SUMMARY OF Ph.D. THESIS.

PRODUCTION OF VIBRATIONS BY MEANS OF SOLID CARBON DIOXIDE

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The thesis gives a concise account of investigations, which are more fully described in the publications which accompany it, on the subject of the production of vibrations by means of solid carbon dioxide.

The circumstances which led to the discovery that pure loud notes could be produced from metallic bodies by means of the substance are described. Certain properties of solid carbon dioxide which are relevant to the investigation are considered.

An account of the technique, which took some months to perfect is given. Frequencies of vibration were measured with a calibrated valve-oscillator. Bodies excited by solid carbon dioxide, besides possessing adequate vibrating properties must be reasonably good thermal conductors. It is possible to produce intense vibrations in quartz crystals and bars in addition to metal objects.

Semi-conductors may be distinguished from insulators by the fact that the former only emit a sound when touched with solid carbon dioxide. The range of vibration frequencies most easily excited depends but little upon the size, shape, mass, material or heating of the excited body; it lies between about 1000 and 4000 c./sec.

A theory to account for the phenomenon is given in the

paper. Metallic bodies, heated to suitable temperatures, have also been set into vibration by means of ice and certain other materials; only small amplitudes of vibration have been obtained in this way.

There are a number of applications of the solid-carbon-dioxide method of excitation. Of these, mention may be made here of the production of Chladni figures on plates and of overtones in bodies of irregular shape.

A simple method of measuring Poisson's ratio in terms of the ratio of two specified natural frequencies of a vibrating free square plate and the results of acoustic studies of metals and alloys under various physical conditions are also included among the publications.

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1. INTRODUCTION.

Origin of the research. The well known fact that solid carbon dioxide will make a bicycle bell emit a chattering ring was brought to the author's notice on July 11th 1932, by an itinerant vendor of ice-creams. On enquiry it was found that the manufacturers also were familiar with the noise or squeak produced when a metal object touches a piece of solid carbon dioxide; the phenomenon however had not been studied in the laboratory.

It is the purpose of this paper to record in a logical order the investigations which were at once entered into as a result of this chance encounter. The publications which accompany the thesis are referred to in the text by means of the numbered titles given on page 30.

Discovery that pure loud notes may be produced and maintained by means of solid carbon dioxide. Miscellaneous objects - hammers, chisels etc. were brought into contact with solid carbon dioxide and in each case a rattling noise or squeak was heard. It was not found possible to produce noise from glass and the conclusion was drawn that this was due to the low thermal conductivity of the material. The next objects tried were tuning-forks of ordinary frequencies, these gave out a chattering noise.

When however high pitched forks, for example 2000 and 3000 c./sec. were tried, it was discovered that if a small area of solid carbon dioxide was brought into light contact with one prong, an intense pure note could be produced and maintained for several minutes. The demonstration of this phenomenon is very striking.

Scope of the Investigations. Further observations have been concerned with the properties of solid carbon dioxide and the vagaries of its behaviour, the perfecting of the technique necessary for the production of pure notes, the types of objects which may be set into vibration and some of their physical properties and the range of frequencies of the vibrations which may be excited. These subjects are dealt with in § 2, § 3, § 4 and § 5 below. Some experiments with mercury are described in the next paragraph. A theory has been developed to account for the phenomenon an account of which will be found in § 7. It has been satisfactory to be able to recognise, though on a very small scale, the same phenomenon of production of vibrations when ice and certain other substances are brought into contact with suitable metals heated to suitable temperatures, see § 8.

Certain applications of the solid-carbon-dioxide method of producing vibrations are mentioned in § 9, and in § 10 attention is drawn to further lines of research which have developed out of the previous work.

2. PROPERTIES OF SOLID CARBON DIOXIDE RELEVANT TO THE INVESTIGATION.

The ease with which vibrations may be produced was found to vary very greatly from day to day. This must be attributed to two causes, namely the hygrometric state of the air and the quality of the solid carbon dioxide. The temperature of solid carbon dioxide is -80°C and on a damp day moisture readily forms on the cooled metal surface and this interferes seriously with the process of excitation. But the nature of the solid carbon dioxide itself is an even more important factor.

In connexion with the production of ripples on mercury (see § 6 below), the question of pressure due to the weight of the block was considered; and when the density of the blocks was ascertained, by weighing and measurement, it was found to vary from day to day. This suggested the cause of the behaviour of different blocks. It is only blocks of high density which will produce sound in metals, and this doubtless is the reason why the phenomenon has not attracted more attention.

Light "snow" carbon dioxide, which floats on water, may be compressed into solid blocks, which vary in density and hardness according to the amount of air left in them. When sufficiently compressed they may behave like "ice" carbon dioxide in producing vibrations, and the density of some blocks the author has handled has been as high as 1·4. It is to be noted that a block made by compressing snow may be denser at the centre than at the outside. The density of ice blocks which have been made by the process of directly freezing with the aid of liquid air also varies; an entirely solid block would have a density of 1·56, but any block consisting of solid ice crystals will be satisfactory. Some blocks contain more oil than others, and the author is inclined to consider that this reduces their efficiency in producing vibrations. Apart from actual determination of the density, it has been found possible to recognize blocks from different factories by viewing them by transmitted light, and also by rubbing a surface and watching how far a snow crust forms as sublimation occurs. Crust does not form on a good ice block.

3. EXPERIMENTAL ARRANGEMENTS.

Some points of technique. Certain points of technique which have been developed during the course of the investigations and which are important for the effective production of vibrations may be summarised as follows:-

(1) The solid carbon dioxide used must be of the high density "ice" variety, see § 2 above, The author always uses Drikold, kindly supplied by Imperial Chemical Industries Ltd. The material may be conveniently stored for some days in a

large thermos-flask.

It is sometimes desirable to discard a piece of carbon dioxide, if it appears sugary or breaks up easily, in favour of a pointed piece broken by means of hammer and chisel from another part of the solid carbon-dioxide block. This last observation is of theoretical as well as of practical importance.

(2) The area of contact between the metal and solid carbon dioxide must be small, and the pressure of application light. It does not appear to be possible to substitute a mechanical device for the hand, and the skill of the operator in sensing the onset of vibration, the amplitude of which grows with remarkable rapidity, increases with practice. For producing Chladni figures the point of application of the solid carbon dioxide (at or near an anti-node) may be either on the upper or the lower surface of the bar or plate and it is often convenient to excite from below.

(3) The atmosphere must be dry. Excitation may be rendered independent of hygroscopic conditions by gently warming the vibrating object and surrounding air by means of a bowl electric radiator, the heat being insufficient to alter either the elastic or damping properties of the material of the object.

Measurement of vibration frequencies. In early experiments a sonometer was used but later the note emitted by a loud-speaker, connected to a calibrated mains-operated oscillator, was adjusted to that of the vibrating object the vibration frequency by this means being given to an accuracy of 0·1%. Actually, the oscillator note was adjusted just above and just below the note to be determined so as to produce the same number of beats in either case, and the mean of the two oscillator readings was taken.

Methods of supporting the vibrating objects. Tuning forks and bells were held in one hand. Bars were suspended from two nodes by means of fine threads of about 12 cm. length.

Several methods of support were employed in connection with the production of Chladni figures on vibrating plates. When figures with a central node were obtained, the plates were supported centrally in the usual manner. In order to obtain figures with antinodal centres the plate under test (preferably one without a central hole) was laid on three equidistant small circular-sectioned pieces of indiarubber, which were placed in appropriate nodal positions on a horizontal table provided with a paper scale graduated in circles and angles, In the case of square plates it was occasionally found useful to suspend them horizontally by means of two parallel loops of fine thread.

4. SOME PHYSICAL PROPERTIES OF THE VIBRATING OBJECTS.

It was evidently necessary in the first place to use a large number of objects of varying size, shape and material in order to determine generally under what conditions pure notes can be produced by means of solid carbon dioxide. Among such objects mention may be made of tuning-forks, brass bars, steel bars, metal discs, tubes, rings, bells, wires and objects of irregular shape such as silver thimbles, aural specula, spoons and teapots, spanners, chisels, and Trevelyan's rocker.

It was early recognised that the range of vibration frequencies which can be excited was limited; that high thermal conductivity is desirable in the object to be set into vibration which must also possess adequate vibrating properties. These last were estimated by determining, under standard conditions, the duration of audibility, of the note which had been excited. Full details regarding these observations have been given elsewhere (1) and it will be sufficient for the present argument to consider table 1 below which concerns a number of vibrating bars made of various materials which are listed in descending order of thermal conductivity. The material of the bar is given in the first column and its dimensions, fundamental frequency of vibration and duration of audibility in the following four columns.

The physical constants quoted in the next three columns - thermal conductivity, Young's modulus and density - are sometimes only approximate and sometimes the mean of several values, but are sufficiently accurate for present purposes; they are in most cases taken from tables. In the case of carbon, Young's modulus was calculated after determining the density, dimensions and the vibration frequency of one of the rods.

Table 1.

Material of bar	Length (cm.)	Diameter d or thickness t (cm.)	Frequency	Duration of audibility (sec)	Thermal conductivity	Young's modulus of elasticity (c.g.s.u. x10^{11})	Density	Remarks
Copper	14.1	1.27 t	2190	16	0.9	12.3	8.9	Very loud, maintained
Aluminium	15.3	1.27 t	2290	15	0.50	7	2.6	"
Duralumin	15.3	1.27 t	2390	30	0.31	6.9	2.8	"
Trevelyan's brass rocker	11.2	Irregular	2420	2	0.26	10	8.4	Difficult to maintain
Zinc	14	1 d	880	2	0.26	8.7	7.1	Loud, difficult to maintain.
Lead	15	1.27 d	510	0	0.08	1.6	11.4	Noise.
Eureka	13.2	1.27 t	(3100 (3260	5	0.05	16.3	8.9	Clear, difficult to maintain.
"	10.3	"	5020	6	"	"	"	"
"	7.1	"	11000	3.5	"	"	"	"
Arc carbon	14.3	0.5 d	670	0.5	0.01	0.9	1.4	No sound produced.
"	18	1.2 d	820	"	"	"	"	"
"	19.2	1.8 d	1060	"	"	"	"	"
"	13.3	1.8 d	2250	"	"	"	"	"
"	8.2	2 d	6580	"	"	"	"	"
Glass	30.5	1.2 d	1970	9	0.001	6	2.9	"

It is to be noted (1) that Trevelyan's rocking brass bar with its irregular cross-section is not such a good vibrator as a simple bar. A purer note can be obtained by means of carbon dioxide than by hammering. The pureness of the notes obtained by excitation by means of carbon dioxide is an interesting feature of the phenomenon. (2) Zinc has poorer vibrating properties than brass and is accordingly difficult to maintain in vibration, though the conductivity is equal to that of brass and the note emitted is clear. (3) The pitch of the lead bar could not be determined either by hammering or by means of carbon dioxide, but considerable noise could be produced by contact with the latter. The pitch was determined in an interesting manner, namely by increasing the vibrating properties of the bar by cooling it in a vacuum flask containing solid carbon dioxide and setting it in vibration the moment it was removed. The result was in agreement with calculation. In the case of two thick lead tubes, not included in the table, it was found possible to recognize the pitches, 980 and 2400 respectively, because of the rather better vibrating properties of the tube, and to get a purer momentary note with the carbon dioxide than with the hammer. (4) Eureka was investigated on account of its particular thermal conductivity, and after the failure to produce any sound from arc carbon rods; it behaved as expected. The longest bar was not very accurately square and its vibrating properties were accordingly worse than they should be. (5) Particular attention was paid to carbon as at the time it was felt that it was the borderland substance, having regard to the thermal conductivity, between those bodies which could and those which could not be made to vibrate. No sound was produced in carbon. Though the damping factor is considerable, the vibrating properties of the rods are sufficiently good for a clear note to be obtained when they are hammered. In later observations it has been shown that sound can be produced in bodies having a far smaller thermal conductivity than carbon; and in particular in a diamond. We must conclude that carbon rods, made as they are by compressing gas carbon, do not behave like such solids as metals or stones, and that the hardness of the material and not the thermal conductivity is the factor which determines that sound cannot be produced in carbon and can be produced in a diamond.

This conclusion was tested later on a specimen of Acheson graphite kindly supplied by Mr. Darling. The thermal conductivity is much greater than that of carbon, and its apparent density is 1.56 while its real density is 2.21; hence its porosity is 28 per cent, which is less than that of an ordinary carbon rod. After copper had been deposited on a carbon rod electrolytically it was found possible to produce notes and maintain them for a few seconds.

Semi-thermal conductors and insulators. Notes on the experiments made on semi-thermal conductors and insulators are given below.

Quartz is a semi-conductor the thermal conductivity of which varies with direction. Its vibrating properties are good. The conductivity along the optic axis is 0.06 and in one direction at right angles to this it is 0.03. These values are intermediate to those of eureka and carbon respectively. Two quartz discs of diameters 5 cm. and 3.8 cm., and thicknesses 1.6 mm. and 1.8 mm. respectively were excited to give out clear notes by means of carbon dioxide, though these were not maintained. The pitch of the larger disc, determined experimentally, was 3800, while that of the smaller one, as calculated from the relative dimensions of the two, was 7400.

A natural quartz crystal, $5\frac{1}{2}$ in. long and of cross-section about $\frac{1}{2}$ in. was tried at the Natural History Museum. A loud maintained note was produced, of frequency just above 4000~. The note was harder to start than in the case of metals, because of the smaller thermal conductivity; on the other hand the loudness increased very rapidly. A crystal, $2\frac{1}{2}$ in. long and of cross-section about $\frac{1}{4}$ in., was investigated in the laboratory, the frequency in this case being about 6000~.

A bar of square cross-section which was cut from a large crystal emitted a piercing note of about 4000 c./sec. Quartz lenses emitted a rattling noise when touched with solid carbon dioxide.

Rattles, squeaks or smaller sounds have been obtained from a number of non-metal semi-thermal conductors. They are enumerated below in descending order of thermal conductivity, as far as this is known, or according to the results of the carbon-dioxide test when it is unknown. Small sounds were heard when the following objects were touched with the solid carbon dioxide: various precious stones including pearls, diamonds, rubies and garnets; crystals, marble, mother of pearl, slate, glass.

No sound was obtained from wood, ebonite, paper, cork, sealing-wax, amber, ivory, or Rochelle salt. Dr. Mandell kindly allowed me to try a crystal of this highly insulating, fragile material.

From all these observations we may conclude that the lower limit of thermal conductivity of a body in which it is possible to produce any sound by contact with solid carbon dioxide is about $\frac{1}{1000}$ that of silver; while the lower limit for which it is possible (provided the other physical properties are suitable) to produce vibrations of definite pitch is probably about $\frac{1}{50}$ that of silver.

The range of vibration frequencies most easily excited lies between about 1000 and 4000 c./sec. In order to determine whether this range is affected by the size, shape and nature of the vibrating object required a further investigation the results of which are considered in the following paragraph.

5. FREQUENCIES OF VIBRATION EXCITED BY SOLID CARBON DIOXIDE.

Provided that the fundamental of a metal object is high enough that is to say about 1000 c./sec., it may be excited by means of solid carbon dioxide; otherwise the material will select an overtone. The limits of frequency within which the solid-carbon-dioxide method of excitation is effective may accordingly be estimated with considerable accuracy by observing which overtones may be excited in objects of low fundamental frequency.

Brass bars of varying length and thickness. The data given in tables 2 and 3 illustrate the fact that the overtones that may be excited by means of solid carbon dioxide become progressively higher as the fundamental

Table 2. Brass bars of varying length.

Thickness, 1.27 cm. Thickness, 0.9 cm.

Length (cm.)	Tone*	Frequency		Length (cm.)	Tone*	Frequency	
		Excited	Fundamental (first tone, 2 nodal lines)			Excited	Fundamental
11.5	1	3358	3358	11.5	1	2156	2156
15.0	1	1964	1964	15.0	1	1263	1263
22.9	2	2301	836	17.9	2	2381	860
30.0	2	1298	469	20.2	2	1876	683
	3	2523	"		3	3700	"
				30.8	3	1588	300
					4	2638	"

* Rayleigh notation.

Table 3. Bars of varying thickness

Length, 60.45 cm.

Thickness (cm.)	Fundamental (to nearest whole number)	Tones excited*	Frequencies
1.302	121	4-7	1098, 1630, 2266, 2977
0.612	57	6-10	(1011) 1476, 1892, 2328, 2875
0.325	32	8-15, 17	(1010) 1280, 1560, 1880, 2260, 2504, 2950, 3400, 4325
0.624§	65	6-10	(1211) 1568, 2080, 2500, 3005

* Add one to obtain number of nodal lines. The frequencies most usually obtained are underlined; brackets denote chattering.
§ Copper.

becomes lower. It will be noted that all the frequencies lie between 1000 and 4325.

Metal objects of varying material, dimensions and temperature. In order to avoid giving lengthy tables, the essential details of one hundred and twenty observations have been collected in figure 4 of paper (6) p.516. The frequencies of each tone excited are plotted on the right. A list of the objects excited, together with notes giving the main purposes of the various observations and the fundamentals are shown in the columns on the left.

Summary of Results. A glance at figure 4 is sufficient to show that the range of frequencies that can be excited by means of solid carbon dioxide is remarkably definite, and does not vary very much from object to object. The results for the first four bars show that a change of conductivity from 0.9 to 0.07 makes no detectable change in this range. The range is definitely lower for the massive fifth bar (938-3468), than for the small rods (1146->4000) or brass plates (1070->4000), 3 mm. thick. It was difficult to establish any change due to temperature variation, but the final observations undertaken on a brass bar, No. 12, of particularly low fundamental frequency, 21, showed that the range was a little lower when the bar was heated to 150°C. The intensity of the vibrations is often very great for the heated bars, provided always that the temperature is not sufficient to impair the vibrating properties of the metal. The latter is very important; it is, for example, easier to produce figures with hard brass than with annealed pure copper in spite of its high conductivity.

[The numerous overtones produced from plates will be noted, Nos. 14 to 16; vibrating plates became later a subject for much research (7,9,10,11,12,13)]

From the above results it may be concluded that the preferred range of frequencies lies between 1000 and 4000 c./sec. and that this range depends but little upon the size, shape, mass, material or heating of the excited metal.

6. EXPERIMENTS WITH MERCURY.

(a) Surface-tension ripples. If a small block of solid carbon dioxide of high density be floated on mercury and remain stationary, it at once gives rise to ripples varying from 1 to 3 mm. in length. The frequency of the ripples may be calculated from Kelvin's formula.

$$v^2 = n^2\lambda^2 = \frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\rho},$$

where ρ and T are the density and surface tension respectively. Taking, as Vincent*did, values of 300 to 400 c.g.s.u. for the surface tension of mercury which has not been specially freed from grease and moisture, the frequencies are found to vary from about 100 to 500. The various effects that may be produced on the surface of mercury depend upon the size, shape and height of the floating block. Under favourable conditions many beautiful experiments may be made, such as the production of ripples, stationery ripples, interference, etc.

The under surface of the block gets worn until it has a flat polished marble-like or ice-like appearance according to whether compressed snow or ice is used; this effect can be seen if the block is looked at directly after removal from the mercury. The formation of ripples usually ceases abruptly when the height of the block has diminished to 3 or 4 mm., corresponding to a pressure due to gravity on the liquid of about 0.5 x gm. wt./cm^2. The ripples may even cease before the height of the block has diminished to the above limit, if the under surface has had time to become quite flat. It is to be noted that there may still be a large horizontal area of contact between the block and the mercury, although the ripples have ceased.

If the block be bulky and be pressed down, commotion and bubbling of the liquid is produced; this may be

*J.H.Vincent, Phil. Mag. 43, 411 (1897).

compared with the noise which is made when a block makes large contact with a metal. As in the case of the production of vibrations, the change of state takes place much more rapidly while ripples are being produced.

The ripples do not cease because of the cooling of the mercury, as the bulk of this is large, and the fall of temperature, as measured by a thermometer, is small. They cease, and that suddenly, when the vertical contact is too slight, corresponding to a flat under surface and small height.

(b) Carbon dioxide boat. An irregularly-shaped block will give out ripples which vary in length in different directions. If it is light, i.e. not high, it will sometimes rotate and sometimes move forward. Thus a broad boat-shaped block may travel quite rapidly, if it is carved out so as to have smooth sides and a concave stern. If the dish containing the mercury be tipped so that the bottom forms as it were a shelving beach, the force with which a moving block is hurled on to the shore is obvious, and the force which must be exerted by the hand to keep it off is a valuable indication of the forces which are productive of sound when applied to massive solid metals. It seems certain that the propelling force is the pressure of the carbon dioxide gas. Visual evidence of the pressure can be obtained by watching the movement of a strip of tissue paper which is held near a concavity in the block which is touching the mercury.

7. THEORY

The production of vibrations by means of solid carbon dioxide, the temperature of which is at about -80°C., depends upon the sublimation which occurs when it touches the metal in the process. A large quantity of gas is produced, as has been shown by means of a Foucault-Toepler Schlieren photograph(2), the pressure of which is most obvious in the experiments made with mercury see §6 above.
The sublimation is greatly accelerated during the production of vibrations.

The source of energy is undoubtedly the heat which is transferred from the body which is at room temperature to the solid carbon dioxide during momentary light contact between a small area of the two bodies.

The fact that the maintenance of vibrations is effected by the communication of heat makes us think of Trevelyan's rocking bar, and there are interesting similarities and differences between the two phenomena. Thus in Trevelyan's experiment it is in the body which loses heat that the vibrations are produced. Again, the production of sound depends upon the difference of temperature between the heated rocker and the block upon which it rests and upon the sufficiently rapid conduction of heat near the points of contact.

Coming to the differences: we notice that the vibrations in the rocking bar are excited by the alternate expansions of the two portions of the lead block which come into contact with the two parallel grooves, and the resulting vibrations are mechanical or gravity vibrations of relatively low frequency, e.g. about one to several hundred c./sec. The vibrations produced by solid carbon dioxide are elastic and of higher frequency, and as has already been stated they are caused by the impulsive pressures of the carbon dioxide as it sublimes.

The mechanism of maintenance may also be compared to that of an ordinary electrically maintained tuning-fork in which a relaxation oscillation is excited by the vibrating body, but maintains the latter. Thus the vibrating metal determines the frequency, and the sudden gaseous pressure at the point of contact of the two bodies maintains the vibrations.

The experimental results can be explained by supposing that the main impulse depends upon the molecular forces of sublimation which are operative at the moment of contact. Each impulse lasts only for a small fraction of the total period and is given, except at the beginning of excitation, while the vibrating object is at one extreme of its swing. The localized pressure is then very great as compared with that which exists during the rest of the vibration.

It is of interest to consider the relation between the mean free path of the CO_2 molecule, which is about 3.9×10^{-6} cm. at normal temperature and pressure, and the amplitudes of vibration. Andrade and Smith* have investigated the latter in connexion with the production of Chladni figures. Thus the motion at an antinode of the vibrating surface may be represented by $a \sin nt$, the maximum acceleration amounting to an^2. In order that figures may be produced on a bar, an^2 must be greater than g, or a must be greater than $g/4\pi^2N^2 > 25/N^2$, where N is the frequency of vibration. The frequency of vibration corresponding to an amplitude of vibration of 3.9×10^{-6} cm. just sufficient to produce motion of the sand under favourable conditions would be 2530. The vibrations being intense, at any rate at the higher frequencies, the amplitudes are no doubt considerably greater than these minimum values.

The mean free path cannot of course be estimated accurately. At the first moment of sublimation while the main impulse is occurring, it should be equal to about $p^{-1}.3{\cdot}9.10^{-6}\sqrt{(193/273)}$ cm., where p is possibly many atmospheres; but half a period later, when the vibrating object is at the other extreme of its swing, if the pressure is nearly atmospheric and the temperature not much below that of the room, the mean free path will not

* Proc. Phys. Soc. 43, 405 (1931)

differ greatly from $3 \cdot 9 \times 10^{-6}$ cm. It is probable therefore that the upper limit of frequency that can be obtained with the carbon-dioxide method of excitation depends upon the vibration amplitude becoming comparable to the mean free path, and the impulse lasting too long in relation to the period to be effective. The lower limit of frequencies is imposed by the increase in the chattering which is no doubt connected with the larger amplitudes of vibration.

It is evidently sufficient to have a material of relatively low thermal conductivity in order to produce enough sublimation to separate the two surfaces, and as mentioned above loud notes have been produced in quartz bars. At the same time the production of gas is essential and no sound can be produced from insulators. In the case of heated metals, the impulses are more intense and the mean free path greater, so that the slight decrease of both the upper and lower limits of frequencies is also accounted for by the theory proposed.

8. VIBRATIONS FROM HEATED METALS PRODUCED BY CONTACT WITH ICE AND OTHER SUBSTANCES.

Ice and heated metal bars. Can the above phenomenon be produced by other materials than solid carbon-dioxide? A trial with ice and a heated brass bar led to the discovery that small momentary notes may be so produced, a fact which if not looked for would otherwise almost certainly have remained unnoticed. The material of the bar must possess adequate vibrating properties at a temperature some degrees above the temperature necessary to produce the spheroidal state, which, is about 140°C. The note is quickly muffled by the water formed and can be heard only in a quiet room. It was amplified for demonstration purposes to an audience (2) by means of ear-phone coils and magnets suitably fixed before a small piece of iron attached to the side of the bar near one of its ends. The coils were connected to amplifying valves and a loud-speaker.

Other substances and heated metal bars. The phenomenon has also been sought for and found with a large number of other substances. The metals used for establishing the conditions necessary for the production of notes in heated bars when brought into contact with the various exciting materials were zinc, aluminium, copper and steel. Certain results have been published in graphical form (2) and the conclusions reached may be summarised as follows:-

(i) Light contact made between two solid bodies which are at different temperatures may result in the production of elastic vibrations of audible frequency in the hotter body.

(ii) The hotter body must be a good thermal conductor, be capable of vibrating at suitable frequencies (about 1000 to 10,000 ~) and possess adequate vibrating properties at temperatures which exceed by some degrees the subliming, boiling or decomposing-temperature of the cooler exciting substance. Vibrations are not produced unless the metal is heated above these temperatures. The upper limit of temperature for which notes may be excited is determined by the loss of vibrating properties of the metal.

(iii) Apart from solid carbon dioxide, which would appear to be unique (at any rate among materials at present available) in its capacity to produce very loud notes, there are a considerable number of materials which may be used to excite soft pure notes of short duration in heated bars. The mechanism of production is identical with, but on a very much smaller scale than, that of solid carbon dioxide.

Such materials are: (a) Those substances which sublime or boil at suitable temperatures when brought into contact with heated metals, for example ice, camphor, iodine, and mercuric chloride. If the sublimation is immediately followed by a return to the solid state in the atmosphere, as in the case of "meta" contact must be heavier and the note produced is not pure.

Again the formation of liquid, as in the case of ice, soon quenches the note. (b) Substances which decompose when brought into contact with the heated metal with the production of gas pressure, as for example oxalic acid, ammonium phosphate, sodium thiosulphate and zinc nitrate. Solid deposit, or a melting or boiling mass on the metal, is a frequent cause of failure to produce notes with certain other compounds.

(iv) Notes have not been produced when certain compounds containing water of crystallization have been dehydrated by contact with the heated metal.

9. APPLICATIONS OF THE SOLID-CARBON-DIOXIDE METHOD OF EXCITATION

There are a number of interesting applications of this new method of exciting vibrations to which attention is drawn in the following paragraphs.

(1) Setting in vibration of metallic bodies which cannot be excited by hammering, bowing or electrical methods, e.g. small rings, small bars of magnetic or non-magnetic material, brittle materials or objects attached to apparatus which must not be jarred.

(2) Recognition of materials and estimation of relative thermal conductivities.

For example, diamonds and pearls may at once be distinguished from their counterfeits, inasmuch as the real substances

emit a rattle or squeak when touched with solid carbon dioxide. Similarly, a quartz lens may at once be distinguished from a glass one.
The geologist also will obtain more or less sound from minerals of greater or less conductivity.

(3) Recognition and accentuation of overtones. Solid carbon dioxide may fulfil the functions of a Helmholtz resonator in a novel way, in picking out the overtones in vibrating bodies. The carbon dioxide will also maintain one overtone exclusively. This property may be of considerable value to musical instrument makers, more especially for tuning percussion instruments such as bells, tubular bells, and those instruments in which graduated metal bars are employed. The overtones of irregularly shaped bodies may similarly be excited. This may sometimes be of value to the engineer in discovering undesirable resonances in small pieces of machinery or in electric plate condensers.

(4) Production of Chladni figures, with wide applications to the study of vibrating plates [6,7,9,10,11,13] and to decorative designs in art and industry (see Catalogue, Physical Society Exhibition 1938, when some examples of such designs in metal work, wood carving, embroidery etc. were shown)

(5) Tests of uniformity of metal plates; detection of flaws; study of recrystallisation phenomena - by observation of distortions in the normal nodal designs.

(6) Rapid estimation of Poisson's ratio, for metals and

alloys [12].

10. FURTHER RESEARCHES.

Moreover there are yet other researches, which, although they do not employ carbon dioxide, have developed naturally from the work described above.

Thus the simple acoustic method of estimating the vibrating properties of solid bodies, more especially in the form of bars vibrating transversely, enables very numerous observations to be made on a structure ~~and~~-sensitive property of the solid state which may vary enormously when either the temperature, the composition, or the mechanical and heat treatment of the material is altered. A special form of thermocouple, which does not interfere with the vibrations of the specimen under test has been devised [4] and some results for heated metals have been published [2,3].

The studies at low temperatures on non- transforming and transforming special-steels, undertaken at the request of Sir Robert Hadfield, are given in another paper [5]. It was for example easy to discover a maximum in Young's modulus at about $-50^{\circ}C$ in the celebrated Hadfield Manganese steel and to study the enormous increase in the internal-damping and the considerable decrease in Young's modulus which accompanies the transformation from the face-centred cubic to the body-centred cubic lattice structure of certain Nickel-steels. The method of observation is very suitable for studying either brief or lengthy secular changes. That it is considered of some practical use to the engineer to whom the subject of "damping capacity" is of great importance, is indicated by the fact that the author was asked to give a confidential report on some steels used in certain large constructions.

Curves, showing how the internal damping of Nickel varies with the degree of magnetisation have also been published [8].

The study of the elastic-vibrations produced by solid carbon dioxide has also aroused an interest in the mechanism of production of the slow gravity-vibrations of the Trevelyan-rocker. More experimental work, including a consideration of the coupling between rocker and base, must be undertaken before this phenomenon, already studied during the nineteenth and twentieth centuries, becomes thoroughly elucidated. It is for example possible to start a visible vibration of period about one second which will persist for a whole hour after the rocker has been heated and until it is perhaps only 30°C hotter than the base.

Finally, yet another phenomenon of interest may be mentioned which was noticed during the study of vibrating plates, namely the sand-striations which sometimes occur on stationary surfaces in the proximity of the plates. The author is of opinion that they are caused by vortex motions in a manner which is similar to the well known striations which occur in Kundt's tube which have recently been so extensively studied by Andrade. Unavoidable circumstances have for the last two years interfered with the investigations which had been begun on this subject.

11. CONCLUSION.

Apart from the intrinsic beauty and interest of the solid-carbon-dioxide method of producing vibrations, the applications mentioned above, suffice to show that the method is not without practical importance. Excitation is very intense and the fact that it can be effected at any point of either plane or curved surfaces and to objects which need not be rigidly supported give it certain unique advantages in the matter of producing free normal vibrations. The solid-carbon-dioxide method evidently provides a peculiarly simple and effective method of obtaining Chladni figures. Finally one other characteristic may be noted namely its capacity to pick out single overtones in objects of irregular shape which emit but a jangle of notes when struck with a hammer.

TITLES AND DATES OF PUBLICATIONS.

1. Vibrations produced in bodies by contact with solid carbon dioxide. 1933.
2. The production of sounds from heated metals by contact with ice and other substances. 1934.
3. Note on the vibrating properties of metals at different temperatures. 1934.
4. Thermocouple for vibrating metal bars. 1935.
5. Acoustic studies of some non-transforming and transforming special steels at low temperatures. 1936.
6. The production of Chladni figures by means of solid carbon dioxide. Part 1: Bars and other metal bodies. 1937.
7. Vibrations of free circular plates.
   Part 1: Normal modes.
   Part 2: Compounded normal modes.
   Part 3: A study of Chladni's original figures. 1938.
8 Magneto-damping in Nickel 1938.
9. Fundamental vibration of a rectangular plate. 1939.
10. Vibrations of free square plates.
    Part 1: Normal vibrating modes. 1939.
11. Vibrations of free square plates.
    Part 2: Compounded normal modes. 1940.
12. A simple method of finding Poisson's ratio. 1940.
13. Vibrations of free plates: Isosceles right angled triangles. 1941.

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VIBRATIONS PRODUCED IN BODIES BY CONTACT WITH SOLID CARBON DIOXIDE

BY

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VIBRATIONS PRODUCED IN BODIES BY CONTACT WITH SOLID CARBON DIOXIDE

BY MARY D. WALLER, B.Sc.

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ABSTRACT. The paper describes the conditions under which very loud notes may be produced and maintained for a considerable time in metal objects capable of vibration, such as tuning forks, bars, discs, rings, and tubes, when brought into contact with a solid carbon dioxide block. Notes have also been sustained in quartz crystals. It is shown that the vibration frequencies normally excited may range from about 1000 to 15000 ~. Lower frequencies have been excited in wires. Surface-tension ripples may be produced on mercury. The vibrations are only produced by solid carbon dioxide of high density. The physical properties of the vibrating body which are of importance in connexion with the phenomenon are considered. Evidence is brought forward in favour of the view that the source of energy for producing the vibrations is the heat which is given up by the metal to the solid carbon dioxide, and that the efficacy of this substance in producing vibrations is determined by the fact that it sublimes and in so doing produces considerable gas-pressures. Some suggestions regarding possible applications of the phenomenon are made.

§ 1. VIBRATIONS OF AUDIBLE FREQUENCY PRODUCED IN VARIOUS BODIES

If a solid block of carbon dioxide be held against a metal, a rattling or singing sound, due to vibrations of the latter, will often be heard. This fact is familiar to some of those who handle carbon dioxide commercially.

In the course of an experimental investigation of the phenomenon it was at first found that no noise could be produced in glass bars and vessels, although they had good vibrating properties, and it was concluded that this was on account of the low thermal conductivity of glass. The next objects tried were tuning forks of the ordinary frequencies. These gave out a chattering noise. When, however, forks of higher frequency were tried, loud pure notes were excited. A demonstration of the phenomenon was given at the York Meeting of the British Association, 1932.

It may be stated, in anticipation of results to be described later, that the physical properties desirable in the body to be sêt into vibration in this manner are a suitable natural period of vibration, high thermal conductivity and elasticity, a small damping factor, and hardness; and, for maintaining vibrations in small bodies of a given size, a high specific heat and specific gravity. The observations that have been made on bodies of various shapes, masses and materials are described below.

(*a*) *Tuning forks.* Range of frequencies: 100 to 10000 ~.

If a block of solid carbon dioxide of high specific gravity (see § 2 below) be brought into contact with the tip of one of the prongs of a tuning fork of high frequency, e.g. 3000 ~, the fork will be set into violent vibration, and, if it is massive, will give out a very loud piercing note. The fork should be held with a corner of one of the prongs lightly touching the block; sometimes a better result is obtained when a small portion of an edge a short way down the prong touches the block. Contact, but contact of a small area only, is essential. If a large area touches the block there may be much noise, which will diminish and cease when the surface of the block has become smooth and flat. The rate at which the solid carbon dioxide sublimes is visibly accelerated greatly during vibration of the fork.

If the fork be held skilfully, and gradually moved over the surface of the block so as to adjust the contact suitably, a loud note can be maintained for a minute or more, but only if there is no chattering.

The results obtained with tuning forks of different pitches are given in table 1, and it is evident that vibrations of higher frequency are more readily excited and maintained than those of lower frequency. Thus, with forks of low pitch it is the first overtone, and not the fundamental, that is excited. It should be noted that the ratio of frequency of the first overtone to that of the fundamental of a fork may vary from 5·8 to 6·6 (Helmholtz)*.

Table 1. Tuning forks

Natural frequency	Vibrations produced		Mass (gm.)	Duration of audibility (sec.)	Remarks
	Fundamental	First overtone			
100	Chattering	610	230	70	$n_2/n_1 = 6{\cdot}08$
256	,,	1600	158	40	$n_2/n_1 = 6{\cdot}21$
512	,,	3200	60	50	$n_2/n_1 = 6{\cdot}21$
1024	1024	Audible	219	24	Difficult to maintain
1024	1024	,,	32	20	,,
2000	2000	,,	17	14	Loud, some chattering, maintained
2040	2040	,,	28	10	Clear, some chattering, difficult to maintain
3000	3000	—	275	28	Very loud, maintained
4000	4000	—	242	16	,,
5000	5000	—	196	11·5	,,
10000	10000	—	159	3	Very loud, difficult to excite and maintain

The masses of the forks and duration of audibility are given in the table because these quantities affect the ease with which the vibrations can be maintained and will be considered in the theory of the phenomenon.

The *duration of audibility* is defined for purposes of the present investigation as the number of seconds during which the vibrations from a body remain audible to the same observer when the body is situated near one ear, after it has been set vibrating vigorously. The figure indicates roughly that which a knowledge of the modulus of decay or time constant of a damped vibration would convey accurately

* J. Tyndall, *On Sound*, 138.

regarding the vibration properties of the body. In view of the nature of the present investigation, it has not been considered necessary to make allowance for the change of sensitivity of the ear for different frequencies.

Thus different bodies, made of the same material, will vary in their resonant properties because of their shape and mass; while bodies which are similar in the latter properties but are made of various materials will have different resonant qualities depending upon the internal damping forces of solid viscosity. The figures given are each the mean of several values which may vary among themselves by from 10 per cent to 20 per cent.

