On w-maximal groups

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Abstract

Let $w = w(x_1, \dots, x_n)$ be a word, i.e. an element of the free group $F = \langle x_1, \dots, x_n \rangle$ on n generators $x_1, \dots, x_n$. The verbal subgroup $w(G)$ of a group G is the subgroup generated by the set $\{w(g_1, \dots, g_n)^{\pm 1} \mid g_i \in G,\ 1 \leq i \leq n\}$ of all w-values in G. We say that a (finite) group G is *w-maximal* if $|G : w(G)| > |H : w(H)|$ for all proper subgroups H of G and that G is *hereditarily w-maximal* if every subgroup of G is w-maximal. In this text we study w-maximal and hereditarily w-maximal (finite) groups.

1 Introduction

Let p be a prime. In [15], Thompson observed that, if G is a finite p-group such that $|G : [G, G]| > |H : [H, H]|$ for all proper subgroups H of G, then the nilpotency class of G is at most 2. This insight prompted Laffey to prove that, for $p > 2$, the minimum number of generators $d(G)$ of a finite p-group G is bounded by r, where p^r is the maximal order of a subgroup of exponent p in G; see [11].

Properties of what are known as d-maximal p-groups form a key ingredient of Laffey's argument; a group G is d-maximal if $d(H) < d(G)$ for all proper subgroups H of G. The minimum number of generators for a finite p-group G is given by $d(G) = \log_p |G : \Phi(G)|$, where $\Phi(G) = G^p[G, G]$ denotes the Frattini subgroup of G. Hence a finite p-group G is d-maximal if $|G : \Phi(G)| > |H : \Phi(H)|$ for all proper subgroups H of G.

In the context of regular representations of finite groups, Kahn proved, for $p > 2$, that every d-maximal finite p-group G has nilpotency class at most 2; see [8]. In fact, he showed that in such a group G the derived subgroup $[G, G]$ is of exponent p and contained in the center $\mathrm{Z}(G)$ of G. Subsequently properties of d-maximal finite p-groups were investigated by Kahn as well as Minh, e.g. see [14]. More recently, a similar class of groups was studied by the first author in order to bound the index of the agemo subgroup of a finite p-group in terms of the number of elements of order p. In particular, he proved for $p > 2$ that, if G is a finite p-group such that $|G : G^p \gamma_{p-1}(G)| > |H : H^p \gamma_{p-1}(H)|$ for every proper subgroup H of G, then the nilpotency class of G is bounded by $p - 1$; see [4].

The aim of this paper is to explore this circle of fruitful ideas in a more general framework. For this we introduce and explore the new concept of a w-maximal group, which is briefly referred to in [4]. Let $w(\mathbf{x}) = w(x_1, \dots, x_n)$ be a word, i.e. an element of the free group $F = \langle x_1, \dots, x_n \rangle$ on n generators $x_1, \dots, x_n$. The verbal subgroup $w(G)$ of a group G is the subgroup generated by the set $\{w(\mathbf{g})^{\pm 1} \mid \mathbf{g} \in G^{(n)}\}$ of all w-values in G.

Definition. We say that a group G is *w-maximal* if $|G : w(G)| > |H : w(H)|$ for all proper subgroups H of G.

The classes of groups referred to above are instances of w-maximal groups for the special words $w = [x, y] = x^{-1}y^{-1}xy$, $w = x^p[y, z]$ and $w = x^p[y_1, \dots, y_{p-1}]$, respectively. Our study of w-maximal groups for more general values of w sheds more light on existing theorems and leads to a number of new results. Indeed, Thompson's original theorem generalises to w-maximal groups for many words w. We also include the outcomes of our study of hereditarily w-maximal groups, i.e. groups with the property that all their subgroups are w-maximal. Finally we suggest a range of questions concerning the structure of w-maximal and hereditarily w-maximal groups, in order to stimulate further research in this direction. We use a mixture of methods, involving, for instance, classical results of Iwasawa [6, 7] and techniques from the theory of finite p-groups and their inverse limits.

The organisation of the paper is the following. In Section 2 we introduce the concept of w-breadth and a partial ordering on the class of w-maximal groups. We show that, if w is a commutator word, then every ascending chain of w-maximal finite groups with respect to the defined ordering becomes stationary; see Corollary 2.4. A key ingredient is Theorem 2.2 which characterises words w admitting only finitely many (isomorphism types of) finite groups of any fixed w-breadth. Our results lead to a natural classification problem. Section 3 focuses on w-maximal finite p-groups. We define and illustrate the concept of interchangeability. Theorem 3.3 shows that, if w is interchangeable in a w-maximal finite p-group G, then the verbal subgroup $w(G)$ is central in G. The proof is based on Thompson's original argument in [15]. We state various questions, of which we highlight here the problem of classifying those w which are interchangeable in every w-maximal finite p-group. We also give some applications of Theorem 3.3. These provide, in any finite p-group, a lower bound for the maximum size of subgroups with certain properties in terms of the size of a quotient with similar properties; e.g. see Propositions 3.7 and Proposition 3.8. Corollary 3.10 is an extension of the result of Laffey mentioned above. In Section 4, we investigate d-maximal finite p-groups by Lie theoretic means. For p odd, the abelianisation $G/[G, G]$ and the commutator subgroup $[G, G]$ of a d-maximal finite p-group G are elementary abelian. We show that, furthermore, $|[G, G]| \le p^{d(G)-2}$ and construct examples which suggest that this bound may be best possible. In Section 5 we study hereditarily w-maximal groups. Theorem 5.3 provides examples of non-trivial hereditarily w-maximal finite groups, i.e. hereditarily w-maximal finite groups G such that $w(G) \neq 1$, for the higher derived words $w = \delta_k$, $k \ge 2$. Finally, in Section 6 we investigate hereditarily d-maximal groups. In order to avoid unwieldy examples, it is natural to restrict attention to the class of residually-finite groups, which, in fact, reduces further to the class of finite groups. Theorem 6.6 provides a complete classification of hereditarily d-maximal finite groups.

