CHARGE TRANSFER IN ALGEBRAIC QUANTUM FIELD THEORY

by

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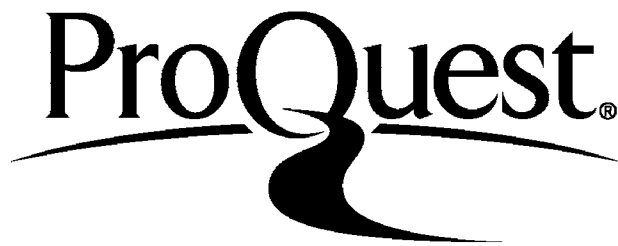University of London

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## ABSTRACT

We discuss aspects of the algebraic structure
of quantum field theory.  We take the view that the
superselection structure of a theory should be determinable
from the vacuum representation of the observable algebra,
and physical properties of the charge.  Hence one
determines the nature of the charge transfer operations:
the automorphisms of the observable algebra corresponding
to the movement of charge along space-time paths.

New superselection sectors are obtained from the
vacuum sector by an automorphism which is a limit of
charge transfer operations along paths with an endpoint
tending to spacelike infinity.  Roberts has shown that
for a gauge theory of the first kind, the charge transfer
operations for a given charge form a certain kind of
1-cocycle over Minkowski space.  The local 1-cohomology
group of their equivalence classes corresponds to the
superselection structure.  The exact definition of the
cohomology group depends on the properties of the charge.

Using displaced Fock representations of free
fields, we develop model field theories which illustrate
this structure.  The cohomological classification of
displaced Fock representations has been elucidated by
Araki.  For more general representations, explicit
determination of the cohomology group is a hard problem.

Using our models, we can illustrate ways in which
fields with reasonable physical properties depart from

the abovementioned structure. In 1+1 dimensions, we use the Streater-Wilde model to illustrate explicitly the representation-dependence of the cohomology structure, and the direction-dependence of the limiting charge transfer operation. The cohomology structure may also be representation-dependent in higher-dimensional theories without strict localization of charge, for example the electromagnetic field. The algebraic structure of the electromagnetic field has many other special features, which we discuss in relation to the concept of charge transfer. We also give some indication of the modifications needed to account for gauge theories of the second kind.

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## CHAPTER 1

### INTRODUCTION

There has been considerable interest recently in
the rôle in quantum field theory of "soliton" solutions
to partial differential equations. The term soliton is
now used to refer to solutions with localized energy
and a degree of stability under scattering, which can
be found for certain non-linear partial differential
equations. Quantum field theories based on such equations
have been constructed, and it seems clear that all their
physically reasonable classical solutions should bear
some relation to the quantum theory. Furthermore,
the stability properties and localized energy of the
so-called soliton solutions invite a particle interpretation.

In many cases, the stability of these solutions
is related to a conserved label they carry, which is
topological in origin. An example is the monopole
solutions of certain equations in 3+1 dimensions.
The labels carried by these solutions are elements of
a homotopy group, and can be considered as charges
which will classify the superselection sectors in the
quantum theory (e.g. Goddard and Olive, 1978).

Now for certain idealized quantum theories (those
satisfying the axioms of Doplicher, Haag and Roberts, (DHR)
1969a), it has been shown by Roberts (1976) that the
superselection sectors can be classified using certain
1-cohomology groups over Minkowski space. It is
believed that this method of classification can be

extended to more general quantum theories. Various
extensions of the method have already been considered
(e.g. Roberts 1977, 1978, Fröhlich 1979). If we
consider that this classification method can be applied
to all quantum field theories, then the topological
classification of monopole sectors should correspond
to a subset of the cohomological superselection sector
classification in the quantum theory.

At the basis of the algebraic approach (Haag and
Kastler 1964) is the idea that if the physical states
of a quantum field are divided into superselection
sectors by grouping together all states containing
the same amount of charge, then the distinct sectors
of states are Hilbert spaces which carry inequivalent
representations of the algebra of observables. The
sector of zero charge is called the vacuum sector, $\mathcal{H}_0$
say, and carries the vacuum representation $\pi_0$ of the
observable algebra $\mathcal{A}$. Doplicher, Haag and Roberts
(1969b) consider to what extent a field theory is
determined by the vacuum representation of its observable
algebra.

The reason why the cohomological classification
might be expected to exist in some form for any elementary
particle theory is that it is arrived at using the
concept of charge transfer to relate the inequivalent
sectors of the theory. The concept of charge transfer
was introduced in an algebraic context by Doplicher,
Haag and Roberts (1971). We indicate how this concept
is made precise in algebraic quantum field theory. To

do this, it is necessary to show how the heuristic concept of charge is related to the mathematical quantity describing an aspect of the algebraic structure of the field theory. This will be done in detail, assuming a particular set of field theory axioms, in chapter 2. Here, we simply set the stage with a descriptive account.

In classical theories and in heuristic descriptions, the physical process by which a state of given charge is obtained from the vacuum is the introduction of that charge "from infinity". This process is also used in the algebraic approach. In fact, properties which a charge is expected to satisfy on physical grounds may be taken as the starting point in describing the algebraic representation structure of a given theory. One then identifies families of operators indexed by paths in Minkowski space, which implement automorphisms having the appropriate properties to be interpreted as charge transfer. Then new sectors (non-vacuum representations of the observable algebra) are obtained by taking a limit of automorphisms from a given family, indexed by a sequence of paths whose initial points tend to spacelike infinity. It turns out that the operators implementing a given family of automorphisms are 1-cocycles over Minkowski space. Those cocycles which give rise to equivalent representations are cohomologous in a sense which depends on the physical properties of the charge in the theory. (In particular, assumptions about localization are crucial. For example, Roberts (1976)

assumes strict locality: for any given charge there is a bounded space-time region outside which it cannot be detected.) Hence the cohomology group labels the superselection structure of the theory.

Suggestions for generalizing the method of classification to deal with a wider class of models include a relaxation of the localization requirement (necessary since e.g. electric charge does not satisfy strict localization, Frölich 1979) and extensions of the fundamental algebraic structure needed to describe gauge theories of the second kind. In the latter case it has been suggested that a 2-cohomology may be needed to describe the algebraic representation structure. (Roberts 1977).

The motivation for studying soliton sectors of solutions of classical field equations (e.g. Parenti, Strocchi and Velo 1977) is that the existence of certain superselection sectors in the quantum theory may be predicted through some sort of correspondence principle from a knowledge of the solutions of the classical equation. Eventually, this must be linked up with the more systematic approach to the sectors of the quantum theory which we have sketched.

With very few exceptions, the detailed work which has been done on models involving soliton sectors has been in a basically classical framework (e.g. Ishikawa 1976). It is therefore still of interest to choose simpler models in order to study the algebraic structure which a quantum theory taking account of soliton

solutions might have. In particular, quantum soliton theories involve inequivalent representations of the observable algebra, and must be interpreted as gauge theories. We shall look at the simplest possible models where this interpretation can be studied. It seems reasonable that inequivalent representations of the free field obtained by adding a c-number function to the field operator should play the same rôle algebraically as the soliton sectors of a non-linear theory. The extent of the parallel can be seen by comparing the models in 2 space-time dimensions which we shall describe in chapter 3: the Streater-Wilde model is built up from inequivalent representations of the free field, while the $P(\varphi)_2$ model we consider is a quantization of a non-linear classical equation with topological soliton solutions, whose non-vacuum sectors have been constructed by Fröhlich (1977).

We shall use the displaced Fock representations to develop models which suggest ways in which the cohomological description of the algebraic structure of gauge theories should be generalized. For example, a relaxation of the localization requirement will be necessary to include quantum electrodynamics, where the charge must satisfy Gauss' law. We find that such a modification also introduces many more possibilities in 1+1 dimensions.

Although displaced Fock representations are not sufficiently general to provide a non-trivial example of the 2-cohomology structure of second type gauge

theories, they do give some indication of how it might arise.

The representations obtained by displacing the free field are also of interest in their own right, as approximations to interacting theories; for example, in the case of the electromagnetic field, we shall interpret the c-number displacement as an external classical current.

Finally, we mention the question of interpretation of soliton sectors. Jackiw (1975), for example, has given several possible descriptions of the rôle which solitons play in physical theories. One possibility he gives is that the particles which have been observed in nature are just those which are described in terms of perturbations of vacuum solutions, and the solitons are heavy gauge particles which are extremely weakly interacting. If this were the case, then the study of solitons would have little hope of contributing to our understanding of elementary particles. Also, this interpretation does not take account of the particle-like properties which originally motivated the study of solitons.

From our point of view, the interpretation of solitons as the known charged particles of the theory seems more reasonable. This follows from the analogy between displaced Fock representations and soliton sectors: in the approximate theory we describe for quantum electrodynamics, where the displacement is

interpreted as an external classical current, the
"soliton" sectors correspond to the usual charged
sectors of quantum electrodynamics.

## CHAPTER 2*

### COHOMOLOGY AND SUPERSELECTION SECTORS

Haag (1957) suggested that the basic structure of
quantum field theory should be a net of local algebras,
indexed by certain bounded open sets $0$ of space-time.
Physically, such a structure arises naturally if each
local observable algebra $\alpha(0)$ is generated by the
quantities that can be measured in $0$. In a structure of
this type, covariance and causality requirements can be
introduced in a natural way. The specific formulation of
these requirements depends on the type of theory being
described. For example, the formulation of Haag and
Kastler (1964), and that in the Doplicher, Haag and Roberts
papers (1969a,b, 1971, 1974), is intended to describe
theories involving gauge transformations of the first
kind, and strictly localizable charges which do not obey
Gauss' law.

We shall not attempt a comprehensive review of the
local algebra approach to quantum field theory, but the
main ideas which lead to the cohomological classification
of superselection sectors will be presented by means of a
model constructed from displaced Fock representations in
a Weyl algebra formulation. The rôle of cohomology groups
has been discussed in several papers by Roberts (1975, 1976,

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*   Much of the material in this chapter, and in chapter 4,
    has already appeared in the paper of Basarab-Horwath,
    Streater and Wright (1979).

1977, 1978) and also by Frohlich (1977, 1979), and is mentioned in Doplicher, Haag and Roberts (1974).

For reference, we state here the basic assumptions for a strictly local algebraic field as formulated by Doplicher, Haag and Roberts (1969a).

1. There is a correspondence between open bounded regions of space-time, $0$, and algebras of operators on a Hilbert space $\mathcal{H} : 0 \to F(0)$.

$F(0)$ is the field algebra corresponding to the region $0$. It is a *-subalgebra of $B(\mathcal{H})$, the algebra of bounded operators on $\mathcal{H}$. $F(0)$ is weakly closed, and the Hilbert space is irreducible with respect to $F$, the norm closure of $\underset{0}{\cup} F(0)$. The correspondence $0 \to F(0)$ satisfies isotony, i.e. if $0_1$ and $0_2$ are 2 bounded regions of space-time and $0_1 \subset 0_2$ then $F(0_1) \subseteq F(0_2)$.

2. $F$ is <u>Poincaré-covariant</u>: i.e., there exists a strongly continuous unitary representation of $P_+^\uparrow$ on $\mathcal{H}$, $L \to V(L)$, for $L$ in $P_+^\uparrow$, satisfying the spectral condition, such that for $F$ in $F(0)$ $\tau_L(F) \equiv V(L) F V(L)^{-1}$ lies in $F(L0)$. There is a unique vector in the zero eigenspace of the generator of time translations: the vacuum vector $\Omega$.

3. There is a <u>compact gauge group</u> $G$, with a faithful strongly continuous unitary representation

$$g \in G \to U(g) \in B(\mathcal{H})$$

which induces a group of automorphisms of $F$:

$$U(g) F U(g)^{-1} = \alpha_g(F).$$

$V(P_+^\uparrow)$ lies in the commutant $U(G)'$ of the gauge group representation and $\alpha_z$ acts in a strictly local way on $F$:

$$\alpha_g (F(0)) = F(0) \quad \text{for all} \quad 0.$$

4.  The local observable algebras are defined by

$$\mathcal{A}(0) = F(0) \cap U(G)' .$$

<u>Relativistic causality</u> is expressed by the requirement

$$\mathcal{A}(0_1) \subseteq \mathcal{A}(0_2)' \quad \text{if} \quad 0_1 \text{ is spacelike to } 0_2.$$

Certain technical assumptions are also introduced, permitting important structural theorems to be proved. The following two assumptions are perhaps suggested by the fact that these properties occur as theorems for Wightman fields.

5.  The vacuum vector $\Omega$ satisfies the <u>cluster property</u>: If $F_i \in F$, $i=1,2,3$, and $x$ denotes a spatial translation, then

$$\lim_{|x| \to \infty} \langle \Omega, F_1 \tau_x (F_2) F_3 \Omega \rangle = \langle \Omega, F_1 F_3 \Omega \rangle \langle \Omega, F_2 \Omega \rangle .$$

6.  If $\Psi$ is an analytic vector for the energy operator, it is cyclic and separating for each $F(0)$. (the <u>Reeh-Schlieder property</u> for analytic vectors.)

Duality, or a similar assumption, is also introduced. Its exact formulation strongly influences the nature of the gauge theory. In terms of the observable algebra, the strict duality assumption is

$$\alpha(0) = \alpha(0')' .$$

where $0'$ is the causal complement of $0$.

In fact it is not usually necessary to deal with nets of algebras defined over all open bounded regions $0$ of Minkowski space. The structure of this net will be determined by the smaller net of algebras

$$\{\alpha(0) : 0 \in K\} ,$$

where $K$ is the set of bounded double cones, defined as follows. First define $K_0$ , the set of bounded double cones centred on the origin, to be all regions

$$0_R = \{(\underline{x}, t) : |\underline{x}| < R - |t|\} \quad \text{(where } c=1\text{)}$$

for $R > 0$. Then $K$ consists of all translates of each element of $K_0$.

Physically it seems reasonable to regard the observable algebra as the given structure, and find a gauge and field algebra corresponding to it which satisfy the above assumptions. In physically important aspects, the observable algebra should determine the field algebra. This approach is the subject of Doplicher, Haag and Roberts (1969b).

In this chapter, we shall first describe Segal's Weyl algebra formulation of algebras of observables, but paying particular attention to the possibility of constructing algebras which admit many physical representations. In the following section, we shall consider a class of representations of Weyl algebras

corresponding to the free field, and develop a
classification of the physical representations in
terms of a certain cohomology group of the covariance
group. We then (in section 2.3) place this in a wider
context by demonstrating that it can be formulated in
terms of the charge transfer cocycles which relate the
physical sectors of any strictly local theory satisfying
the Doplicher, Haag, Roberts axioms.

General descriptions of the structure of field
theory usually involve idealizations which are not valid
for many well-known physical theories. An important
example is the assumption of strict locality. Our
initial development of local cohomology (following
Roberts 1976) is based on this assumption. We shall
later use alternative locality requirements. More
general concepts of cohomology groups are needed in
order to classify representations in the theories
corresponding to any given localization. This will
equip us to deal with some (though not all) of the
deviations from the Doplicher, Haag, Roberts axioms
encountered in the examples and applications of later
chapters.

## 2.1  Segal quantization

The axioms of Doplicher, Haag and Roberts specify required properties for a quantum field theory, which are sufficiently precise to lead to a number of rigorous results about the algebraic structure which a theory satisfying the axioms must have.

In Segal's approach, the quantum theory corresponding to a given classical equation is a representation of the Weyl algebra of a symplectic space of solutions of the equation. The requirement that this algebra should be represented as a Weyl system and that the representation space should contain an invariant vacuum vector is often sufficient to guarantee uniqueness, even in infinite dimensions:  an extension of the Stone-von Neumann uniqueness theorem. We shall use the Weyl algebra formalism to construct a field theory algebra satisfying a version of the Doplicher, Haag, Roberts axioms. As we wish to obtain a gauge theory, we shall be interested in cases where the uniqueness result fails and more than one physically acceptable representation is available.

For the present, we shall concentrate on the quantization of the free field. The Weyl algebra formulation may be developed from the form of the canonical commutation relations (CCR) in terms of bounded operators. In the heuristic form of a free canonical field theory, if $\varphi$ and $\pi$ are the singular field and its conjugate momentum, (properly regarded as

distributions), then they satisfy

$$[\varphi(x), \pi(x')] = i\delta(x-x') \quad (\hbar=1) \qquad (2.1.1)$$

(For the present, we assume for simplicity of notation that $\varphi$ has spin zero.)

If bounded operators $U(f)$, $V(g)$ can be defined by

$$U(f) = \exp(i\int\varphi(\underline{x})\ f(\underline{x})\ d\underline{x}),$$

$$V(g) = \exp(i\int\pi(\underline{x})\ g(\underline{x})\ d\underline{x}),$$

for $f$ and $g$ in some test function space such as Schwartz space, $S = S(\mathbb{R}^3)$ in $3+1$ dimensions, then from (2.1.1) follows the commutation rule

$$U(f)\ V(g) = e^{i(f,g)}\ V(g)\ U(f)$$

where

$$(f,g) = \int_{\mathbb{R}^3} f(\underline{x})\ g(\underline{x})\ d\underline{x}\ .$$

If $W(f,g)$ is defined by

$$W(f,g) = U(f)\ V(g)\ e^{\frac{1}{2}i(f,g)} \qquad (2.1.2)$$

then $W$ satisfies

$$W(f,g)\ W(f',g') = e^{\frac{1}{2}i\ [(f,g')-(f',g)]}W(f+f',g+g'). \qquad (2.1.3)$$

Pairs of functions $(f,g)$ may be set in correspondence with certain solutions $\Phi$ of the free wave equation

$$(\Box+m^2)\varphi = 0 \qquad (2.1.4)$$

by choosing to correspond to (f,g) the solution $\varphi = \Phi$ whose Cauchy data are given by

$$\Phi(\underline{x},0) = f(\underline{x}) ,$$

$$\dot{\Phi}(\underline{x},0) = g(\underline{x}) .$$

Then W may be defined to act on solutions of (2.1.4) whose Cauchy data lie in $S \times S$, i.e., if $(f_i,g_i)$ are the Cauchy data of $\Phi_i$, we rename $W(f_i,g_i)$ as $W(\Phi_i)$. (2.1.3) becomes

$$W(\Phi_1)W(\Phi_2) = e^{\frac{1}{2}i \, Im <\Phi_1,\Phi_2>} W(\Phi_1+\Phi_2) \qquad (2.1.5)$$

provided

$$Im\langle \Phi_1,\Phi_2\rangle = \int (\Phi_1(\underline{x},0)\dot{\Phi}_2(\underline{x},0) - \dot{\Phi}_1(\underline{x},0)\Phi_2(\underline{x},0))\,d\underline{x}.$$

In fact, to form a Hilbert space of solutions of (2.1.4), the inner product $\langle \Phi_1,\Phi_2\rangle$ may be defined by

$$\langle \Phi_1,\Phi_2\rangle = \int [(B\Phi_1)(\underline{x},0)\Phi_2(\underline{x},0) + (B^{-1}\Phi_1)(\underline{x},0)\Phi_2(\underline{x},0)]\,d\underline{x}$$

$$+ i\int [\Phi_1(\underline{x},0)\dot{\Phi}_2(\underline{x},0) - \dot{\Phi}_1(\underline{x},0)\Phi_2(\underline{x},0)]\,d\underline{x}, \qquad (2.1.6)$$

where B is the unique positive square root of

$$B^2 = m^2 - \nabla^2 ,$$

and the space may be closed in the corresponding norm.

More generally, in the case of higher spin, the field and hence the test functions will be vector valued. The test functions now correspond to solutions $(\Phi_\alpha)$ $(\alpha=1,\ldots,n)$ of

$$(\square + m^2)\varphi_\alpha = 0, \quad D^{\alpha\beta}\varphi_\beta = 0 \qquad (2.1.7)$$

$$\alpha, \beta = 1, \ldots, n \ ,$$

where $D$ is a tensor of differential operators chosen
to remove unwanted spin components. A set of equations
of this form will be used, for example, in defining a
Weyl algebra for the electromagnetic field. (See
chapter 4.) An inner product can be defined on some
pre-Hilbert space of solutions of (2.1.7) which can
then be closed to form the one-particle space, $K$. If
a Hilbert space is chosen as the domain of the Weyl
map $W$, the imaginary part of the inner product
specifies the multiplication law (2.1.5). More generally,
the domain of a Weyl map must be a symplectic space:
a real locally convex topological vector space, $M$ say,
over which a non-degenerate bilinear antisymmetric
(i.e. symplectic) form $B$, continuous in the topology of
$M$, is defined.

For a given symplectic space $M$, a representation
of the Weyl relation is a set of unitary operators $W(\phi)$
for $\phi$ in $M$, acting on some Hilbert space $H$, and
satisfying (2.1.5). If, in addition, $W(t\phi)$ is strongly
continuous in $t$, for $t$ in $\mathbb{R}$, the representation
is called a Weyl system over $M$. The self-adjoint
generator of this one-parameter unitary group is
usually written $R(\phi)$, i.e.

$$W(t\phi) = e^{i \, t \, R(\phi)} \ . \qquad (2.1.8)$$

The existence of such a representation is guaranteed if $M$ has a complex pre-Hilbert space structure, or more generally:

<u>Theorem</u>: If a positive definite symmetric form $S$ can be defined on $M$, relative to which $B$ is continuous, i.e.

$$|B(u,v)|^2 \leqslant S(u,u)S(v,v) \quad \text{for} \quad u,v \quad \text{in} \quad M,$$

then there exists a Weyl system over $(M,B)$.
(This result is due to Segal, 1959).

In the case where $M$ is a pre-Hilbert space, we choose

$$B(u,v) = \text{Im}\langle u,v\rangle$$

and $S$ may be chosen as

$$S(u,v) = |\langle u,v\rangle| .$$

We now take $K$ to be the one-particle space: a Hilbert space of solutions of (2.1.7) on which the representation $U$ of the Poincaré group of appropriate mass and spin is realized, and satisfies the spectral condition $((P_0)^2 \geqslant \underline{P}^2$ , $P_0 \geqslant 0$.) Weyl systems over dense Lorentz invariant subsets $M$ of $K$ will be used in our models.

