)(1-x^6) \cdots (1-x^{3n})$$

$\therefore$ R.H.S.

$$= \prod_{m=1}^{\infty} \frac{1}{(1-x^m)} \left[1 + \sum_{n=1}^{\infty} (-1)^n (1+x^{3n}) x^{\frac{1}{2}(27n^2-3n)} \right]$$

$$= \prod_{m=1}^{\infty} \frac{1}{(1-x^m)} \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}(27n^2+3n)}$$

Hence, using Jacobi's formula

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} z^n = \prod_{n=1}^{\infty} (1-zq^{2n-1})(1-q^{2n-1}/z)(1-q^{2n})$$

with $q = x^{\frac{27}{2}}$ & $z = x^{\frac{3}{2}}$ we have

$$1 + \sum_{n=1}^{\infty} \frac{x_{2n-1}! \, x^{n^2}}{x_n! \, x_{2n-1}!} = \prod_{n=1}^{\infty} \frac{(1-x^{27n-15})(1-x^{27n-12})(1-x^{27n})}{(1-x^n)}$$

(9.2B) is proved as follows.

Putting $p_1 = +\sqrt{ax}$ and letting $p_2 \to \infty$ (9.2) becomes

$$1 + \sum_{n=1}^{\infty} \frac{(\sqrt{ax})_n \{ax^3\}_{n-1} (\sqrt{ax})^n (-1)^n x^{\frac{n}{2}(n-1)}}{x_n! (ax)_{2n-1}}$$

$$= \prod_{m=1}^{\infty} \left[\frac{(1-a^{\frac{1}{2}}x^{m-\frac{1}{2}})}{(1-ax^m)} \right] \times \left[1 + \sum_{n=1}^{\infty} \frac{\{ax^3\}_{n-1}(1-ax^{6n}) a^{4n} x^{\frac{3n}{2}(n+1)} x^{\frac{3n}{2}(n-1)}}{x_n! \, (\sqrt{ax})^{3n}} \right],$$

(9.2D)

and putting $a = x^3$, we obtain

$$1 + \sum_{n=1}^{\infty} \frac{(x^2)_n \{x^4\}_{n-1} (-1)^n x^{\frac{n^2}{2} + \frac{3n}{2}}}{x_n! \, (x^4)_{2n-1}} = \prod_{m=1}^{\infty} \frac{(1-x^{m+1})}{(1-x^{m+3})} \left[1 + \sum_{n=1}^{\infty} \frac{\{x^6\}_{n-1} (1-x^{6n+3}) x^{9n^2+12n}}{x_n^3! \, (x^2)^{3n}} \right]$$

Now, divide both sides by $(1 - x^2)$

$$\text{R.H.S.} = (1-x^3)\left[1 + \sum_{n=1}^{\infty} \frac{(1-x^{6n+3}) x^{9n^2+6n}}{(1-x^3)} \right]$$

$$= \left[1 - x^3 + \sum_{n=1}^{\infty} x^{9n^2+6n} - \sum_{n=1}^{\infty} x^{9n^2+12n+3} \right]$$

$$= \sum_{n=0}^{\infty} x^{9n^2+6n} - \sum_{n=1}^{\infty} x^{9n^2-6n} \qquad\qquad \text{(putting } n+1 = N \text{ in the second sum)}$$

$$\text{L.H.S.} = \frac{1}{(1-x^2)} + \sum_{n=1}^{\infty} \frac{(x^2)_n \{x^6\}_{n-1} (-1)^n x^{\frac{n^2}{2}+\frac{3n}{2}}}{x_n! \, (1-x^2)(x^4)_{2n-1}}$$

$$= \frac{1}{(1-x^2)} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{\frac{1}{2}(n^2+3n)} x_n^3! \, (1-x^{n+1})}{(1-x)(1-x^2)(1-x^3)(x^4)_{2n-1}}$$

$$= \frac{1}{(1-x^2)} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{\frac{1}{2}(n^2+3n)} x_n^3!}{x_{2n+1}! \, (1+x^{n+1})}$$

$$\therefore \sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{1}{2}(n^2+3n)} x_n^3!}{(1+x^{n+1}) x_{2n+1}!} = \sum_{n=0}^{\infty} x^{9n^2+6n} - \sum_{n=1}^{\infty} x^{9n^2-6n}$$

$\underline{(9.2C) \text{ is } \underline{\text{proved as follows}}}$

In (9.2) take $p_1 = +\sqrt{ax}$ and let $p_2 \to \infty$ then we obtain (9.2D). In this take $a = 1$, then

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n (x^{\frac{1}{2}})_n \{x^{\frac{3}{2}}\}_{n-1} x^{\frac{n^2}{2}}}{x_n! \, (x)_{2n-1}} = \prod_{m=1}^{\infty} \frac{(1-x^{m-\frac{1}{2}})}{(1-x^m)} \left[1 + \sum_{n=1}^{\infty} \frac{\{x^3\}_{n-1} (1-x^{6n}) x^{\frac{9n^2}{2}}}{x_n^3! \, (x^{\frac{1}{2}})^{3n}} \right]$$

Now change x into x^2 and we obtain,

$$\text{R.H.S.} = \prod_{m=1}^{\infty} \frac{(1-x^{2m-1})}{(1-x^{2m})} \left[1 + \sum_{n=1}^{\infty} (1+x^{6n}) x^{9n^2-3n} \right]$$

$$= \prod_{m=1}^{\infty} \frac{(1-x^{2m-1})}{(1-x^{2m})} \left[\sum_{n=0}^{\infty} x^{9n^2+3n} + \sum_{n=1}^{\infty} x^{9n^2-3n} \right]$$

$$= \frac{(1-x)(1-x^3)(1-x^5)\cdots}{(1-x^2)(1-x^4)(1-x^6)\cdots} \left[\sum_{n=0}^{\infty} x^{9n^2+3n} + \sum_{n=1}^{\infty} x^{9n^2-3n} \right]$$

$$= \frac{x_\infty!!}{x_\infty^2!} \left[\sum_{n=0}^{\infty} x^{9n^2+3n} + \sum_{n=1}^{\infty} x^{9n^2-3n} \right]$$

When we change x into x^2

$(x^{\frac{1}{2}})_n$ becomes $(1-x)(1-x^3)(1-x^5)\ldots\ldots(1-x^{2n-1})$

$(x)_{2n-1}$ becomes $(1-x^2)(1-x^4)(1-x^6)\ldots\ldots(1-x^{4n-2})$

$\{x^3\}_{n-1}$ becomes $(1-x^6)(1-x^{12})(1-x^{18})\ldots(1-x^{6n-6})$ i.e $x^6_{n-1}!$

$\therefore (x^{\frac{1}{2}})_n / (x)_{2n-1}$ becomes $1/x_n!!\, x^4_{n-1}!$

$\therefore$ changing x into x^2, the L.H.S. becomes $1+\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n x^6_{n-1}!\, x^{n^2}}{x^2_n!\, x_n!!\, x^4_{n-1}!}$

$\therefore$ we obtain the identity

$$\sum_{n=0}^{\infty} x^{9n^2+3n} + \sum_{n=1}^{\infty} x^{9n^2-3n} = \dfrac{x^2_\infty!}{x_\infty!!}\left[1+ \sum_{n=1}^{\infty}\dfrac{(-1)^n x^6_{n-1}!\, x^{n^2}}{x^2_n!\, x_n!!\, x^4_{n-1}!}\right]$$

Results of the third type, but of even greater complexity, may be obtained by taking $p_1 = -\sqrt{a}, \rho_2 = +\sqrt{ax}, a = x^3$; $\rho_1 = -\sqrt{ax}, \rho_2 = +\sqrt{ax}, a = 1$; $a = x^3$; $\rho_1 = \sqrt{a}, \rho_2 = -\sqrt{a}$ & $a = x^3$.

(9.3)

Using the formula (9.3) the following results have been obtained.

Identities of the Rogers-Ramanujan Type

$p_1 \to \infty$, a = 1 gives

$$\sum_{n=0}^{\infty}\dfrac{x^{n^2+n}\, x_n!^3}{x_n!\, x_{2n+1}!} + 1 + \sum_{n=1}^{\infty}\dfrac{x^{n^2}\, x_{n-1}!^3}{x_n!\, x_{2n-1}!} = \prod_{n=1}^{\infty}\left(\dfrac{1-x^{9n}}{1-x^n}\right) + \prod_{n=1}^{\infty}\dfrac{(1-x^{27n-15})(1-x^{27n-12})(1-x^{27n})}{(1-x^n)},$$

$p_1 \to \infty$, $a = x^3$ gives

$$\sum_{n=0}^{\infty}\dfrac{x^{n^2+3n}\, x_n!^3}{x_n!\, x_{2n+2}!} = \prod_{n=1}^{\infty}\dfrac{(1-x^{27n-21})(1-x^{27n-6})(1-x^{27n})}{(1-x^n)},$$

$p_1 = -\sqrt{ax}$, $a = x^3$ gives

$$\sum_{n=0}^{\infty}\dfrac{x^{\frac{1}{2}(n^2+n)}\, x_n!^3}{(x_n!)^2\, x_{n-1}!!} + x\sum_{n=0}^{\infty}\dfrac{x^{\frac{1}{2}(n^2+3n)}\, x_n!^3}{x_n!\, x_{n-1}!\, x_{n-1}!!} = \prod_{n=1}^{\infty}\dfrac{(1-x^{6n})}{(1-x^n)(1-x^{2n-1})} + x\prod_{n=1}^{\infty}\dfrac{(1-x^{18n-15})(1-x^{18n-3})(1-x^{18n})}{(1-x^n)(1-x^{2n-1})}, \quad -(9.3A)$$

$p_1 = -\sqrt{a}$, $a = 1$ gives

$$2\sum_{n=0}^{\infty}\dfrac{x^{\frac{1}{2}(n^2+n)}\, x_n!^3}{(x_n!)^2\, x_{2n+1}!!} + 1 + 2\sum_{n=1}^{\infty}\dfrac{x^{\frac{1}{2}(n^2+n)}\, x_{n-1}!^3}{x_{n-1}!\, x_n!\, x_n!!} = 2\prod_{n=1}^{\infty}\dfrac{(1-x^{6n})}{(1-x^n)(1-x^{2n-1})} + \prod_{n=1}^{\infty}\dfrac{(1-x^{9n})(1-x^{18n-9})}{(1-x^n)(1-x^{2n-1})},$$

$p_1 = -\sqrt{a}$, $a = x^6$ gives

$$\sum_{n=0}^{\infty}\dfrac{x^{n^2+n}\, x_n!^6}{x_{2n+1}!\, x_{n+1}!^4} = \prod_{n=1}^{\infty}\dfrac{(1-x^{18n-9})(1-x^{36n})}{(1-x^{2n-1})(1-x^{4n})}.$$

Identities of false ϑ series type

$p_1 = +\sqrt{ax}$, $a = x^3$ gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{1}{2}(n^2+n)} 3!}{x_{2n+1}!} - x\sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{1}{2}(n^2+3n)} 3!}{(1+x^{n+1})x_{2n+1}!} = \sum_{n=0}^{\infty} x^{9n^2+3n} - \sum_{n=0}^{\infty} x^{9n^2-3n} - \sum_{n=0}^{\infty} x^{9n^2+6n+1} + \sum_{n=1}^{\infty} x^{9n^2-6n+1}$$

The only other results which have been obtained are linear combinations of identities as follows:—

$p_1 = -\sqrt{ax}$, $a = 1$ gives

$$\sum_{n=0}^{\infty} \frac{x^{n^2+2n} 6!}{x_{2n+1}! x_n^4!} + (1+x)\left[1 + \sum_{n=1}^{\infty} \frac{x^{n^2} 6!}{x_n^4! x_{n+1}^4!}\right] = \prod_{n=1}^{\infty} \frac{(1-x^{18n-9})(1-x^{3n})}{(1-x^{4n-1})(1-x^{4n})} + (1+x)\frac{1}{x}\prod_{n=1}^{\infty} \frac{(1-x^{3n-2})(1-x^{3n-15})(1-x^{3n})}{(1-x^{4n-1})(1-x^{4n})}$$

For $p_1 = -\sqrt{a}\,x$ $a = 1$, we obtain a linear combination of

$$\sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}(n^2+3n)} 3!}{x_n! x_{n+1}! x_{n+1}!} = \prod_{n=1}^{\infty} \frac{(1-x^{18n-15})(1-x^{18n-3})(1-x^{18n})}{(1-x^n)(1-x^{2n-1})} \quad,$$

$$\sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}(n^2+n)} 3!}{(x_n!)^2 x_{n+1}!!} = \prod_{n=1}^{\infty} \frac{(1-x^{6n})}{(1-x^n)(1-x^{2n-1})} \quad,$$

$$1 + 2\sum_{n=1}^{\infty} \frac{x^{\frac{1}{2}(n^2+n)} 3!}{x_{n-1}! x_n! x_n!!} = \prod_{n=1}^{\infty} \frac{(1-x^{9n})(1-x^{18n-9})}{(1-x^n)(1-x^{4n-1})} \quad.$$

If we take $p_1 = +\sqrt{a}$, $a = x^3$, we obtain a linear combination of two results of the third type, and similarly for $p_1 = +\sqrt{ax}$, $a = 1$.

To shew how these linear combinations of results arise, we obtain the formula (9.3A). In (9.3) put $p_1 = -\sqrt{ax}$, then we have

$$1 + \sum_{n=1}^{\infty} \frac{\{ax^3\}_{n-1}(-\sqrt{ax})_n a^n x^{\frac{1}{2}n(n-1)}}{x_n! (ax)_{2n-1} (\sqrt{ax})^n}$$

$$= \prod_{m=1}^{\infty} \frac{(1+a^{\frac{1}{2}}x^{m-\frac{1}{2}})}{(1-ax^m)}\left[1 + a + a^{\frac{1}{2}}x^{-\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{\{ax^3\}_{n-1}(1-ax^{6n})(1+ax^{6n}+a^2 x^{3n-\frac{1}{2}})a^{4n} x^{3n(3n-1)}}{x_n^3! (-1)^{3n}(\sqrt{ax})^{3n}}\right],$$

and putting $a = x^3$ this becomes

$$1 + \sum_{n=1}^{\infty} \frac{\{x^6\}_{n-1}(-x^2)_n x^{\frac{1}{2}n(n+1)}}{x_n! (x^4)_{2n-1}}$$

$$= \prod_{m=1}^{\infty} \frac{(1+x^{m+1})}{(1-x^{m+3})}\left[1 + x^3 + x + \sum_{n=1}^{\infty} \frac{\{x^6\}_{n-1}(1-x^{6n+3})(-1)^n(1+x^{6n+3}+x^{3n+1})x^{9n^2+3n}}{x_n^3!}\right]$$

R.H.S.

$$= \frac{1}{(1-x^3)} \prod_{m=1}^{\infty} \left(\frac{1+x^{m+1}}{1-x^{m+3}}\right)\left[1-x^6+x(1-x^3) + \sum_{n=1}^{\infty}(-1)^n(1+x^{3n+1}-x^{9n+4}-x^{12n+6})x^{9n^2+3n}\right]$$

$$= \frac{1}{(1-x^3)} \prod_{m=1}^{\infty} \left(\frac{1+x^{m+1}}{1-x^{m+3}}\right)\left[1-x^6+x(1-x^3)+\sum_{n=1}^{\infty}(-1)^n x^{9n^2+3n}-\sum_{n=1}^{\infty}(-1)^n x^{(n+1)(9n+6)}+x\sum_{n=1}^{\infty}(-1)^n x^{9n^2+6n}-x\sum_{n=1}^{\infty}(-1)^n x^{(n+1)(9n+3)}\right]$$

$$= \frac{1}{(1-x^3)} \prod_{m=1}^{\infty}\left(\frac{1+x^{m+1}}{1-x^{m+3}}\right)\sum_{n=-\infty}^{\infty} x^{9n^2+3n} + \frac{x}{(1-x^3)}\prod_{m=1}^{\infty}\left(\frac{1+x^{m+1}}{1-x^{m+3}}\right)\sum_{n=-\infty}^{\infty} x^{9n^2+6n}$$

$$= (1-x)^2 \prod_{n=1}^{\infty} \frac{(1-x^{6n})}{(1-x^n)(1-x^{6n-1})} + x(1-x)^2 \prod_{n=1}^{\infty} \frac{(1-x^{18n-15})(1-x^{18n-3})(1-x^{18n})}{(1-x^n)(1-x^{6n-1})}.$$

For the L.H.S.

$$1 + \sum_{n=1}^{\infty} \frac{\{x^6\}_{n-1}(-x^2)_n x^{\frac{1}{2}(n^2+n)}}{x_n! (x^4)_{2n-1}}$$

$$= 1 + (1-x)^2 \sum_{n=1}^{\infty} \frac{x^{\frac{1}{2}(n^2+n)} x_n!^3}{x_n! x_{n+1}! x_{n+1}!!} = (1-x)^2 \sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}(n^2+n)} x_n!^3}{x_n! x_{n+1}! x_{n+1}!!}[1-x^{n+1}+x^{n+1}]$$

$$= (1-x)^2 \sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}(n^2+n)} x_n!^3}{(x_n!)^2 x_{n+1}!!} + x(1-x)^2 \sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}(n^2+3n)} x_n!^3}{x_n! x_{n+1}! x_{n+1}!!}.$$

and dividing both sides by $(1-x)^2$ we obtain (9.3A)

Three other methods of obtaining linear combinations of known results are as follows

(1) In (9.2) by taking $p_1 \to -\infty$, $p_2 \to \infty$, $a = x^6$.
(2) In (9.2) by taking $p_1 = p_2 = x^6$, $a = x^{12}$.
(3) By taking $\alpha_n = (-1)^n(1-ax^{2n})(ax)_{n-1} a^n x^{\frac{1}{2}n(3n-1)}/x_n!$, $\beta_n = (ax)_{3n}/x_n! \{a^3 x^3\}_{2n}$

in (9.4) and using (9.6) to give the values of δ_n & β_n.

Results obtainable from Sears formula.

We now consider a formula which has been proved by D.B.Sears.
The formula is as follows:—

$$P\ S_1(a_1)_{2M}W_{2M-1}(a_1;-x) = a_2\,Q(a_2)S_1(a_1/a_2^2)_{2M}W_{2M-1}(a_2;-x) + idem\ (a_2;a_3,\ \ldots,a_{M+1}) - (10.1)$$

where the parameters are $a_1,a_2,\ \ldots,a_{2M}$ and $x = (pa_1)^n(a_1\ldots a_{2M})^{-1}$.

Also

$$P = \left\{\prod_{M+2}^{2M} G(pa_1/a_r,\,p/a_r)\right\}\left\{\prod_{1}^{M+1}{}'G(a_r,\,a_r/a_1)\right\}^{-1},$$

$$Q(a_2) = \left\{\prod_{M+2}^{2M} G(pa_2/a_r,\,pa_1/a_2a_r)\right\}\left\{\prod_{1}^{M+1}{}'G(a_r/a_2,\,a_1a_r/a_1)\right\}^{-1},$$

$$S_1(x) = \prod_{r=0}^{\infty}(1-xp^{2r})(1-x^{-1}p^{2r+2}),$$

$$G(a) = \prod_{r=0}^{\infty}(1-ap^r),$$

$$G(a_1,\ \ldots,a_M;b_1,\ \ldots b_N;) = \prod_{r=1}^{M}G(a_r)\Big/\prod_{r=1}^{N}G(b_r),$$

$$_{M+1}W_M(a_1;x) = {}_{M+1}\Phi_M\left[\begin{array}{ccccc} a_1, & a_2, & \ldots\ldots, & a_{M+1}; & x \\ pa_1/a_2, & \ldots\ldots, & pa_1/a_{M+1} \end{array}\right],$$

$$_{M+1}W_M(a_r;x) = {}_{M+1}\Phi_M\left[\begin{array}{cccccc} a_r^2/a_1, & a_r, & a_ra_2/a_1, & \ldots,a_ra_{r-1}/a_1,a_ra_{r+1}/a_1, & \ldots\ a_ra_{M+1}/a_1; & x \\ pa_r/a_1,pa_r/a_2, & \ldots\ldots\ldots\ldots\ldots & pa_r/a_{M+1} \end{array}\right]$$

for r = 2 to $M + 1$.

For the meaning of idem, the sum

$$\sum_{i=1}^{N} g(a_1,a_2,\ \ldots a_{i-1},a_i,a_{i+1},\ \ldots\ a_M)$$

is written as

$$g(a_1,a_2,\ \ldots\ a_M) + idem\ (a_1;\ a_2,a_3,\ \ldots\ a_N).$$

The second term of the sum coming from the first by interchange of $a_1 \leftrightarrow a_2$ etc.

We shall follow Sears and replace the q used in the notation of basic hypergeometric series by a p.