(*b*) *Brass bars*. Range of frequencies: 1950 to 15000 ~.

The results obtained with a series of twelve rectangular bars, of square cross-section ($1{\cdot}27 \times 1{\cdot}27$ cm^2) excited in a similar fashion to the tuning forks, are given in table 2. The note generally given out by such relatively short stout bars corresponds to a transverse vibration with two nodes. The bars were suspended by threads from the nodes (about $\frac{1}{4}$ of the length from either end); the positions of these could be made apparent by sprinkling sand on the upper surface, and then setting the bars in vibration.

Table 2. Brass bars

Length (cm.)	Frequency		Duration of audibility (sec.)	Remarks
	Measured with sonometer	Calculated		
15·1	1950	1950	20	Very loud, maintained
12·7	2620	2780	17	
12	3200	3100	13	
11·5	3300	3390	11	
10·9	3500	3780	9	
10·2	4200	4300	10·8	Sometimes excited, very loud, maintained
9·5	4700	4970	7	
9·1	—	5400	4·9	
8·8	—	5800	4	Difficult to excite
8·5	—	6200	4·4	
8·1	—	6800	3·8	
7·7	—	7600	3·8	
6·5	—	10000	2	
5·25	—	15000	0·4	

The frequencies as determined practically with a sonometer and forks agree reasonably with the values calculated from the formula given by Rayleigh* for a free-free bar of rectangular cross-section. Thus for brass, the frequency

$$n = \tfrac{2}{3} \times 538400 \times t/l^2,$$ *n*

where t is the thickness, and l the length of the bar. *t, l*

With some blocks of carbon dioxide frequencies up to 4300, and with others frequencies up to 6200, were easily excited and maintained, by touching the bars either at an end or at the centre. The difficulty of exciting higher frequencies is

* Rayleigh, *Theory of Sound*, **1**, 279.

probably partly due to the smallness of the time constant rather than to the high value of the frequency. The longest bar weighed 200 gm. The figures given and statements made in the tables are of a somewhat general nature on account of the varying behaviour of different samples of solid carbon dioxide (see § 2).

Table 3 shows how the time constant diminishes with increasing cross-section of bars of equal length, and how this property partly controls the ease with which vibrations may be excited with solid carbon dioxide.

Table 3. Brass bars of square cross-section, and of length 11·5 cm.

Thickness (cm.)	Frequency	Duration of audibility (sec.)	Remarks
1·27	3330	10	Very loud, maintained
2	5230	5	Very loud, difficult to excite, maintained
2·5	7100	4	Very difficult to excite or maintain

(*c*) *Steel bars*. Range of frequencies: 4440 to supersonic.

Twenty-one steel bars of circular cross-section and of radius 1 cm. were available. The first fifteen bars, ranging from 15·1 cm. down to 7·7 cm. in length, were suspended and excited in the same manner as the brass bars (*b*). The frequencies as calculated ranged from 4450 to 18000 ~, and the duration of audibility diminished from 5·5 down to 1·8 sec. The bars were harder to excite than the brass bars on account probably of the smaller time constants, but piercing and maintained notes were sometimes obtained, especially when the bars were first of all struck with a hammer so that they were already vibrating before application of the carbon dioxide. The highest bar maintained was the tenth of the series having a (calculated) frequency of 11200 ~.

(*d*) *Longer bars*. Transverse overtones and longitudinal vibrations.

It would be of interest to carry out more observations on longer bars, which often vibrate in three or more audible modes simultaneously. The results so far obtained will not be described in detail, but two remarks will be made. (1) As in the case of tuning forks, a low-frequency bar is inclined to chatter, but an overtone (transverse vibration) can usually be produced. (2) It is possible to excite longitudinal vibrations. For example, in the case of three brass bars of lengths 100, 38 and 23 cm. respectively, the corresponding longitudinal vibrations of 1760, 4900 and 8000 ~ were produced, but were not maintained for any considerable length of time. Similarly an aluminium bar of length 91 cm. vibrated longitudinally with a frequency of 2660 ~. It will be noted that the bars have been so chosen that the frequencies are comparable with those of the transverse vibrations produced in short bars; the longitudinal vibrations of the latter were of course supersonic.

(*e*) *Metal plates*. Chladni sand figures.

Thin plates, of either rectangular or circular cross-section, e.g. the usual Chladni plates 4, 8 or 12 in. in diameter and 2 mm. in thickness, which give numerous sand patterns when bowed, could generally only be excited to give out a confused noise with the block of carbon dioxide, though overtones could sometimes be picked out. On the other hand, loud notes were obtained with smaller and thicker discs which are not easily excited by bowing. Chladni sand figures may be produced, and a photograph of one of these is shown in the plate.

The data relating to six brass blanks are given in table 4, which also includes observations made on copper and zinc discs. It is to be noted that although copper conducts heat three times as well as brass and has a modulus of elasticity and density nearly equal to that of brass, its vibrating properties are inferior on account of internal friction. On the whole its capacity for vibrating in contact with carbon dioxide is about equal to that of brass. The frequencies were determined by means of a sonometer except in the case of no. 7, the frequency of which was compared with those of the series of brass and steel rods; see (*b*) and (*c*) above. Its frequency was also calculated from the approximate relation that the frequency of the discs varies directly as the thickness and inversely as the square of the radius. The discs were held centrally between finger and thumb. Irregular shaped thin discs, such as brass cymbals, may be excited to give out noise, while, e.g., a thick silver dish may give out a noise that has been likened to that of a pneumatic street drill!

Table 4. Metal discs

Number of disc	Material	Radius (cm.)	Thickness (mm.)	Frequency	Duration of audibility (sec.)	Remarks
1	Brass	10	5	1600	21	Very loud, maintained
2	,,	8·8	5	2200	14	
3	,,	7·6	5	2960	11	
4	,,	6·9	5	3260	8	
5	,,	5	5	7750	2	Difficult to excite
6	,,	2·5	5	Supersonic	—	—
7	,,	5	2	3100	5	Loud, maintained
8	Copper	7·6	5	2970	2	Very similar to no. 3
9	,,	7·6	3	3820	2	Clear, maintained
10	Zinc	7·6	3	4000	0·5	,,

(*f*) *Bars of various materials.*

Several materials were tried in the form of brass bars suspended as at (*b*) above, attention being paid to various physical properties. In table 5 the bars are arranged in order of decreasing thermal conductivity. The cross-sections of the bars were either square or circular, except in the case of Trevelyan's brass bar with the wooden handle removed.

The physical constants quoted in the table are sometimes only approximate and sometimes the mean of several values, but are sufficiently accurate for present purposes; they are in most cases taken from tables. In the case of carbon, Young's modulus was calculated after determining the density, dimensions and the vibration frequency of one of the rods.

Table 5. Bars of various materials

Material	Length (cm.)	Diameter *d* or thickness *t* (cm.)	Frequency	Duration of audibility (sec.)	Thermal conductivity	Young's modulus of elasticity (c.g.s.u. × 10^{11})	Density	Result
Copper	14·1	1·27 *t*	2190	16	0·9	12·3	8·9	Very loud, maintained
Aluminium	15·3	1·27 *t*	2290	15	0·50	7	2·6	,,
Duralumin	15·3	1·27 *t*	2390	30	0·31	6·9	2·8	,,
Trevelyan's brass rocker	11·2	Irregular	2420	2	0·26	10	8·4	Difficult to maintain
Zinc	14	1 *d*	880	2	0·26	8·7	7·1	Loud, difficult to maintain
Lead	15	1·27 *d*	510	0	0·08	1·6	11·4	Noise
Eureka	13·2	1·27 *t*	{3100, 3260	5	0·05	16·3	8·9	Clear, difficult to maintain
,,	10·3	,,	5020	6	,,	,,	,,	,,
,,	7·1	,,	11000	3·5	,,	,,	,,	,,
Arc carbon	14·3	0·5 *d*	670	0·5	0·01	0·9	1·4	No sound produced
,,	18	1·2 *d*	820	,,	,,	,,	,,	,,
,,	19·2	1·8 *d*	1060	,,	,,	,,	,,	,,
,,	13·3	1·8 *d*	2250	,,	,,	,,	,,	,,
,,	8·2	2 *d*	6580	,,	,,	,,	,,	,,
Glass	30·5	1·2 *d*	1970	9	0·001	6	2·9	,,

It is to be noted (1) that Trevelyan's rocking brass bar with its irregular cross-section is not such a good vibrator as a simple bar. A purer note can be obtained by means of carbon dioxide than by hammering. The pureness of the notes obtained by excitation by means of carbon dioxide is an interesting feature of the phenomenon. (2) Zinc has poorer vibrating properties than brass and is accordingly difficult to maintain in vibration, though the conductivity is equal to that of brass and the note emitted is clear. (3) The pitch of the lead bar could not be determined either by hammering or by means of carbon dioxide, but considerable noise could be produced by contact with the latter. The pitch was determined in an interesting manner, namely by increasing the vibrating properties of the bar by cooling it in a vacuum flask containing solid carbon dioxide and setting it in vibration the moment it was removed. The result was in agreement with calculation. In the case of two thick lead tubes, not included in the table, it was found possible to recognize the pitches, 980 and 2400 respectively, because of the rather better vibrating properties of the tube, and to get a purer momentary note with the carbon dioxide than with the hammer. (4) Eureka was investigated on account of its particular thermal conductivity, and after the failure to produce any sound from arc carbon rods; it behaved as expected. The longest bar was not very accurately square and its vibrating properties were accordingly worse than they should be. (5) Particular attention was paid to carbon as at the time it was felt that it was the borderland substance, having regard to the thermal conductivity, between those bodies which could and those which could not be made to vibrate. No sound has been produced in carbon. Though the damping factor is considerable, the vibrating properties of the rods are sufficiently good for a clear note to be obtained when they are hammered. In later observations, see (*i*) below, it has been shown that sound can be produced in bodies having a far smaller thermal conductivity than carbon; and in particular in a diamond. We must conclude that carbon rods, made as they are by compressing gas carbon, do not behave like such solids as metals or stones, and that the hardness of the material and not the thermal conductivity is the factor which determines that sound cannot be produced in carbon and can be produced in a diamond.

(*g*) *Tubes, rings and bells.*

(1) Large thick brass tubes. Several large brass tubes have been investigated. For example, one of them had an outer diameter of 6·3 cm., the walls were 3·5 mm. thick, the length was 10 cm., and the duration of audibility was 8 sec. The principal note which could be excited and maintained by means of carbon dioxide had a frequency of 1950 ~. The tube was sawn into two parts one-third and two-thirds of its length respectively, and the same note was excited in each of them. The notes doubtless corresponded to flexural vibrations with the production of four nodal lines.

(2) Small thick brass tubes. Thick brass tubes have better vibrating properties than either thin tubes or solid rods. They exhibit very strikingly the fact, already noted more than once, that overtones rather than a low fundamental vibration will be excited by solid carbon dioxide. Similar results have been obtained in tubular bells.

(3) Small rings. If a wedding ring be placed on a flat surface of a carbon dioxide block, a small high note of definite pitch may be produced, lasting a few moments. This observation is recorded for two reasons. Firstly because it indicates a means of exciting vibrations in small objects which cannot conveniently be set vibrating by other means, and secondly because it demonstrates practically that it is only while the metal is losing heat to the block that the vibrations occur. The ring leaves an imprint on the block. In the case of small objects the specific heat and specific gravity are physical factors that will affect the time for which the sound will continue.

(4) Brass bells. Several hand bells have been investigated. For example, a bell that gave out sounds of frequencies 270 and 1460 with its hammer gave out the latter frequency only when excited with carbon dioxide.

(*h*) *Wires.*

(1) Sonometer. Only a very small amount of chattering can be produced in sonometer wires. We note that the mass of metal is small, and the frequencies rather low. The persistence of vibration in a particular sonometer was about a third of that of a small upright piano with the sustaining pedal down.

(2) Piano. The wires of a small upright piano were investigated with the sustaining pedal down. The unexpected result was obtained that the fundamental

tones and not the overtones are excited by the carbon dioxide down to a frequency as low as about 100 ~. Notes of frequencies exceeding 1000 are pure, clear and maintained; as the frequencies get smaller chattering gradually increases, but it is still the fundamental that is produced and maintained. In the case of the lowest wires, corresponding to frequencies from about 30 to 180 ~, noise, including the fundamental and the overtone three octaves above (240 to 800 ~), was usually heard. The duration of audibility of the notes ranges from about 50 sec. for the longest, to 5 sec. for the shortest wire. Wires are the only vibrating bodies in which low frequencies have been excited; we may suppose that the difference in this respect between wires and other vibrating bodies is due to the fact that the carbon dioxide gas under pressure flows past the wire on both sides of it.

(*i*) *Miscellaneous objects.*

(1) Silver objects. As had been expected on account of its high conductivity, clear and sometimes beautiful tones can be produced in silver objects. According to Honda and Konno* the solid viscosity of silver lies between that of brass and steel. Thimbles, aural specula (about 3000 ~) and even irregularly shaped objects can be excited so as to give out a single note. A small tea pot was made to emit a pure tone (2860 ~) when touched at a particular place on the bottom. This picking out of a note when an object is touched at the right spot is very characteristic.

The observations made on a table spoon, dessert spoon and tea spoon respectively, touched by the blocks as indicated in table 6, are given because they emphasize the possible application of the carbon-dioxide method of excitation to picking out the resonant vibrations of irregularly shaped objects. Pure, piercing tones were sometimes obtained.

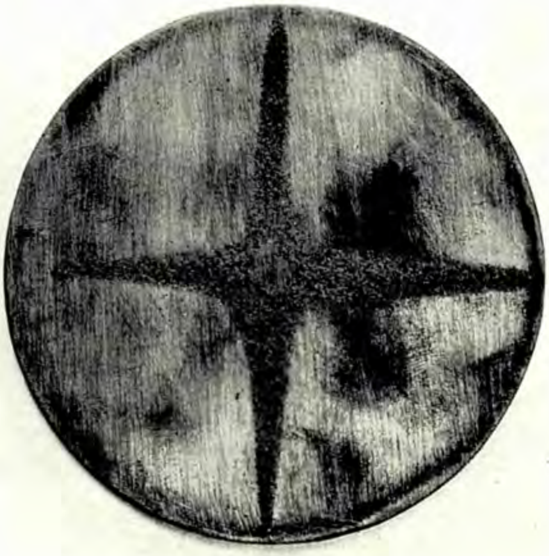
Table 6

End of handle	Between handle and centre of gravity	Near centre of gravity	On bowl
850	570	2750	Various
1150	800	4000	,,
1740	1170	—	,,

(2) Quartz. Quartz is a semi-conductor the thermal conductivity of which varies with direction. Its vibrating properties are good. The conductivity along the optic axis is 0·06 and in one direction at right angles to this it is 0·03. These values are intermediate to those of eureka and carbon respectively. Two quartz discs of diameters 5 cm. and 3·8 cm., and thicknesses 1·6 mm. and 1·8 mm. respectively were excited to give out clear notes by means of carbon dioxide, though these were not maintained. The pitch of the larger disc, determined experimentally, was 3800, while that of the smaller one, as calculated from the relative dimensions of the two, was 7400.

When a block of carbon dioxide was held against a quartz convex lens considerable noise was produced. This result was compared with that obtained with a

* *Phil. Mag.* **42**, 115 (1921).



Chladni sand figure on brass disc, produced by contact with solid carbon dioxide.
Diameter, 3 in.; thickness, 5 mm.; frequency, 2960 ~.

To face page 109

glass lens and a pure block of carbon dioxide ice (see § 2 below), when for the first time a very slight noise was detected with glass. It is thus just possible under the most favourable conditions, i.e. using a material with good vibrating properties and a very efficient block of carbon dioxide, to detect sound (not to produce sustained vibrations) in a material the conductivity of which is about $\frac{1}{1000}$ that of silver. Quartz is the only non-metal in which a note of definite frequency has so far been produced.

§ 2. SOME PROPERTIES OF SOLID CARBON DIOXIDE RELEVANT TO THE INVESTIGATION

It was soon noticed that the ease with which vibrations could be excited varied from day to day, and that when a fork was being excited water sometimes condensed on it near the point of contact with the carbon dioxide block. This latter fact is presumably caused by a cooling of air below the dew point. Solid carbon dioxide sublimes at about $-80°$ C.

The presence of dew led to the suspicion that changes in weather might be responsible for the differences in the results obtained on different days. However, experiments conducted (1) in a dried closed space, (2) in a moist closed space, (3) near an electric stove, and (4) in the cool atmosphere of an electric refrigerator showed that these varying external conditions produced no effect that could be appreciated, either in hindering the vibrations if these were easily excited, or in assisting their production on occasions when they were difficult to produce. In connexion with the production of ripples on mercury (see § 4 below), the question of pressure due to the weight of the block was considered; and when the density of the blocks was ascertained, by weighing and measurement, it was found to vary from day to day. This suggested the cause of the behaviour of different blocks. It is only blocks of high density which will produce sound in metals, and this doubtless is the reason why the phenomenon has not attracted more attention.

Light snow carbon dioxide, which floats on water, may be compressed into solid blocks, which vary in density and hardness according to the amount of air left in them. When sufficiently compressed they behave like ice carbon dioxide in producing vibrations, and the density of some blocks the author has handled has been as high as 1·4. It is to be noted that a block made by compressing snow may be denser at the centre than at the outside. The density of ice blocks which have been made by the process of directly freezing with the aid of liquid air also varies; an entirely solid block would have a density of 1·56, but any block consisting of solid ice crystals will be satisfactory. Some blocks contain more oil than others, and the author is inclined to consider that this reduces their efficiency in producing vibrations. Apart from actual determination of the density, it has been found possible to recognize blocks from different factories by viewing them by transmitted light, and also by rubbing a surface and watching how far a snow crust forms as sublimation occurs. Crust does not form on a good ice block.

There is some evidence of selectivity in the range of frequencies best excited by different blocks. Thus, a very efficient block excited loud notes between 2000 and 15000 ~, while a lighter and less efficient block excited the 1024 and 2000 ~ forks most easily and also excited the 1960 ~ brass bar. The production of gas-pressure is probably slower, and evidently less efficient, with blocks of low density than with those of high density.

The author is indebted to Dr Ezer Griffiths for the following data. The sublimation temperature of carbon dioxide may vary from $-78°$ C. down to $-93°$ C. In passing from the solid to the gaseous state 1 gm. takes up 138 calories. The specific heat of solid carbon dioxide at $-78°$ C. is 0·31. The mean specific heat of the gas is 0·19. The total refrigerating effect of 1 gm. from solid to gas at 0° C. is 153 cal.

It is to be noted that on account of the large heat of sublimation, solid carbon dioxide can be conveniently stored for a considerable time in a vacuum flask.

§ 3. MISCELLANEOUS OBSERVATIONS

(*a*) *Sound produced in thermal semi-conductors.* In early observations there was no question of producing any sound from non-metal solids. We have noted however that with a pure ice block, a minute sound was detected in a glass lens.

Other non-metallic solids have accordingly been investigated. They are enumerated below in descending order of thermal conductivity, as far as this is known, or according to the results of the carbon-dioxide test when it is unknown. Small sounds were heard when the following objects were touched with the solid carbon dioxide: various precious stones including pearls, diamonds, rubies and garnets; crystals, marble, mother of pearl, slate, glass.

No sound was obtained from wood, ebonite, paper, cork, sealing-wax, amber, ivory, or Rochelle salt. Dr Mandell kindly allowed me to try a crystal of this highly insulating, fragile material.

From these observations and from those described in the previous section we may conclude that the lower limit of thermal conductivity of a body in which it is possible to produce any sound by contact with solid carbon dioxide is about $\frac{1}{1000}$ that of silver; while the lower limit for which it is possible (provided the other physical properties are suitable) to produce vibrations of regular pitch is probably about $\frac{1}{50}$ that of silver.

(*b*) *Vibrations in heated bars.* The vibrations produced in a brass bar are little affected by heating it, either by means of a bunsen flame or by means of an electric oven (in the latter case to about 180° C.). The intensity of sound which can be obtained is however reduced in a bar which is more strongly heated by means of a glass-blowing flame. The reduced effect is due to the diminishing vibrating properties of the brass.

(*c*) *Liquid air.* It was not found possible to produce any sound (due to vibrations of the bar) by holding the tip of a brass bar against the surface of liquid air contained in a vacuum flask.

(*d*) *Vibrations in heated metal bars caused by contact with ice.* Clear, feeble, momentary notes, which are quenched at once by the formation of water, have actually been produced by contact with ice in heated brass bars under certain conditions. It is hoped to describe these observations when they are complete.

§ 4. EXPERIMENTS WITH MERCURY

(*a*) *Surface-tension ripples.* If a small block of solid carbon dioxide of high density be floated on mercury and remain stationary, it at once gives rise to ripples varying from 1 to 3 mm. in length. The frequency of the ripples may be calculated from Kelvin's formula

$$v^2 = n^2\lambda^2 = \frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\rho},$$

where ρ and T are the density and surface tension respectively. Taking, as Vincent* did, values of 300 to 400 c.g.s.u. for the surface tension of mercury which has not been specially freed from grease and moisture, the frequencies are found to vary from about 100 to 500. The various effects that may be produced on the surface of mercury depend upon the size, shape and height of the floating block. Under favourable conditions many beautiful experiments may be made, such as the production of ripples, stationary ripples, interference, etc.

The under surface of the block gets worn until it has a flat polished marble-like or ice-like appearance according to whether compressed snow or ice is used; this effect can be seen if the block is looked at directly after removal from the mercury. The formation of ripples usually ceases abruptly when the height of the block has diminished to 3 or 4 mm., corresponding to a pressure due to gravity on the liquid of about 0·5 × gm. wt./cm.^{2} The ripples may even cease before the height of the block has diminished to the above limit, if the under surface has had time to become quite flat. It is to be noted that there may still be a large horizontal area of contact between the block and the mercury, although the ripples have ceased.

If the block be bulky and be pressed down, commotion and bubbling of the liquid is produced; this may be compared with the noise which is made when a block makes large contact with a metal. As in the case of the production of vibrations, the change of state takes place much more rapidly while ripples are being produced.

The ripples do not cease because of the cooling of the mercury, as the bulk of this is large, and the fall of temperature, as measured by a thermometer, is small. They cease, and that suddenly, when the vertical contact is too slight, corresponding to a flat under surface and small height.

(*b*) *Carbon dioxide boat.* An irregularly-shaped block will give out ripples which vary in length in different directions. If it is light, i.e. not high, it will sometimes rotate and sometimes move forward. Thus a broad boat-shaped block may travel quite rapidly, if it is carved out so as to have smooth sides and a concave stern. If the dish containing the mercury be tipped so that the bottom forms as it were a shelving beach, the force with which a moving block is hurled on to the shore is obvious, and the force which must be exerted by the hand to keep it off is a valuable indication of the forces which are productive of sound when applied to massive solid metals. It seems certain that the propelling force is the pressure of the carbon dioxide gas. Visual evidence of the pressure can be obtained by watching the

* J. H. Vincent, *Phil. Mag.* **43**, 411 (1897).

movements of a strip of tissue paper which is held near a concavity in the block which is touching the mercury.

(*c*) *Mercury hygrometer.* If mercury be progressively cooled by contact with solid carbon dioxide, dew eventually forms on the surface. The dew forms suddenly, but owing to the ripples it appears far below the dew point. Thus on a warm damp day when the room temperature was 68° F., dew only appeared at 37° F., but then appeared copiously. On the other hand when the block was removed at the right moment leaving the surface still, dew formed at 60° F. as compared with 59° F. on the surface of a polished calorimeter containing water to which pieces of ice were added.

Thus, were it not for the ripples, mercury would offer an excellent polished surface for dew-point determinations, and solid carbon dioxide an easy method of cooling it. There might also be objections, however, from the point of view of hygrometry to the presence of carbon dioxide gas.

§ 5. THEORY

Gathering up the significant experimental facts, and at the same time developing a theory of the production of vibrations in solids by contact with solid carbon dioxide, we note:

(1) The source of energy is undoubtedly the heat which is transferred from the body to the carbon dioxide during momentary contacts of the two bodies. The thermal conductivity and mass of the body must be sufficient to ensure an adequate supply of heat, although the area of contact must be small. The conditions are well fulfilled with tuning forks, metal bars, thick tubes and thick metal discs, and vibrations may be maintained in these bodies for a considerable period. With the exception of the semi-conductor quartz, metals are the only materials that have been excited to give out sustained notes. Other crystals of suitable conductivity could doubtless be set in vibration. With bodies of small mass, the source of energy and consequently the vibrations cease almost immediately, because of the fall of temperature.

(2) The efficacy of solid carbon dioxide in producing vibrations is undoubtedly due to the fact that it sublimes, with the production of considerable gaseous pressures. These pressures are made evident in the experiments with mercury. The sublimation is greatly accelerated during the production of vibrations. Vibrations cannot be produced by contact of a metal with liquid air, and although under certain conditions feeble momentary notes have lately been detected when ice comes into contact with heated metals, the formation of water quenches the vibrations, and these observations confirm the view that it is the fact of sublimation that is essential to the production of loud notes. The process reminds us of the vibrations of the reed of a wind instrument. We may even speculate as to whether the contact of the gas with the metal does not produce eddies as it does in the cases of Aeolian and edge tones, and whether it would be possible to obtain magnified stroboscopic photographs by the shadow method showing how the gas is moving!

(3) The fact that the maintenance of vibrations is effected by the communication of heat makes us think of Trevelyan's rocking bar, and there are interesting similarities and differences between the two phenomena. Thus in Trevelyan's experiment it is in the body which loses heat that the vibrations are produced. Again, the production of sound depends upon the difference of temperature between the heated rocker and the block upon which it rests, and upon the sufficiently rapid conduction of heat near the points of contact.

Coming to the differences: we notice that the vibrations in the rocking bar are excited by the alternate expansions of the two portions of the lead block which come into contact with the two parallel grooves, and the resulting vibrations are mechanical or gravity vibrations of relatively low frequency, e.g. forty to several hundred*. The vibrations produced by solid carbon dioxide are elastic and of higher frequency, and as has already been stated they are caused by the impulsive pressures of the carbon dioxide as it sublimes.

(4) It is of interest that, with the exception of wires, the range of frequencies excited should be so high, for example, the overtone 3200 ~ and not the fundamental of a 512 ~ tuning fork will be excited by contact with solid carbon dioxide. The necessary passage of heat and sublimation must accordingly occur in an extremely short interval of time. If we try to excite too low a frequency there will be chattering because the pressure and volume of gas sublimated are so great. On the other hand it has so far been found impossible to excite frequencies above 15000. One hesitates to ascribe this to want of gas-pressure; it may be due to the inevitably small time constant of the vibrators in question, as they must either be made very thick which increases the damping factor, or very small, which diminishes the energy content, and we have seen that there is difficulty in maintaining vibrations unless the time constant is sufficient. We may well imagine that rather different conditions prevail in the case of wires and that the gas probably flows past them on both sides. Noise is obtained if the area of contact is not small. We may suppose that the metal is receiving a number of impulses from different parts of the block which do not synchronize. Once the block is worn flat the noise ceases.

(5) Finally the mechanism of maintenance may be compared to that of an ordinary electrically maintained tuning fork in which a relaxation oscillation† is excited by the vibrating body, but maintains the latter. Thus the vibrating metal determines the frequency, and the sudden gaseous pressure produced in a small cavity or groove at the point of contact of the two bodies maintains the vibrations.

A highly damped vibrator is not excited by contact with solid carbon dioxide. The author suggests that the reason why large Chladni plates as a rule only give out various noises when excited is that there are generally some four dozen possible audible modes of vibration, and that while the lower ones will not be excited, no single one among the higher modes has the necessary time constant to govern the situation and set up vibrations of one particular frequency. It is otherwise when

* E. G. Richardson, *Phil. Mag.* **45**, 976 (1923). S. Bhargava and R. N. Ghosh, *Phys. Rev.* 517 (1923).

† van der Pol, *Phil. Mag.* **51**, 978 (1926).

such a plate is bowed, as the time constant is not of the same significance, and the note given out by the plate is not maintained. On the other hand it is to be expected that Chladni sand figures will be easily produced in thick metal discs with blocks of carbon dioxide.

§ 6. SUGGESTED APPLICATIONS AND EXPERIMENTS

In conclusion some suggestions will be made regarding possible applications of the carbon dioxide method of exciting vibrations.

(1) *Loud maintained standard frequencies.* Stout forks of frequencies ranging from 2000 to 6000 ~, or even above this, are not easy to excite electrically, but if they are excited as described above, § 1 (*a*), loud notes may be maintained for a minute or more. These might be useful in certain wireless investigations. For some purposes an inexpensive standard would be a small thick brass tube suspended from the two nodes by threads.

(2) *Setting in vibration of metal objects* which cannot be excited by hammering, bowing or electrical methods, e.g. small rings, small bars of magnetic or non-magnetic material, or objects attached to apparatus which must not be jarred.

(3) *Recognition and accentuation of overtones.* Solid carbon dioxide may fulfil the functions of a Helmholtz resonator in a novel way, in picking out the overtones in vibrating bodies. The carbon dioxide will also maintain one overtone exclusively. This property may be of considerable value to musical instrument makers, more especially for tuning percussion instruments such as bells, tubular bells, and those instruments in which graduated metal bars are employed. The overtones of irregularly shaped bodies may similarly be excited. This may sometimes be of value to the engineer in discovering undesirable resonances in small pieces of machinery or in electric plate condensers. The damping in the case of stretched metal diaphragms is probably too great for the method to be applicable, but it would be interesting to search for resonances in these.

(4) *Rapid estimation of relative thermal conductivities in solid non-metals.* Solid carbon dioxide which has been prepared by freezing and which is free of oil may be considered to have standard properties in producing vibrations. With such a block, more or less noise may be produced by contact with solid substances which are semi-conductors, as for example quartz, marble and other stones, and it is only in substances with high insulating properties such as sealing-wax, ebonite, amber or cork that no sound whatsoever can be detected. These facts may be of interest to the geologist. Again quartz lenses can be immediately distinguished from glass lenses by bringing them into contact with solid carbon dioxide.

(5) *Setting in vibration of piezo-electric crystals of audible frequency**. No experi-

* Since the above was written a natural quartz crystal, $5\frac{1}{2}$ in. long and of cross-section about $\frac{1}{2}$ in., has been tried at the Natural History Museum. A loud maintained note was produced, of frequency just above 4000 ~. The note is harder to start than in the case of metals, because of the smaller thermal conductivity; on the other hand the loudness increases very rapidly. A crystal, $2\frac{1}{2}$ in. long and of cross-section about $\frac{1}{4}$ in., has been investigated in the laboratory, the frequency in this case being about 6000 ~. It is hoped shortly to try a bar of square cross-section which is being cut from a large crystal.

mental work has been done on this subject, but in a crystal such as quartz which has considerable thermal conductivity one would expect interesting results since it was found possible to produce a note in a quartz disc. The method has been found useless in the case of crystals of low thermal conductivity, such as Rochelle salt.

(6) *Study of ripples in mercury* (see § 4 above).

(7) *Mercury hygrometer* (see § 4 above).

DISCUSSION

Mr G. G. Blake asked whether the wave-length of the ripples produced on mercury was related to the size of the piece of carbon dioxide employed. He had found that when solid carbon dioxide was allowed to form bubbles in a liquid, the speed of bubbling increased as the size of the solid was diminished, and depended also on the nature of the liquid, large bubbles being produced at a slow rate in thick oil.

Principal S. Skinner said that the phenomenon described was analogous to the spheroidal state of water.

Mr C. R. Darling suggested that graphite might behave similarly to metals towards solid carbon dioxide. He thought that the traditional explanation of the singing kettle needed revision, because a glass beaker did not give the same result as a copper kettle. The conductivity of the material had to be taken into account.

Mr R. Appleyard suggested, in the light of an experiment by Prof. Boys, that the mercury ripples might be due to breakdown of the surface tension of the mercury. It was possible to project a demonstration of the spheroidal state optically by running a drop of water down a vertical platinum wire on to a hot plate, and projecting the end of the wire; it would be interesting if the author could project one of her experiments in a somewhat similar way. Had the phenomenon anything to do with cavitation? Was the surface of the metal roughened?

Mr F. C. Mead suggested that the failure of carbon to vibrate might be due to its absorbing the gas.

Dr J. E. R. Constable: The author mentioned that the period of decay of her vibrating bars when struck depended but slightly upon the weight of the hammer used for the striking. I would suggest that this can be simply explained by the logarithmic nature of the decay of the oscillations, and does not need the hypothesis of variation of internal viscosity with amplitude which was advanced. Judging from the intensity of the sound generated by the bars exhibited by Miss Waller I should say that the initial amplitude is probably about 10^4 times the final amplitude and, since the time of decay of the sound is proportional to the logarithm of the initial amplitude, a factor of 2 or 3 in this amplitude will produce only a comparatively small change in the decay period.

Prof. A. O. Rankine said that the last speaker's suggestion was borne out by Dr A. H. Davis's experience in using a tuning fork as a standard of comparison in noise-measurement. Small variations in the initial amplitude of the fork did not seriously impair the consistency of the results.

Author's reply. In reply to Mr Blake: Factors which increase the gas-pressure between the block of carbon dioxide and the mercury diminish the wave-length of the ripples produced. Thus a higher block (within limits) or a concavity at the side of the block near the surface of the mercury diminishes the wave-length. See also § 4 of the paper.

In reply to Principal Skinner: The minimum difference between the temperature of the metal plate and the boiling point of water necessary to produce the spheroidal state is about 40° C. This is of the same order of minimum temperature-difference as that which must exist between metal and carbon dioxide if vibrations are to be produced. For example, a brass bar, cooled by means of solid carbon dioxide, gave no sound when touched with carbon dioxide until its temperature had risen from − 80° C. to about − 30° C. A practical difficulty in getting very exact results arises from the fact that hoar-frost forms on the metal.

I am much obliged to Mr Darling for very kindly sending me a specimen of Acheson graphite. Noise is produced when it is brought into contact with solid carbon dioxide; moreover the frequencies of vibration of several bars have been recognized. The notes cannot be maintained for any considerable period. The thermal conductivity of graphite is much greater than that of carbon, and its apparent density is 1·56 while its real density is 2·21; hence its porosity is 28 per cent, which is less than that of carbon.

In further reply to Mr Darling and in reply to Mr Mead: I find that notes can be produced and maintained for a few seconds in a carbon rod on which copper has been deposited electrically. Mr G. E. H. Rawlins suggested this experiment. It would seem that the porosity is the determining factor in preventing the production of vibrations in carbon rods.

In reply to Mr Appleyard: An attempt to project a demonstration in an optical lantern was not successful. I have not found that metals are in any way physically affected by the gas-pressure of the carbon dioxide.

I agree that the constancy of the results obtained by hand-hammering is sufficiently accounted for by the logarithmic nature of the decay of the vibrations, as suggested by Dr Constable. In experiments with brass bars of the same frequency, 2070 ∼, but of different masses, and hammers of 120 and 5 gm. weight respectively, I find that with bars whose masses are greater than about 100 gm., the heavier hammer, while with bars less than 100 gm. in mass, the lighter hammer, is the more efficacious. With bars of about 100 gm. it matters little which hammer is employed. The fact that hand and ear can give such consistent figures is of considerable interest.

2.3.

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THE PRODUCTION OF SOUNDS FROM HEATED METALS BY CONTACT WITH ICE AND OTHER SUBSTANCES

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ABSTRACT. Experiments in which very loud notes were produced as a result of contact of metal bodies with solid carbon dioxide have been described in a previous communication*. The present paper is concerned with the conditions under which soft notes may be produced from heated metal bars brought into contact with ice and a number of other substances. These substances must either sublime or boil or decompose with the evolution of gas, at temperatures for which the metal to be excited still retains adequate vibrating properties. The experiments confirm the theory regarding the mechanism of the phenomenon which was developed in the previous paper. They establish the general conditions under which elastic vibrations of audible frequency may be excited in a metallic body, when contact with another cooler solid substance results in the production of gas from the latter.

§ 1. INTRODUCTION

The conditions under which loud sustained notes may be produced by bringing metal bodies into contact with solid carbon dioxide have been fully described in a previous communication†. This phenomenon, as has already been mentioned in that paper, led to the discovery that small momentary notes may be produced from heated metals when brought into contact with ice, a fact which would otherwise almost certainly have remained unnoticed.

Some of the conclusions reached in the above communication will now be reviewed, and the observations which have been made on several heated metal bars—ice and a number of other materials being used to set them vibrating—will then be described.

(i) Solid carbon dioxide excites metal bars whose natural vibration frequencies may vary from about 1000 to 15000 ~. In preliminary experiments a few of these bars differing widely in frequency were heated, and it was found possible to set them into vibration by contact with ice. There is thus in the case of excitation by ice (or presumably by the other substances used) a wide range of audible frequencies which can be produced.

(ii) Variously shaped bodies—bars, rings, tubes, bells, discs, etc.—can be set into vibration by contact with solid carbon dioxide. It has been considered sufficient to restrict the present study to metal bars, and to assume that bodies of other shapes could be excited if heated suitably.

* *Proc. Phys. Soc.* **45**, 101 (1933). † *Ibid.*

(iii) The phenomenon depends upon the transference of heat from metal to solid carbon dioxide. If metal bars are to be excited by ice, etc. they must be raised to a suitable temperature.

(iv) The body must possess adequate vibrating properties. In the case of excitation by ice or other suitable materials, the variation of this property with temperature has therefore had to be considered; see § 3 below.

(v) The mechanism of the excitation of vibrations, depending upon the sublimation of solid carbon dioxide with the production of considerable gas pressures when it is brought into light contact with the metal, would appear to be somewhat similar to the production of edge tones.