We have described the appropriate definition of a Weyl system corresponding to a free field. We also have a result which guarantees the existence of a Weyl system for this case. An explicit construction of the

C*-algebra generated by the $W(\Phi)$ is given by Manuceau (1968). (See also Emch 1972). It is then possible to find concrete representations which have sufficient continuity properties to be Weyl systems.

More generally, for a field specified by non-linear classical field equations, Segal's scheme of quantization involves the selection of a space of solutions of the field equations, and a symplectic form on the space, such that a Weyl system over this symplectic space exists; then the construction of a Weyl system. All of these steps are usually non-trivial.

For the finite-dimensional case, we have the von Neumann uniqueness result for the representation of the CCR (von Neumann 1931, Stone 1932). For this case we first define a Schrödinger Weyl system.

Definition: Given an n-dimensional complex inner product space $M = \mathbb{R}^n + i\mathbb{R}^n$, a Schrödinger Weyl system over $M$ is a set of operators $W_s(\underline{x} + i\underline{y})$ for $\underline{x} + i\underline{y}$ in $M$, defined on the representation space $L^2(\mathbb{R}^n)$ by

$$W_s(\underline{x} + i\underline{y})f(\underline{u}) = e^{\frac{1}{2}i\,\underline{x} \cdot \underline{y}}\, e^{i\,\underline{u} \cdot \underline{x}}f(\underline{u} + \underline{y}),$$

$$\text{for all } \underline{u}, \underline{v}, \underline{x} \in \mathbb{R}^n.$$

Then we have:

Theorem: If $M$ is an n-dimensional complex inner product space and $W$ is a Weyl system over $M$, then $W$ is unitarily equivalent to a direct sum (possibly

not countable) of copies of a Schrödinger Weyl system over $M$.

To obtain analogous results in infinite dimensions, it is necessary to impose certain restrictions.

Firstly the domain of the Weyl relations must be a Hilbert space, $K$, say. In this case, a distinguished Weyl system exists. This Segal calls the conventional free field system, and one realization is the Fock-Cook particle representation (Fock 1932, Cook 1953). It has the following structure:

1) a distinguished vector $v$ in the representation space $\mathcal{H}$, which is cyclic under the $W(\Phi)$,

2) a continuous unitary or anti-unitary representation $\Gamma_0$ on $\mathcal{H}$ of the group of unitary and antiunitary operators $U$ on $K$, such that

$$\Gamma_0(U)W(\Phi)\Gamma_0(U)^{-1} = W(U\Phi) \quad \text{for all} \quad \Phi \quad \text{in} \quad K, \quad \text{and all} \quad U,$$

and

$$\Gamma_0(U)v = v, \quad \text{for all} \quad U.$$

The uniqueness theorem can then be stated (Segal 1962):

Theorem: Suppose $W$ is a Weyl system over $(K, \text{Im}\langle\cdot,\cdot\rangle)$ with representation space $\mathcal{H}$, and $\{U(t): t \in \mathbb{R}\}$ is a continuous one-parameter group of unitary operators on $K$, whose self-adjoint generator $h$ is positive definite. Suppose there exists a continuous one-parameter unitary group, $\{\Gamma(t): t \in \mathbb{R}\}$ on $\mathcal{H}$ with positive definite self adjoint generator $H$, such that

$$\Gamma(t) W(\Phi) \Gamma(t)^{-1} = W(U(t)\Phi).$$

Then if $\mathcal{K}$ contains a vector $v$ which is cyclic for the $W(\Phi)$ and invariant under the $\Gamma(t)$, and if also $(v, W(\Phi)v)$ is a continuous function of $\Phi$, then the system defined by $(K, W, \mathcal{K}, v)$ is unitarily equivalent to the conventional free-field system. Under this equivalence, $\Gamma(t)$ coincides with $\Gamma_0(t)$.

Now if $(K, \langle \cdot, \cdot \rangle)$ is a one-particle space, time translations are implemented in $K$, and generated by a positive definite self adjoint operator. Thus in this case, any irreducible Weyl system over $(K, \mathrm{Im}\langle \cdot, \cdot \rangle)$ on which time translations are implemented, and with a time-translation-invariant cyclic vacuum vector, and positive energy, is unitarily equivalent to the Fock representation. In particular, therefore, all Lorentz transformations are implemented and leave the vacuum invariant.

As we are studying the classification of inequivalent representations, we shall be interested in cases where one of these restrictions fails and uniqueness does not hold. If we consider the Weyl relations defined over $(M, \mathrm{Im}\langle \cdot, \cdot \rangle)$ where $M$ is chosen to be properly smaller than $K$ (but dense) this introduces the possibility of many vacuum sectors. But because we regard the vacuum as unique in elementary particle physics, we shall choose $M$ in such a way that these vacuum sectors need not be regarded as distinct. Nonetheless, the

existence of sectors labelled by topological quantum numbers is closely associated with these different vacua.

We shall also relax the requirement that the representation space should contain an invariant state. This requirement is not necessary from a physical point of view if the representation is to be interpreted as a charged sector. This is the interpretation of the non-vacuum sectors which have a place in the Doplicher, Haag, Roberts framework.

For some purposes, we shall also consider representations whose covariance group $C$ is smaller than the Poincaré group, although it will always be required to contain the space-time translations, $T$.

The generating functional corresponding to a Weyl system $W$ over $(M,B)$ with cyclic vector $v$ is given by

$$\rho(\Phi) = (v, W(\Phi)v) \quad \text{for } \Phi \in M.$$

There is a correspondence between cyclic Weyl systems and suitable generating functionals: corresponding to a functional $\rho$ over $M$ which satisfies

1) $\rho(0) = 1$

2) $\rho(\lambda\Phi+\Phi')$ is continuous in $\lambda \in \mathbb{R}$ for all $\Phi,\Phi' \in M$

3) for all $\Phi_i$ in $M$ and for all complex numbers $\alpha_i$, $i$ ranging over some finite index set $F$,

$$\sum_{i,j \in F} \rho(\Phi_i - \Phi_j)\alpha_i \bar{\alpha}_j \exp -\tfrac{1}{2}iB(\Phi_i, \Phi_j) \geq 0,$$

there exists a Weyl system with cyclic vector  v  such
that

$$\rho(\Phi) = (v, W(\Phi)v).$$

It is unique up to unitary equivalence.

In particular, this result characterizes those
cyclic representations of the  C*-algebra of the CCR
which are Weyl systems.  The representation may be
recovered from  $\rho$  by the GNS construction (Gelfand and
Naimark 1943, Segal 1947).

## 2.2 Displaced Fock representations

Given a one-particle space $K$, carrying an action $U$ of the Poincaré group, suppose $M$ is a dense subspace of $K$ satisfying

$$U(L)M \subseteq M \quad \text{for each} \quad L \in P_+^\uparrow \;.$$

The Weyl algebra over $(M, \text{Im}\langle \cdot, \cdot \rangle)$ has a Fock representation determined by the generating functional

$$\rho_F(\Phi) = \exp(-\tfrac{1}{4} \| \Phi \|_K^2)$$

for $\Phi \in M$.

Probably the simplest way of constructing inequivalent representations of the algebra $\mathcal{a}$ is to multiply the Fock functional $\rho_F$ by a phase factor:

$$\rho_F(W(\Phi)) \rightarrow \rho_{\Psi^x}(W(\Phi)) = e^{i \, \text{Im}\langle \Psi^x, \Phi \rangle} \rho_F(W(\Phi)) \qquad (2.2.1)$$

and form the GNS representation, $\pi_{\Psi^x}$ say, corresponding to $\Psi^x$. At this stage, the functional $\Psi^x$ may be chosen as any element of the algebraic dual, $M^x$ of $M$. Formally, the transformation

$$\pi_F(A) \rightarrow \pi_{\Psi^x}(A)$$

leads to the same result as adding $\Psi^x$ to the field $\varphi$ in the Fock representation. Representations constructed in this way appear frequently in the literature (e.g. Klauder (1970), Roepstorff (1970)). This is partly because they provide a first approximation to an inter-

acting field; the displacement is the classical external field or the c-number current.

We now formulate certain conditions that a representation $\pi$ of $\mathcal{O}$ must satisfy to be physically interesting. The action $U(L)$ of $L$ on the space $M$ induces an action $\tau_L$ on $\mathcal{O}$, determined by

$$W(\Phi) \rightarrow \tau_L(W(\Phi)) = W(U(L)\Phi). \qquad (2.2.2)$$

For each $L$, $\tau_L$ must be implemented by a unitary operator $V_\pi(L)$ on the representation space of $\pi$, and $V_\pi(L)$ must be able to be chosen to be continuous in $L$. In other words, $\pi$ must be covariant. It must also satisfy the spectral condition.

(In fact, we shall also consider representations for which the covariance group $C$ is smaller than $P_+^\dagger$, although it must always include space-time translations, $T$. In certain models, such representations find a use as irreducible components of a fully covariant theory.) There are also other criteria a representation must satisfy to be physically interesting, but these depend on the type of charge sectors described by the theory, (in particular, the form of localization obeyed by the charge) and will be discussed later.

There is no constructive way of finding all equivalence classes of covariant representations, or all physically interesting representations of a given observable algebra. At present, it is not easy to decide

when the list of such representations is complete, although an equivalent problem is to construct a certain cohomology group of the covariance group. If we restrict attention to those representations which can be obtained by a specific construction, such as displaced Fock representations, the problem becomes tractable.

The displaced Fock representations $\pi_{\Psi_1^x}, \pi_{\Psi_2^x}$ corresponding to the functionals $\Psi_1^x, \Psi_2^x$ in $M^x$, may be specified by their action on elements of the form $W(\Phi)$ for $\Phi$ in $M$. They are equivalent if and only if the mapping

$$\pi_{\Psi_1^x}(W(\Phi)) \rightarrow \pi_{\Psi_2^x}(W(\Phi)) \,,$$

i.e.

$$e^{i \, Im \langle \Psi_1^x, \Phi \rangle} \pi_F(W(\Phi)) \rightarrow e^{i \, Im \langle \Psi_2^x, \Phi \rangle} \pi_F(W(\Phi))$$

where $\pi_F$ is the Fock representation), is unitarily implemented. By a simple application of Shale's criterion, (Shale (1962)) this holds if and only if $(\Psi_1^x - \Psi_2^x)$ is a continuous functional on $M$, i.e. exactly when $(\Psi_1^x - \Psi_2^x)$ lies in $M*$, the topological dual of $M$. (In our examples, $M*$ is just the one-particle Hilbert space $K$, in which $M$ is densely imbedded.)

Now, in order for a Poincaré transformation $L$ to be spatial in $\pi_{\Psi^x}$, the mapping

$$\tau_L : \pi_{\Psi^x}(W(\Phi)) \rightarrow \pi_{\Psi^x}(W(U(L)\Phi)),$$

or

$$e^{i \, \text{Im} \langle \Psi^x, \Phi \rangle} \pi_F(W(\Phi)) \to e^{i \, \text{Im} \langle \Psi^x, U(L)\Phi \rangle} \pi_F(W(U(L)\Phi)) \quad (2.2.3)$$

must be unitarily implemented.

If $U^x(L)$ is defined on $M^x$ by duality,

$$\langle U^x(L)\Psi^x, \Phi \rangle = \langle \Psi^x, U(L)\Phi \rangle \ ,$$

the change in phase may be written

$$i\text{Im} \langle \Psi^x - U^x(L)\Psi^x, \Phi \rangle$$

and again by Shale's criterion (2.2.3) is implemented if and only if

$$\Psi_L \equiv \Psi^x - U^x(L)\Psi^x \quad (2.2.4)$$

lies in $M^*$.

Now for any $L$, $M$ in $P_+^\uparrow$

$$U^*(M)\Psi_L = U^x(M)(\Psi^x - U^x(L)\Psi^x)$$

$$= U^x(M)\Psi^x - U^x(LM)\Psi^x$$

$$= -(\Psi^x - U^x(M)\Psi^x) + (\Psi^x - U^x(LM)\Psi^x)$$

$$= \Psi_{LM} - \Psi_M \ ,$$

i.e. the mapping defined on $P_+^\uparrow$ by $L \to \Psi_L$ satisfies

$$U^*(M)\Psi_L - \Psi_{LM} + \Psi_M = 0 \quad (2.2.5)$$

It is also continuous and therefore, in the sense of Araki (1970) it is a topological cocycle of $P_+^\uparrow$ with coefficients in $M^*$.

So far, we have established that the functionals $\Psi^x$ in $M^x$ that give rise via (2.2.4) to 1-cocycles with coefficients in $M^*$ are exactly those that also give rise to covariant representations via a GNS construction from the state $\rho_{\psi x}$, defined in (2.2.1). We now investigate the division into equivalence classes of the representations and of the cocycles. We have already established that $\Psi_1^x$ and $\Psi_2^x$ give rise to equivalent displaced Fock representations if and only if they differ by an element of $M^*$. Two 1-cocycles $\psi_{1_L}$ and $\psi_{2_L}$ are said to be equivalent cocycles, or cohomologous, if they differ by a 1-coboundary,

$$\psi_{1_L} - \psi_{2_L} = \Lambda - U^*(L)\Lambda \, , \qquad (2.2.6)$$

for some $\Lambda$ in $M^*$.

Now if $\psi_{i_L} = \Psi_i^x - U^x(L)\Psi_i^x$, $i=1,2$, are topological cocycles and if $(\Psi_1^x - \Psi_2^x)$ lies in $M^*$, then $\psi_{1_L} - \psi_{2_L}$ has the form (2.2.6) with $(\Psi_1^x - \Psi_2^x) = \Lambda$, i.e. $\psi_{1_L}$ and $\psi_{2_L}$ are cohomologous.

We might look for a converse result. If for all $L$ in $P_+^\uparrow$,

$$\psi_{1_L} - \psi_{2_L} = \Lambda - U^*(L)\Lambda \qquad (2.2.7)$$

for some $\Lambda$ in $M^*$, does it follow that $\Psi_1^x - \Psi_2^x$ lies in $M^*$? In fact, under a certain condition on $M$, this will hold.

Given (2.2.7), form the element of $M^x$, $(\Psi_1^x - \Psi_2^x - \Lambda)$. Then for all $L$ in $P_+^\uparrow$,

$$0 = \psi_{1_L} - \psi_{2_L} - (\Lambda - U^*(L)\Lambda)$$

$$= (\Psi_1^x - \Psi_2^x - \Lambda) - U^*(L)(\Psi_1^x - \Psi_2^x - \Lambda).$$

Hence $(\Psi_1^x - \Psi_2^x - \Lambda)$ is Poincaré invariant, and by choosing $M$ to lie in the kernel of all Poincaré invariant functionals on $K$, we find that

$$\langle \Psi_1^x - \Psi_2^x - \Lambda, \Phi \rangle = 0$$

for all $\Phi$ in $M$. Thus, as functionals on $M$, $\Psi_1^x - \Psi_2^x = \Lambda$, i.e. $(\Psi_1^x - \Psi_2^x)$ lies in the topological dual of $M$.

This kind of restriction on $M$ occurs in another context: it gives rise to the possibility of spontaneous symmetry breaking. This will be discussed further in the next chapter where we introduce the Streater-Wilde model.

We summarize the results of this section so far:

<u>Theorem</u>: If all $P_+^\uparrow$-invariant functionals vanish on $M$, then there is a 1-1 correspondence between the equivalence classes of displaced Fock representations in which the Poincaré transformations are implemented and the first cohomology group of the Poincaré group with coefficients in $M^*: H^1(P_+^\uparrow, M^*)$.

Araki's treatment of topological cocycles provides a characterization of the subspace of $M^x$ whose elements $\Psi^x$ give rise to cocycles via

$$\Psi^x \rightarrow \psi : L \rightarrow \psi_L = \Psi^x - U^x(L)\Psi^x .$$

(Araki 1970). In the course of a more general discussion of factorizable representations of current algebras, he shows that to give rise to a 1-cocycle of $C$ with coefficients in $M*$, $\Psi^x$ must lie in the subspace $\bar{D}^+(C)$ of $M^x$, defined as follows. For each generator, $\ell$, of $C$, (i.e. for each $\ell$ in the basis of the corresponding Lie algebra) let

$$K(\ell) = 1 - \int h(t) e^{i t \ell} dt \qquad (2.2.8)$$

where $h(t)$ is a function in $D$ whose Fourier transform

$$\tilde{h}(\lambda) = \int e^{i \lambda t} h(t) dt$$

satisfies

$$\tilde{h}(0) = 1, \; 1 > \tilde{h}(\lambda) \geqslant 0, \; \lambda \neq 0$$

and $\tilde{h}''(0) \neq 0$ . $\qquad (2.2.9)$

Let $K = \Sigma K(\ell)$. Then $D^+(C)$ is the range of the operator $K^{-\frac{1}{2}}$ acting on $K$, and $\bar{D}^+(C)$ is its closure in the norm $\| . \|_+$ defined by

$$\|\varphi\|_+ = \| K^{\frac{1}{2}} \varphi \|_K . \qquad (2.2.10)$$

In general, in addition to topological cocycles, (those which are constructed from elements $\Psi^x$ of $\bar{D}^+$ via (2.2.4)), there may also be algebraic cocycles. Since the representation of any covariance group containing $T$ in the one-particle space never contains

the trivial representation, all cocycles in $Z^1(C, M^*)$ are topological (Araki 1970, Lemma 7.2). In fact from Araki's theorem 7.3 it also follows that for any cocycle of $C, \psi^x$ can be chosen to lie in $\bar{D}^+(T)$, since $T$ is an invariant abelian subalgebra of $C \subset P_+^\uparrow$.

The covariance of the displaced Fock representation corresponding to the topological cocycles can be demonstrated constructively by checking that the Poincaré transformations $W(\Phi) \rightarrow W(U(L)\Phi)$ are implemented by operators $V_{\psi x}(L)$ defined by

$$V_{\psi x}(L) = V_F(L) W_F(-\psi_L) \qquad (2.2.11)$$

where the $V_F(L)$ are the operators implementing Poincaré transformations in the Fock representation. Thus using the Weyl multiplication rule (2.5) we have

$$V_{\psi x}(L) W_{\psi x}(\Phi) V_{\psi x}(L)^{-1}$$

$$= V_F(L) [W_F(-\psi_L) e^{i \operatorname{Im} \langle \psi^x, \Phi \rangle} W_F(\Phi) W_F(\psi_L)] V_F(L)^{-1}$$

$$= e^{i \operatorname{Im} \langle \psi^x, \Phi \rangle} e^{\frac{1}{2}i \operatorname{Im} \langle -\psi_L, \Phi \rangle} e^{\frac{1}{2}i \operatorname{Im} \langle \Phi - \psi_L, \psi_L \rangle} V_F(L) W_F(\Phi) V_F(L)^{-1}$$

$$= e^{i \operatorname{Im} \langle \psi^x, U(L)\Phi \rangle} W_F(U(L)\Phi) = W_{\psi x}(U(L)\Phi). \qquad (2.2.12)$$

Provided $L \rightarrow \psi_L$ is a continuous mapping, $L \rightarrow V_{\psi x}(L) = V_F(L) W_F(-\psi_L)$ will also be continuous, since $V_F$ is, and hence $V_{\psi x}$ is a continuous multiplier representation of $P_+^\uparrow$ :

$$V_{\psi x}(L) V_{\psi x}(M) = e^{-\frac{1}{2}i \operatorname{Im} \langle \psi_L, \psi_{M-1} \rangle} V_{\psi x}(LM) . \qquad (2.2.13)$$

Conversely, continuity of $V_{\psi x}(L)$ implies that

$$\text{Re } \log\langle \Omega_F, V_{\psi x}(L)\Omega_F\rangle = \tfrac{1}{4}\| \psi_{L^{-1}} \|^2$$

is continuous in $L$.  Hence for $L, M \in P_+^\uparrow$ ,

$$\| \psi_L - \psi_M \| = \| \psi_{LM^{-1}{}_M} - \psi_M \|$$

$$= \| U^*(M)\psi_{LM^{-1}} \| \quad \text{by the cocycle condition}$$
$$(2.2.5),$$

$$= \| \psi_{LM^{-1}} \|$$

$$\to 0 \quad \text{as} \quad L \to M.$$

We have shown how cohomological techniques can be used to characterize the displacements giving rise to representations in which the covariance group is unitarily implemented.  So far, we have not mentioned the other axioms required of a representation in a DHR field theory:  in particular, the spectral condition, locality requirements, and the relationship between the inequivalent representations and the gauge structure of the field theory.  In section 2.3, the more general description of cocycles as charge transfer operators will lead to the formulation of localization requirements; the connection with gauge structure will be developed through examples in later chapters.  In the case of displaced Fock representations, we now look into the requirement that the spectral condition should hold.

Let $\hat{\varphi}^F$ and $\hat{\pi}^F$ be the Fock field and conjugate momentum. The transformation to the displaced Fock representation corresponding to $\Phi^x$ in $\bar{D}^+$ carries these operators to $\hat{\varphi}^F + \Phi^x$, $\hat{\pi}^F + \Pi^x$, on a domain on which all operators involved make sense.

For simplicity, we consider a scalar field with Hamiltonian

$$H(\varphi, \pi) = \tfrac{1}{2}\int d^3x \, [\pi(\underline{x}, t)^2 + \nabla\varphi(\underline{x}, t)^2 + m^2\varphi(\underline{x}, t)^2] \quad (2.2.14)$$

Then the Fock Hamiltonian is given by

$$H^F = \, :H(\hat{\varphi}^F, \hat{\pi}^F): \, , \quad\quad\quad (2.2.15)$$

and at least formally, the Hamiltonian in the new representation may be expected to have the form

$$H(\hat{\varphi}^F + \Phi^x, \hat{\pi}^F + \Pi^x) = H^F + H(\Phi^x, \Pi^x) + C \quad\quad (2.2.16)$$

where

$$C = \int [\hat{\pi}^F(\underline{x}, t)\Pi^x(\underline{x}, t) + \nabla\hat{\varphi}^F(\underline{x}, t)\nabla\Phi^x(\underline{x}, t)$$
$$+ m^2\hat{\varphi}^F(\underline{x}, t)\Phi^x(\underline{x}, t)] d^3x. \quad (2.2.17)$$

In order for (2.2.16) to make sense, all the operators involved must have a common domain. So it is necessary that $H(\Phi^x, \Pi^x)$, the classical energy of $\Phi^x$, should be finite. Now $H^F$ is self-adjoint and bounded below on its domain. We now discuss what can be said about C.