Sear's formula gives a relationship between (M+1) well-poised $_{2M}\Phi_{2M-1}$'s of perfectly general type. If however, we introduce the special form of second and third parameters into the basic series, the number of series is reduced by 2 and we obtain a relationship between (M-1) well-poised series $_{2M}\Phi_{2M-1}$ of the type

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The formula is as follows:—

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where the parameters are $a_1 a_2, \ldots , a_{2M} \ 4 \ x = (pa_1)^n (a_1 \ldots a_{2M})^{-1}$.
Also

$$P = \left\{ \prod_{M+2}^{2M} G(pa_1/a_r , p/a_r) \right\} \left\{ \prod_1^{M+1} G(a_r , a_r/a_1) \right\}^{-1} ,$$

$$Q(a_2) = \left\{ \prod_{M+2}^{2M} G(pa_2/a_r , pa_1/a_2 a_r) \right\} \left\{ \prod_1^{M+1} G(a_r/a_2 , a_1 a_r/a_1) \right\}^{-1} ,$$

$$S_1(x) = \prod_{r=0}^{\infty} (1 - x p^{2r})(1 - x^{-1} p^{2r+2}) ,$$

$$G(a) = \prod_{r=0}^{\infty} (1 - a p^r) ,$$

$$G(a_1, \ldots , a_M ; b_1, \ldots b_N ;) = \prod_{r=1}^{M} G(a_r) \Big/ \prod_{r=1}^{N} G(b_r) ,$$

$$_{M+1} W_M (a_1 ; x) = {}_{M+1}\Phi_M \left[\begin{matrix} a_1, & a_2, & \ldots \ldots, & a_{M+1} ; & x \\ pa_1/a_2, & \ldots \ldots, & pa_1/a_{M+1} & \end{matrix} \right] ,$$

$$_{M+1} W_M (a_r ; x) = {}_{M+1}\Phi_M \left[\begin{matrix} a_r^2/a_1, & a_r, & a_r a_2/a_1, & \ldots , a_r a_{r-1}/a_1, a_r a_{r+1}/a_1, & \ldots & a_r a_{M+1}/a_1 ; & x \\ pa_r/a_1, & pa_r/a_2, & \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots & pa_r/a_{M+1} & \end{matrix} \right]$$

for $r = 2$ to $M+1$.

For the meaning of idem, the sum

$$\sum_{i=1}^{N} q^{(a_1, a_2, \ldots a_{i-1}, a_i, a_{i+1}, \ldots a_M)}$$

is written as

$$g(a_1, a_2, \ldots a_M) + \text{idem}\,(a_1 ; a_2, a_3, \ldots a_N).$$

The second term of the sum coming from the first by interchange
of $a_1 \leftrightarrow a_2$ etc.

We shall follow Sears and replace the q used in the
notation of basic hypergeometric series by a p.

Sear's formula gives a relationship between (M+1) well-
poised $_{2M}\Phi_{2M-1}$'s of perfectly general type. If however, we
introduce the special form of second and third parameters into
the basic series, the number of series is reduced by 2 and we
obtain a relationship between (M—1) well-poised series
$$_{2M}\Phi_{2M-1} \qquad \text{of the type}$$

$${}_{2M}\Phi_{2M-1}\left[\begin{array}{c} a_1,\ p\sqrt{a_1},\ -p\sqrt{a_1},\ a_2,\ \ldots\ ,a_{2M-2}\ ;\ p^{M-2}\dfrac{a_1^{M-1}}{a_2\cdots a_{2M-2}} \\ \sqrt{a_1},\ -\sqrt{a_1},\ \dfrac{pa_1}{a_2},\ \ldots\ ,\ \dfrac{pa_1}{a_{2M-2}} \end{array}\right]$$

We are concerned here only with basic series,but analogous results for ordinary series could be obtained by considering the integral

$$\frac{1}{2\pi i}\int \frac{(\tfrac{1}{2}a_1+\zeta)\Gamma(a_1+\zeta)\Gamma(a_2+\zeta)\ldots\Gamma(a_{M-1}+\zeta)\Gamma(-\zeta)\Gamma(a_2-a_1-\zeta)\ldots\Gamma(a_{M-1}-a_1-\zeta)e^{i\pi\zeta}\,d\zeta}{\sin\pi(\tfrac{1}{2}a_1+\zeta)\Gamma(1-a_M-\zeta)\ldots\Gamma(1-a_{2M-2}+\zeta)\Gamma(1+a_1-a_M-\zeta)\ldots\Gamma(1+a_1-a_{2M-2}-\zeta)}$$

taken round a large circle which avoids the poles of the integrand. By considering residues at the poles of the integrand,we obtain a relationship between $(M-1)$ well-poised ${}_{2M-1}F_{2M-2}$ of the type:

$${}_{2M-1}F_{2M-2}\left[\begin{array}{c} a_1,\ 1+\tfrac{1}{2}a_1,\ a_2,\ \ldots\ ,\ a_{2M-2}\ ;\ 1 \\ \tfrac{1}{2}a_1,\ 1+a_1-a_2,\ \ldots\ ,\ 1+a_1-a_{2M-2} \end{array}\right].$$

This method was given by Whipple (16).

It should be noted that there is no relationship between the parameters in either of these cases. All the results connecting more than two $_rF$'s and those connecting 2 or more $_qF_s$'s given in the Tract, involve relations between the parameters.

The formulae given by Sears and Whipple are useful in the consideration of bilateral series of hypergeometric type, series which are infinite in both directions. These series are defined as follows

$$_pH_p\begin{bmatrix}\alpha_1, \alpha_2, \cdots, \alpha_p; \gamma \\ \rho_1, \rho_2, \cdots \rho_p\end{bmatrix} = \sum_{n=-\infty}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n \gamma^n}{(\rho_1)_n (\rho_2)_n \cdots (\rho_p)_n}$$

where $(a)_n$ & $(a)_{-n}$ have the usual interpretations viz:—

$$(a)_n = a(a+1)\cdots(a+n-1) \quad \& \quad (a)_{-n} = (-1)^n/(1-a)_n .$$

(As in the case of generalised hypergeometric series, z is omitted when it has the value 1.)

$$\& \quad _r\psi_r\begin{bmatrix}\alpha_1, \alpha_2, \cdots, \alpha_r; \gamma \\ \rho_1, \rho_2, \cdots, \rho_r\end{bmatrix} = \sum_{n=-\infty}^{\infty} \frac{(\alpha_1)_{p,n}(\alpha_2)_{p,n} \cdots (\alpha_r)_{p,n}\gamma^n}{(\rho_1)_{p,n}(\rho_2)_{p,n} \cdots (\rho_n)_{p,n}}$$

where $(a)_{p,n} = (1-a)(1-ap)\cdots(1-ap^{n-1})$ & $(a)_{p,-n} = 1/(1-a/p)(1-a/p^2)\cdots(1-a/p^n)$.

The bilateral series by which $_pH_p$ and $_r\psi_r$ are defined can be divided into two parts, one consisting of terms for which n is positive and zero, and the other of terms for which n is negative. Thus, we have immediately that

$$_pH_p\begin{bmatrix}\alpha_1, \alpha_2, \cdots \alpha_p; \gamma \\ \rho_1, \rho_2, \cdots \rho_p\end{bmatrix} = _{p+1}F_p\begin{bmatrix}1, \alpha_1, \alpha_2, \cdots \alpha_p; \gamma \\ \rho_1, \rho_2, \cdots \rho_p\end{bmatrix} + \frac{(1-\rho_1)\cdots(1-\rho_p)}{(1-\alpha_1)\cdots(1-\alpha_p)} \, _{p+1}F_p\begin{bmatrix}1, 2-\rho_1, 2-\rho_2, \cdots 2-\rho_p; \frac{1}{\gamma} \\ 2-\alpha_1, 2-\alpha_2, \cdots 2-\alpha_p\end{bmatrix} \quad (10.3)$$

The analogues result for basic bilateral series is easily seen to be

$$_r\psi_r\begin{bmatrix}\alpha_1, \alpha_2, \cdots, \alpha_r; \gamma \\ \rho_1, \rho_2, \cdots, \rho_r\end{bmatrix} = _{r+1}\Phi_r\begin{bmatrix}p, \alpha_1, \alpha_2, \cdots, \alpha_r; \gamma \\ \rho_1, \rho_2, \cdots, \rho_r\end{bmatrix}$$

$$+\gamma\frac{(1-p/\rho_1)\cdots(1-p/\rho_r)}{(1-p/\alpha_1)\cdots(1-p/\alpha_r)} \, _{r+1}\Phi_r\begin{bmatrix}p, p^2/\rho_1, p^2/\rho_2, \cdots p^2/\rho_r; \rho_1\cdots\rho_r \\ p^2/\alpha_1, p^2/\alpha_2, \cdots p^2/\alpha_r \quad \alpha_1\cdots\alpha_r\gamma\end{bmatrix}. \quad — (10.4)$$

Thus we see, that using Sear's formula and (10.4), we can express a well-poised $_{2M-2}\psi_{2M-2}$, with the special form of second and third parameters, in terms of $(M-3)$ well-poised $_{2M-2}\Phi_{2M-3}$. Similarly, Whipple's formula and (10.3) give a well-poised $_{2M-3}H_{2M-3}$, with the special form of second parameter, in terms of $(M-3)$ well-poised $_{2M-3}H_{2M-4}$.

These bilateral series do not seem to have received much investigation. In 1907, Dougall (8) gave the sums of a $_2H_2$ and $_5H_5$, though Dougall gave the results in terms of sums of products of gamma functions and not in terms of bilateral hypergeometric series. In 1935, Professor Bailey[(11)] gave the connection between Dougall's formula and the more general results for ordinary hypergeometric series, of which Dougall's results were special cases. In the same paper, Professor Bailey gave a method of obtaining transformations of bilateral series from transformations of terminating series both of the ordinary and the basic type. This method yielded several new results as well as those given by Dougall.

(10.1)

We first take $M = 5$ in Sear's formula, the parameters being $a_1, a_2, \ldots \ldots, a_{10}$. Since we are concerned with series which have the special forms of second and third parameters, we take $a_3 = p\sqrt{a_1}, a_4 = -p\sqrt{a_1}$, then, $S_1\left(\frac{a_1}{a_3}\right) = S_1\left(\frac{a_1}{a_4}\right) = 0$ and Sear's formula gives a relation between 4 well-poised $_{10}\Phi_9$'s viz :—

$$\prod_{r=0}^{\infty} \frac{(1-a_1 p^{r+1}/a_1)(1-a_1 p^{r+1}/a_8)(1-a_1 p^{r+1}/a_9)(1-a_1 p^{r+1}/a_{10})(1-p^{r+1}/a_7)(1-p^{r+1}/a_8)(1-p^{r+1}/a_9)(1-p^{r+1}/a_{10})}{(1-a_1 p^{r+1})(1-a_2 p^r)(1-a_3 p^r)(1-a_4 p^r)(1-a_2 p^r/a_1)(1-a_3 p^r/a_1)(1-a_4 p^r/a_1)}$$

$$\times \quad {}_{10}\Phi_9\left[\begin{array}{c} a_1, \ p\sqrt{a_1}, -p\sqrt{a_1}, \ a_2, \ a_3, \ a_4, \ a_7, \ a_8, \ a_9, \ a_{10} \ ; \quad x \\ \sqrt{a_1}, -\sqrt{a_1}, \ pa_1/a_2, pa_1/a_3, pa_1/a_4, pa_1/a_7, pa_1/a_8, pa_1/a_9, pa_1/a_{10} \end{array}\right]$$

$$= \prod_{r=0}^{\infty} \frac{a_2(1-a_1/a_2^2)(1-a_1 p^{r+1}/a_7)(1-a_1 p^{r+1}/a_8)(1-a_1 p^{r+1}/a_9)(1-a_1 p^{r+1}/a_{10})(1-a_1 p^{r+1}/a_2 a_7)(1-a_1 p^{r+1}/a_2 a_8)(1-a_1 p^{r+1}/a_2 a_9)(1-a_1 p^{r+1}/a_2 a_{10})}{(1-a_1 p^r/a_2^2)(1-a_3 p^r/a_2)(1-a_4 p^r/a_2)(1-a_2 p^r)(1-a_2 a_3 p^r/a_1)(1-a_2 a_4 p^r/a_1)}$$

$$\times \quad {}_{10}\Phi_9\left[\begin{array}{c} a_2^2/a_1, \ p a_2/\sqrt{a_1}, -p a_2/\sqrt{a_1}, \ a_2, \ a_2 a_3/a_1, \ a_2 a_4/a_1, \ a_2 a_7/a_1, \ a_2 a_8/a_1, \ a_2 a_9/a_1, \ a_2 a_{10}/a_1 \ ; \quad x \\ a_2/\sqrt{a_1}, -a_2/\sqrt{a_1}, p a_2/a_1, p a_2/a_3, p a_2/a_4, p a_2/a_7, p a_2/a_8, p a_2/a_9, p a_2/a_{10} \end{array}\right]$$

$$+ \ \mathrm{idem}(a_2; a_3, a_4)$$

where $x = p^3 a_1^3/a_2 a_3 a_4 a_7 a_8 a_9 a_{10}$.

Now, in this formula, take $a_2 = p$, then using (10.4) the first $_{10}\Phi_9$ on the right hand side combines with the $_{10}\Phi_9$ on the left hand side to give an $_8\psi_8$ and the other two $_{10}\Phi_9$'s reduce to $_8\Phi_7$'s. Thus,

$$\prod_{r=0}^{\infty} \frac{(1-a_1\rho^{r+1}/a_1)(1-a_1\rho^{r+1}/a_8)(1-a_1\rho^{r+1}/a_9)(1-a_1\rho^{r+1}/a_{10})(1-\rho^{r+1}/a_7)(1-\rho^{r+1}/a_8)(1-\rho^{r+1}/a_9)(1-\rho^{r+1}/a_{10})}{(1-a_1\rho^{r+1})(1-\rho^{r+1})(1-a_4\rho^r)(1-a_4\rho^r)(1-\rho^{r+1}/a_1)(1-a_3\rho^r/a_1)(1-a_4\rho^r/a_1)}$$

$$\times \, _8\psi_8 \left[\begin{array}{c} p\sqrt{a_1}, -p\sqrt{a_1}, a_3, a_4, a_7, a_8, a_9, a_{10} \quad ; \quad p^2 a_1^3/a_3a_4a_7a_8a_9a_{10} \\ \sqrt{a_1}, -\sqrt{a_1}, pa_1/a_3, pa_1/a_4, pa_1/a_7, pa_1/a_8, pa_1/a_9, pa_1/a_{10} \end{array} \right]$$

$$= \frac{a_3(1-a_1)}{a_1^2} \prod_{r=0}^{\infty} \frac{(1-a_3\rho^{r+1}/a_7)(1-a_3\rho^{r+1}/a_8)(1-a_3\rho^{r+1}/a_9)(1-a_3\rho^{r+1}/a_{10})(1-a_1\rho^{r+1}/a_3a_7)(1-a_1\rho^{r+1}/a_3a_8)(1-a_1\rho^{r+1}/a_3a_9)(1-a_1\rho^{r+1}/a_3a_{10})}{(1-a_1\rho^r/a_3)(1-\rho^{r+1}/a_3)(1-a_4\rho^r/a_3)(1-a_3\rho^r)(1-a_3\rho^{r+1}/a_1)(1-a_3^2\rho^r/a_1)(1-a_3a_4\rho^r/a_1)}$$

$$\times \, _8\Phi_7 \left[\begin{array}{c} a_3^2/a_1, pa_3/\sqrt{a_1}, -pa_3/\sqrt{a_1}, a_3a_4/a_1, a_3a_7/a_1, a_3a_8/a_1, a_3a_9/a_1, a_3a_{10}/a_1 \quad ; \quad p^2a_1^3/a_3a_4a_7a_8a_9a_{10} \\ a_3/\sqrt{a_1}, -a_3/\sqrt{a_1}, pa_3/a_4, pa_3/a_7, pa_3/a_8, pa_3/a_9, pa_3/a_{10} \end{array} \right]$$

$$+ \; idem \; (a_3; a_4). \tag{10.1A}$$

This formula, expresses a well-poised $_8\psi_8$, in terms of two well-poised $_8\Phi_7$'s . It is a generalisation of the result

$$\chi \left[a; b, c, d, e, f \right]$$

$$= \prod_{n=1}^{\infty} \left[\frac{(1-aq^n/cf)(1-aq^n/df)(1-aq^n/ef)(1-bq^{n-1})(1-aq^n/b)(1-bq^{n-1}/a)}{(1-q^n/c)(1-q^n/d)(1-q^n/e)(1-bq^{n-1}/a)(1-bq^n/b)(1-bq^{n-1}/f)} \right] \times \chi \left[\frac{f^2}{a}; \frac{bf}{a}, \frac{cf}{a}, \frac{df}{a}, \frac{ef}{a}, f \right] \tag{10.1B}$$

$+$ idem (b; f)

where

$$\chi \left[a; b, c, d, e, f \right] = \prod_{n=1}^{\infty} \left[\frac{(1-aq^n/b)(1-aq^n/c)(1-aq^n/d)(1-aq^n/e)(1-aq^n/f)(1-a^2q^{n+1}/bcdef)}{(1-aq^n)} \right]$$

$$\times \, _8\Phi_7 \left[\begin{array}{c} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \quad ; \quad a^2q^2/bcdef \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f \end{array} \right].$$

Putting $a_{10} = a_1$ in (10.1A) we obtain (10.1B). The formula (10.1B) was first obtained by Professor Bailey (17), who used a method entirely analogous to that used by Whipple when he obtained the corresponding formula for ordinary series.

The formula (10.1A) immediately suggests the problem of expressing an $_8\psi_8$ in terms of one $_8\Phi_7$. No method of doing this in the general case has been discovered, but it is perhaps worth while noting that certain particular $_8\psi_8$'s can be expressed in terms of one $_8\Phi_7$. ~~If~~ ~~in (10.1A)~~ ~~then there is a zero in the infinite product in front of~~ ~~the first~~ ~~, and we obtain an~~ ~~in which one~~

~~parameter is p times, another, in terms of an~~
~~Also,~~ If the $_8\psi_8$ terminates below i.e. if $a_3 = a_1 p^{-n}$;
then $\prod\limits_{r=0}^{\infty}(1 - a_3 p^r/a_1)$ = 0 and we see that an $_8\psi_8$ which terminates
below can be expressed in terms of an $_8\Xi_7$. This result can
be proved from first principles and is true in the general
case, i.e. an $_nH_n$ which terminates below can be expressed
in terms of an $_nH_{n-1}$. Thus, any formula giving a
transformation of a hypergeometric series, may be regarded
as giving a transformation of a bilateral series which
terminates below.

If in (10.1A) we take $a_{30} = a_1 p/a_3$ we obtain a relation
between a $_6\psi_6$ and a $_6\Xi_5$ viz :—

$$_6\psi_6 \left[\begin{array}{cccccc} p\sqrt{a_{11}}, & -p\sqrt{a_{11}}, & a_4 , & a_7 , & a_8, & a_9 ; & pa_1^2/a_4 a_7 a_8 a_9 \\ \sqrt{a_{11}}, & -\sqrt{a_{11}}, & pa_1/a_4, & pa_1/a_7, & pa_1/a_8, & pa_1/a_9 \end{array} \right]$$

$$= \prod_{r=0}^{\infty} \frac{(1-a_4 p^{r+1}/a_7)(1-a_4 p^{r+1}/a_8)(1-a_4 p^{r+1}/a_9)(1-a_1 p^{r+1}/a_4 a_7)(1-a_1 p^{r+1}/a_4 a_8)(1-a_1 p^{r+1}/a_4 a_9)(1-a_1 p^{r+1})(1-p^{r+1})(1-p^{r+1}/a_1)}{(1-a_1 p^{r+1}/a_4)(1-p^{r+1}/a_4)(1-a_4^2 p^{r+1}/a_1)(1-a_1 p^{r+1}/a_7)(1-a_1 p^{r+1}/a_8)(1-a_1 p^{r+1}/a_9)(1-p^{r+1}/a_7)(1-p^{r+1}/a_8)(1-p^{r+1}/a_9)}$$

$$\times {}_6\Xi_5 \left[\begin{array}{cccccc} a_4^2/a_1, & pa_4/\sqrt{a_{11}}, & -pa_4/\sqrt{a_{11}}, & a_4 a_7/a_1, & a_4 a_8/a_1, & a_4 a_9/a_1 ; & pa_1^2/a_4 a_7 a_8 a_9 \\ a_4/\sqrt{a_{11}}, & -a_4/\sqrt{a_{11}}, & pa_4/a_7, & pa_4/a_8, & pa_4/a_9 \end{array} \right] .$$

$$(10.1C)$$

Thus, we see that a general well-poised $_6\psi_6$ can be expressed
in terms of a $_6\Xi_5$. The $_6\Xi_5$ on the right of (10.1C) can be
summed and we obtain the formula

$$_6\psi_6 \left[\begin{array}{cccccc} p\sqrt{a_{11}}, & -p\sqrt{a_{11}}, & a_4 , & a_7 , & a_8, & a_9 , \quad \cdots ; & pa_1^2/a_4 a_7 a_8 a_9 \\ \sqrt{a_{11}}, & -\sqrt{a_{11}}, & pa_1/a_4, & pa_1/a_7, & pa_1/a_8, & pa_1/a_9 \end{array} \right]$$

$$= \prod_{r=0}^{\infty} \frac{(1-a_1 p^{r+1})(1-a_1 p^{r+1}/a_4 a_7)(1-a_1 p^{r+1}/a_4 a_8)(1-a_1 p^{r+1}/a_4 a_9)(1-a_1 p^{r+1}/a_7 a_9)(1-a_1 p^{r+1}/a_7 a_9)(1-a_1 p^{r+1}/a_8 a_9)(1-p^{r+1})(1-p^{r+1}/a_1)}{(1-a_1 p^{r+1}/a_4)(1-a_1 p^{r+1}/a_7)(1-a_1 p^{r+1}/a_8)(1-a_1 p^{r+1}/a_9)(1-p^{r+1}/a_4)(1-p^{r+1}/a_7)(1-p^{r+1}/a_8)(1-p^{r+1}/a_9)(1-a_1^2 p^{r+1}/a_4 a_7 a_8 a_9)} ,$$

which is a known result.