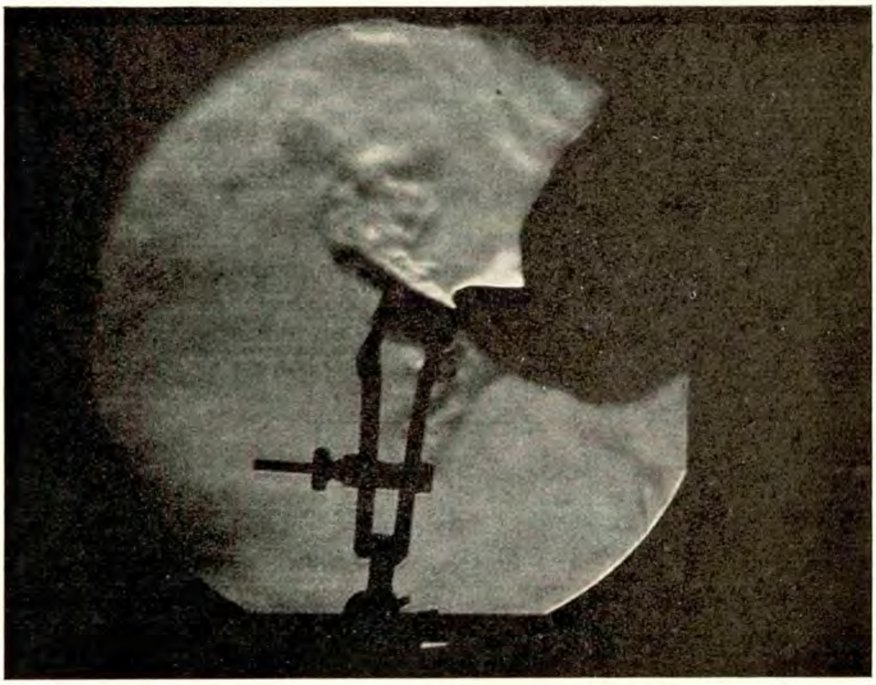

Figure 1. Schlieren photograph of carbon-dioxide sublimation caused by contact with edge of tuning fork.

(By courtesy of the Research Laboratory, General Electric Company, Ltd.)

Through the courtesy of the Research Laboratory of the General Electric Company, Ltd., the author has been able to study the effect on the quantity of gas produced of varying the degrees of contact between a metal bar or tuning fork and a block of solid carbon dioxide, by the Schlieren method, and a photograph very kindly taken by that Company is reproduced in figure 1. The photograph is of a fork viewed end-on, so that the shadows of the prongs are seen in cross section and merge into one. The solid block of carbon dioxide is touching an edge near the tip of one of the prongs, and two streams of carbon dioxide can be seen emerging in definite directions at right angles to the length of the fork. The block was pressing slightly on the edge, and the fork was chattering as well as vibrating regularly. When the contact is light and a pure loud note is being emitted, so little gas is produced that the two gas streams are barely visible. This fact is of interest and is to be expected, for the contact is slight and intermittent, the condition is one of resonance, and since the damping of the fork is small little energy is required to maintain the vibration. The frequency of the fork being 3000 ~ and photographic exposure $\frac{1}{50}$ sec., sixty contacts take place during exposure, and individual puffs of gas cannot of course be observed.

The Schlieren method of observing the phenomenon thus confirms the theory of the production of the vibrations previously advanced, but shows how little sublimation occurs when loud pure notes are being produced. It would be of interest to observe the Schlieren image through stroboscopic apparatus in order to determine whether the puffs are on alternate sides as they are in the case of edge tones.

§ 2. EXPERIMENTAL ARRANGEMENTS

The bars used were 15 cm. long and of $\frac{1}{2}$ in. square cross-section. They were suspended from two nodes so as to vibrate transversely as free-free bars. The dimensions chosen are convenient for purposes of heating by means of either a blow-pipe flame or an electric oven. Such bars cool at a convenient rate, their vibrations persist for a suitable length of time, and lastly their frequency of vibration is in the region of greatest sensitivity of the ear, which is especially desirable as the notes produced are very soft.

Measurement of temperatures. In preliminary experiments the temperatures were roughly estimated by dropping the bars into a calorimeter containing a known mass of water. It was found for example with a brass bar, that the range of temperature over which notes can be produced by contact with ice was roughly from 100° to 200° C. The notes are accompanied by the hissing of the steam and are quickly quenched by water. At lower temperatures the metal eats its way silently into the ice block, producing water.

In some experiments in which the bars were heated in an electric oven the temperatures were measured by means of a mercury thermometer placed in the oven. It is to be noted that the loudness of the notes produced is not appreciably altered, although the surrounding air as well as the metal is now hot.

Another method for measuring temperatures, used in the case of a brass bar, was to solder a brass and constantan wire at the two nodes respectively, so that the wires fulfilled the double function of suspending the bar and enabling its temperature to be determined thermoelectrically by connecting the two wires with a shunted galvanometer. The wires were soldered into small holes bored into the upper surface of the bar, and interfered but slightly with its vibrating properties.

In most experiments two identical bars, one the vibrator and one the measurer of temperatures, were used. They touched along one edge while being heated, and were subsequently separated so that the former could be conveniently set vibrating. With a bar 15 cm. long it is easy to ensure that the temperature along it is practically uniform. In experiments with a bar, the temperature of which was measured at the centre and one end by boring two holes and inserting two thermometers in them, the best routine for heating by means of a gas flame was established, and it was found that the temperature of the two thermometers tended to become sensibly equal fairly rapidly even if one was 5° or 10° C. above the other to begin with. The arrangement proved very satisfactory and experimental errors are generally much less than 10 per cent.

Detection of sounds produced. The sounds produced by means of any of the substances tested, other than solid carbon dioxide, are soft and of short duration. They can be heard in a quiet room, and the results given below were detected by ear alone. The author is indebted to Mr G. E. H. Rawlins for constructing a stand for holding the bar and a suitable arrangement for amplifying the sounds, for purposes of demonstration. A light microphone attached to the bar was eventually replaced by a more satisfactory arrangement of ear-phone coils and magnets which were fixed opposite a small piece of iron attached to a side of the bar near one of its ends. The coils were connected to amplifying valves and a loud-speaker.

By this means it would doubtless be possible to extend the list of compounds found to produce notes in heated bars, for the arrangement not only amplifies the note but also eliminates extraneous noises due to boiling, etc., which otherwise tend to mask it.

§ 3. VIBRATING PROPERTIES OF METALS AT DIFFERENT TEMPERATURES

A note on this subject follows the present paper*. The vibrating properties of metals usually decrease with temperature, and the upper limit of temperature for which it is possible to produce notes by contact of a metal bar with ice or other exciting substance is usually determined by this physical factor.

The results obtained for commercially pure copper, aluminium and zinc are shown in figure 2 and for medium steel in figure 3, where the vibrating properties, as determined by the duration in seconds of audibility of a note emitted by the vibrating bar, are plotted against the corresponding temperatures. The bars were struck with a hammer and the time for which the sound remained audible was noted by means of a stop-watch.

It will be noticed that the duration of audibility becomes too small to be measured by means of a stop-watch in the case of the particular samples of zinc, aluminium and copper at about 50°, 150° and 250° C. respectively.

Thereafter the melodiousness of the note decreases with rising temperature until at a higher temperature the pitch is no longer recognizable. It is to be noted that the frequency of the note varies but slightly at different temperatures, since the temperature variations of elasticity and density are both small. The vibrating properties of mild steel reach a minimum at about 120° C. and a maximum at about 240° C., after which there is a gradual decrease with rising temperature, the pitch however being still recognizable at a dull red heat.

* Page 124 of this volume.

§4. EXPERIMENTAL RESULTS

The metals used for establishing the conditions necessary for the production of notes in heated bars when brought into contact with various exciting materials were zinc, aluminium, copper, mild steel (or steel), and a number of the results

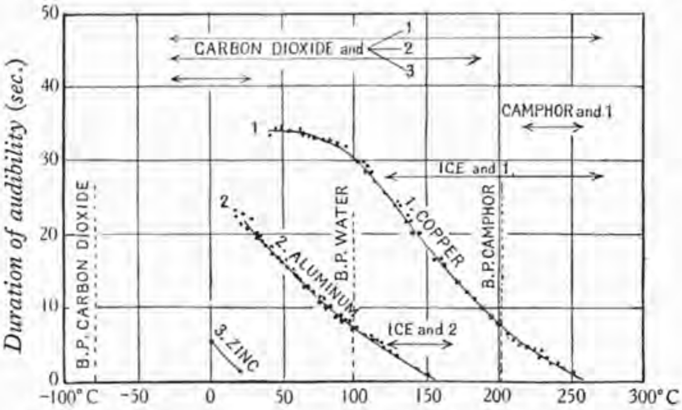

Figure 2. Vibrating properties (as measured by the duration of audibility of notes) of (1) copper, (2) aluminium, (3) zinc at different temperatures. The horizontal lines show the range of temperatures for which notes may be obtained from these metals when brought into contact with various exciting substances.

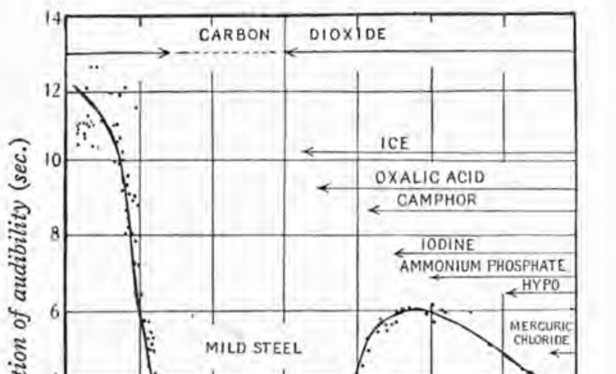

Figure 3. Excitation of notes from heated mild steel by contact with various exciting substances.

obtained are shown graphically in figures 2 and 3. The horizontal lines show the approximate range of bar-temperatures necessary for the production of notes with various pairs of metals and exciting substances. We proceed to consider some of the exciting materials used.

Ice. It is impossible to excite notes from zinc by means of ice, whereas carbon dioxide will produce loud notes. Aluminium, copper and most metals can however be excited to give soft notes if they are heated above 120° C. and brought into contact with a block of ice. The lower limit of temperature, about 120° C., does not differ very greatly from the temperature necessary to produce the spheroidal state, which is about 140° C.

As is shown graphically in figure 2, the upper limit of temperature is determined by the diminution in the vibrating properties of the metal and is about 150° C. for aluminium and 250° C. for copper. Reference to figure 3 shows that in the case of mild steel the lower limiting temperature, about 160°, depends upon the poor vibrating properties at that temperature and not on the boiling-point of water. The upper limit is not much below the temperature of red heat.

Contact between the ice and metal should be light. The note is somewhat masked by the hissing of the steam (unless it be listened to through the amplifier), but while it lasts it is pure and the phenomenon is similar to that of excitation by solid carbon dioxide, though on a very much smaller scale. The rapid formation of water soon quenches the soft sound.

Camphor, $C_{10}H_{16}O$, passes through a pasty condition in passing from the solid to the gaseous state, its melting-point being 170° C. and boiling-point 204° C. Care must be taken to prevent a flow of hot camphor over the fingers. As is shown in figure 2, notes cannot be produced in zinc or aluminium and only over a very limited range of temperatures, from about 220° C. to 260° C., in copper. Mild steel will be excited between about 220° C. and a much higher temperature.

Iodine, the melting-point of which is 116° C. and boiling-point 184° C., is harder to manipulate on account of the small size of the crystals. Notes have however been produced in mild steel above 230° C. (see figure 3) by holding the crystals by means of forceps.

We have seen that vibrations are produced in a metal bar on account of the gas pressure which results from the sublimation or boiling of the several exciting materials so far considered. It would seem probable, therefore, that materials which decompose with the evolution of gas when brought into contact with heated metals would also be capable of exciting vibrations in heated bars. Again, it might be possible to produce vibrations by means of crystals which give up their water of crystallization when brought into contact with the heated bar.

An exhaustive search on these lines has not been undertaken, but the results obtained with a number of crystalline compounds show that gas evolved as a result of chemical decomposition may produce soft notes over suitable ranges of temperature, whereas the vapour produced as a result of the liberation of water of crystallization is not sufficiently vigorous in its action to produce audible vibrations.

The experiments were generally made on a heated mild steel or steel bar, since these possess good vibrating properties at higher temperatures than the other metals used. The principal results obtained are summarized below and in some cases appear graphically in figure 3.

Oxalic acid, $C_2H_2O_4, 2H_2O$, produced notes when brought into contact with the bar heated to temperatures above about 175° C., on account of sublimation and decomposition with the liberation of carbon-dioxide gas.

Mercuric chloride, $HgCl_2$, produced notes when the bar-temperature exceeded 330° C. Mercuric chloride is generally said to sublime, its melting-point being 277° C. and boiling-point 304° C. Wet drops which evaporate rapidly form on the bar when this is touched on its upper surface.

Ammonium phosphate, $(NH_4)_2 . HPO_4$, produced notes when the bar-temperature exceeded 250° C. These are due to decomposition with the liberation of ammonia gas. A boiling deposit which forms on the bar damps the vibrations.

Sodium thiosulphate or "*hypo*," $Na_2S_2O_3 . 5H_2O$, produced notes, which are probably due to sulphur vapour, above 300° C.

Zinc nitrate, $Zn(NO_3)_2 . 6H_2O$, produced notes when the bar-temperature exceeded 150° C., probably on account of nitrous fumes and oxygen.

Meta or metaldehyde, which sublimes between 112° and 115° C., will produce notes in mild steel above about 240° C. The block has to be pressed against the heated metal and the note produced, though easily heard at higher temperatures, is not pure. In the case of meta we have passed a long way from the production of notes by solid carbon dioxide, in which (i) contact must be the lightest possible; (ii) the solid carbon dioxide offers a hard background; (iii) the sublimed carbon dioxide remains in the gaseous condition instead of immediately condensing into flakes; (iv) the notes produced are pure, loud and sustained.

Examples of substances which have failed to excite audible vibrations are: benzoic acid, potash alum, manganese sulphate, sodium sulphate, copper sulphate, borax, sodium carbonate, potassium nitrate, ammonium chloride, and ammonium carbonate. The results obtained will not be described in detail; each material behaves individually, and there may be hissing or silence, a deposit due to dehydration or decomposition which sticks on the bar, or a mass which gradually melts. Again if chemical decomposition occurs at a high temperature, for example in the case of potassium bromide, there is no effect, just as a material with an elevated boiling-point would produce no effect. Finally it is necessary that the crystals should be of a reasonable size: notes cannot be produced by means of powders.

§5. CONCLUSIONS

(i) Light contact made between two solid bodies which are at different temperatures may result in the production of elastic vibrations of audible frequency in the hotter body.

(ii) The hotter body must be a good thermal conductor, be capable of vibrating at suitable frequencies (about 1000 to 10,000 ~) and possess adequate vibrating properties at temperatures which exceed by some degrees the subliming, boiling or decomposing-temperature of the cooler exciting substance. Vibrations are not produced unless the metal is heated above these temperatures. The upper limit of temperature for which notes may be excited is determined by the loss of vibrating properties of the metal.

(iii) Apart from solid carbon dioxide, which would appear to be unique (at any rate among materials at present available) in its capacity to produce very loud notes, there are a considerable number of materials which may be used to excite soft pure notes of short duration in heated bars. The mechanism of production is identical with, but on a very much smaller scale than, that of solid carbon dioxide.

Such materials are: (*a*) Those substances which sublime or boil at suitable temperatures when brought into contact with heated metals, for example ice, camphor, iodine, and mercuric chloride. If the sublimation is immediately followed by a return to the solid state in the atmosphere, as in the case of "meta," contact must be heavier and the note produced is not pure. Again the formation of liquid, as in the case of ice, soon quenches the note. (*b*) Substances which decompose when brought into contact with the heated metal with the production of gas pressure, as for example oxalic acid, ammonium phosphate, sodium thiosulphate and zinc nitrate. Solid deposit, or a melting or boiling mass on the metal, is a frequent cause of failure to produce notes with certain other compounds.

(iv) Notes have not been produced when certain compounds containing water of crystallization have been dehydrated by contact with the heated metal.

(v) If a more comprehensive study were contemplated of chemical compounds which will produce notes from heated metals it would be desirable (*a*) to amplify the notes and isolate them from extraneous noises due to boiling, etc., as described in the paper; and (*b*) to use a bar made of a metal which will vibrate at high temperatures, for example steel or monel metal or invar.

DISCUSSION

See p. 126.

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NOTE ON THE VIBRATING PROPERTIES OF METALS AT DIFFERENT TEMPERATURES

By MARY D. WALLER, B.Sc., F.Inst.P.

Received July 1, 1933. *Read November* 3, 1933.

ABSTRACT. A simple acoustical method of studying the vibrating properties of metals at different temperatures is described. Since the damping of the vibrations, which is mainly due to internal friction or solid viscosity, is greatly altered either by previous heat or mechanical treatment and by impurities, i.e. is not constant at any given temperature, such a method capable of giving numerous comparative data over wide ranges of temperature should be of value. Furthermore, there is promise that on account of the large variations of the vibrating properties with temperature it will be possible to obtain, by observations of irregularities in these variations, much interesting information regarding the state of metals and alloys at different temperatures.

THE variation with temperature in the vibrating properties of several metals has been studied incidentally in the preceding paper* and a few results are there shown in figures 2 and 3.

The simple hand-and-ear method of observation by which these results have been obtained is now being used to carry out a survey of the vibrating properties of pure metals and alloys of known type between the temperatures of liquid air and red heat.

The author since developing the method has learnt that in 1911 F. Robin† used an essentially similar acoustical method for studying the vibrating properties of various metals between room temperature and red heat, and he was the first to draw attention to what he called the "aphonia" of carbon steels at about 120° C.‡. He used a standard impulse to set the bars vibrating, but the author finds that on account of the logarithmic nature of the damping, the magnitude of the impulse given to the bar may vary over wide limits without appreciably altering the duration of audibility of the note.

The method as used by the author consists in suspending a bar of square or circular cross-section, of side or diameter 0·5 in. and length 15 cm., from two nodes so that it may vibrate transversely as a free-free bar, heating it to known temperatures§, striking it a smart blow with a hammer held in the hand, and determining, by means of a stop-watch, the number of seconds for which the resulting note remains audible to the same observer listening near the bar with the same ear. Properly employed the method is satisfactory, and the degree of accuracy can be gauged by

* Page 116 of this volume.

† Iron and Steel Institute, *Carnegie Scholarship Memoirs*, 3, 125 (1911).

‡ See figure 3 of the preceding paper. § *Ibid.* § 2.

the lie of the experimental points on the curves. The irregularities may prove to be of significance, and not merely due to experimental errors. Losses of energy to the supports, provided these are suitably chosen, or on account of radiation of sound are of secondary importance, but as the research develops they may have to be considered. The most serious damping is due to formation of water drops below the dew-point, and to a lesser extent to the formation of hoar-frost in bars which are cooled to low temperatures in liquid oxygen. If comparisons between different bars are to be made, the frequency of the bar must be taken into account in order to determine the damping per cycle and to allow for varying sensitivity of the ear to different frequencies.

Again, no comparisons or specific constants of different metals can be given unless the exact composition, the degree and length of heating, annealing, cooling, etc., and the amount of working on the metal are known. Thus, for example, commercially pure copper bars purchased from two different dealers, and practically identical as regards dimensions, elasticity and density, vibrated for 3 and 20 seconds respectively at room temperature.

The field that may be investigated by the method would appear to be a wide one. The vibrating properties vary by very large amounts as the temperature changes, whereas many physical properties change but slightly with temperature. If relations can be established between the vibrating properties and certain physical conditions which are hard to detect, the method may be of practical use in the workshop. Properties which depend not only on the composition but on the treatment of the material, such as hardness, tensile strength, fatigue limit, etc., may bear some relation to the vibrating properties.

It is interesting to compare figure 3 of the author's paper cited above with Cuthbertson's* curve for the fatigue limit at different temperatures of medium carbon steel, which shows a very sharp minimum in the fatigue limit at 123° C. Again, the variation of the specific heat with temperature is such that a sharp maximum occurs at about 120° C. in the curve given by Dearden†. The curves given by Thompson and Whitehead‡ for the variations in specific resistance and thermoelectric power include maxima at this same temperature. Sauveur and Lee's§ figure 1 relating tensile strength and temperature of iron and steel shows a marked minimum at 100° C. for electrolytic iron. Herbert in the discussion on Cuthbertson's paper|| compares the working properties of iron at 120° C. to those of stiff putty or plasticine.

These considerations are sufficient to indicate that a study of the vibrating properties by the method indicated, simple and merely approximate though it may be, is worth while. Its chief virtue lies in the fact that very numerous observations may be made on a property which varies greatly when either the temperature, the composition or the treatment of the material is varied. Irregularities obtained in

* *J. Iron and Steel Inst.* **126**, 237 (1932).

† Iron and Steel Institute, *Carnegie Scholarship Memoirs*, **17**, 89 (1928).

‡ *Proc. R.S.* A **102**, 587 (1923).

§ *J. Iron and Steel Inst.* **112**, 324 (1925).

|| *Loc. cit.* p. 260.

curves promise to yield much interesting information in view of the present rapidly increasing knowledge of the metallic state.

It may also be possible to compare results with those given by more refined methods for the viscosity of various metals. These are limited in number and often contradictory, see for example the summary given by Erk*. The contradictions are not surprising once it is fully realized that whereas the viscosity of gases or liquids has a definite value for any given temperature, the viscosity of a metal can be enormously altered by mechanical and heat treatment, and by the addition of very small amounts of impurity. Ultimately no study of the damping forces or viscosity of metals will be complete without a knowledge of the crystalline condition of the particular specimen.

DISCUSSION

Prof. Martin Knudsen. I should like to point out the existence of the radiometer force which exists between two bodies in a gas when they are at different temperatures. Thus, suppose that we have two bodies at temperatures θ_1 and θ_2 in a gas at a pressure p, then the radiometer force exerted on every square centimetre of the opposite faces of each is equal to $\frac{1}{2}p\,\{\sqrt{(\theta_1/\theta_2)}-1\}$, provided that their distance apart is negligible compared with the mean free path of the gas.

Suppose now that $\theta_1 = 800^\circ$ K. and $\theta_2 = 200^\circ$ K. The radiometer force is then equal to $\frac{1}{2}p$, so that if the pressure be atmospheric a force equal to half an atmosphere is exerted on every square centimetre of the opposite faces of each body, and thus if one body be fixed the other body will be forced away from it until the radiometer force becomes small, when, if there be a restoring force, the body will return to be forced away again by the radiometer force. Thus vibrations depending on the vibrational properties of the body will be set up.

If, as in the present case, gas be evolved, this will give an additional force which may be sufficient for the explanation of the vibrations. I wish only to emphasize that the radiometer force should also be taken into account in the explanation.

Mr C. R. Darling. I should like to ask whether Miss Waller is of opinion that the vibrations are not in any way due to alternate contraction and expansion, as in the case of the Trevelyan rocker. Can invar, which is practically non-expansive, be made to vibrate by touching with a cold solid?

The extension of this work to the investigation of the properties of metals at different temperatures may prove of great service, as the experimental work is simpler than that involved in other methods having this object, and the results are equally certain.

Mr T. Smith. I should like to call attention to a misleading word which enters into Miss Waller's paper by the merest chance. About twenty years ago one of my colleagues at the National Physical Laboratory had occasion in a paper to mention Töpler's Schlieren method. Shortly afterwards the late Lord Rayleigh referred to

* *Z. f. Metallkunde*, **6**, 185 (1929).

this, and pointed out that the method should be known as Foucault's. Since that time I have endeavoured, though not with unvarying success, to persuade members of the N.P.L. staff who apply this method in various ways to adopt Lord Rayleigh's advice. It would help towards a correct understanding of the history of physics, which is so difficult nowadays for students to acquire, if we avoided using terms which tend to mislead, of which many examples besides "schlieren" might be mentioned. I suggest that the Physical Society should, as opportunities occur, call attention to widespread errors of this kind, and endeavour to bring about their general correction.

AUTHOR's reply. I should like to thank Prof. Knudsen for his interesting remarks regarding the magnitude of the radiometer pressure and its possible contribution to the production of vibrations. It is not, however, possible to get any sign of vibration when a heated metal is touched with another cold body which does not emit gas. The mean free path of air being only about 10^{-5} cm. at atmospheric pressure, and the contact being necessarily light and between two small areas, one of which is irregular, it would appear that the radiometer force must be very small.

Mr Darling raises an important point. I have repeated the experiments, and do find that it is easier to produce notes in iron than in invar, though the pure sustained notes are about equal in loudness. The two materials are comparable as regards thermal conductivity, elasticity, density and vibrating properties, and having regard to experiments made on other metals I am inclined to think that a large coefficient of expansion is a contributory though not a principal cause of the production of vibrations.

With regard to Mr T. Smith's remarks I agree that scrupulous care should be taken to see that honour is given where it is due. A. Toepler introduced the term *Schlierenapparat** to describe the apparatus which was used to detect the *Schlieren* or streaks in imperfect glass. He has added footnotes in both publications, from which we learn that Kirchhoff drew his attention to Foucault's work on the examination of optical surfaces published in the *Annales de l'Observatoire Impérial de Paris*, **5**, 203. He recognizes that his and Foucault's methods are very similar in principle. The word Schlieren does not occur in most German dictionaries; its meaning is given as "streaks (in glass and igneous rocks)" in Patterson's *German-English Dictionary for Chemists*. Since therefore the word is not an ordinary one and the method promises to be used commercially, and since Foucault and Toepler's work was independent and devised for different purposes, it would seem better to speak of the Foucault-Toepler method.

* *Ann. d. Phys.* **128**, 126 (1860); **131**, 35 (1867).

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THERMOCOUPLE FOR VIBRATING METAL BARS.

By MARY D. WALLER, B.Sc., F.Inst.P., Physics Department, London (Royal Free Hospital) School of Medicine for Women.

[*MS. received 2nd July*, 1935.]

A note on an acoustical method of studying the vibrating properties of metals at different temperatures has already been published by the author.* In these experiments a bar of suitable dimensions, for example 6 in. long and ½ in. diameter, is suspended horizontally at two nodes (0·224 of the length from either end) by means of two fine wire stirrups, each consisting of a single turn. The bars are excited to transverse vibration by means of a hammer, or in the case of delicate or brittle materials by the author's "solid-carbon dioxide" method of excitation.†

There is difficulty in measuring the temperature of the bar without interfering with the vibrations, and small thermocouples either applied to the surface or soldered at the nodes have proved unsatisfactory. A thermocouple has now been devised by making the supporting wires of different metals and connecting their further ends to a mirror galvanometer with suitable resistances. Wires made of copper and constantan respectively (34 s.w.g.) have been employed for measuring temperatures between that of liquid oxygen and 300° C. Bars weighing not less than 50 gm. make such excellent contact with the supporting wires that it is impossible to detect any galvanometer deflexion when a heated or cooled bar is given a sharp blow with a hammer. The metal of which the bar is composed does not affect the deflexion, which depends only on the metals of the wires themselves and on the temperature. Both the bars and the wires are easily replaceable.

Calibration has been effected above room temperature using a copper bar in which a hole had been drilled for the insertion of the bulb of a mercury thermometer, and the calibration curve has been extended to low temperatures using the melting-point of ice, the temperatures of solid carbon dioxide (−78·5° C.), and liquid oxygen (−183° C.).

* *Proc. Phys. Soc.* **46** 1934 (124). † *Proc. Phys. Soc.* **45** 1933 (101).

5

ACOUSTIC STUDIES OF SOME NON-TRANSFORMING AND TRANSFORMING SPECIAL STEELS AT LOW TEMPERATURES

By

M. D. Waller, B.Sc., F.Inst.P.

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Acoustic Studies of Some Non-Transforming and Transforming Special Steels at Low Temperatures

By Mary D. Waller, B.Sc., F.Inst.P.

(*Communicated by Sir Robert Hadfield, F.R.S.—Received* 8 *November*, 1935)

1—Introduction

The present investigation on some special steels and on Swedish charcoal iron was undertaken at the request of Sir Robert Hadfield, following my published note* describing a simple acoustic method of studying the persistence of vibration of transversely vibrating metal bars at different temperatures. In the present work the vibration frequencies have also been determined. The two series of observations give information regarding the internal damping and Young's elastic modulus respectively. Secular as well as temperature changes have been recorded in transforming nickel steels. The experiments have been made between −183° C and room temperature, and the results, whenever possible, compared with the Hadfield,† Dewar-Hadfield,‡ and De Haas-Hadfield§ low temperature researches on the maximum stress, elongation, yield point, reduction of area, and Brinell hardness of some of the same steels (*see* Table III).

A list of the materials investigated is given in Table I, together with the data supplied by Sir Robert Hadfield regarding their composition, treatment, Brinell hardness, and specific magnetism.

Historical Note—As early as 1904 Beilby|| studied the frequencies of heated vibrating metal reeds by tuning their notes to those of an adjustable organ pipe. Robin¶ made acoustic experiments in 1911 on the persistence of the notes emitted by transversely vibrating bars above room temperature which are essentially similar in principle to the present ones. Orowan** has recently shown how greatly the persistence of the note emitted by vibrating mica sheets may be affected by the presence of crevices

* 'Proc. phys. Soc., Lond.,' vol. 46, p. 124 (1934).
† 'J. Iron Steel Inst.,' vol. 1, p. 147 (1905).
‡ 'Proc. Roy. Soc.,' A, vol. 74, p. 326 (1904).
§ 'Phil. Trans.,' A, vol. 232, p. 297 (1933).
|| "Aggregation and Flow of Solids," p. 170 (1921).
¶ 'Carnegie Schol. Mem.,' vol. 3, p. 125 (1911).
** 'Z. Physik,' vol. 87, p. 749 (1934).

at their edges and the subject of vibration damping was discussed at the International Conference on Physics in 1934.*

2—Apparatus and Method

Duration of Audibility of Vibrating Bars—Bars of uniform size (6 inches long, ½-inch square cross-section) were suspended horizontally at the two nodes (0·224 of the length from either end) from a rigid support by means of 34 S.W.G. wires, 12 cm long. The duration of audibility of the notes emitted by the bars, after striking their centres with a hammer so that they vibrated transversely in the free-free mode, was determined by ear and stop-watch. The impulse of the blow may vary over wide limits without appreciably altering the duration of audibility. This is on account of the logarithmic nature of the damping and the very great disparity between the initial and final vibration amplitude. Provided the ear was adjusted to a standard position (using for the purpose a mirror fixed at the side of the supporting stand), the error in the observations was of the order of 5%. The only serious external damping was caused by the hoar frost which formed on the surface of the bars below 0° C. The damping suddenly increased at 0° C when the hoar frost melted. Repeated experiments on numerous bar surfaces eventually made it possible to apply a reasonable correction for this source of error in the observations.

Vibration Frequency and Calculation of Young's Modulus of Elasticity—The note emitted by a loud-speaker, connected to a calibrated mains-operated oscillator, was adjusted to that of the vibrating bar; the vibration frequency by this means being given to an accuracy of 0·1%. Actually, the oscillator note was adjusted just above and just below the bar note so as to produce the same number of beats in either case, and the mean of the two oscillator readings was taken. Young's modulus E was calculated in c.g.s. units from the frequency n by means of the equation:—

$$E = \frac{mn^2l^3}{1 \cdot 0575t^4},$$

where m = mass, l = length, t = thickness in the plane of vibration of a thin bar of rectangular cross-section which is vibrating transversely in the "free-free" fundamental mode. This formula gave satisfactory values of E when used in connexion with a number of materials of known elasticities, all the quantities on the right-hand side being measured to

* "Inter. Conf. Phys., London," vol. 2 (1935).

four significant figures. The contraction of the bars at low temperatures was neglected; the error so involved amounting, for example, for iron at liquid oxygen temperature, to 0·2%.

Cooling and Measurement of Temperature—The bars were cooled in liquid oxygen (−183° C) and the observations were made as they warmed to room temperature. The suspending wires were made of copper and constantan respectively and were employed as a thermocouple. The arrangement, devised so as not to interfere with the vibrations, has been described elsewhere.*

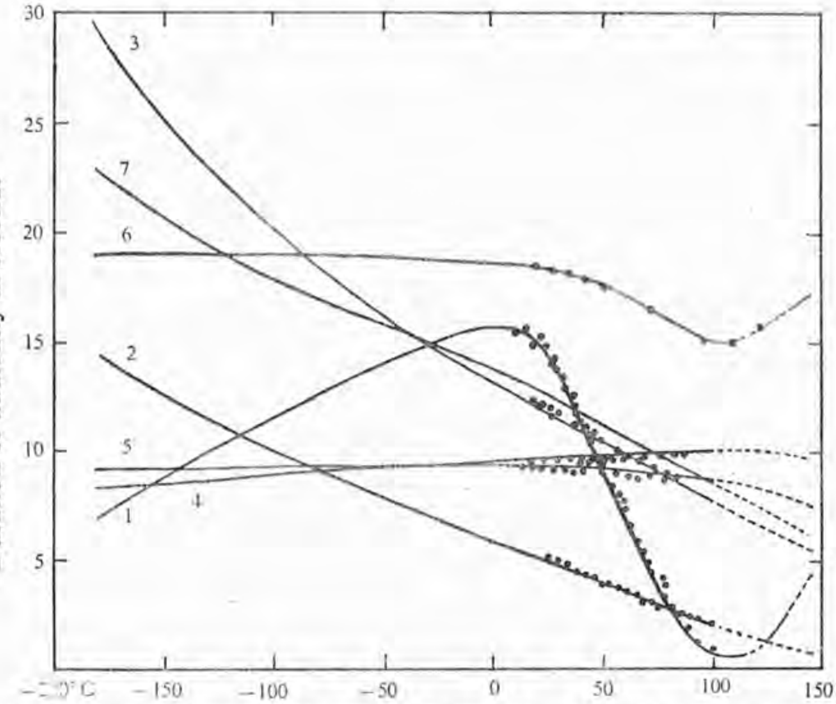

FIG. 1—Duration of audibility and temperature. (1) S.C. iron. Steels:—(2) 2·4 Cr; (3) 24·3 Ni, 6 Mn; (4) 57·5 Ni; (5) 18·1 Cr, 8·5 Ni; (6) Elinvar; (7) 12·7 Mn. A correction has been applied for hoar frost damping below 0° C.

3—NON-TRANSFORMING STEELS AND IRON

The duration of audibility—and vibration frequency—temperature curves of the first seven bars of Table I are shown in figs. 1 and 2 respectively. (The curves were continued above room temperature with the purpose of obtaining more satisfactory corrections for hoar frost damping in fig. 1.) Each curve is characteristic of the given material, indicating that no permanent change has occurred as a result of its immersion in

* Waller, 'J. sci. Instr.,' vol. 12, p. 300 (1935).

TABLE I

No.	Mark	Type	Analysis % by weight C	Si	Mn	Cr	Ni	Fe	Heat treatment °C	Brinell hardness	Spec. magnetism
1	S.C.I.	Swedish charcoal iron	0·03	0·01	0·04	—	—	99·89	As forged	102/92	100
2	3055	2½% Cr steel	0·92	0·16	0·32	2·41	—	—	900 F 850 O	821/811*	87
3	1414B	Ni-Mn steel	1·18	—	6·05	—	24·3	—	1050 W	168/168	0·1
4	5277	57% Ni steel	0·34	0·14	1·31	—	57·5	—	As forged	164/164	92
5	3754	18% Cr, 8% Ni steel	0·12	0·19	0·46	18·10	8·50	—	1150 W	152	<0·10
6	5513/2	Representing "Elinvar"†	0·63	0·16	1·47	10·80	32·80	50·10	1000 W	240	55
7	1010	Manganese steel	1·27	0·12	12·69	—	—	—	1000 W	214/218	0
8	1449A	31% Ni steel	0·70	—	0·82	—	31·40	—	800 F	156	55
9	3916	25% Ni steel	0·07	0·08	0·78	—	25·50	—	800 F	138, 302 (before and after experiments)	23, 86 (before and after experiments)

* Diamond pyramid hardness. † Also 3·5 W 0·17 Cu. F, cooled in furnace. O, cooled in oil. W, cooled in water.

liquid oxygen. One partial exception must be made to this statement since bar 5 (18·10% Cr, 8·5 Ni) showed traces of transformation when examined after the experiments.

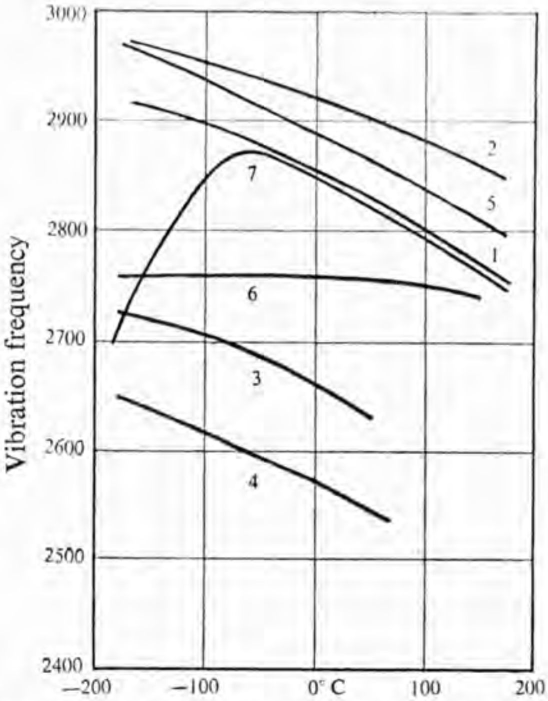

FIG. 2—Vibration frequency and temperature. (1) S.C. iron. Steels:—(2) 2·4 Cr; (3) 24·3 Ni, 6 Mn; (4) 57·5 Ni; (5) 18·1 Cr, 8·5 Ni; (6) Elinvar; (7) 12·7 Mn. The numerous observations used for constructing the curves are not shown individually.