Notation: Short explanations of possibly non-standard notation and terminology are given in the text. Let G be a group. For $n \in \mathbb{N}$ we write $G^{(n)}$ for the nth cartesian power of G, whereas G^n refers to the subgroup generated by all nth powers of elements of G. If G is a topological group, then invariants such as $d(G)$ or $\mathrm{br}_w(G)$ are tacitly defined in terms of closed subgroups; e.g. $d(G)$ is the minimum number of topological generators.

2 Basic properties of w-maximal groups

Throughout this section, let $n \in \mathbb{N}$ and let $w = w(\mathbf{x})$ be a fixed word, i.e. an element of the free group $F = \langle x_1, \ldots, x_n \rangle$. Our first lemma collects two elementary, but useful properties of w-maximal groups.

Lemma 2.1. *Let w be a word and let G be a finite group.*

(a) *If $w(G) = 1$, then G is w-maximal.*

(b) *If G is w-maximal and $N \trianglelefteq G$ with $N \subseteq w(G)$, then G/N is w-maximal.*

Proof. These properties are clear. □

Based on Lemma 2.1 (b), we introduce a partial ordering on the class of w-maximal groups. Let G and H be w-maximal groups. We say that H precedes G, in symbols $H \preceq_w G$, if there exists an epimorphism $f : G \to H$ such that $\ker f \leq w(G)$. The binary relation $\preceq_w$ defines a partial ordering on the class of w-maximal groups.

For a group G, we define the *w-breadth* of G as

$$\mathrm{br}_w(G) := \sup\{|H : w(H)| \mid H \leq G\}.$$

The next theorem shows that, if w is a commutator word, then groups of bounded w-breadth can be controlled.

Theorem 2.2. *Let $w = w(\mathbf{x})$ be an element of the free group $F = \langle x_1, \ldots, x_n \rangle$. Then the following assertions are equivalent:*

(1) *w is a commutator word, i.e. $w \in [F, F]$;*

(2) *for every natural number m there are only finitely many (isomorphism classes of) finite groups G with $\mathrm{br}_w(G) \leq m$.*

Proof. First suppose that w is a commutator word, and let m be a natural number. If G is a group with $\mathrm{br}_w(G) \leq m$, then the prime divisors of $|G|$ are less than or equal to m. Indeed, if S is a non-trivial Sylow p-subgroup of G, then $p \leq |S : w(S)| \leq m$.

Consider now a prime number p less than or equal to m. We claim that there exist only finitely many (isomorphism classes of) p-groups P with $\mathrm{br}_w(P) \leq m$. This will show that there is a uniform bound on the sizes of the Sylow p-subgroups of a finite group G with $\mathrm{br}_w(G) \leq m$. Consequently, there is a uniform bound on the order of a finite group G with $\mathrm{br}_w(G) \le m$, and there are only finitely many finite groups with this property.

For a contradiction, assume that there exist an infinite number of pairwise non-isomorphic finite p-groups with w-breadth bounded by m. These groups and the possible epimorphisms between them give rise to an inverse system. Since the inverse limit of a non-empty system of finite sets is non-empty, one obtains an infinite pro-p group G with $\mathrm{br}_w(G) \le m$. Since $w \in [F, F]$, we have $d(H) \le \log_p |H : w(H)| \le \log_p m$ for every closed subgroup H of G. Consequently the group G has finite rank, equivalently G is p-adic analytic; cf. [1, Chapters 8 and 9]. Therefore there exists a uniformly powerful open subgroup U of G; see [1, Corollary 4.3]. But in this case, $|U^{p^k} : w(U^{p^k})| \ge |U^{p^k} : [U^{p^k}, U^{p^k}]|$ tends to infinity as k tends to infinity. This is in contradiction to $\mathrm{br}_w(G) \le m$.

Conversely, suppose that w is not a commutator word, i.e. $w \notin [F, F]$. Let p be a prime and consider a free $\mathbb{Z}_p$-module M with basis $\{x_1, \ldots, x_d\}$, where $\mathbb{Z}_p$ denotes the ring of p-adic integers. Then $w(M)$ is a non-trivial characteristic subgroup of M and hence $w(M) = p^r M$ for some $r \in \mathbb{N}_0$. In fact, for all submodules U of M we have $w(U) = p^r U$ and consequently $|U : w(U)| \le |M : w(M)|$. Set $m := p^{rd} = |M : w(M)|$. Then for every open submodule K of M with $K \subseteq w(M)$, the finite quotient M/K has w-breadth bounded by m. This shows that there are infinitely many (isomorphism classes of) finite abelian groups G with $\mathrm{br}_w(G) \le m$. □

Motivated by the proof of Theorem 2.2, we record

Proposition 2.3. *Let $w = w(\mathbf{x})$ be an element of the free group $F = \langle x_1, \ldots, x_n \rangle$. Let $m \in \mathbb{N}$ and p a prime. Let $\mathcal{G}_m(p)$ denote the class of all pro-p groups G with $\mathrm{br}_w(G) \le m$.*

(1) *If $w \in [F, F]$, then $\mathcal{G}_m(p)$ consists of finitely many isomorphism classes of finite p-groups.*

(2) *If $w(F)F^p[F, F] = F$, then $\mathcal{G}_m(p)$ consists of all pro-p groups.*

(3) *Suppose that $w(F)F^{p^{k+1}}[F, F] = F^{p^k}[F, F]$ with $k \in \mathbb{N}$. Then $\mathcal{G}_m(p)$ consists of isomorphism classes of p-adic analytic pro-p groups. Conversely, if G is a p-adic analytic pro-p group of dimension d and if $m \ge p^{kd}$, then there exists an open subgroup U of G such that all subgroups of U are contained in $\mathcal{G}_m(p)$.*

Proof. (1) This follows immediately from Theorem 2.2.