We can obtain a more general result than (10.1C) by
taking $a_3 = pa_1/a_4$ in (10.1A). Then, the $_8\psi_8$ reduces to
a $_6\psi_6$ and the two $_8\Xi_7$'s reduce to $_7\Xi_6$'s which combine
to give a $_6\psi_6$. Thus, we have

$$_6\psi_6\left[\begin{matrix} p\sqrt{a_1}, -p\sqrt{a_1}, a_7, a_8, a_9, a_{10} \; ; \; pa_1^2/a_7a_8a_9a_{10} \\ \sqrt{a_{11}}, -\sqrt{a_{11}}, pa_1/a_7, pa_1/a_8, pa_1/a_9, pa_1/a_{10} \end{matrix}\right]$$

$$= \prod_{r=0}^{\infty} \frac{(1-a_1p^{r+1})(1-p^{r+1}/a_1)(1-a_4p^{r+1}/a_7)(1-a_4p^{r+1}/a_8)(1-a_4p^{r+1}/a_9)(1-a_4p^{r+1}/a_{10})(1-a_1p^{r+1}/a_{a_7})(1-a_1p^{r+1}/a_{a_8})(1-a_1p^{r+1}/a_{a_9})(1-a_1p^{r+1}/a_{a_{10}})}{(1-a_4p^{r+2}/a_1)(1-a_4p^{r+1}/a_1)(1-a_1p^{r+1}/a_7)(1-a_1p^{r+1}/a_8)(1-a_1p^{r+1}/a_9)(1-a_1p^{r+1}/a_{10})(1-p^{r+1}/a_7)(1-p^{r+1}/a_8)(1-p^{r+1}/a_9)(1-p^{r+1}/a_{10})}$$

$$\times \; _6\psi_6\left[\begin{matrix} pa_4/\sqrt{a_{11}}, -pa_4/\sqrt{a_{11}}, a_4a_7/a_1, a_4a_8/a_1, a_4a_9/a_1, a_4a_{10}/a_1 \; ; \; pa_1^2/a_7a_8a_9a_{10} \\ a_4/\sqrt{a_{11}}, -a_4/\sqrt{a_{11}}, pa_4/a_7, pa_4/a_8, pa_4/a_9, pa_4/a_{10} \end{matrix}\right], \qquad (10.1D)$$

putting $a_{10} = a_4$ in (10.1D), we obtain (10.1C).

Probably the most useful formula obtained in connection with basic series is that which expresses an $_8\Phi_7$ in terms of 2 Saalschützian $_4\Phi_3$'s . We have seen that the form which this result takes when the $_8\Phi_7$ terminates was used by Professor Watson in his proof of the Rogers-Ramanujan identities and he has also used it to give alternative definitions of the mock theta functions of the third order. Consequently, if a similar result could be obtained for an $_8\psi_8$, the limiting cases of such a formula might be of interest. Unless there is a relation between the parameters, we have been unable to obtain a formula giving an $_8\psi_8$ in terms of 2 Saalschützian $_4\Phi_3$'s . The simplest formula (10.1F), gives an $_8\psi_8$ in terms of 3 Saalchutzian $_4\Phi_3$'s , but the formula (10.1G), in which an $_8\psi_8$ is expressed in terms of 4 Saalschützian $_4\Phi_3$'s, appears to be more useful when we generalise the two and three term relationships between $_8\Phi_7$'s . We therefore give both results.

If in the formula (10.1A) we use the formula

$$_8\Phi_7\left[\begin{matrix} a, p\sqrt{a}, -p\sqrt{a}, d, e, f, g, h \; ; \; a^2p^2/defgh \\ \sqrt{a}, -\sqrt{a}, ap/d, ap/e, ap/f, ap/g, ap/h \end{matrix}\right]$$

$$= \prod_{r=0}^{\infty} \frac{(1-ap^{r+1})(1-ap^{r+1}/fg)(1-ap^{r+1}/fh)(1-ap^{r+1}/gh)}{(1-ap^{r+1}/f)(1-ap^{r+1}/g)(1-ap^{r+1}/h)(1-ap^{r+1}/fgh)} \times \; _4\Phi_3\left[\begin{matrix} ap/de, f, g, h \; ; \; p \\ ap/d, ap/e, fgh/a \end{matrix}\right]$$

$$+ \prod_{r=0}^{\infty} \frac{(1-ap^{r+1})(1-ap^{r+1}/de)(1-fp^r)(1-gp^r)(1-hp^r)(1-a^2p^{r+2}/defgh)(1-a^2p^{r+2}/efgh)}{(1-ap^{r+1}/a)(1-ap^{r+1}/e)(1-ap^{r+1}/f)(1-ap^{r+1}/g)(1-ap^{r+1}/h)(1-a^2p^{r+2}/defgh)(1-fghp^{r+1}/a)}$$

$$\times \; _4\Phi_3\left[\begin{matrix} ap/gh, ap/fh, ap/fg, a^2p^2/defgh \; ; \; p \\ ap^2/fgh, a^2p^2/dfgh, a^2p^2/efgh \end{matrix}\right], \qquad (10.1E)$$

to transform the ${}_8\Phi_7$'s into ${}_4\Phi_3$'s , we obtain 3 or 4 Saalschützian series according as to whether we take a_3a_4/a_1 to be e or d, or f, g, or h.

Taking e $= a_3a_4/a_1$, d $= a_3a_7/a_1$, f $= a_3a_8/a_1$, g $= a_3a_9/a_1$, h $= a_3a_{10}/a_1$, a $= a_3^2/a_1$,

to transform the first ${}_8\Phi_7$ and the same values with idem $(a_3; a_4)$ to transform the second ${}_8\Phi_7$, we obtain

$$\prod_{r=0}^{\infty}\frac{(1-a_1p^{r+1}/a_7)(1-a_1p^{r+1}/a_8)(1-a_1p^{r+1}/a_9)(1-a_1p^{r+1}/a_{10})(1-p^{r+1}/a_7)(1-p^{r+1}/a_8)(1-p^{r+1}/a_9)(1-p^{r+1}/a_{10})}{(1-a_1p^{r+1})(1-p^{r+1})(1-a_3p^{r})(1-a_4p^{r})(1-p^{r+1}/a_1)(1-a_3p^{r}/a_1)(1-a_4p^{r}/a_1)}$$

$$\times\; {}_8\psi_8\left[\begin{array}{c} p\sqrt{a_1},\; -p\sqrt{a_1},\; a_3,\; a_4,\; a_7,\; a_8,\; a_9,\; a_{10}\;;\; p^2a_1^3/a_3a_4a_7a_8a_9a_{10} \\ \sqrt{a_1},\; -\sqrt{a_1},\; pa_1/a_3,\; pa_1/a_4,\; pa_1/a_7,\; pa_1/a_8,\; pa_1/a_9,\; pa_1/a_{10} \end{array}\right]$$

$$=\; -\frac{a_1}{a_3}\prod_{r=0}^{\infty}\frac{(1-a_1p^{r+1}/a_3a_7)(1-a_1p^{r+1}/a_3a_8)(1-a_1p^{r+1}/a_3a_9)(1-a_1p^{r+1}/a_3a_{10})}{(1-a_1p^{r}/a_3)(1-p^{r+1}/a_3)(1-a_4p^{r}/a_3)(1-a_3p^{r})(1-a_3p^{r}/a_1)(1-a_3a_4p^{r}/a_1)}$$

$$\times\;\left[\prod_{r=0}^{\infty}\frac{(1-a_1p^{r+1}/a_8a_9)(1-a_1p^{r+1}/a_9a_{10})(1-a_1p^{r+1}/a_8a_{10})(1-a_3p^{r+1}/a_7)}{(1-a_1^2p^{r+1}/a_7a_8a_9a_{10})}\; \times\; {}_4\Phi_3\left[\begin{array}{c} a_1p/a_4a_7,\; a_3a_8/a_1,\; a_3a_9/a_1,\; a_3a_{10}/a_1\;;\; p \\ a_3p/a_4,\; a_3p/a_7,\; a_3a_8a_9a_{10}/a_1^2 \end{array}\right]\right.$$

$$+\;\prod_{r=0}^{\infty}\frac{(1-a_1p^{r+1}/a_4a_7)(1-a_3a_8p^{r}/a_1)(1-a_3a_9p^{r}/a_1)(1-a_3a_{10}p^{r}/a_1)(1-a_1^2p^{r+1}/a_4a_8a_9a_{10})(1-a_1^2p^{r+1}/a_7a_8a_9a_{10})}{(1-a_3p^{r+1}/a_4)(1-a_1^2p^{r+2}/a_3a_4a_8a_9a_{10})(1-a_3a_8a_9a_{10}p^{r-1}/a_1^2)}$$

$$\times\;\left.{}_4\Phi_3\left[\begin{array}{c} a_1p/a_9a_{10},\; a_1p/a_8a_{10},\; a_1p/a_8a_9,\; a_1^3p^2/a_3a_4a_7a_8a_9a_{10}\;;\; p \\ a_1^2p^2/a_3a_8a_9a_{10},\; a_1^2p^2/a_4a_8a_9a_{10},\; a_1^2p^2/a_7a_8a_9a_{10} \end{array}\right]\right]$$

$+$ idem $(a_3; a_4)$.   (10.1F)

The second ${}_4\Phi_3$ on the right of (.10.1F) being symmetrical in a_3 and a_4 we see that the formula (10.1F) gives an ${}_8\psi_8$ in terms of 3 Saalschützian ${}_4\Phi_3$'s. Unfortunately, the two infinite products in front of the second ${}_4\Phi_3$ do not appear to be expressible as a single product. When $a_3 = a_4$, (10.1F) reduces to (10.1E).

If we take a_3a_4/a_1 as f (say) in (10.1E) we obtain

$$\prod_{r=0}^{\infty} \frac{(1-a_1 p^{r+1}/a_7)(1-a_1 p^{r+1}/a_8)(1-a_1 p^{r+1}/a_9)(1-a_1 p^{r+1}/a_{10})(1-p^{r+1}/a_7)(1-p^{r+1}/a_8)(1-p^{r+1}/a_9)(1-p^{r+1}/a_{10})}{(1-a_1 p^{r+1})(1-p^{r+1})(1-a_3 p^r)(1-a_4 p^r)(1-p^{r+1}/a_1)(1-a_3 p^r/a_1)(1-a_4 p^r/a_1)}$$

$$\times \; {}_8\psi_8 \left[\begin{array}{c} p\sqrt{a_1},\; -p\sqrt{a_1},\; a_3,\; a_4,\; a_7,\; a_8,\; a_9,\; a_{10} \quad ; \quad p^2 a_1^3/a_3 a_4 a_7 a_8 a_9 a_{10} \\ \sqrt{a_1},\; -\sqrt{a_1},\; pa_1/a_3,\; pa_1/a_4,\; pa_1/a_7,\; pa_1/a_8,\; pa_1/a_9,\; pa_1/a_{10} \end{array} \right]$$

$$= -\frac{a_1}{a_3} \prod_{r=0}^{\infty} \frac{(1-a_1 p^{r+1}/a_3 a_7)(1-a_1 p^{r+1}/a_3 a_8)(1-a_1 p^{r+1}/a_3 a_9)(1-a_1 p^{r+1}/a_3 a_{10})}{(1-a_1 p^r/a_3)(1-p^{r+1}/a_3)(1-a_4 p^r/a_3)(1-a_3 p^r)(1-a_3 p^r/a_1)(1-a_3 a_4 p^r/a_1)(1-a_3 p^{r+1}/a_4)(1-a_1^3 p^{r+2}/a_3 a_4 a_7 a_8 a_9 a_{10})}$$

$$\times \; \left[\prod_{r=0}^{\infty} \frac{(1-a_1 p^{r+1}/a_4 a_7)(1-a_1 p^{r+1}/a_4 a_8)(1-a_1 p^{r+1}/a_7 a_8)(1-a_3 p^r/a_9)(1-a_3 p^{r+1}/a_{10})(1-a_1^3 p^{r+1}/a_3 a_4 a_7 a_8 a_9 a_{10})}{(1-a_1^2 p^{r+1}/a_3 a_4 a_7 a_8)} \right.$$

$$\times \; {}_4\Phi_3 \left[\begin{array}{c} pa_1/a_9 a_{10},\; a_3 a_4/a_1,\; a_3 a_7/a_1,\; a_3 a_8/a_1 \;;\; \\ pa_3/a_9,\; pa_3/a_{10},\; a_3 a_4 a_7 a_8/a_1^2 \end{array} \right]$$

$$+ \prod_{r=0}^{\infty} \frac{(1-a_1 p^{r+1}/a_9 a_{10})(1-a_3 a_9 p^r/a_1)(1-a_3 a_8 p^r/a_1)(1-a_3 a_9 p^r/a_1)(1-a_1^2 p^{r+2}/a_4 a_8 a_{10})(1-a_1^2 p^{r+2}/a_4 a_7 a_8 a_9)}{(1-a_3 a_4 a_7 a_8 p^{r+1}/a_1^2)} \times {}_4\Phi_3 \left[\begin{array}{c} pa_1,\; pa_1,\; pa_1,\; p^2 a_1^3/a_7 a_8 a_9 a_{10},\; \dfrac{p^2 a_1^3}{a_3 a_4 a_7 a_8 a_9 a_{10}} \;; p \\ \dfrac{p^2 a_1^2}{a_3 a_4 a_7 a_8},\; \dfrac{p^2 a_1^2}{a_4 a_7 a_8 a_{10}},\; \dfrac{p^2 a_1^2}{a_4 a_7 a_8 a_9} \end{array} \right] \Bigg]$$

$$+ \text{ idem } (a_3 : a_4). \qquad (10.1G)$$

It does not appear to be possible to combine any of the ${}_4\Phi_3$'s which occur in (10.1F) unless there is a relationship between the parameters. The third ${}_4\Phi_3$ however comes from the first by interchange of a_4 and a_3. Now, it is known that (14) :—

$$\prod_{r=0}^{\infty} \frac{(1-ap^{r+1}/de)(1-ap^{r+1}/df)(1-ap^{r+1}/ef)(1-ap^{r+1}/b)(1-ap^r/c)(1-a^2 p^{r+2}/bcdef)}{(1-ap^{r+1}/def)}$$

$$\times \; {}_4\Phi_3 \left[\begin{array}{c} ap/bc,\; d,\; e,\; f \;;\; p \\ ap/b,\; ap/c,\; def/a \end{array} \right]$$

$$+ \prod_{r=0}^{\infty} \frac{(1-ap^{r+1}/bc)(1-ap^r)(1-ep^r)(1-fp^r)(1-a^2 p^{r+2}/bdef)(1-a^2 p^{r+1}/cdef)}{(1-def p^{r+1}/a)}$$

$$\times \; {}_4\Phi_3 \left[\begin{array}{c} ap/ef,\; ap/df,\; ap/de,\; a^2 p^2/bcdef \;;\; p \\ ap^2/def,\; a^2 p^2/bdef,\; a^2 p^2/cdef \end{array} \right]$$

$$= \prod_{r=0}^{\infty} \frac{(1-ap^{r+1}/cd)(1-ap^{r+1}/ce)(1-ap^{r+1}/cf)(1-ap^r/b)(1-bp^r)(1-a^2 p^{r+2}/bdef)}{(1-bp^r/c)} \times {}_4\Phi_3 \left[\begin{array}{c} ap/bd,\; ap/be,\; ap/bf,\; c \;;\; p \\ a^2 p^2/bdef,\; ap/c,\; pc/b \end{array} \right]$$

$$+ \text{ idem } (c;b) \text{ ——— } (10.1H)$$

Now, if in (10.1H) the parameters satisfy the relation $a^2 p^2 = bcdef$, then $\prod_{r=0}^{\infty}\left(1-\dfrac{a^2 p^{r+2}}{bcdef}\right) = 0$, and the second ${}_4\Phi_3$ on the left has sum unity. Thus if $a^2 p^2 = bcdef$ there is a relationship which involves only infinite products and the two ${}_4\Phi_3$'s on the right.

$\therefore$ In (10.1G) taking $a = \dfrac{a_1 p}{a_1^2}$ $\quad b = \dfrac{a_1 p}{a_3 a_7}$ $\quad c = \dfrac{a_1 p}{a_4 a_7}$ $\quad d = \dfrac{a_1 p}{a_7 a_8}$ $\quad e = \dfrac{a_1 p}{a_7 a_9}$

$f = \dfrac{a_1 p}{a_7 a_{10}}$ $\quad$ where $a_3 a_4 a_7 a_8 a_9 a_{10} = a_1^3 p$, $\quad$ we obtain

$$\prod_{r=0}^{\infty} (1-a_1 p^{r+1}/a_7 a_8)(1-a_1 p^{r+1}/a_7 a_9)(1-a_1 p^{r+1}/a_7 a_{10})(1-a_3 a_8 a_9 a_{10} p^r/a_1^2)(1-a_4 a_8 a_9 a_{10} p^r/a_1^2)$$

$$= \prod_{r=0}^{\infty} \frac{(1-a_4 a_8 p^r/a_1)(1-a_4 a_9 p^r/a_1)(1-a_4 a_{10} p^r/a_1)(1-a_3 p^{r+1}/a_7)(1-a_1 p^{r+1}/a_3 a_7)(1-a_3 a_8 a_9 a_{10} p^r/a_1^2)}{(1-a_4 p^r/a_3)}$$

$$\times \ _2\Phi_2 \left[\begin{array}{c} a_2 a_8/a_1, \ a_3 a_9/a_1, \ a_3 a_{10}/a_1 ; \ p \\ a_3 p/a_7, \ a_3 p/a_4 \end{array} \right]$$

$$+ \ \text{idem}\ (a_3 ; a_4) .$$

$$\text{—— (10.1I)}$$

$\therefore$ If the parameters satisfy the relationship $a_1^3 p = a_3 a_4 a_7 a_8 a_9 a_{10}$ we can eliminate the series $\ _2\Phi_2 \left[\begin{array}{c} a_4 a_8/a_1, \ a_4 a_9/a_1, \ a_4 a_{10}/a_1 ; \ p \\ a_4 p/a_7, \ a_4 p/a_3 \end{array} \right]$ between