Using integration by parts, and the fact that $\Phi^x$ satisfies the wave equation, we obtain

$$C = \int [\hat{\pi}^F (\underline{x},t) \Pi^x (\underline{x},t) - \hat{\varphi}^F (\underline{x},t) (\frac{\partial^2}{\partial t^2} + m^2) \Phi^x (\underline{x},t)$$

$$+ m^2 \hat{\varphi}^F (\underline{x},t) \Phi^x (\underline{x},t)] d^3 x$$

$$= \int [\hat{\pi}^F (\underline{x},t) \Pi^x (\underline{x},t) - \hat{\varphi}^F (\underline{x},t) \frac{\partial^2}{\partial t^2} \Phi^x (\underline{x},t)] d^3 x$$

$$= \int [\hat{\pi}^F (\underline{x},t) \Pi^x (\underline{x},t) - \hat{\varphi}^F (\underline{x},t) \frac{\partial}{\partial t} \Pi^x (\underline{x},t)] d^3 x. \quad (2.2.18)$$

Now since $\Phi^x$ is an element of $\bar{D}^+$, and so gives rise to a cocycle taking values in $K$, we have $(1-U(t)) \Phi^x \in K$. The infinitesimal form of this condition is

$$\frac{\partial}{\partial t} \Phi^x \in K$$

i.e.

$$\Pi^x \in K.$$

This infinitesimal form will hold if $\Phi^x$ is analytic for the group generators, and this can always be arranged. (Pinczon and Simon, 1975. We quote their exact results in chapter 3, where they will be used to prove properties of a specific model.) Then $(\Pi^x (\underline{x},0), \dot{\Pi}^x (\underline{x},0))$ are the Cauchy data on the surface $t = 0$ for an element of $K$. (2.2.18) has the correct form for the self adjoint generator $R(\Pi^x)$ of the one-parameter unitary group $W(\lambda \Pi^x)$, as defined in (2.1.8).

Now we use Kato's criteria for $H^F + C$ to be self adjoint and bounded below when $H^F$ is self adjoint

and bounded below.  It is sufficient that  C   should
be symmetric and Kato small with respect to  $H^F$,   i.e.

$$\| C\Psi \| \leqslant a\| H^F \Psi \| + b\| \Psi \| \quad \text{for all} \quad \Psi \quad \text{in the domain of} \quad H^F$$

for some  a, b,  with  $0 < a < 1, 0 < b < \infty$ .

(This criterion follows from the Kato-Rellich theorem,
e.g. theorem  X.12  of Reed and Simon (1975).)

For the case of positive mass, the result that  C
is in fact Kato small with respect to  $H^F$  is shown by
Emch, (1972) for an operator essentially the same as  C.   For
the case  m = 0,  the result is contained in Cook (1961).

## 2.3  Charge transfer cocycles

The additive 1-cocycles which have provided a classification of displaced Fock representations can be set in correspondence with certain charge transfer cocycles of the kind described, for example, in Doplicher, Haag and Roberts (1971).

Consider a free field of given mass and spin, and suppose $\Psi^x$ is a displacement giving rise to a sector with covariance group $C$. (As before, $T \subseteq C \subseteq P_+^\uparrow$.) In other words, $\Psi^x$ lies in $\bar{D}^+(C)$.

For $L$ in $C$, let $V_F(L)$ and $V(L)$ be unitary operators implementing the action $\tau_L$ of $L$ on $\mathcal{O}$, in $\pi_F$ and $\pi_{\Psi^x}$ respectively, with $V_F$ a true representation. Then by (2.2.12), if

$$\psi_L = \Psi^x - U^x(L)\Psi^x ,$$

then

$$V(L) = e^{i\,\lambda(L)}V_F(L)W_F(-\psi_L) \qquad (2.3.1)$$

where $\lambda(L)$ is an arbitrary real number. The condition that $V(L)$ should be a true representation of the covariance group is

$$\lambda(L) + \lambda(M) - \lambda(LM) = \tfrac{1}{2}\mathrm{Im}\langle\psi_L, \psi_{M^{-1}}\rangle \qquad (2.3.2)$$

Using $U(L)\psi_L = -\psi_{L^{-1}}$, (2.3.1) can be rewritten

$$V(L) = e^{i\,\lambda(L)}W_F(\psi_{L^{-1}})V_F(L) \qquad (2.3.3)$$

$$= \Gamma(L)V_F(L), \quad \text{where} \quad \Gamma(L) = e^{i\,\lambda(L)}W_F(\psi_{L^{-1}}) .$$

If (2.3.2) holds, then from the fact that the map
$L \to \psi_L$ satisfies the additive cocycle identity (2.2.5),
we obtain

$$\Gamma(L)\tau_L(\Gamma(M)) = \Gamma(LM) \qquad (2.3.4)$$

where

$$\tau_L(.) = V_F(L).V_F(L)^* ,$$

which is the 1-cocycle identity in multiplicative form.
This also follows directly from the fact that $V_F$ and
$V$ are true representations. The operators $\Gamma(L)$ act
on the representation $V_F$ in the vacuum sector to give
rise to the new representation $V$ of the covariance
group. In the case where $V$ is a multiplier representation,

$$V(L)V(M) = e^{i\,\omega(L, M)}V(LM), \quad \text{say}, \qquad (2.3.5)$$

we have

$$\Gamma(L)\tau_L(\Gamma(M)) = e^{i\,\omega(L, M)}\Gamma(LM). \qquad (2.3.6)$$

For the present, we shall assume that all representations
are true, so that the identity (2.3.4) holds.

By finding equivalence classes of these multiplicative
cocycles which correspond to equivalence classes of
representations, we shall now show that the cohomological
classification of displaced Fock representations is an
example of a structure that may be used to classify
more general superselection sectors. We shall also find

that certain multiplicative 1-cocycles for the space-time translation group ($\Gamma$ restricted to $T$ is an example) with coefficients in the unitary operators of $\mathcal{A}$, $U(\mathcal{A})$, have a natural interpretation as charge transfer operators.

Doplicher, Haag and Roberts (1974, equation 2.3) define cocycles corresponding to a given representation $\pi$ in the way we have described them: as the "difference" between the representation of the covariance group in the representation spaces of $\pi$ and of the vacuum representation $\pi_0$. (Essentially, in our notation, they define an operator $\Gamma_\pi(L) = V_\pi(L)V_{\pi_0}(L)^{-1}$.) The properties they require of the representation $\pi$ should lead to some restrictions on the cocycles $\Gamma_\pi$. In order to pursue the study of $\Gamma_\pi$ from this point of view, we first review their characterization of "representations interesting for particle physics". (This review is a selection of results from the four papers of Doplicher, Haag and Roberts, 1969a,b, 1971, 1974.) Certain aspects of the discussion given here apply only in more than one space-dimension. The 1+1-dimensional case has many qualitatively different features which will be reviewed in chapter 3.

Our starting point is a local field theory satisfying the assumptions 1-6 described at the beginning of the chapter. If, in addition, strict duality holds, then any representation $\pi$ in such a theory is strongly locally equivalent to the vacuum representation $\pi_0$. In other words,

$$\pi \Big|_{\alpha \, (O')} = \pi_0 \Big|_{\alpha \, (O')} \qquad\qquad (2.3.7)$$

for sufficiently many double cones $O$. The set of double cones for which (2.3.7) holds must contain all translates of some fixed double cone.

For convenience, we refer all representations of interest to the representation space $\mathcal{H}_0$ of $\pi_0$. Then $\pi$ is characterized by a morphism $\rho$ for which

$$\pi = \pi_0 \circ \rho \qquad\qquad (2.3.8)$$

Suppose $\pi$ is a representation acting on $\mathcal{H}_\pi$ which is strongly locally equivalent to $\pi_0$, and $O$ and all its translates $O(x)$ are cones for which (2.3.7) holds. The condition (2.3.7) for $O$ is equivalent to the condition that there exists a <u>localized morphism</u> $\rho$ of $\alpha$ with localization region $O$ (i.e. $\rho(A) = A$ for $A \in \alpha \, (O')$), such that (2.3.8) holds. (Doplicher, Haag and Roberts 1971, proposition 1.2.) Hence for each $x, y$, there exist localized morphisms $\rho_x, \rho_y$ with localization regions $O(x)$, $O(y)$ for which

$$\pi \cong \pi_0 \circ \rho_x \cong \pi_0 \circ \rho_y \, .$$

Two localized morphisms $\rho_1$ and $\rho_2$ are said to be equivalent if $\pi_0 \circ \rho_1$ and $\pi_0 \circ \rho_2$ are unitarily equivalent. This holds exactly when

$$\rho_1 = \sigma\rho_2 \qquad \qquad (2.3.9)$$

for $\sigma$ lying in $I$, the set of inner automorphisms which can be implemented by a unitary operator in some $\mathcal{O}\iota(0)$. So for all x,y, there exists a $\sigma_{xy} \in I$ such that

$$\rho_x = \sigma_{xy}\rho_y \quad .$$

To summarize, for each representation $\pi$ which is "interesting for particle physics", there is a "locally transportable" morphism $\rho$, with localization cone $0$ (i.e., for each x, there is an equivalent morphism with localization cone $0(x)$.)

The set of locally transportable morphisms with localization cone $0$ is denoted by $\Delta_t(0)$, and the union of these sets is called $\Delta_t$. By (2.3.9), equivalence classes of these morphisms are given by $\Delta_t/I$. These equivalence classes are in one to one correspondence with the unitary equivalence classes of physically interesting representations.

In a theory in which the gauge group is abelian, $\pi_0 \circ \rho$ can always be chosen to be irreducible; if only standard Bose and Fermi statistics are allowed, $\rho$ will always be an automorphism.

Thus, Doplicher, Haag and Roberts describe the properties of interesting representations in terms of corresponding morphisms. They find (1971, section III) that a locally transportable morphism $\rho$, with localization cone $0$ can be written in the form

$$\rho(A) = \lim_{k \to \infty} U_k^* \, A \, U_k \qquad\qquad (2.3.10)$$

where for each $k$, $U_k$ is a unitary in $[\alpha(0_k') \cap \alpha(0')]'$ (this must be modified in $1+1$ dimensions. See section 3.3) and $0_k$ is a sequence of translates of $0$ which are eventually spacelike to any given bounded double cone. ($U_k$ is just the local unitary which must implement $\rho \to \rho_k$, where $\rho_k$ is localized in $0_k$.)

This is interpreted as follows. $U_k$ implements charge transfer from $0_k$ to $0$, and the limit (2.3.10) is the morphism corresponding to the transfer of charge from infinity to $0$, and hence the transformation from the vacuum sector $\pi_0$ to the charged sector $\pi = \pi_0 \circ \rho$.

We shall now use these results in order to translate the properties which Doplicher, Haag and Roberts formulate in terms of the morphism corresponding to a given representation into conditions which must be satisfied by the 1-cocycle corresponding to the representation.

In a relativistic theory, the requirement that the morphism should be locally transportable is subsumed in the requirement that $(\pi_0 \circ \rho)$ should carry a representation $V_\pi$ of the Poincaré group. So in particular, if $x \in T$,

$$(\pi_0 \circ \rho)(A) \to (\pi_0 \circ \rho)(\tau_{-x}(A))$$

is unitarily implemented. Also, from the covariance

of $\pi_0$,

$$\pi_0(\rho \circ \tau_{-x}(A)) \to \pi_0(\tau_x(\rho \circ \tau_{-x})(A))$$

is unitarily implemented. Now $(\tau_x \circ \rho \circ \tau_{-x})$ is a local morphism, since firstly

$$(\rho \circ \tau_{-x})(A) = \tau_{-x}(A) \quad \text{for} \quad \tau_{-x}(A) \in \mathcal{O}t(0'),$$

where $0$ is the localization cone of $\rho$. Also, $\tau_{-x}(A) \in \mathcal{O}t(0')$ is equivalent to the requirement that $A \in \mathcal{O}t(0(x)')$. So for $A \in \mathcal{O}t(0(x)')$,

$$(\tau_x \circ \rho \circ \tau_{-x})(A) = A. \qquad (2.3.11)$$

Thus $\rho_x \equiv \tau_x \circ \rho \circ \tau_{-x}$ is a local morphism equivalent to $\rho$, with localization cone $0(x)$.

If $V_0$ and $V_\pi$ implement the covariance group in $\pi_0$ and $\pi$ respectively, then

$$\pi_0 \circ (\tau_x \circ \rho \circ \tau_{-x})(A) = V_0(x)\pi_0((\rho \circ \tau_{-x})(A))V_0(x)^{-1}$$

$$= V_0(x)\pi(\tau_{-x}(A))V_0(x)^{-1}$$

$$= V_0(x)V_\pi(-x)\pi(A)V_\pi(-x)^{-1}V_0(x)^{-1}$$

$$= V_0(x)V_\pi(x)^*\pi(A)V_\pi(x)V_0(x)^*$$

$$= \Gamma_\pi(x)^*\pi(A)\Gamma_\pi(x) \qquad (2.3.12)$$

where $\Gamma_\pi(x) = V_\pi(x)V_0(x)^*$. If $V_0$ and $V$ are true representations, $\Gamma_\pi(x)$ satisfies the cocycle identity

(essentially (2.3.4)) and it is the 1-cocycle corresponding
to $\pi$.

If $A \in \alpha(0') \cap \alpha(0(x)')$ then $\rho(A) = \rho_x(A) = A$,
and so from (2.3.12), $\Gamma_\pi(x)$ commutes with $A$. Thus
$\Gamma_\pi(x)$ lies in $[\alpha(0') \cap \alpha(0(x)')]'$, and is said to
have localization cone $0$.

From (2.3.11) and (2.3.12), for any fixed $A$ in
a local algebra $\alpha(0_1)$, $0_1$ some bounded double cone,

$$\lim_{|x| \to \infty} \Gamma_\pi(x) \pi_0(A) \Gamma_\pi(x)^* = (\pi_0 \circ \rho)(A) \qquad (2.3.13)$$

where the limit is taken in such a way that $0(x)$ is
eventually spacelike to any given bounded double cone.
Thus $\rho$ can be recovered from a knowledge of $\Gamma_\pi$,
and any unitary parametrized by $x$, with the same
localization as $\Gamma_\pi(x)$ corresponds to the local morphism
of a space-time translation covariant representation.

(2.3.13) is again interpreted as charge transfer
from infinity, and $\Gamma_\pi$ is called a charge transfer
cocycle.

We now seek a definition of equivalence of cocycles
which coincides with unitary equivalence of the
corresponding representations.

For any unitary operator, $V$ in $\alpha$, the
operator $\Gamma(x) = V^{-1} \tau_x(V)$ satisfies the cocycle
identity (2.3.4). Now if $V$ lies in $\alpha(0)$ for some
bounded double cone $0$, $\Gamma(x)$ is a local cocycle with
localization cone $0$.

The unitary operator  U,  say, which intertwines the representations, must be able to be chosen to lie in  $\alpha(0')'$,  i.e. in  $\alpha(0)$,  for some  $0$,  since both representations satisfy (2.3.7); and  U  gives rise to the coboundary relating the cocycles of the representations.

This description is essentially equivalent to that which Roberts (1976) uses to show the relationship between charge transfer and local  1-cocycles.  He uses more standard cohomology notation, which relates more naturally to the generalizations needed in a theory involving gauge transformations of the second kind.

Consider first the operation of moving a given charge along a finite path,  b,  from points  $a_1$  to  $a_2$  in Minkowski space.  This operation should be strictly localized about the path  b,  should not change the total charge, and the final state should not depend on the path taken.  These requirements are tailored to strictly local theories which fit into the Doplicher, Haag, Roberts scheme; for example, to include electric charge, the localization condition must be modified to be consistent with Gauss' law, and complete path independence can no longer be required.

Since the total charge is unchanged, the operation gives rise to an equivalent sector, and must be implemented by a unitary operator,  z(b),  say.  (In

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fact, Roberts defines his cocycles to take values
among the inner automorphisms of $\mathcal{A}$. We find it
more convenient to work with the unitary operators
which implement these automorphisms.) The localization
is expressed by requiring $z(b)$ to lie in the algebra
$\mathcal{A}(b+O)$ for some bounded double cone $O$, where the
set $b+O$ is defined by

$$b+O = \{x+a : a \text{ lies on } b \text{ and } x \in O\} .$$

Path independence is equivalent to the requirement
that charge transfer along a path $b_2$ from $a_0$ to $a_1$
followed by transfer along $b_0$ from $a_1$ to $a_2$ is
equivalent to charge transfer along any path $b_1$ from
$a_0$ to $a_2$. In the language of cohomology, the paths
$b_0$, $b_1$ and $b_2$ determine a 2-simplex $c$ in Minkowski
space, $\mathbb{R}^{s+1}$, $c:\Delta^2 \to \mathbb{R}^{s+1}$, where

$$\Delta^2 = \{(t_0,t_1,t_2) \in \mathbb{R}^3 : t_i \geqslant 0, \sum_{i=0}^{2} t_i = 1\} .$$

The paths $b_i$ are given by $b_i = \partial_i c$, and the above
requirement becomes

$$z(\partial_1 c) = z(\partial_0 c) z(\partial_2 c), \qquad (2.3.14)$$

which is just the 1-cocycle condition.
($\partial_i$ is defined by $(\partial_i f)(t_0 \ldots t_{n-1}) = f(t_0 \ldots t_{i-1}, 0, t_i \ldots t_{n-1})$
where $f$ is an $n$-simplex.)

We now take $\Gamma(x)$ to be $z(b)$ where $\partial_0 b$ is the
origin of space-time and $\partial_1 b = x$. Then $\tau_x(\Gamma(y))$

corresponds to $z(b')$ where $\partial_0 b' = x$ and $\partial_1 b' = y+x$.
Then (2.3.4), with L and M given by translation
through x and y respectively, corresponds exactly
to (2.3.14).

We now define a local 1-cocycle to be a mapping
z, from the set of paths in Minkowski space to $\mathcal{U}(\mathcal{O}\!\!\!l)$
such that for some bounded double cone $\mathcal{O}$, and for all
paths b, z(b) lies in $\mathcal{O}\!\!\!l(b+\mathcal{O})$, and (2.3.14) holds
for all 2-simplexes c with values in Minkowski space.
A 1-coboundary is a mapping w from 1-simplexes (paths)
in Minkowski space to $\mathcal{U}(\mathcal{O}\!\!\!l)$ for which there is a
function $y: \mathbb{R}^{s+1} \to \mathcal{U}(\mathcal{O}\!\!\!l)$ such that for every path b,
$w(b) = y(\partial_0 b) y(\partial_1 b)^{-1}$. If y can be chosen such that
y(a) lies in $\mathcal{O}\!\!\!l(a+\mathcal{O})$ for each point a, then w is
a local 1-coboundary. Since Minkowski space is
contractible, all 1-cocycles are 1-coboundaries. (For
a proof, see, for example, Roberts 1976, lemma A2.)
However, local 1-cocycles need not in general be local
1-coboundaries. The cohomology class containing a given
1-cocycle z is defined to consist of all local
1-cocycles z' which satisfy

$$z(b)y(\partial_1 b) = y(\partial_0 b)z'(b)$$

for some $y: \mathbb{R}^{s+1} \to \mathcal{U}(\mathcal{O}\!\!\!l)$, such that y(a) lies in
$\mathcal{O}\!\!\!l(a+\mathcal{O})$ for each point a.

Since all 1-cocycles are 1-coboundaries, we
deduce in particular that all local 1-cocycles have
the form

$$z(b) = y(\partial_0 b) y(\partial_1 b)^{-1}$$

(where there is now no restriction on the localization of y.) It follows that if b' is a path with endpoints $\partial_i b' = \partial_i b$, i=0,1, then

$$z(b) = z(b'). \qquad (2.3.15)$$

In more than 2 space-time dimensions, this property can be used to localize z(b) more precisely. (Roberts 1976, lemma 2.1.)

Suppose $O_1$ is a bounded double cone such that

$$O_1 \subset (O + \partial_0 b)' \cap (O + \partial_1 b)' \, ,$$

where $O$ is the localization cone of z. Then $\partial_0 b$ and $\partial_1 b$ lie in the causal complement of the set

$$O_1 + O \overset{\text{def}}{=} \underset{x_1 \in O_1}{\cup} (x_1 + O).$$

Since this is a bounded region, then in more than 2 space-time dimensions, its causal complement, $(O_1 + O)'$ is path connected. Thus there is a path b' with $\partial_0 b' = \partial_0 b$, $\partial_1 b' = \partial_1 b$, lying entirely in $(O_1 + O)'$. Now by definition of z as a local cocycle, $[z(b'), A] = 0$ for $A \in \mathcal{O}(O_1)$. But since the endpoints of b and b' are the same, (2.3.15) holds, and $[z(b), A] = 0$. That is,

$$z(b) \in [\mathcal{O}((O + \partial_0 b)' \cap (O + \partial_1 b)')]' \quad (2.3.16)$$

$$= \mathcal{O}((O + \partial_0 b) \cup (O + \partial_1 b))$$

since we are assuming strict duality.

So far, we have discussed a type of local cohomology which corresponds to strictly local field theories. Less stringent localization requirements give rise to different versions of local cohomology, and the examples considered in later chapters use some of these.

Leyland and Roberts (1978) discuss the techniques necessary for calculating local cohomology groups. Although these groups fit into the definition of sheaf cohomology, considerable generalization of the techniques so far available will be needed to calculate the groups which are relevant for field theory. Leyland and Roberts have done analogous calculations for cohomology groups whose coefficients lie in an abelian group, rather than the non-abelian algebra which arises in field theory. If the cohomology groups which arise in field theory can be calculated, then the correspondence with superselection sectors will lead to a major break-through in the problem of finding all physically relevant representations of a field algebra.