(2) Suppose that $w(F)F^p[F, F] = F$. Then $w(H) = H$ for any finite p-group H, because the w-values in H generate H modulo the Frattini subgroup $H^p[H, H] = \Phi(H)$. The claim follows.

(3) Let G be in $\mathcal{G}_m(p)$. Since $w(F) \subseteq F^p[F, F]$, we have $d(H) \le \log_p |H : w(H)| \le \log_p m$ for every closed subgroup H of G. Thus G has finite rank and is p-adic analytic.

Conversely, let G be a p-adic analytic pro-p group of dimension d and suppose that $m \ge p^{kd}$. Take for U a uniformly powerful open subgroup of G. A simple collection process allows us to write $w = x_1^{e_1} \cdots x_r^{e_r} c$, where $e_1, \ldots, e_r \in \mathbb{Z}$ and $c \in [F, F]$. Since $w \notin [F, F]$, at least one of the exponents e_i is non-zero, and we put $e := \gcd\{e_1, \ldots, e_r\}$. Then $F^e \subseteq w(F)$ and $w(F)[F, F] = F^e[F, F]$. From $w(F)F^{p^{k+1}}[F, F] = F^{p^k}[F, F]$ it follows that p^k is the highest p-power dividing e. Hence for any closed subgroup H of U one has $H^{p^k} = H^e \subseteq w(H)$. Moreover, for any closed subgroup H of U, one has $\mu\{h^{p^k} \mid h \in H\} = p^{-k}\mu(H)$, where μ denotes the Haar measure on H; see [10, Lemma 3.4]. Thus $|H : w(H)| \leq |H : H^{p^k}| \leq p^{kd}$ for any open and, by passing to the appropriate limit, for any closed subgroup H of U. $\square$

From Lemma 2.1 and Theorem 2.2 one can easily deduce

Corollary 2.4. *Let $w = w(\mathbf{x})$ be a commutator word in a free group $F = \langle x_1, \ldots, x_n\rangle$, i.e. let $w \in [F, F]$. Then every ascending chain*

$$H_1 \preceq_w H_2 \preceq_w \ldots \preceq_w H_i \preceq_w \ldots$$

of w-maximal finite groups is eventually stationary.

In particular, if H is a w-maximal finite group, then there exists a w-maximal finite group G such that $H \preceq_w G$ and G is maximal with respect to the partial ordering $\preceq_w$.

Proof. We argue by contradiction. If

$$H_1 \preceq_w H_2 \preceq_w \ldots \preceq_w H_i \preceq_w \ldots$$

is a strictly increasing chain of w-maximal groups, then there exist infinitely many finite groups whose w-breadth is bounded by $|H_1 : w(H_1)|$ in contradiction to Theorem 2.2. $\square$

This leads to a natural classification problem.

Question 2.5. Let F be the free group on generators $x_1, \ldots, x_n$, and let w be a commutator word in F, i.e. let $w \in [F, F]$. Can one classify w-maximal groups of a given w-breadth m which are maximal with respect to the partial ordering $\preceq_w$?

3 Properties of w-maximal p-groups and applications

In this section we focus on w-maximal finite p-groups which are somehow easier to control than w-maximal finite groups. The following concept will be useful throughout this section. Let $w = w(\mathbf{x})$ be a word and let G be a finite p-group. We say that w is *interchangeable* in G if for every normal subgroup N of G,

$$[w(N), G] \leq [N, w(G)] \cdot [w(G), G]^p [w(G), G, G].$$

Observe that, if w is interchangeable in G, then w is interchangeable in every quotient of G. The next lemma provides a considerable supply of words which are interchangeable in every finite p-group.

Lemma 3.1. *Let G be a finite p-group, and let w be equal to one of the group words*

(i) $\gamma_k = [y_1, \ldots, y_k]$ *for some $k \in \mathbb{N}$,*

(ii) $x^{p^i}[y_1, \ldots, y_k]$ *for some $i, k \in \mathbb{N}$ with $k \leq p - 1$,*

(iii) $x^{p^i}[y_1, \dots, y_p]$ *for some* $i \in \mathbb{N}$ *with* $i \geq 2$.

Then w *is interchangeable in* G.

Proof. Let N be a normal subgroup of G.

(i) We start with the case $w = \gamma_k = [y_1, \dots, y_k]$, where $k \in \mathbb{N}$. Then, by [9, Lemma 4-9], one has

$$[\gamma_k(N), G] \leq [G,_k N] \leq [\gamma_k(G), N] = [N, \gamma_k(G)]. \tag{3.1}$$

(ii) Next suppose that $w = x^{p^i}[y_1, \dots, y_k]$, where $i, k \in \mathbb{N}$ with $k \leq p-1$. By (i), we have $[\gamma_k(N), G] \leq [N, \gamma_k(G)]$. For p^ith powers [2, Theorem 2.4] yields

$$[N^{p^i}, G] \equiv [N, G^{p^i}] \pmod{\gamma_{p+1}(G)}.$$

Therefore we conclude that

$$[N^{p^i}\gamma_k(N), G] \leq [N, G^{p^i}\gamma_k(G)] \cdot [\gamma_k(G), G, G].$$

(iii) Finally suppose that $w = x^{p^i}[y_1, \dots, y_p]$, where $i \in \mathbb{N}$ with $i \geq 2$. Again $[\gamma_p(N), G] \leq [N, \gamma_p(G)]$. Since $i \geq 2$, we conclude from [2, Theorem 2.4] that

$$[N^{p^i}, G] \equiv [G^{p^i}, N] \pmod{\gamma_{p+1}(G)^p\gamma_{p+2}(G)}.$$

This yields

$$[N^{p^i}\gamma_p(N), G] \leq [G^{p^i}\gamma_p(G), N] \cdot [\gamma_p(G), G]^p \cdot [\gamma_p(G), G, G].$$

$\square$

Example 3.2. *Let* p *be a prime, and let* $H = \langle\alpha\rangle \ltimes A$ *be the pro-*p *group of maximal class, where* $\langle\alpha\rangle \cong C_p$, $A = \langle x_1, \dots, x_{p-1}\rangle \cong \mathbb{Z}_p^{p-1}$, *and the action of* α *on* A *is given by*

$$[x_i, \alpha] = x_{i+1} \textit{ for } 1 \leq i \leq p-2, \quad \textit{and} \quad [x_{p-1}, \alpha] = \prod_{j=1}^{p-1} x_j^{-\binom{p}{j}}.$$

Let $G := H/[H^p, H]^p[H^p, H, H]$. *Then* G *is a finite* p*-group of order* p^{p+2} *and the word* x^p *is not interchangeable in* G.