(10.1I) and (10.1F) and we obtain

$$\prod_{r=0}^{\infty} \frac{\left(1-a_1 p^{r+1}/a_7\right)\left(1-a_1 p^{r+1}/a_8\right)\left(1-a_1 p^{r+1}/a_9\right)\left(1-a_1 p^{r+1}/a_{10}\right)\left(1-p^{r+1}/a_7\right)\left(1-p^{r+1}/a_8\right)\left(1-p^{r+1}/a_9\right)\left(1-p^{r+1}/a_{10}\right)}{\left(1-a_1 p^{r+1}\right)\left(1-p^{r+1}\right)\left(1-a_3 p^r\right)\left(1-a_4 p^r\right)\left(1-p^r/a_1\right)\left(1-a_3 p^r/a_1\right)\left(1-a_4 p^r/a_1\right)} \times {}_8\psi_8\left[\begin{matrix} p\sqrt{a_1}, -p\sqrt{a_1}, a_3, a_4, a_7, a_8, a_9, a_{10}; & p \\ \sqrt{a_1}, -\sqrt{a_1}, \frac{pa_1}{a_3}, \frac{pa_1}{a_4}, \frac{pa_1}{a_7}, \frac{pa_1}{a_8}, \frac{pa_1}{a_9}, \frac{pa_1}{a_{10}} \end{matrix}\right]$$

$$= \left[\frac{a_1}{a_4}\prod_{r=0}^{\infty}\frac{\left(1-a_1 p^{r+1}/a_8 a_9\right)\left(1-a_1 p^r/a_8 a_{10}\right)\left(1-a_1 p^{r+1}/a_9 a_{10}\right)\left(1-a_1 p^r/a_7\right)\left(1-a_1 p^{r+1}/a_4 a_1\right)\left(1-a_1 p^r/a_4 a_8\right)\left(1-a_1 p^r/a_4 a_9\right)\left(1-a_1 p^{r+1}/a_4 a_{10}\right)\left(1-a_4 a_8 p^r/a_1\right)\left(1-a_4 a_9 p^r/a_1\right)\left(1-a_4 a_{10} p^r/a_1\right)}{\left(1-a_3 a_4 p^r/a_1\right)\left(1-a_4 p^r/a_3\right)\left(1-a_1 p^r/a_4\right)\left(1-p^{r+1}/a_4\right)\left(1-a_4 p^r\right)\left(1-a_4 p^r/a_1\right)\left(1-a_3 a_1 p^r/a_1\right)\left(1-a_3 a_8 p^r/a_1\right)\left(1-a_3 a_9 p^r/a_1\right)\left(1-a_3 a_{10} p^r/a_1\right)}\right.$$

$$-\frac{a_1}{a_3}\prod_{r=0}^{\infty}\frac{\left(1-a_1 p^{r+1}/a_8 a_9\right)\left(1-a_1 p^{r+1}/a_8 a_{10}\right)\left(1-a_1 p^{r+1}/a_9 a_{10}\right)\left(1-a_3 p^{r+1}/a_7\right)\left(1-a_1 p^{r+1}/a_3 a_7\right)\left(1-a_1 p^{r+1}/a_3 a_8\right)\left(1-a_1 p^{r+1}/a_3 a_9\right)\left(1-a_1 p^{r+1}/a_3 a_{10}\right)}{\left(1-a_3 a_4 p^r/a_1\right)\left(1-a_4 p^r/a_3\right)\left(1-a_1 p^r/a_3\right)\left(1-p^{r+1}/a_3\right)\left(1-a_3 p^r\right)\left(1-a_3 p^{r+1}/a_1\right)\left(1-a_4 a_7 p^r/a_1\right)} \right]$$

$$\times {}_3\psi_2\left[\begin{matrix} a_3 a_8/a_1, a_3 a_9/a_1, a_3 a_{10}/a_1; & p \\ a_3 p/a_7, a_3 p/a_4 \end{matrix}\right]$$

$$+\left[\frac{-a_1}{a_3}\prod_{r=0}^{\infty}\frac{\left(1-a_1 p^{r+1}/a_3 a_7\right)\left(1-a_1 p^{r+1}/a_4 a_7\right)\left(1-a_3 a_4 p^r/a_1\right)\left(1-a_1 p^r/a_4 a_8\right)\left(1-a_1 p^{r+1}/a_3 a_9\right)\left(1-a_1 p^{r+1}/a_3 a_{10}\right)\left(1-a_3 a_7 p^r/a_1\right)\left(1-a_3 a_8 p^r/a_1\right)\left(1-a_3 a_4 p^r/a_1\right)\left(1-a_3 a_9 p^r/a_1\right)}{\left(1-a_3 a_4 p^r/a_1\right)\left(1-a_4 p^r/a_3\right)\left(1-a_3 p^{r+1}/a_4\right)\left(1-p^{r+1}\right)\left(1-a_1 p^r/a_3\right)\left(1-p^{r+1}/a_3\right)\left(1-a_3 p^r\right)\left(1-a_3 p^{r+1}/a_1\right)\left(1-a_1 p^r/a_4 a_7\right)}\right.$$

$$\left. + \text{idem} \; (a_3; a_4) \right]$$

$$\times {}_4\psi_3\left[\begin{matrix} p, a_1 p/a_9 a_{10}, a_1 p/a_8 a_{10}, a_1 p/a_8 a_9; & p \\ a_4 a_7 p/a_1, a_3 a_7 p/a_1, a_3 a_4 p/a_1 \end{matrix}\right]$$

$$\frac{-\dfrac{a_1}{a_4}\displaystyle\prod_{r=0}^{\infty}\left(1-\dfrac{a_1 p^{r+1}}{a_4 a_7}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_4 a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_4 a_9}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_4 a_{10}}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_8 a_9}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_8 a_{10}}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_9 a_{10}}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_7 a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_7 a_{10}}\right)\left(1-a_4 p^r\right)}{\displaystyle\prod_{r=0}^{\infty}\left(1-\dfrac{a_1 p^r}{a_4}\right)\left(1-\dfrac{p^{r+1}}{a_4}\right)\left(1-a_3 p^r\right)\left(1-a_4 p^r\right)\left(1-\dfrac{a_4 p^{r+1}}{a_1}\right)\left(1-\dfrac{a_3 a_4 p^r}{a_1}\right)\left(1-\dfrac{a_3 a_7 p^r}{a_1}\right)\left(1-\dfrac{a_3 a_8 p^r}{a_1}\right)\left(1-\dfrac{a_3 a_9 p^r}{a_1}\right)\left(1-\dfrac{a_3 a_{10} p^r}{a_1}\right)} \tag{10.1J}$$

The equation (10.1J) expresses a general well-poised Saalschützian ${}_8\psi_8$ in terms of a Saalschützian ${}_4\psi_3$ and a Saalschützian ${}_3\psi_2$. If $a_3 = a_4$ (10.1J) reduces to the form which (10.1E) takes when the ${}_8\psi_7$ is Saalschützian and well-poised.

We now write (10.1G) in the form

$$\frac{\prod_{r=0}^{\infty}\left(1-a_1\frac{p^{r+1}}{a_4}\right)\left(1-a_1\frac{p^{r+1}}{a_8}\right)\left(1-a_1\frac{p^{r+1}}{a_9}\right)\left(1-a_1\frac{p^{r+1}}{a_{10}}\right)\left(1-\frac{p^{r+1}}{a_4}\right)\left(1-\frac{p^{r+1}}{a_8}\right)\left(1-\frac{p^{r+1}}{a_9}\right)\left(1-\frac{p^{r+1}}{a_{10}}\right)\left(1-a_3a_4p^{r}\right)\left(1-\frac{a_1^3p^{r+2}}{a_3a_4a_7a_8a_9a_{10}}\right)\left(1-\frac{p^{r+1}}{a_3}\right)}{\prod_{r=0}^{\infty}\left(1-a_1p^{r+1}\right)\left(1-p^{r+1}\right)\left(1-a_4p^{r}\right)\left(1-\frac{p^{r+1}}{a_1}\right)\left(1-\frac{a_3p^{r}}{a_1}\right)\left(1-\frac{a_4p^{r}}{a_1}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_7}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_8}\right)}$$

$$\times\ _8\psi_8\left[\begin{matrix} p\sqrt{a_1},\ -p\sqrt{a_1},\ a_3,\ a_4,\ a_7,\ a_8,\ a_9,\ a_{10} & ; & \dfrac{p^2a_1^3}{a_3a_4a_7a_8a_9a_{10}}\\[4pt] \sqrt{a_1},\ -\sqrt{a_1},\ \dfrac{pa_1}{a_3},\ \dfrac{pa_1}{a_4},\ \dfrac{pa_1}{a_7},\ \dfrac{pa_1}{a_8},\ \dfrac{pa_1}{a_9},\ \dfrac{pa_1}{a_{10}} \end{matrix}\right]$$

$$=\frac{-a_1}{a_3}\cdot\frac{\prod_{r=0}^{\infty}\left(1-\frac{a_1p^{r+1}}{a_3a_9}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_{10}}\right)}{\prod_{r=0}^{\infty}\left(1-\frac{a_1p^{r}}{a_3}\right)\left(1-\frac{a_4p^{r}}{a_3}\right)\left(1-\frac{a_3p^{r+1}}{a_4}\right)\left(1-\frac{a_3p^{r+1}}{a_1}\right)}\qquad\qquad \mathrm{x}\ (A)$$

$$-\frac{a_1}{a_4}\cdot\frac{\prod_{r=0}^{\infty}\left(1-\frac{a_1p^{r+1}}{a_4a_7}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_9}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_{10}}\right)\left(1-a_3p^{r}\right)\left(1-\frac{p^{r+1}}{a_3}\right)}{\prod_{r=0}^{\infty}\left(1-\frac{a_1p^{r}}{a_4}\right)\left(1-\frac{p^{r+1}}{a_4}\right)\left(1-\frac{a_3p^{r}}{a_4}\right)\left(1-a_4p^{r}\right)\left(1-\frac{a_4p^{r+1}}{a_1}\right)\left(1-\frac{a_4p^{r+1}}{a_3}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_7}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_8}\right)}\qquad \mathrm{x}\ (B)$$

where (A) and (B) can be obtained from (10.1G).

Now, in the above formula, change a_1 to $\dfrac{pa_1^3}{a_3a_4a_7a_8}$, a_3 to $\dfrac{pa_1^2}{a_4a_7a_8}$, a_4 to $\dfrac{pa_1^2}{a_3a_7a_8}$, a_7 to $\dfrac{pa_1^2}{a_3a_4a_7}$, a_8 to $\dfrac{pa_1^2}{a_3a_4a_8}$, leaving a_9 & a_{10} unaltered, and we obtain

$$\frac{\prod_{r=0}^{\infty}\left(1-a_1\frac{p^{r+1}}{a_7}\right)\left(1-a_1\frac{p^{r+1}}{a_8}\right)\left(1-\frac{a_1^3p^{r+2}}{a_3a_4a_7a_8a_{10}}\right)\left(1-\frac{a_1^3p^{r+2}}{a_3a_4a_7a_8a_9}\right)\left(1-\frac{a_3a_4a_7p^{r}}{a_1^2}\right)\left(1-\frac{a_3a_4a_8p^{r}}{a_1^2}\right)\left(1-\frac{a_4a_7a_8p^{r}}{a_1^2}\right)\left(1-\frac{p^{r+1}}{a_9}\right)\left(1-\frac{p^{r+1}}{a_{10}}\right)\left(1-\frac{a_1p^{r+1}}{a_7a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_9a_{10}}\right)}{\prod_{r=0}^{\infty}\left(1-\frac{a_1^3p^{r+2}}{a_3a_4a_7a_8}\right)\left(1-p^{r+1}\right)\left(1-\frac{a_1^3p^{r+1}}{a_3a_4a_8}\right)\left(1-\frac{a_3a_4a_7a_8p^{r}}{a_1^3}\right)\left(1-\frac{a_3p^{r}}{a_1}\right)\left(1-\frac{a_4p^{r}}{a_1}\right)\left(1-\frac{a_4a_7p^{r}}{a_1}\right)\left(1-\frac{a_4a_8p^{r}}{a_1}\right)}.$$

$$\times\ _8\psi_8\left[\begin{matrix} p\sqrt{\dfrac{pa_1^3}{a_3a_4a_7a_8}},\ -p\sqrt{\dfrac{pa_1^3}{a_3a_4a_7a_8}},\ \dfrac{pa_1^2}{a_4a_7a_8},\ \dfrac{pa_1^2}{a_3a_7a_8},\ \dfrac{pa_1^2}{a_3a_4a_7},\ \dfrac{pa_1^2}{a_3a_4a_8},\ a_9,\ a_{10} & ; & \dfrac{pa_1}{a_9a_{10}}\\[6pt] \sqrt{\dfrac{pa_1^3}{a_3a_4a_7a_8}},\ -\sqrt{\dfrac{pa_1^3}{a_3a_4a_7a_8}},\ \dfrac{pa_1}{a_3},\ \dfrac{pa_1}{a_4},\ \dfrac{pa_1}{a_8},\ \dfrac{pa_1}{a_9},\ \dfrac{p^2a_1^3}{a_3a_4a_7a_8a_9},\ \dfrac{p a_1^3}{a_3a_4a_7a_8a_{10}} \end{matrix}\right]$$

$$=\frac{-a_1}{a_3}\cdot\frac{\prod_{r=0}^{\infty}\left(1-\frac{a_1p^{r+1}}{a_3a_9}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_{10}}\right)}{\prod_{r=0}^{\infty}\left(1-\frac{a_1p^{r}}{a_3}\right)\left(1-\frac{a_4p^{r}}{a_3}\right)\left(1-\frac{a_3p^{r+1}}{a_4}\right)\left(1-\frac{a_3p^{r+1}}{a_1}\right)}\qquad \mathrm{x}\ (A)$$

$$-\frac{a_1}{a_4}\cdot\frac{\prod_{r=0}^{\infty}\left(1-\frac{a_3a_7p^{r}}{a_1}\right)\left(1-\frac{a_3a_8p^{r}}{a_1}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_9}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_{10}}\right)\left(1-\frac{a_1^2p^{r+1}}{a_4a_7a_8}\right)\left(1-\frac{a_4a_7a_8p^{r}}{a_1^2}\right)}{\prod_{r=0}^{\infty}\left(1-\frac{a_1p^{r}}{a_4}\right)\left(1-\frac{a_3a_4a_8p^{r}}{a_1^2}\right)\left(1-\frac{a_3p^{r}}{a_4}\right)\left(1-\frac{a_1p^{r+1}}{a_4}\right)\left(1-\frac{a_4p^{r+1}}{a_1}\right)\left(1-\frac{a_4p^{r+1}}{a_3}\right)\left(1-\frac{a_4a_7p^{r}}{a_1}\right)\left(1-\frac{a_4a_8p^{r}}{a_1}\right)}.$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{x}\ (B)$$

Subtracting these two results and replacing the two $_4\Phi_3$'s
by an $_8\bar{\Phi}_7$ we obtain, after a little rearrangement of terms:—

$$\frac{\prod_{r=0}^{\infty}\left(1-\frac{a_1p^{r+1}}{a_9}\right)\left(1-\frac{a_1p^{r+1}}{a_{10}}\right)\left(1-\frac{p^{r+1}}{a_7}\right)\left(1-\frac{p^{r+1}}{a_8}\right)\left(1-a_3a_4p^{r}\right)\left(1-\frac{a_1^3p^{r+2}}{a_3a_4a_7a_8a_9a_{10}}\right)\left(1-\frac{p^{r+1}}{a_3}\right)}{\prod_{r=0}^{\infty}\left(1-a_1p^{r+1}\right)\left(1-a_4p^{r}\right)\left(1-\frac{p^{r+1}}{a_1}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_7}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_8}\right)}$$

$$\times\ _8\psi_8\left[\begin{array}{c} p\sqrt{a_1},\ -p\sqrt{a_1},\ a_3,\ a_4,\ a_7,\ a_8,\ a_9,\ a_{10}\ ;\ \dfrac{p^2a_1^3}{a_3a_4a_7a_8a_9a_{10}} \\[2mm] \sqrt{a_1},\ -\sqrt{a_1},\ \dfrac{pa_1}{a_3},\ \dfrac{pa_1}{a_4},\ \dfrac{pa_1}{a_7},\ \dfrac{pa_1}{a_8},\ \dfrac{pa_1}{a_9},\ \dfrac{pa_1}{a_{10}} \end{array}\right]$$

$$=\frac{\prod_{r=0}^{\infty}\left(1-\frac{a_1^3p^{r+2}}{a_3a_4a_7a_8a_9}\right)\left(1-\frac{a_1^3p^{r+2}}{a_3a_4a_7a_8a_{10}}\right)\left(1-a_3a_4a_7p^{r}\right)\left(1-a_3a_4a_8p^{r}\right)\left(1-a_4a_7a_8p^{r}\right)\left(1-\frac{a_1p^{r+1}}{a_7a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_9a_{10}}\right)}{\prod_{r=0}^{\infty}\left(1-\frac{a_1^3p^{r+2}}{a_3a_4a_7a_8}\right)\left(1-\frac{a_1^2p^{r+1}}{a_3a_7a_8}\right)\left(1-a_3a_4a_7a_8p^{r}\right)\left(1-\frac{a_4a_7p^{r}}{a_1}\right)\left(1-\frac{a_4a_8p^{r}}{a_1}\right)}$$

$$\times\ _8\psi_8\left[\begin{array}{c} p\sqrt{\dfrac{pa_1^3}{a_3a_4a_7a_8}},\ -p\sqrt{\dfrac{pa_1^3}{a_3a_4a_7a_8}},\ \dfrac{pa_1^2}{a_4a_7a_8},\ \dfrac{pa_1^2}{a_3a_7a_8},\ \dfrac{pa_1^2}{a_3a_4a_7},\ \dfrac{pa_1^2}{a_3a_4a_8},\ a_9,\ a_{10}\ ;\ \dfrac{pa_1}{a_9a_{10}} \\[2mm] \sqrt{\dfrac{pa_1^3}{a_3a_4a_7a_8}},\ -\sqrt{\dfrac{pa_1^3}{a_3a_4a_7a_8}},\ \dfrac{pa_1}{a_3},\ \dfrac{pa_1}{a_4},\ \dfrac{pa_1}{a_8},\ \dfrac{pa_1}{a_9},\ \dfrac{p^2a_1^3}{a_3a_4a_7a_8a_9},\ \dfrac{p^2a_1^3}{a_3a_4a_7a_8a_{10}} \end{array}\right]$$

$$+\left[\frac{\prod_{r=0}^{\infty}\left(1-\frac{a_3a_4p^{r}}{a_1}\right)\left(1-a_3a_8p^{r}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_7a_8}\right)\left(1-\frac{a_4a_7p^{r}}{a_1^2}\right)}{\prod_{r=0}^{\infty}\left(1-\frac{a_3a_7a_8p^{r}}{a_1^2}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_7a_8}\right)\left(1-\frac{a_4a_7p^{r}}{a_1}\right)\left(1-\frac{a_4a_8p^{r}}{a_1}\right)}\ -\ \frac{\prod_{r=0}^{\infty}\left(1-\frac{a_1p^{r+1}}{a_4a_7}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_8}\right)\left(1-a_3p^{r}\right)\left(1-\frac{p^{r+1}}{a_3}\right)}{\prod_{r=0}^{\infty}\left(1-\frac{p^{r+1}}{a_4}\right)\left(1-a_4p^{r}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_7}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_8}\right)}\right]$$

$$\times\frac{a_1}{a_4}\frac{\prod_{r=0}^{\infty}\left(1-\frac{a_1p^{r+1}}{a_4a_9}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_{10}}\right)\left(1-p^{r+1}\right)\left(1-\frac{a_3p^{r}}{a_1}\right)\left(1-\frac{a_4p^{r}}{a_1}\right)\left(1-\frac{a_7p^{r+1}}{}\right)\left(1-a_8p^{r+1}\right)\left(1-a_9p^{r+1}\right)\left(1-a_{10}p^{r+1}\right)\left(1-\frac{a_1^3p^{r+2}}{a_3a_4a_7a_8a_9a_{10}}\right)}{\prod_{r=0}^{\infty}\left(1-\frac{a_1p^{r}}{a_4}\right)\left(1-\frac{a_3p^{r}}{a_4}\right)\left(1-\frac{a_4p^{r+1}}{a_1}\right)\left(1-\frac{a_1p^{r+1}}{a_7}\right)\left(1-\frac{a_1p^{r+1}}{a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_9}\right)\left(1-\frac{a_1p^{r+1}}{a_{10}}\right)\left(1-\frac{a_4p^{r+1}}{a_1}\right)}$$

$$\times\ _8\bar{\Phi}_7\left[\begin{array}{c} \dfrac{a_4^2}{a_1},\ \dfrac{pa_4}{\sqrt{a_1}},\ \dfrac{-pa_4}{\sqrt{a_1}},\ \dfrac{a_3a_4}{a_1},\ \dfrac{a_4a_7}{a_1},\ \dfrac{a_4a_8}{a_1},\ \dfrac{a_4a_9}{a_1},\ \dfrac{a_4a_{10}}{a_1}\ ;\ \dfrac{p^2a_1^3}{a_3a_4a_7a_8a_9a_{10}} \\[2mm] \dfrac{a_4}{\sqrt{a_1}},\ -\dfrac{a_4}{\sqrt{a_1}},\ \dfrac{pa_4}{a_3},\ \dfrac{pa_4}{a_7},\ \dfrac{pa_4}{a_8},\ \dfrac{pa_4}{a_9},\ \dfrac{pa_4}{a_{10}} \end{array}\right]$$

The expression in square brackets can be expressed as
a single product using the formula [m]

$$\prod_{r=0}^{\infty}\left(1-\frac{a p^{r+1}}{b}\right)\left(1-\frac{b p^{r}}{a}\right)\left(1-\frac{a p^{r+1}}{ef}\right)\left(1-\frac{ef p^{r}}{a}\right)\left(1-\frac{a p^{r+1}}{df}\right)\left(1-\frac{df p^{r}}{a}\right)\left(1-\frac{a p^{r+1}}{bda}\right)\left(1-\frac{bda p^{r}}{a}\right)$$

$$=\prod_{r=0}^{\infty}\left(1-\frac{a p^{r+1}}{f}\right)\left(1-\frac{f p^{r}}{a}\right)\left(1-\frac{a p^{r+1}}{ba}\right)\left(1-\frac{ba p^{r}}{a}\right)\left(1-\frac{a p^{r+1}}{bda}\right)\left(1-\frac{bda p^{r}}{a}\right)\left(1-\frac{a p^{r+1}}{daf}\right)\left(1-\frac{daf p^{r}}{a}\right).$$

$$-\frac{b}{a}\prod_{r=0}^{\infty}\left(1-d p^{r}\right)\left(1-\frac{p^{r+1}}{d}\right)\left(1-e p^{r}\right)\left(1-\frac{p^{r+1}}{e}\right)\left(1-\frac{b p^{r}}{f}\right)\left(1-\frac{f p^{r}}{b}\right)\left(1-\frac{a^2 p^{r+1}}{bdaf}\right)\left(1-\frac{bdaf p^{r}}{a^2}\right)$$

with $\dfrac{a}{b}=\dfrac{a_1}{a_4 a_7}$, $\dfrac{a}{ef}=\dfrac{a_1}{a_4 a_8}$, $\dfrac{a}{df}=\dfrac{a_1^2}{a_3 a_7 a_8}$, $\dfrac{a}{bda}=\dfrac{1}{a_3}$.