In section 2.2, where the construction of displaced Fock representations was discussed, the local algebra structure of $\mathfrak{a}$ was not mentioned, and so the localization properties of the cocycle corresponding to $\psi^x$ in $\bar{D}^+$ could not be established. The local structure of $\mathfrak{a}$ will be specified in the models involving displaced Fock representations in later chapters.

The cocycles $\psi_L = \Psi^x - U^x(L)\Psi^x$, where $\Psi^x \in \bar{D}^+$

correspond via (2.3.3) to multiplicative cocycles $\Gamma_{\Psi^x}$

which will certainly be localized, if not strictly, at

least in an asymptotic sense, since $\psi_L$ must be an

$L^2$ function.

To check that the theorem in section 2.2. on the

classification of displaced Fock representations is a

particular case of the more general description of

superselection sectors in terms of local cohomology,

we must explain the requirement that $M$ should lie in

the kernel of Poincaré invariant functionals on $K$.

This condition on $M$ amounts to identifying all vacuum

representations, which are related by the spontaneously

broken symmetry

$$\varphi \to \varphi + \eta, \quad \eta \quad \text{constant.}$$

(This point will be discussed further in chapter 3,

where we consider the Streater-Wilde model.)

The Doplicher, Haag, Roberts approach requires us

not to distinguish between these representations, as

their picture of elementary particle physics calls for

one vacuum representation, and charged sectors obtained

from the vacuum by localized morphisms. As the

automorphisms which give rise to vacuum sectors are

not localized, they cannot be described by local charge

transfer cocycles, so we expect the local cohomology

group classification to apply only to charged sectors.

CHAPTER 3

1+1 - DIMENSIONAL MODELS

Many of the special properties of models in
2-dimensional space-time are due to the separation of
spacelike left and right infinity. Because of this,
in a model involving broken symmetry and multiple
vacua, there is an obvious way to create inequivalent
sectors, by constructing states which become
indistinguishable from the vacuum at spacelike infinity,
but tend to different vacua at spacelike left and
spacelike right. Then the energy of the state is
localized and may be finite, but there is no vacuum
state from which it is obtained by a small perturbation.
If there are $n$ vacua, $n(n-1)$ "soliton" sectors may
be obtained in this way. They are sometimes referred
to as topological soliton sectors.

Models which involve sectors of this type include
the Streater-Wilde model (Streater and Wilde, 1970)
which is built up from displaced Fock representations
of the massless scalar boson field in 2 dimensions, and
the $P(\varphi)_2$ model with potential

$$g\varphi(x)^4 - \tfrac{1}{4}\varphi(x)^2 + 1/(64g) .$$

Soliton sectors of this model have been introduced by
Streater (1976), and Fröhlich (1976, 1977).

In higher dimensions, nontrivial classical field
configurations at infinity, related to the multiple

vacua of a broken symmetry, are labelled by a homotopy group. In two dimensions the homotopy classification just reduces to the choice of different values at left and right infinity described above.

The homotopy classification in higher dimensions may also suggest certain non-vacuum sectors of the quantum theory, and form a subset of the local cohomology classification. The interpretation of certain solutions in 3+1 dimensions as monopoles anticipates such a development in the quantum theory.

The classical $P(\varphi)_2$ theory with potential (3.1) has "topological" soliton solutions, and the soliton sectors of the quantum theory correspond to these. In all the other examples which we shall consider, the superselection sectors will be inequivalent representations of the free field, appropriate to the study of an approximate quantum theory, obtained by introducing a classical external field. The sectors of the $P(\varphi)_2$ model give some indication of the extent to which the same structure might occur in a true interacting theory. However it must be borne in mind that the structure of this model is not typical. In particular, the super-selection structure of field theories in 1+1 dimensions is different in many respects from those in higher dimensions.

We first review the Streater-Wilde model, which will share the features characteristic of topological superselection sectors in 1+1 dimensions.

This demonstrates how displaced Fock representations
share the structural features of true non-linear
theories.  We then examine the general structure of
models in this number of dimensions, illustrating
certain aspects by reference to the $P(\varphi)_2$ model
mentioned above.

### 3.1 The Streater-Wilde Model

In 1+1 dimensions, there are non-trivial equivalence classes of Poincaré-covariant displaced Fock representations corresponding to the free massless scalar boson field, whose field equation is

$$\Box \varphi = 0 \qquad (3.1.1)$$

Solutions $\xi$ of this equation may be specified by their Cauchy data $(f,g)$, where

$$f(x) = \dot{\xi}(x,0),$$
$$\qquad \qquad (3.1.2)$$
$$g(x) = \xi(x,0).$$

We consider solutions whose Cauchy data are smooth functions of compact support: $f,g \in \mathcal{D}$. In order to avoid the infrared problem, we also impose the condition

$$\int \dot{\xi}(x,0) \ dx = 0. \qquad (3.1.3)$$

The subset of $\mathcal{D}$ determined by this requirement is denoted $\mathcal{D}_0$. Thus we consider a class $M$ of solutions of (3.1.1) which may be identified with $\mathcal{D}_0 \times \mathcal{D}$.

The one-particle space carrying the mass zero, spin zero representation of the Poincaré group in 1+1 dimensions may be realized as

$$K = L^2(\mathbb{R}, \frac{dp}{|p|}) \qquad (3.1.4)$$

and $M$ is identified with a dense subspace of this Hilbert space, by the correspondence

$$\xi = (f,g) \in M \leftrightarrow \tilde{\psi} \in K ,$$

such that

$$\tilde{\psi}(p) = |p|^{\frac{1}{2}}\tilde{g}(p) - i|p|^{-\frac{1}{2}}\tilde{f}(p) . \qquad (3.1.5)$$

Thus $M$ is a pre-Hilbert space with

$$\text{Im}\langle \xi_1, \xi_2 \rangle = \int [\dot{\xi}_1(x,0)\xi_2(x,0) - \xi_1(x,0)\dot{\xi}_2(x,0)]dx. \quad (3.1.6)$$

The condition (3.1.3) amounts to regarding $\nabla\varphi$, and not $\varphi$ itself, as observable, or regarding descriptions of the field related by the transformation

$$\varphi \rightarrow \varphi + \eta, \quad \eta \quad \text{constant,}$$

as indistinguishable. It is therefore just the restriction needed to ensure that all Poincaré invariant functionals vanish on $M$, so we may expect the one-to-one correspondence described in section 2.2 to hold.

Streater and Wilde construct the Fock representation $\pi_F$ acting on $\mathcal{H}_F$, say, of the Weyl algebra $\mathcal{A}$ over $(M, \text{Im}\langle \cdot, \cdot \rangle)$, and find inequivalent sectors of the topological kind mentioned at the beginning of this chapter. They are obtained as displaced Fock representations, where the displacements $\theta^x$ lie in the set $N$ of waves which satisfy (3.1.1) and $\dot{\theta}^x \in M$. It is easy to check that such waves have finite energy and belong to $\bar{D}^+$.

Since the addition of a global constant is not observable, $\theta^x(-\infty, 0)$ can be set to zero without loss of generality. The equivalence classes of sectors obtained by a displacement of this kind may be labelled

by two parameters $(\alpha, \beta)$, which can be chosen in various ways. For example, for $\theta^x \in N$, we may write

$$\theta^x(\underline{x}, t) = \theta_1(\underline{x}+t) + \theta_2(\underline{x}-t)$$

where $\theta_i(-\infty) = 0$, and $\dfrac{d\theta_i}{dx} \in \mathcal{D}$, $i=1,2$.

Then $\alpha$ and $\beta$ may be chosen as

$$\alpha = \theta_1(\infty),$$
$$\beta = \theta_2(\infty).$$

The equivalence class of the displaced Fock representation corresponding to the displacement $\theta^x$ may be denoted by $\pi_{\alpha\beta}$. We denote the elements generating the Weyl algebra $\mathcal{A}$ by $W(\xi)$, for $\xi$ in $M$. For convenience of notation, we also write

$$\pi_F(W(\xi)) = W_F(\xi),$$

$$\pi_{\alpha\beta}(W(\xi)) = W_{\alpha\beta}(\xi).$$

A local structure can be defined on $M$, and hence on $\mathcal{A}$ and $N$. Given a bounded double cone $0$ in $\mathbb{R}^2$, $\xi$ lies in $M(0)$ if there exists a spacelike line

$$\ell_{v, t_0} = \{(x,t) : t=x/v+t_0, \text{ for some } v, t_0 \in \mathbb{R}, |v| > 1\},$$

and an interval

$$I = \{(x,t) \in \ell_{v, t_0} : x_1 \leqslant x \leqslant x_2\}$$

such that $I \subset 0$, and $(\dot{\xi}, \xi)\Big|_{\ell_{v, t_0}}$ is zero outside $I$.

Then $\mathcal{A}(0)$ is the von Neumann algebra generated by $\{W(\xi): \xi \in M(0)\}$, and $\mathcal{A}$ is the c*-algebra generated by $\underset{0}{\cup} \mathcal{A}(0)$. This is a consistent definition of the localization of $W(\xi)$, because although there are many cones $0$ to which it can be assigned, they are all causally connected.

We define

$$N(0) = \{\Phi^x \in N: \dot{\Phi}^x \in M(0)\}.$$

Then the charge transfer cocycle corresponding to $\Psi^x \in N(0)$ via (2.3.3) is a local cocycle with localization cone $0$.

Representations $\pi_{\alpha\beta}$ corresponding to different parameters will be inequivalent, but each is strongly locally equivalent to the Fock representation.

From the discussion of displaced Fock representations in chapter 2, we find that the charge transfer operator $\Gamma_{\alpha\beta}$ corresponding to the sector $\pi_{\alpha\beta}$ is given by

$$\Gamma_{\alpha\beta}(x) = W_F(\eta(-x))$$

where $\eta(L) = \Theta^x - U^x(L)\Theta^x$.

The limiting procedure, (corresponding to (2.3.13)), for obtaining $\pi_{\alpha\beta}(A)$ from $\pi_F(A)$, $A \in \mathcal{A}$, will be

$$\pi_{\alpha\beta}(A) = \lim_{\substack{x \to \infty \\ \text{spacelike}}} \Gamma_{\alpha\beta}(x) \pi_0(A) \Gamma_{\alpha\beta}(x)^*$$

and the resulting operator acts on $\mathcal{H}_{\alpha\beta} \cong \mathcal{H}_F$.

In fact this automorphism looks slightly different, depending whether $\underline{x}$ tends to spacelike left or right

infinity. To see this consider the mapping

$$W_F(\Phi) \to \Gamma_{\alpha\beta}(\underline{y}) W_F(\Phi) \Gamma_{\alpha\beta}(\underline{y})*$$

$$= e^{-i \, \text{Im} \, \langle \, \Phi, \, \Theta^x - U^x(-\underline{y}) \, \Theta^x \rangle} W_F(\Phi).$$

This is a displacement of the field $\hat{\varphi}$ by the function $\Theta^x - U^x(-\underline{y})\Theta^x = \eta(-\underline{y})$. As $\underline{y} \to +\infty$, $\eta(-\underline{y}) \to \Theta^x - \alpha - \beta$.
As $\underline{y} \to -\infty$, $\eta(-\underline{y}) \to \Theta^x$.
Depending on the direction we choose, we obtain two possible automorphisms

$$\hat{\varphi} \to \hat{\varphi} + \Theta^x$$

or $$\hat{\varphi} \to \hat{\varphi} + \Theta^x - \alpha - \beta.$$

The results differ by an internal symmetry: the addition of a constant to $\hat{\varphi}$. Since we have chosen an observable algebra which is unaffected by the addition of a constant, these automorphisms give rise to equivalent sectors.

In many respects, the algebra $\mathcal{O}$ behaves like the observable algebra of a Doplicher, Haag, Roberts field theory. In the spirit of DHR (1969b), we should be able to reconstruct the gauge group and field algebra associated with the limited class of representations $\pi_{\alpha\beta}$ of the observable algebra. To achieve this, a standard function $\Psi^x_{\alpha\beta}$ is chosen for each $(\alpha, \beta)$ which gives rise to a representation in the equivalence class $\pi_{\alpha\beta}$. Let $\mathcal{H}_{\alpha\beta}$ be a Hilbert

space isomorphic to $\mathcal{H}_0$, on which $\pi_{\alpha\beta}$ is realized, and let $\psi^*_{\alpha\beta}$ be an isometric mapping of $\mathcal{H}_0$ onto $\mathcal{H}_{\alpha\beta}$, with $\psi^*_{00}$ the identity on $\mathcal{H}_0$. Let $\gamma_{\alpha\beta}$ be the automorphism of $\mathcal{A}$ determined by $\psi^x_{\alpha\beta}$:

$$\gamma_{\alpha\beta}: W(\xi) \to e^{i \, \text{Im} \, \langle \psi^x_{\alpha\beta}, \, \xi \rangle} W(\xi).$$

Then

$$\pi_{\alpha\beta}(A) = \psi^*_{\alpha\beta}(\pi_0 \circ \gamma_{\alpha\beta})(A)\psi_{\alpha\beta},$$

where $\psi_{\alpha\beta}$, is the inverse of $\psi^*_{\alpha\beta}$.

Define $\mathcal{H} = \underset{\alpha,\beta}{\oplus} \mathcal{H}_{\alpha\beta}$, on which the reducible representation, $\pi = \underset{\alpha,\beta}{\oplus} \pi_{\alpha\beta}$ of $\mathcal{A}$, acts in the obvious way. $\psi^*_{\alpha\beta}$ is determined on $\mathcal{H}$ by its action on each $\mathcal{H}_{\alpha'\beta'}$:

$$\psi^*_{\alpha\beta}\Big|_{\mathcal{H}_{\alpha'\beta'}} = \psi^*_{\alpha+\alpha'\,\beta+\beta'}\psi_{\alpha'\beta'}\Big|_{\mathcal{H}_{\alpha'\beta'}}.$$

The field algebra $F$ is now defined in a concrete representation on $\mathcal{H}$, as the c*-algebra generated by $\pi(\mathcal{A})$ together with the charged fields $\psi^*_{\alpha\beta}, \psi_{\alpha\beta}$. The gauge group is the commutant of the reducible representation $\pi(\mathcal{A})$, which consists of multiples of the identity on each $\mathcal{H}_{\alpha\beta}$. It is therefore given by a product of copies of the circle group:

$$\underset{\alpha,\beta\in\mathbb{R}}{X} T_{\alpha\beta}.$$

By considering certain inequivalent representations of the free field in a Doplicher, Haag, Roberts framework, we have found a structure which admits a gauge

theory involving the interaction of two fields.

Certain properties which Doplicher, Haag and Roberts prove for a field theory satisfying their axioms in 3+1 space-time dimensions are violated by this example in 1+1 dimensions. For example, the $\psi$-fields need not obey the standard commutation or anticommutation rules; in fact, in general,

$$\psi^*_{\alpha_1\beta_1}\psi^*_{\alpha_2\beta_2} = e^{i\nu}\psi^*_{\alpha_2\beta_2}\psi^*_{\alpha_1\beta_1}$$

where $\nu = \pm\alpha_1\alpha_2\pm\beta_1\beta_2$ .

But discounting the special properties of field theories in 1+1 dimensions, the model is an example of the idealized description of a strictly local elementary particle theory given by DHR (1969b). They suggest that the full field algebra $F$ can be deduced from the vacuum sector $\pi_0(\mathcal{O})$ of the observable algebra. This is achieved by finding all equivalence classes of localized automorphisms $\rho$ which give rise to physical sectors, $\pi = (\pi_0 \circ \rho)$. This is equivalent to finding all cohomology classes of multiplicative cocycles (or charge transfer operators) $\Gamma_\pi$ which give rise to automorphisms $\rho$ via (2.3.10). This is a much harder problem than finding all representations of displaced Fock type which satisfy the axioms. Leyland and Roberts (1978) and Roberts (1978) have begun an investigation of the necessary cohomological techniques.

The relationship between multiplicative 1-cohomology classes and equivalence classes of representations is somewhat altered in 1+1 dimensions. However, the analysis of displaced Fock representations remains valid, and further sectors can be obtained by relaxing the localization requirement. We specify a milder localization requirement in terms of the charge transfer cocycles, and exhibit displaced Fock representations which satisfy this requirement. The requirement is of the type suggested by Roberts (1975).

For simplicity of notation, let us consider the case of space-time translation cocycles. We wish to impose the condition that charge transfer along a path b from x to x+y say, which is sufficiently space-like distant from 0, has negligible effect on $\alpha(0)$, i.e. the cocycle almost commutes with elements of $\alpha(0)$ in this limit. Thus we require that for $A \in \alpha(0)$,

$$\tau_x(\Gamma(y))A - A\tau_x(\Gamma(y)) \to 0$$

as x and x+y tend to infinity in a spacelike direction. Such an asymptotic condition will come up again in the next chapter where we shall specify in what topology the limit should be taken.

In the present example $\bar{D}^+$ contains certain functions which give rise to non-trivial cocycles, which obey the asymptotic localization condition. That is, by appropriate choice of the form of $\tilde{\phi}^x$ for small

$p$, we can arrange that $\phi^x \notin K$ (i.e. $\tilde{\phi}^x \notin L(\mathbb{R}, \frac{dp}{p})$) but $\phi^x \in \bar{D}^+$. Specifically, we show:

<u>Proposition</u>: Suppose $\tilde{\phi}^x(p)$ is given by one of the following functions in a neighbourhood $(-c,c)$ of the origin:

$$\tilde{\phi}^x(p) = u_\alpha(|p|) \equiv (-\log|p|)^{\frac{1}{2}\alpha}, \quad 0 < \alpha < 1, \quad (3.1.14)$$

or $\qquad \tilde{\phi}^x(p) = v(|p|) \equiv \log(-\log|p|), \qquad (3.1.15)$

and agrees with a bounded $C^\infty$ function of compact support on the set $\mathbb{R} \setminus (-\frac{1}{2}c, \frac{1}{2}c)$.

Then $\phi^x$ lies in $\bar{D}^+$ but not in $K$, and $\phi^x(x) \to 0$ as $x \to \pm\infty$.

Before proving this proposition, we first note certain results, proved by Pinczon and Simon (1975), on the analyticity of $\phi^x$. They have shown that for each equivalence class in $\bar{D}^+$, there is a representative, $\phi^x$ which is analytic in the group generators, and satisfies the algebraic cocycle requirement

$$\ell \phi^x \in K$$

for each generator $\ell$ of $P_+^\uparrow$. That is, there is an element $\varphi_L = \phi^x - U^x(L)\phi^x$ in each cohomology class in $H^1(P_+^\uparrow, K)$ such that $\phi^x$ satisfies these requirements. Furthermore, any $\phi^x$ satisfying these requirements is in $\bar{D}^+$.

<u>Proof of Proposition</u>

To check that $\phi^x$ is an element of $\bar{D}^+$, we show that the infinitesimal conditions hold:

$$|p|\frac{\partial}{\partial p}\tilde{\phi}^x(p), \quad p\tilde{\phi}^x(p) \in L^2(\mathbb{R},\frac{dp}{|p|})$$

i.e. $J^{01}\tilde{\phi}^x, \quad P^0\tilde{\phi}^x \in L^2(\mathbb{R},\frac{dp}{|p|})$,

where $J^{01}$ is the infinitesimal generator of Lorentz boosts.

It is sufficient to show that these functions must be square integrable with respect to $\frac{dp}{|p|}$ in a neighbourhood of $p=0$.

If $\tilde{\phi}^x$ is given by (3.1.14), the small $|p|$ contribution to $\|J^{01}\tilde{\phi}^x\|^2$ is

$$\int_0^\epsilon \left||p|\frac{\partial}{\partial p}\tilde{\phi}(p)\right|^2\frac{dp}{p} = \tfrac{1}{4}\alpha^2\int_0^\epsilon(-\log|p|)^{\alpha-2}\frac{dp}{p}$$

$$= \tfrac{1}{4}\alpha^2\int_{-\infty}^{\log\epsilon}(-\ell)^{\alpha-2}d\ell$$

$$= -\tfrac{1}{4}\alpha^2\left[\frac{(-\ell)^{\alpha-1}}{\alpha-1}\right]_{-\infty}^{\log\epsilon}$$

$$= -\tfrac{1}{4}\frac{\alpha^2}{\alpha-1}(-\log\epsilon)^{\alpha-1} \quad (\text{as } \alpha-1 < 0)$$

$$< \infty$$

and the small $|p|$ contribution to $\|P^0\tilde{\phi}^x\|^2$ is

$$\int_0^\epsilon \left|p\tilde{\phi}^x(p)\right|^2\frac{dp}{|p|} = \int_0^\epsilon p(-\log|p|)^\alpha dp$$

$$= \int_{-\infty}^{\log\epsilon}e^{2\ell}(-\ell)^\alpha d\ell \quad \text{where} \quad \ell=\log p$$

$$< \infty.$$

But $\phi^x$ does not lie in $K$; $\|\phi^x\|^2$ includes

$$\int_0^\varepsilon (-\log p)^\alpha \frac{dp}{|p|} = \int_{-\infty}^{\log \varepsilon} (-\ell)^\alpha d\ell$$

which diverges.

Now consider the case

$$\tilde{\phi}^x(p) = v(|p|) = \log(-\log|p|)$$

$\|P^0 \tilde{\phi}^x(p)\|^2$ includes the term

$$\int_0^\varepsilon p^2 (\log(-\log p))^2 \frac{dp}{|p|}$$

$$= \int_{-\infty}^{\log \varepsilon} e^{2\ell} (\log(-\ell))^2 d\ell$$

$$= \int_{-\log \varepsilon}^\infty e^{-2\ell} (\log \ell)^2 d\ell < \infty .$$

The term which must be checked for $\|J^{01} \tilde{\phi}^x(p)\|^2$ is

$$\int_0^\varepsilon \left[ p \frac{\partial}{\partial p} (\log(-\log p)) \right]^2 \frac{dp}{|p|}$$

$$= \int_0^\varepsilon \frac{p^2}{(p\log p)^2} \frac{dp}{|p|}$$

$$= \int_{-\infty}^{\log \varepsilon} \frac{d\ell}{\ell^2} < \infty .$$

To see that $v(|p|) \notin L^2(\mathbb{R}, \frac{dp}{|p|})$,

$$\int_0^\varepsilon \left| v(|p|) \right|^2 \frac{dp}{p} = \int_0^\varepsilon (\log(-\log p))^2 \frac{dp}{p}$$

$$= \int_{-\infty}^{\log \varepsilon} |\log(-\ell)|^2 d\ell$$

$$= \int_{-\log \varepsilon}^{\infty} |\log \ell|^2 d\ell$$

which diverges.