Indeed, consider the image N *of* A *in* G. *Since* $|G/N^p| = p^p$, *we have* $\gamma_p(G/N^p) = 1$. *Hence* G/N^p *is regular and* G/N^p *has exponent* p. *This shows that* $N^p = G^p$. *Now one easily verifies that* $[N^p, G] = [G^p, G]$ *is a non-trivial central cyclic subgroup of* G. *This shows that*

$$[N^p, G] \not\leq 1 = [N, G^p][G^p, G]^p[G^p, G, G].$$

We are ready to prove the main result of this section.

Theorem 3.3. *Let* w *be a word, and let* G *be a* w*-maximal finite* p*-group such that* w *is interchangeable in* G. *Then one has* $w(G) \leq \mathrm{Z}(G)$.

Proof. For a contradiction, assume that G is a minimal counterexample. Then $[w(G), G]$ is cyclic of order p and contained in the centre $\mathrm{Z}(G)$ of G, i.e. $[w(G), G]^p[w(G), G, G] = 1$. Consider the following characteristic subgroups of G:

$$N_1 := \{x \in G \mid [x, w(G)] = 1\} = \mathrm{C}_G(w(G)),$$
$$N_2 := \{x \in w(G) \mid [x, G] = 1\} = \mathrm{Z}(G) \cap w(G).$$

We observe that $N_2 \leq N_1$. Since w is interchangeable in G, we conclude that $[w(N_1), G] \leq [N_1, w(G)] = 1$. In particular $w(N_1) \leq N_2$.

Next we define $\langle , \rangle : G/N_1 \times w(G)/N_2 \longrightarrow [w(G), G] \cong \mathbb{F}_p$ given by $\langle xN_1, yN_2 \rangle = [x, y]$. This map $\langle , \rangle$ is a pairing of abelian p-groups, and hence $|G : N_1| = |w(G) : N_2|$. Therefore $|N_1 : w(N_1)| \geq |N_1 : N_2| = |G : w(G)|$, and G is not w-maximal, the desired contradiction. □

Of course, Question 2.5 specialises to

Question 3.4. Let F be the free group on generators $x_1, \ldots, x_n$, and let w be a commutator word in F, i.e. let $w \in [F, F]$. Can one classify w-maximal p-groups of a given w-breadth m which are maximal with respect to the partial ordering $\preceq_w$?

Other natural questions arising from our discussion are

Question 3.5. Characterise words which are interchangeable in all w-maximal p-groups.

Question 3.6. Let w be a word and let G be a w-maximal group. If w is interchangeable in all w-maximal p-groups, we know from Theorem 3.3 that $w(G)$ is contained in the center $\mathrm{Z}(G)$ of G. Can one describe properties of the inclusion of $w(G) \subseteq G$ in other situations?

Next we give some applications of w-maximal p-groups which allow us to translate, in any finite p-group G, information about the size of quotients with certain properties to information about the maximal size of subgroups with similar properties. Indeed, Proposition 3.7 guarantees the existence of large subgroups of comparatively small nilpotency class, i.e. of order at least $|G : \gamma_c(G)|$ and class at most c. In a similar vein, Proposition 3.8 shows that there exist subgroups of nilpotency class 2 and exponent p^i which are of size at least $|G : G^{p^i}[G, G]|$. Corollary 3.10 is an extension of the result of Laffey, which we quoted in the introduction; see [11].

Proposition 3.7. *Let G be a finite p-group and $c \in \mathbb{N}$. Then there exists a subgroup H of G of nilpotency class at most c such that $|H| \geq |G : \gamma_c(G)|$.*

Proof. Clearly, there exists a γ_c-maximal subgroup H of G such that $|G : \gamma_c(G)| \leq |H : \gamma_c(H)|$. From Lemma 3.1 and Theorem 3.3 we deduce that $\gamma_{c+1}(H) = 1$. □

Proposition 3.8. *Let G be a finite p-group and let $i \in \mathbb{N}$ with $i \geq 2$ if $p = 2$. Put $\epsilon := 0$ if p is odd, and $\epsilon := 1$ if $p = 2$. Then there exists a subgroup H of G of nilpotency class at most 2 and exponent $p^{i+\epsilon}$ such that $|H| \geq |G : G^{p^{i+\epsilon}}[G, G]|$.*

Proof. Put $w = x^{p^i}[y,z]$. By induction on the order of G we may assume that G is w-maximal. Recall the notation $\Omega_i(G) := \langle x \in G \mid x^{p^i} = 1 \rangle$. Consider first the case when p is odd. Lemma 3.1 and Theorem 3.3 show that $[G,G,G] = 1$. Therefore G is a regular p-group, and

$$|G : G^{p^i}[G,G]| \le |G : G^{p^i}| = |\Omega_i(G)|,$$

where $H := \Omega_i(G) = \{x \in G \mid x^{p^i} = 1\}$ is a subgroup of G of exponent p^i and nilpotency class at most 2; see [5, Kapitel III §10].