By analogy with the definitions of ψ [a; b, c, d, e, f] and χ [a; b, c, d, e, f] given by Whipple and Professor Bailey[(19)] we define

$$\Omega\left[a_1; a_3, a_4, a_7, a_8, a_9, a_{10}\right]$$

$$=\prod_{r=0}^{\infty}\left(1-\frac{a_1 p^{r+1}}{a_3}\right)\left(1-\frac{a_1 p^{r+1}}{a_4}\right)\left(1-\frac{a_1 p^{r+1}}{a_7}\right)\left(1-\frac{a_1 p^{r+1}}{a_8}\right)\left(1-\frac{a_1 p^{r+1}}{a_9}\right)\left(1-\frac{a_1 p^{r+1}}{a_{10}}\right)\left(1-\frac{a_1^3 p^{r+2}}{a_3 a_4 a_7 a_8 a_9 a_{10}}\right)\Big/\prod_{r=0}^{\infty}\left(1-a_1 p^{r+1}\right)\left(1-p^{r+1}\right)$$

$$\times {}_8\psi_8\left[\begin{array}{c} p\sqrt{a_1},\ -p\sqrt{a_1},\ a_3,\ a_4,\ a_7,\ a_8,\ a_9,\ a_{10}\ ;\ \dfrac{p^2 a_1^3}{a_3 a_4 a_7 a_8 a_9 a_{10}} \\ \sqrt{a_1},\ -\sqrt{a_1},\ \dfrac{p a_1}{a_3},\ \dfrac{p a_1}{a_4},\ \dfrac{p a_1}{a_7},\ \dfrac{p a_1}{a_8},\ \dfrac{p a_1}{a_9},\ \dfrac{p a_1}{a_{10}} \end{array}\right].$$

We thus obtain the formula

$$\dfrac{\displaystyle\prod_{r=0}^{\infty}\left(1-\frac{p^{r+1}}{a_7}\right)\left(1-\frac{p^{r+1}}{a_3}\right)\left(1-\frac{p^{r+1}}{a_8}\right)\left(1-\frac{a_3 a_4 p^{r}}{a_1}\right)}{\displaystyle\prod_{r=0}^{\infty}\left(1-\frac{p^{r+1}}{a_1}\right)\left(1-\frac{a_1 p^{r+1}}{a_3 a_7}\right)\left(1-\frac{a_1 p^{r+1}}{a_3 a_8}\right)\left(1-a_4 p^{r}\right)}\ \times\ \Omega\left[a_1; a_3, a_4, a_7, a_8, a_9, a_{10}\right]$$

$$=\dfrac{\displaystyle\prod_{r=0}^{\infty}\left(1-\frac{a_3 a_4 a_8 p^{r}}{a_1^2}\right)\left(1-\frac{a_4 a_4 a_9 p^{r}}{a_1^2}\right)\left(1-\frac{a_4 a_7 a_8 p^{r}}{a_1^2}\right)\left(1-\frac{a_1 p^{r+1}}{a_7 a_8}\right)}{\displaystyle\prod_{r=0}^{\infty}\left(1-\frac{a_4 a_9 p^{r}}{a_1}\right)\left(1-\frac{a_4 a_8 p^{r}}{a_1}\right)\left(1-\frac{a_1^2 p^{r+1}}{a_3 a_7 a_8}\right)\left(1-\frac{a_3 a_4 a_7 a_8 p^{r}}{a_1^2}\right)}\ \times\ \Omega\left[\dfrac{p a_1^3}{a_3 a_4 a_7 a_8}; \dfrac{p a_1^2}{a_4 a_7 a_8}, \dfrac{p a_1^2}{a_3 a_7 a_8}, \dfrac{p a_1^2}{a_3 a_4 a_7}, \dfrac{p a_1^2}{a_3 a_4 a_8}, a_9, a_{10}\right]$$

$$-\dfrac{\dfrac{a_4 a_7}{a_1}\displaystyle\prod_{r=0}^{\infty}\left(1-\frac{a_1 p^{r}}{a_7}\right)\left(1-\frac{a_1 p^{r+1}}{a_1}\right)\left(1-\frac{a_2 p^{r}}{a_4}\right)\left(1-\frac{a_4 p^{r+1}}{a_3}\right)\left(1-\frac{a_1 p^{r}}{a_8}\right)\left(1-\frac{a_8 p^{r+1}}{a_1}\right)\left(1-\frac{a_1^2 p^{r}}{a_3 a_4 a_7 a_8}\right)\left(1-\frac{a_3 a_4 a_3 a_8 p^{r+1}}{a_1^2}\right)}{\displaystyle\prod_{r=0}^{\infty}\left(1-a_4 p^{r}\right)\left(1-\frac{p^{r+1}}{a_4}\right)\left(1-\frac{a_1 p^{r}}{a_3 a_7}\right)\left(1-\frac{a_1 p^{r+1}}{a_3 a_8}\right)\left(1-\frac{a_4 a_9 p^{r}}{a_1}\right)\left(1-\frac{a_4 a_8 p^{r}}{a_1}\right)\left(1-\frac{a_3 a_7 a_8 p^{r}}{a_1^2}\right)\left(1-\frac{a_1^2 p^{r+1}}{a_3 a_7 a_8}\right)}$$

$$\times\ \dfrac{\displaystyle\prod_{r=0}^{\infty}\left(1-\frac{a_1 p^{r+1}}{a_4 a_{10}}\right)\left(1-\frac{a_1 p^{r+1}}{a_4 a_9}\right)\left(1-\frac{a_4 p^{r+1}}{a_3}\right)\left(1-\frac{a_3 p^{r+1}}{a_1}\right)}{\displaystyle\prod_{r=0}^{\infty}\left(1-\frac{p^{r+1}}{a_{10}}\right)\left(1-\frac{p^{r+1}}{a_9}\right)\left(1-\frac{a_3 p^{r}}{a_4}\right)\left(1-\frac{a_1 p^{r+1}}{a_3}\right)}\ \times\ \chi\left[\dfrac{a_4^2}{a_1}; \dfrac{a_3 a_4}{a_1}, \dfrac{a_4 a_7}{a_1}, \dfrac{a_4 a_8}{a_1}, \dfrac{a_4 a_9}{a_1}, \dfrac{a_4 a_{10}}{a_1}\right].$$

$$\text{———(10.1K)}$$

This formula (10.1K) is a generalisation of the formula

$$\chi \left[a; \; b, \; c, \; d, \; e, \; f \right] = \chi \left[a^2 p/def; \; b, c, ap/de, ap/df, ap/ef \right]$$

which can be obtained by putting $a_7 = a_1$ in the above result.

We now proceed to obtain from (10.1G) the generalisation of the basic analogue of Whipple's fundamental three term relationship.

Multiply the formula (10.1G) by $\dfrac{a_3}{a_1} \prod_{r=0}^{\infty} \left(\tfrac{1-a_1 p^r}{a_3} \right)\left(1-\tfrac{p^{r+1}}{a_3}\right)\left(1-a_3 p^r\right)\left(1-\tfrac{a_3 p^{r+1}}{a_1}\right)$ and write it as

[complex displayed equation]

$$\times \; {}_8\psi_8 \left[p\sqrt{a_1}, \; -p\sqrt{a_1}, \; a_3, \; a_4, \; a_7, \; a_8, \; a_9, \; a_{10} \; ; \; \dfrac{p^2 a_1^3}{a_3 a_4 a_7 a_8 a_9 a_{10}} \right]$$

$$= (c)$$

[complex displayed equation] $\quad \times \; (D)$

Replacing a_1 by $a_3^2 a_{10}/a_1^3$, a_3 by $a_3^2 a_{10}/a_1^2$, a_4 by $a_3 a_4 a_{10}/a_1^2$, a_7 by $a_3 a_7 a_{10}/a_1^2$, a_8 by $a_3 a_8 a_{10}/a_1^2$, a_9 by $a_3 a_9 a_{10}/a_1^2$, a_{10} by $a_3 a_{10}^2/a_1^2$, we have

[complex displayed equation]

$$\times \; {}_8\psi_8 \left[p\sqrt{\tfrac{a_3^2 a_{10}}{a_1^3}}, \; -p\sqrt{\tfrac{a_3^2 a_{10}}{a_1^3}}, \; \tfrac{a_3 a_{10}}{a_1^2}, \; \tfrac{a_3 a_4 a_{10}}{a_1^2}, \; \tfrac{a_3 a_7 a_{10}}{a_1^2}, \; \tfrac{a_3 a_8 a_{10}}{a_1^2}, \; \tfrac{a_3 a_9 a_{10}}{a_1^2}, \; \tfrac{a_3 a_{10}^2}{a_1^2} \; ; \right]$$

$$= (c)$$

[complex displayed equation] $\quad \times \; (D)$

Subtracting these two equations, we obtain

$$\frac{a_3}{a_1}\prod_{r=0}^{\infty}\left(1-a_1\frac{p^{r+1}}{a_7}\right)\left(1-a_1\frac{p^{r+1}}{a_8}\right)\left(1-a_1\frac{p^{r+1}}{a_9}\right)\left(1-a_1\frac{p^{r+1}}{a_{10}}\right)\left(1-\frac{p^r}{a_7}\right)\left(1-\frac{p^{r+1}}{a_8}\right)\left(1-\frac{p^{r+1}}{a_9}\right)\left(1-\frac{p^r}{a_{10}}\right)\left(1-\frac{a_1 p^r}{a_3}\right)\left(1-\frac{p^{r+1}}{a_3}\right)\left(1-a_3 p^r\right)\left(1-a_3\frac{p^{r+1}}{a_1}\right)$$

$$\overline{\prod_{r=0}^{\infty}\left(1-a_1 p^{r+1}\right)\left(1-p^{r+1}\right)\left(1-a_3 p^r\right)\left(1-a_4 p^r\right)\left(1-\frac{p^{r+1}}{a_1}\right)\left(1-a_3\frac{p^r}{a_1}\right)\left(1-a_4\frac{p^r}{a_1}\right)}$$

$$\times\ _8\psi_8\left[\begin{matrix}p\sqrt{a_1},\ -p\sqrt{a_1},\ a_3,\ a_4,\ a_7,\ a_8,\ a_9,\ a_{10}\ ;\ \dfrac{p^2 a_1^3}{a_3 a_4 a_7 a_8 a_9 a_{10}}\\ \sqrt{a_1},\ -\sqrt{a_1},\ \dfrac{p a_1}{a_3},\ \dfrac{p a_1}{a_4},\ \dfrac{p a_1}{a_7},\ \dfrac{p a_1}{a_8},\ \dfrac{p a_1}{a_9},\ \dfrac{p a_1}{a_{10}}\end{matrix}\right]$$

$$-\frac{a_1}{a_{10}}\prod_{r=0}^{\infty}\frac{\left(1-\frac{a_{10}a_1 p^{r+1}}{a_1 a_7}\right)\left(1-\frac{a_3 a_{10}p^{r+1}}{a_1 a_8}\right)\left(1-\frac{a_3 a_{10}p^{r+1}}{a_1 a_9}\right)\left(1-\frac{a_{10}p^{r+1}}{a_1}\right)\left(1-\frac{a_1 p^{r+1}}{a_3 a_7 a_{10}}\right)\left(1-\frac{a_1^2 p^{r+1}}{a_3 a_8 a_{10}}\right)\left(1-\frac{a_1^2 p^{r+1}}{a_3 a_9 a_{10}}\right)\left(1-\frac{a_1^2 p^{r+1}}{a_3 a_{10}^2}\right)\left(1-\frac{a_{10}p^r}{a_1}\right)\left(1-\frac{a_1^2 p^{r+1}}{a_3^2 a_{10}}\right)\left(1-\frac{a_3 a_{10}p^r}{a_1}\right)\left(1-\frac{a_1 p^{r+1}}{a_{10}}\right)}{\prod_{r=0}^{\infty}\left(1-\frac{a_3^2 a_{10}p^{r+1}}{a_1^3}\right)\left(1-p^{r+1}\right)\left(1-\frac{a_3 a_{10}p^r}{a_1^2}\right)\left(1-\frac{a_3 a_4 a_{10}p^r}{a_1^2}\right)\left(1-\frac{a_1 p^{r+1}}{a_3^2 a_{10}}\right)\left(1-\frac{a_1 p^r}{a_{10}}\right)\left(1-\frac{a_1 a_4 p^r}{a_3 a_{10}}\right)}$$

$$\times\ _8\psi_8\left[\begin{matrix}p\sqrt{\dfrac{a_3^2 a_{10}}{a_1^3}},\ -p\sqrt{\dfrac{a_3^2 a_{10}}{a_1^3}},\ \dfrac{a_3^2 a_{10}}{a_1^2},\ \dfrac{a_3 a_4 a_{10}}{a_1^2},\ \dfrac{a_3 a_7 a_{10}}{a_1^2},\ \dfrac{a_3 a_8 a_{10}}{a_1^2},\ \dfrac{a_3 a_9 a_{10}}{a_1^2},\ \dfrac{a_3 a_{10}}{a_1^2}\ ;\ \dfrac{p^2 a_1^3}{a_3 a_4 a_8 a_9 a_{10}}\\ \sqrt{\dfrac{a_3^2 a_{10}}{a_1^3}},\ -\sqrt{\dfrac{a_3^2 a_{10}}{a_1^3}},\ \dfrac{p a_{10}}{a_1},\ \dfrac{p a_3 a_{10}}{a_1 a_4},\ \dfrac{p a_3 a_{10}}{a_1 a_7},\ \dfrac{p a_3 a_{10}}{a_1 a_8},\ \dfrac{p a_3 a_{10}}{a_1 a_9},\ \dfrac{p a_3}{a_1}\end{matrix}\right]$$

$$=\left[\frac{a_3}{a_4}\prod_{r=0}^{\infty}\frac{\left(1-\frac{a_1 p^{r+1}}{a_4 a_7}\right)\left(1-\frac{a_1 p^{r+1}}{a_4 a_8}\right)\left(1-\frac{a_1 p^{r+1}}{a_4 a_9}\right)\left(1-\frac{a_1 p^{r+1}}{a_4 a_{10}}\right)\left(1-\frac{a_{10}p^r}{a_1}\right)\left(1-\frac{a_1^2 p^{r+1}}{a_3^2 a_{10}}\right)\left(1-\frac{a_3 a_{10}p^r}{a_1^2}\right)\left(1-\frac{a_1 p^{r+1}}{a_{10}}\right)}{\prod_{r=0}^{\infty}\left(1-\frac{a_3 a_{10}p^r}{a_1 a_4}\right)\left(1-\frac{a_1^2 p^{r+1}}{a_3 a_4 a_{10}}\right)\left(1-\frac{a_3 p^r}{a_4}\right)\left(1-\frac{a_3 a_4 a_{10}p^r}{a_1^2}\right)\left(1-\frac{a_1 a_4 p^{r+1}}{a_3 a_{10}}\right)\left(1-\frac{a_3 a_4 p^r}{a_1}\right)\left(1-\frac{a_4 p^{r+1}}{a_1}\right)\left(1-\frac{a_1^3 p^{r+2}}{a_3 a_4 a_7 a_8 a_9 a_{10}}\right)}\right.$$

$$-\frac{a_3}{a_4}\prod_{r=0}^{\infty}\frac{\left(1-a_1\frac{p^{r+1}}{a_4 a_7}\right)\left(1-a_1\frac{p^{r+1}}{a_4 a_8}\right)\left(1-a_1\frac{p^{r+1}}{a_4 a_9}\right)\left(1-a_1\frac{p^{r+1}}{a_4 a_{10}}\right)\left(1-a_1\frac{p^r}{a_3}\right)\left(1-\frac{p^{r+1}}{a_3}\right)\left(1-a_3 p^r\right)\left(1-a_3\frac{p^{r+1}}{a_1}\right)}{\prod_{r=0}^{\infty}\left(1-a_1\frac{p^r}{a_4}\right)\left(1-\frac{p^{r+1}}{a_4}\right)\left(1-a_3\frac{p^r}{a_4}\right)\left(1-a_4 p^r\right)\left(1-a_4\frac{p^{r+1}}{a_1}\right)\left(1-a_3 a_4\frac{p^r}{a_1}\right)\left(1-a_4\frac{p^{r+1}}{a_3}\right)\left(1-\frac{a_1^3 p^{r+2}}{a_3 a_4 a_7 a_8 a_9 a_{10}}\right)}$$

$$\times\ (D)$$

$$(10.1L)$$

Now, multiply (10.1G) by

$$\frac{\dfrac{a_3}{a_1}\displaystyle\prod_{r=0}^{\infty}\left(1-\dfrac{a_1 p^r}{a_3}\right)\left(1-\dfrac{p^{r+1}}{a_3}\right)\left(1-a_3 p^r\right)\left(1-\dfrac{a_3 p^{r+1}}{a_1}\right)}{\displaystyle\prod_{r=0}^{\infty}\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_7}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_9}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_{10}}\right)}$$

and

write the resulting formula as

$$\frac{\dfrac{a_3}{a_1}\displaystyle\prod_{r=0}^{\infty}\left(1-\dfrac{a_1 p^{r+1}}{a_7}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_9}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_{10}}\right)\left(1-\dfrac{p^{r+1}}{a_7}\right)\left(1-\dfrac{p^{r+1}}{a_8}\right)\left(1-\dfrac{p^{r+1}}{a_9}\right)\left(1-\dfrac{p^{r+1}}{a_{10}}\right)\left(1-\dfrac{a_1 p^r}{a_3}\right)\left(1-\dfrac{p^{r+1}}{a_3}\right)\left(1-a_3 p^r\right)\left(1-\dfrac{a_3 p^{r+1}}{a_1}\right)}{\displaystyle\prod_{r=0}^{\infty}\left(1-a_1 p^{r+1}\right)\left(1-p^{r+1}\right)\left(1-a_3 p^r\right)\left(1-a_4 p^r\right)\left(1-\dfrac{p^r}{a_1}\right)\left(1-\dfrac{a_4 p^r}{a_1}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_7}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_9}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_{10}}\right)}$$

$$\times \; {}_8\psi_8\left[\begin{array}{c}p\sqrt{a_1},\,-p\sqrt{a_1},\,a_3,\,a_4,\,a_7,\,a_8,\,a_9,\,a_{10}\\[4pt]\sqrt{a_1},\,-\sqrt{a_1},\,\dfrac{pa_1}{a_3},\,\dfrac{pa_1}{a_4},\,\dfrac{pa_1}{a_7},\,\dfrac{pa_1}{a_8},\,\dfrac{pa_1}{a_9},\,\dfrac{pa_1}{a_{10}}\end{array}\right]$$

$$= (E)$$

$$\frac{-\dfrac{a_3}{a_4}\displaystyle\prod_{r=0}^{\infty}\left(1-\dfrac{a_1 p^{r+1}}{a_4 a_7}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_4 a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_4 a_9}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_4 a_{10}}\right)\left(1-\dfrac{a_1 p^r}{a_3}\right)\left(1-\dfrac{p^{r+1}}{a_3}\right)\left(1-a_3 p^r\right)\left(1-\dfrac{a_3 p^{r+1}}{a_1}\right)}{\displaystyle\prod_{r=0}^{\infty}\left(1-\dfrac{a_4 p^r}{a_1}\right)\left(1-\dfrac{p^{r+1}}{a_4}\right)\left(1-\dfrac{a_3 p^r}{a_4}\right)\left(1-a_4 p^r\right)\left(1-\dfrac{a_4 p^{r+1}}{a_1}\right)\left(1-\dfrac{a_3 a_4 p^r}{a_1}\right)\left(1-\dfrac{a_4 p^{r+1}}{a_1}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3}\right)\left(1-\dfrac{a_1^2 p^{r+1}}{a_3 a_4 a_7 a_8 a_9 a_{10}}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_7}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_9}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_{10}}\right)}\times (D)$$