TABLE II

	1—Duration of audibility in seconds			2—Vibration frequency		
	20° C	−183°	20°	20° C	−183°	20°
1	16	7	16	2842	2910	2842
2	6	15	6	2910	2965	2910
3	13	30	12	2654	2725	2654
4	8·5	8	8·5	2574	2650	2574
5	9	9	9	2877	2965	2877
6	19	19	19	2754	2754	2754
7	14	23	14	2846*	2694	2846
8	17	2	2–0–11	2568	2343 (1) 2488 (2)	2467
9	20	3	4–2–27	2706	2590 (1) 2627 (2)	2570

* 2862 at −50° C. (1) Immediately. (2) Later.

The unusual maximum in the vibration frequency of the Hadfield steel bar (12·69% Mn, 1·27 C) in the neighbourhood of −50° C corresponds closely with one previously found with carbon and Acheson graphite electrodes (*see* fig. 3), and it disappears when the carbon content of this steel is low. The iron curve, which has been inserted in the right-hand diagram, is typical of many of the metallic elements, and may be contrasted with the other curves of the figure.

Connexions between the vibrating properties and the mechanical properties collected in Table III are not obvious, but the fact that Hadfield

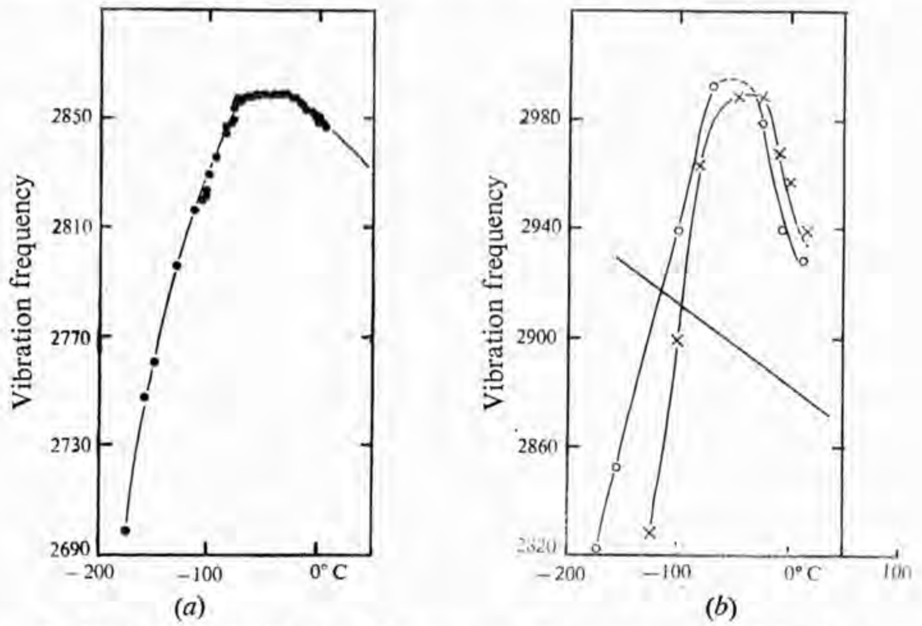

FIG. 3—Variation of vibration frequency with temperature. (The bars were of different sizes.) (*a*) ● 12·7% Mn steel; (*b*) ○ carbon; × graphite; — S.C. iron.

manganese steel becomes brittle at low temperatures may be noted in relation with what has been said above.

4—TRANSFORMING NICKEL STEELS

The last two steels of Table I, containing 31·4% and 25·5% of nickel respectively, transformed from the austenitic (γ, face-centred cubic crystal lattice), to the martensitic (α, body-centred cubic lattice) structure as a result of immersion in liquid oxygen.

The duration of audibility of the vibrating bars was temporarily much reduced. Fig. 4 shows that it fell from 17 seconds to a fraction of a

TABLE III

(Collected to two significant figures from Dewar-Hadfield and De Haas-Hadfield researches)

No.	Maximum stress tons/sq in			Elongation %			Yield point tons/sq in		Reduction of area %		Brinell hardness at normal temperature	
	20° C	−183° C	−252° C	20° C	−183° C	−252° C	20° C	−252° C	20° C	−252° C	Before cooling	After cooling
1	23	52	—	25	0	—	—	—	—	—	122	—
1	23	—	52	25	—	0	19	53	81	0	104	106
3	51	84	—	60	67	—	—	—	—	—	173	—
3	55	—	87	51	—	26	26	81	60	50	191	182
4	48	—	73	31	—	35	32	48	60	54	177	187
5	52	—	120	56	—	25	26	56	53	30	176	170
7	56	61	—	30	2	—	—	—	—	—	198	—
7	66	—	65	44	—	0	34	65	39	0	227	245
8	41	111	—	30	10	11	—	—	—	—	—	—
8	43	—	118	29	—	—	30	96	45	16	162	317

second for the 31·4% Ni steel bar. This observation may be compared with that of Wassermann* on a 30·33% Ni steel wire, to the effect that transformation from the γ- to the α-structure was accompanied by plastic flow under tension and a decrease in the plastic strength. By means of

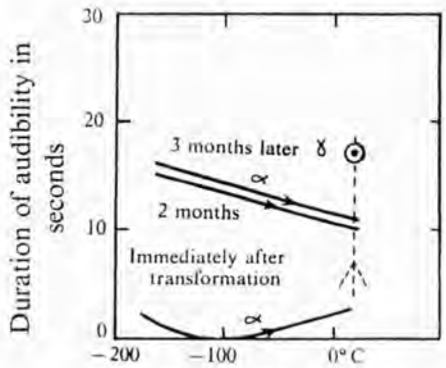

FIG. 4—31·4% Ni

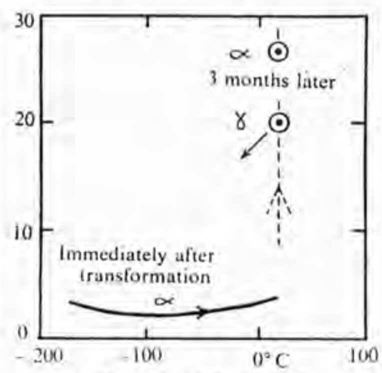

FIG. 5—25·5% Ni

FIGS. 4 and 5—Showing the great increase in internal damping which accompanies the transformation from austenitic (γ) to martensitic (α) nickel steels.

the acoustic observations, the subsequent decrease in the vibration damping was followed until, as shown in fig. 6, a new stable condition, corresponding to a duration of audibility of 11 seconds, was reached some three months later. Re-immersions in liquid oxygen of the stable α steel

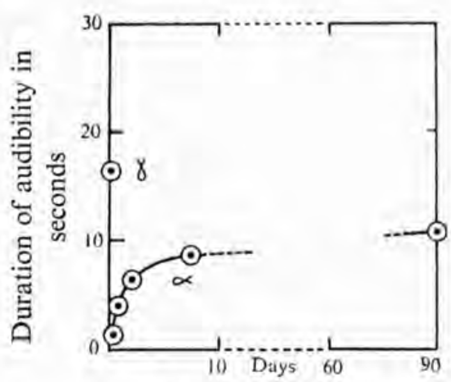

FIG. 6—31·4% Ni

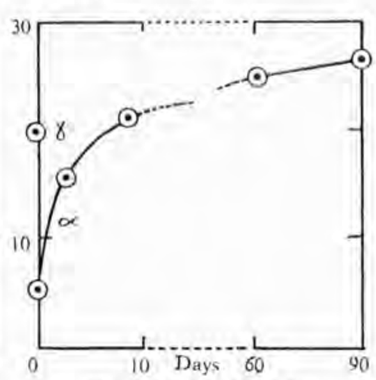

FIG. 7—25·5% Ni

FIGS. 6 and 7—Showing the secular decreases in internal damping of newly formed martensitic (α) nickel steels.

were not accompanied by abnormal damping, and the curves included in fig. 6 are comparable with those of the non-transforming steels in fig. 1.

The results obtained for the 25·5% Ni steel bar, which also transformed, are given in figs. 5 and 7 and are clear without further comment. The

* 'Arch. Eisenhüttenw.,' vol. 8, p. 347 (1932–1933).

steel is very similar to the 1287L (23·7% Ni, 0·10% C, 0·30% Si, 0·50% Mn) steel of the De Haas-Hadfield* researches which was not permanently affected by low temperature. The transformation detected acoustically was confirmed in the Hadfield laboratories, where it was found that the Brinell hardness of the 25·5% Ni steel had increased from 138 to 302 and the specific magnetism from 23 to 86 (*see* Table I). It is in accordance with the constitutional diagram of nickel steel,† and the slight differences in composition account for the great differences in the room temperature properties of the two steels.

The *vibration frequencies* decreased both temporarily and permanently, as a result of the $\gamma \rightarrow \alpha$ transformation. Fig. 8 shows that the frequency of the 31·4% Ni steel bar fell from 2568 to 2343, which indicates a temporary decrease of 16·7% in Young's modulus. The final frequency 2467

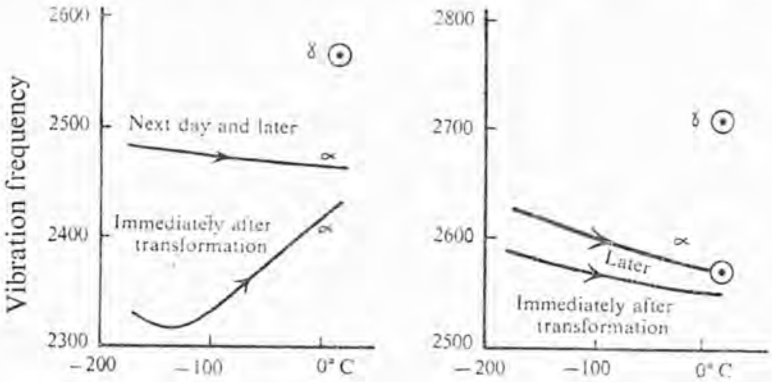

FIG. 8—31·4% Ni FIG. 9—25·5% Ni

FIGS. 8 and 9—Permanent and temporary decrease in vibration frequency accompanying the transformation of austenitic (γ) to martensitic (α) nickel steels.

was attained some hours later and denotes a permanent decrease of 8·4% in Young's modulus, or of about 9·1%, if account be taken of the increase in volume which accompanies the transformation. A vibration frequency-temperature curve obtained with the α steel is also shown in fig. 8; it is comparable with the curves of the non-transforming steels in fig. 2.

The vibration frequency-temperature curves of the 25·5% Ni steel bar are given in fig. 9 in which it will be noted that the transformation is again accompanied by temporary and permanent decreases in the elastic modulus.

In Table IV all the materials examined have been arranged in decreasing

* De Haas and Hadfield, ' Phil. Trans.,' A, vol. 232, p. 297 (1934).

† *See* T. H. Burnham, " Special Steels," Guillett, vol. 53, p. 103 (1933).

order of Young's modulus. The low values for the transforming steels, as compared with either iron or nickel, will be noted.

5—Conclusion

Each steel, between −183° C and room temperature, behaves in its own characteristic way as regards the temperature variation of either the internal damping or of Young's modulus of elasticity; but these properties are not obviously correlated with the structure sensitive properties such as hardness, strength, or ductility.

Table IV

1	2	3	4	5
Number	Type	Young's modulus At 20° C	Young's modulus At −183° C	% change between 3 and 4
		× 10^{11} dynes/sq cm		
2	2·4 Cr	21·80	22·62	3·8
5	18·1 Cr, 8·5 Ni	20·12	21·36	6·2
1	S.C.I.	19·46	20·41*	4·8
7	12·7 Mn	19·97†	17·82	−10·3
6	Elinvar	18·47	18·47	0
4	57·5 Ni	17·88	18·95	5·9
3	24·3 Ni, 6 Mn	16·42	17·31	5·4
9	25·5 Ni	17·84 (γ)	16·35 (*t*)	−8·4
		15·97 (α)	16·82 (α)	5·2
8	31·4 Ni	15·81 (γ)	13·17 (*t*)	−16·7
		14·59 (α)	14·85 (α)	1·7

* 20·45 after correction for contraction; the figures in column 4 have not been corrected for contraction (*see* § 2).

† A maximum, 20·10 at −50° C.

(γ) austenitic, (α) martensitic, (*t*) immediately after transformation.

An enormous, but temporary, increase in the internal damping occurs when the γ (face-centred cubic crystal lattice) structure of a nickel steel transforms to the α (body-centred cubic lattice) structure. The transformation is also accompanied by a considerable temporary, and some permanent, reduction in Young's modulus. Whereas the damping continues to vary for several months, the secular changes in Young's modulus last only a few hours.

A maximum in Young's modulus exists at about −50° C for Hadfield steel (12·69% Mn, 1·27% C), which corresponds closely with that found for carbon, and which is absent when the carbon content of this steel is low.

These results have been obtained by simple observations which appear to be very suitable for studying either brief or lengthy secular changes in metals, and by means of which rapid comparisons of Young's modulus may be made.

I should like to express my great appreciation and thanks to Sir Robert Hadfield, F.R.S., for his interest in this acoustic method of study, and for the opportunity of experimenting on some of the same special steels that have been used in his own low temperature researches.

I have pleasure also in stating that the oscillator used for measuring the frequencies is the property of the Augustus and Alice Waller Memorial Research Trust.

6—Summary

An acoustic method used for determining the persistence of vibration and vibration frequency of transversely vibrating bars at different temperatures, is described.

Some Hadfield special steels have been examined by the method between $-183°$ C and room temperature. During the transformation from austenitic, (γ) to martensitic, (α) nickel steel, the internal damping is greatly increased and Young's modulus of elasticity is considerably decreased. The modulus of α nickel steel is permanently less than that of nickel γ steel. Hadfield steel (12·69% Mn, 1·27% C) has a maximum modulus at about $-50°$ C, which corresponds closely with a maximum found for carbon, and which is not present when the carbon content of this steel is low.

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6

THE PRODUCTION OF CHLADNI FIGURES BY MEANS OF SOLID CARBON DIOXIDE. PART 1: BARS AND OTHER METAL BODIES

BY

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THE PRODUCTION OF CHLADNI FIGURES BY MEANS OF SOLID CARBON DIOXIDE. PART 1: BARS AND OTHER METAL BODIES

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Received 31 *March* 1937. *Read, with demonstration,* 11 *June* 1937

ABSTRACT. Chladni figures can be very readily produced by means of solid carbon dioxide: some photographs of results obtained on metal objects of various shapes are shown. The frequencies of the vibrations excited in objects of low fundamental tone usually lie between about 1000 and 4000. The theoretical implications of this fact are considered.

§1. INTRODUCTION

A GENERAL theory of the production of pure loud notes from metal bodies by means of solid carbon dioxide has been given previously[1], and the circumstances which led to the discovery have also been recorded[2]. The production of Chladni figures, more especially of the nodal lines that may be produced on metal bars, has been undertaken in a further study of the phenomenon. At the same time it has been established that the method of excitation provides a rapid and simple way of obtaining these figures on bodies both of regular and of irregular shape.

By means of these overtones, the limits of frequency within which the carbon-dioxide method of excitation is effective can be estimated with considerable accuracy. It appears that this preferred range depends relatively little upon the size, shape, mass, material or temperature of the excited metal. Conversely, a knowledge of the preferred range enables bodies of suitable dimensions to be chosen on which to produce Chladni figures corresponding to given overtones. The observations that are recorded below serve therefore two purposes. The one concerns the theoretical interpretation of the excitation phenomenon; the other a new method of obtaining Chladni figures that may be useful in the future.

§2. TECHNIQUE

The following points of technique, some of which have been mentioned previously, are important for producing effective vibrations.

(1) The solid carbon dioxide used must be of the high-density "ice" variety. I am indebted to Imperial Chemical Industries Ltd., for kindly supplying the solid carbon dioxide known as "Drikold" which was used in the present investigation.

(2) The area of contact between the metal and solid carbon dioxide must be small, and the pressure of application light. Figure 1 and the inset of figure 8 show pointed pieces of carbon dioxide which are suitable for purposes of excitation. It does not appear to be possible to substitute a mechanical device for the hand, and the skill of the operator in sensing the onset of vibration, the amplitude of which grows with remarkable rapidity, increases with practice. The point of application of the solid carbon dioxide (at or near an anti-node) may be either on the upper or the lower surface of the bar; for producing sand patterns it is often convenient to excite from below.

(3) The atmosphere should be dry. It was originally stated [1] that humidity did not affect the production of vibrations. This conclusion was based on observations made on a brass bar which vibrated so readily in the free-free fundamental mode

Figure 1. Production of Chladni nodal lines by means of solid carbon dioxide. Frequency, 1317 c./sec.

of 2000 c./sec. that the difficulties subsequently encountered in moist weather with other vibrators were not anticipated. These may be partially overcome by warming the bars before use, and by sprinkling dried sand on to the cleaned surface at the last moment before excitation. Moisture from the air formed on the surface of the solid carbon dioxide cannot however be avoided; and when one is investigating the higher tones of the larger bars, especially if their vibration frequencies lie above or below those which are most easily excited by the solid carbon dioxide, it is well to experiment on a dry day. It is also sometimes desirable to discard a piece of carbon dioxide, if it appears sugary or breaks up easily, in favour of a piece broken by means of hammer and chisel from another part of the solid carbon-dioxide block. This last observation is of theoretical as well as of practical importance.

§ 3. FREQUENCIES OF VIBRATIONS EXCITED BY SOLID CARBON DIOXIDE

Provided that the fundamental of a metal object is high enough, that is to say about 1000 c./sec., this will be excited by means of solid carbon dioxide; it has been found possible to produce the two nodal lines on bars vibrating in the free-free manner up to 7000 c./sec. and to detect notes up to much higher frequencies.* The observations that have been made on objects of low fundamental are described below.

Brass bars of varying length and thickness. Figures 2 and 3, together with the data given in tables 1 and 2, illustrate the fact that the overtones that may be excited by means of solid carbon dioxide become progressively higher as the fundamental

Table 1. Brass bars of varying length

Thickness, 1·27 cm. Thickness, 0·9 cm.

Length (cm.)	Tone*	Frequency		Length (cm.)	Tone*	Frequency	
		Excited	Fundamental (first tone, 2 nodal lines)			Excited	Fundamental
11·5	1†	3358	3358	11·5	1†	2156	2156
15·0	1†	1964	1964	15·0	1†	1263	1263
22·9	2†	2301	836	17·9	2†	2381	860
30·0	2	1298	469	20·2	2	1876	683
	3†	2523	,,		3†	3700	,,
				30·8	3	1588	300
					4†	2638	,,

* Rayleigh notation. † See figure 2.

Table 2. Bars of varying thickness

Length, 60·45 cm.

Thickness (cm.)	Fundamental (to nearest whole number)	Tones excited*	Frequencies†
1·302	121	4–7	1098, **1630**‡, 2266, 2977
0·612	57	6–10	(1011) 1476, **1892**, **2328**‡, 2875
0·325	32	8–15, 17	(1010) 1280, 1560, **1880**‡, **2260**, 2504, 2950, 3400, 4325
0·624§	65	6–10	(1211) 1568, **2080**, **2500**, 3005‡

* Add one to obtain number of nodal lines. † The frequencies most usually obtained are in black type; brackets denote chattering. ‡ See figure 3. § Copper.

becomes lower. This latter varies inversely as the length squared in each of the series of bars of figure 2, and directly as the thicknesses of the brass bars shown in figure 3. It will be noted that all the frequencies lie between 1000 and 4325.

Metal objects of varying material, dimensions and temperature. In order to avoid giving lengthy tables, the essential details of a number of observations have been collected in figure 4. The frequencies of each tone excited are plotted on the right. A list of the objects excited, together with notes giving the main purposes of the various observations and the fundamentals are shown in the columns on the left.

Summary of results. A glance at figure 4 is sufficient to show that the range of frequencies that can be excited by means of solid carbon dioxide is remarkably definite, and does not vary very much from object to object. The results for the first

* Frequencies were determined by means of a mains-operated calibrated valve oscillator, the property of the Augustus and Alice Waller Memorial Trust.

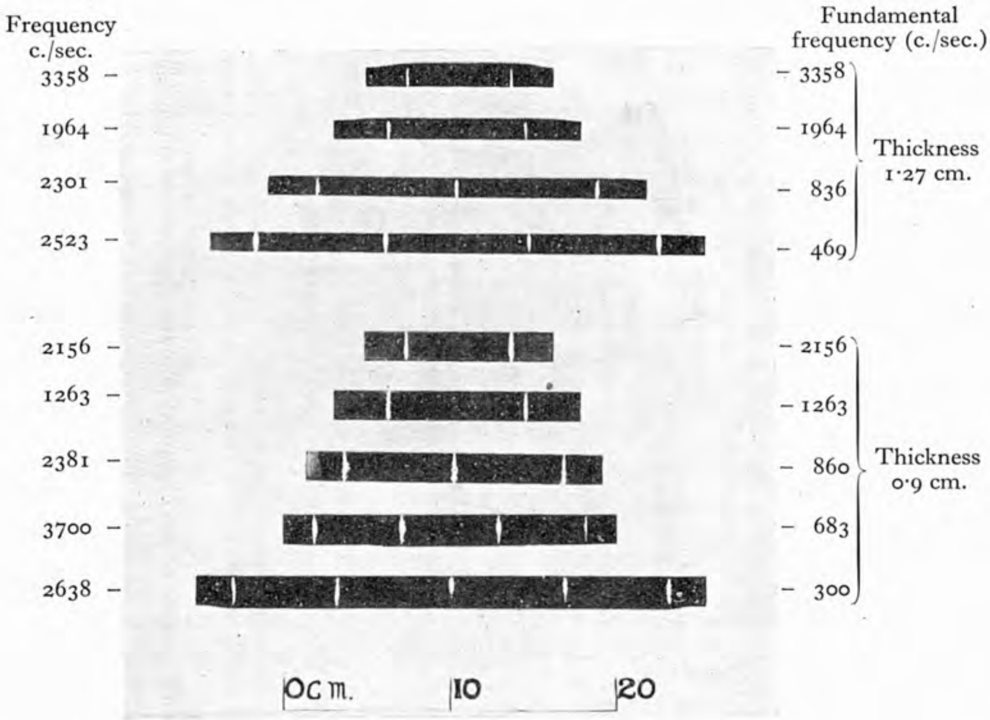

Figure 2. Varying length.

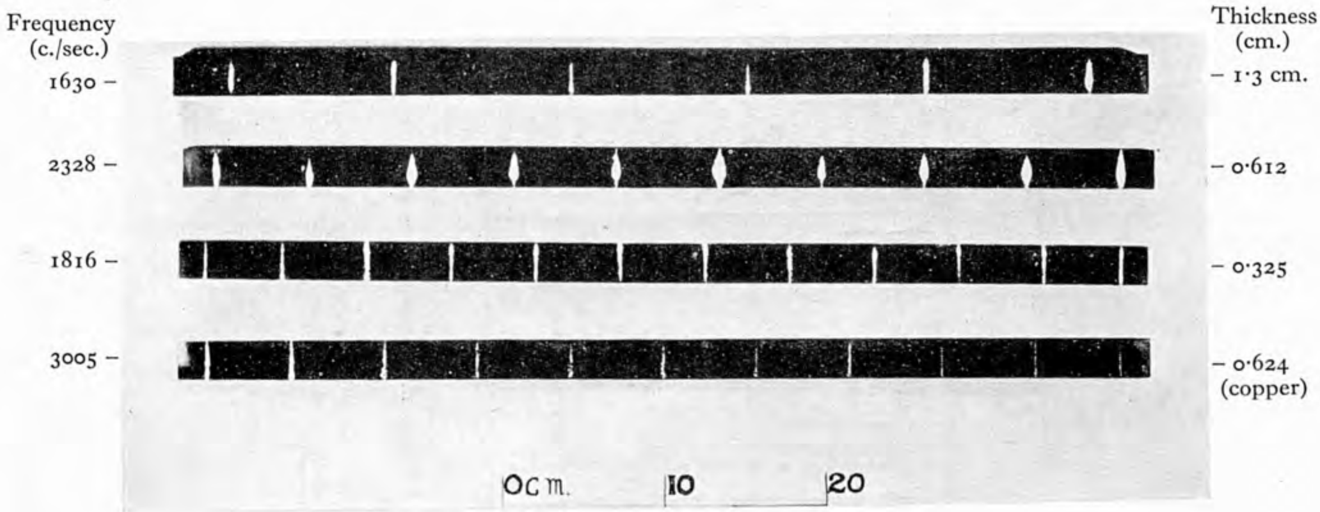

Figure 3. Varying thickness.

	Vibrating object	Notes	Fundamental (first tone)	Tones excited
1	Copper bar	60·7 × 1·27 × 1·27 cm.	130	4–8
2	Mild steel bar	K = 0·9, 0·1	177	3–6
3	Copper bar	44·8 × 1·27 × 1·27	206	3–5
4	German silver bar	K = 0·9, 0·07	198	3–5
5	Brass bar, figure 1	m = 3·29 kg.	71	5–11
6	Aluminium bar	120·8 × 1·27 × 1·27	43	7–12
7	,,	l = 122. Diameter = 1·27	38	7–13
8	Mild steel bar	92·5 × 1·27 × 1·27	78	5–8
9	Brass bar	l = 91·35. Diameter = 1·27	84	5–8
10	,,	42·5 × 1·27 × 1·27	206	3–6
11	,,	20° C.	50	7–12
	,,	150° C.	48·5	7–12
12	,,	20° C.	21	10–20
	,,	150° C.	20	9–19
13	Narrow rods	Diameter = 4 to 6 mm.	—	Misc.
14	Brass plate	Diameter = 13 cm.	—	,,
15	,,	Diameter = 20 cm.	—	,,
16	,,	Diameter = 30 cm.	—	,,

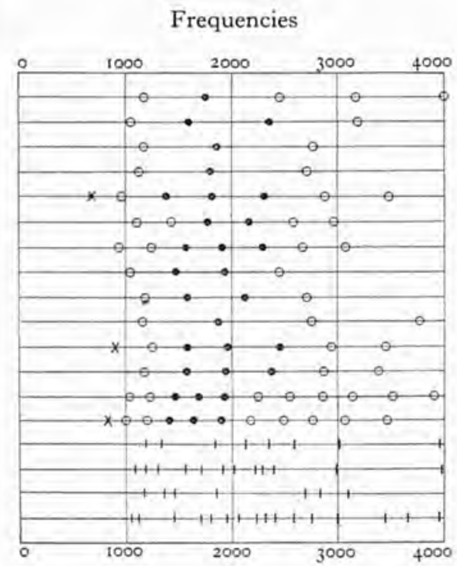

Figure 4. Frequencies of vibrations excited by solid carbon dioxide. K, thermal conductivity; •, most usual frequencies; ×, accompanied by chattering; |, miscellaneous overtones. The frequencies, N_1, given in the fourth column were calculated by means of the relation

$$N_1 = N_n\, 9{\cdot}061/(2n+1)^2.$$

The tones excited in bars of circular cross-section were inferred from the ratios of the experimental frequencies which approximate to the squares of the odd numbers.

four bars show that a change of conductivity from 0·9 to 0·07 makes no detectable change in this range. The range is definitely lower for the massive fifth bar (938–3468), a photograph of which was given in figure 1, than for the small rods (1146–>4000) or brass plates (1070–>4000), 3 mm. thick (see also figure 9). It was difficult to establish any change due to temperature variation, but the final observations undertaken on a brass bar, no. 12, of particularly low fundamental frequency, 21, showed that the range was a little lower when the bar was heated to 150° C.* The intensity of the vibrations is often very great for the heated bars, provided always that the temperature is not sufficient to impair the vibrating properties of the metal. The latter is very important; it is, for example, easier to produce figures with hard brass than with annealed pure copper in spite of its high conductivity. The theoretical significance of these results will be considered in § 5.

§ 4. VIBRATIONS OF METAL OBJECTS OF IRREGULAR SHAPE

Excitation by means of solid carbon dioxide is very intense, and the fact that it can be effected at any point of the metal surface makes it possible to excite single overtones in objects of irregular shape which emit but a jangle of sound when struck with a hammer. A few illustrations of the possibilities of studying resonances by the method are given in figures 5 to 8. Incidentally the frequencies of the vibrations confirm the results of the previous paragraph.

Figure 5 (triangle) gives an idea of how far the ordinary transverse vibrations in the plane of the thickness are affected by breadth, while figure 6 (wedge) shows the manner of vibration when the thickness varies (from approximately zero to 8 mm.). The vibration frequencies of the triangle are from 7 to 10 per cent greater than those of the bar from which it was cut, and the nodal lines are very slightly displaced towards the base. The frequencies of the wedge are very roughly double those of the parent bar, and the adjustment of nodal distances to varying thickness and the large displacements of the nodal lines towards the thin end of the wedge will be noted.

The familiar Trevelyan rocker is shown, as it vibrated with and without its handle, in figure 8; these elastic vibrations are of course much higher than the gravity vibrations of the bar when it is rocking as a whole about its two ridges, and the figure is of interest in connexion with recent discussion concerning this ancient rocker(3).

Figure 7 shows the nodal lines produced on some spanners and a chisel, and suggests the possibility of studying resonances in pieces of machinery. The vibrations of large pieces could be investigated by means of small-scale models.

§ 5. MECHANISM OF PRODUCTION OF VIBRATIONS BY MEANS OF SOLID CARBON DIOXIDE

The production of vibrations by means of solid carbon dioxide, the temperature of which is at about −80° C., depends upon the sublimation which occurs when it touches the metal in the process. A large quantity of gas is produced, as has been

* The slight shift of the frequencies that will be noticed in figure 4 for the heated bars is of course due to the temperature-diminution in Young's modulus.

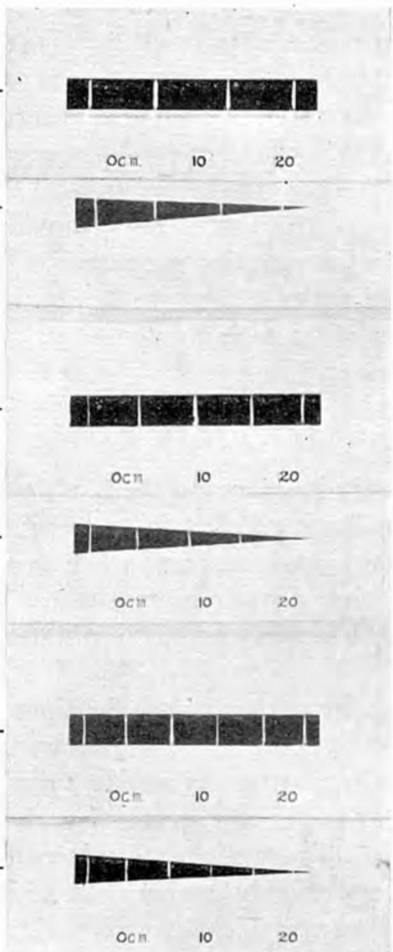

Figure 5. Bar and triangle. Thickness 6·84 mm.

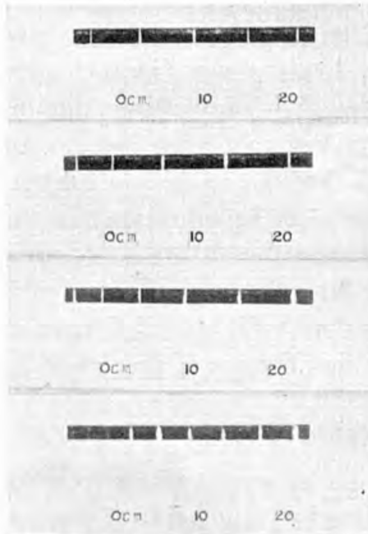

Figure 6. Bar and wedge.

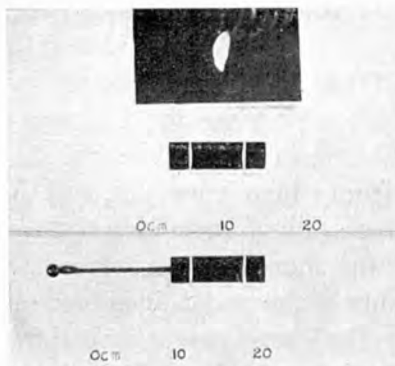

Figure 8. Trevelyan rocker (inset solid carbon dioxide).

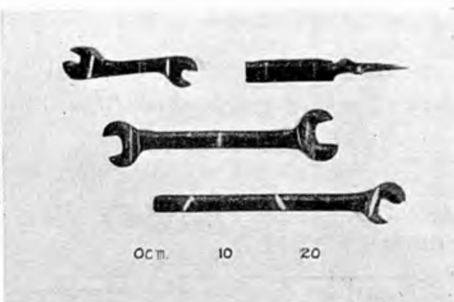

Figure 7. Spanners and chisel.

Figure 9. Plate.

Figures 5–9. Study of vibrations of metal objects by means of Chladni figures produced by solid carbon dioxide.

shown by means of a Foucault-Toepler or Schlieren photograph[4], the pressure of which is most obvious in the experiments made with mercury[1].

The experimental results that have been summarized in § 4 can be explained by supposing that the main impulse depends upon the irresistible molecular forces of sublimation which are operative at the moment of contact. Each impulse lasts only for a small fraction of the total period and is given except at the beginning, while the vibrating object is at one extreme of its swing. The localized pressure is then very great as compared with that which exists during the rest of the vibration.

It is of interest to consider the relation between the mean free path of the CO_2 molecule, which is about $3 \cdot 9 \times 10^{-6}$ cm. at normal temperature and pressure, and the amplitudes of vibration. Andrade and Smith[5] have investigated the latter in connexion with the production of Chladni figures. Thus the motion at an antinode of the vibrating surface may be represented by $a \sin nt$, the maximum acceleration amounting to an^2. In order that figures may be produced on a bar, an^2 must be greater than g, or a must be greater than $g/4\pi^2N^2 > 25/N^2$, where N is the frequency of vibration. The frequency of vibration corresponding to an amplitude of vibration of $3 \cdot 9 \times 10^{-6}$ cm. just sufficient to produce motion of the sand under favourable conditions would be 2530. The vibrations being intense, at any rate at the higher frequencies, the amplitudes are no doubt considerably greater than these minimum values.

The mean free path cannot of course be estimated accurately. At the first moment of sublimation while the main impulse is occurring, it should be equal to about $p^{-1} . 3 \cdot 9 . 10^{-6} \sqrt{(193/273)}$ cm., where p is possibly many atmospheres; but half a period later, when the vibrating object is at the other extreme of its swing, if the pressure is nearly atmospheric and the temperature not much below that of the room, the mean free path will not differ greatly from $3 \cdot 9 \times 10^{-6}$ cm. It is probable therefore that the upper limit of frequency that can be obtained with the carbon-dioxide method of excitation depends upon the vibration amplitude becoming comparable to the mean free path, and the impulse lasting too long in relation to the period to be effective.* The lower limit of frequencies is imposed by the increase in the chattering which is no doubt connected with the larger amplitudes of vibration.

It is evidently sufficient to have a material of relatively low thermal conductivity in order to produce enough sublimation to separate the two surfaces, and loud notes have been produced in a quartz bar cut from a large crystal, the conductivity of which is 0·06 or 0·03 according to direction[1]. At the same time the production of gas is essential and no sound can be produced from insulators. Incidentally this provides a convenient way of distinguishing quartz from glass lenses since the former only, emit a rattle when touched with the carbon dioxide.

In the case of heated metals the impulses are more intense and the mean free path greater, so that the slight decrease of both the upper and lower limits of fre-

* The theory suggested experimenting with a very narrow, thin, long strip of brass ($1 \cdot 276 \times 0 \cdot 333 \times 86 \cdot 64$ cm.) and exciting it near one end. In such a case the range of frequencies excited should be higher. Clear nodal lines were obtained for all tones from the 15th to the 29th corresponding to frequencies ranging from 1690 to 6000. The fundamental can be calculated with accuracy and is found to be 15·61.

quencies is also accounted for by the theory proposed. It might be possible to investigate this matter further with some alloy for which the internal damping remains small at high temperatures[6]. One of the difficulties of producing maintained vibrations at higher temperatures is, however, that the shape of the solid carbon dioxide alters rapidly on account of the increased rate of sublimation.

§ 6. CONCLUSION

The production of overtones in metal bodies by means of solid carbon dioxide has enabled the mechanism of the production of vibrations by this method to be studied in considerable detail. At the same time it has become apparent that solid carbon dioxide provides a peculiarly simple and effective method of obtaining Chladni figures. Powerful excitation can be immediately produced and maintained for some seconds at any point of either plane or curved metal surfaces, and it is possible by this means to study the resonances of objects of irregular shape.

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DISCUSSION

Dr O. Kantorowicz. I should like to ask whether it is possible to control the mode of vibration of a body, if this body has several possible modes in the frequency-range available. Does the author consider the method a suitable one for estimating the damping-capacity of metals, or does damping by sound-radiation mask the effect of internal damping so as to make its observation impossible? Even a rough method would be quite valuable.

Mr J. H. Awbery asked why the author expected to find a simple relation between the mean free path of carbon dioxide and the amplitude of vibration of the metal. He pointed out that in any event air would be mixed with the carbon dioxide. He suggested that local stresses might be set up in the metal as a result of contraction and expansion with changes of temperature.

Mr H. R. Calvert asked whether the author had tried ammonium chloride and other substances which sublimate under attainable conditions.

Author's reply. In reply to Dr Kantorowicz: The desired partials can be obtained by using very hard-pointed pieces of solid carbon dioxide for the higher notes and blunter pieces for the lower notes, the vibrating object being suspended from, or supported at, two nodal positions. The method of measuring the vibration-damping has been described in a previous paper[6]; the air resistance can be neglected to a first approximation. Vibration-damping is probably connected with damping-capacity (as defined by Föppl) though the latter deals with conditions outside the elastic limit.

In reply to Mr Calvert: Vibrations have been produced in heated metals by means of a number of substances which either sublime, or boil, or decompose, with the evolution of gas[2]. I did not succeed in detecting the vibrations when using ammonium chloride. The notes produced are feeble and transitory, and the phenomenon, except in the case of solid carbon dioxide, would not be noticed unless looked for.