Now suppose that $p = 2$. Since G is w-maximal, Lemma 3.1 and Theorem 3.3 yield $[G,G,G] = 1$ and $[G,G]^{2^i} = [G^{2^i}, G] = 1$. By the Hall-Petrescu identity we have $(xy)^{2^{i+1}} = x^{2^{i+1}} y^{2^{i+1}}$ for all $x, y \in G$; see [5, Kapitel III, Satz 9.4]. Therefore $\varphi : G \to G^{2^{i+1}}$, $x \mapsto x^{2^{i+1}}$ is a surjective homomorphism with kernel $\Omega_{i+1}(G) = \{x \in G \mid x^{2^{i+1}} = 1\}$. Hence $|\Omega_{i+1}(G)| = |G : G^{2^{i+1}}| \ge |G : G^{2^{i+1}}[G,G]|$, and $\Omega_{i+1}(G)$ is a subgroup of G of exponent 2^{i+1} and nilpotency class at most 2. □

A result of Glauberman [3] allows us to deduce the existence of normal subgroups with similar properties.

Corollary 3.9. *Let G be a finite p-group and let $i \in \mathbb{N}$. Suppose that $p \ge 7$. Then there exists a normal subgroup H of G of nilpotency class at most 2 and exponent p^i such that $|H| = \min\{|G : G^{p^{i+\epsilon}}[G,G]|, p^{\lfloor (2p+4)/3 \rfloor}\}$.*

Corollary 3.10. *Let G be a finite p-group. Let p^k be the maximal order of a subgroup of G of nilpotency class 2 and exponent p, if p is odd, nilpotency class 2 and exponent 8, if $p = 2$. Then G can be generated by k elements, i.e. $d(G) \le k$.*

Proof. The claim follows from $d(G) = \log_p |G : G^p[G,G]|$ and Proposition 3.8. □

4 d-Maximal finite p-groups and $\mathbb{Z}_p$-Lie rings

For any group G, let $d(G)$ denote the minimal number of elements required to generate G, possibly ∞. Throughout this section let p be a prime. As noted in the introduction a finite p-group G is d-maximal, i.e. satisfies $d(H) < d(G)$ for all proper subgroups H of G, if and only if it is w-maximal for $w = x^p[y,z]$. The following proposition (cf. [8]) is an easy consequence of Theorem 3.3.

Proposition 4.1. *Let p be an odd prime and let G be a d-maximal finite p-group. Then $G/[G,G]$ and $[G,G]$ are elementary abelian p-groups. If $|G| > p$, then $|[G,G]| \le p^{d(G)-2}$.*

Proof. Since G is d-maximal, $G/[G,G]$ is also d-maximal and therefore an elementary abelian p-group. By Lemma 3.1 and Theorem 3.3, one has $[G,G,G] = 1$ and $[G,G]^p = [G^p, G] = 1$. Hence $[G,G]$ is an elementary abelian p-group.

Suppose that $|G| > p$, and put $d := d(G)$. Since G is d-maximal, we have $|[G,G]| \le p^{d-1}$. For a contradiction, assume that $|[G,G]| = p^{d-1}$. The map $\pi : G/[G,G] \to [G,G]$, $x \mapsto x^p$ is a homomorphism between elementary abelian p-groups. Since $|G/[G,G]| > |[G,G]|$, the map π is not injective. Suppose that $1 \ne y \in \ker \pi$. Then y is an element of order p in G and $y \notin [G,G]$. Put $H = \langle \{y\} \cup [G,G] \rangle$. Then H is an elementary abelian p-group of order p^d which is strictly contained in G, a contradiction. □

For p odd, Theorem 3.3 and Proposition 4.1 show that every d-maximal finite p-group G has nilpotency class at most 2 and exponent p^2. Conversely, finite p-groups of nilpotency class at most 2 and exponent p need not be d-maximal; for instance, the Heisenberg group over the finite field $\mathbb{F}_p$ is not d-maximal. Interesting examples of d-maximal finite p-groups can be constructed by Lie methods as follows.

The Lazard correspondence, which operates via functors exp and log, is a correspondence between finite p-groups of nilpotency class smaller than p and finite $\mathbb{Z}_p$-Lie rings of nilpotency class smaller than p; see [9, § 10.2]. Suppose that p is odd, and consider a finite p-group G whose nilpotency class is bounded by 2. Put $L := \log(G)$. Then the group G is d-maximal if and only if L is d-maximal, in the sense that for any Lie subring M of L one has $|M : pM + [M, M]_{\mathrm{Lie}}| < |L : pL + [L : L]_{\mathrm{Lie}}|$.

Example 4.2. *Let $L = \mathrm{span}\langle x, y, z \mid px = py = p^2 z = 0\rangle \cong C_p \times C_p \times C_{p^2}$ be and abelian p-group, and extend $[x, z]_{Lie} = [y, z]_{Lie} = 0$ and $[x, y]_{Lie} = pz$ bi-additively to L. Then $(L, +, [,]_{Lie})$ is a finite d-maximal $\mathbb{Z}_p$-Lie ring. The finite p-group $G = \exp(L)$ is d-maximal such that $d(G) = 3$ and $|[G, G]| = p$.*

We continue to work under the hypothesis that $p > 2$ and consider a finite p-group G of nilpotency class 2 and of exponent p. Then $L := \log(G)$ is an $\mathbb{F}_p$-Lie algebra and, as an $\mathbb{F}_p$-vector space, L decomposes as $L = V \oplus [L, L]_{\mathrm{Lie}}$. Put $k := \dim_{\mathbb{F}_p}[L, L]_{\mathrm{Lie}}$. Since $[L, L]_{\mathrm{Lie}} \subseteq \mathrm{Z}(L)$, the Lie product on L is determined by its restriction to $V \times V$, which can be written as

$$V \times V \to [L, L]_{\mathrm{Lie}}, \quad [v, w]_{\mathrm{Lie}} = f_1(v, w)z_1 + \ldots + f_k(v, w)z_k, \tag{4.1}$$

where $\{z_1, \ldots, z_k\}$ is an $\mathbb{F}_p$-basis of $[L, L]_{\mathrm{Lie}}$ and f_i, $1 \leq i \leq k$, is a collection of antisymmetric bilinear forms on the vector space V.