Hence $\tilde{\phi}^x(p) = v(|p|)$ is in $\bar{D}^+ \setminus K$.
This completes the proof.

To check that $\phi^x(\underline{x}, x_0) \to 0$ as $\underline{x} \to \pm\infty$ (any fixed $x_0$), we note that the behaviour of $\phi^x$ for large $|\underline{x}|$ is determined by the behaviour of $\tilde{\phi}^x$ at its singularity. Both $u_\alpha$ and $v$ have a singularity at $|p| = 0$, where they tend to $+\infty$. But both are dominated as $|p| \to 0$ by $\log|p|$. The Fourier transform

$$F(\log|p|) \sim \frac{1}{2x} \quad \text{as} \quad x \to \infty$$

(e.g. Lighthill 1958).
Hence also $\phi^x(x) \to 0$ as $\underline{x} \to \infty$.

It then follows that the charge transfer operator

$$\Gamma_{\phi^x}(y) = W_F(\phi^x - U^x(-y)\phi^x)$$

is asymptotically localized in the sense described above.

Roberts (1976) has stressed the importance of broken symmetries to the existence of non-vacuum sectors in 1+1-dimensional theories. Fröhlich (1976) has given a method of construction of such sectors which

depends on multiple vacua, although not on broken symmetry.

Our example demonstrates that other kinds of non-vacuum sectors are possible if the strict localization requirement is relaxed. However, the soliton sectors which have been constructed in  1+1  dimensions arise from strictly local cocycles and depend on multiple vacua. Therefore in section 3.2, we re-impose the requirement of strict localization in order to discuss charge transfer and  superselection sectors in  1+1 dimensions.

## 3.2  Charge transfer operators in  1+1  dimensions

The principal reason for taking an interest in
the 1+1-dimensional case is that it has been possible
to construct the soliton sectors of a non-linear
theory in this number of dimensions.  This has been
achieved (Fröhlich 1977) for the  $(g\varphi^4)_2$  model, with
interaction Hamiltonian of the form

$$H_I(\varphi(x)) = g\varphi(x)^4 + \tfrac{1}{2}\sigma\varphi(x)^2 + \sigma^2/(16g) \qquad (3.2.1)$$

with  $\sigma = -\tfrac{1}{2}$,  and  $0 < g \ll 1$.

For these values of  $\sigma$  and  $g$,  the Hamiltonian
gives rise to two distinct vacua, connected by the
broken symmetry

$$\varphi \to -\varphi . \qquad (3.2.2)$$

The non-vacuum superselection sectors which have been
constructed for this model are those associated with
the classical soliton solutions which interpolate
between the two vacua.  Thus they arise from the broken
symmetry in the same way as the "soliton" sectors of
the Streater-Wilde model.  Sectors which arise in this
way are sometimes regarded as owing their stability to
a topological quantum number:  the difference between
the expectation value of the field at positive and
negative spatial infinity.  In principle, this is the
same kind of superselection rule as those which describe
vortex and monopole sectors in higher dimensions.

In attempting to identify the properties of such models which might be expected to apply to any elementary particle theory, we must endeavour to recognise aspects of the structure which depend on the "topological" nature of the superselection sectors, and also those aspects which are peculiar to 1+1 dimensions.

In our discussion of charge transfer in chapter 2, we concluded from the physical nature of charge transfer along a path b, that the charge transfer operator $z(b)$ should lie in $\alpha(b+0)$, for some bounded double cone $0$. In more than one space dimension, $z(b)$ must actually satisfy the stronger condition

$$z(b) \in \alpha[(\partial_0 b+0)' \cap (\partial_1 b+0)']' \qquad (3.2.3)$$

(analogous to the condition specified by Doplicher, Haag and Roberts, for the operator $U_k$ which we introduced in equation (2.3.7)). In a theory satisfying strict duality, this becomes

$$z(b) \in \alpha((\partial_0 b+0) \cup (\partial_1 b+0)). \qquad (3.2.4)$$

But the argument which establishes this property (3.2.3) (Roberts, 1976, lemma 2.1) depends on the path-connectedness of the causal complement of any finite region of Minkowski space. This fails in 1+1 dimensions. As a result, the superselection structure in 1+1 dimension is qualitatively quite different from the structure in higher dimensions. For the case of 1+1 dimensions, we must therefore

rewrite the analysis, valid for higher dimensions, given in section 2.3. The only localization property we have is

$$z(b) \in \mathcal{O}(b+\mathcal{O}),$$

for some bounded double cone $\mathcal{O}$. For a cocycle $\Gamma$ defined on the space-time translations and related to $z$ as described in chapter 2, (after equation (2.3.14)) this localization property may be written

$$\Gamma(x) \in \mathcal{O}(\mathcal{O}_x)$$

where $\mathcal{O}_x$ is the smallest convex connected union of bounded open double cones containing $\mathcal{O}$ and $\mathcal{O}+x$. (Following the notation of Fröhlich 1977).

We have seen previously that in higher dimensions, the limit of the mapping

$$A \to \Gamma(x)A\Gamma(x)*$$

as $x \to \infty$ in a spacelike direction is a *-morphism of $\mathcal{O}$ independent of the direction. But in the Streater-Wilde model, we found that the limits as $\underline{x} \to \infty$ and $\underline{x} \to -\infty$ differed by an internal symmetry. This will always be the case when $\Gamma$ is a charge transfer operator for a topological charge in 1+1 dimensions.

The equivalence and triviality analysis of the local cocycles in higher dimensions also depended on the localization about the endpoints of the charge

transfer path. Hence this analysis must be modified
in 1+1 dimensions, and the equivalence classes become
representation-dependent. A charge transfer operator
$\Gamma$ will be said to be trivial in the representation
$\pi$ if there exists a unitary operator $V \in \pi(\mathcal{A})''$ such
that

$$\Gamma(x) = V^* \tau_x (V) \qquad (3.2.5)$$

(following Fröhlich 1977). We shall use examples to
show why it becomes necessary to specify the representation
in which such a V can be found.

The $(g\varphi^4)_2$ model is one of the few theories
where sectors corresponding to the soliton solutions
of a non-linear equation have been rigorously
constructed in the quantum theory. The construction
of these sectors is made possible by the special
properties of cocycles in 1+1 dimensions which we
have mentioned. The charge transfer cocycles corresponding
to the topological charge of the $(g\varphi^4)_2$ model are
trivial in the Fock representation, but non-trivial
in the charged sectors. Fröhlich has exploited this
property to construct the charge transfer cocycles.

The charged sectors are discussed by Bonnard and
Streater (1976) and Fröhlich (1976, 1977). For the
values of $\sigma$ and $g$ we have specified, equation
(3.2.1) becomes

$$H_I (\varphi(x)) = g\varphi(x)^4 - \tfrac{1}{4}\varphi(x)^2 + \frac{1}{64g} .$$

Since we do not have general methods for calculating the full superselection structure, we look for indications of the possibility that charged sectors of certain types might exist.

Given that there are $n=2$ vacua, $n(n-1) = 2$ soliton sectors may be expected to exist, whose states tend to different vacua at left and right spatial infinity.

These sectors can also be predicted on other grounds. We can exhibit four finite-energy classical field configurations corresponding to the given Hamiltonian:

$$\varphi_+(x) = \frac{1}{\sqrt{8g}} , \quad \varphi_-(x) = \frac{-1}{\sqrt{8g}}$$

are the two distinct vacuum solutions, corresponding to quantum vacuum states $\omega_+$ and $\omega_-$, and

$$\varphi_s(x) = \frac{1}{\sqrt{8g}}\tanh(\frac{x}{\sqrt{8g}}) , \quad \varphi_{\bar{s}}(x) = -\frac{1}{\sqrt{8g}}\tanh(\frac{x}{\sqrt{8g}})$$

are the soliton solutions.

We might expect that inequivalent non-vacuum sectors $\pi_s, \pi_{\bar{s}}$ should exist in the quantum theory, generated by states $\omega_s, \omega_{\bar{s}}$ such that $\omega_s(\hat{\varphi}(x))$ and $\omega_{\bar{s}}(\hat{\varphi}(x))$ are approximated by $\varphi_s$ and $\varphi_{\bar{s}}$ respectively.

Fröhlich achieves the construction of these sectors by constructing the corresponding translation cocycles. (They may be extended to Poincaré cocycles.)

We cannot reproduce the details of this construction here, as it relies on a great deal of information about the construction of the vacuum sectors. (These are discussed, for example, by Simon 1974, and Fröhlich 1977).

The method of construction appears to apply quite generally to topological charge transfer cocycles in 1+1 dimensions. The essential characteristic of 1+1 dimensions which is used in the construction is the representation-dependence of the cohomology structure, and the reason why this comes about is perhaps clearer in the Streater-Wilde model. We re-examine the charge transfer cocycle for a sector of this model to see whether it is a local coboundary in either representation.

For simplicity we consider a sector $\pi_{\alpha 0}$ corresponding to a displacement

$$\theta_\alpha^x(\underline{x}, t) = \theta_\alpha(\underline{x}+t)$$

for some $\theta_\alpha$ with $\theta_\alpha(-\infty) = 0$, $\theta_\alpha(\infty) = \alpha$, $\frac{d\theta_\alpha}{dx} \in \mathcal{D}$ so that $\theta_\alpha \in N(0)$ for some $0$. For definiteness, we choose $0$ to be the bounded double cone with time zero cross-section $[0,1]$. (A similar analysis can be done with $\beta \neq 0$.)

The corresponding charge transfer operator

$$\Gamma_{\alpha 0}(x) = W(\eta_\alpha(-x)) ,$$

where $\eta_\alpha(x) = \theta_\alpha^x - U^x(x)\theta_\alpha^x$, is represented in the Fock representation by

$$W_F(\eta_\alpha(-x)) = W_F(\theta_\alpha^x - U^x(x)\theta_\alpha^x).$$

Define a sequence of $C^\infty$ functions by

$$f_n(\underline{x}, t) = \theta^{(n)}(\underline{x}+t), \quad n \geqslant 2,$$

where

$$\theta^{(n)}(\underline{x}) = \theta_\alpha(\underline{x}) \quad \text{for} \quad \underline{x} \leqslant 1,$$

$$= \alpha \quad \text{for} \quad 1 < \underline{x} \leqslant n$$

$$= 0 \quad \text{for} \quad \underline{x} > n+1$$

and the values between $n$ and $n+1$ are chosen such that the function is $C^\infty$. Then each $f_n$ is a solution of the wave equation, belonging to $M$, and so $W(f_n)$ lies in $\mathscr{A}(O_1)$ for some bounded double cone $O_1$.

We have

$$(\text{pointwise}) \lim_{n \to \infty} f_n(\underline{x}, t) = \theta_\alpha^x(\underline{x}, t)$$

and consequently $w\text{-}\lim_{n \to \infty} \pi_F(W(f_n))$ is well-defined. (It is straightforward to check this using the constructive definition of the Fock representation in terms of rigorously defined annihilation and creation operators, which is given for this case by Streater and Wilde.)

We write $w\text{-}\lim_{n \to \infty} \pi_F(W(f_n)) = U_0^* \in \pi_F(\mathscr{A})''$.

Put $g_n(\underline{x}, t) = -\theta^{(n)}(\underline{x}+t)$.

Then $\eta(-x) = (\text{pointwise}) \lim_{n \to \infty} (f_n + U(x)g_n)$

and $U_0^* \tau_x(U_0) = W_F(\eta_x(-x)) = \pi_F(\Gamma_{\alpha 0}(x))$.

So $\Gamma_{\alpha 0}$ is trivial in the Fock representation (in the sense specified in (3.2.5)).

We now examine $\pi_{\alpha_0}(\Gamma_{\alpha_0}(x)) = \pi_{\alpha_0}(W(\eta(-x)))$

$$= e^{i \, \text{Im} \, \langle \Theta_{\underline{\alpha}}^{x}, \, \eta(-x) \rangle} W_F(\eta(-x)) \, .$$

We find

$$\text{Im} \langle \Theta_{\alpha}^{x}, f_n \rangle \ = \ -\alpha^2$$

and for $\underline{x}$ large and positive

$$\text{Im} \langle \Theta_{\alpha}^{x}, U(x) g_n \rangle \ = \ 0$$

but for $\underline{x}$ large and negative, and for large $n$,

$$\text{Im} \langle \Theta_{\alpha}^{x}, U(x) g_n \rangle \ = \ 2\alpha^2 \, .$$

So that for $\underline{x}$ large and positive

$$\Gamma_{\alpha_0}(x) \ = \ e^{i \, \alpha^2} U_0^* \tau_x (U_0)$$

and for $\underline{x}$ large and negative

$$\Gamma_{\alpha_0}(x) \ = \ e^{-i \, \alpha^2} U_0^* \, \tau_x (U_0) \, .$$

We conclude that $\pi_{\alpha_0}(\Gamma_{\alpha_0}(x))$ is not of the form

$$V^* \tau_x (V)$$

where $V \in \tau_{\alpha_0}(\mathcal{O}l)''$ is independent of $x$, and $\Gamma_{\alpha_0}$ is therefore non-trivial in the displaced representation.

To summarize, in the Streater-Wilde model, we can write down directly the morphism giving rise to any soliton sector, and the corresponding cocycle. We can then express the cocycle as a trivial cocycle in the Fock representation, but not in the soliton representation.

In more complicated (1+1 dimensional) examples, Fröhlich has given a construction for the morphism $\rho$ of a sector $\pi_s$ which interpolates between two vacua. (1976, §5; 1977). This construction should work for any topological soliton in 1+1 dimensions. For the $g\varphi^4{}_2$ model, Fröhlich (1977) has also demonstrated, using results of Buchholz (1974), the existence of a $U_0 \in \pi_F(\mathit{ot})''$ such that

$$\pi_F(\Gamma(x)) = U_0^* \tau_x(U_0)$$

is the Fock representation of a charge transfer operator giving rise to the new sector $\pi_s$. $\Gamma(x)$ is therefore a local coboundary in the Fock representation. The fact that it is not a local coboundary in the new representation $\pi_s$ then follows because the sector it generates, $\pi_s$, is inequivalent to the vacuum sector.

From our experience with the Streater-Wilde model, we expect that any attempt to write $\Gamma$ as a local coboundary in $\pi_s$ will result in different expressions for $\underline{x}$ large and positive or large and negative. Thus the representation- dependence appears to be the result of the topological nature of the new sector.

CHAPTER 4

## MODELS IN 3+1 DIMENSIONS

Our programme of using displaced Fock representations
to obtain "superselection sectors" of the algebras
corresponding to free fields cannot be carried over to
higher dimensions as it stands. The principal reason
is the following result:

Theorem. In 3+1 space-time dimensions, $H^1(P_+^\uparrow, K) = 0$,
for any irreducible representation of $P_+^\uparrow$ of mass
$m \geqslant 0$ and spin $s \geqslant 0$.

Proof. 1. The case of positive mass. (For this case
the triviality result holds for any number of space-
time dimensions.)

We show that the norm $\|\cdot\|_+$ of $\bar{D}^+$ is equivalent
to the norm $\|\cdot\|_K$ of the one-particle space $K$. It
follows that the spaces $\bar{D}^+$ and $K$ coincide. Since
each topological cocycle $\psi_L$ in $H^1(P_+^\uparrow, K)$ corresponds
to an element $\Psi^x$ of $\bar{D}^+$ via (2.2.4), and those $\psi_L$
for which $\Psi^x$ can be chosen in $K$ are coboundaries,
it follows that all topological cocycles are coboundaries.
As we saw in section 2.2, a result of Araki (1970, lemma
7.2) applied to the present case implies that there are
no algebraic cocycles. Hence the 1-cohomology group is
trivial.

The norm $\|\cdot\|_+$ is defined, for $\varphi$ in $D^+$, by

$$\|\varphi\|_+ = \|K^{\frac{1}{2}}\varphi\|_K \qquad (4.1)$$

with $K = \Sigma K(\ell)$, where the sum is over all generators
$\ell$ of Poincaré transformations, and $K(\ell)$ is defined
by (2.2.8). Thus each $K(\ell)$ is non-negative. In
particular, one generator is the energy,

$$\ell_E = (p^2 + m^2)^{\frac{1}{2}} \geqslant m > 0.$$

It follows that $K(\ell_E)$, hence $K$, is bounded below
by a positive number, and so both $K^{\frac{1}{2}}$ and $K^{-\frac{1}{2}}$ exist
as bounded operators. We then have the sequence of
inequalities:

$$\| \varphi \|_+ = \| K^{\frac{1}{2}} \varphi \|_K \leqslant C \| \varphi \|_K = C \| K^{-\frac{1}{2}} K^{\frac{1}{2}} \varphi \|_K$$

$$\leqslant CC' \| K^{\frac{1}{2}} \varphi \|_K = C'' \| \varphi \|_+$$

so that the norms $\| \cdot \|_+$ and $\| \cdot \|_K$ are equivalent.

## 2. The case $m=o$, $s > 0$

Since the restriction of a cocycle for a group $G$
to any subgroup $H$ is also a cocycle for $H$, any
cocycle for $P_+^\uparrow$ in 3 space-dimensions is also a cocycle
for the rotation group $SO(3)$. But $SO(3)$ is a compact
group and therefore has trivial cohomology (Araki 1970,
theorem 7.1). So any $\phi^x \in \bar{D}^+$ giving rise to a cocycle
for $P_+^\uparrow$ may be chosen to be rotation invariant. (If it
is not, then $\varphi(R)$ defined by

$$\varphi(R) = \phi^x - U^x(R)\phi^x, \quad R \in SO(3),$$

is a possibly non-zero element of $K$, which can be expressed as

$$\varphi(R) = \Psi - U^x(R)\Psi \quad \text{with} \quad \Psi \in K,$$

since the cohomology of the rotation group is trivial. Thus $(\Phi^x - \Psi)$ is rotation invariant, and gives rise to a cohomologous cocycle for $P_+^\uparrow$.)

Now Lomont and Moses (1967, theorem 6) show that the generators of $SO(3)$ in a representation corresponding to zero mass, may be realized as

$$J_1 = s - i(\underline{p} \times \underline{V})_1$$

$$J_2 = \frac{p_2 s}{(p+p_1)} - i(\underline{p} \times \underline{V})_2 \qquad (4.2)$$

$$J_3 = \frac{p_3 s}{p+p_1} - i(\underline{p} \times \underline{V})_3$$

(where $p = |\underline{p}|$).

Since $\Phi^x$ may be chosen rotation invariant, each $J_i$ satisfies

$$J_i \Phi^x = 0 . \qquad (4.3)$$

It follows that $\Phi^x = 0$. For, by (4.2), the linear combination

$$p_2 [J_2 - \frac{p_2}{p+p_1} J_1] - p_3 [J_3 - \frac{p_3}{p+p_1} J_1] = p(p+p_1)(\underline{p} \times \underline{V})_1 ,$$

that is, from (4.3),

$$p(p+p_1)(\underline{p} \times \underline{V})_1 \Phi^x = 0$$

so that $(\underline{p} \times \underline{V})_1 \Phi^x = 0$, which combined with $J_1 \Phi^x = 0$, gives $s\Phi^x = 0$, hence $\Phi^x = 0$.

3.    The case  m = s = 0.

Up to a rotation, a Lorentz boost transformation acts on elements $\tilde{\phi}^x = \tilde{\phi}^x(p_1,\ldots,p_d)$ of the one-particle space $K = L^2(\mathbb{R}^d, d^d p/|\underline{p}|)$ (d = number of space-dimensions) by

$$\tilde{\phi}^x(p_1,\ldots,p_d) \rightarrow \tilde{\phi}^x(p_1 \cosh \eta + |\underline{p}| \sinh \eta, p_2,\ldots,p_d) \quad (4.5)$$

where $\eta$ is defined by $\tanh \eta = v/c$, with $v$ the velocity in the 1-direction.

A nontrivial cocycle is determined by $\phi^x \in \bar{D}^+$, of the form

$$\tilde{\phi}^x(\underline{p}) = u(|\underline{p}|) \quad (4.6)$$

such that $\tilde{\phi}^x \notin K$, but $p^0 \tilde{\phi}^x \in K$ and $J^{01} \tilde{\phi}^x \in K$. $\quad$ (4.7)

Acting on rotation invariant functionals, (4.6), these generators are given by

$$J^{01} = \frac{\partial}{\partial \eta}\Big|_{\eta=0} = |\underline{p}| \frac{\partial}{\partial p_1} = p_1 \frac{\partial}{\partial |\underline{p}|} \quad,$$

$$p^0 = p.$$

In order to translate the conditions (4.7) into useful restrictions on $u$, we use the following technical results (Pinczon and Simon 1975):

(a)    Let $U$ be a continuous representation of a connected Lie group $G$ on a Banach space $K$. Then $H^1(G,K) = H^1_\omega(G,K)$, where $H^1_\omega$ consists of 1-cocycles which are analytic in the group parameters about the

identity. Elements of $H^1_\omega$ are valued in the set $K_\omega$ of analytic vectors for the group representation.

(b)   If $g$ is the Lie algebra of $G$, the map $\Delta$, defined by

$$\frac{d}{dt}\psi(\exp tX)_{t=0} = (\Delta\psi)(X)$$

maps $H^1_\omega(G,K)$ into $H^1(g,K_\omega)$. If $G$ is simply connected, then $H^1_\omega(G,K) = H^1(g,K_\omega)$.