In reply to Mr Awbery: The experiments just quoted, together with the fact that vibrations cannot be produced in the absence of evolved gas, show that radiation pressure due to the presence of air does not play any detectable part in the production of the vibrations. Nor do I think that local stresses are responsible, since the contact is so slight and the fall in temperature of the metal small. On the other hand the pressure produced by the carbon-dioxide gas is very obvious. The theory developed in connexion with the amplitude of vibration and mean free path of the carbon-dioxide molecule offers some explanation of the fact that the range of frequencies that can be obtained is so definite and varies so little from metal to metal. Longitudinal vibrations also have been produced[1], and further experiments with these might reveal the existence of higher zones of excitable vibrations.

7

VIBRATIONS OF FREE CIRCULAR PLATES

PART 1: NORMAL MODES

PART 2: COMPOUNDED NORMAL MODES

PART 3: A STUDY OF CHLADNI'S ORIGINAL FIGURES

BY

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VIBRATIONS OF FREE CIRCULAR PLATES. PART 1: NORMAL MODES

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Received 15 *October* 1937. *Read* 26 *November* 1937

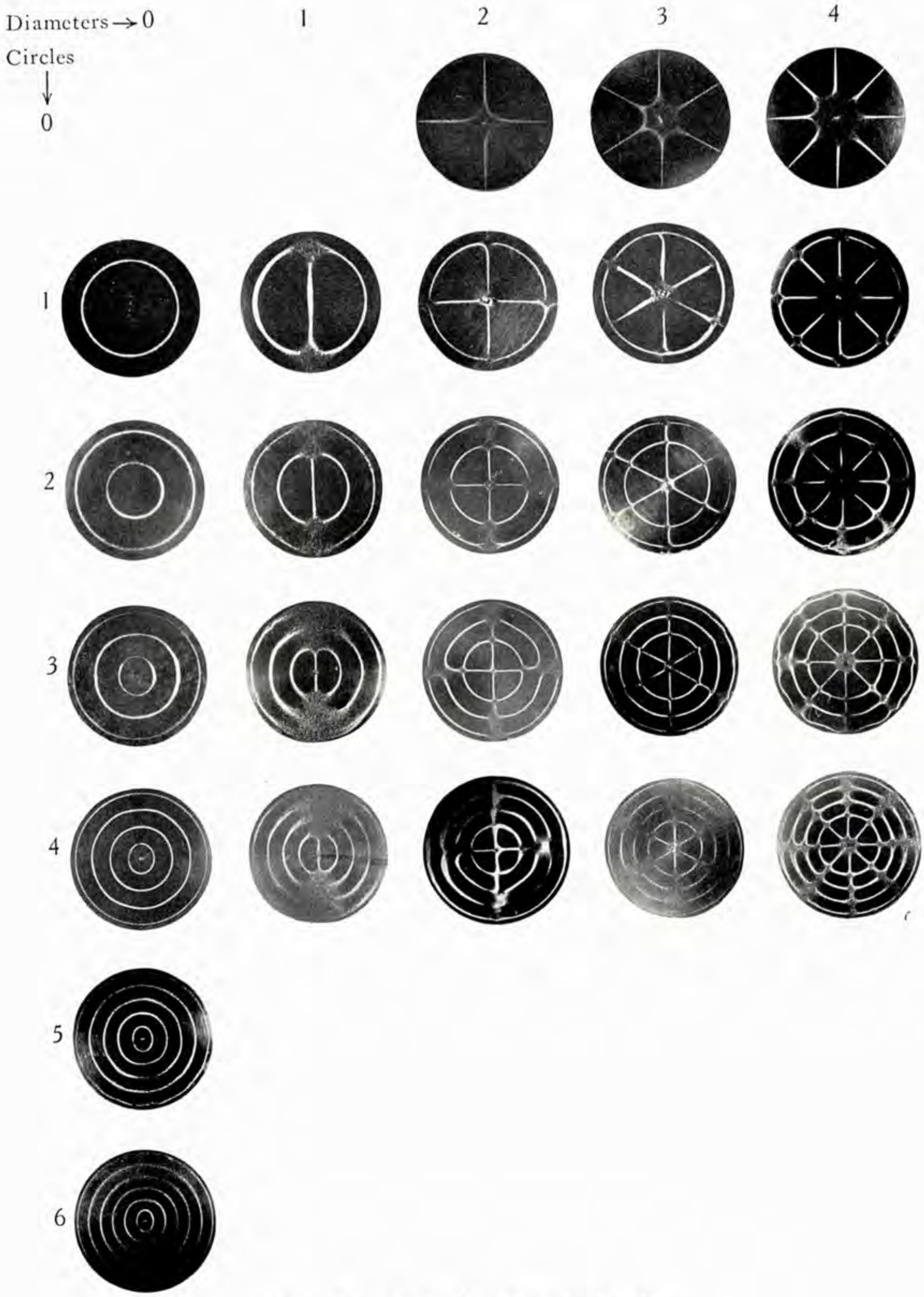
ABSTRACT. The normal vibrating modes of free circular plates have been determined over an interval of more than six octaves, the solid-carbon-dioxide method of excitation being used to produce the nodal figures, and a valve oscillator to measure their frequencies. It is concluded that the nodal system of ideally uniform and entirely free plates would, in accordance with theory, consist only of circles, diameters and combinations of circles and diameters. It is found that although each simple mode has its characteristic frequency, cases occur in which the difference between two or even three of these is very small, and the number of nodal diameters, which as regards raising the pitch are approximately equivalent to one nodal circle, increases from two to five, in the interval investigated, in passing from figures with relatively more circles to those with relatively more diameters. The expression, based on approximate theory, for calculating the frequencies of the higher modes, in which it is assumed that the addition of two nodal diameters increases the frequency by as much as one nodal circle, is accordingly not applicable in ordinary practice.

It is suggested that the simple and rapid solid-carbon-dioxide method of exciting free vibrations might be employed for testing the uniformity of plates, for detecting internal flaws, for studying recrystallization phenomena, and for obtaining comparative values of Poisson's ratio of metals and alloys.

§ 1. INTRODUCTION

The main facts regarding the vibrations of free circular plates, both theoretical and practical, are given in Rayleigh's treatise on *Sound*, vol. **1**, chapter 10 (§§ 218–23). The critical discussions and suggestions made not only in this chapter, but also in the preceding one which deals with vibrating membranes, have been much used in connexion with the present work.

The nodal figures of free circular plates, as also those of any other system of revolution, should, according to theory, consist of diameters symmetrically distributed round the centre, but otherwise arbitrary, together with concentric circles. In practice, however, it is well known that many other designs are frequently produced, which have never been systematically studied or adequately explained. The solid-carbon-dioxide method of producing vibrations[1] possesses peculiar advantages for doing this since vibration frequencies other than those natural to the plate cannot be produced and the excitation can be made on any part of the surface. It appears that the most extensive list of observed normal vibration frequencies of free circular plates is still that of Chladni[2]. His results are given in the notation of the chromatic scale, and in view of the apparatus at his disposal they are necessarily



Normal vibrating modes of free circular plate.

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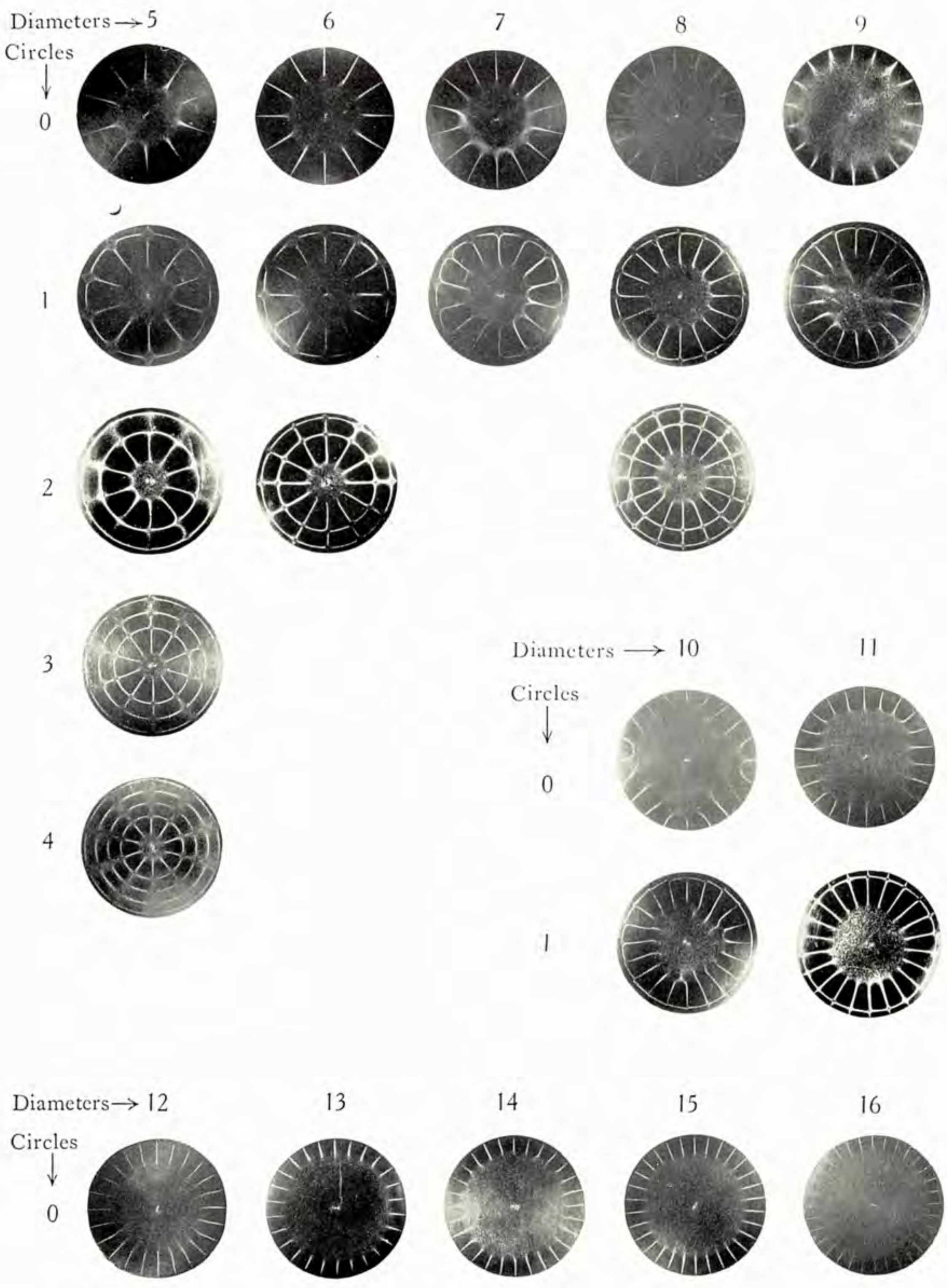

Plate 2. Normal vibrating modes of free circular plate (*cont.*).

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only approximate. They do not extend far enough to explain the greater number of his own nodal patterns nor those produced during the present investigation. Strehlke's[3] measurements of the radii of nodal circles will be referred to in due course.

Kirchhoff's theoretical frequencies, which on account of the lengthy calculations involved are restricted to the graver tones, will be considered later, and it will then be shown that the approximate expression for calculating the higher partials is not generally applicable in practice.

The present part of the paper describes and discusses the observations that have been made to obtain the frequencies and nodal systems of the normal vibrating modes over an interval of about six octaves. Part 2 will be concerned with a large number of nodal designs, other than those consisting of circles and diameters. As a result of these studies it has been found possible to interpret Chladni's original figures on circular plates, and these are considered in part 3.

§ 2. EXPERIMENTAL ARRANGEMENTS

Frequencies. The frequencies of the vibrations were determined by the method of beats by means of a mains operated calibrated valve oscillator.

Plates. A considerable number of plates of different sizes were employed, but final measurements were restricted as far as possible to large and thin plates. The diameters of two of these, for example, were 30·47 and 25·71 cm. respectively, the thickness in each case being approximately 2 mm. The importance of using thin plates is emphasized by some recent work of A. B. Wood[4] in which he shows how the frequencies of the two gravest tones of free circular plates gradually diminish as the ratio of thickness to diameter increases.

Supports and manner of excitation. When figures with a central node were obtained, the plates were supported centrally in the usual manner. The diameter of the hole in the plate was 5 mm. In order to obtain the nodal circles, the plate under test (preferably one without a central hole) was laid on three equidistant small circular-sectioned pieces of indiarubber, which were placed in appropriate nodal positions on a horizontal table provided with a paper scale graduated in circles and angles. With the first arrangement of support, the solid carbon dioxide was usually applied to the under surface of the plate, and with the second arrangement to the upper surface. A second translucent graduated scale, which could be laid on the surface of the plate, was often found useful in deciding on the point of excitation. While the surface was being sprinkled with sand this position was protected by means of a small proof plane which was afterwards removed. A desired nodal figure was also sometimes encouraged by gently touching the plate at one or two appropriate nodal positions by means of a divider. In order to obtain the best figures the experiments were made on dry days[1], dried sand being sprinkled just before the excitation on polished plates from which any grease had been removed with xylol. The Drikold solid carbon dioxide used for exciting the plates was very kindly given by Imperial Chemical Industries Ltd.

Photographs. Since it was desirable, for purposes of comparison, that all nodal figures should be given on the same scale, some of the photographs shown in the plates are enlargements of the original photographs. The two to eight diameter-only figures, obtained in the first instance on small vibrating plates by means of solid carbon dioxide, have been replaced by figures produced by bowing on the larger vibrating plates.

Radii of nodal circles. The radii of the nodal circles were measured directly on the vibrating plates and the results confirmed subsequently on the photographs, either a travelling microscope or a projection lantern being used.

§ 3. EXPERIMENTAL RESULTS

The nodal systems extending over an interval of nearly six octaves are shown in plates 1 and 2, in which each column contains figures with a given number of nodal diameters and each row figures with a constant number of nodal circles. In most of the figures, the small want of uniformity in thickness which is generally present in large metal plates cannot be detected, but in the symmetrical figures of the first column a variation in thickness which was found to be less than 1 per cent has been sufficient to cause a considerable distortion in the innermost circles of the 4/0, 5/0 and 6/0 figures. The longer diameter corresponds with the direction of smaller thickness. At the 0/10 figure there is a very significant break in the regularity of the diameter figures of the first row. It was subsequently found possible to obtain figures without this irregularity on some particularly uniform steel plates kindly lent to me by Professor Andrade.

For the frequencies of the vibrations the various nodal systems are expressed as multiples of the gravest, 0/2 mode (i.e. the mode characterized by 0 circles and 2 diameters), and are given in table 1. It will be seen that although each frequency is

Table 1. Relative frequencies of normal vibrating modes of free circular brass plate

Circles \ Diameters	0	1	2	3	4	5	6	7	8	9	10	11	12
0			1*	2·29	4·10	6·19	8·80	11·7	15·1	18·8	23·0	27·6	33·0
1	1·70	3·99	6·79	10·3	13·8	18·1	23·1	28·6	34·7	40·8	47·5	54·7	63·0
2	7·51	11·7	16·1	21·2	27·1	33·4	40·5	47·7	56·0	*65*	*75*	*86*	*100*
3	16·4	22·7	29·4	36·3	43·0	50·5	*59*	*69*	*80*	*90*	*(104)*	*(118)*	*(135)*
4	29·1	37·0	43·2	53·1	63·0	*74*	*86*	*99*	*(120)*	*(145)*			
5	47·0	56·0	*66*	*77*	*92*	*(110)*	*(135)*						
6	72·7	*83*	*100*	*(127)*	*(147)*								
7	(110)	*(132)*	*(160)*										
Circles \ Diameters	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	
0	38·0	44·5	51·0	58·0	65·5	74·3	83·5	90·9	100	109	118	127	
1	*72·5*	(81)	(90)	(97)	*(108)*	*(120)*	*(135)*						
2	*(115)*												

The figures in italics are approximate and those in brackets have been estimated by extrapolation only or by rough measurements on a large plate 46·7 cm. in diameter.

* Actual frequencies of the plates 30·47 and 25·7 cm. in diameter were 80·9 and 113·3 c./sec. respectively.

distinct, cases occur in which there is very little difference between two or even three of the numbers. In particular the 0/10 relative frequency, 23·0, is very near to that of the 1/6 mode, 23·1. It becomes evident, then, that the imperfect 0/10 nodal figure is due to the double circumstance of a departure from perfect symmetry in the plate and a near coincidence of natural periods. We shall return to this important result in part 2. Other small irregularities will also then receive an adequate explanation.

Table 1 also includes some further frequencies which have been obtained either by extrapolation or from recent observations which have been made without photographing the nodal systems. Some of the approximate numbers of the highest partials are in italics; they will be needed in parts 2 and 3. Measurements of the radii of the nodal circles, given relatively to the plate's radius, are shown in table 2.

Table 2. Radii of nodal circles relative to that of the disc, taken as 1000

Circles \ Diameters	0	1	2	3	4	5	6	7	8	9	10	11
1	680	781	823	843	859	871	880	889	897	903	909	912
2	391	497	562	604	635	662	681	702	715			
	843	867	887	898	906	915	922	927	932			
3	257	349	415	461	505	531						
	591	643	681	706	728	745						
	895	902	913	919	925	933						
4	190	269	328	374	411	443						
	441	495	540	571	596	623						
	692	726	748	764	779	794						
	918	928	934	938	941	944						
5	154											
	351											
	548											
	753											
	956											
6	131											
	292											
	456											
	624											
	794											
	958											

§ 4. COMPARISON WITH PREVIOUS RESULTS AND WITH THEORY

Chladni's law. It is evident from table 1 (see also the last column of table 1, part 2) that Chladni's experimental conclusion (2), which states that the addition of 2 nodal diameters raises the pitch by approximately the same amount as that of 1 nodal circle, has not been substantiated. This number increases from 2 to 5 in the interval investigated, being least for the vibrating modes shown on the lower left-hand side and greatest for those shown on the upper right-hand side of the table. Since the approximate expression (see below) for calculating the frequencies of higher partials also makes the assumption made by Chladni, it has been necessary to examine the evidence very carefully.

A study of Chladni's work[2] shows that he was more interested in producing nodal circles than in producing nodal diameters, and that whenever possible he supported and controlled his plates by hand. His table of frequencies extends only to 8 nodal diameters, but as far as 6 nodal circles which characterize roughly the same pitch as 18 diameters. It appears that he must have formulated his law with reference to the frequencies at the lower left hand of table 1 of the present paper. This is all the more curious since his own table gives evidence of three diameters being equivalent to one circle. For example the pitch g' is given both for figure 0/5 and for 1/2, b'' for 0/8 and 2/2, and g''' for 1/7 and 2/4. His nodal figures, which are the subject of part 3, also provide much evidence of more than two diameters being equivalent to one circle.

According to the approximate expression[3] for calculating the higher roots of Kirchhoff's free-plate equation the frequencies of the normal vibrating modes are proportional to $(n+2h)^2$, where n is the number of diameters and h is an integer. It will be best to quote Rayleigh's own words, in which the expression in question is quoted as (4)*: "It appears by a numerical comparison that h is identical with the number of circular nodes.... *Within the limits of application of* (4), we see also that the pitch is approximately unaltered, when any number is subtracted from h, provided twice that number be added to n. This law, of which *traces* appear in the following table, may be expressed by saying that towards raising the pitch nodal circles have twice the effect of nodal diameters. It is probable that, strictly speaking, *no two components have exactly the same pitch.*"

It appears from the present observations, that Chladni's law applies to very few of the partials in the first six octaves. When it is remembered that Kirchhoff's theory was formulated for thin plates and how the nodal diameters crowd up to the centre of the plate it is scarcely to be expected that any constant relation will exist between the number of nodal diameters and nodal circles which must be added in order to raise the vibration frequency by the same amount.

Before leaving this subject it is interesting to note that A. B. Wood[4] found a tendency for his 1/0 observations to yield somewhat higher values of velocity of sound than those derived from 0/1 observations.

Poisson's ratio. In the case of transversely vibrating plates, both Poisson's ratio and Young's modulus are factors determining the frequency of vibration, which, other conditions being constant, should vary with the material of the plate.

Since Chladni preferred glass to metal plates, it may fairly be assumed that his observations correspond to a value of Poisson's ratio σ such that $\sigma \doteqdot \frac{1}{4}$. A few vibration frequencies obtained with plates made of brass, for which $\sigma \doteqdot \frac{1}{3}$, and steel, for which $\sigma \doteqdot \frac{1}{4}$, are compared with Chladni's results in table 3, in which also some calculated frequencies due to Kirchhoff[3] for values $\frac{1}{3}$ and $\frac{1}{4}$ of σ are included.

It will be seen that an alteration of σ from 1/4 to 1/3 produces nearly a 7 per cent increase in the relative frequency of the 1/1 mode. These results are sufficiently striking to make further observations on plates of identical dimensions but of varying

* The italics are mine, and the table to which he refers extends only as far as 2 circles and 3 diameters.

Table 3. Comparison of results. Frequencies relative to that of the gravest mode, 0/2

Circles	Diameters	Material of plate. Observed relative frequencies			Kirchhoff's calculation[3]	
		Brass $\sigma \doteqdot \frac{1}{3}$	Steel $\sigma \doteqdot \frac{1}{4}$	Chladni glass* $\sigma \doteqdot \frac{1}{4}$	$\sigma = \frac{1}{3}$	$\sigma = \frac{1}{4}$
1	0	1·7	—	1·60	1·73	1·61
1	1	3·99	3·77 3·82	3·75	3·91	3·70
0	4	4·10	4·12 4·03	4·0	4·05	—

* The measurements were almost certainly made on glass plates[2].

material desirable. Table 4 may also be examined in connexion with this subject. In it Strehlke's[3] measurements on glass plates have been compared with the present measurements made on brass plates and with Kirchhoff's calculated values for $\sigma = \frac{1}{3}$ and $\frac{1}{4}$ respectively. It is perhaps just possible to detect the influence of Poisson's ratio in the later figures.

Table 4. Comparison of results. Radii of circular nodes (radius of disc = 1000)

Circles	Diameters	Observed on brass $\sigma \doteqdot \frac{1}{3}$	Strehlke Observed on glass $\sigma \doteqdot \frac{1}{4}$	From Kirchhoff's theory	
				$\sigma = \frac{1}{3}$	$\sigma = \frac{1}{4}$
1	0	680	678	—	681
2	0	391 843	391 841	— —	391 842
3	0	257 591 895	256 591 894	— — —	257 591 894
1	1	781	782	780·8	781·4
1	2	823	810	822·7	821·9
1	3	843	840	846·8	845·2
2	1	497 867	490 869	497·1 870·1	497·7 870·6

Measurements made on Chladni's drawings are not included since the results are variable.

§ 5. CONCLUSIONS

(1) By means of the solid-carbon-dioxide method of producing vibrations in metal objects it has been possible to obtain all the normal vibrating modes of free circular plates which occur in an interval of over six octaves.

(2) The nodal system of ideally uniform and perfectly free circular plates would consist, in agreement with theory, only of circles, diameters, or combinations of circles and diameters.

(3) The vibration frequencies of the normal vibrating modes are all distinct. Occasionally the frequencies of two or even three modes are very nearly equal.

(4) The number of nodal diameters which must be added to raise the frequency by as much as one nodal circle is not constant, but increases from two to five in the interval investigated. This increase occurs in passing from figures with relatively more circles to those with relatively more diameters. Approximate expressions for calculating the higher partials, which assume that this number is always equal to two, are accordingly inapplicable in practice.

(5) The relative frequencies of the overtones are appreciably affected by the value of Poisson's ratio in a manner which is in general agreement with theory.

§ 6. SUGGESTED APPLICATIONS

The following are some possible future applications of the solid-carbon-dioxide method of producing vibrations:

(1) *Rapid tests of uniformity of metal plates and detection of flaws.* Any departure from symmetry of a circular plate is at once apparent in the innermost nodal circle of the symmetrical nodal figures and this provides a sensitive method of detecting want of uniformity in plates. In this connexion it may be mentioned that a magnetostriction oscillator has recently been employed by Hayes[5] for the detection and location of laminations in steel plates.

(2). *Study of recrystallization phenomena.* It would be interesting to study alterations in the nodal figures, more especially perhaps in nodal circles, which would occur when a metal plate which had been rolled in one direction only was heated until recrystallization occurred. This subject has already been studied by Tammann and his collaborators[6,7] principally on the two-diameter and three-diameter nodal figures produced by bowing.

(3) *Comparative measurements of Poisson's ratio of metals and alloys.* It has been shown in table 3 that an alteration in the value of Poisson's ratio has an appreciable effect on the relative frequencies of the normal vibrating modes, which is possibly sufficient to provide a simple means of obtaining comparative values of Poisson's ratio for different metals.

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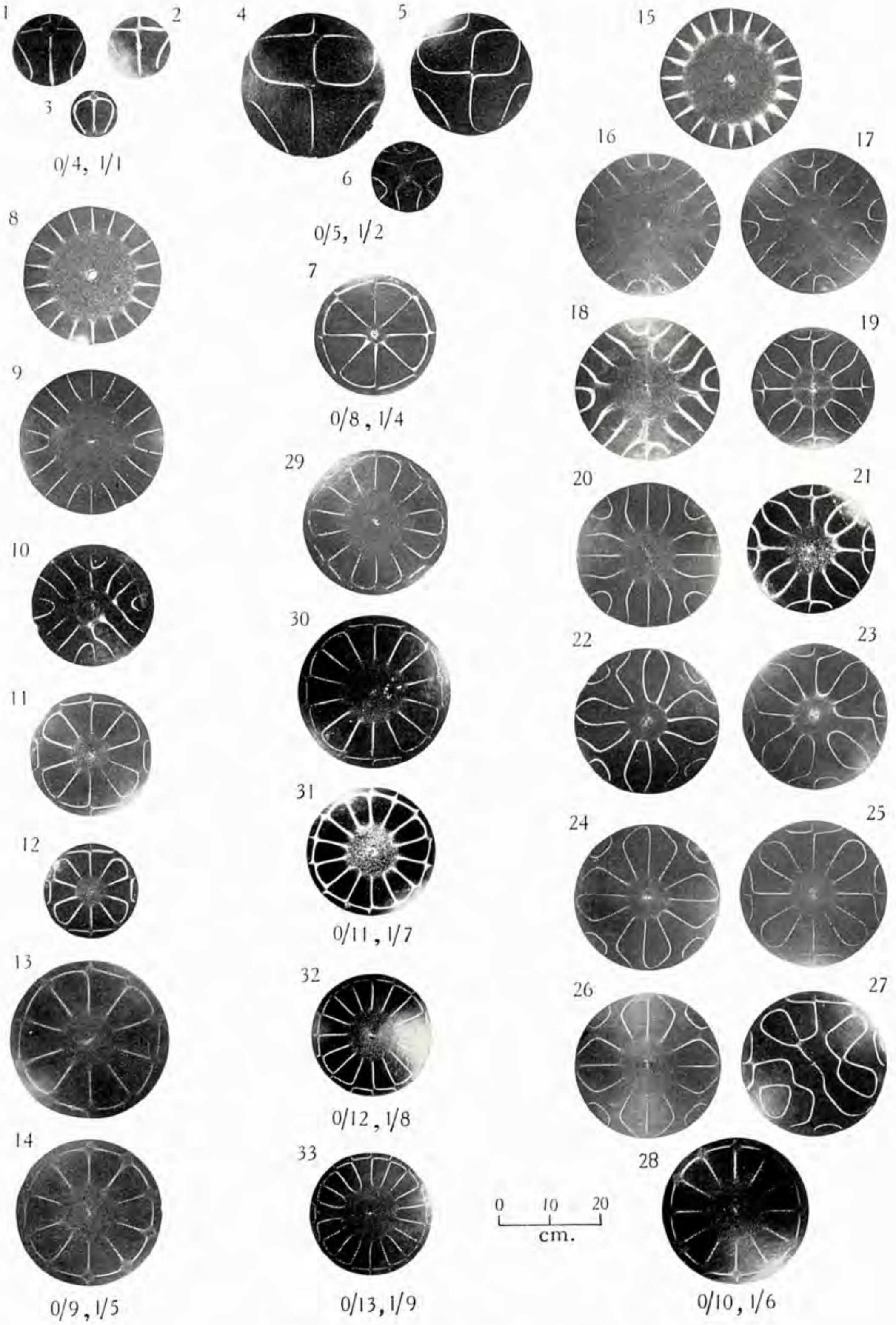

Combination of two vibrating modes.

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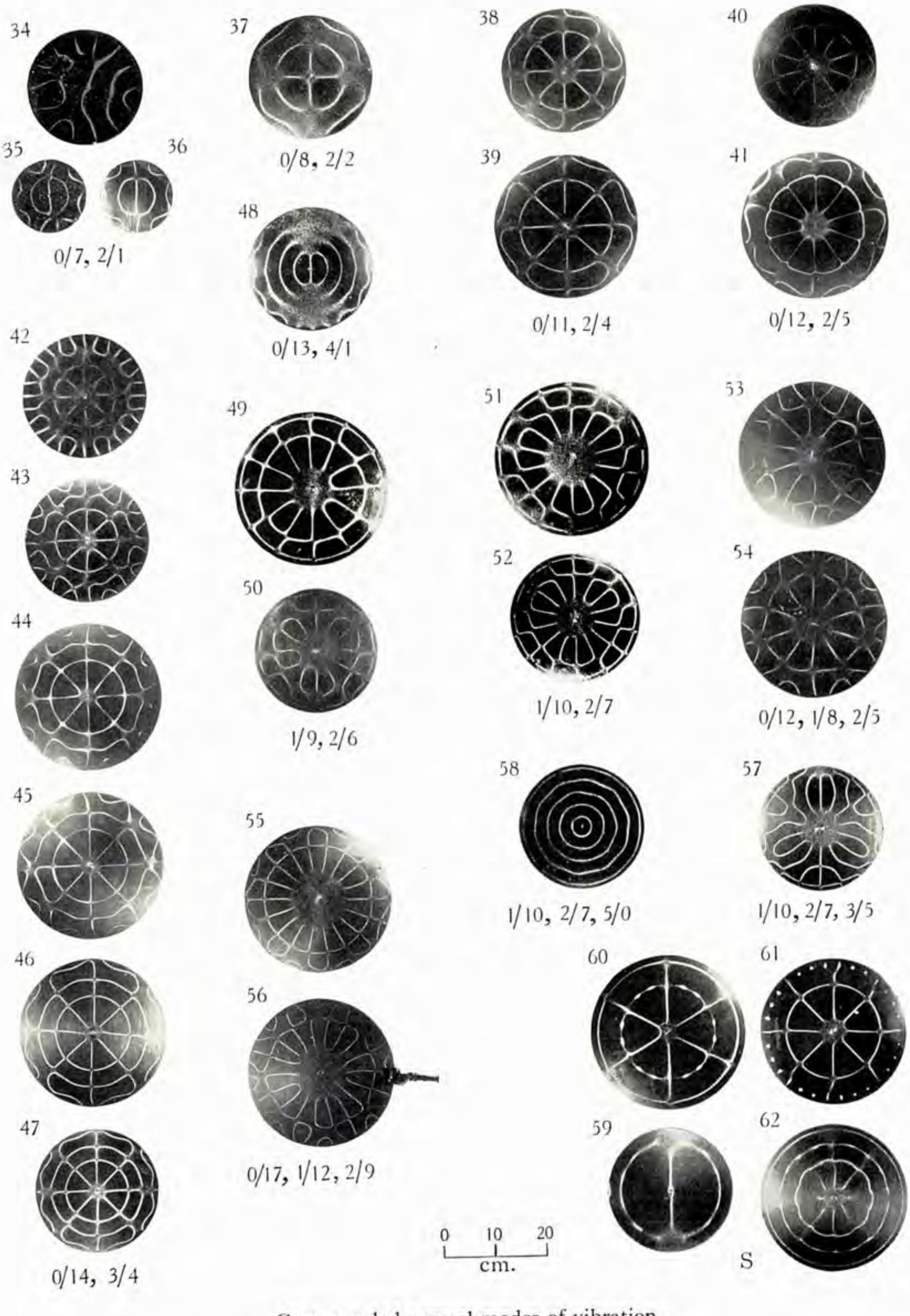

Compounded normal modes of vibration.

VIBRATIONS OF FREE CIRCULAR PLATES. PART 2: COMPOUNDED NORMAL MODES

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ABSTRACT. The many nodal designs (other than those consisting solely of circles and diameters) which are obtained on free circular plates are shown to consist of two or more compounded normal modes of vibration, the periods of which are very nearly equal. The vibration frequencies corresponding to the designs are intermediate between those of the combining modes. The figures are produced as a result of the slight divergence of free periods which exists in all but ideally uniform plates. An increase in damping, either internal or arising from the manner of support or of excitation, also favours the production of these designs, since it renders the resonance less sharp. Compounded modes of a somewhat similar character may be expected to occur in other systems of revolution, such as circular plates clamped at the edge, circular membranes, rings and cymbals, in all of which the normal nodal system consists of combinations of circles and diameters.

§ 1. INTRODUCTION

One of the earliest nodal designs produced by means of solid carbon dioxide was that shown in figure 21 of the present paper. It was noticed that whereas 12 nodal lines can be counted at the centre, there are 20 round the circumference. The figure appears to be a variant of either a 0/10 (no circles, ten diameters) figure or of a 1/6 (one circle, six diameters) figure, or really a combination of these two simple figures. This conclusion, arrived at visually, was subsequently confirmed by aural means when it was found that the vibration frequencies of the two simple modes of the compounded mode were practically identical.

This early observation supplied the key to the explanation of all the other figures that have been produced. The grouping of these figures is shown in the two plates; it was not completed until the knowledge regarding the normal vibrating modes which has been described in part 1 was available.

The combination of two simple modes is made possible by a slight want of uniformity in the plate under observation. Thus in the present experiments the thickness of the 30·47-cm.-diameter plate was found to vary from 2·040 to 2·020 mm. and that of the 25·71-cm.-diameter plate from 2·017 to 2·030 mm. This departure from exact symmetry causes a divergence of free periods[1a] and renders determinate the positions of the nodal diameters. The nodal figure will therefore vary according to the position of excitation, and it will be possible for two or more normal modes of neighbouring periods to coexist in any proportions. This compounding of modes may be compared with that which is always possible in the case of square plates[1b] where two fundamental modes, at least, have equal periods.

§ 2. TECHNIQUE

The plates were treated in the manner described in part 1, § 2. Both the bowing and the solid-carbon-dioxide methods of producing vibrations were used, the former for exciting at the edge, and the latter for exciting at other points on the surface of the plate. The figures shown at 1, 2, 4–6, 9, 17, 20, 23, 24, 27, 34–36, 48 and 55–58 in plates 1 and 2 were produced by bowing.

§ 3. COMPOUNDED NORMAL MODES

The groups of figures, arranged as far as possible in ascending order of tone, are shown in the two plates, and a summary of results is given in table 1.

Table 1. Compounded normal modes of vibration, figures 1 to 58

		Frequencies relative to o/2 tone = 1	R*
Plate 1:	Two modes		
Figures 1 to 3	o/4, 1/1	4·10, 3·99	3
4 to 6	o/5, 1/2	6·19, 6·79	3
7	o/8, 1/4	15·1, 13·8	4
8 to 14	o/9, 1/5	18·8, 18·1	4
15 to 28	o/10, 1/6	23·0, 23·1	4
29 to 31	o/11, 1/7	27·6, 28·6	4
32	o/12, 1/8	33·0, 34·7	4
33	o/13, 1/9	38·0, 40·0	4
—	o/17, 1/12	65·5, 63·0	5
Plate 2:			
Figures 34 to 36	o/7, 2/1	11·7, 11·74	3
37	o/8, 2/2	15·1, 16·1	3
38, 39	o/11, 2/4	27·6, 27·1	3·5
40, 41	o/12, 2/5	33·0, 33·4	3·5
—	o/16, 2/8	58·0, 56·0	4
42 to 47	o/14, 3/4	44·5, 43·0	3·3
48	o/13, 4/1	38·0, 37·0	3
49, 50	1/9, 2/6	40·8, 40·5	3
51, 52	1/10, 2/7	47·5, 47·7	3
	Three modes		
53, 54	o/12, 1/8, 2/5	33·0, 34·7, 33·4	4, 3
55, 56	o/17, 1/12, 2/9	65·5, 63·0, 65·0	5, 3
57	1/10, 2/7, 3/5	47·5, 47·7, 50·5	3, 2
58	1/10, 2/7, 5/0	47·5, 47·7, 47·0	3, 2·3

* Number of nodal diameters which raise the pitch by approximately the same amount as one nodal circle.

The o/10, 1/6 group (figures 15–28), of which figure 21 has already been mentioned, will first be considered in some detail.

It will be seen that the two simple modes, 0/10 (obtained on Prof. Andrade's uniform plate mentioned in part 1) and 1/6, are shown in figures 15 and 28 respectively. Between these figures there are two columns of compounded nodal figures in which it is interesting to compare the two different developments of design. The distorting effect of an additional clamp at the edge is shown in figure 27, which is only just recognizable as belonging to the group.

For the recognition of the basic normal modes of more complicated figures, it will be useful to record from observations made on this group that (1) A hoop at the circumference, figure 20, is an indication of 2 diameters in one of the constituent normal modes and of a circle in the other. (2) A loop from the centre, figure 25, indicates 2 diameters and 1 circle of one of the modes. (3) A bend in a nodal radius, figure 22, is evidence of the presence of a nodal circle in one of the modes.