In order to check whether the p-group G is d-maximal, it is enough to check whether the $\mathbb{F}_p$-Lie algebra L is d-maximal. Clearly, for this is suffices to check whether for any Lie subalgebra M of L containing $[L, L]_{\mathrm{Lie}}$ one has $|M : [M, M]_{\mathrm{Lie}}| < |L : [L, L]_{\mathrm{Lie}}|$. Equivalently, one needs to test whether for any proper subspace W of V,

$$\dim(W) + k - \dim_{\mathbb{F}_p}\left(\mathrm{span}\langle\sum_{i=1}^{k} f_i(v, w)z_i \mid v, w \in W\rangle\right) < \dim_{\mathbb{F}_p} V. \tag{4.2}$$

One can easily compute the dimension of $\mathrm{span}\langle\sum_{i=1}^{k} f_i(v, w)z_i \mid v, w \in W\rangle$ by studying the space generated by the antisymmetric bilinear forms $f_1, \ldots, f_k$ in the exterior algebra $W \wedge W$.

Lemma 4.3. *Let W be an $\mathbb{F}_p$-vector space, let $f_1, \ldots, f_k$ be a collection of antisymmetric bilinear forms on W, and let $\{z_1, \ldots, z_k\}$ be a basis of $\mathbb{F}_p^k$. Let U denote the vector subspace generated by $f_1, \ldots, f_k$ in the exterior algebra $W \wedge W$. Then*

$$\dim_{\mathbb{F}_p} U = \dim_{\mathbb{F}_p}\left(\mathrm{span}\langle\sum_{i=1}^{k} f_i(v, w)z_i \mid v, w \in W\rangle\right). \tag{4.3}$$

Proof. For $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_k) \in \mathbb{F}_p^k$ define $\Phi_{\boldsymbol{\lambda}} : \mathbb{F}_p^k \to \mathbb{F}_p$, $\sum_{i=1}^k x_i z_i \mapsto \sum_{i=1}^k \lambda_i x_i$. Then $\sum_{i=1}^k \lambda_i f_i = 0$ is a linear dependency relation in $W \wedge W$ if and only if for all $v, w \in W$, $\Phi_{\boldsymbol{\lambda}}(\sum_{i=1}^k f_i(v,w) z_i) = 0$. $\square$

Example 4.4. *Suppose that p is odd. Let $V = \operatorname{span}\langle e_1, \ldots, e_4\rangle$ be the standard 4-dimensional $\mathbb{F}_p$-vector space and consider the antisymmetric bilinear forms f_1 and f_2 on V, represented by the matrices*

$$F_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \textit{and} \quad F_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \tag{4.4}$$

with respect to the basis $(e_1, \ldots, e_4)$. Then the 6-dimensional $\mathbb{F}_p$-Lie algebra $L = V \oplus \operatorname{span}\langle z_1, z_2\rangle$, defined by $[v,w]_{Lie} = f_1(v,w)z_1 + f_2(v,w)z_2$, is of nilpotency class 2. Based on Lemma 4.3, a short computation shows that L is d-maximal. The finite p-group $G = \exp(L)$ is d-maximal such that $d(G) = 4$ and $|[G,G]| = p^2$.

Examples 4.2 and 4.4 suggest that, perhaps, the bound in Proposition 4.1 is the best possible. We record

Question 4.5. Does there exist for every integer $k > 2$ a d-maximal finite p-group G such that $|G : \Phi(G)| = p^k$ and $|[G,G]| = p^{k-2}$?

5 Hereditarily w-maximal groups

Let $w = w(\mathbf{x})$ be a fixed word, i.e. an element of the free group $F = \langle x_1, \ldots, x_n\rangle$ on n generators. We say that a group G is *hereditarily w-maximal* if all subgroups of G are w-maximal.

Proposition 5.1. *Let w be an element of the free group $F = \langle x_1, \ldots, x_n\rangle$ and let G be a hereditarily w-maximal finite group.*

(1) *If G is a p-group, then $w(G) = 1$.*

(2) *If w is equal to x^m or $\gamma_k = [x_1, \ldots, x_k]$, for some $m, k \in \mathbb{N}$, then $w(G) = 1$*

Proof. (1) We argue by induction on the order of G. Suppose that G is a non-trivial p-group. Let H be a maximal subgroup of G, i.e. a subgroup of index p in G. Since H is hereditarily w-maximal, induction shows that $w(H) = 1$. From $|G : w(G)| > |H : w(H)| = |H|$, we conclude that $w(G) = 1$.

(2) If $w = x^m$ for some $m \in \mathbb{N}$, then for every $g \in G$, the cyclic subgroup $\langle g\rangle$ is x^m-maximal, and hence $g^m = 1$.

Next suppose that $w = \gamma_k = [x_1, \ldots, x_k]$ for some $k \in \mathbb{N}$. If G is nilpotent, then it is a direct product of its Sylow p-subgroups and the claim follows from (1). For a contradiction, assume that G is not nilpotent. By induction, we may assume that all proper subgroups of G are nilpotent of class at most $k-1$. By a classical result of

Iwasawa (see [6]), $G \cong C \ltimes Q$ where C is a cyclic p-group and Q is a q-group with p and q prime. There are two cases.
Case 1: $[G,G] \lneqq Q$. Then there exist subgroups H_1 and H_2 of index p and q, and these are nilpotent of class at most $k-1$. Since $|G : \gamma_k(G)| > |H_i : \gamma_k(H_i)| = |H_i|$ for $i \in \{1,2\}$, we conclude that $\gamma_k(G) = 1$.
Case 2: $[G,G] = Q$. In this case, since C is cyclic, $[G,G] = [G,Q] = Q$. Therefore $\gamma_k(G) = Q$ and $|G : \gamma_k(G)| = |G : Q| = |C| = |C : \gamma_k(C)|$, a contradiction. $\square$

The following example shows that there is no direct analogue of Proposition 5.1 for the second commutator word $w = [[x_1,x_2],[y_1,y_2]]$.