(The cocycles of $g$ are elements $\phi^x$ of $K^x_\omega$ such that $K(\ell^x)\phi^x \in K$.)  So in the present case we are assured that, without loss of generality, $\phi^x$ may be chosen such that

$$(J^{01})^n \phi^x = (p\frac{\partial}{\partial p})^n \phi^x \in K, \quad n=1,2,\dots . \qquad (4.8)$$

So $u$ may be chosen to be a $C^\infty$ function, and from (4.8) with $n=1$,

$$\int_0^\infty (u')^2 p^d \, dp < \infty .$$

Redheffer (1970) supplies the inequality

$$(\frac{d-1}{2})^2 \int_0^\infty |u|^2 p^{d-2} \, dp \leqslant \int_0^\infty \left|\frac{du}{dp}\right|^2 p^d \, dp$$

where $d$ is the number of space dimensions. For the case $d=3$, it follows that

$$\int_{-\infty}^\infty |\tilde{\phi}^x(\underline{p})|^2 d^3p < \infty, \quad \text{i.e.} \quad \phi^x \in K.$$

Thus the cocycle corresponding to $\phi^x$ is trivial.

Nonetheless, if we relax the requirement of full Poincaré covariance, representations can be obtained which are useful in constructing physically interesting models.

A basic requirement for an interesting representation will always be that space-time translations are implemented: without this, it could not even be interpreted as a charged sector, obtained as a limit of a charge transfer operation. The above proof shows that the existence of non-trivial translation covariant displaced Fock representations is only possible if $m=0$. It is also clear that only if $m = s = 0$ can rotation covariance also be obtained. Now it turns out that the extra structure introduced by forming an induced representation to obtain Lorentz boost covariance is very helpful to the interpretation of "soliton" sectors in terms of interacting fields. To illustrate this we shall therefore first concentrate on the free scalar field with $m = s = 0$ in $3+1$ dimensions, forming a model using displaced Fock representations of this field.

The failure of Lorentz boost covariance is familiar from attempts to quantize the free electromagnetic field in a Hilbert space formulation. In this case too we shall attempt to apply the soliton sector interpretation.

## 4.1    The kinetic charge model

Let $\mathcal{A}$ be the Weyl algebra over a space $M$ of smooth solutions of compact support, of the equation

$$\Box\varphi = 0, \qquad (4.1.1)$$

where $\varphi$ is defined on 4-dimensional Minkowski space, and $M$ is a dense Poincaré invariant subspace of the Hilbert space $K$ with inner product

$$\langle \Phi, \Psi \rangle_K = \int \Phi(\underline{x},0)(-\nabla^2)^{\frac{1}{2}}\Psi(\underline{x},0)\,d^3x + \int \dot{\Phi}(\underline{x},0)(-\nabla^2)^{-\frac{1}{2}}\dot{\Psi}(\underline{x},0)\,d^3x$$

$$+ i\int(\Phi\dot{\Psi}-\dot{\Phi}\Psi)\,d^3x \ . \qquad (4.1.2)$$

Assume also that $M$ lies in the kernel of all Poincaré invariant functionals on $K$, so that the cohomological classification of covariant sectors applies. Thus the symplectic form determining the Weyl algebra is supplied by the imaginary part of the inner product,

$$B(\Phi,\Psi) = \int(\Phi\dot{\Psi}-\dot{\Phi}\Psi)\,d^3x \ . \qquad (4.1.3)$$

Denote the Fock representation of the Weyl algebra $\mathcal{A}$ by $\pi_F$, acting on $\mathcal{H}_F$. Let $\varphi_F$ be the Fock representation of the field. We initially construct one displaced Fock representation, $\pi_0$, of the algebra, corresponding to the transformation

$$\varphi_F \rightarrow \varphi_0 = \varphi_F + \Psi^x \ , \qquad (4.1.4)$$

where $\Psi^x$ is specified by the Cauchy data

$$\Psi^x (\underline{x},0) = \lambda(|\underline{x}|)$$

$$\dot{\Psi}^x (\underline{x},0) = 0 \qquad\qquad (4.1.5)$$

and $\lambda$ is a $C^\infty$ function satisfying

$$\lambda(r) = q/r, \qquad r \geqslant r_0 \qquad\qquad (4.1.6)$$

for some $q$, $r_0$ real positive constants.

For $r > r_0$, $\nabla^2 \Psi^x (\underline{x},0) = 0$, and since $\Psi^x$ must also satisfy the wave equation, it follows that

$$\ddot{\Psi}^x (\underline{x},0) = 0 , \qquad\qquad (4.1.7)$$

so that at $t=0$ , the wave is stationary outside $r = r_0$, and the energy associated with $\Psi^x$ is localized in some sense. We verify that $\Psi^x$ gives rise to a non-trivial cocycle of $\mathbb{R} \times \mathbb{E}^3$, and establish in what sense the corresponding charge transfer cocycle is localized.

For non-triviality, $\Psi^x$ must not be an element of $K$. We show that the contribution to the form $\langle \Psi^x, \Psi^x \rangle_K$ from the region where $|\underline{x}| > r_0$ diverges. Since in this region $\dot{\Psi}^x (\underline{x},0)$ vanishes, the contribution reduces to

$$\int_{|\underline{x}|>r_0} \Psi^x (\underline{x},0) (-\nabla^2)^{\frac{1}{2}} \Psi^x (\underline{x},0) \, d^3x. \qquad (4.1.8)$$

The asymptotic behaviour of this integral is determined by the behaviour of the corresponding integral of Fourier

transforms near its singularity $k=0$. Therefore we consider the integral in terms of the 3-dimensional Fourier transform $\tilde{\psi}^x$ of $\psi^x$:

$$\int_0 \overline{\tilde{\psi}^x}(\underline{k}) |\underline{k}| \tilde{\psi}^x(\underline{k}) d^3k.$$

As $|\underline{k}| \to 0$, $\tilde{\psi}^x(\underline{k}) \sim q/|\underline{k}|^2$, so that the integral is given by

$$q^2 \int_0 (1/k^2) k (1/k^2) k^2 dk \quad \text{where} \quad k = |\underline{k}|$$

$$= q^2 \int_0 (1/k) dk \qquad (4.1.9)$$

which is logarithmically divergent.

Thus the displacement $\psi^x$ does not belong to $K$, and must give rise to an inequivalent representation of the algebra.

In an analogous calculation for $P^\mu \psi^x$, two extra powers of $k$ will appear, so that $\langle P^\mu \psi^x, P^\mu \psi^x \rangle$ is finite, and $P^\mu \psi^x$ is an element of $K$. Thus if $\psi^x$ has been chosen to be analytic for the generators $P^\mu$,

$$\psi^x - U^x(x) \psi^x \in K,$$

for any translation $x = (x^\mu)$, and space-time translations are implemented in $\pi_0$. Similarly, $\langle \psi^x, P^0 \psi^x \rangle_K$ is finite, so that $\psi^x$ is a wave of finite energy.

We now construct charge transfer cocycles. The unitary operator $\Gamma_0(x)$ for each translation $x$, may be defined by

$$\Gamma_0(x) = W_F(\psi_{-x}) . \qquad (4.1.11)$$

Then if $V_F$ is a true representation of the translation group implementing translations in the Fock representation, $V_0(x)$ defined for each $\underline{x}$ in $\mathbb{R}^{3+1}$ by

$$V_0(x) = \Gamma_0(x)V_F(x) \qquad (4.1.12)$$

will implement translations in the displaced representation, defined by

$$\pi_0(W(\varphi)) = e^{i \operatorname{Im} \langle \Psi^x, \varphi \rangle} W_F(\varphi) .$$

In accordance with our general discussion of displaced Fock representations, we find as in (2.2.13) that $V_0$ is a multiplier representation of the translation group:

$$V_0(x)V_0(y) = e^{-\frac{1}{2}i \operatorname{Im} \langle \psi_x, \psi_{-y} \rangle} V_0(x+y), \qquad (4.1.13)$$

but from the initial data, (4.1.5), it is clear that in this case the multiplier is identically equal to 1.

From the definition of $\Psi^x$ it is clear that the mapping

$$x \to \psi_x = \Psi^x - U^x(x)\Psi^x, \quad x \in \mathbb{R}^{3+1}$$

is continuous, and this is sufficient to guarantee the continuity of the representation $V_0$ and of $\Gamma_0$.

$\Gamma_0$ satisfies the cocycle condition:

$$\Gamma_0(x)\tau_x(\Gamma_0(y)) = \Gamma_0(x+y) . \qquad (4.1.14)$$

(The definition of the new representation of $\sigma\iota$ as a limit of unitary automorphisms of the Fock representation implemented by the $\Gamma_0(x)$, as described in (2.3.13) would remain valid even if the multiplier were non-trivial and (4.1.14) had the form of (2.3.6), since this limit is independent of the phase.)

For the interpretation of $\Gamma_0$ as a charge transfer operator to make sense, the morphism must be localized, so that it corresponds to the transfer of charge from one region to another. To formulate this requirement, we need a local structure for the algebra $\sigma\iota$. This is defined in essentially the same way as the 1+1-dimensional case. Let $M(O)$ be the subset of $M$ consisting of those elements $\phi$ whose Cauchy data on some spacelike hyperplane $P$ have support in $P \cap O$. Then $\sigma\iota(O)$ is generated by $\{W(\phi):\phi \in M(O)\}$, and the C*-algebra generated by $\underset{O}{\cup} \sigma\iota(O)$ is equal to $\sigma\iota$.

If $\psi^x$ were constant outside some bounded set, then $\psi_{-x}$ would have bounded support and $\Gamma_0$ as defined in (4.1.11) would be strictly local, i.e., for some bounded double cone $O$, each $\Gamma_0(x)$ would be in the commutant of $\sigma\iota(O') \cap \sigma\iota(O(x)')$. However, the functional $\psi^x$ specified in (4.1.5) and (4.1.6) does not give rise to a strictly local charge transfer operator, but obeys only an asymptotic localization condition. The condition of strict localization would require $\Gamma_0(x)$ to commute with all operators associated with regions which are spacelike relative to some cone of points about each endpoint of its charge transfer path. An asymptotic

generalization of this will say in some sense that at
sufficient spacelike distances, the effect of a charge
is arbitrarily small. Thus we require that the commutant
between $\Gamma_0(x)$ and elements of $\alpha(O_1)$ can be made
arbitrarily small by choosing $O_1$ sufficiently spacelike
distant from the points 0 and x. So in some topology,
it is required that for any fixed A in $U\alpha(O)$

$$\tau_x(\Gamma_0(y))A - A\tau_x(\Gamma_0(y)) \qquad (4.1.15)$$

tends to zero as both x and x+y tend to infinity in
a spacelike direction. Of course this means that we are
no longer demanding strong local equivalence between the
new representation and the vacuum representation, but it
will still be a basic physical requirement that all
representations of interest should be locally normal.
We now formalize the new conditions for charge transfer
cocycles:

Definition: (quasilocal cocycle) Let $\alpha$ be a net of
local algebras over $K$ and let $\omega$ be a vacuum state
on $\alpha$ giving rise to a representation $\pi_\omega$ on the
Hilbert space $\mathcal{H}_\omega$. A translation cocycle $\Gamma$ on $\alpha$ is
given by a unitary operator $\Gamma(x)$ on $\mathcal{H}_\omega$ for each x
in $\mathbb{R}^{3+1}$ satisfying the cocycle condition (2.3.4). It
is called quasilocal if for all locally normal representations
$\pi$, (i.e. $\pi(\alpha(O))'' \cong \pi_\omega(\alpha(O))''$ for all O in $K$)

(1)    the morphism $\rho$ given by

$$(\pi \circ \rho)(A) = \lim_{\substack{x \to \infty \\ (\text{spacelike})}} \Gamma(x)\pi(A)\Gamma(x)^* \qquad (4.1.16)$$

(cf. (2.3.13)) exists as a weak limit, the result is independent of the spacelike direction in which the limit is taken, and $\rho$ gives rise to a locally normal representation, and

(2) $\quad \text{weak-lim}_{\substack{x \to \infty \\ x+y \to \infty \\ (\text{spacelike})}} [\tau_{-x}(\Gamma(y)(\pi \circ \tau_x)(A)\Gamma(y)^*) - \pi(A)] = 0 \quad (4.1.17)$

Because of the possibility of multiplier representations, we shall also speak of <u>quasilocal charge transfer operators</u>, which satisfy the above definition, except that the cocycle condition may be replaced by a condition of the form (2.3.6). The choice of topology in which to take the limit is not crucial in the present model. The definition we have given will turn out to be appropriate for the electromagnetic field also.

To summarize, having finite localized energy, the classical wave $\psi^x$ has certain particle-like properties, in the same sense that certain stationary solutions of non-linear field equations do. $\psi^x$ determines an operator $\Gamma_0$ via (4.1.11) which has the appropriate localization properties to be a charge transfer operator. Thus $\psi^x$ is associated with a charged particle in the quantum theory also, and the sector $\pi_0$ may be obtained as the limit of the charge transfer operation implemented by $\Gamma_0$, and consists of all states carrying the charge associated with $\psi^x$.

In addition to the space-time translations, rotations $R$ are clearly implemented in the new representation since

$$\psi_R \equiv \psi^x - U^x(R)\psi^x$$

vanishes for any rotation $R$. Therefore the sector $\pi_0$ has covariance group at least $\mathbb{R} \times \mathbb{E}^3$ where $\mathbb{E}^3$ is the Euclidean group in 3 dimensions, and $\mathbb{R}$ parametrizes time translations. We now show that Lorentz boosts are not implemented in $\pi_0$ by checking that

$$J^{01}\psi^x \notin K$$

where $J^{01}$ is the generator of Lorentz boosts in the $x_1$ direction:

$$J^{01} = k_1 \frac{\partial}{\partial k}.$$

For small $k$, we have $\psi^x \sim q/(k^2)$, and

$$\int_0 \|J^{01}\psi^x\|^2 \frac{d^3k}{k} = \int_0 (k \cos \theta)^2 \left|\frac{\partial}{\partial k}\left(\frac{q}{k^2}\right)\right|^2 k \, dk \, d\sigma(\theta) \qquad (4.1.18)$$

The $k$-integration goes as $4q^2 \int_0 dk/(k^3)$, which diverges.

Therefore the covariance group is exactly $\mathbb{R} \times \mathbb{E}^3$. It is implemented by

$$V_0(L) = \Gamma_0(L)V_F(L) \quad \text{for } L \text{ in } \mathbb{R} \times \mathbb{E}^3.$$

Now we must deal with the fact that insufficient physical symmetry is present in the sector $\pi$. In this

model and also in the case of the electromagnetic field, the additional symmetry may be obtained by using Mackey's induced representation construction. (Mackey 1968, 1976.)

We therefore describe the construction in an example which is sufficiently general to include both cases. In the course of this, we shall make use of the following theorem on physical symmetries. (See, for example, Streater and Wightman (1964), theorem 1.1. The first part is Wigner's theorem, (Wigner 1931).)

Theorem. Let $\pi(\mathcal{O})$ be a representation on a Hilbert space $\mathcal{H}$ of the algebra $\mathcal{O}$ of a physical theory, and suppose $\pi(\mathcal{O})'$ is commutative. (i.e. the hypothesis of commutative superselection rules holds.) Then

(1)     if a physical symmetry, regarded as a mapping between rays of $\mathcal{H}$, $\underline{\Phi} \rightarrow \underline{\Phi}'$, leaves coherent subspaces (superselection sectors) of $\mathcal{H}$ invariant, there exists for each coherent subspace $\mathcal{H}_q$, a unitary or antiunitary operator $V:\mathcal{H}_q \rightarrow \mathcal{H}_q$, such that for all physically realizable states of that subspace, $\underline{\Phi}' = V\underline{\Phi}$. V is uniquely determined up to a phase.

(2)     if the symmetry does not leave coherent subspaces invariant, then, restricted to a coherent subspace $\mathcal{H}_q$, it is a one-to-one mapping onto another coherent subspace $\mathcal{H}_{q'}$, unitary or antiunitary, and unique up to a phase.

In the light of this theorem, we now consider what symmetry structure should be expected in a relativistic

theory. In dealing with the Poincaré group, the anti-unitary case does not arise. The relativistic requirement implies that for $L, M \in P_+^\uparrow$, there must be unitary operators $U(L)$, $U(M)$ and $U(LM)$ which implement $L, M$, and $LM$ respectively, and so $U(L)U(M)$ implements the same symmetry as $U(LM)$. We conclude from the theorem that in the case where the symmetries do not leave the coherent subspaces invariant, $U(L)U(M)$ may differ from $U(LM)$ by an element of $\pi(\mathcal{O})'$. Thus $U$ need not even be a ray representation of $P_+^\uparrow$.

Let $\mathcal{O}$ be a C*-algebra with irreducible representation $\pi_0$ on $\mathcal{H}_0$. Suppose the Poincaré group acts as a group of automorphisms on $\mathcal{O}$, but only the subgroup $\mathbb{R} \times \mathbb{E}^3$ is implemented in $\pi_0$. Denote the unitary operator implementing $R \in \mathbb{R} \times \mathbb{E}^3$ in $\pi_0$ by $U_0(R)$. Then according to Wigner's theorem, $U_0$ will be a multiplier representation of $\mathbb{R} \times \mathbb{E}^3$: for $R_1, R_2 \in \mathbb{R} \times \mathbb{E}^3$,

$$U_0(R_1)U_0(R_2) = \sigma_0(R_1, R_2)U_0(R_1 R_2) \qquad (4.1.19)$$

where each $\sigma_0(R_1, R_2)$ is a complex number of modulus 1. Since $\mathbb{R} \times \mathbb{E}^3$ is a closed subgroup of $P_+^\uparrow$, the quotient group $V$ defined by

$$V = P_+^\uparrow / (\mathbb{R} \times \mathbb{E}^3) \qquad (4.1.20)$$

is a $P_+^\uparrow$-space under the natural action. It is equipped with an invariant measure $\mu$. To find an expression for this measure, we use the fact that $V$ may be parametrized

by the coordinates of the points on the mass shell corresponding to any given mass. We choose the mass shell of mass $m$, so that $V$ may be labelled by the set of points in momentum space:

$$\{p = (p_\mu): \; \underline{p}^2 = m^2, \; p_0 > 0\} \; .$$

Alternatively, each point $p = (p_\mu)$ may be labelled by the velocity vector

$$v = (p_0/m, \; \underline{v})$$

with

$$\underline{p} = m\underline{v}/(1 - \underline{v}^2)^{\frac{1}{2}} \quad (c=1)$$

(4.1.21)

The invariant measure $\mu$ is given by

$$d\mu(v) = d^3v/(1 - v^2)^2 = (1/m^2)d^3p/(p^2 + m^2)^{\frac{1}{2}} \; . \quad (4.1.22)$$

We first define the induced representation $U_1$ of $P_+^\uparrow$ corresponding to $U_0$ in the case where $U_0$ is a true representation, i.e. $\sigma_0 \equiv 1$.

A function $f: P_+^\uparrow \to \mathcal{H}_0$ is said to be <u>left covariant</u> with respect to $U_0$ if, for $R \in \mathbb{R} \times \mathbb{E}^3$ and $L \in P_+^\uparrow$,

$$f(RL) = U_0(R)f(L) \qquad (4.1.23)$$

Given any two left covariant functions $f$ and $g$, since $U_0$ is a unitary representation, the inner products in $\mathcal{H}_0$

$$\langle f(L), g(L)\rangle_{\mathcal{H}_0} \text{ and } \langle f(RL), g(RL)\rangle_{\mathcal{H}_0} = \langle U_0(R)f(L), U_0(R)g(L)\rangle_{\mathcal{H}_0}$$

are equal, and therefore the mapping

$$P_+^\uparrow \to \mathbb{C}$$

$$L \mapsto \langle f(L), g(L)\rangle_{\mathcal{H}_0}$$

is constant on cosets of $\mathbb{R} \times \mathbb{E}^3$ in $P_+^\uparrow$ (elements v of V). It follows that if, for each $v \in V$ an arbitrary $L(v) \in v$ is selected, the value of $\| f(L(v)) \|^2$ is independent of the choice of $L(v)$ and hence the condition

$$\int_V \| f(L(v)) \|^2 d\mu(v) \; < \; \infty \qquad (4.1.24)$$

makes sense as a condition on f. We denote the space of left covariant functions $f: P_+^\uparrow \to \mathcal{H}_0$ satisfying (4.1.24) by $\mathcal{H}_1$, and define the scalar product by

$$\langle f, g \rangle_{\mathcal{H}_1} = \int_V d\mu(v) \langle f(L(v)), g(L(v)) \rangle_{\mathcal{H}_0}$$

Define an action $U_1$ of $P_+^\uparrow$ on $\mathcal{H}_1$ by

$$(U_1(M)f)(L) = f(LM) \quad \text{for} \quad L, M \in P_+^\uparrow . \quad (4.1.26)$$

By the invariance of $\mu$ this is unitary, and clearly

$$U_1(LM) = U_1(L) U_1(M)$$

so that $U_1$ is a unitary representation of $P_+^\uparrow$, continuous if $U_0$ is.

We now replace the assumption that the multiplier $\sigma_0$ in (4.1.19) is identically 1 by the weaker assumption that it is the restriction to $\mathbb{R} \times \mathbb{E}^3$ of a multiplier $\sigma$ for $P_+^\uparrow$. To include this case, we make the following alterations. The definition (4.1.23) of a left covariant function is modified to

$$f(RL) = \sigma(R,L) U_0(R) f(L) \qquad (4.1.23)'$$

and the definition (4.1.26) of the induced representation becomes

$$(U_1(M)f)(L) = \frac{1}{\sigma(L,M)} f(LM) \qquad (4.1.26)'$$

For the case of true representations, $U_1$ can be obtained in an alternative form, which we shall find has a more natural physical interpretation.

We select as a distinguished element of $V$ the point $v_0 = (1,0,0,0)$ which is invariant under $\mathbb{R} \times \mathbb{E}^3$. For each $v$ in $V$, a Poincaré transformation $b(v)$ can be chosen, such that

$$b(v)v = v_0 \qquad (4.1.27)$$

Of course $b(v)$ is arbitrary up to the little group $\mathbb{R} \times \mathbb{E}^3$ of $v_0$.