Passing now to the other figures shown in plates 1 and 2, it may be remarked that the crown-like figure 3 (0/4, 1/1) is frequently obtained when a small plate is held at a node between the thumb and first finger and the solid carbon dioxide is applied in the middle of one of the two vibrating central portions; the plate is not quite free, and both this figure and figure 2 may be compared to the inverted Chladni figure 101 *b*, part 3. An eccentric circular clamp was used in the production of figure 1. Figures 4 to 6, in the 0/5, 1/2 group, are again typical of plates which are unsupported so as to allow of entirely free vibration. The first two, if inverted, may be compared with Chladni's figure 102 *b*. The component modes of figures 8 to 14 (in the 0/9, 1/5 group) are easily recognized, but the flatness of the hoop in figure 13 should be noted as this makes it easy to recognize the 0/8, 1/4, figure 7, and the component parts of several other nodal patterns.

The varying designs in the 0/14, 3/4 group, figures 42 to 47, may be noted, and especially the meaning of such wavy lines as are found in the last two figures, some of which indicate 3 and some 4 diameters. It is then possible to recognize the component modes of figures 37 and 38 or 39. Figure 48 (characterized as 0/13, 4/1) is the most important nodal design in connexion with the study of Chladni's own drawings, part 3, and shows the *Biegungen* or bendings which are so characteristic of many of them. Comparison with figures 36, 45, 44, 43 and 42 shows that each bending from crest to crest indicates two radii of one of the combining modes. Figure 58 should also be studied in this connexion; the waviness of the lines is typical of bow-excitation, and from them traces of the 1/10 and 2/7 modes can be recognized superposed on the predominating 5/0 mode. Three simple modes are combined in figure 54, of which 2/5 can be recognized by looking at the central portion of the plate 0/12 and by looking at the circumference, while the presence of 1/8 is seen by omitting diameters (indicated by 4 hoops at the circumference) and adding a circle in their place. The plate was clamped as in figure 56 (0/17, 1/12, 2/9), and excited by bowing. The double bend of some of the radii in figure 57 (1/10, 2/7, 3/5), indicative of two circles, may be noted; it is also possible to see a trace of the 3/5 mode in this nodal design.

The above conclusions regarding the combination of simple modes, which have been arrived at mainly by inspection of the nodal systems, are confirmed by finding that the vibration frequencies of the designs belonging to any one group are intermediate between the two nearly equal frequencies of the parent modes. The summary of results which is given in table 1 includes also several later observations on higher compound nodal designs which were not photographed. The numbers in the last column give the number of nodal diameters which raise the frequency of vibration by as much as does one nodal circle, and are of importance in connexion with the argument set out in § 4 of part 1.

Superposed sand figures. The four figures, 59 to 62, shown at the bottom of the right-hand side of plate 2, differ in character from those already considered, and were obtained by chance when it happened that the plate was excited a second time by means of solid carbon dioxide in such a way as to produce a second sand pattern, which because of its geometrical similarity did not disturb much of the first pattern. Figure 59 appears to be a 0/6 nodal pattern superposed on a 1/1 pattern, and figure 60 a 1/9 pattern superposed on a 2/3 pattern. Figure 61 is 0/9 superposed on 2/4, assuming the amplitude of the second vibration to have been insufficient to disturb the sand of the inner circle. Figure 62 is possibly made up of 0/9 with a compound vibration 1/10, 3/4 superposed on it. The vibration frequencies (which are unequal) of the two patterns do not of course enter into the elucidation of such figures.

§ 4. FURTHER REMARKS

Comparison with nodal designs of damped and forced vibrations. It is interesting to compare the present results with a large number of drawings made by Elsas[2] showing the nodal figures obtained on damped circular membranes, the vibrations of which were forced. In such a case, as was shown subsequently by Debye[3], it is possible for two neighbouring natural modes to combine because of the flatness of resonance. In fact a simple mode is never produced singly. Franke[4] has followed up this work with a study of the forced vibrations of clamped circular plates, and a paper by Shünemann[5] may also be mentioned in this connexion. Though the figures are of course not the same as those given by free circular plates, yet since they are also those of a system of revolution they often bear a strong resemblance to some of them. There is, I think, evidence of forcing or at any rate external damping in many of the figures obtained by Colwell[6] on free circular plates by means of a magnetostriction oscillator. The exciter has its own period, and a very nice adjustment of frequency is necessary in order to avoid an element of forcing. The method of support also affects the result. The somewhat square circle and the dumb-bell-shaped central nodal curve in some of Colwell's figures can be found also among the nodal figures yielded by Elsas's forced vibrations.

Vibrations of circular membranes. The principle of composition that has been developed in the present paper is evidently applicable to circular membranes, and assuming that the relative actual frequencies are the same as those given theoretically by a table of the roots of Bessel functions of the first kind, the following combinations among the lower partials appear probable: 0/2, 2/0; 0/5, 2/2; 3/2, 4/0. As in the case of plates it would seem that no two modes have exactly the same period, but since the overtones are very numerous (as may be seen from table 2, where the number of overtones in given intervals of several different vibrating systems have been compared) and the vibrations are relatively damped, compounded modes should occur fairly frequently. Any inequality of tension over the surface of the membrane will cause distortion, and the recent figures due to Schiller(7) have the irregular features of figure 27 of the present paper, which was distorted by means of a clamp. The following compound modes can however possibly be recognized in his figures 4*a*, 4*b* and 4*c* respectively: 0/4, 1/2; 0/5, 2/2; 0/6, 2/2; and 0/3, 1/0 in his figure 5.

Table 2. Number of overtones in given intervals

Octaves	1	2	3	4	5	6	7	Source of information
Strings	2	4	8	16	32	64	128	
Free plates	1	4	8	14	27	51	84	Present observations.
Clamped plates	1	4	7	18				Tables and Franke.
Membranes	2	10	44	*c.* 157				From roots of Bessel Functions of first kind.

Comparison with circular plates clamped round the edge. Similarly it is possible to surmise which vibrating modes should combine in plates which are clamped round the edge, as is done in table 3, in which a few suggested combinations are placed by the side of the actual combinations that have been obtained with free circular plates. It may be noted, for example, that the 0/5 figure which we have seen compounded with a 1/2 mode in figures 4 to 6 above should, in the case of the clamped plate, combine with a 2/0 mode.

Table 3. Compounded normal vibrating modes

Free circular plate, experimental		Clamped circular plate, suggested	
0/4 (4·10),	1/1 (3·99)		
0/5 (6·19),	1/2 (6·79)	0/5 (8·88),	2/0 (8·72)
		0/6 (11·27),	2/1 (11·8)
0/7 (11·7),	2/1 (11·74)	0/7 (13·8),	1/4 (13·7)

The frequencies of the free plate are given relatively to that of its gravest 0/2 mode, taken as 1, and those of the clamped plate are given relatively to that of its gravest 0/0 mode, taken as 1. It has been calculated(8) that the frequency of the 0/0 mode of the clamped plate is approximately 1·92 times the frequency of the 0/2 mode of the free plate.

Cymbals. A few preliminary experiments with cymbals, kindly lent by Messrs Boosey and Hawkes, show that both the normal modes, in which the radii of the nodal circles and relative frequencies depend upon the type of cymbal, and also compounded modes can be produced. It will be interesting to continue these experiments, since the results could scarcely be obtained by analytical methods. The main difficulty is to obtain a powder which, though sufficiently mobile, will not slide off the plates.

§ 5. CONCLUSIONS

All nodal designs, other than those corresponding to the normal modes of vibration, which can be produced on free circular plates or on other systems of revolution result from the combination of two or more normal modes of nearly equal period. This compounding of modes is made possible by the want of uniformity which generally exists in actual plates, and it is also made easier by an increase in the damping of the vibrations. The most usual cause of want of uniformity is a small variation in thickness. Another possible cause is variation of elasticity of the material in plates which have been rolled in one direction.

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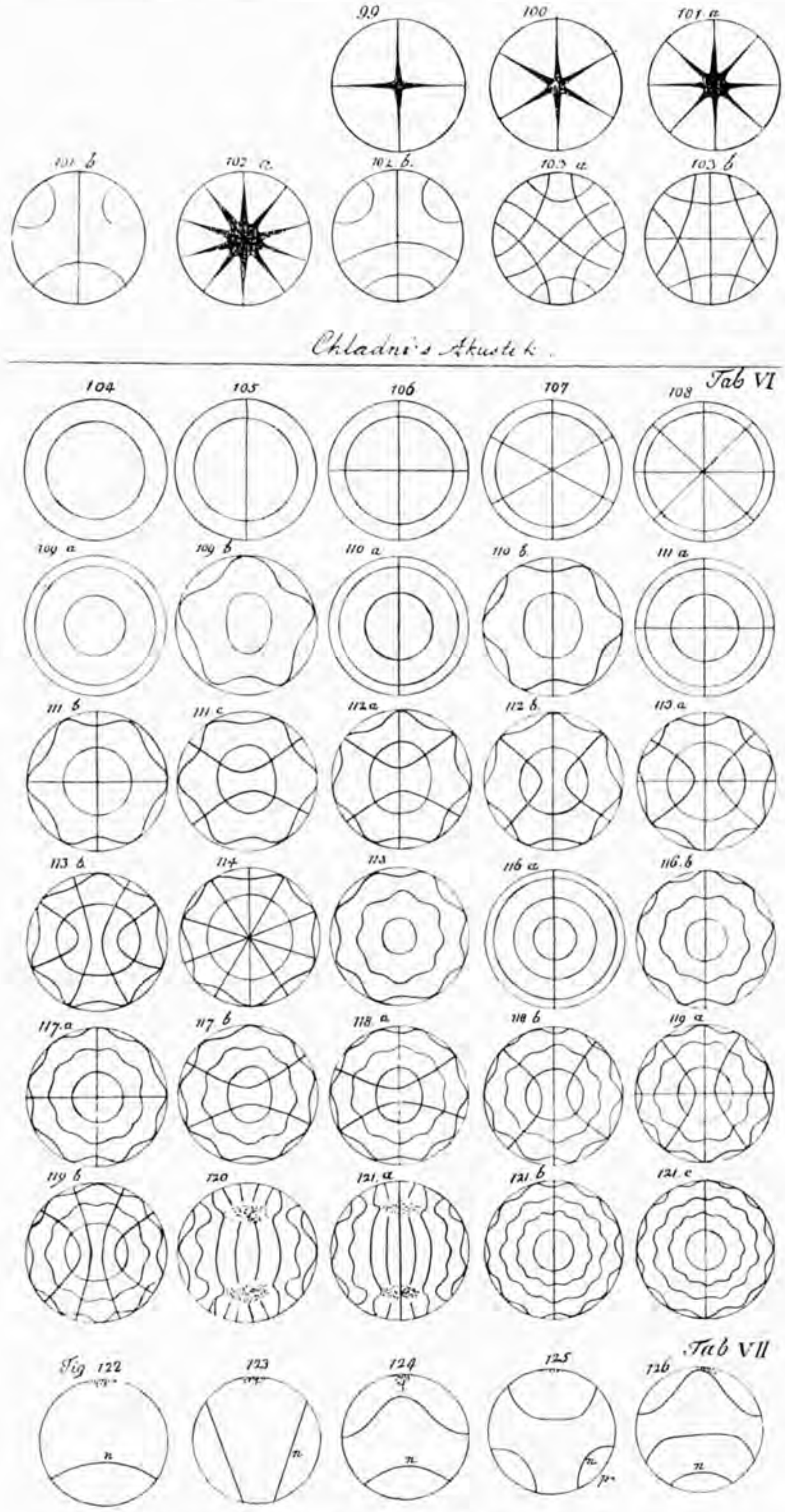

Figures 99–126 from Chladni, *Die Akustik*.

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VIBRATIONS OF FREE CIRCULAR PLATES. PART 3: A STUDY OF CHLADNI'S ORIGINAL FIGURES

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Received 15 *October* 1937. *Read* 26 *November* 1937

ABSTRACT. The complete set of figures which Chladni produced on circular plates has been examined. It has been found that a second subsidiary normal vibrating mode, the nodal system of which consists of diameters, is in many of the figures compounded with a principal figure of nearly identical period. The manner of support and the manner of excitation make such combinations likely.

§ 1. CHLADNI'S ORIGINAL FIGURES ON CIRCULAR PLATES

THE *Entdeckungen über die Theorie des Klanges* which was published by Ernst Florens Friedrich Chladni in 1787 contains seventy-five copper-plate figures of his drawings of nodal lines produced on circular plates. Forty-three of these figures are reproduced in *Die Akustik* (1802, second edition 1830). They are shown in plates 1 and 2 of the present paper and were obtained at the Royal Institution through the kindness of Mr R. Cory the Librarian. The rest of the figures in the *Entdeckungen über die Theorie des Klanges* are, with but few exceptions to which reference will be made below, similar in character to those shown in plates 1 and 2. The study that has already been described in parts 1 and 2 of the present paper make it possible to see that most of the figures are a combination of two normal modes of vibration having nearly equal periods.

Of the single simple modes the highest number of diameters shown is five. Chladni, of course, produced a greater number of diameters than this, and his table of frequencies extends as far as eight diameters. He gives drawings for one and for two nodal circles; this is probably quite as much as could be managed by means of a bow applied at the edge of the plate. The highest single mode shown is that of figure 116*a*, which is of the 3/1 type. The relative frequency of this mode, according to table 1, of part 1, is 22·7.* There is considerable distortion in some of the earliest and latest figures, and this is no doubt due to a want of freedom in the manner in which the plate is supported.

The remaining figures all show a greater or smaller number of *Biegungen* or bendings in the nodal circles. Chladni paid special attention to these and gave a table[1] of the number generally associated with each figure, which number can also be counted directly on the figures reproduced in the present paper.

* It has been shown in part 1, § 4, that the relative frequencies are affected by the value of Poisson's ratio, so that the figures quoted here and below apply only approximately to glass plates.

Figure 48 of part 2 of the present paper, which is a 0/13, 4/1 figure, was produced by bowing. It is of the same type as Chladni's, and each bend, or distance from crest to crest, in the wavy nodal circles evidently indicates the presence of two diameters of one of the combining modes, so that counting round the whole circle the number of bends in a figure such as 115 shows that an 8-diameter mode has combined with a 3-circle mode. In the case of figures which include obvious diameters, whether straight or curved, it has been found that the *Biegungen* diameters must be added to these in order to obtain the second vibrating mode.

Table 1 has been compiled according to this rule, the predominating mode being printed in black numerals while the frequency relative to that of the gravest mode, 0/2 (obtained from table 1 of part 1), is given in brackets. References to figures given in part 2 are followed by the letter *W*.

Table 1. Composition of Chladni's figures on circular plates

Figure	Modes and relative frequencies	Remarks
101*b*	0/4 (4·1), 1/1 (3·99)	Cf. figure 2, *W*, inverted, and also the less regular figures 125, 126 below
102*b*	0/5 (6·19), 1/2 (6·79)	Cf. figures 4–6, *W*, inverted
103*a*, *b*	0/8 (15·1), 2/2 (16·1)	Cf. 37, *W*
109*b*	**2/0** (7·50), 0/5 (6·19)	Chladni states that this was the only case in which he noted that the pitch was lower than that of the corresponding regular figure, here 109*a*. There are 6 instead of 5 *Biegungen* in the corresponding *Entdeckungen* figure, **2/0** (7·50), 0/6 (8·8)
110*b*	**2/1** (11·74), 0/6+1=0/7 (11·7)	Cf. 36, *W*
111*b*, *c*	**2/2** (16·1), 0/8 (15·1)	Cf. 103*a*, *b* and 37, *W*
112*a*, *b*	**2/3** (21·2), 0/7+3=0/10 (23)	
113*a*, *b*	**2/4** (27·1), 0/7+4=0/11 (27·6)	Cf. 38, 39, *W*
114	**2/5** (33·4), 0/7+5=0/12 (33)	Cf. 40, 41, *W*
115	**3/0** (16·4), 0/8 (15·1)	
116*b*	**3/1** (22·7), 0/9+1=0/10 (23)	
117*a*, *b*	**3/2** (29·4), 0/9+2=0/11 (27·6)	
118*a*, *b*	**3/3** (36·3), 0/10+3=0/13 (38)	
119*a*, *b*	**3/4** (43), 0/10+4=0/14 (44·5)	Cf. 42–47, *W*
120	4/0 (29·1), 0/12 (33)	In this and several neighbouring figures, and in those showing hyperbolic diameters, the imperfect freedom of the vibrations is specially manifest
121*a*, *b*, *c*	**4/1** (37), 0/13 (38)	Cf. 48, *W*
122	Less free, distorted 0/2	
123, 124	Distorted 0/3 (2·29)	
125, 126	Distorted 0/4 (4·10) and 1/1 (3·99)	

Additional figures in the Entdeckungen über die Theorie des Klanges. There are several more variants of the 0/5 (6·9), 1/2 (6·79) nodal figure to be seen in Chladni's earlier publication. The remaining figures are similar to those already considered but are compounded of higher partials, the last of which is a 7/0, 0/22 (110) combination evidently made on a plate which was not quite free.

1

2

Figures 1, 2. Plate excited at the edge with a bow.

3

4

5

6

Figures 3 to 6. Plate excited between the centre and the edge with solid carbon dioxide; see table 2.

To face page 85

§ 2. NOTE ON METHODS OF PRODUCING VIBRATION IN PLATES

The six figures of plate 2 which were produced on a centrally supported plate of diameter 46·2 cm. and thickness 2 mm. illustrate the possibilities of producing higher partials by means of a bow or of solid carbon dioxide. The first two figures were made with a small violin bow plentifully supplied with powdered resin, and the last four with solid carbon dioxide which was applied between the centre and circumference of the plate. Data regarding these figures are given in table 2. The essential freedom of these two methods of excitation is shown by the single-mode nodal designs of figures 1, 3 and 4.

Table 2

Figure	Nodal design	Approximate frequency (c./sec.)	Approximate frequency relative to that of 0/2 mode
	Single mode		
1	0/24	4200	127
3	7/2	5300	160
4	3/13	5100	154
	Compounded modes		
2	0/23, 3/11	3900	118
5	5/2, 1/12, etc.	2080	65
6	3/12 (0/25)	4360	130

The difference in character between the decorative designs produced by excitation at the edge and near the centre respectively, is specially noticeable in figures 2 and 6. In figure 2 the 0/23 mode is combined with a 3/11 mode of smaller amplitude, while in figure 6 a 3/12 mode is combined with a 0/25 mode of very much smaller amplitude. There is a threefold motif in both figures which are very interesting to compare.

It is evident that by varying the point of application of the solid carbon dioxide, and by the appropriate use of a divider to introduce constraints and damping, it should be possible to obtain a large number of very beautiful designs.

The figures may be compared with one of Colwell's high-frequency figures[2] produced with a magnetostriction oscillator, in which an 8/1 or 9/1 component is, I think, present. The frequency of this figure is stated to be some submultiple of 15,000 c./sec.; the size of the circular plate used is not specified.

An exhaustive reference to electrical methods of producing Chladni figures cannot be made here, but a brief mention of some purposes for which they have been employed when the two simpler methods could not be used may not be out of place. Thus Wood and Smith[3] and Colwell and Hall[4] have employed a magnetostriction oscillator to produce large numbers of nodal circles, the former for the purpose of determining the velocity of sound in non-metallic as well as in metallic circular sheets. Wood[5] has also used the same method to excite small circular discs. Franke[6] has produced forced oscillations in clamped circular plates. Andrade and Smith[7] have used an electromagnet in order to control the amplitude of vibration of steel plates in their study of the manner of formation of sand figures. In this last connection it is interesting to compare such figures as 1 and 3 of plate 2, where in the first figure the amplitude of vibration is small except round the edge, while in figure 3, the plate is vibrating vigorously over its entire surface.

The main purpose of this concluding section, however, is to draw attention to the very simple and effective methods of excitation which continue to be available for most general purposes—the bowing method for producing diameter figures and decorative figures of a specified type, and the solid-carbon-dioxide method for producing not only these but also the circle and circle-diameter figures and the large variety of designs which can be obtained by altering the position of excitation over the surface of the plate.

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Magneto-damping in nickel

Demonstration given on 26 November 1937 by Mary D. Waller, B.Sc., F.Inst.P.

The subject of this demonstration was the outcome of some acoustic studies made on metals and special alloy steels[1,2] in order to determine how the internal damping is affected by various physical conditions.

A pure hard nickel rod, kindly supplied by Messrs Henry Wiggin and Co., Ltd., was suspended horizontally at the two nodes (0·224 of the length from either end) by means of two fine threads, and struck with a hammer. The note emitted remained audible for some seconds. This procedure was repeated after the specimen had been placed in a magnetizing field of about 100 gauss, when it was found that the note died out three or four times more rapidly. For purposes of demonstration in a large room, a rod about 12 cm. long and about 1 cm. in diameter, which vibrated at 2570 c./sec., was readily audible, but the phenomenon was also demonstrated with a thinner and with a thicker rod.

A typical set of observations, made with the ear in a standard position and using a stop watch is shown in figure 1, in which the abscissa shows the strength of the magnetic field to which the specimen was subjected, while the ordinate shows the subsequent duration of audibility in seconds. The magnetic condition of the rod in figure 1 can be inferred from the hysteresis curve given in figure 2. Points to be noted in connexion with figure 1 are the horizontal part of the curve at A for weak fields, the steep part of the curve (corresponding to the steep part of the AB hysteresis curve), the impossibility of restoring the vibrating properties except by thoroughly demagnetizing the specimen, and the two maxima at fields which are near to, but exceed the coercivity.

Similar observations have been made with a more highly damped annealed pure nickel rod, and with rods made of commercial nickel, iron, cobalt (kindly given by the Brandhurst Co., Ltd.) and a number of Sir Robert Hadfield's special alloy

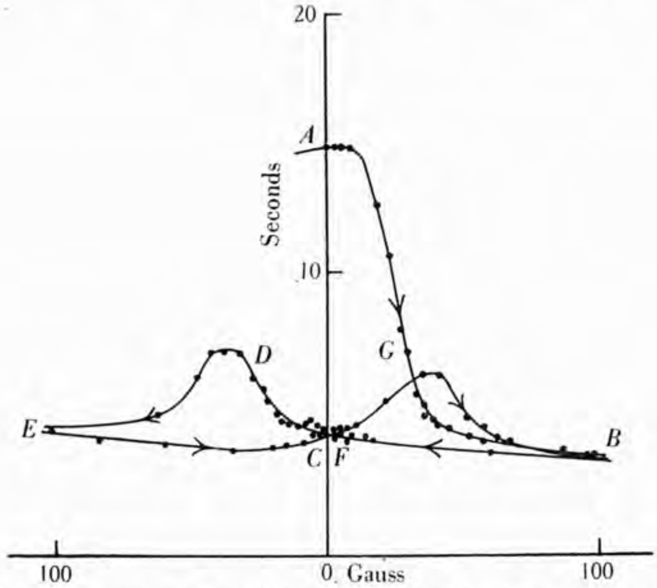

Figure 1. Variation of duration of audibility of vibrating nickel bar with intensity of magnetic field previously used to magnetize it.

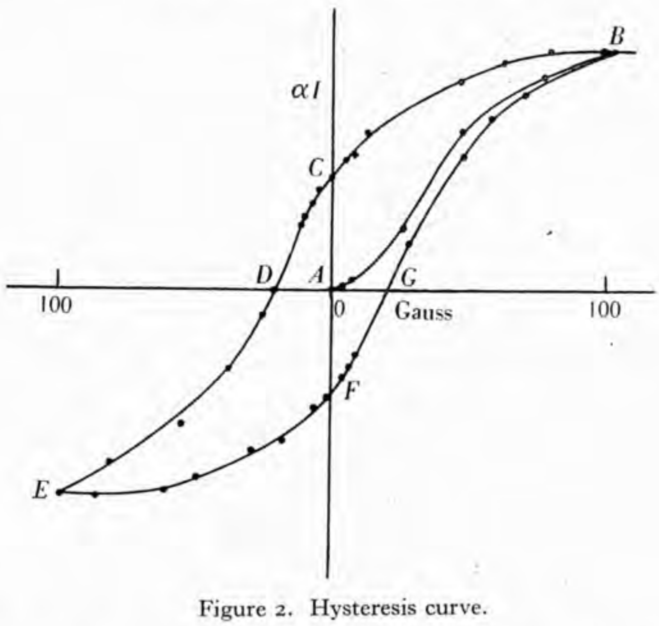

Figure 2. Hysteresis curve.

steels[2]. The effect is greatest with nickel and greater with cobalt than with iron. Kerston[3] has explained the increase of damping which occurs in the magnetized condition as being due to eddy currents occurring in the oscillating body. I have found that the effect is too small to be detected in some of the magnetic alloys investigated. The increase in damping in the case of nickel is comparable with that which can be produced by suitable mechanical and heat treatment.

Since vibration damping is of considerable importance in engineering practice[4], and since it is a structure-sensitive property which depends upon so many physical conditions, a simple method of observing it, such as the above, should be of value.

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Fundamental Vibration of a Rectangular Plate

THE *higher overtones* of vibrating plates sometimes combine when it so happens that the periods of two or more of them are very nearly equal. This phenomenon, which is of academic rather than of applied interest, has been fully studied for a free circular plate[1].

That the *graver tones* of the transverse vibrations may occasionally combine, would appear to be of importance in engineering resonance problems ; the rectangular plate is a case in point. Two familiar sand nodal figures are shown in Figs. 1 and 2, of which the first is fundamental to a narrow, and the second is fundamental to a wide free rectangular plate. The frequencies for the particular brass plate employed were very nearly equal, 558 and 548·8 c./sec. respectively.

(1)



(2)

(3)



(4)

(5)



(6)

By filing the longer side of the plate, the two periods were adjusted to equality with the aid of a valve oscillator. This resulted in a profound change in the form of the vibrating surface (see Fig. 4), the former nodal systems being no longer obtainable. A pattern which is the mirror image of Fig. 4 can also be obtained. The sharpness of resonance, which is an inverse measure of the 'damping capacity' of engineering practice, is considerable. Thus a change of only 1 c./sec. one way or the other in the tone, produced the modified nodal patterns of Figs. 3 and 5.

That these curved nodal lines are due, neither to a want of uniformity in the plate, nor to absence of freedom in the vibration, is proved by the fact that as soon as the same plate is excited in another mode (see Fig. 6), a characteristic single nodal pattern is obtained. The ratio of length to breadth in Fig. 4 is 1·93. I estimate that a combination between the nodal systems of Figs. 2 and 6 should occur when this ratio is about 3·9.

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Dec. 5.

[1] Waller, M. D., *Proc. Phys. Soc.*, **50**, 77 (1938).

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10

VIBRATIONS OF FREE SQUARE PLATES: PART 1. NORMAL VIBRATING MODES

BY

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VIBRATIONS OF FREE SQUARE PLATES: PART 1. NORMAL VIBRATING MODES

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ABSTRACT. After a brief historical note, the paper describes the result of systematic observations which have been made on the normal vibrations of free square plates. The nodal systems are divided into seven classes and their further recognition is effected by the application of given rules. The natural vibration frequencies found by measurement are in fair agreement with Ritz's calculations. The relative frequencies of the higher partials agree with Chladni's observation that they are proportional to m^2+n^2. For values of m and n exceeding three, it is found that the frequencies, relatively to the gravest $1|1$ tone taken as unity, are given roughly by $\frac{3}{4}(m^2+n^2)$.

§ 1. HISTORICAL NOTE

VIBRATING plates have now been a subject of study for one and a half centuries and this historical note is supplementary to the well-known accounts of the subject given by Rayleigh[14] and Love[15].

Chladni's early experiments on free square plates were described in his first book *Entdeckungen über die Theorie des Klanges*[1]. The majority of his nodal figures are given again in *Die Akustik*[2]. The largest and best arranged collection of figures can however only be seen in a later book, *Neue Beiträge zur Akustik*[3], which is now very rare, or else in Wheatstone's paper of the year 1833[8]. In this important work, the whole of Chladni's set of normal nodal drawings precedes the great collection of Wheatstone's own constructed diagrams, and an account is also given of Chladni's final conclusions on the subject. It is interesting to find that Chladni's later notation, $\overline{m|n}$ and $\underline{m|n}$ (see § 3·1 below) is immediately comparable with the notation of the later approximate theory, and that his vibration frequencies, which extend as far as the $9|9$th mode, are expressed in numbers instead of in the notation of the chromatic scale.

Chladni's last paper was written in 1825[5], not, as stated by Melde[13], in criticism of foreign publications, but actually because of the curiously distorted figures and wrong conclusions of an early paper by Strehlke[4]. Chladni died in 1827 and Strehlke published a second paper in 1830[6] in which his wrong conclusions are maintained. Faraday's[7] and Wheatstone's[8] papers appeared a few years later, and Strehlke's accurate drawings and measurements of some of the simpler nodal figures were first given in 1839[10]. It is interesting to read the publications of this period in chronological order. The criticism mentioned by Melde, or at any rate one that sounds very like it, is contained in Strehlke's last short paper which was published as late as 1872[12].

The approximate solution for the free square plate was given by Ritz[16] in 1909. Ritz compared the results of his theory with the nodal figures of Chladni and Strehlke, and with Chladni's frequencies, given in the notation of the chromatic scale. The Chladni figures said to be missing are given in the *Neue Beiträge zur Akustik*. Since Ritz's work is not mentioned in Rayleigh's *Treatise*, it is well to draw attention, as has been done by Temple and Bickley[17] and Southwell[18], to Rayleigh's paper of 1911 entitled *On the calculation of Chladni's figures for a Square Plate*[19]. Commenting on Ritz's work, Rayleigh remarks, "The general method of approximation is very skilfully applied but I am surprised that Ritz should have regarded the method itself as new."

§ 2. SCOPE OF THE PRESENT OBSERVATIONS AND NOTES ON TECHNIQUE

It was evidently desirable that a systematic experimental study of the vibrations of free square plates should be undertaken with the resources of modern apparatus. This has already been done for the free circular plate[23], and the experimental details given in connexion with these need not be described again in detail. The vibrations were produced by the solid carbon dioxide method of excitation[20,21,22] or else with a bow, and the same arrangements of support for the plates as in the previous work were generally employed. It is necessary to vary these according to the type of nodal figure which it is desired to produce, and for square plates an additional arrangement, in which the plates are suspended horizontally by means of two parallel loops of fine thread, was occasionally found convenient. The frequencies were measured in the usual manner by means of a calibrated valve oscillator. The observations were rendered independent of hygroscopic conditions by gently warming the plate and surrounding air by means of a bowl electric radiator, the heat being insufficient to alter either the elastic or damping properties of the material of the plate. A considerable number of *thin*[24] plates varying in size and material were employed. Details of some of these are given in table 3.

Most of the results described in the present paper are for brass plates, and a more detailed consideration of the dependence of both the actual and relative frequencies on Poisson's ratio is deferred to a later occasion.

It is well known that the variety of nodal figures which may be formed on square plates is very great, and one of the main purposes of the present work was to exhibit the results in a readily accessible form. To this end, the nodal figures of the normal vibrating modes have been arranged on one diagram, plate 2, so as to exhibit the natural groups into which they may be divided. The further identification of any nodal figure then becomes a very simple matter and can be effected by means of the rules given below. The natural relative vibration frequencies are

   

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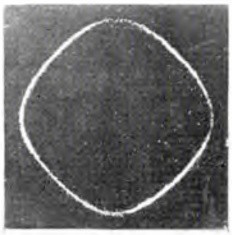
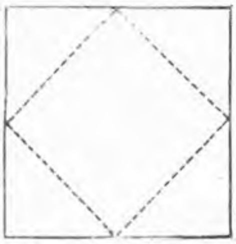
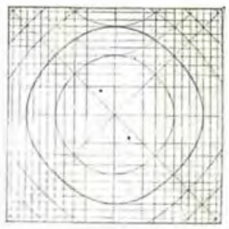
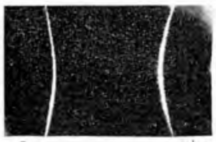
   

5   6 2|0+   7 2|0+   8 2|0+



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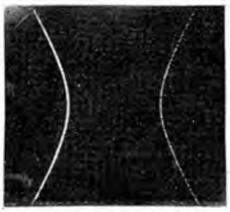
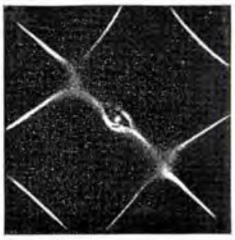
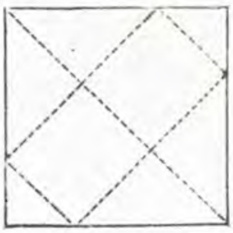
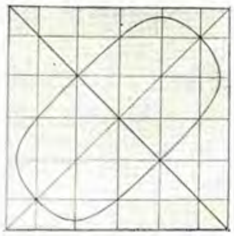
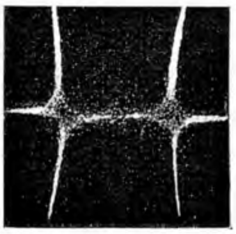
   

10 7|0   11 2|0   12 2|0   13 2|0

14 3|0   15 3|0   16 3|0

17 2|1   18 1|2   19 2|1−   20

See § 3. Figures 7 and 15 are hypothetical free membranes and figures 8 and 16 are vibrating free-bar constructions from Rayleigh. Figures 4 and 20 are superposed photographs.

shown in table 2, which corresponds in its arrangement with that of the diagram. Before describing the results and comparing them with those of other observers and with theory, we shall, however, discuss several preliminary matters and useful formulae. These are the subject of the several paragraphs of the next section.

§ 3. PRELIMINARY REMARKS

(3·1) *Considerations of symmetry and comparison with vibrating membrane* (see plate 1). In the first place it is interesting to follow Rayleigh[14] in tracing the continuity of the nodal figures as the form of the plate is gradually altered from a circle to a square. The indeterminate two nodal diameters of the circle, figure 1, plate 1, may give rise to one of the two nodal systems, figures 2 and 3, while the circle of figure 5 becomes the closed nodal system of figure 6. Systems similar to 2 and 3 may evidently be produced on any vibrating square surface. A closed system, comparable to 6, is also possible, although its exact shape and size will be determined by the nature of the restoring force (tension or elasticity) and by the prescribed boundary conditions (free, fixed or supported). Normal nodal figures, which are approximately those of a free square plate, may therefore be inferred from those of a hypothetical free membrane, for example from figure 7 or figure 15. Rayleigh has discussed the matter in some detail in his well-known chapters on membranes and plates[14]. The appropriate equation for constructing the approximate nodal systems of a plate of side l is

$$\omega = \cos\frac{m\pi x}{l}\cos\frac{n\pi y}{l} \pm \cos\frac{n\pi x}{l}\cos\frac{m\pi y}{l} = 0 \qquad \text{......(1)},$$

where ω is the displacement of the plate at the point x, y, but the more general free-membrane relation

$$\omega = A\cos\frac{m\pi x}{l}\cos\frac{n\pi y}{l} \pm B\cos\frac{n\pi x}{l}\cos\frac{m\pi y}{l} = 0 \qquad \text{......(2)},$$

can be used when one of the numbers m, n is odd while the other is even (class 7 of table 1). In this case, the ratio of amplitudes $A \mid B$ of the superposed *single* vibrations may have any value. The physical meaning of the numbers m and n is made evident by the consideration that, when B is zero, the nodal system consists of m lines lying parallel to the y axis and n lines parallel to the x axis, the origin being taken at the lower left corner of the plate; see for example the $2 \mid 1$ system, figure 17. In other cases, as was so often emphasized by Chladni, there are, for fixed values of m and n, only two nodal figures, $\overline{m \mid n}$ and $\underline{m \mid n}$, the vibration frequencies of which are unequal. They correspond with the plus and minus signs in equation (1) and are referred to below as the $m \mid n+$ and $m \mid n-$ modes respectively. Tanaka[25], who gives equation (2), should not have applied it to the $7 \mid 3$ mode of a plate, and Colwell[26], who has constructed a number of nodal systems with the same formula, has also used it for the $5 \mid 3$ and $7 \mid 3$ modes where, m and n being both odd, it is not applicable.

The fact that equation (2) should not be used when the values of m and n are both even can be realized for the 2|0 mode, by considering the rectangular-plate figures, 9, 11–13. A narrow rectangle may vibrate like a bar with two straight nodal lines situated 0·224 of the length from either end. The manner in which these lines are modified when the rectangle is wider is shown in figures 9 and 11. Finally, when the rectangle becomes a square, the lines meet and form the diagonals of figure 3. Similarly, the figures of the wide rectangles 13 and 12 become, on a square plate, the closed nodal system of 6. This same matter has also received attention in Pavlík's[27] recent paper dealing with rectangular plates.

The reason why the above facts are sometimes overlooked is probably that it is so easy to obtain distorted figures unless the experimental conditions are perfect. Distorted examples of the 7|3 mode just mentioned are shown in plate 4, figure 3 and in plate 6, figure 8. These may be compared with plate 4, figure 2. There are at least four possible causes for such results, namely (1) a want of uniformity in the length of the sides or the thickness of the plate, (2) a directional difference in its elastic properties which is often caused by mechanical rolling, (3) a want of freedom in the method of support, (4) an element of forcing in the method of excitation; see also § 4·2.

(3·2) *Graphical construction of nodal systems by the superposition of vibrations of free bars.* It was early recognized that, just as the vibrations of square membranes could be derived from those of parallel wires vibrating at right angles, equation (2), so those of a square plate might, assuming Poisson's ratio to be zero, be constructed by the superposition of vibrating bars. Two of Rayleigh's[14] constructions are reproduced in figures 8 and 16 of plate 1, the method being that of Maxwell. It will be seen that they are much nearer to the actual photographs, figures 6 and 14, than are the membrane diagrams. They are in fact simple cases of the approximate constructions obtainable when two terms only of Ritz's series equation are retained, thus:

$$\omega = u_m(x)\, u_n(y) \pm u_n(x)\, u_m(y) = 0 \qquad \ldots\ldots(3),$$

[16]

where the functions u are those proper to a free bar of length equal to the side of the plate. As before, when one of the values m, n, is odd while the other is even, additional figures are given by

$$\omega = Au_m(x)\, u_n(y) \pm Bu_n(x)\, u_m(y) = 0 \qquad \ldots\ldots(4).$$

[16]

For greater accuracy it is necessary to retain a larger number of terms of Ritz's equation, as was done by Lemke[28] for the 2|0+ and 3|3 modes; her results are too lengthy to be quoted here.