Now replacing a_4 by $\dfrac{a_1 a_7 a_8}{a_9 a_{10}}$, a_3 by $\dfrac{pa_1^2}{a_4 a_9 a_{10}}$, a_4 by $\dfrac{pa_1^2}{a_3 a_9 a_{10}}$,

a_9 by $\dfrac{a_7 a_8}{a_{10}}$, a_{10} by $\dfrac{a_7 a_8}{a_9}$ leaving a_7 and a_8 unaltered,

the above becomes

$$\frac{\dfrac{pa_1}{a_4 a_7 a_8}\displaystyle\prod_{r=0}^{\infty}\left(1-\dfrac{a_1 a_8 p^{r+1}}{a_9 a_{10}}\right)\left(1-\dfrac{a_1 a_7 p^{r+1}}{a_9 a_{10}}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_{10}}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_9}\right)\left(1-\dfrac{p^{r+1}}{a_7}\right)\left(1-\dfrac{p^{r+1}}{a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_7 a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_7 a_8}\right)\left(1-\dfrac{a_1 a_7 a_8 p^{r-1}}{a_1}\right)\left(1-\dfrac{a_1 a_7 a_8 p^r}{a_1^2}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_1^2}\right)\left(1-\dfrac{a_1^2 p^{r+2}}{a_4 a_9 a_{10}}\right)}{\displaystyle\prod_{r=0}^{\infty}\left(1-\dfrac{a_1 a_7 a_8 p^{r+1}}{a_9 a_{10}}\right)\left(1-p^{r+1}\right)\left(1-\dfrac{a_1^2 p^{r+1}}{a_4 a_9 a_{10}}\right)\left(1-\dfrac{a_1^2 p^{r+1}}{a_3 a_9 a_{10}}\right)\left(1-\dfrac{a_9 a_{10} p^{r+1}}{a_1 a_7 a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_4 a_7 a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_3 a_7 a_8}\right)\left(1-\dfrac{a_1 p^{r+1}}{a_1}\right)\left(1-\dfrac{a_4 p^r}{a_1}\right)\left(1-\dfrac{a_4 a_9 p^r}{a_1}\right)\left(1-\dfrac{a_4 a_9 p^r}{a_1}\right)\left(1-\dfrac{a_4 a_9 p^r}{a_1}\right)}$$

$$\times \; {}_8\psi_8\left[\begin{array}{c}p\sqrt{\dfrac{a_1 a_7 a_8}{a_9 a_{10}}},\,-p\sqrt{\dfrac{a_1 a_7 a_8}{a_9 a_{10}}},\,\dfrac{pa_1^2}{a_4 a_9 a_{10}},\,\dfrac{pa_1^2}{a_3 a_9 a_{10}},\,a_7,\,a_8,\,\dfrac{a_7 a_8}{a_{10}},\,\dfrac{a_7 a_8}{a_9};\,\dfrac{a_3 a_4}{a_1}\\[6pt]\sqrt{\dfrac{a_1 a_7 a_8}{a_9 a_{10}}},\,-\sqrt{\dfrac{a_1 a_7 a_8}{a_9 a_{10}}},\,\dfrac{a_4 a_9 a_{10}}{a_1},\,\dfrac{a_3 a_7 a_8}{a_1},\,\dfrac{a_1 a_8}{a_9 a_{10}},\,\dfrac{a_1 a_7}{a_9 a_{10}},\,\dfrac{pa_1}{a_9},\,\dfrac{pa_1}{a_{10}}\end{array}\right]$$

$$= (E)$$

$$\frac{-\dfrac{a_3}{a_4}\displaystyle\prod_{r=0}^{\infty}\left(1-\dfrac{a_3 a_9 p^r}{a_1}\right)\left(1-\dfrac{a_3 a_8 p^r}{a_1}\right)\left(1-\dfrac{a_3 a_{10} p^r}{a_1}\right)\left(1-\dfrac{a_3 a_9 p^r}{a_1}\right)\left(1-\dfrac{a_4 a_9 a_8 p^{r-1}}{a_1}\right)\left(1-\dfrac{a_4 a_9 a_4 p^r}{a_1}\right)\left(1-\dfrac{a_1^2 p^{r+1}}{a_1^2}\right)\left(1-\dfrac{a_1 p^{r+2}}{a_4 a_9 a_{10}}\right)}{\displaystyle\prod_{r=0}^{\infty}\left(1-\dfrac{a_3 a_7 a_8 p^{r-1}}{a_1}\right)\left(1-\dfrac{a_3 a_9 a_{10} p^r}{a_1}\right)\left(1-\dfrac{a_3 p^r}{a_4}\right)\left(1-\dfrac{a_1^2 p^{r+1}}{a_1^2}\right)\left(1-\dfrac{a_1 p^{r+2}}{a_1^2}\right)\left(1-\dfrac{a_1^3 p^{r+2}}{a_3 a_4 a_7 a_8 a_9 a_{10}}\right)\left(1-\dfrac{a_4 p^r}{a_3}\right)\left(1-\dfrac{a_3 a_4 p^r}{a_1}\right)\left(1-\dfrac{a_4 a_9 p^r}{a_1}\right)\left(1-\dfrac{a_4 a_9 a_8 p^r}{a_1}\right)\left(1-\dfrac{a_4 a_{10} p^r}{a_1}\right)}$$

$$\times (D).$$

Subtracting these two results, we obtain

$$\frac{\frac{a_3}{a_1}\prod\limits_{r=0}^{\infty}\left(1-\frac{a_1p^{r+1}}{a_7}\right)\left(1-\frac{a_1p^{r+1}}{a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_9}\right)\left(1-\frac{a_1p^{r+1}}{a_{10}}\right)\left(1-\frac{p^{r+1}}{a_7}\right)\left(1-\frac{p^{r+1}}{a_8}\right)\left(1-\frac{p^{r+1}}{a_9}\right)\left(1-\frac{p^{r+1}}{a_{10}}\right)\left(1-a_3p^{r}\right)\left(1-\frac{p^{r}}{a_3}\right)\left(1-a_1p^{r}\right)\left(1-\frac{a_3p^{r+1}}{a_1}\right)}{\prod\limits_{r=0}^{\infty}\left(1-a_1p^{r+1}\right)\left(1-p^{r+1}\right)\left(1-a_3p^2\right)\left(1-a_4p^{r}\right)\left(1-\frac{p^{r+1}}{a_1}\right)\left(1-\frac{a_3p^{r}}{a_1}\right)\left(1-\frac{a_1p^{r}}{a_1}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_7}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_9}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_{10}}\right)}$$

$$\times\ {}_8\psi_8\left[\begin{array}{c} p\sqrt{a_1},\ -p\sqrt{a_1},\ a_3,\ a_4,\ a_7,\ a_8,\ a_9,\ a_{10}\ ;\ \dfrac{p^2a_1^3}{a_3a_4a_7a_8a_9a_{10}}\\[4pt] \sqrt{a_1},\ -\sqrt{a_1},\ \dfrac{pa_1}{a_3},\ \dfrac{pa_1}{a_4},\ \dfrac{pa_1}{a_7},\ \dfrac{pa_1}{a_8},\ \dfrac{pa_1}{a_9},\ \dfrac{pa_1}{a_{10}} \end{array}\right]$$

$$\frac{-\frac{pa_1}{a_4a_7a_8}\prod\limits_{r=0}^{\infty}\left(1-\frac{a_1a_8p^{r+1}}{a_9a_{10}}\right)\left(1-\frac{a_1a_7p^{r+1}}{a_9a_{10}}\right)\left(1-\frac{a_1p^{r+1}}{a_9}\right)\left(1-\frac{a_1p^{r}}{a_{10}}\right)\left(1-\frac{p^{r+1}}{a_7}\right)\left(1-\frac{p^{r+1}}{a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_7a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_7a_8}\right)\left(1-a_4a_7a_8p^{r-1}\right)\left(1-a_4a_9a_{10}p^{r}\right)\left(1-\frac{a_1p^{r+1}}{a_1^2}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_9a_{10}}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_7a_8}\right)}{\prod\limits_{r=0}^{r+1}\left(1-\frac{a_1a_4p^{r+1}}{a_9a_{10}}\right)\left(1-p^{r+1}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_9a_{10}}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_9a_{10}}\right)\left(1-\frac{a_9a_{10}p^{r+1}}{a_1a_7a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_7a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_7a_8}\right)\left(1-\frac{a_4a_7p^{r}}{a_1}\right)\left(1-\frac{a_4a_8p^{r}}{a_1}\right)\left(1-\frac{a_4a_9p^{r}}{a_1}\right)\left(1-\frac{a_4a_{10}p^{r}}{a_1}\right)}$$

$$\times\ {}_8\psi_8\left[\begin{array}{c} p\sqrt{\dfrac{a_1a_7a_8}{a_9a_{10}}},\ -p\sqrt{\dfrac{a_1a_7a_8}{a_9a_{10}}},\ \dfrac{pa_1^2}{a_4a_9a_{10}},\ \dfrac{pa_1^2}{a_3a_9a_{10}},\ a_7,\ a_8,\ \dfrac{a_7a_8}{a_{10}},\ \dfrac{a_7a_8}{a_9}\ ;\ \dfrac{a_3a_4}{a_1}\\[6pt] \sqrt{\dfrac{a_1a_7a_8}{a_9a_{10}}},\ -\sqrt{\dfrac{a_1a_7a_8}{a_9a_{10}}},\ \dfrac{a_4a_7a_8}{a_1},\ \dfrac{a_3a_7a_8}{a_1},\ \dfrac{a_1a_8}{a_9a_{10}},\ \dfrac{a_1a_7}{a_9a_{10}},\ \dfrac{pa_1}{a_9},\ \dfrac{pa_1}{a_{10}} \end{array}\right]$$

$$\text{——— (10.1M)}$$

$$\frac{=\frac{a_3}{a_4}\left[\prod\limits_{r=0}^{\infty}\left(1-\frac{a_3a_7p^{r}}{a_1}\right)\left(1-\frac{a_3a_8p^{r}}{a_1}\right)\left(1-\frac{a_3a_9p^{r}}{a_1}\right)\left(1-\frac{a_3a_{10}p^{r}}{a_1}\right)\left(1-\frac{a_4a_7a_8p^{r-1}}{a_1}\right)\left(1-\frac{a_4a_9a_{10}p^{r}}{a_1^2}\right)\left(1-a_1p^{r+1}\right)\left(1-\frac{a_1p^{r+2}}{a_4a_9a_{10}}\right)\left(1-\frac{a_1p^{r}}{a_4a_7a_8}\right)\right]}{\prod\limits_{r=0}^{\infty}\left(1-\frac{a_3a_7p^{r-1}}{a_1}\right)\left(1-\frac{a_3a_4a_9a_{10}p^{r}}{a_1}\right)\left(1-\frac{a_1p^{r}}{a_4}\right)\left(1-\frac{a_1p^{r+1}}{a_1^2}\right)\left(1-\frac{a_1p^{r+1}}{a_1^3}\right)\left(1-\frac{a_1p^{r+2}}{a_3a_4a_7a_8a_9a_{10}}\right)\left(1-\frac{a_4a_8p^{r}}{a_3}\right)\left(1-\frac{a_4a_7p^{r}}{a_1}\right)\left(1-\frac{a_4a_9p^{r}}{a_1}\right)\left(1-\frac{a_4a_{10}p^{r}}{a_1}\right)}$$

$$-\frac{\prod\limits_{r=0}^{\infty}\left(1-\frac{a_1p^{r+1}}{a_4a_7}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_9}\right)\left(1-\frac{a_1p^{r+1}}{a_4a_{10}}\right)\left(1-\frac{a_1p^{r}}{a_3}\right)\left(1-\frac{p^{r+1}}{a_3}\right)\left(1-a_3p^{r}\right)\left(1-\frac{a_3p^{r+1}}{a_1}\right)}{\prod\limits_{r=0}^{\infty}\left(1-\frac{a_1p^{r}}{a_4}\right)\left(1-\frac{p^{r+1}}{a_4}\right)\left(1-a_3p^2\right)\left(1-a_4p^{r}\right)\left(1-\frac{a_1p^{r+1}}{a_1}\right)\left(1-\frac{a_3a_4p^{r}}{a_1}\right)\left(1-\frac{a_4p^{r}}{a_3}\right)\left(1-\frac{a_1p^{r+2}}{a_3a_4a_7a_8a_9a_{10}}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_7}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_8}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_9}\right)\left(1-\frac{a_1p^{r+1}}{a_3a_{10}}\right)}$$

$$\times\ (D).$$

Now eliminate (D) between (10.1M) and (10.1L) and we obtain ~~a generalisation~~ of

$$\prod_{n=1}^{\infty}\left(1-aq^{n}/def\right)\left(1-defq^{n-1}/a\right)\left(1-bdq^{n-1}/a\right)\left(1-beq^{n-1}/a\right)\left(1-q^{n}/c\right)\left(1-bfq^{n-1}/a\right)\times\ \chi\ \big[a;\ b,c,d,e,f\big]$$

$$=\prod_{n=1}^{\infty}\left(1-aq^{n}/b\right)\left(1-bq^{n-1}/a\right)\left(1-aq^{n}/ef\right)\left(1-aq^{n}/df\right)\left(1-aq^{n}/de\right)\left(1-a^2q^{n+1}/bcdef\right)\times\chi\ \big[ef/c\ ;\ e,f,\ aq/bc,\ aq/cd,\ ef/a\big]$$

$$+\prod_{n=1}^{\infty}\left(1-aq^{n}/bc\right)\left(1-dq^{n-1}\right)\left(1-eq^{n-1}\right)\left(1-fq^{n-1}\right)\left(1-a^2q^{n}/bdef\right)\left(1-bdefq^{n-1}/a^2\right)\times b/a\times\chi\ \big[b^2/a\ ;\ b,\ bc/a,\ bd/a,\ be/a,\ bf/a\big]$$

viz:-

$$\frac{a_3}{a_4}\left[\frac{\prod_{r=0}^{\infty}\left(1-a_3a_7\frac{p^r}{a_1}\right)\left(1-a_3a_8\frac{p^r}{a_1}\right)\left(1-a_3a_9\frac{p^r}{a_1}\right)\left(1-a_3a_{10}\frac{p^r}{a_1}\right)\left(1-a_1\frac{p^{r+2}}{a_4a_7a_8}\right)\left(1-a_4a_7a_{10}\frac{p^r}{a_1^2}\right)\left(1-a_1^2\frac{p^{r+1}}{a_4a_7a_9a_{10}}\right)\left(1-a_4a_7a_8\frac{p^{r-1}}{a_1}\right)}{\prod_{r=0}^{\infty}\left(1-a_3a_7a_8\frac{p^{r-1}}{a_1}\right)\left(1-a_3a_9a_{10}\frac{p^r}{a_1}\right)\left(1-a_1^2\frac{p^{r+1}}{a_3a_9a_{10}}\right)\left(1-a_1\frac{p^{r+2}}{a_3a_7a_8}\right)\left(1-a_4a_7\frac{p^r}{a_1}\right)\left(1-a_4a_8\frac{p^r}{a_1}\right)\left(1-a_4a_9\frac{p^r}{a_1^2}\right)\left(1-a_4\frac{p^r}{a_1^2}\right)}\right.$$

$$\left.-\frac{\prod_{r=0}^{\infty}\left(1-a_1\frac{p^{r+1}}{a_4a_7}\right)\left(1-a_1\frac{p^{r+1}}{a_4a_8}\right)\left(1-a_1\frac{p^{r+1}}{a_4a_9}\right)\left(1-a_1\frac{p^{r+1}}{a_4a_{10}}\right)\left(1-a_1\frac{p^r}{a_4}\right)\left(1-a_1^2\frac{p^{r+1}}{a_1}\right)\left(1-a_3a_{10}\frac{p^r}{a_1^2}\right)\left(1-a_1\frac{p^{r+1}}{a_{10}}\right)}{\prod_{r=0}^{\infty}\left(1-a_3a_{10}\frac{p^r}{a_1a_4}\right)\left(1-a_1^2\frac{p^{r+1}}{a_3a_4a_{10}}\right)\left(1-a_3a_4a_{10}\frac{p^{r+1}}{a_1^2}\right)\left(1-a_1a_4\frac{p^{r+1}}{a_3a_{10}}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_7}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_8}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_9}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_{10}}\right)}\right]$$

$$\times\frac{a_3}{a_1}\frac{\prod_{r=0}^{\infty}\left(1-a_1\frac{p^{r+1}}{a_7}\right)\left(1-a_1\frac{p^{r+1}}{a_8}\right)\left(1-a_1\frac{p^{r+1}}{a_9}\right)\left(1-a_1\frac{p^{r+1}}{a_{10}}\right)\left(1-\frac{p^{r+1}}{a_7}\right)\left(1-\frac{p^{r+1}}{a_8}\right)\left(1-\frac{p^{r+1}}{a_9}\right)\left(1-\frac{p^{r+1}}{a_{10}}\right)\left(1-a_3\frac{p^r}{a_1}\right)\left(1-\frac{p^{r+1}}{a_3}\right)}{\prod_{r=0}^{\infty}\left(1-a_1p^{r+1}\right)\left(1-a_4p^r\right)\left(1-\frac{p^{r+1}}{a_1}\right)\left(1-a_4\frac{p^r}{a_1}\right)}$$

$$\times\;_8\psi_8\left[\begin{array}{c}p\sqrt{a_1},\;-p\sqrt{a_1},\;a_3,\;a_4,\;a_7,\;a_8,\;a_9,\;a_{10}\;;\;\dfrac{p^2a_1^3}{a_3a_4a_7a_8a_9a_{10}}\\[4pt]\sqrt{a_1},\;-\sqrt{a_1},\;\dfrac{pa_1}{a_3},\;\dfrac{pa_1}{a_4},\;\dfrac{pa_1}{a_7},\;\dfrac{pa_1}{a_8},\;\dfrac{pa_1}{a_9},\;\dfrac{pa_1}{a_{10}}\end{array}\right]$$

$$-\frac{a_1}{a_4}\frac{\prod_{r=0}^{\infty}\left(1-a_1\frac{p^{r+1}}{a_4a_7}\right)\left(1-a_1\frac{p^{r+1}}{a_4a_8}\right)\left(1-a_1\frac{p^{r+1}}{a_4a_9}\right)\left(1-a_1\frac{p^{r+1}}{a_4a_{10}}\right)\left(1-a_3a_{10}\frac{p^r}{a_1^2}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_{10}}\right)\left(1-a_3a_4\frac{p^r}{a_1}\right)\left(1-a_1\frac{p^r}{a_3a_4}\right)\left(1-a_4\frac{p^r}{a_1}\right)\left(1-a_3\frac{p^r}{a_4}\right)\left(1-a_1\frac{p^r}{a_3a_{10}}\right)\left(1-a_3a_{10}\frac{p^r}{a_1}\right)}{\prod_{r=0}^{\infty}\left(1-a_1\frac{p^r}{a_4}\right)\left(1-\frac{p^{r+1}}{a_4}\right)\left(1-a_4p^r\right)\left(1-a_4\frac{p^{r+1}}{a_1}\right)\left(1-a_3a_{10}\frac{p^r}{a_1}\right)\left(1-a_1a_4\frac{p^{r+1}}{a_3a_{10}}\right)\left(1-a_1^2\frac{p^{r+1}}{a_3a_4a_{10}}\right)\left(1-a_3a_4a_{10}\frac{p^r}{a_1^2}\right)}$$

$$\times\frac{pa_1}{a_4a_7a_8}\frac{\prod_{r=0}^{\infty}\left(1-a_1a_8\frac{p^{r+1}}{a_9a_{10}}\right)\left(1-a_1a_9\frac{p^{r+1}}{a_9a_{10}}\right)\left(1-a_1\frac{p^{r+1}}{a_9}\right)\left(1-\frac{p^{r+1}}{a_9}\right)\left(1-\frac{p^{r+1}}{a_7}\right)\left(1-\frac{p^{r+1}}{a_8}\right)\left(1-a_{10}\frac{p^{r+1}}{a_7a_8}\right)\left(1-a_8\frac{p^{r+1}}{a_7a_8}\right)\left(1-a_1\frac{p^{r+2}}{a_4a_7a_8}\right)\left(1-a_4a_7a_8\frac{p^{r-1}}{a_1^2}\right)\left(1-a_4a_9a_{10}\frac{p^r}{a_1^2}\right)}{\prod_{r=0}^{\infty}\left(1-a_1a_7a_8\frac{p^{r+1}}{a_9a_{10}}\right)\left(1-a_1^2\frac{p^{r+1}}{a_3a_9a_{10}}\right)\left(1-a_9a_{10}\frac{p^{r+1}}{a_1a_7a_8}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_7a_8}\right)\left(1-a_4a_7a_8\frac{p^{r+1}}{a_1}\right)\left(1-a_4a_7\frac{p^r}{a_1}\right)\left(1-a_4a_8\frac{p^r}{a_1}\right)\left(1-a_4a_9\frac{p^r}{a_1}\right)\left(1-a_4a_{10}\frac{p^r}{a_1}\right)}$$