Example 5.2. *Consider the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ of order 8 and a cyclic group $C_3 = \langle \alpha \rangle$ of order 3. Take the semidirect product $G = C_3 \ltimes Q_8$ with respect to the natural action, given by $i^\alpha = j$, $j^\alpha = k$ and $k^\alpha = i$. We have $G^{(2)} = [[G,G],[G,G]] = \{\pm 1\}$. This shows that $|G : G^{(2)}| = 12$, and since G has no proper subgroup of order larger than or equal to 12, the group G is $[[x_1,x_2],[y_1,y_2]]$-maximal. But G is not metabelian.*

We show that the special example 5.2 generalises to higher commutator words. The *standard derived words* are defined recursively as

$$\delta_1 := [x,y] \quad \text{and} \quad \delta_{k+1} := [\delta_k(x_1,\ldots,x_{2^k}), \delta_k(y_1,\ldots,y_{2^k})] \text{ for } k \in \mathbb{N}.$$

Accordingly, a group G is soluble of derived length at most k if and only if $\delta_k(G) = 1$.

Theorem 5.3. *Let $k \in \mathbb{N}$. Then there exists a finite group G which is δ_{k+1}-maximal, soluble of derived length $k+2$, but satisfies $\delta_{k+1}(G) \neq 1$.*

Proof. Let p, q be prime numbers with $2^k < p < q$, and let m be the order of p in $(\mathbb{Z}/q\mathbb{Z})^*$. Then $\mathbb{F}_p(\zeta) \cong \mathbb{F}_{p^m}$, where ζ is a qth root of unity.

Let L be the free nilpotent Lie algebra of nilpotency class 2^k over $\mathbb{F}_p$ on $d := 2m$ generators $x_1, \ldots, x_d$. Recall that the derived series of L is a subseries of the lower central series: $\delta_j(L) = \gamma_{2^j}(L)$ for all $j \in \mathbb{N}_0$. Hence L has derived length $k+1$, with $\delta_k(L) = \gamma_{2^k}(L) \neq 0$ central in L.

Let $V := \mathbb{F}_p x_1 + \ldots + \mathbb{F}_p x_d$ so that $L = V \oplus \gamma_2(L)$ as an $\mathbb{F}_p$-vector space. Write $V = V_1 \oplus V_{-1}$ with $V_i \cong \mathbb{F}_p(\zeta)$ for $i \in \{1,-1\}$ and choose $z_V \in \mathrm{GL}(V)$ of prime order q, acting on V_i as multiplication by ζ^i for $i \in \{1,-1\}$. Since L is free, any vector space automorphism of V lifts uniquely to a Lie algebra automorphism of L. We denote the lift of z_V to $\mathrm{Aut}(L)$ by z_L. Clearly, ζ and ζ^{-1} are among the eigenvalues of the automorphism z_V of V. For later use we observe that the $\mathbb{F}_p\langle z_V\rangle$-module V is completely reducible with irreducible submodules V_1 and V_{-1}. In particular, V does not admit any z_V-invariant subspaces of co-dimension 1.

Clearly, z_L acts on the $\mathbb{F}_p$-vector space $\delta_k(L)$ and we denote the restriction of z_L to $\delta_k(L) = \gamma_{2^k}(L)$ by $z_{\delta_k(L)}$. The eigenvalues of $z_{\delta_k(L)}$ are products of length 2^k in the eigenvalues of z_V; among the latter are ζ and ζ^{-1}. This shows that 1 is an eigenvalue of $z_{\delta_k(L)}$. Since the $\mathbb{F}_p\langle z_{\delta_k(L)}\rangle$-module $\delta_k(L)$ is completely reducible, we find a subalgebra

Z of co-dimension 1 in $\delta_k(L)$, which is z_L-invariant. Since $\delta_k(L)$ lies in the centre of L, the subalgebra Z is, in fact, an ideal of L.

As $p > 2^k$, Lazard's correspondence yields a finite p-group $N := \exp(L/Z)$ (of exponent p) with a natural action of $\langle z \rangle \cong C_q$ on N, where $z = z_L$. Under this correspondence, $V \cong (V + Z)/Z$ is isomorphically mapped to a complement of $\gamma_2(N)$ in N; we denote this complement also by V so that $N = V \ltimes \gamma_2(N)$. We claim that the semidirect product $G := \langle z \rangle \ltimes N$ is δ_{k+1}-maximal, while $\delta_{k+1}(G) \neq 1$.

Indeed, since z_V does not admit 1 as an eigenvalue, we have $N \supseteq \delta_1(G) \supseteq [V, \alpha] = V$. As $N = \langle V \rangle$, it follows that $\delta_1(G) = N$, and hence $\delta_{k+1}(G) = \delta_k(N)$, which is associated under the Lazard correspondence to $\delta_k(L/Z)$, is a cyclic group of order p and hence non-trivial. This implies that $|G : \delta_{k+1}(G)| = |G : N||N : \delta_k(N)|= |G|/p$.

Now suppose that H is a subgroup of G with $|H : \delta_{k+1}(H)| \geq |G : \delta_{k+1}(G)|$. A fortiori we have $|H| \geq |G|/p$. Since $q > p$ is prime, this implies that $H \not\subseteq N$ and $HN = G$. Then $|N : H \cap N| \leq p$, and consequently $(H \cap N)\gamma_2(N)$ has index at most p in $N/\gamma_2(N) \cong V$. Since V does not admit any z-invariant subspaces of co-dimension 1, we must have $(H \cap N)\gamma_2(N) = N$. This implies that $H \cap N = N$, and $H = G$. Thus G is δ_{k+1}-maximal. □

We finish this section by proving that, for many words w, finite groups which are hereditarily w-maximal are necessarily soluble. In particular, the proposition shows that a hereditarily δ_k-maximal groups are solvable.