An idea of how soon we may expect the simpler membrane construction from equation (1) to give reasonably accurate results for the nodal systems of upper partials, except near the edges, may be gained by looking at figure 10, which shows how nearly, in the 7|0 bar system, the nodal spacing approaches to the equal spacing of a vibrating wire. Attention is also drawn to the superposed prints of figures 20 and 4. The points of intersection of the nodal lines of the superposed vibrations, Strehlke's "poles", are always nodal[8, 10]. This applies even in cases like 2|0, + or −, where, although the *single* vibrations are not possible, the poles are almost coincident with the nodes of free bars. Thus in figure 4 the poles are situated 0·228 of the length of side away from the edge, a distance which is only 2 per cent greater than for a bar, 0·224. These two numbers would presumably coincide if the value of Poisson's ratio were zero. The measurements are found to be in excellent agreement with those given by Strehlke[9] at the end of his paper on vibrating bars.

(3·3) *Calculation of the vibration frequencies.* The natural frequencies f are given by

$$\lambda = \frac{192\pi^2 f^2 (1-\sigma^2)\,\rho}{Et^2}\ ^{(16)} \qquad \text{......(5)}$$

for a plate of thickness t and length unity. E is Young's modulus, σ Poisson's ratio, and ρ the density. The values of the constant λ have been calculated by Ritz to the 6|6 mode for $\sigma = 0{\cdot}225$ and by Lemke[28] for some other values of σ. It follows that for a plate of side l, the fundamental frequency

$$f = 0{\cdot}6577\ (\text{or } 0{\cdot}6277)\ c\,\frac{t}{l^2};\quad \sigma = \tfrac{1}{4}\ (\text{or } \tfrac{1}{3}) \qquad \text{......(6)}$$

where c is the velocity of sound appropriate to the material. Thus the given increase in Poisson's ratio produces a decrease of frequency of 4·8 per cent. This result is convenient for calculations in connexion with materials such as glass or steel ($\sigma \doteq \frac{1}{4}$) as compared with brass ($\sigma \doteq \frac{1}{3}$). It is also to be noted that since the variation of λ with σ is approximately linear[28], frequencies corresponding to other values of Poisson's ratio are easy to estimate with reasonable accuracy. When, for example, σ is $\frac{1}{5}$, the frequency should be equal to about $\frac{2}{3}.c.t/l^2$.

Relative frequencies. Ritz states that the natural frequencies are roughly proportional to

$$\sqrt{\{m^4 + n^4 + 2(1-\sigma)\,m^2n^2\}} \qquad \text{......(7).}$$

This relation is evidently meant for higher partials where the distinction between plus and minus systems can be neglected. It is seen also that according to equation (7), when m and n are equal, the relative frequencies are unaffected by the value of Poisson's ratio. In view of these facts it is simpler to use Chladni's expression

$$m^2 + n^2 \qquad \text{......(8)}$$

for obtaining the frequencies of upper partials in relation to one another.

According to the present observations (table 4) their values relatively to the gravest 1|1 tone taken as unity are given approximately by

$$\tfrac{3}{4}(m^2 + n^2) \qquad \text{......(9)}$$

for values of m and n exceeding 3.

We may also note that when either m or n is zero, the sequence of tones according to Chladni is nearly that of a free bar, $(3{\cdot}011)^2, 5^2, 7^2, \ldots$. This relation seems to fit best when the mean value of the 2|0+ and 2|0− frequencies is used for the first member of the series.

§4. EXPERIMENTAL RESULTS: I. THE NODAL SYSTEMS

(4·1) *The seven classes of nodal symmetry.* In plate 2 the normal nodal systems are arranged on a square diagram which exhibits the seven classes into which they may be divided according to the nodal design at or near the centre. The systems for which the values of m and n are equal occupy the squares which extend diagonally from the top left corner to the bottom right corner of the page, the $6|6$ space remaining unfilled. The $m|n-$ systems occupy the triangular space to the right of this diagonal, and the $m|n+$ systems the triangular space to the left of it. A general study of alternate figures and of the development which occurs in passing along the rows, columns or diagonals, results in the following classification.

Table 1. The seven classes of nodal symmetry of free square plates, see plate 2

e denotes that the values of m and n are even, and o that they are odd

Specification of vibrating mode	Condition at centre	Further details of nodal lines at or near centre	Rotational symmetry	Class abbreviation symbol	Further examples see plates
$m=n$					
1 $o\|o$	Node	Parallel to edges	90°	+	
2 $e\|e$	Antinode	Parallel to edges	90°	□	
$m \neq n$					
3 $o\|o-$	Node	Diagonals and medians	90°	✳	4
4 $e\|e-$	Node	Diagonals	90°	×	5
5 $o\|o+$	Node	Medians	90°	+	6
6 $e\|e+$	Antinode	Closed figure	90°	○	6
7 $o\|e+$ or $-$	Node	One diagonal	180°	╱ or ╲	7, 8

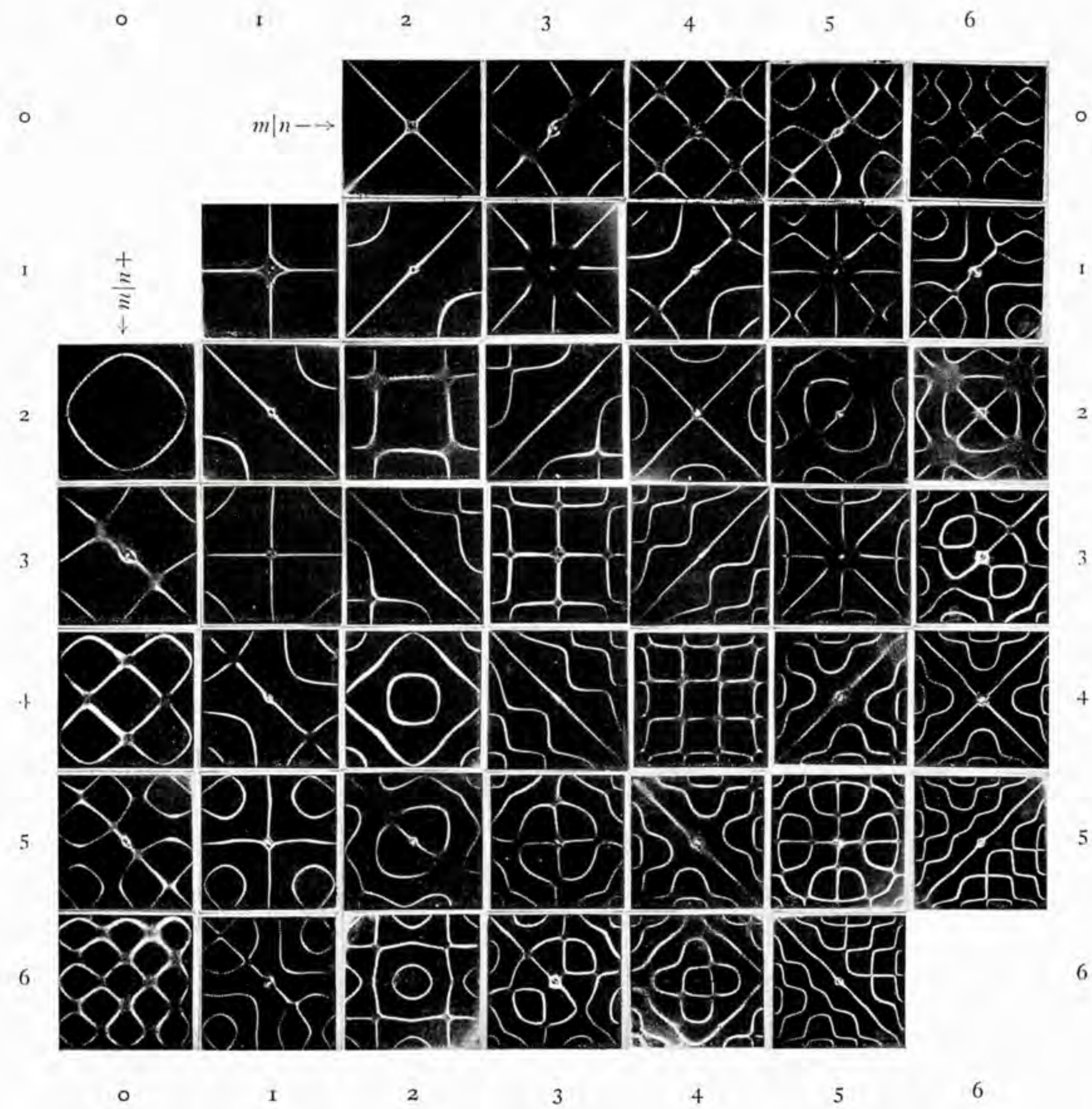
Equations (2) and (4) can be used for class 7 only.

The division into seven classes, arrived at by inspection of the nodal system, is of course consistent with Ritz's analysis. Ritz distinguishes only five classes, the first and second of table 1 being included in the "plus" fifth and sixth classes respectively. Since however classes 1 and 2 have distinctive nodal systems and occupy a peculiar position in plate 2 it is convenient to think of them separately. It will be noted that there are no corresponding special cases for the "minus" classes 3 or 4; analytically the plate remains at rest, when m and n are equal, and in plate 2 all the spaces are already accounted for.

(4·2) *Recognition of the nodal systems. Higher overtones.* The recognition of a normal nodal system can be effected with certainty and ease if it is first of all placed in one of the classes of table 1. We then already know whether the values of m and n are odd or even, and it only remains to determine their actual values.

When m and n are equal, classes 1 and 2, recognition is immediate since the nodal lines, numbering m, run approximately parallel to the sides.

In the remaining classes one of the numbers m, n (say m) is counted either across the plate or more generally along one edge. The manner of finding n then varies according to the system, but, except in class 3, which will be considered last, n also can be determined in a few moments. Thus, in classes 5, 6 and 7, the



Normal nodal systems of free square plate, see equations (1) and (3). The nodal designs are distinct for given values of m and n except when one of the values of m, n is even when the other one is odd. See table 1.

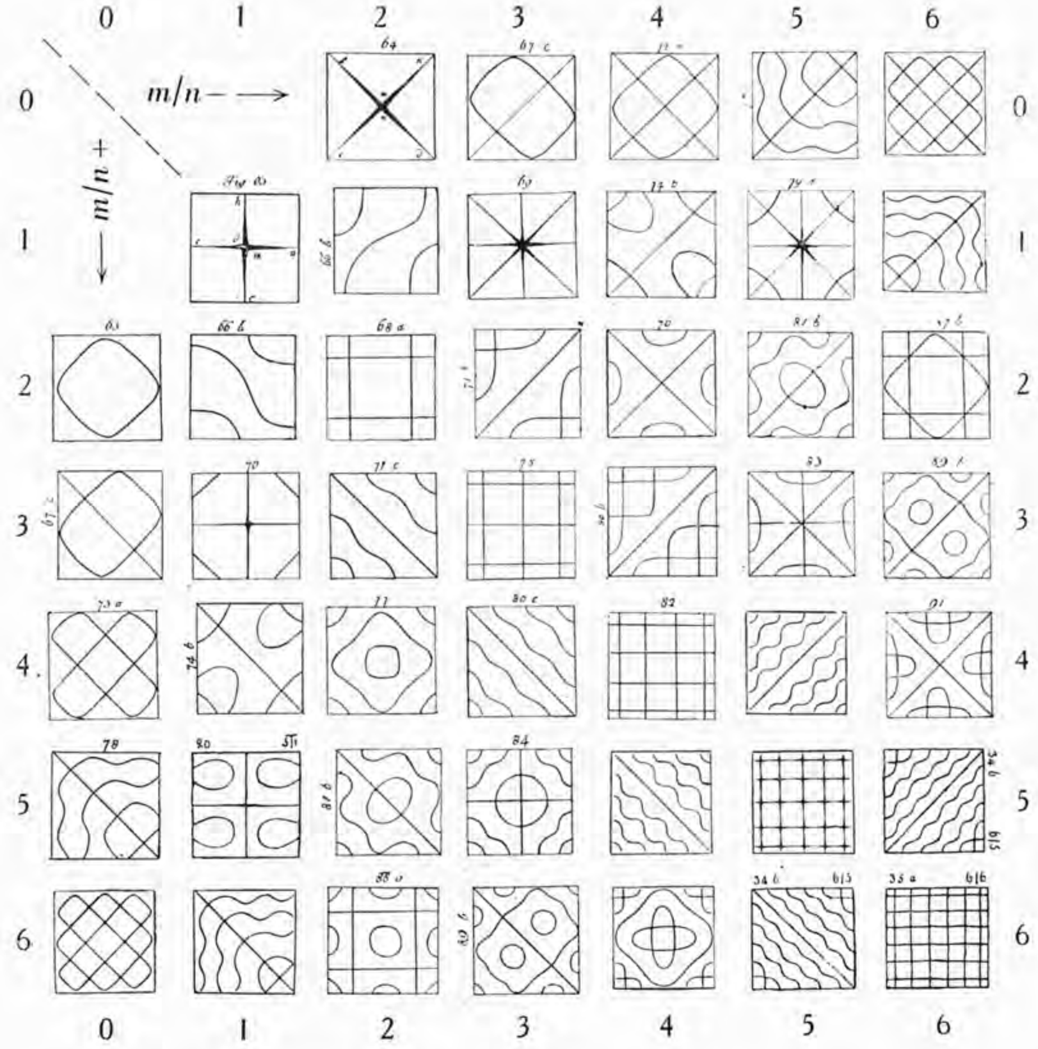


Plate 3. Selection of Chladni's drawings arranged as in plate 2, see § 5.

number of nodal lines which cut a diameter which is not nodal always equals $m+n$, a relation which can be verified on plate 2 and also on plates 6–8.

The nodal systems of classes 4 and 6 are divisible (if the edge effects are neglected) into four smaller squares for which m and n have half their former values; see plates 2, 5 and 6, and the examples given by Chladni[(2)]. His observation may be generalized, for it is evident that when m and n have a common factor s the figure can be divided into s^2 smaller squares for which $m'=m/s$ and $n'=n/s$. It may be noted that when higher nodal designs are composed from lower ones, plus systems result in a larger plus system and minus systems in a larger minus system, but $o|e$ systems can be joined together to produce either plus or minus systems. Consider, for example, how either a $2|4+$ or a $2|4-$ nodal system is derived from a $2|1$ system. In this connexion attention may be drawn to photographs 1 to 4 of plate 7, which have been cut in half and fitted together edge to edge. These photographs will also give an idea of the importance of the edge effects.

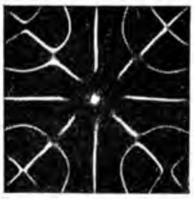
All the photographs shown in plates 4 to 8 are labelled, and accordingly it is only necessary to add a few remarks concerning them.

Plate 6 (1). The antinodal centres show that the systems belong to class 6, $e|e+$. They are identified immediately by the m and the $m+n$ count. They are also divisible into four smaller squares, and in figures 1 and 4, where s is four, into sixteen squares. Figure 6 is included in order to show that in practice designs are sometimes produced which are at first sight puzzling. It shows a high overtone which has been distorted by the central screw.

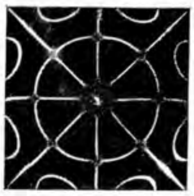
Plate 6 (2). The figures are placed in the $o|o+$ class 5 by looking at the centre, and are then identified as above. In figure 10, since s is three, the $9|3+$ system is divisible into nine $3|1$ systems.

Plate 5. The systems belong to class 4, $e|e-$, for which both the diagonals are nodal, but by division into four smaller squares the systems can be identified as above unless they belong to class 3. In figure 2, $8|2$ is not theoretically perfect since the amplitude A is greater than B; see equations (1) to (4). A better example is given in figure 4. Such distortions are not uncommon, for in addition to the four possible causes for their production already mentioned in § 3·1, there is, for such a high overtone, scarcely any difference between the frequencies of the plus and minus systems.

Plate 4. It now remains to consider the class 3, $o|o-$ systems shown in this plate, which cannot all be recognized by means of one comprehensive rule and need, therefore, to be looked at in more detail than the figures of the other classes. The value of m is found as before. All the systems for which $m-n=2$ present the typical appearance seen already in plate 2, and now in figures 6, 7 and 10, and $m+n$ can be determined by counting from one corner to the opposite corner along two adjacent sides. Another way of stating this rule is that m is given by counting along one side including the diameters, and n is obtained by counting along the next side excluding the diameters. In applying this relation to the systems for which $m-n=4$, see plate 2, $6|2-$, and figures 2 and 5, the *loops* count as two along the first side and as one along the second side. This result may be compared

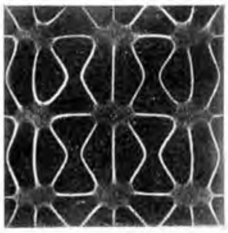


1 $7|1-$

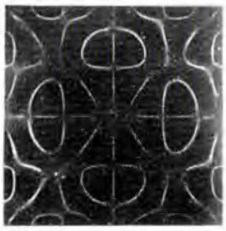


2 $7|3-$

3 $7|3-$

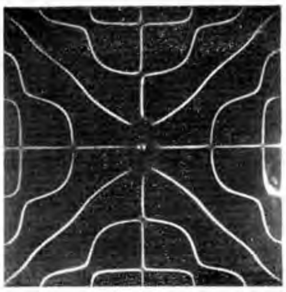


4 $9|3-$

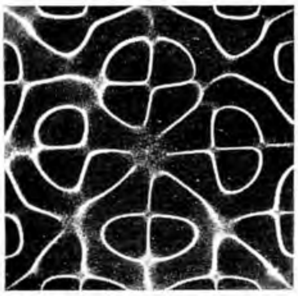


5 $9|5-$

6 $9|7-$



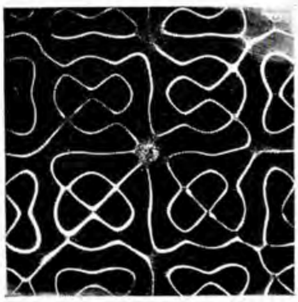
7 $7|5-$



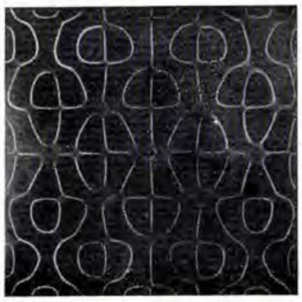
8 $9|5-$

9 $11|3-$

10 $11|9-$



11 $13|3-$



12 $13|7-$

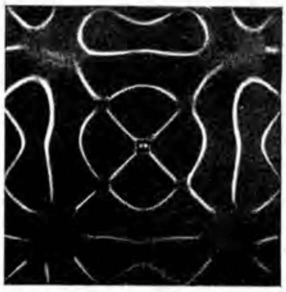
Nodal systems of $o|o-$, class 3, see table 1 and § 4·2.

To face page 838

1 8|0−

2 8|2−

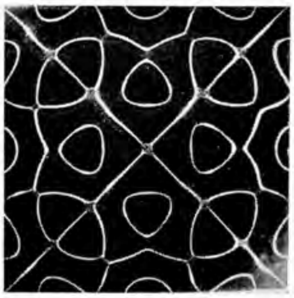
3 8|6−



4 8|2−

5 8|4−

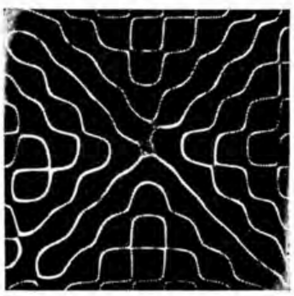
6 10|2−



7 10|4−

8 10|6−

9 10|6−



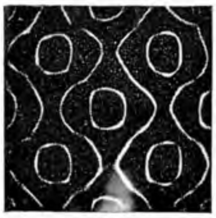
10 10|8−

11 14|4−

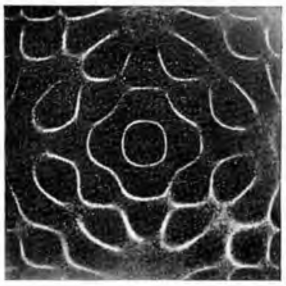
12 14|4

Nodal systems of $e\,|\,e-$, class 4, see table 1 and § 4·2.

(1)

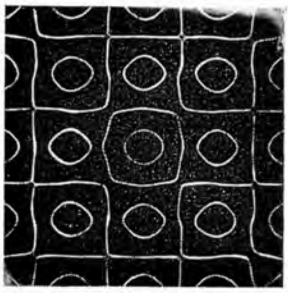


1 8|4+



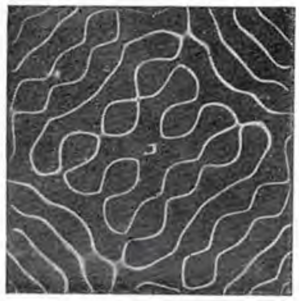
2 8|6+

3 8|6+



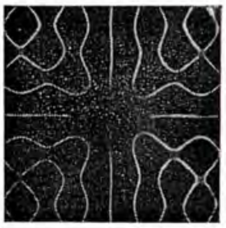
4 12|4+

5 12|10+



6 10|8+

(2)



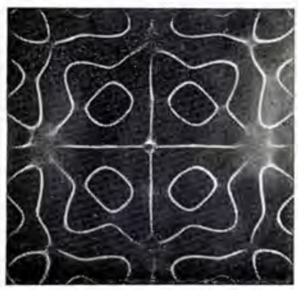
7 9|1+

8 7|3+

9 7|5+

10 9|3+

11 9|5+



12 11|3+

Nodal systems of (1) $e|e+$, class 6, (2) $o|o+$, class 5, see table 1 and § 4·2.

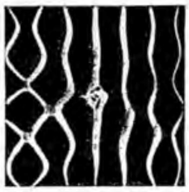
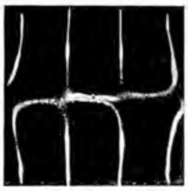
(1)




1 2 3 4

(2)

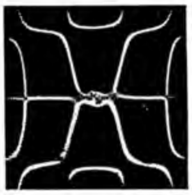


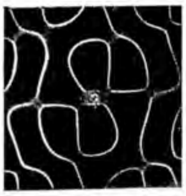



5 3|0 6 5|0 7 7|0 8 4|1



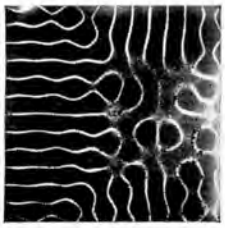

9 4|1 10 8|1 11 3|2 12 5|2





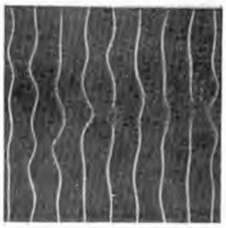

13 4|3 14 6|3 15 6|3 16 6|3



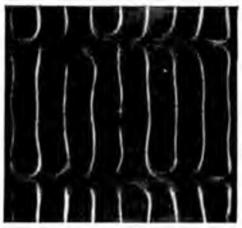



17 6|1 18 6|0— 19 7|4 20 13|0

(1) Showing edge effect, see § 4·2. (2) Nodal systems of $o\,|\,e$, class 7 (except figure 18), see table 1 and § 4·3



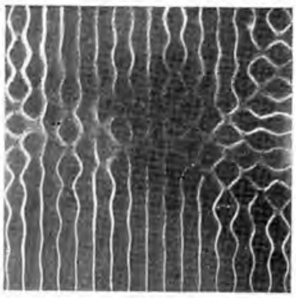
1 9|0



2 9|2

3 10|3

4 9|6



5 17|0

6 10|5

7 9|4

8 9|4

9 13|0

10 9|8

11 10|9

12 12|3

Nodal systems of $o\,|\,e$, class 7, see table 1 and § 4·3 with somewhat similar conclusions in connexion with the compounded modes of circular plates[(23b)]. A second distorted version of 9|5− is given in figure 8 in order that the three-line prongs which are frequently met with in practice may be made clear. It is interesting to find a nodal circle in the 7|3− figure; it is the only example so far obtained. Another point occurs in connexion with the systems for which $m-n=6$, see 7|1−, figure 1, and the slightly distorted 13|7−, figure 12. After m has been counted as usual along one of the sides it is seen that the lines at right angles to diameters are not included again in the n count. The 9|3− mode incidentally, where $s=3$, is divisible into nine 3|1 systems. The lattice appearance of the figure may be noticed, since it is sometimes stated that nodal lines approximately parallel to the sides only occur when m and n are equal. Further examples may be found in other plates. The lattice is sometimes of considerable assistance in identifying the class-3 systems, as for example in figures 9 and 11.

(4·3) *Further observations on the o|e nodal systems, plates 7 and 8.* As has been mentioned in § 3, when one of the numbers m, n is odd while the other is even, equations (2) or (4) are applicable. There is no distinction (except a formal one of rotation through a right angle, see plate 2) between the plus and the minus nodal systems, and there is no limit to the variety of nodal figures which may be obtained for fixed values of m and n. *Single* vibrations, corresponding to zero values of the second term of equations (2) or (4), are shown in some of the photographs of plates 7 and 8. The 3|0, figure 5, is similar to one of Chladni's, and one and a half *Biegungen* or bendings are visible in the nodal lines. Had Chladni multiplied all his data regarding these bendings by two it is possible that he would have recognized their significance. In the present case the three half-bends indicate that a trace of the 0|3 system is present. This matter has been fully discussed in the paper on free circular plates[(23b)]. One $e|e$ distorted nodal system is included in plate 7, figure 18. It will be noted that were the bendings slightly more pronounced we should obtain the normal 6|0− mode of plate 2. The closed nodal lines of 7|4, figure 19, should join, as can be seen from the diagonal count; cf. Chladni's version[(3)]. Attention may be drawn to the characteristic single nodal line through the centre and the 180° rotational symmetry which persists in all the figures of plate 8. Broken patterns, such as those shown in plate 7, figure 20 and plate 8, figure 5, are quite commonly produced on large plates which are difficult to obtain uniform.

§ 5. COMPARISON OF RESULTS: I. THE NODAL SYSTEMS

A selection of Chladni's drawings, arranged as in plate 2, are shown in plate 3. Most of them are from *Die Akustik*, those with no numbering are taken from Tyndall's *Sound*, while the 5|1+, 6|5 and 6|6 gaps have lately been filled from the collection given in *Neue Beiträge zur Akustik*.

The nodal systems of plates 2 and 3 may be conveniently compared along the diagonal series of figures for which the values of m and n are equal and thereafter on either side of this. The remarks which follow also include a comparison where necessary with the diagrams given in Ritz's paper.

$m=n$. Except for 1 | 1, the actual figures vary considerably from the idealized lattice drawings given by Chladni. Strehlke's version (given by Ritz) of the 2 | 2 system is essentially more correct, though it tends to over-emphasize the curves of the four nodal lines. Lemke's[28] elaborate construction of the 3 | 3 system is practically identical with the present photograph, and shows that when sufficient terms of the Ritz equation are retained, the theory is in good agreement with practice. The 5 | 5 photograph is of special interest and will, it is hoped, be referred to again in a later communication on the subject of compounded modes: a trace of a second 7 | 1 tone, of near period, can be detected in it. Chladni[3] gives a suggestive distortion of the 6 | 6 lattice shown in plate 3, which consists of 12 parallel wavy lines similar to the 11 wavy lines of the correct 6 | 5 figure.

$m-n=1$. In Chladni's version of the 2 | 1 mode A and B are unequal; the more typical figure is shown in the photograph. The peaks of the wavy lines in the photographs are more pointed than those given by Chladni or by Ritz.

$m-n=2$. This familiar series of nodal systems is very easily produced. Examples as far as 11 | 9 − may be found on plates 2, 4 or 5, and as far as 12 | 10 + (except 9 | 7 +) on plates 2 and 6.

$m-n=3$. The 3 | 0 system shown in the photograph has been produced many times. I have never obtained the figure as it is given by Chladni and Strehlke. Other differences in detail may be noted in this series.

$m-n=4$. The 4 | 0 photograph differs considerably from the versions given by Chladni and Ritz. The two nodal diameters which are missing in Chladni's 6 | 2 − drawing are included in his later book[3]. None of the drawings really convey the character of this familiar figure. The 5 | 1 mode tends to combine with the 5 | 0 system as shown in the photograph and in Chladni's later drawings[3].

§ 6. EXPERIMENTAL RESULTS: II. THE NATURAL FREQUENCIES

The natural frequencies, relative to that of the 1 | 1 tone taken as unity, are given in table 2, which corresponds in its arrangement with the nodal systems of plate 2. The table extends over an interval of about 140, that is to say, over more than seven octaves. The higher figures are included so as to be available in a subsequent study of compounded normal modes. The difference between the frequencies of the "plus" and "minus" modes, which are printed in black type, are to be noted, and also the manner in which this difference rapidly decreases with the higher tones. The frequencies of the plus modes should not be greater than those of the minus modes were Poisson's ratio equal to zero.

The actual 1 | 1 frequencies for some of the plates employed, together with other particulars, are given in table 3. The observed frequencies of the fourth column may be compared with those, calculated by means of equation (6), which are given in the last column. Considering that, except for brass, the values for c and σ are assumed, the agreement between the two columns is very satisfactory.

For brass, c was measured from figures 1 or 5, plate 8, by means of the relation used by A. B. Wood and F. D. Smith [29],

$$c = \frac{2\sqrt{3}}{\pi}\frac{\lambda}{t}c_t \qquad \text{......(10)}$$

in which c_t is the velocity, and λ the wave-length of the transverse wave. The result, $35{\cdot}6 \times 10^4$ cm./sec. for the particular brass used, is in near agreement with the value given by Kaye and Laby, $36{\cdot}5 \times 10^4$.

Table 2. Relative frequencies of normal vibrating modes of free square brass plate

$m \mid n-$ / $m \mid n+$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
0	—	—	**1·52**	5·10	**9·14**	15·8	**23·0**	32·5	43	55·2	70	*84*	*101*	*119*	141	0
1	—	1*	2·71	**5·30**	10·3	**15·8**	23·9	**32·2**	43	55·8	71	86·1	102	*121*	(*143*)	1
2	**1·94**	2·71	4·81	8·52	**12·4**	19·0	**26·4**	34	46·6	59	73	89	105	*124*	(*146*)	2
3	5·10	**6·00**	8·52	11·8	16·6	**22·6**	30·0	**39·5**	50·5	63·4	77·5	92·4	110	128	(*151*)	3
4	**9·9**	10·3	**13·2**	16·6	21·5	28·7	**35·5**	45·4	55·9	69·7	82·9	99	116	*132*	(*155*)	4
5	15·8	**16·6**	19·0	**23·3**	28·7	35	43	**52·1**	64·5	75·9	90	106	*122*	136	(*161*)	5
6	**23·8**	23·9	**27·1**	30·0	**35·9**	43	*51*	61·7	*73*	84	99	115	*130*	(*147*)	—	6
7	32·5	**32·4**	34	**39·8**	45·4	**53**	61·7	*70·3*	84	93	*108*	*124*	(*140*)	(*161*)	—	7
8	43·0	43·0	46·6	50·5	57·2	64·5	*73*	84	*94·4*	106	120	*136*	(*153*)	(*175*)	—	8
9	*55·2*	55·8	59	63·4	69·7	76·2	84	93·2	106	*120*	*133*	(*150*)	(*168*)	—	—	9
10	70	71	73	77·5	82·9	90	99	*108*	120	*133*	(*149*)	(*165*)	—	—	—	10
11	*84*	86·1	89	92·4	99	106	115	*124*	136	(*150*)	(*165*)	(*178*)	—	—	—	11
12	101	102	105	110	116	*122*	*130*	(*140*)	(*153*)	(*168*)	(*185*)	—	—	—	—	12
13	*119*	*121*	*124*	128	132	136	*147*	(*161*)	(*175*)	—	—	—	—	—	—	13
14	141	(*143*)	(*146*)	(*151*)	(*155*)	(*161*)	—	—	—	—	—	—	—	—	—	14
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	

* See table 3 for actual fundamental frequencies.

The numbers in italics are approximate only, and those in brackets have been determined by extrapolation. Any circular arc whose centre is at the top left corner will pass through nearly equal frequencies.

§ 7. COMPARISON OF RESULTS: II. THE NATURAL FREQUENCIES

It has already been shown by the numbers given in the fourth and last columns of table 3 that the experimental values of the fundamental frequencies agree well with those obtained by calculation. With regard to overtones, in order to avoid lengthy tables it has been considered sufficient to restrict the comparison of results to the series of partials for which m and n are equal, as shown in table 4.

Table 3. Fundamental 1 | 1 frequencies

Material of plate	Dimensions of plate (cm.)		Observed frequency (c./sec.)	Velocity c of sound (cm./sec. $\times 10^4$)	Poisson's ratio, σ	Calculated frequency*
	Length of side	Thickness				
Brass	6·00	0·201	**1222**	35·6	$\frac{1}{3}$	**1248**
	10·16	0·2035	**432**	,,	,,	**442**
	14·89	0·2135	**206·5**	,,	,,	**208**
	20·28	0·488	**260**	,,	,,	**264**
	19·89	0·1848	**104**	,,	,,	**104**
	20·44	0·164	**87·7**	,,	,,	**87**
	30·6	0·197	**†46**	,,	,,	**44**
	40·71	0·204	**†27**	,,	,,	**25**
Glass	20·4	0·3081	**256**	50	$\frac{1}{4}$	**243**
Steel	20·38	0·1935	**157**	50	$\frac{1}{4}$	**153·2**
Aluminium	20·49	0·2016	**158**	51	$\frac{1}{3}$	**153·8**

* By equation (6). † Inferred from second or third tone.
The values for c and σ are from Kaye and Laby's tables except that, for brass, c was determined from figures 1 or 5, plate 8[29].

Table 4. Comparison of results. Relative frequencies

$m \| n$	1 | 1	2 | 2	3 | 3	4 | 4	5 | 5	6 | 6	7 | 7	8 | 8	9 | 9	10 | 10
Chladni, observed, glass*	1	4·58	10·8	18·5	30	42·7	60	80	102	—
Ritz, calculated, $\sigma = 0{\cdot}225$†	1	4·63	11·2	21·0	33·2	48·1	—	—	—	—
Pavlík, observed, stainless steel	1	4·34	9·33	—	—	—	—	—	—	—
Brass plates										
Lemke, calculated‡	1	4·87	11·8	—	—	—	—	—	—	—
observed	1	4·81	11·8	—	—	—	—	—	—	—
Waller, observed	**1**	**4·81**	**11·8**	**21·5**	**35**	**51**	**70·3**	**94·4**	**120**	**149**
$\frac{3}{4}(m^2+n^2)$§	—	—	—	24	37	54	73	96	121	151

* The numbers given in *Neue Beiträge zur Akustik* have been divided by six.
† From Ritz's values of λ, see equation (5). ‡ From Lemke's actual frequencies.
§ A convenient approximate formula for higher overtones.

Since frequencies of the lower tones are somewhat dependent on the value of Poisson's ratio, it is not to be expected that the early numbers given in the first three rows will be the same as those given later for plates made of brass. Pavlík's[27] numbers appear to be too low. This may be due to his experiments having been made on small plates in which the length of side, about 1 cm., was only ten times greater than the thickness; such plates can scarcely be regarded as thin[24].

The relative frequencies of the higher tones as given by Chladni are seen to be considerably less than those which are required by the theory or obtained in the present observations. It must have been difficult to determine the higher frequencies accurately with the means at Chladni's disposal.

The approximate agreement in the numbers of the last two rows is to be noted. This indicates that a rough estimation of the frequencies of the higher tones can be made using the formula (9) suggested earlier in this paper.

§8. CONCLUSIONS

1. Chladni's maturest work on the subject of vibrating square plates are given only in his last book *Neue Beiträge zur Akustik*. His complete set of normal nodal drawings and observed natural frequencies are reproduced by Wheatstone[(8)] (§ 1).

2. The Ritz approximate method of solution of the problem of the vibrating free square plate is an extension of the Rayleigh method (§ 1).

3. The normal nodal systems may be arranged in a systematic manner on one diagram, plate 2, which shows that there are seven classes of nodal symmetry. These may be recalled by means of descriptive abbreviation symbols, see table 1.

4. Any nodal figure after being placed in its class, may be identified by means of simple rules (§ 4·2). For example, except when both diameters are nodal, the value $m+n$ is obtained by counting the number of lines across a diameter, and the value of m by counting along an edge or across the plate.

When m and n have a common factor s, the nodal system is divisible into s^2 smaller systems for which $m'=m/s$ and $n'=n/s$.

5. There are a variety of causes, see § 3·1 and § 4·3, for the distorted nodal figures which are in practice so commonly produced.

6. When the figures are being constructed graphically, it is important to recall that it is only when one of the values of m, n is odd while the other is even that an indefinitely large number of nodal figures may be constructed with fixed values of m and n (§ 3·1).

7. Since it is impossible for more than four nodal lines to pass through the centre of the square plate, the vibration amplitude for any mode is usually sufficient to move the sand over most of the surface. This result may be contrasted with the circular plate for which, when the number of nodal diameters are numerous, the sand remains quiescent except near the edge[(23)]. The subject is of interest in connexion with the radiation of sound.

There is great variation both in the persistence of vibration and in the loudness of the various overtones.

8. The natural frequencies found by measurement are in fair agreement with Ritz's calculations, see § 3·3 and tables 3 and 4. The frequency of the gravest tone, when Poisson's ratio is $\frac{1}{6}$, is given approximately by $\frac{2}{3}\,ct/l^2$ for a plate of thickness t and length of side l where c is the velocity of sound appropriate to the material. The frequency decreases slowly and approximately linearly with increasing value of Poisson's ratio, see § 3·3.