$$\times\;_8\psi_8\left[\begin{array}{c}p\sqrt{\dfrac{a_1a_7a_8}{a_9a_{10}}},\;-p\sqrt{\dfrac{a_1a_7a_8}{a_9a_{10}}},\;\dfrac{pa_1^2}{a_4a_9a_{10}},\;\dfrac{pa_1^2}{a_3a_9a_{10}},\;a_7,\;a_8,\;\dfrac{a_7a_8}{a_{10}},\;\dfrac{a_4a_8}{a_9}\;;\;\dfrac{a_3a_4}{a_1}\\[6pt]\sqrt{\dfrac{a_1a_7a_8}{a_9a_{10}}},\;-\sqrt{\dfrac{a_1a_7a_8}{a_9a_{10}}},\;\dfrac{a_4a_7a_8}{a_1},\;\dfrac{a_3a_7a_8}{a_1},\;\dfrac{a_1a_8}{a_9a_{10}},\;\dfrac{a_1a_7}{a_9a_{10}},\;\dfrac{pa_1}{a_9},\;\dfrac{pa_1}{a_{10}}\end{array}\right]$$

$$+\frac{a_3}{a_4}\frac{\prod_{r=0}^{\infty}\left(1-a_1\frac{p^{r+1}}{a_4a_7}\right)\left(1-a_1\frac{p^{r+1}}{a_4a_8}\right)\left(1-a_1\frac{p^{r+1}}{a_4a_9}\right)\left(1-a_1\frac{p^{r+1}}{a_4a_{10}}\right)\left(1-a_3\frac{p^r}{a_1}\right)\left(1-a_3\frac{p^{r+1}}{a_1}\right)\left(1-a_3p^r\right)\left(1-\frac{p^{r+1}}{a_3}\right)}{\prod_{r=0}^{\infty}\left(1-a_1\frac{p^r}{a_4}\right)\left(1-a_4\frac{p^{r+1}}{a_1}\right)\left(1-a_4p^r\right)\left(1-\frac{p^{r+1}}{a_4}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_7}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_8}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_9}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_{10}}\right)}$$

$$-\frac{\prod_{r=0}^{\infty}\left(1-a_3a_7\frac{p^r}{a_1}\right)\left(1-a_3a_8\frac{p^r}{a_1}\right)\left(1-a_3a_9\frac{p^r}{a_1}\right)\left(1-a_3a_{10}\frac{p^r}{a_1}\right)\left(1-a_4a_9a_{10}\frac{p^r}{a_1^2}\right)\left(1-a_1^2\frac{p^{r+1}}{a_4a_9a_{10}}\right)\left(1-a_4a_7a_8\frac{p^{r-1}}{a_1}\right)\left(1-a_1\frac{p^{r+2}}{a_4a_7a_8}\right)}{\prod_{r=0}^{\infty}\left(1-a_3a_7a_8\frac{p^{r-1}}{a_1}\right)\left(1-a_1\frac{p^{r+2}}{a_3a_7a_8}\right)\left(1-a_3a_9a_{10}\frac{p^r}{a_1}\right)\left(1-a_1^2\frac{p^{r+1}}{a_3a_9a_{10}}\right)\left(1-a_4a_7\frac{p^r}{a_1}\right)\left(1-a_4a_8\frac{p^r}{a_1}\right)\left(1-a_4a_{10}\frac{p^r}{a_1}\right)}$$

$$\times\frac{\prod_{r=0}^{\infty}\left(1-a_3a_{10}\frac{p^{r+1}}{a_1a_7}\right)\left(1-a_3a_{10}\frac{p^{r+1}}{a_1a_8}\right)\left(1-a_3a_{10}\frac{p^{r+1}}{a_1a_9}\right)\left(1-a_3\frac{p^r}{a_1}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_7a_{10}}\right)\left(1-a_1^2\frac{p^{r+1}}{a_3a_8a_{10}}\right)\left(1-a_1^2\frac{p^{r+1}}{a_3a_9a_{10}}\right)\left(1-a_1\frac{p^r}{a_3a_{10}^2}\right)\left(1-a_1^2\frac{p^{r+1}}{a_1}\right)}{\prod_{r=0}^{\infty}\left(1-a_3^2a_{10}\frac{p^{r+1}}{a_1^2}\right)\left(1-a_1\frac{p^{r+1}}{a_3a_{10}}\right)\left(1-a_3a_4a_{10}\frac{p^r}{a_1^2}\right)\left(1-a_1a_4\frac{p^r}{a_3a_{10}}\right)}$$

$$\times\;_8\psi_8\left[\begin{array}{c}p\sqrt{\dfrac{a_3^2a_{10}}{a_1^3}},\;-p\sqrt{\dfrac{a_3^2a_{10}}{a_1^3}},\;\dfrac{a_3a_{10}}{a_1^2},\;\dfrac{a_3a_4a_{10}}{a_1^2},\;\dfrac{a_3a_7a_{10}}{a_1^2},\;\dfrac{a_3a_8a_{10}}{a_1^2},\;\dfrac{a_3a_9a_{10}}{a_1^2},\;\dfrac{a_3a_{10}^2}{a_1^2}\;;\;\dfrac{p^2a_1^3}{a_3a_4a_7a_8a_9a_{10}}\\[6pt]\sqrt{\dfrac{a_3^2a_{10}}{a_1^3}},\;-\sqrt{\dfrac{a_3^2a_{10}}{a_1^3}},\;\dfrac{pa_{10}}{a_1},\;\dfrac{pa_3a_{10}}{a_1a_4},\;\dfrac{pa_3a_{10}}{a_1a_7},\;\dfrac{pa_3a_{10}}{a_1a_8},\;\dfrac{pa_3a_{10}}{a_1a_9},\;\dfrac{pa_3}{a_1}\end{array}\right].$$

Some miscellaneous results

(11)

The sum of a particular nearly-poised $_3F_2$ with argument -1

The method used to sum this series was suggested by Professor Watson's proof of Dixon's Theorem. (20)

$$\frac{\Gamma(A)\Gamma(b)\Gamma(c)}{\Gamma(1+a-b)\Gamma(1+a-c)} \times {}_3F_2\left[\begin{array}{c} A, b, c ; -1 \\ 1+a-b, 1+a-c \end{array}\right] = \sum_{n=0}^{\infty} \frac{\Gamma(A+n)\Gamma(b+n)\Gamma(c+n)(-1)^n}{n!\,\Gamma(1+a-b+n)\Gamma(1+a-c+n)}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(A+n)\Gamma(b+n)\Gamma(c+n)(-1)^n}{n!\,\Gamma(1+a+2n)\Gamma(1+a-b-c)} \times {}_2F_1\left[\begin{array}{c} b+n, c+n ; \\ 1+a+2n \end{array}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(A+n)\Gamma(b+n+m)\Gamma(c+n+m)(-1)^n}{n!\,m!\,\Gamma(1+a+2n+m)\Gamma(1+a-b-c)}$$

$$= \sum_{p=0}^{\infty} \frac{\Gamma(A)\Gamma(b+p)\Gamma(a+p)}{p!\,\Gamma(1+a+p)\Gamma(1+a-b-c)} \times {}_2F_1\left[\begin{array}{c} A, -p ; \\ 1+a+p \end{array}\, 1\,\right]$$

$$= \sum_{p=0}^{\infty} \frac{\Gamma(A)\Gamma(b+p)\Gamma(c+p)\Gamma(1+a-A+2p)}{p!\,\Gamma(1+a+2p)\Gamma(1+a-b-c)\Gamma(1+a-A+p)}$$

$$= \frac{\Gamma(A)\Gamma(b)\Gamma(c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \times {}_4F_3\left[\begin{array}{c} b, c, \frac{1+a-A}{2}, 1+\frac{a-A}{2} ; \\ 1+a-A, \frac{1+a}{2}, 1+\frac{a}{2} \end{array}\right]$$

now taking $b = 1+\frac{1}{2}a$ and $A = 2c-a+2$, we obtain

$$_3F_2\left[\begin{array}{c} 2c-a+2, 1+\frac{a}{2}, c ; -1 \\ \frac{a}{2}, 1+a-c \end{array}\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)} \times {}_3F_2\left[\begin{array}{c} c, a-c, a-c-\frac{1}{2} ; \\ \frac{1+a}{2}, 2a-c-1 \end{array}\right]$$

The $_3F_2$ on the right can be summed using Watson's Theorem (Tract § 3.3 (1)). and we obtain

$$_3F_2\left[\begin{array}{c} 2c-a+2, 1+\frac{a}{2}, c ; -1 \\ \frac{a}{2}, 1+a-c \end{array}\right] = \frac{2^{-c}\,\Gamma(\frac{1}{2})\Gamma(1+a-c)}{a\,\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(a-\frac{3c}{2})}, \qquad\text{—— (11.1)}.$$

provided that the series on the left converges i.e. if $R(2a-4c-1) > 0$.

If c is a negative even integer, the series terminates and we obtain

$$_3F_2\left[\begin{array}{c} 2-a-2n,\ 1+\frac{a}{2},\ -2n\ ;\ -1 \\ \frac{a}{2},\ 1+a+2n \end{array}\right] = \frac{(-4)^n\,(a)_{2n}\,(\frac{1}{2})_n}{(a)_{3n}}\ \text{for all } a. - (11.2)$$

Professor Bailey has pointed out that this result can be obtained from Tract § 4.6 (3) where we have the transformation

$$_3F_2\left[\begin{array}{c} a,\ b,\ c\ ;\ -1 \\ K-b, K-c \end{array}\right] = \frac{\Gamma(K-b)\Gamma(K-c)}{\Gamma(K)\Gamma(K-b-c)} \times\ _4F_3\left[\begin{array}{c} b,\ c,\ \frac{1}{2}(K-a),\ \frac{1}{2}(1+K-a)\ ; \\ K-a,\ \frac{1}{2}K,\ \frac{1}{2}(K+1) \end{array}\right]$$

Taking b = 1 + $\frac{1}{2}$A. K = 1 + A, the $_4F_3$ on the right reduces to a $_3F_2$ which can be summed by Watson's Theorem if a = 2c − A + 2, hence (11.1), but the existence of a direct proof may be of interest.

The sum of a particular non terminating Saalschützian ${}_8\Phi_7$

In the formula

$$_8\Phi_7\left[\begin{array}{c} a,\; q\sqrt{a},\; -q\sqrt{a},\; d,\; e,\; f,\; g,\; h\;;\; a^2q^2/defgh \\ \sqrt{a},\; -\sqrt{a},\; aq/d,\; aq/e,\; aq/f,\; aq/g,\; aq/h \end{array}\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1-aq^n/fg)(1-aq^n/fh)(1-aq^n/gh)}{(1-aq^n/f)(1-aq^n/g)(1-aq^n/h)(1-aq^n/fgh)}\right] \times {}_4\Phi_3\left[\begin{array}{c} aq/de,\; f,\; g,\; h\;;\; q \\ aq/d,\; aq/e,\; fgh/a \end{array}\right]$$

$$+ \prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1-aq^n/de)(1-fq^{n-1})(1-gq^{n-1})(1-hq^{n-1})(1-a^2q^{n+1}/dfgh)(1-a^2q^{n+1}/efgh)}{(1-aq^n/d)(1-aq^n/e)(1-aq^n/f)(1-aq^n/g)(1-aq^n/h)(1-a^2q^{n+1}/defgh)(1-fghq^{n-1}/a)}\right]$$

$$\times {}_4\Phi_3\left[\begin{array}{c} aq/gh,\; aq/fh,\; aq/fg,\; a^2q^2/defgh\;;\; q \\ aq^2/fgh,\; a^2q^2/dfgh,\; a^2q^2/efgh \end{array}\right]$$

take $h = aq/d$ and $e = a/fg$.

Then, we have :—

$$_6\Phi_5\left[\begin{array}{c} a,\; q\sqrt{a},\; -q\sqrt{a},\; a/fg,\; f,\; g\;;\; q \\ \sqrt{a},\; -\sqrt{a},\; fgq,\; aq/f,\; aq/g \end{array}\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1-aq^n/fg)(1-dq^{n-1}/fg)(1-dq^{n-1}/g)}{(1-aq^n/f)(1-aq^n/g)(1-dq^{n-1})(1-dq^{n-1}/fg)}\right] \times {}_2\Phi_1\left[\begin{array}{c} f,\; g\;;\; q \\ fgq \end{array}\right]$$

$$+ \left[\prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1-fgq^n/d)(1-fq^{n-1})(1-gq^{n-1})(1-aq^n/d)(1-aq^n/fg)(1-dq^n)}{(1-aq^n/d)(1-fgq^n)(1-aq^n/f)(1-aq^n/g)(1-dq^{n-1})(1-q^n)(1-fgq^{n-1}/d)}\right]\right]$$

$$\times {}_3\Phi_2\left[\begin{array}{c} d/g,\; d/f,\; q\;;\; q \\ dq/gf,\; dq \end{array}\right]$$

Now the ${}_6\Phi_5$ and the ${}_2\Phi_1$ can be summed using known results and we obtain

$$\frac{\prod_{n=1}^{\infty}(1-aq^n)(1-aq^n/fg)(1-gq^n)(1-fq^n)}{\prod_{n=1}^{\infty}(1-aq^n/f)(1-aq^n/g)(1-fgq^n)(1-q^n)}$$

$$=\frac{\prod_{n=1}^{\infty}(1-aq^n)(1-aq^n/fg)(1-aq^{n-1}/g)(1-aq^{n-1}/f)(1-fq^n)(1-gq^n)}{\prod_{n=1}^{\infty}(1-aq^n/f)(1-aq^n/g)(1-aq^{n-1})(1-aq^{n-1}/fg)(1-q^n)(1-fgq^n)}$$

$$+\frac{\prod_{n=1}^{\infty}(1-aq^n)(1-fgq^n/a)(1-fq^{n-1})(1-gq^{n-1})(1-aq^n/fg)(1-aq^n)}{\prod_{n=1}^{\infty}(1-fgq^n)(1-aq^n/f)(1-aq^n/g)(1-aq^{n-1})(1-q^n)(1-fgq^{n-1}/a)}$$

$$\times \, {}_3\Phi_2\left[\begin{array}{c} q, \ a/g, \ a/f \ ; \ q \\ aq, \ aq/gf \end{array}\right]$$

$$\therefore \, {}_3\Phi_2\left[\begin{array}{c} q, \ a/g, \ a/f \ ; \ q \\ aq, \ aq/gf \end{array}\right]$$

$$=\frac{(1-a)(1-fg/a)}{(1-g)(1-f)} - \frac{(1-fg/a)}{(1-f)(1-g)}\prod_{n=1}^{\infty}\frac{(1-aq^{n-1}/f)(1-aq^{n-1}/g)}{(1-aq^n)(1-aq^{n-1}/fg)}$$

and with a change of notation, this formula can be
written

$$ {}_3\Phi_2\left[\begin{array}{c} a, \ b, \ q \ ; \ q \\ c, \ abq^2/c \end{array}\right] =\frac{(1-c/q)(1-c/abq)}{(1-c/aq)(1-c/bq)} - \frac{(1-c/abq)}{(1-c/aq)(1-c/bq)}\prod_{n=1}^{\infty}\frac{(1-aq^{n-1})(1-bq^{n-1})}{(1-cq^{n-1})(1-abq^n/c)}. \qquad (12.1)$$

In the case of ordinary series, the corresponding
result can be obtained from the formula

$$ {}_3F_2\left[\begin{array}{c} a, \ b, \ e+f-a-b-1 \ ; \\ e, \ f \end{array}\right] = \frac{\Gamma(e)\Gamma(f)\Gamma(e-a-b)\Gamma(f-a-b)}{\Gamma(e-a)\Gamma(e-b)\Gamma(f-a)\Gamma(f-b)}$$

$$+\frac{1}{a+b-e}\frac{\Gamma(e)\Gamma(f)}{\Gamma(a)\Gamma(b)\Gamma(e+f-a-b)} \times \, {}_3F_2\left[\begin{array}{c} e-a, \ e-b \ , \ 1 \ ; \\ e-a-b+1, \ e+f-a-b \end{array}\right]$$

Tract § 3.8 (2)

taking a = 1, the ${}_3F_2$ on the right reduces to a ${}_2F_1$
which can be summed using the Gauss' Theorem.

$$\frac{\prod_{n=1}^{\infty}(1-aq^n)(1-aq^n/fg)(1-gq^n)(1-fq^n)}{\prod_{n=1}^{\infty}(1-aq^n/f)(1-aq^n/g)(1-fgq^n)(1-q^n)}$$

$$=\frac{\prod_{n=1}^{\infty}(1-aq^n)(1-aq^n/fg)(1-aq^{n-1}/g)(1-aq^{n-1}/f)(1-fq^n)(1-gq^n)}{\prod_{n=1}^{\infty}(1-aq^n/f)(1-aq^n/g)(1-aq^{n-1})(1-aq^{n-1}/fg)(1-q^n)(1-fgq^n)}$$

$$+\frac{\prod_{n=1}^{\infty}(1-aq^n)(1-fgq^n/a)(1-fq^{n-1})(1-gq^{n-1})(1-aq^n/fg)(1-aq^n)}{\prod_{n=1}^{\infty}(1-fgq^n)(1-aq^n/f)(1-aq^n/g)(1-aq^{n-1})(1-q^n)(1-fq^{n-1}/a)}$$

$$\times \; {}_3\Phi_2\left[\begin{array}{c} q, \; a/g, \; a/f \; ; \; q \\ aq, \; aq/gf \end{array}\right]$$

$$\therefore \; {}_3\Phi_2\left[\begin{array}{c} q, \; a/g, \; a/f \; ; \; q \\ aq, \; aq/gf \end{array}\right]$$

$$=\frac{(1-a)(1-fg/a)}{(1-g)(1-f)} - \frac{(1-fg/a)}{(1-f)(1-g)}\prod_{n=1}^{\infty}\frac{(1-aq^{n-1}/f)(1-aq^{n-1}/g)}{(1-aq^n)(1-aq^{n-1}/fg)}$$

and with a change of notation, this formula can be written

$${}_3\Phi_2\left[\begin{array}{c} a, \; b, \; q \; ; \; q \\ c, \; abq^2/c \end{array}\right]=\frac{(1-c/q)(1-c/abq)}{(1-c/aq)(1-c/bq)} - \frac{(1-c/abq)}{(1-c/aq)(1-c/bq)}\prod_{n=1}^{\infty}\frac{(1-aq^{n-1})(1-bq^{n-1})}{(1-cq^{n-1})(1-abq^n/c)} \; . \qquad (12.1)$$

In the case of ordinary series, the corresponding result can be obtained from the formula

$${}_3F_2\left[\begin{array}{c} a, \; b, \; e+f-a-b-1 \; ; \\ e, \; f \end{array}\right] = \frac{\Gamma(e)\Gamma(f)\Gamma(e-a-b)\Gamma(f-a-b)}{\Gamma(e-a)\Gamma(e-b)\Gamma(f-a)\Gamma(f-b)}$$

$$+\frac{1}{a+b-e}\frac{\Gamma(e)\Gamma(f)}{\Gamma(a)\Gamma(b)\Gamma(e+f-a-b)} \times {}_3F_2\left[\begin{array}{c} e-a, \; e-b \; , \; 1 \; ; \\ e-a-b+1, \; e+f-a-b \end{array}\right]$$

taking a = 1, the ${}_3F_2$ on the right reduces to a ${}_2F_1$ which can be summed using ~~the~~ Gauss' Theorem.

The sum of the general Saalschützian $_2\psi_2$

This result follows immediately from the previous result.

For,
$$_2\psi_2\left[\begin{array}{cc} a, & b\ ; \\ c, & abq^2/c \end{array}\ q\right]$$

$$= {}_3\bar{\Phi}_2\left[\begin{array}{c} a,\ b,\ q\ ; \\ c,\ abq^2/c \end{array}\ q\right] + \frac{q(1-q/c)(1-c/abq)}{(1-q/a)(1-q/b)} \times {}_3\bar{\Phi}_2\left[\begin{array}{c} q,\ q^2/c,\ c/ab\ ; \\ q^2/a,\ q^2/b \end{array}\ q\right]$$

and both these $_3\bar{\Phi}_2$'s can be summed using (12.1).