To each function $f$ of $\mathcal{H}_1$, we associate a function $\varphi: V \to \mathcal{H}_0$ by

$$\varphi(v) = f(b(v)^{-1}) \qquad (4.1.28)$$

and denote by $\mathcal{H}$ the space of functions $\varphi$ which are of the form (4.1.28) for some $f$ in $\mathcal{H}_1$. It is clear from the condition (4.1.24) on $f$ in $\mathcal{H}_1$ that in fact $\mathcal{H}$ is the direct integral space:

$$\mathcal{H} = \int_{\oplus V} d\mu(v) \mathcal{H}_v, \quad \text{where} \quad \mathcal{H}_v = \mathcal{H}_0 \quad \text{for all} \quad v.$$

From the definition (4.1.26) of $U_1$, we deduce a corresponding action $U$ of $P_+^\uparrow$ on $\mathcal{H}$:

$$(U(M)\varphi)(v) = (U_1(M)f)(b(v)^{-1})$$

$$= f(b(v)^{-1}M)$$

$$= f((b(v)^{-1}M\,b(M^{-1}v))(b(M^{-1}v))^{-1})$$

$$= U_0(b(v)^{-1}M\,b(M^{-1}v))\,f(b(M^{-1}v)^{-1})$$

(since $b(v)^{-1}M\,b(M^{-1}v) \in \mathbb{R} \times \mathbb{E}^3$. This can be checked by showing that it lies in the little group of $v_0$.) Thus we find

$$(U(M)\varphi)(v) = U_0(b(v)^{-1}Mb(M^{-1}v))\varphi(M^{-1}v) \quad . \quad (4.1.29)$$

If we assume $U_0$ is a true representation of $\mathbb{R} \times \mathbb{E}^3$, then $U$ is a true representation of $P_+^\dagger$ :

$$(U(L)U(M)\varphi)(v) = U_0(b(v)^{-1}Lb(L^{-1}v))\,(U(M)\varphi)(L^{-1}v)$$

$$= U_0(b(v)^{-1}L\,b(L^{-1}v))\,U_0(b(L^{-1}v)^{-1}M\,b(M^{-1}L^{-1}v))\varphi((LM)^{-1}v)$$

$$= U_0(b(v)^{-1}LM\,b(LM)^{-1}v))\varphi((LM)^{-1}v)$$

$$= (U(LM)\varphi)(v). \qquad\qquad (4.1.30)$$

In fact, the two definitions $U$ and $U_1$ for true representations are equivalent since the mapping of $\mathcal{H}_1$ onto $\mathcal{H}$ is unitary and implements the equivalence.

Now suppose $U_0$ is a multiplier representation satisfying (4.1.19). Whether or not $\sigma_0$ is the restriction of a multiplier for $P_+^\dagger$, we retain the definition (4.1.29) of $U(M)$ exactly as it was in the

case of true representations. Then we find, similarly
to (4.1.30),

$$(U(L)U(M)\varphi)(v) = \lambda(L,M,v)(U(LM)\varphi)(v) \qquad (4.1.31)$$

where

$$\lambda(L,M,v) = \sigma_0(b(v)^{-1} L\, b(L^{-1}v), b(L^{-1}v)^{-1} Mb((LM)^{-1}v)). \qquad (4.1.32)$$

U will be a multiplier representation of $P_+^\uparrow$ only if
$\lambda(L,M,v)$ in independent of v. Now if we define a direct
integral representation $\pi(\mathcal{O}t)$ on the direct integral
space $\mathcal{H}$ by

$$\pi = \int_{\oplus V} d\mu(v)\, \pi_v \quad ,$$

where $\pi_v(A) = \pi_0(b(v)A)$ for all A in $\mathcal{O}t$ and v
in V, then U implements the Poincaré transformations
of $\mathcal{O}t$ in $\pi(\mathcal{O}t)$, i.e., we can show that for any A in
$\mathcal{O}t$ and L in $P_+^\uparrow$,

$$\pi(LA) = U(L)\pi(A)U(L)^{-1} . \qquad (4.1.33)$$

First we need an expression for $U(L)^{-1}$. Since U is
not a representation of $P_+^\uparrow$ in general, it will not be
simply $U(L^{-1})$, but we find that the operator V defined
on $\varphi \in \mathcal{H}$ by the requirement that

$$(V\varphi)(L^{-1}v) = \lambda^{-1}(L,L^{-1},v)(U(L^{-1})\varphi)(L^{-1}v) \qquad (4.1.34)$$

is the inverse $U(L)^{-1}$ of $U(L)$. For,

$$(U(L)V\varphi)(v) = U_0(b(v)Lb^{-1}(L^{-1}v))(V\varphi)(L^{-1}v)$$

$$= \lambda^{-1}(L,L^{-1},v)U_0(b(v)Lb^{-1}(L^{-1}v))[(U(L^{-1})\varphi)(L^{-1}v)]$$

$$= \lambda^{-1}(L,L^{-1},v)U_0(b(v)Lb^{-1}(L^{-1}v))U_0(b(L^{-1}v)L^{-1}b^{-1}(LL^{-1}v))\varphi(v)$$

$$= \lambda^{-1}(L,L^{-1},v)\lambda(L,L^{-1},v)\varphi(v)$$

$$= \varphi(v) \; ;$$

and

$$(VU(L)\varphi)(v) = (VU(L)\varphi)(L^{-1}v') \quad \text{where} \quad v' = Lv$$

$$= \lambda^{-1}(L,L^{-1},L^{-1}v')(U(L^{-1})U(L)\varphi)(L^{-1}v')$$

$$= \lambda^{-1}(L,L^{-1},L^{-1}v')\lambda(L^{-1},L,L^{-1}v')\varphi(L^{-1}v')$$

$$= \varphi(v).$$

Now

$$[U(L)\pi(A)U(L)^{-1}\varphi](v) = U_0(b(v)Lb^{-1}(L^{-1}v))[\pi(A)U(L)^{-1}\varphi](L^{-1}v)$$

$$= U_0(b(v)Lb^{-1}(L^{-1}v))\pi_0(b(L^{-1}v),A)(U(L)^{-1}\varphi)(L^{-1}v)$$

$$= U_0(b(v)Lb^{-1}(L^{-1}v))\pi_0(b(L^{-1}v)A)\lambda^{-1}(L,L^{-1},v)(U(L^{-1})\varphi(L^{-1}v)$$

$$= \lambda^{-1}(L,L^{-1},v)\pi_0(b(v)LA)U_0(b(v)Lb^{-1}(L^{-1}v))U_0(b(L^{-1}v)L^{-1}b^{-1}(v))\varphi(v)$$

$$= \pi_0(b(v)LA)\varphi(v)$$

$$= (\pi(LA)\varphi)(v).$$

Thus the representation $\pi$ satisfies the conditions
to be a covariant representation given in section 2.2, and
so the representation determines a good relativistic

theory. But since U is not actually a representation of $P_+^\uparrow$ in any usual sense, $\pi$ is not a so-called "explicitly covariant" representation of $\mathcal{O}$. In fact, although U(L)U(M) and U(LM) implement the same symmetry, we see from (4.1.31) that they differ by an element of $\pi(\mathcal{O})'$. That is, we have an example in which certain Poincaré symmetries mix the coherent sectors (in this case components in a direct integral rather than subspaces of $\mathcal{H}$) and so the most general description of the relationship between operators implementing Poincaré transformations is needed.

We now return to the displaced Fock representations of the scalar massless boson field, and apply the construction we have developed in order to obtain a relativistic theory. We now consider arbitrary $\mathbb{R} \times \mathbb{E}^3$-covariant representations $\pi_0$; not just the one associated with $\psi^x$ as defined in (4.1.5). Then, in general, $V_0$ defined in (4.1.12), is a multiplier representation of the covariance group $\mathbb{R} \times \mathbb{E}^3$ of $\pi_0$, so we expect to construct a representation $\pi$ in which all Poincaré symmetries are implemented, but the unitary operators implementing the symmetries satisfy a condition of the form (4.1.31), rather than the more restrictive requirement of forming a multiplier representation.

First, for $v \in V$, and for $b(v) \in P_+^\uparrow$, defined such that (4.1.27) holds, we define

$$\pi_v(A) = \pi_0(b(v)A), \quad \text{for } A \in \mathcal{O},$$

the Weyl algebra over $M$.

Then in particular,

$$\pi_v(W(\phi)) = \pi_0(W(U(b(v))\phi))$$

$$= e^{i \, Im \langle \Psi^x, \, U(b(v))\phi \rangle} W_F(U(b(v))\phi) \quad (4.1.35)$$

where  U  implements Poincaré transformations in  M.

$\pi_v$  is equivalent to the displaced Fock representation corresponding to the displacement

$$\Psi^x_v = U^x(b(v))\Psi^x \quad ,$$

where  $U^x$  is the dual action of  $P^\uparrow_+$  on  $M^x$.  (This follows from the fact that all Poincaré transformations are implemented in the Fock representation of  $\mathcal{A}$ .)

Form the direct integral of these inequivalent displaced Fock representations,

$$\pi = \int_{\oplus V} d\mu(v) \, \pi_v$$

which acts on the direct integral space  $\mathcal{H}$  given by

$$\mathcal{H} = \int_{\oplus V} d\mu(v) \, \mathcal{H}_v \quad (4.1.36)$$

where each  $\mathcal{H}_v$  is isomorphic to  $\mathcal{H}_0$.  Then for  $L \in P^\uparrow_+$, the operator  $V(L): \mathcal{H} \to \mathcal{H}$,  defined for  $\phi \in \mathcal{H}$  by

$$(V(L)\phi)(v) = V_0(b(v)^{-1} L \, b(L^{-1}v))\phi(L^{-1}v) \quad (4.1.37)$$

implements  L  in  $\pi(\mathcal{A})$.

Now there is a natural isomorphism between  $\mathcal{H}$  as defined in (4.1.36) and the space

$$H = L^2(\mathbb{R}, d\mu) \otimes \mathcal{H}_F . \qquad (4.1.38)$$

If we write the measure $d\mu$ in terms of $p$,

$$d\mu(p) = \frac{1}{m^2} d^3p/(\underline{p}^2+m^2)^{\frac{1}{2}} ,$$

we see that the $L^2$-space is the one-particle space for a particle of mass $m$. We shall return to this point in section 4.2 where we construct a space similar to $H$ for the electromagnetic field, and interpret it in terms of an approximate scattering theory.

We now consider how to define the charge transfer operator in this model. First, for the charge of the representation $\pi_0$, or in general $\pi_v$, $v \in V$, the associated charge transfer operator is as described in chapter 2. We define

$$\Gamma_v(x) = W_F(\psi_v(-x)) \qquad (4.1.39)$$

where $\psi_v(x) = \Psi_v^x - U^x(x)\Psi_v^x$, and $\Psi_v^x = U^x(b(v))\Psi^x$ and as in (2.3.6), this satisfies the modified 1-cocycle identity

$$\Gamma_v(x)\tau_x(\Gamma_v(y)) = e^{-\frac{1}{2}i \, Im \langle \psi_v(x), \psi_v(-y)\rangle} \Gamma_v(x+y). \qquad (4.1.40)$$

The c-number function

$$\omega_v(x,y) = e^{-\frac{1}{2}i \, Im \langle \psi_v(x), \psi_v(-y)\rangle} \qquad (4.1.41)$$

is also the multiplier for the representation $V_v$ of the translation group, defined by

$$V_v(x) = \Gamma_v(x)V_F(x),\qquad\qquad (4.1.42)$$

and it is well-known that such a multiplier satisfies
the 2-cocycle identity

$$\omega_v(x+y,z)\,\omega_v(x,y) = \omega_v(x,y+z)\,\omega_v(y,z)\qquad (4.1.43)$$

For different $v$, the $\Gamma_v$ can be regarded as implementing
the same charge transfer in different velocity frames.
If we regard each component $\mathcal{H}_v$ of $\mathcal{H}$ as providing the
states corresponding to a different velocity frame of
reference, then the transfer of charge associated with $\pi$
should be implemented by the operators $\Gamma(x)$ acting on
$\mathcal{H}$, defined by

$$\Gamma(x) = \int_{\oplus V} d\mu(v)\,\Gamma_v(x).\qquad\qquad (4.1.44)$$

It satisfies

$$\Gamma(x)\,\tau_x(\Gamma(y)) = \omega(x,y)\,\Gamma(x+y)\qquad\qquad (4.1.45)$$

where $\qquad \omega(x,y) = \displaystyle\int_{\oplus V} d\mu(v)\,\omega_v(x,y)$

is a 2-cocycle with values in the unitary operators of
$\pi(\mathcal{A})'$.

For a classification of the superselection
sectors obtained as direct integrals of displaced Fock
representations, we need to know which displacements
$\psi^x$ will give rise, via the construction described above,
to equivalent sectors. The available $\mathbb{R} \times \mathbb{E}^3$-covariant

displaced Fock representations $\pi_0$ from which to develop the inducing construction are determined by elements $\psi^x$ of $\bar{D}^+(\mathbb{R} \times \mathbb{E}^3)$. If $\psi^x_{(1)}$ and $\psi^x_{(2)}$ are in $\bar{D}^+(\mathbb{R} \times \mathbb{E}^3)$ and

$$\psi^x_{(1)} - \psi^x_{(2)} \in K ,$$

then $\psi^x_{(1)}$ and $\psi^x_{(2)}$ give rise to equivalent representations $\pi_0^{(1)}$ and $\pi_0^{(2)}$. The inducing construction based on $\pi_0^{(1)}$ and $\pi_0^{(2)}$ will therefore lead to equivalent representations $\pi^{(1)}$ and $\pi^{(2)}$. If

$$\psi^x_{(1)} - \psi^x_{(2)} \notin K ,$$

then certainly $\pi_0^{(1)}$ is inequivalent to $\pi_0^{(2)}$, but we may have, for some $v$,

$$\psi^x_{(1)} - U^x(b(v))\psi^x_{(2)} \in K , \qquad (4.1.46)$$

so that $\pi_0^{(1)}$ is equivalent to $\pi_v^{(2)}$. Then the inducing construction will again give rise to equivalent representations $\pi^{(1)}$ and $\pi^{(2)}$. Therefore the 1-cohomology classification of displaced Fock representations does not extend to the induced representations constructed from them.

In analogy with the local 1-cohomology structure for theories involving gauge transformations of the first kind, Roberts (1977) has suggested that a local 2-cohomology structure might exist for gauge theories. He defines a 2-cocycle which is related to the charge

transfer operator in the way we have described for $\omega$, (4.1.45). We consider the information which a local 2-cohomology analysis might yield in the present model.

We must first analyse the multiplier more closely. Putting $v=0$ in (4.1.41), we have

$$\omega_0(x,y) = e^{-\frac{1}{2}i \, \text{Im} \langle \psi(x), \psi(-y) \rangle} \qquad (4.1.47)$$

as the multiplier for the representation $\pi_0$ corresponding to the displacement $\Psi^x$ with

$$\psi(x) = \Psi^x - U^x(x)\Psi^x .$$

Formally, we can write

$$\text{Im}\langle \psi(x), \psi(-y) \rangle = \zeta(x) + \zeta(y) - \zeta(x+y) \qquad (4.1.48)$$

where

$$\zeta(x) = \text{Im}\langle \Psi^x, U^x(x)\Psi^x \rangle , \qquad (4.1.49)$$

and if it can be shown that this formal expression for $\zeta$ has a well-defined finite value for each $x$, the 2-cocycle $\omega_0$ will be a 2-coboundary. It will then be possible to replace $V_0(x)$ with a true representation $e^{i\zeta(x)} V_0(x)$, and the corresponding charge transfer operator $\Gamma_0$ will satisfy

$$\Gamma_0(x) \tau_x (\Gamma_0(y)) \Gamma_0(x+y)^* = 1. \qquad (4.1.50)$$

Now from Araki (1970) theorem 8.6, we can deduce that a $\zeta$ satisfying (4.1.48) can always be found for $\Psi^x \in \bar{D}^+$

(although (4.1.49) may need to be modified.)  However,
we shall see that  $\zeta(x)$  need not tend to zero as  $x \to \infty$
in a spacelike direction, so that the corresponding
local 2-cocycle is not necessarily asymptotically local.

We emphasize that in the example of the massless
scalar boson field, a non-trivial local 2-cohomology
structure can only arise through "bad" choices for  $\Psi^x$.
This is because although  $\Psi^x \in \bar{D}^+(\mathbb{R} \times \mathbb{E}^3)$  is not
necessarily rotation invariant, we saw at the beginning
of this chapter that there is always a rotation-invariant
$\phi^x \in \bar{D}^+$  such that

$$\phi^x = \Psi^x + \Lambda, \quad \Lambda \in K,$$

giving rise to an equivalent representation.  And the
multiplier  $\omega_0$  corresponding to a rotation-invariant
displacement is identically  1.  We show this as
follows:

Write

$$\text{Im}\langle \psi(x),\psi(-y)\rangle = \zeta(x) + \zeta(y) - \zeta(x+y)$$

where, formally at least,

$$\zeta(x) = \text{Im}\langle \Psi^x, U^x(x)\Psi^x\rangle.$$

Suppose  $\Psi^x$  is rotation invariant.
$\text{Im}\langle \Psi^x, U^x(x)\Psi^x\rangle$  may be expressed in terms of the Fourier
transform  $\tilde{\Psi}^x$  which is defined as a function on the
light cone by

$$\tilde{\psi}^x(\underline{k}) = \frac{1}{(2\pi)^{3/2}} \int (k_0 f(\underline{x}) - i\, g(\underline{x})) e^{-i\,\underline{k}\cdot\underline{x}} d^3x$$

where

$$f(\underline{x}) = \psi^x(\underline{x},0)$$

$$g(\underline{x}) = \dot{\psi}^x(\underline{x},0).$$

Using this and the inverse relationship

$$\psi^x(x) = \int \frac{d^3k}{2|\underline{k}|}(\tilde{\psi}^x(\underline{k}) e^{-i\,k\cdot x} + \overline{\tilde{\psi}^x(\underline{k})} e^{i\,k\cdot x})$$

(where $k.x = |\underline{k}|x - \underline{k}.\underline{x}$)

we find

$$\text{Im}\langle \psi^x, U^x(x)\psi^x \rangle = \int (\psi^x(\underline{x}',0)\frac{d}{dt}(U^x(\underline{x})\psi^x(\underline{x}',0)) \cdot$$

$$- \dot{\psi}^x(\underline{x}',0)U^x(x)\psi^x(\underline{x}',0)d^3x'$$

$$= \int \frac{d^3k}{|\underline{k}|} \tilde{\psi}^x(\underline{k})\overline{\tilde{\psi}^x(\underline{k})}\sin k.x$$

$$= \int \omega\, d\omega \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, \tilde{\psi}^x(\underline{k})\overline{\tilde{\psi}^x(\underline{k})}\sin(\omega r \cos\theta). \quad (4.1.51)$$

where $|\underline{x}| = r$, and the polar coordinates $\omega, \theta, \varphi$ for $\underline{k}$ are chosen such that $\theta$ is the angle between $\underline{k}$ and $\underline{x}$.

Since

$$\int_0^{\pi} \sin\theta\, \sin(\omega r \cos\theta) d\theta = 0$$

and $\psi^x$ is rotationally symmetric, we find

$$\text{Im}\langle \psi^x, U^x(x)\psi^x \rangle = 0$$

and hence

$$\text{Im}\langle \psi(x), \psi(-y) \rangle = 0.$$

Multiplier representations are still of interest, for the following reasons. First, in the case of non-zero spin, $\mathbb{R} \times \mathbb{E}^3$ covariant displaced Fock representations cannot be obtained, and we have the possibility of a non-trivial 2-cohomology. Also, if we allow more general kinds of representations, the multiplier structure may again become interesting. To investigate how this might work, we continue to use the simpler case of the scalar field. We seek conditions under which the multiplier associated with $\Psi^x \in \bar{D}^+(\mathbb{R}^{3+1})$ is a local 2-coboundary. We say that $\omega_0(x,y)$ is a local 2-coboundary if

$$\omega_0(x,y) = \zeta(x) + \zeta(y) - \zeta(x+y)$$

with $\zeta(x) \to 0$ as $x \to \infty$ (spacelike).
In the present case, where $\omega_0(x,y)$ is given by

$$\omega_0(x,y) = \text{Im}\langle \psi(x), \psi(-y)\rangle \ ,$$

Araki's results (1970, theorem 8.6) show that $\zeta(x)$ has the form

$$\zeta(x) = \text{Im}\langle \Psi^x, U^x(x)\Psi^x - \Lambda(x)\rangle \qquad (4.1.52)$$

where $\Lambda$ is linear in $x$.

If $\text{Im}\langle \Psi^x, U^x(x)\Psi^x\rangle$ is a well-defined quantity, we may choose $\Lambda \equiv 0$.

We attempt to characterize $\zeta$ in the cases $\Psi^x \in K$, and $\Psi^x \in \bar{D}^+(\mathbb{R}^{3+1}) \backslash K$.

Choose polar coordinates $(\omega, \theta, \varphi)$ for $\underline{k}$-space, and suppose that for small $\omega$,

$$\tilde{\psi}^x(\underline{k}) \sim \omega^{-\alpha}\Gamma(\theta,\varphi) \qquad (4.1.53)$$

Assuming that the only possible divergence problems occur for small $\omega$, we can give conditions on $\alpha$ which correspond to $\psi^x \in K$, $\psi^x \in \bar{D}^+\backslash K$ etc.