9. It is found experimentally that the frequencies of the higher partials relatively to the gravest tone taken as unity are given roughly by $\frac{3}{4}(m^2+n^2)$, see table 4. When the frequencies are arranged as in table 2, any circular arc with centre at the top left corner passes close to numbers which are very equal.

§ 9. ACKNOWLEDGEMENTS

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VIBRATIONS OF FREE SQUARE PLATES: PART II, COMPOUNDED NORMAL MODES

BY

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VIBRATIONS OF FREE SQUARE PLATES: PART II, COMPOUNDED NORMAL MODES

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ABSTRACT. The nodal designs obtainable on free square plates which cannot be included among the seven normal classes of figures (see *Proc. Phys. Soc.* **51**, 831 (1939)) usually consist of two compounded modes of vibration of near period for which the sum m^2+n^2 is approximately equal. Such combination of modes is possible because the internal damping of the material of the plate reduces the sharpness of resonance and because the material and dimensions of the plate are seldom ideally uniform. The observations are in close agreement with the theory proper to a thin elastic plate, a conclusion which may be contrasted with the case of the circular plate (see *Proc. Phys. Soc.* **50**, 70 (1938)) where, for reasons given, the observations made on higher tones are not in agreement with this theory.

§ 1. INTRODUCTION

COMPLICATED nodal designs sometimes occur on vibrating free square plates which cannot be recognized as belonging to any of the normal classes. It is shown below that such designs generally consist of two compounded modes of vibration, the periods of which are nearly equal.

Before establishing this conclusion it was necessary to be familiar with the normal nodal figures, and to have available a table of the corresponding natural frequencies. A full account of the systematic observations which have been made on these subjects has been published recently[1], in which the main facts regarding the classes of nodal symmetry were collected in a short summary which for convenience of reference is reproduced below, in table 1. The notation employed in the previous paper, $m\,|\,n+$ and $m\,|\,n-$, whereby any particular vibrating mode may be specified, is used again, the letters o and e indicating whether the numbers m and n are odd or even. The connexion between this notation and the equation giving the approximate nodal systems,

$$\omega = Au_m(x)u_n(y) \pm Bu_n(x)u_m(y) = 0 \quad \text{[2], [3], [4]} \qquad \ldots\ldots(1),$$

is obvious.

In this expression ω is the displacement of the plate at the point (x, y) and the functions u are those proper to a free bar of length equal to the side of the plate[2]. Except when one of the values m, n is odd and the other is even, the amplitude constants A and B must be equal to one another.

The rules whereby any nodal figure, after being placed in its appropriate class, may be further identified are also given in the former paper. It may be recalled that, except when both diameters are nodal, the value of $m+n$ may be obtained by counting the number of nodal lines across a diameter, while the value of m (or n) is usually found by counting along an edge of the plate.

According to observation and theory (except in the case of the few gravest tones where changes in Poisson's ratio have an appreciable effect on the relative frequencies) those vibrating modes for which the sum m^2+n^2 is identical have approximately, but not exactly, the same vibration frequencies. We may therefore suppose that nodal figures which do not correspond to any of the seven normal classes of table 1 are compounded from two or more such normal modes. This

Table 1. The seven classes of nodal symmetry of free square plates

e denotes that the values of m and n are even, and *o* that they are odd

Specification of vibrating mode	Condition at centre	Further details of nodal lines at or near centre	Rotational symmetry	Class abbreviation symbol
$m=n$				
1 $o\|o$	Node	Parallel to edges	90°	+
2 $e\|e$	Antinode	Parallel to edges	90°	□
$m\neq n$				
3 $o\|o-$	Node	Diagonals and medians	90°	✳
4 $e\|e-$	Node	Diagonals	90°	×
5 $o\|o+$	Node	Medians	90°	+
6 $e\|e+$	Antinode	Closed figure	90°	○
7 $e\|o+$ or $-$	Node	One diagonal	180°	╱ or ╲

compounding of modes of near period often occurs in the case of the free circular plate (5), and the reason it is possible is because the sharpness of resonance is reduced by the internal damping of the material of the plate. Any want of uniformity in either the material or the dimensions of the plate may also be favourable to the production of compounded modes.

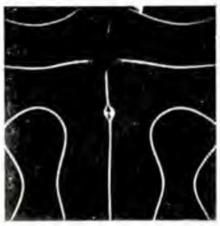
The only cases, as far as I am aware, of classified compounded nodal figures on free square plates are the two drawings of Chladni (6), which are reproduced in figures 4 and 5 of plate 1. Their recognition is due to Wheatstone (7).

§ 2. EXPERIMENTAL RESULTS

The number of nodal designs which may be obtained on square plates is limitless. Those which cannot be placed among the normal figures belong to compounded modes of vibration. In practice, also, it is not unusual to find figures which have been distorted by the central support and, in passing to higher overtones, complicated figures which are caused by irregularities in the texture and dimensions of the plate. A representative collection of such designs is given in


1


2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4


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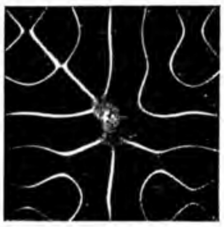
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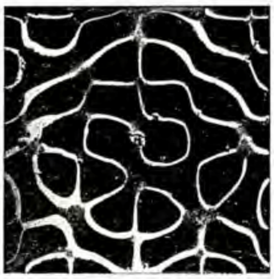
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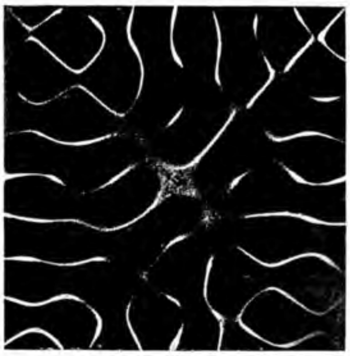
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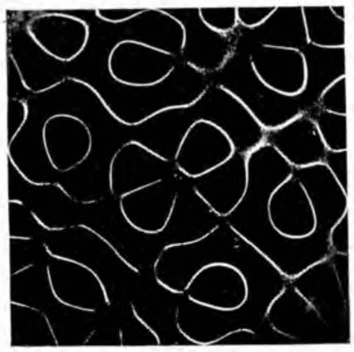
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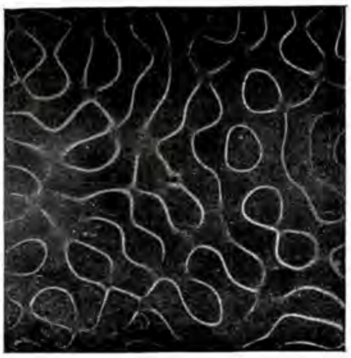
15

Plate 1.

16

17

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Plate 2.

figures 1 to 28. They may be examined in connexion with the data of table 2, where the classification $m_1 | n_1$, $m_2 | n_2$ is given in the second column and the corresponding frequencies (obtained by experiment[1] and expressed in relation to

Table 2
Some compounded normal modes of vibration of free square plates

Figures	Component modes $m_1 \| n_1$, $m_2 \| n_2$	Experimental frequencies relative to 1|1 tone=1	$m_1^2+n_1^2$	$m_2^2+n_2^2$
		Plate 1		
1	3|0, 3|1	5·1, 5·3	9	10
2	5|0, 5|1	15·8, 15·8	25	26
3	5|1, 2 5	16·6, 19	26	29
4	5|1, 2|5	16·6, 19	26	29
5	3|5, 6|1	23·3, 23·9	34	37
6	7|0, 7|1	32·5, 32·4	49	50
7	5|5, 7|1	35, 32·4	50	50
8	5|5, 7|1	35, 32·4	50	50
9	8|3, 7|5	50·5, 53	73	74
10	9|0, 9|1	55·2, 55·8	81	82
11	9|2, 7|6	59, 61·7	85	85
12	11|1, 11|0	86·1, 84	122	121
13	10|5, 11|2	90, 89	125	125
14	12|4, 11|6	116, 115	160	157
15	13|2, 12|5	124, 122	173	169
	Plate 2. Figures showing distortion due to central screw			
16	4|2	13·1	20	
17	6|2	27·1	40	
18	8|2	46·6	68	
19	4|4, 6|0	21·5, 23	32	36
20	10|2, 8|6	73, 73	104	100
21	10|4	82·9	116	
22	12|2, 10|6	105, 99	148	136
23	13|0, 12|3	119, 110	169	153
		Irregular figures		
24	11|6 etc.	115	157	
25	13|2, 11|7	124, 124	173	170
26	11|7, 13|2	124, 124	170	173
27	13|4, 11|8	132, 136	185	185
28	14|0, 14|1	141, 143	196	197

the fundamental 1 | 1 tone taken as unity) are given in the third column. From the last two columns of the table it will be seen that $m_1^2+n_1^2$ is approximately, and in some cases exactly, equal to the corresponding value of $m_2^2+n_2^2$.

Figures 1 to 15 consist of two compounded normal modes which have been recognized by inspection of the nodal lines assisted by a knowledge of the vibration frequencies. The $m | o$, $m | 1$ design is a common one, as may be seen from figures 1, 2, 6, 10 and 12; the symmetry when both component modes belong to the same class may be noted. Thus the $o|o$, $o|o$, figure 8, retains the two nodal lines through the centre and the rotational symmetry of 90° which is typical of the $o|o$ class, see table 1, and the $o|e$, $o|e$ figures 11 and 13 retain the one nodal diameter and rotational symmetry of 180°. In other combinations the rotational symmetry is generally lost—for example see figures 1, 3, 4 and 5; the mirror symmetry of these figures, however, is interesting.

Figures 16 to 22 were, after some consideration, classified as $e|e+$ designs which have been distorted by the central support. If this surmise is correct, they are not actually due to free vibrations of the plate, but their regularity—the rotational symmetry is one of 90°—and the fact that they are familiar designs are justification for including them in this paper.

Figures 24 to 28 are typical of the irregular designs which are sometimes found for higher overtones; the corresponding frequencies are as much as 115 to 143 times that of the fundamental tone. The classification of the more complicated figures is a little uncertain.

§ 3. CONCLUDING REMARKS

Comparison with circular plate. It is noteworthy that whereas the observed natural frequencies on square plates are in agreement with theory, it is otherwise for a circular plate[8] where in passing to higher overtones the discrepancy between observation and theory becomes increasingly marked. It is not difficult to see why this is so, for whereas the maximum number of nodal lines which may pass through the centre of a square plate is four, an indefinitely large number of nodal diameters is possible for a circular plate. Now the Kirchhoff theory[2] of the circular plate and the Rayleigh-Ritz theory[2, 4] of the square plate apply only to plates in which the thickness is small in comparison with the distance between adjacent nodal lines. It is evident then that the circular plate ceases to vibrate according to the laws of thin plates when the number of nodal lines through the centre becomes considerable. A square plate, on the other hand, of the dimensions generally employed for producing Chladni designs, will obey these laws except for much higher overtones than those which have been considered above.

Poisson's ratio. We have, in this paper, been concerned with the higher rather than with the lower overtones of vibrating brass plates. The systematic experimental study of free square plates is not yet complete. It remains to obtain more precise measurements of the graver tones on plates made of different materials and to explore the possibilities of finding Poisson's ratio by an acoustical method.

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A SIMPLE METHOD OF FINDING POISSON'S RATIO

BY

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A SIMPLE METHOD OF FINDING POISSON'S RATIO

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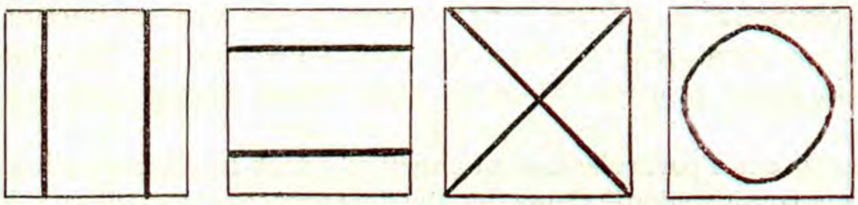
ABSTRACT. The paper describes how Poisson's ratio can be measured in terms of the ratio of two specified natural frequencies of a vibrating free square plate. Results for glass, steel, brass, copper and aluminium are given. It is suggested that such a simple method of estimating this elastic constant might be of value in the mechanical testing of metals and alloys.

§ 1. PRINCIPLE OF THE METHOD

IMAGINE a free square plate made of a material which has a zero value of Poisson's ratio [1]. The simple vibrating modes having the nodal systems 2|0, 0|2 shown in figure 1, *a* and *b*, would then be possible. These modes



a 2|0 *b* 0|2 *c* 2|0− *d* 2|0+

Figure 1.

could combine, either in opposite or like phase, to produce also the 2|0− and 2|0+ nodal systems of *c* and *d*. The periods of the four modes would all be equal. In practice, since Poisson's ratio is never zero, and on account therefore of the anticlastic curvatures of the plate, the simple vibrations *a* and *b* can never

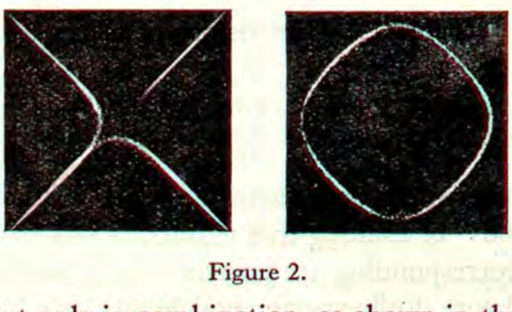


Figure 2.

be obtained singly but only in combination, as shown in the two photographs of figure 2. The natural frequencies of these two modes diverge from one another as Poisson's ratio increases, that of the 2|0+ mode being the greater.

The calculations of Ritz[2] and of Lemke[3] have been used to construct the two lines shown in figure 3, one of which gives the frequencies (in arbitrary units) of the 2|0+, and the other the frequencies of the 2|0 − mode, for different values of Poisson's ratio. The variation of frequency with Poisson's ratio is approximately linear[3], and the lines should intersect for a zero value of the latter quantity. Thus Poisson's ratio must be proportional to the difference between these frequencies, and if the relative frequencies are known, Poisson's ratio can be found.

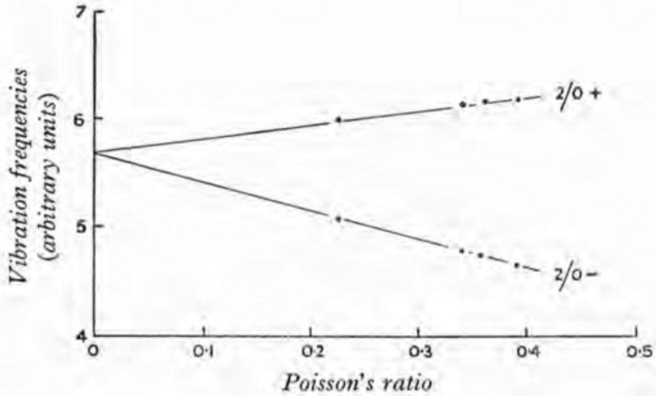


Figure 3.

It is convenient in practice to find Poisson's ratio from the interval or ratio between the frequencies of the 2|0+ and 2|0− modes. This interval is plotted in figure 4 against Poisson's ratio, values having been taken from figure 3.

The above is a particular case of a method which might be applied to other modes, but consideration[4] shows that the dependence of frequency on Poisson's ratio becomes rapidly less as we pass to higher overtones.

§ 2. PRELIMINARY TESTING OF THE METHOD

This method of finding Poisson's ratio was tested for glass, steel, brass, copper and aluminium. The dimensions of the plates used (except number 6) are given in table 1, since, although it is unnecessary to know these, it is convenient, from the experimental standpoint, to have them recorded. Either bowing or the solid carbon dioxide method[5] of exciting free vibrations was employed to obtain the nodal systems, the corresponding frequencies being measured by means of a calibrated valve oscillator. Full experimental details have been given in previous papers[4], [5].

Table 1. Dimensions of plates

Specimen	1	2	3	4	5	7	8	9	10
Length (cm.)	20·4	8·14	20·4	5·10	20·3	5·11	5·10	20·5	5·08
Thickness (mm.)	3·08	1·39	1·94	1·50	4·88	1·64	1·58	2·02	1·66

The results of the observations are recorded in table 2, in which the 2|0+ and 2|0− frequencies of the various plates are given in the third and fourth columns respectively. The ratios of frequencies are contained in the fifth column,

Table 2

Speci-men	Material of plate	Frequencies (c./sec.)		Ratio of frequencies	Poisson's ratio	
		2 | 0+	2 | 0−		From figure 4	From Kaye & Laby
1	Glass, crown	430	377	1·141	**·20**	·20 ·27
2	,,	1251	1091	1·147	**·21**	
3	Steel	267·5	231	1·158	**·22**	·25–·33
4	,,	3330	2660	1·251	**·32**	
5	Brass	502	390·5	1·285	**·36**	·34–·40
6	,,	2359	1840	1·282	**·35**	
7	,,	2677	2056	1·302	**·37**	
8	Copper	2714	2058	1·318	**·38**	·34
9	Aluminium	308	234	1·316	**·38**	·34
10	,,	3925	2967	1·323	**·39**	

while the values of Poisson's ratio, as deduced from figure 4, are shown in the sixth column of the table. These last may be compared with the values of Poisson's ratio, taken from tables, given in the seventh column. Considering that the

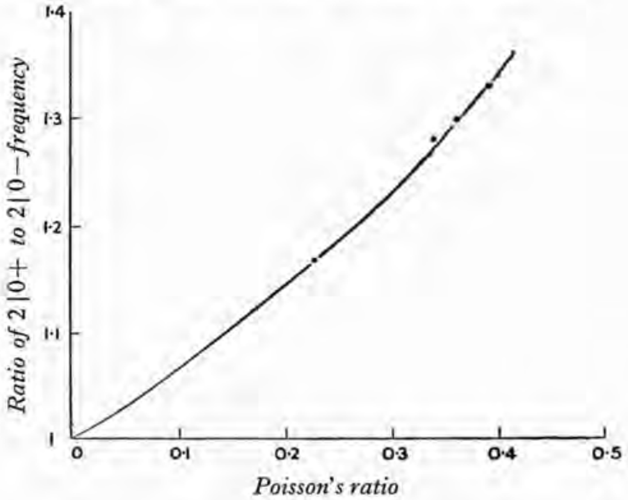


Figure 4.

ratio is affected by the purity of the specimens, and that, for the metals, it depends also on the previous mechanical and heat treatments, the two columns of figures are in satisfactory accordance.

§ 3. CONCLUDING REMARKS

In view of the results described above it would be interesting, when circumstances permit, to resume this study on specimens which could also be tested by an independent method. The square plates, for example, could be cut from such bars as were used by Ferguson and Andrews [6] in their study of anticlastic bending. At a time when the mechanical properties of metals and numberless alloys are receiving so much attention, such a simple method of finding Poisson's ratio might prove to be of practical value.

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VIBRATIONS OF FREE PLATES: ISOSCELES RIGHT-ANGLED TRIANGLES

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VIBRATIONS OF FREE PLATES: ISOSCELES RIGHT-ANGLED TRIANGLES

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ABSTRACT. The paper describes the results of systematic observations made on the vibrating modes of a free plate which is in the shape of a right-angled isosceles triangle. Any mode of vibration which is possible to this plate corresponds to some natural mode of the square plate, but the converse statement is only true in a restricted number of cases. Thus the only nodal systems of the square plate which have their counterpart on the triangular plate are given by $\omega = u_m(x)u_n(y) + u_n(x)u_m(y) = 0$. The triangular nodal systems differ markedly from the figures which are obtained by cutting the square nodal systems in halves diagonally; the natural frequencies of the triangular plate are appreciably lower than the corresponding frequencies of the square plate, these differences being caused by the changed boundary conditions.

§ 1. INTRODUCTION

CHLADNI (1802) expected that the nodal systems of a plate which is in the shape of a right-angled isosceles triangle would resemble the figures which are obtained by cutting the nodal systems of a square plate in halves diagonally. He found that they were very different. Such differences must be ascribed to the fact that the diagonal of the square becomes a free edge for the triangle.

There appear to be no records of either the nodal systems or natural frequencies of the right-angled triangular plate. As a comprehensive experimental study of the vibrating square plate, using for the purpose a new method of producing free vibrations, namely by means of solid carbon dioxide (Waller, 1933, 1934, 1937), had lately been undertaken (Waller, 1939), it was decided to obtain observations of the triangular plate also with a view to recording and classifying the modes of vibration in a systematic manner. As the experimental arrangements have been described in earlier papers, it will suffice to mention here that the vibration frequencies were measured by means of a calibrated valve oscillator, and that the plates were sometimes set into vibration with a bow instead of with solid carbon dioxide in cases where the desired nodal system could be obtained efficiently by excitation at the edge of the plate.

§ 2. THE NODAL SYSTEMS

Preliminary considerations. Although any mode of vibration possible to the right-angled isosceles triangular plate corresponds to some natural mode of the square plate, the converse statement is not always true. In order to recognize and classify the triangular figures, it is desirable first to know which of the square nodal systems may be expected to have their counterpart on the triangular plate.

The nodal systems of the square plate are given approximately by the Rayleigh-Ritz equation (Rayleigh, 1894 and 1911; Ritz, 1909),

$$\omega = Au_m(x)u_n(y) \pm Bu_n(x)u_m(y) = 0, \qquad \ldots\ldots(1)$$

in which ω denotes the displacement at the point x, y, and $u(x)u(y)$ are the normal functions of a free bar which is equal in length to the side of the plate. Except when the sum $m+n$ is odd, the amplitude constants A and B must be equal. The abbreviated notation $m|n+$ and $m|n-$, which is a modification of Chladni's later notation (Chladni, 1817), may be conveniently used for specifying the nodal figures.

We proceed to show that the only square figures which have their counterpart on the triangular plate are given by

$$\omega = u_m(x)u_n(y) + u_n(x)u_m(y) = 0, \qquad \ldots\ldots(2)$$

where m and n may have any integral values except unity.

In the first place it is seen from conditions of symmetry that there can be no triangular nodal system corresponding to the fundamental $1|1$ system shown in figure 2 of plate 2. The second $2|0-$ tone, with two nodal diameters, will also evidently be missing for the triangle. The third square nodal system, $2|0+$, half of which is given in figure 4 of plate 2, is the first square system which has a counterpart on the triangular plate.*

In order to arrive at general conclusions regarding higher overtones, we may now examine the classified photographs of the square nodal systems previously published (Waller, 1939). These are arranged on a single diagram which exhibits the essential symmetry of the various classes of nodal figures in a systematic manner. In case this diagram is not immediately available, figure 1 of the present paper may be studied instead. In it the various classes of symmetry are represented by descriptive abbreviation-symbols, the meanings of which are explained immediately below the diagram. It will be seen that the systems for which m and n are equal occupy the squares which extend diagonally from the top left corner to the bottom right corner of the diagram. The $m|n+$ systems are situated in the triangular space to the left of, and the $m|n-$ systems are situated in the triangular space to the right of, this diagonal. Both diameters are nodal in the *minus* portion of the diagram when the values of m and n are either both odd or both even ; such figures can have no counterpart on the triangle. Remembering that the systems with one nodal diameter, in the *minus* and *plus* parts of the diagram, are only formally distinct from one another, it may be concluded that the triangular systems may be arranged along the diagonal and in the triangular space to the left of this. Conditions of symmetry on the triangle are sufficient to show that additional

* These conclusions may be compared with the simple case of a free bar for which the first tone of the half bar (with two nodal lines) corresponds to the third tone of the whole bar (with four nodal lines).

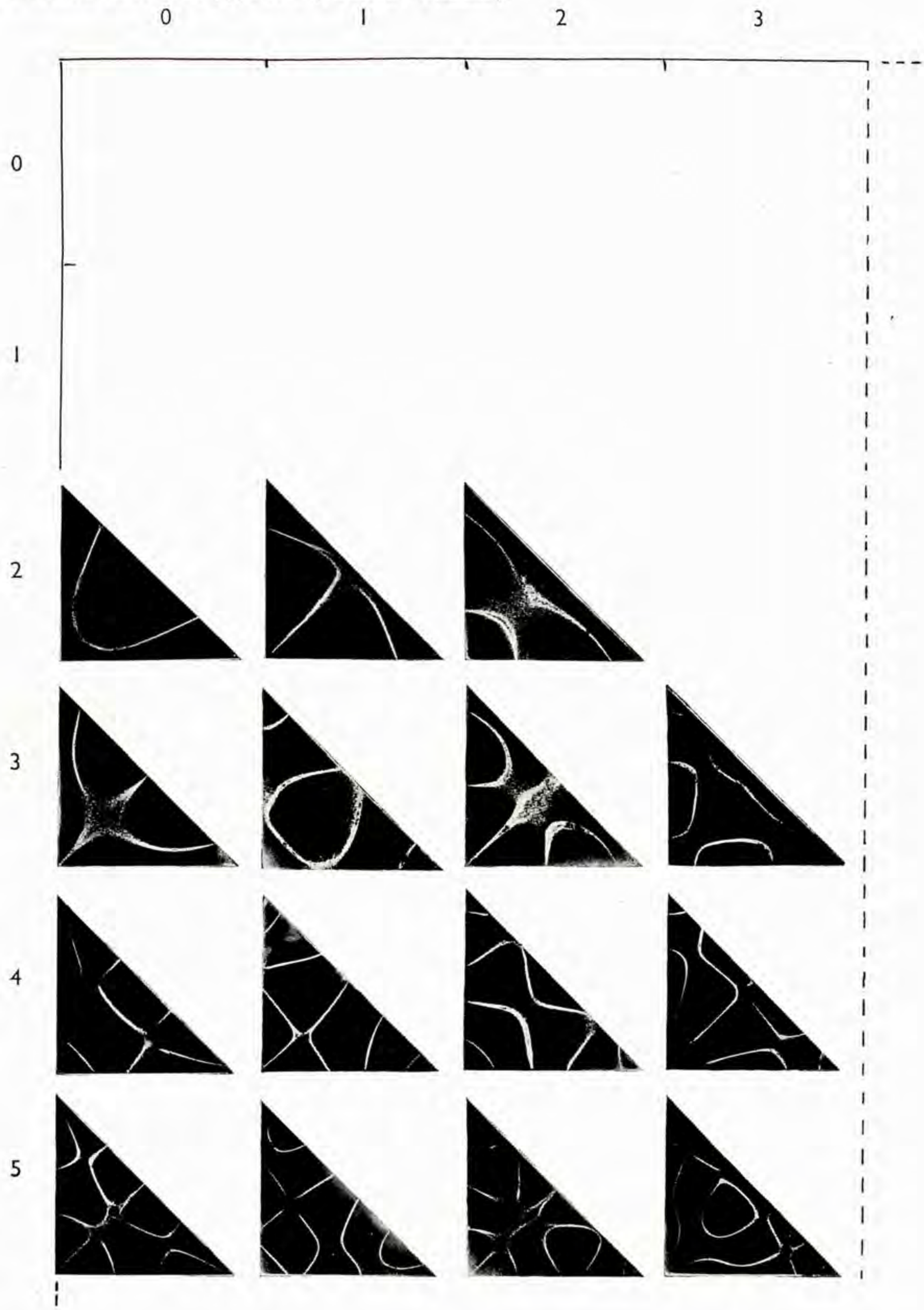

Plate 1. Nodal systems of right-angled isosceles triangular plate.

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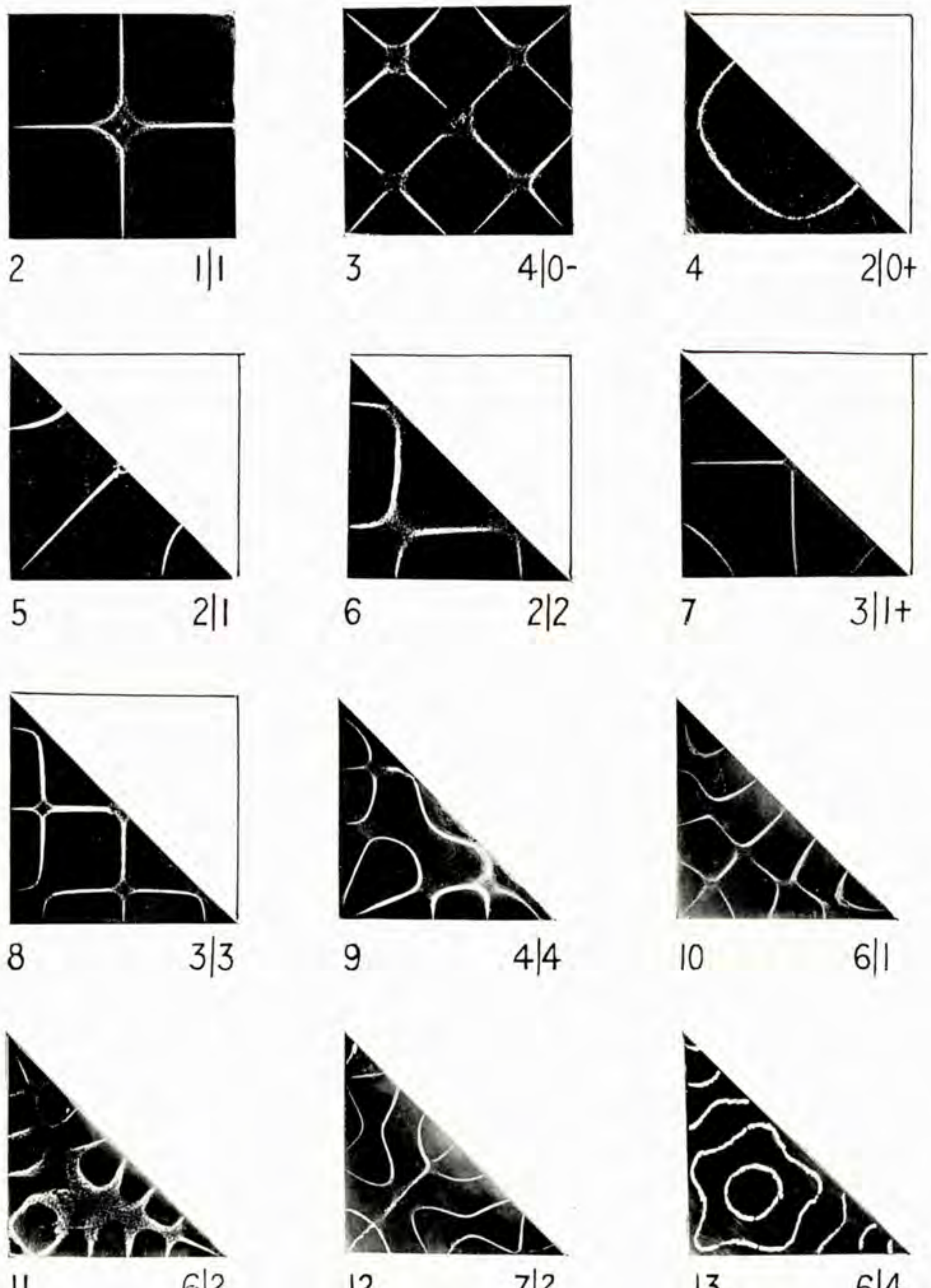

Plate 2. Figures 4 to 8 are *half square* nodal systems to be compared with the corresponding triangular systems in plate 1. Figures 9 to 13, some classified higher overtones.

figures corresponding to unequal values of A and B in equation (1) will not be possible. Thus the only possible triangular figures corresponding to the square figures are given by equation (2).

Experimental results. The photographs of fifteen triangular nodal systems, arranged according to the above plan, are shown in plate 1. In order to obtain some idea of the differences between the actual triangular systems and the bisected-

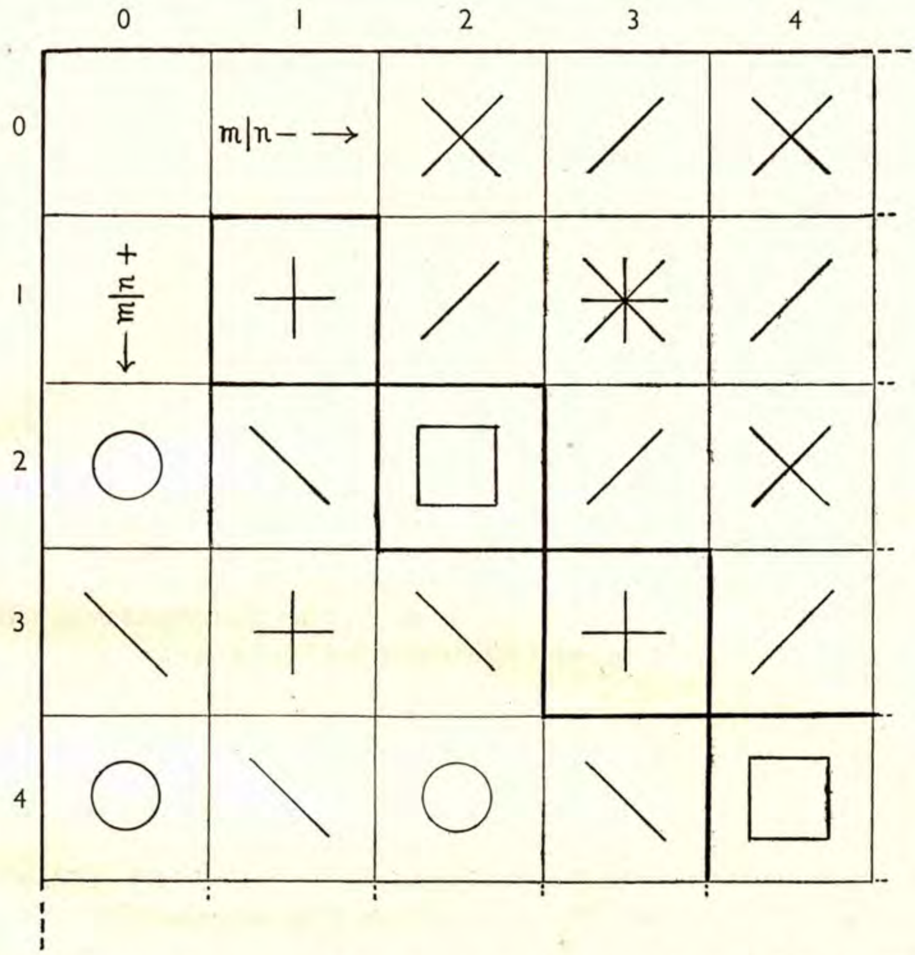

Figure 1. Symbolic classification of nodal systems of free square plate. ○, □ antinodal centre; ⁄ one nodal diameter; + medians nodal; × diameters nodal; ✳ medians and diameters nodal.

square figures, some classified examples of the latter, given in figures 4 to 8 of plate 2, may be compared with the relevant triangular systems of plate 1.

It is seen that in some cases, for example the $2|0+$ mode, the shape of the nodal line only is modified, and the figure is immediately recognizable. In other cases, however, for example the $3|1+$ mode, the lines sometimes meet different edges from those of the half-square figures. It is noteworthy that whereas the mid-point of the hypotenuse is always antinodal, except when one of the values m, n is odd and the other even, the centre of the square is never antinodal except in the $m|n+$

class with m and n both even. The fact that all the nodal lines are curved, excepting only the straight bisecting line which occurs when the sum of $m+n$ is odd, may also be noted.

Recognition of the nodal systems. In spite of the above differences, the triangular nodal systems may be recognized in terms of the m, n values of the square plate by using the following rules :—

1. $m=n$. No nodal lines meet the hypotenuse edge ; m (or n) lines cut the shorter edge.

2. *m odd ; n even, or vice versa.* The triangle is bisected symmetrically by a nodal line.

3. *m, n, both even.* $\frac{1}{2}(m+n)$ lines are encountered in traversing the median line to the centre of the hypotenuse.

4. *m, n, both odd.* All figures not included in the above.

5. The sum $m+n$ (except when $m=n$) is usually equal to the number of lines which cut the hypotenuse.

6. The value of m is usually equal to the number of lines which cut a shorter side.

7. In certain cases, rules 5 and 6 must be extended to include curved nodal lines which do not actually cut the edge, but which, if they did so, would cut it twice. (See, for example, the 3|2 and 3|0 systems of plate 1.)

Finally, when the natural frequencies are known, (§ 3), they are also of assistance in classifying the figures.

Higher overtones. Attention is drawn to the examples of nodal systems of some higher overtones shown in figures 9 to 13 of plate 2 (13 has been retouched). They were classified by means of the rules which have just been given.

§ 3. THE NATURAL FREQUENCIES

Preliminary remarks. The frequencies of the vibrating modes of a square or triangular plate are proportional to $\frac{1}{a^2}\sqrt{\frac{D}{m}}$, in which a is the length of side, m is the mass per unit area and D is the flexural rigidity ($=Eh^3/12(1-\sigma^2)$), where E is Young's modulus, h the thickness, and σ is Poisson's ratio (Timoshenko, 1937).

Although the relative frequencies of corresponding modes of the square and triangular plate may be expected to be roughly comparable, the actual frequencies of the triangular plate should be lower, since the constraint to which a vibrating surface of given area is subject is evidently reduced by making the diagonal of the square a free edge of the triangle.

Experimental results. The observed natural frequencies and the relative frequencies (in italics) of one of the brass plates employed are shown in the table, which corresponds in its arrangements with the nodal systems of plate 1. The length of the shorter side of this plate was 22·48 cm. and the thickness 2·58 mm.

The fundamental frequency of 162 c./sec. is, as expected, appreciably lower than the corresponding frequency for the square plate which, according to calculation, is 227.

Table. Actual (c./sec.) and relative natural frequencies (in italics) of a right-angled isosceles triangular brass plate arranged to correspond with the nodal systems of plate 1

	0	1	2	3
2	162 *1*	227 *1·4*	380 *2·36*	
3	414 *2·56*	590 *3·65*	710 *4·39*	1090 *6·8*
4	862 *5·32*	1078 *6·62*	1350 *8·36*	1690 *10·4*
5	1380 *8·54*	1670 *10·3*	2000 *12·4*	2490 *15·4*

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