We have $\therefore$

$$_2\psi_2\left[\begin{array}{cc} a, & b\ ; \\ c, & abq^2/c \end{array}\ q\right]$$

$$= \frac{(1-c/q)(1-c/abq)}{(1-c/aq)(1-c/bq)} - \frac{(1-c/abq)}{(1-c/aq)(1-c/bq)}\prod_{n=1}^{\infty}\frac{(1-aq^{n-1})(1-bq^{n-1})}{(1-cq^{n-1})(1-abq^n/c)}$$

$$+ \frac{q(1-q/c)(1-c/abq)}{(1-q/a)(1-q/b)}\left[\frac{(1-q/b)(1-a/q)}{(1-c/bq)(1-aq/c)} - \frac{(1-a/q)}{(1-c/bq)(1-aq/c)}\prod_{n=1}^{\infty}\frac{(1-q^{n+1}/c)(1-cq^{n-1}/ab)}{(1-q^{n+1}/b)(1-q^n/a)}\right]$$

$$= \frac{q(c-abq)}{(aq-c)(bq-c)}\left[\prod_{n=1}^{\infty}\frac{(1-aq^{n-1})(1-bq^{n-1})}{(1-cq^{n-1})(1-abq^n/c)} + \frac{b(c-q)}{q(b-q)}\prod_{n=1}^{\infty}\frac{(1-q^{n+1}/c)(1-cq^{n-1}/ab)}{(1-q^{n+1}/b)(1-q^n/a)}\right]$$

$$= \frac{q(c-abq)}{(aq-c)(bq-c)}\left[\frac{\prod_{n=1}^{\infty}(1-aq^{n-1})(1-q^n/a)(1-bq^{n-1})(1-q^n/b) - \prod_{n=1}^{\infty}(1-abq^n/c)(1-cq^{n-1}/ab)(1-q^{n+1}/c)(1-cq^{n-1})}{\prod_{n=1}^{\infty}(1-cq^{n-1})(1-abq^n/c)(1-q^n/b)(1-q^n/a)}\right].$$

$$\text{------ (12.2)}$$

So far as is known, this result is a new one, the most general $_2\psi_2$ which has been summed being a well-poised one.

In the case of ordinary series the corresponding result would only be a particular case of the sum of the series

$$_2H_2 \begin{bmatrix} a, & b & ; \\ c, & d & \end{bmatrix}. \qquad -\ (8)$$

————

(13) A limiting case of the formula (10.1A)

Atkin & Swinnerton—Dyer, working on theory of numbers at Cambridge, established the following formula:—

$$\sum_{n=-\infty}^{\infty} (-1)^n w^{\frac{3n}{2}(n+1)} \left[\frac{\varsigma^{-3n}}{1-\gamma\varsigma^{-1}w^n} + \frac{\varsigma^{3n+3}}{1-\gamma\varsigma w^n} \right]$$

$$= \frac{\varsigma P(\varsigma^2,w)}{P(\varsigma,w)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n w^{\frac{3n}{2}(n+1)}}{1-\gamma w^n} + \frac{P(\varsigma,w)P(\varsigma^{-1},w)\{\prod_{r=1}^{\infty}(1-w^r)\}^2}{P(\gamma\varsigma^{-1},w)P(\gamma,w)P(\gamma\varsigma,w)} ,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (13.1)$$

where $P(z,w) = \prod_{r=0}^{\infty}(1-\gamma w^r)(1-\gamma^{-1}w^{r+1})$ $|w| < 1$.

They have used this formula to prove many of the results suggested by Dyson in his paper "Some guesses in the theory of partitions" (21) in addition to new results. Their formula may be regarded as giving the partial fractions of $\{ P(\gamma\varsigma^{-1},w)P(\gamma,w)P(\gamma\varsigma,w) \}^{-1}$ and it is really a particular case, though heavily disguised, of a formula giving the partial fractions of $e^{(2r-1)\gamma} / \vartheta_1(\gamma-d)\vartheta_1(\gamma-\beta)\vartheta_1(\gamma-\gamma)$ which was established by Professor Watson and used to derive numerous identities involving mock theta functions of the third order.(18)

Atkin & Swinnerton—Dyer arrived at their result by conjecture and then proved it to be true as follows:—
Writing (13.1) in the form $f(z,\varsigma,w) = 0$ they proved that
$f(w\gamma,\varsigma,w) + \gamma^3 f(\gamma,\varsigma,w) = 0$. If $|w| < |\varsigma| < 1, f$, qua function of z, has at most poles at $z = 1$, $z = \varsigma$, $z = w\varsigma^{-1}$ in $|w| < |\gamma| \le 1$ and evidently $z = 1$ is not a pole. Hence $f \equiv 0$. The result for general ς follows by analytic continuation.

We now prove that (13.1) is a limiting case of (10.1A).   In (10.1A), take $a_1=a$, $a_3 = \sqrt{\alpha p}$, $a_4 = z\sqrt{\alpha p}$, $a_7 = b$, $a_8 = c$, $a_9 = d$, $a_{10} = \sqrt{\alpha p}/\gamma$ and let b, c, $d \rightarrow \infty$. Then, since $a/a_{10}=a_4$, there is a parameter 1 in the numerator of the second $_8\Phi_7$ which therefore has unit sum. The $_8\psi_8$ reduces to a $_7\psi_7$ because $a_3 = \alpha p/a_3$

∴ we have

$$\prod_{r=0}^{\infty} \frac{(1-\zeta a^{\frac{1}{2}}p^{r+\frac{1}{2}})(1-\zeta p^{r+\frac{1}{2}}/a^{\frac{1}{2}})(1-a)}{(1-a p^r)(1-p^{r+1})(1-a^{\frac{1}{2}}p^{r+\frac{1}{2}})(1-\zeta a^{\frac{1}{2}}p^{r-\frac{1}{2}})(1-p^{r+1}/a)(1-p^{r+\frac{1}{2}}/a^{\frac{1}{2}})(1-\zeta p^{r-\frac{1}{2}}/a^{\frac{1}{2}})}$$

$$\times \lim_{b,c,d\to\infty} {}_{4}\psi_{4}\left[\begin{array}{c} p\sqrt{a},\ -p\sqrt{a},\ \zeta\sqrt{a/p},\ \sqrt{ap}/\zeta,\ b,\ c,\ d\ ;\ (ap)^{\frac{3}{2}}/bcd \\ \sqrt{a},\ -\sqrt{a},\ p\sqrt{ap}/\zeta,\ \zeta\sqrt{ap},\ ap/b,\ ap/c,\ ap/d \end{array}\right]$$

$$= \prod_{r=0}^{\infty} \frac{\sqrt{ap}(1-\zeta p^{r+1})(1-1/p)}{(1-a^{\frac{1}{2}}p^{r-\frac{1}{2}})(1-p^{r+\frac{1}{2}}/a^{\frac{1}{2}})(1-\zeta p^{r-1})(1-a^{\frac{1}{2}}p^{r+\frac{1}{2}})(1-p^{r+\frac{1}{2}}/a^{\frac{1}{2}})(1-p^{r+1})}$$

$$\times \lim_{b,c,d\to a} {}_{8}\psi_{4}\left[\begin{array}{c} p,\ p^{\frac{3}{2}},\ -p^{\frac{3}{2}},\ \zeta,\ bp^{\frac{1}{2}}/a^{\frac{1}{2}},\ cp^{\frac{1}{2}}/a^{\frac{1}{2}},\ dp^{\frac{1}{2}}/a^{\frac{1}{2}},\ p/\zeta\ ;\ (ap)^{\frac{3}{2}}/bcd \\ p^{\frac{1}{2}},\ -p^{\frac{1}{2}},\ p^{\frac{1}{2}}/\zeta,\ p^{\frac{3}{2}}a^{\frac{1}{2}}/b,\ p^{\frac{3}{2}}a^{\frac{1}{2}}/c,\ p^{\frac{3}{2}}a^{\frac{1}{2}}/d,\ p\zeta \end{array}\right]$$

$$+ \prod_{r=0}^{\infty} \frac{\zeta(\sqrt{a/p})(1-\zeta p^r)(1-p^{r+1})(1-p/\zeta^2)}{(1-p^{r+\frac{3}{2}}/a^{\frac{1}{2}})(1-a^{\frac{1}{2}}p^{r+\frac{1}{2}}/\zeta)(1-p^{r+1}/\zeta)(1-\zeta a^{\frac{1}{2}}p^{r-\frac{1}{2}})(1-\zeta p^r)(1-\zeta p^{r+\frac{1}{2}}/a^{\frac{1}{2}})(1-\zeta p^{r-1})}.$$

Letting b, c, d $\longrightarrow \infty$ and rearranging the products we have

$$\sum_{n=-\infty}^{\infty} (-1)^n p^{\frac{3n}{2}(n+1)}\left[\frac{p^{-\frac{3n}{2}}a^{\frac{3n}{2}}}{(1-\zeta a^{\frac{1}{2}}p^{n-\frac{1}{2}})} + \frac{p^{\frac{3n}{2}}a^{-\frac{3n}{2}}}{(1-\zeta p^{n+\frac{1}{2}}/a^{\frac{1}{2}})}\right]$$

$$= \sqrt{\frac{p}{a}}\prod_{r=0}^{\infty}\frac{(1-ap^r)(1-p^{r+1}/a)}{(1-p^{r+\frac{1}{2}}/a^{\frac{1}{2}})(1-a^{\frac{1}{2}}p^{r+\frac{1}{2}})}\sum_{n=-\infty}^{\infty}\frac{(-1)^n p^{\frac{3n}{2}(n+1)}}{(1-p^n\zeta)}$$

$$+ \prod_{r=0}^{\infty}\frac{(1-p^{r+\frac{1}{2}}/a^{\frac{1}{2}})(1-a^{\frac{1}{2}}p^{r+\frac{1}{2}})(1-ap^r)(1-p^{r+1}/a)\{\prod_{r=0}^{\infty}(1-p^{r+1})\}^2}{(1-\zeta p^r)(1-p^{r+1}/\zeta)(1-\zeta p^{r+\frac{1}{2}}/a^{\frac{1}{2}})(1-a^{\frac{1}{2}}p^{r+\frac{1}{2}}/\zeta)(1-\zeta p^{r-\frac{1}{2}}a^{\frac{1}{2}})(1-p^{r+\frac{1}{2}}/\zeta a^{\frac{1}{2}})},$$

putting w = p and $\sqrt{p/a} = \zeta$ we obtain (13.1)

This proof is of interest, because it gives a method of obtaining moregeneral formula than (13.1) from generalised hypergeometric series. In fact, we have a method of obtaining formulae which correspond to (13.1), with 3 replaced by (2m + 1). It is apparent that such formulae become more and more complex as m increases. The case in which m = 2 is obtained as follows.

As the parameters in Sear's formula fall into two different groups, in order to have 5 parameters which can tend to infinity without introducing awkward limits, it is necessary to start from Sear's formula with M = 7 and not with M = 6 as might be supposed. Taking M = 7, $a_2 = p$, $a_1 = p\sqrt{a_1}$, $a_8 = -p\sqrt{a_1}$, where the parameters are $a_1, a_2, \ldots, a_{14}$, Sear's formula becomes:—

$$\prod_{r=0}^{\infty} \left(1-a_1 p^{r+1}\right)\left(1-a_1 p^{r+1}\over a_9\right)\left(1-a_1 p^{r+1}\over a_{10}\right)\left(1-a_1 p^{r+1}\over a_{11}\right)\left(1-a_1 p^{r+1}\over a_{12}\right)\left(1-a_1 p^{r+1}\over a_{13}\right)\left(1-a_1 p^{r+1}\over a_{14}\right)\left(1-p^{r+1}\over a_9\right)\left(1-p^{r+1}\over a_{10}\right)\left(1-p^{r+1}\over a_{11}\right)\left(1-p^{r+1}\over a_{12}\right)\left(1-p^{r+1}\over a_{13}\right)\left(1-p^{r+1}\over a_{14}\right)$$

$$\prod_{r=0}^{\infty} \left(1-a_1 p^{r+1}\right)\left(1-p^{r+1}\right)\left(1-a_3 p^r\right)\left(1-a_4 p^r\right)\left(1-a_5 p^r\right)\left(1-a_6 p^r\right)\left(1-p^{r+1}\over a_1\right)\left(1-a_3 p^r\over a_1\right)\left(1-a_4 p^r\over a_1\right)\left(1-a_5 p^r\over a_1\right)\left(1-a_6 p^r\over a_1\right)$$

$$\times {}_{12}\psi_{12}\left[\begin{matrix} p\sqrt{a_1}, -p\sqrt{a_1}, a_3, a_4, a_5, a_6, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14} \\ \sqrt{a_1}, -\sqrt{a_1}, \dfrac{pa_1}{a_3}, \dfrac{pa_1}{a_4}, \dfrac{pa_1}{a_5}, \dfrac{pa_1}{a_6}, \dfrac{pa_1}{a_9}, \dfrac{pa_1}{a_{10}}, \dfrac{pa_1}{a_{11}}, \dfrac{pa_1}{a_{12}}, \dfrac{pa_1}{a_{13}}, \dfrac{pa_1}{a_{14}} \end{matrix} ; \dfrac{p^4 a_1^5}{a_3 a_4 a_5 a_6 a_9 a_{10} a_{11} a_{12} a_{13} a_{14}}\right]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\,(\,13.2\,)$$

$$= \prod_{r=0}^{\infty} a_3\left(1-a_1\over a_3\right)\left(1-a_1 p^{r+1}\over a_9\right)\left(1-a_1 p^{r+1}\over a_{10}\right)\left(1-a_1 p^{r+1}\over a_{11}\right)\left(1-a_1 p^{r+1}\over a_{12}\right)\left(1-a_3 p^r\over a_{13}\right)\left(1-a_3 p^r\over a_{14}\right)\left(1-a_1 p^{r+1}\right)\left(1-a_1 p^{r+1}\right)\left(1-a_1 p^{r+1}\over a_3 a_9\right)\left(1-p^{r+1}\over a_3 a_{10}\right)\left(1-p^{r+1}\over a_3 a_{11}\right)\left(1-p^{r+1}\over a_3 a_{12}\right)\left(1-p^{r+1}\over a_3 a_{13}\right)\left(1-p^{r+1}\over a_3 a_{14}\right)$$

$$\over \prod_{r=0}^{\infty} \left(1-a_1 p^r\over a_3\right)\left(1-p^{r+1}\over a_3\right)\left(1-a_4 p^r\over a_3\right)\left(1-a_5 p^r\over a_3\right)\left(1-a_6 p^r\over a_3\right)\left(1-a_3 p^r\right)\left(1-a_3 p^{r+1}\right)\left(1-a_3 a_4 p^r\over a_1\right)\left(1-a_3 a_5 p^r\over a_1\right)\left(1-a_3 a_6 p^r\over a_1\right)$$

$$\times {}_{12}\Phi_{11}\left[\begin{matrix} \dfrac{a_3^2}{a_1}, \dfrac{pa_3}{\sqrt{a_1}}, -\dfrac{pa_3}{\sqrt{a_1}}, \dfrac{a_3 a_4}{a_1}, \dfrac{a_3 a_5}{a_1}, \dfrac{a_3 a_6}{a_1}, \dfrac{a_3 a_9}{a_1}, \dfrac{a_3 a_{10}}{a_1}, \dfrac{a_3 a_{11}}{a_1}, \dfrac{a_3 a_{12}}{a_1}, \dfrac{a_3 a_{13}}{a_1}, \dfrac{a_3 a_{14}}{a_1} \\ \dfrac{a_3}{\sqrt{a_1}}, -\dfrac{a_3}{\sqrt{a_1}}, \dfrac{pa_3}{a_4}, \dfrac{pa_3}{a_5}, \dfrac{pa_3}{a_6}, \dfrac{pa_3}{a_9}, \dfrac{pa_3}{a_{10}}, \dfrac{pa_3}{a_{11}}, \dfrac{pa_3}{a_{12}}, \dfrac{pa_3}{a_{13}}, \dfrac{pa_3}{a_{14}} \end{matrix} ; \dfrac{p^4 a_1^5}{a_3 a_4 a_5 a_6 a_9 a_{10} a_{11} a_{12} a_{13} a_{14}}\right]$$

+ idem (a_3; a_4, a_5, a_6).

Now, in (13.2) taking
$a_1 = a$, $a_3 = \sqrt{a p}$, $a_4 = z\sqrt{a/p}$, $a_5 = x\sqrt{a p}$, $a_6 = \sqrt{a p}/x$
$a_{14} = \sqrt{a p}/y$ and letting $a_9, a_{10}, a_{11}, a_{12}, a_{13} \to \infty$, we obtain, in the notation of Atkin & Swinnerton-Dyer, the formula (13.3) viz:—

$$\sum_{n=-\infty}^{\infty} (-1)^n w^{\frac{Sn}{2}(n+1)} \left[\frac{\zeta^{-Sn}}{1-\zeta\varsigma^{-1}w^n} + \frac{\zeta^{Sn+S}}{1-\zeta\varsigma w^n} \right]$$

$$= \frac{\varsigma^2 P(\varsigma^2, w) P(x\varsigma, w) P(x\varsigma^{-1}, w)}{P(\varsigma, w)\left[P(x, w)\right]^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n w^{\frac{Sn}{2}(n+1)}}{(1-\zeta w^n)}$$

$$+ \frac{P(\varsigma, w) P(\varsigma^2, w) P(x\varsigma, w) P(\varsigma/x, w) \left(\prod_{r=1}^{\infty}(1-w^r)^2\right)}{P(\zeta\varsigma^{-1}, w) P(\zeta, w) P(\zeta\varsigma, w) P(\zeta/x, w) P(x\zeta, w)}$$

$$+ \frac{P(\varsigma, w) P(\varsigma^2, w)}{P(x, w) P(x^2, w)} \frac{\varsigma}{x} \sum_{n=-\infty}^{\infty} (-1)^n w^{\frac{Sn}{2}(n+1)} \left[\frac{x^{-Sn}}{1-\zeta x^{-1}w^n} + \frac{x^{Sn+S}}{1-\zeta x w^n} \right].$$

$$\underline{\qquad} \quad (13.3)$$

We can also regard (13.3) as giving the partial
fractions of $\left\{ P(\zeta\varsigma^{-1}, w) P(\zeta, w) P(\zeta\varsigma, w) P(\zeta/x, w) P(x\zeta, w) \right\}^{-1}$.

(14)

A generalisation of the formula (10.1A)

If, in (13.2) we take $a_3 = pa_1/a_4$, $a_5 = pa_1/a_6$, we obtain

$$_8\psi_8\left[p\sqrt{a_1},-p\sqrt{a_1},\ a_9,\ a_{10},\ a_{11},\ a_{12},\ a_{13},\ a_{14}\ ;\ \frac{p^2a_1^3}{\sqrt{a_1},-\sqrt{a_1},\frac{pa_1}{a_9},\frac{pa_1}{a_{10}},\frac{pa_1}{a_{11}},\frac{pa_1}{a_{12}},\frac{pa_1}{a_{13}},\frac{pa_1}{a_{14}}}\ \ \frac{}{a_9a_{10}a_{11}a_{12}a_{13}a_{14}}\right]$$

$$= \frac{(a_1-a_4^2)}{(a_1-a_1^2)}\frac{\prod\limits_{r=0}^{\infty}(1-a_1p^r)(1-\frac{p^{r+1}}{a_1})(1-a_4\frac{p^{r+1}}{a_1})(1-\frac{p^{r+1}}{a_4})(1-a_6p^r)(1-a_6p^r)\prod\limits_{r=0}^{\infty}(1-\frac{a_4p^{r+1}}{a_n})(1-\frac{a_1p^{r+1}}{a_6a_n})}{\prod\limits_{r=0}^{\infty}(1-\frac{a_1p^{r+1}}{a_4^2})(1-\frac{a_4p^r}{a_1})(1-\frac{a_1p^{r+1}}{a_4a_6})(1-\frac{a_6p^r}{a_4})(1-a_4a_6p^{r+1})(1-\frac{a_1a_6p^r}{a_1})\prod\limits_{r=0}^{\infty}(1-\frac{a_1p^{r+1}}{a_n})(1-\frac{p^{r+1}}{a_n})}$$

$$\times\ _8\psi_8\left[\frac{pa_4}{\sqrt{a_1}},\frac{-pa_4}{\sqrt{a_1}},\frac{a_4a_9}{a_1},\frac{a_4a_{10}}{a_1},\frac{a_4a_{11}}{a_1},\frac{a_4a_{12}}{a_1},\frac{a_4a_{13}}{a_1},\frac{a_4a_{14}}{a_1}\ ;\ \frac{p^2a_1^3}{\frac{a_4}{\sqrt{a_1}},\frac{-a_4}{\sqrt{a_1}},\frac{pa_4}{a_9},\frac{pa_4}{a_{10}},\frac{pa_4}{a_{11}},\frac{pa_4}{a_{12}},\frac{pa_4}{a_{13}},\frac{pa_4}{a_{14}}}\ \frac{}{a_9a_{10}a_{11}a_{12}a_{13}a_{14}}\right]$$

$$- (14.1)$$

$+$ idem $(a_4;\ a_6)$.

This is a relation connecting 3 well-poised $_8\psi_8$'s of general type. (10.1A) is obtained from (14.1) by taking $a_9 \doteqdot a_4$ and $a_{13} = a_6$.

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