Proposition 5.4. *Let w be an element of the free group $F = \langle x_1, \ldots, x_n \rangle$ and let G be a hereditarily w-maximal finite group. Then w vanishes on every composition factor of G.*

Proof. Consider a descending chain $G = N_0 \trianglerighteq N_1 \trianglerighteq \ldots \trianglerighteq N_k = 1$ of subnormal subgroups of G such that N_{i-1}/N_i is simple for each $i \in \{1, \ldots, k\}$. By induction on the composition length it is enough to prove that w vanishes on G/N_1. For a contradiction, assume that w does not vanish on G/N_1 so that, in particular $G = w(G)N_1$. Then $|G : w(G)| = |N_1 : N_1 \cap w(G)| \leq |N_1 : w(N_1)|$, which contradicts the w-maximality of G. □

Question 5.5. Is there a uniform bound on the derived length of a hereditarily δ_2-maximal group, where $\delta_2 = [[x_1, x_2], [y_1, y_2]]$?

6 Hereditarily d-maximal groups

More definitive results can be obtained for hereditarily d-maximal groups which we define as follows. A group G is said to be *hereditarily d-maximal*, if every subgroup H of G is d-maximal. Of course, a finite p-group G is hereditarily d-maximal if and only if G is hereditarily w-maximal for $w = x^p[y, z]$.

Lemma 6.1. *Let G be a d-maximal group. Then G is finitely generated, and for every maximal subgroup M of G we have $d(G) = d(M) + 1$.*

Proof. If G is trivial there is nothing to show. Now suppose that M is a maximal subgroup of G. Since G is d-maximal, the inequality $d(G) \geq d(M) + 1$ holds and in particular $d(M)$ is finite. On the other hand, if $g \in G \setminus M$, then the maximality of M implies $\langle M \cup \{g\} \rangle = G$, thus $d(G) \leq d(M) + 1$. □

Lemma 6.2. *Let G be a hereditarily d-maximal group. Then G is equichained of finite length, i.e. there exists a finite chain $1 = M_0 \lneqq M_1 \lneqq \ldots \lneqq M_r = G$, where M_i is maximal in M_{i+1} for all indices i, and all such chains have the same length. Moreover this length is equal to $d(G)$.*

Proof. This follows by induction from Lemma 6.1. □

The existence of so-called Tarski groups, i.e. infinite groups all of whose non-trivial proper subgroups have prime order p, indicates that it may be difficult to classify hereditarily d-maximal groups in general; see [12, 13] for a construction of such groups.

In order to avoid these problems we choose to restrict our attention to residually-finite groups. Note that every non-trivial residually-finite group has at least one maximal subgroup of finite index. Thus, if G is a residually-finite d-maximal group, then Lemma 6.1 and an easy induction on $d(G)$ show that G is in fact finite. It remains to present a classification of *finite* hereditarily d-maximal groups.

For $n \in \mathbb{N}$ let $\nu(n)$ denote the number of prime divisors, counting repetitions, of n. By a classical result of Iwasawa, a finite group is equichained if and only if it is supersoluble; see [5, Satz VI.9.7] or [7]. This gives

Proposition 6.3. *Let G be a finite group. Then G is hereditarily d-maximal if and only if $d(G) = \nu(|G|)$.*

Proof. Suppose that G is d-maximal. Then Iwasawa's characterisation of finite equichained groups and Lemma 6.2 imply that G is supersoluble and $d(G) = \nu(|G|)$.

Now suppose that $d(G) = \nu(|G|)$. Then for every subgroup $H \leq G$ we have $d(H) = \nu(|H|)$. It follows that G is hereditarily d-maximal. □

Corollary 6.4. *Every quotient of a finite hereditarily d-maximal group is hereditarily d-maximal.*

Lemma 6.5. *Let G be a finite nilpotent group. Then G is hereditarily d-maximal if and only if G is elementary abelian.*

Proof. Suppose that G is hereditarily d-maximal. Being nilpotent, the group G is the direct product of its Sylow subgroups, $G = P_1 \times \ldots \times P_r$ say. Then $d(G) = \max\{d(P_i) \mid 1 \leq i \leq r\}$, and since G is d-maximal, this implies that G is a p-group for a suitable prime p. Proposition 6.3 asserts that $\log_p|G/\Phi(G)| = d(G) = \nu(|G|) = \log_p|G|$, so G is equal to its Frattini quotient, thus an elementary p-group.

It is clear that elementary abelian groups are hereditarily d-maximal. □

Theorem 6.6. *Let G be a finite group. Then G is hereditarily d-maximal if and only if one of the following holds:*

(1) *there exist primes p, q such that $G = \langle x \rangle \ltimes Q$ where $\langle x \rangle \cong C_p$, Q is an elementary q-group and x acts on Q as a non-trivial scalar;*

(2) *there exists a prime p such that G is an elementary p-group.*

Proof. It is clear that the groups given in the list are hereditarily d-maximal. For the opposite direction, suppose that G is hereditarily d-maximal. In the proof of Proposition 6.3 it was seen that G is supersoluble. This implies that the derived subgroup $D := [G, G]$ of G is nilpotent. Corollary 6.4 and Lemma 6.5 show that G/D and D are elementary abelian, i.e. $G/D \cong C_p^r$ and $D \cong C_q^s$ for primes p, q and $r, s \in \mathbb{N}_0$.

If $D = 1$, there is nothing more to prove. So assume that $r, s \geq 1$. Choose generators $x_1, \ldots, x_r$ for G modulo D. Proposition 6.3 gives $d(G) = \nu(|G|) = r+s$, which has strong consequences:

(i) every non-trivial element of G has order p or q;

(ii) the group $H := \langle x_1, \ldots, x_r \rangle$ has order p^r, in particular $G = H \ltimes D$ is a split extension of D by H;

(iii) for every $y \in D \setminus \{1\}$ the group $\langle y \rangle \cong C_q$ is normal in G and its centraliser $\mathrm{C}_H(\langle y \rangle)$ in H is trivial; consequently $H = \langle x_1 \rangle \cong C_p$ and x_1 acts as a non-trivial scalar on D.

□

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