For $\psi^x \in K$, we require

$$\int_0 |\tilde{\psi}^x|^2 \frac{d^3k}{|\underline{k}|} < \infty$$

i.e.
$$\int_0 \frac{1}{\omega^{2\alpha}}(\omega^2 \frac{d\omega}{\omega}) < \infty$$

i.e.
$$\int_0 \frac{1}{\omega^{2\alpha-1}} d\omega < \infty$$

so that $\qquad \alpha < 1.$

The infinitesimal condition that $\psi^x \in \bar{D}^+$ is

$$\int_0 |\tilde{\psi}^x|^2 \omega^2 \cdot (\omega^2 \frac{d\omega}{\omega}) < \infty$$

i.e.
$$\int_0 \frac{1}{\tilde{\omega}^{2\alpha-3}} d\omega < \infty$$

so that $\qquad \alpha < 2.$

In fact, if we also require $\psi^x$ to have finite classical energy, we find $\alpha < 3/2$.

In summary, covariant non-trivial displaced Fock representations correspond to $1 \leqslant \alpha < 2$, and for those with finite energy, $1 \leqslant \alpha < 3/2$. If the small $\omega$ behaviour of $\psi^x$ is as specified in (4.1.53) then the formal expression (4.1.51) for $\zeta(x) = \text{Im}\langle \psi^x, U^x(x)\psi^x\rangle$

has the following form:

$$\zeta(\underline{x}) \sim \int d^3k \; \omega^{-2\alpha-1} q(\omega,\theta,\varphi) \sin \underline{k}.\underline{x}, \qquad (4.1.54)$$

where $q$ is continuous at $\omega=0$.

(Since we will require the limit as $x$ tends to infinity in a spacelike direction, we assume for simplicity $x_0=0$.)

Again we take $\theta$ to be the angle between $\underline{k}$ and $\underline{x}$, and (4.1.54) may be written

$$\zeta(\underline{x}) = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \int_0^\infty d\omega \; \omega^2 \sin\theta \frac{1}{\omega^{2\alpha+1}} q(\omega,\theta,\varphi) \sin(\omega r\cos\theta)$$

(where $r = |\underline{x}|$.)

$$= \int_{-1}^1 dz \int_0^{2\pi} d\varphi \int_0^\infty d\omega \; \frac{1}{\omega^{2\alpha+1}} \sin(\omega rz) q(\omega,z,\varphi) \qquad (4.1.55)$$

where $z = \cos\theta$.

Let $\zeta_0$ be the contribution to $\zeta$ from the $0$ to $\varepsilon$ integration range of $\omega$, and $\zeta_\infty$ be the contribution from $\omega=\varepsilon$ to $\infty$. Consider first $\zeta_0$. For small $\omega$, we have

$$\sin \omega rz \sim \omega rz, \quad \text{and by}$$

continuity,

$$q(\omega,z,\varphi) \approx q(0,z,\varphi),$$

so that

$$\zeta_0(\underline{x}) \approx \int_{-1}^1 dz \int_0^{2\pi} d\varphi \int_0^\varepsilon d\omega \; \frac{1}{\omega^{2\alpha-2}} zrq(0,z,\varphi), \qquad (4.1.56)$$

which is convergent provided $\alpha < 3/2$.

The contribution from the $\varepsilon$ to $\infty$ integration is finite, since

$$|\zeta_\infty(\underline{x})| \leq \int_{-1}^{1} dz \int_{0}^{2\pi} d\varphi \int_{\varepsilon}^{\infty} d\omega \; \omega^{1-2\alpha} |q(\omega,z,\varphi)| < \infty, \quad (4.1.57)$$

so that $\zeta_\infty(\underline{x})$ is a bounded function of $\underline{x}$.

Hence for $\alpha < 3/2$: that is, for finite energy $\psi^x$, $\zeta(\underline{x}) = \text{Im}\langle \psi^x, U^x(\underline{x})\psi^x \rangle$ is well-defined, and we may take $\Lambda = 0$ in (4.1.52).

To determine for which values of $\alpha$ the multiplier $\omega$ is a local coboundary, we now look at the behaviour of $\zeta$ as a function of $\underline{x}$.

We have

$$\zeta_0(\underline{x}) \approx \int_{-1}^{1} dz \int_{0}^{2\pi} d\varphi \int_{0}^{\varepsilon} d\omega \; \omega^{1-2\alpha} \sin(\omega r z) q(0,z,\varphi)$$

$$= r^{2\alpha-2} \int_{-1}^{1} dz \int_{0}^{2\pi} d\varphi \int_{0}^{r\varepsilon} d\lambda \; \lambda^{1-2\alpha} \underline{q}(0,z,\varphi) \sin\lambda z$$

where $\lambda = r\omega$.

Since the integral is convergent, for large $r$ the integral from $0$ to $r\varepsilon$ may be replaced by an integral from $0$ to $\infty$, and

$$\zeta_0(\underline{x}) \approx r^{2\alpha-2} \int_{-1}^{1} dz \int_{0}^{2\pi} d\varphi \int_{0}^{\infty} d\lambda \; \lambda^{1-2\alpha} q(0,z,\varphi) \sin(\lambda z)$$

$$= \text{constant.} \quad r^{2\alpha-2} \qquad\qquad (4.1.58)$$

Using the Riemann-Lebesgue Lemma, we deduce that for large $|\underline{x}|$, $\zeta_\infty(\underline{x}) \to 0$. Hence, for $\alpha \geq 1$ the asymptotic behaviour of $\zeta$ is given by (4.1.58).

It is also clear that for $\alpha < 1$, $\zeta(\underline{x}) \to 0$ as $|\underline{x}| \to \infty$.

For the constant in (4.1.58) to vanish in every $\underline{x}$-direction, rotational symmetry would be required.

In the case where rotational covariance does not hold, we draw the following conclusions. For $\Psi^x \in K$, the multiplier defined in (4.1.47) is a local 2-coboundary, and if $\Psi^x \in \bar{D}^+ \setminus K$ and $\langle \Psi^x, P^0 \Psi^x \rangle < \infty$, $\omega_0$ is a coboundary but it is not asymptotically local. The latter result also holds for any $\Psi^x \in \bar{D}^+$, but if $\Psi^x$ does not have finite energy, a term linear in $\underline{x}$ must be introduced into $\zeta$ (cf. (4.1.52)) in order that it should be well-defined.

We have shown that if rotational covariance does not hold, the local 2-cohomology classes of $\omega_0$ correspond to equivalence classes of representations in the same way that 1-cohomology classes of $\psi(x) = \Psi^x - U^x(x)\Psi^x$ do.

The classification of $\omega$ defined by

$$\omega(x,y) = \int d\mu(v)\, \omega_v(x,y)$$

presents no further problems, since $U^x(b(v))\Psi^x$ lies in $K$ or $\bar{D}^+$ respectively if and only if $\Psi^x$ is in $K$ or $\bar{D}^+$ respectively. Hence each component $\omega_v$ is a local 2-coboundary exactly when $\omega_0$ is.

Hence in the very simple structure we have described, the local 2-cohomology classification of 2-cocycles $\omega$ corresponding to the charge transfer operators is related to the classification of super-selection sectors, but does not provide any new information

which was not present in the 1-cohomology analysis. A
richer structure could be expected in a case where the
2-cocycles took values in a non-commutative algebra.

## 4.2    The electromagnetic field

We now attempt to apply the above construction to the electromagnetic field. We use displaced Fock representations where the displacement corresponds to an external classical current, and interpret the model as an approximate interacting theory for interaction with an asymptotic or classical external field.

The quantum theory of the free electromagnetic field can be formulated as a Weyl system along the lines of the quantization procedure described in chapter 2. Symbolically, the field is usually described in terms of the four component potential $A^\mu$ ($\mu=0,1,2,3$). In terms of this potential, the classical equations of motion are

$$\Box A^\mu = 0, \tag{4.2.1a}$$

subject to

$$\partial_\mu A^\mu = 0. \tag{4.2.1b}$$

There are only two independent components, since only the field strengths

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{4.2.2}$$

are observable. One way of eliminating unphysical degrees of freedom is to choose test functions

$$f = (f_\mu)$$

so that the smeared fields $A(f)$, formally given by

$$A(f) = \int A^{\mu}(x) f_{\mu}(x) \, dx \qquad (4.2.3)$$

do not distinguish between potentials giving rise to the same values of $F^{\mu\nu}$. Thus they are chosen to be of the form

$$f_{\mu} = \partial^{\nu} f_{\mu\nu} \qquad (4.2.4)$$

where $f_{\mu\nu}$ is antisymmetric (and of course with appropriate smoothness and support properties.) The Weyl algebra of the electromagnetic field is then defined over certain equivalence classes of these test functions. (Heuristically we may think of the Weyl operators as $e^{i A(f)}$ and the symplectic form is deduced from the commutation relations of the $A^{\mu}$.) For example Roepstorff (1970) uses this approach in a discussion of the infrared sectors. It is probably the quantization of the electromagnetic field which follows the procedure discussed in chapter 2 most closely. However we wish to discuss the gauge and covariance structure of the theory and we need a formulation which shows this structure clearly; in particular, for this purpose it will be necessary to have an algebra of operators associated with the unrestricted electromagnetic potential. We therefore adopt a more complex algebraic description of the quantized electromagnetic field as a Weyl system developed by Carey, Gaffney and Hurst (1977).

In this approach we first define a Weyl algebra corresponding to the unrestricted four-vector potential,

and obtain a description of the electromagnetic field
by finding a representation of a quotient algebra of
a subalgebra of this algebra of the vector potential.

The algebra of the vector potential will correspond
in some sense to the smeared potentials

$$A(f) = \int d^4x A^\mu(x) f_\mu(x) \qquad (4.2.5)$$

where there are now no symmetry restrictions on the test
functions $(f_\mu)$, each $f_\mu$ belonging to Schwartz space
$S(\mathbb{R}^4)$. Initially we write the exponential

$$U(f) = e^{i A(f)}, \quad f_\mu \in S(\mathbb{R}^4) \quad \mu=0,1,2,3 \quad (4.2.6)$$

as describing the operators which should generate the
Weyl algebra of the vector potential. But again equivalence
classes of functions $f$ give rise to the same operator
$U(f)$, and these equivalence classes may be identified
with real solutions of the wave equation,

$$\Box\hat{\varphi}_\mu = 0 \quad (\mu=0,1,2,3) \qquad (4.2.7)$$

by $f \to \hat{\varphi}$, where $\hat{\varphi}$ has components

$$\hat{\varphi}_\mu = D*f_\mu . \qquad (4.2.8)$$

We denote the corresponding $U(f)$ by $W(\varphi)$.

From the singular commutation relations

$$[A^\mu(x), A^\nu(x')] = -ig^{\mu\nu}D(x-x') \qquad (4.2.9)$$

where  D  is the solution of  $\Box D = 0$  with initial data

$$D(\underline{x},0) = 0, \quad \frac{\partial}{\partial t} D(\underline{x},t)\Big|_{t=0} = -\delta(\underline{x}),$$

we expect the multiplication rule for the Weyl algebra
of the vector potential

$$W(\hat{\varphi})W(\hat{\varphi}') = e^{-\frac{i}{2}B(\hat{\varphi},\hat{\varphi}')} W(\hat{\varphi}+\hat{\varphi}') \qquad (4.2.10)$$

where

$$B(\hat{\varphi},\hat{\varphi}') = -\int d^3x (\hat{\varphi}^\mu(\underline{x},0)\dot{\hat{\varphi}}'_\mu(\underline{x},0) - \dot{\hat{\varphi}}^\mu(\underline{x},0)\hat{\varphi}'_\mu(\underline{x},0))$$

For technical reasons associated with the
construction of the quotient algebra, the Manuceau
C*-algebra formulation of Weyl algebras (Manuceau 1968,
see also Emch 1972, p. 236) is used to form the C*-algebra
of the electromagnetic potential. We now summarize the
construction, following Carey and Hurst (1979).
(Results quoted without proof are drawn from this paper
or references therein.)

We first select a suitable space of test functions
over which to define a Weyl algebra for the vector
potential. We construct  $\mathbb{R}^4$  valued solutions  $\hat{\varphi}$  of
the wave equation

$$\Box\hat{\varphi} = 0 \qquad (4.2.11)$$

corresponding to the Cauchy data

$$\hat{\varphi}(\underline{x},0) = f(x)$$
$$\dot{\varphi}(\underline{x},0) = g(\underline{x}) \qquad (4.2.12)$$

where $f,g$ lie in the set $\mathcal{D}$ of $C^\infty$ functions with compact support in $\mathbb{R}^3$, taking values in $\mathbb{R}^4$. $\hat{\varphi}$ corresponding to $(f,g)$ may be defined as follows: first define $\varphi: X_0^+ \to \mathbb{C}^4$ by

$$\varphi(\underline{p}) = \frac{1}{(2\pi)^{3/2}} \int (\omega f(\underline{x}) - i\, g(\underline{x})) e^{-i\,\underline{p}\cdot\underline{x}} d^3x \qquad (4.2.13)$$

where, as in the previous section, $X_0^+ = \{p \in \mathbb{R}^4 : p_0 > 0, p^2 = 0\}$, and $\omega = |\underline{p}| = p_0$. Then

$$\hat{\varphi}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p / (2\omega) (\varphi(\underline{p}) e^{-i\,p\cdot x} + \varphi(\underline{p}) e^{i\,p\cdot x}) \qquad (4.2.14)$$

where $p.x = \omega x_0 - \underline{p}.\underline{x}$, satisfies (4.2.5) and has the appropriate Cauchy data (4.2.12). The space of solutions $\hat{\varphi}$ of the wave equation with Cauchy data in $\mathcal{D} \times \mathcal{D}$ is denoted by $M_0$. $M_0$ has a symplectic form, as in (4.2.10),

$$B(\hat{\varphi}, \hat{\varphi}') = \frac{1}{i} \int \frac{d^3p}{2\omega} (\varphi^\mu(\underline{p}) \overline{\varphi'_\mu(\underline{p})} - \overline{\varphi^\mu(\underline{p})} \varphi'_\mu(\underline{p})) \qquad (4.2.15)$$

Defining a complex structure $J_F$ on $M$ by

$$(J_F \varphi)(\underline{p}) = -ig\varphi(\underline{p}) \qquad (4.2.16)$$

(where $g = \mathrm{diag}(1,-1,-1,-1)$), we obtain an inner product of which $B$ is the imaginary part:

$$\langle \hat{\varphi}, \hat{\varphi}' \rangle = B(\hat{\varphi}, J_F \hat{\varphi}') + iB(\hat{\varphi}, \hat{\varphi}') \qquad (4.2.17)$$

The completion of $M_0$ in the corresponding norm is a

Hilbert space, denoted by M. M and $M_0$ are invariant under the natural action of the Poincaré group: for $(a,\Lambda) \in P_+^\uparrow$,

$$(U(a,\Lambda)\hat{\varphi})(x) = \Lambda\hat{\varphi}(\Lambda^{-1}(x-a)). \qquad (4.2.18)$$

This action is not unitary as the inner product is not invariant, although B is.

A Weyl algebra over $(M_0,B)$ is defined in the Manuceau formulation as follows. For $\hat{\varphi} \in M_0$, let $W(\varphi)$ be the function on $M_0$ defined by

$$(W(\varphi))(\varphi') = 1 \quad \text{if} \quad \varphi = \varphi' \qquad (4.2.19)$$
$$0 \quad \text{otherwise.}$$

We define a *-algebra generated by the $W(\varphi)$ with the following specification of conjugation and products:

$$(\sum_i \lambda_i W(\varphi_i))^* = \sum_i \bar{\lambda}_i W(-\varphi_i) \qquad (4.2.20)$$

$$W(\varphi)W(\varphi') = e^{-\frac{1}{2}i\, B(\varphi,\, \varphi')} W(\varphi+\varphi') \qquad (4.2.21)$$

and closing it in a suitable norm to form the C*-algebra of the CCR over $M_0$, denoted $\Delta_c(M_0)$. (The requirement that it be a C*-algebra determines the norm, which is given, for example, in Carey, Gaffney and Hurst 1977.)

$\Delta_c(M_0)$ is the algebra of the vector potential. Its Fock representation $\pi_F$ is determined by the generating functional

$$\sigma_F(\varphi) = \exp -\tfrac{1}{4}B(\varphi,J_F\varphi). \qquad (4.2.22)$$

It turns out that the Poincaré invariant subspaces
of $M_0$ are related to the supplementary condition and
gauge requirements which deal with the unphysical degrees
of freedom of the electromagnetic potential. The
invariant subspaces are

$$N_0 = \{\hat{\varphi} \in M_0 : \partial^\mu \hat{\varphi}_\mu(x) = 0\} ,$$

$$T_0 = \{\hat{\varphi} \in M_0 : \hat{\varphi}_\mu(x) = \partial_\mu \chi(x) \quad \text{for some} \quad \chi : \mathbb{R}^4 \to \mathbb{R}\} \qquad (4.2.23)$$

and the corresponding Poincaré invariant closed subspaces
of $M$ are

$$N = \{\hat{\varphi} \in M : p^\mu \varphi_\mu(\underline{p}) = 0\}$$

$$T = \{\hat{\varphi} \in M : \varphi_\mu(\underline{p}) = p_\mu \chi(\underline{p}) \quad \text{where} \quad \chi : X_0^+ \to C\} \qquad (4.2.24)$$

Algebras $\Delta_c(N_0)$ and $\Delta_c(T_0)$ are generated by the
appropriate $W(\varphi)$ in the same way as $\Delta_c(M_0)$.

We see that $T$ provides the Lorentz gauge
transformations (heuristically $A_\mu \to A_\mu + \partial_\mu \chi$ with
$\Box\chi = 0$) by the following calculation. For $\psi \in T$,
$W(\psi)$ implements an inner automorphism of $\Delta_c(M)$.
Consider in particular the image of elements $W(\varphi)$
under this automorphism:

$$W(\varphi) \to W(\psi) W(\varphi) W(-\psi) = e^{-i B(\varphi, \psi)} W(\varphi) \qquad (4.2.25)$$

To make contact with the heuristic formalism, we write

$$W(\varphi) = e^{i A(f)}$$

for some $f$ such that $\varphi_\mu = D * f_\mu$,

where $A(f) = \int A^\mu(x) f_\mu(x) dx$.

Since $\psi \in T$, we have

$$\hat{\psi}_\mu = \partial_\mu \chi \quad \text{for some solution} \quad \chi \quad \text{of} \quad \Box \chi = 0.$$

The exponent of (4.2.25) is given by

$$-B(\varphi, \psi) + \int A^\mu(x) f_\mu(x) dx . \qquad (4.2.26)$$

Now

$$B(\varphi, \psi) = \int d^3x (\dot{\varphi}_\mu(\underline{x}, 0) \psi^\mu(\underline{x}, 0) - \varphi_\mu(\underline{x}, 0) \dot{\psi}^\mu(\underline{x}, 0))$$

$$= \int dx' \int d^3x (\dot{D}(\underline{x}-\underline{x}', 0) f_\mu(x') \partial^\mu \chi(\underline{x}, x_0') - D(\underline{x}-\underline{x}', 0) f_\mu(x') \partial^\mu \dot{\chi}(\underline{x}, x_0'))$$

$$= \int dx' \int d^3x (-\delta(\underline{x}-\underline{x}') f_\mu(x') \partial^\mu \chi(\underline{x}, x_0'))$$

$$= -\int d^4x \, f_\mu(x) \partial^\mu \chi(x), \qquad (4.2.27)$$

so that (4.2.26) has the form

$$\int (A^\mu(x) + \partial^\mu \chi(x)) f_\mu(x) dx,$$

a gauge transformation of $A(f)$.

Furthermore, we find that $\Delta_c(T_0)$ is the algebra of the supplementary condition operators. For, if

$$\hat{\psi}_\mu = \partial_\mu \chi \in T_s,$$

$$W(\psi) = \exp i A(f)$$

where $\hat{\varphi}_\mu = D * f_\mu$, and hence $\chi = D * g$, with

$$\frac{\partial g}{\partial x^\mu} = f_\mu \ .$$

Then $A(f) = \int dx \ A^\mu(x) \frac{\partial g}{\partial x^\mu}(x)$

$$= \int dx \ \partial_\mu A^\mu(x) \ g(x) \ ,$$

and $\partial_\mu A^\mu$ is the supplementary condition in the heuristic formulation.

The algebra $\Delta_c(N_0)$ is the commutant of $\Delta_c(T_0)$ in $\Delta_c(M_0)$, and $\Delta_c(T_0)$ is the centre of $\Delta_c(N_0)$. Since physical operators must commute with the supplementary condition, we expect all physically relevant quantities to be elements of $\Delta_c(N_0)$. We must therefore restrict attention to this algebra when seeking a representation which describes the physical electromagnetic field. A representation of $\Delta_c(N_0)$ may still contain non-physical degrees of freedom, since the algebra contains elements which differ by a multiple of the supplementary condition operator. To deal with this, we shall choose a representation in which elements of the supplementary condition algebra, $\Delta_c(T_0)$, act as scalars. Such a representation is found as follows. The Fock representation $\pi_F$ of $\Delta_c(M_0)$ restricted to $\Delta_c(N_0)$ may be realized as a direct integral which diagonalizes $\pi_F(\Delta_c(T_0))''$, that is, the von Neumann algebra generated by $\Delta_c(T_0)$. Then, as required, elements of $\Delta_c(T_0)$ will act as scalars in component representations of the direct integral decomposition.

To achieve this, we need a direct integral over the orthogonal complement $T^\perp$ of $T$ in $M$. The component

representation $\pi_\zeta(\Delta_c(N_0))$ corresponding to $\zeta \in T^\perp$ is specified by the generating functional $\pi_\zeta$ on $\Delta_c(N_0)$ determined by its action on $W(\varphi)$, $\varphi \in N_0$:

$$\pi_\zeta(W(\varphi)) = \exp(-\tfrac{1}{2}C(\varphi,\varphi) + iB(\zeta,\varphi)) \qquad (4.2.28)$$

where
$$C(\varphi,\varphi) = \int \overline{\hat{\varphi}^\mu(\underline{k})} \, \hat{\varphi}_\mu(\underline{k}) \, \frac{d^3k}{2|\underline{k}|} \; .$$

If $\pi_0$ is determined by $\pi_0(W(\varphi)) = \exp(-\tfrac{1}{2}C(\varphi,\varphi))$, it follows that the representation $\pi_\zeta$, for each $\zeta$ in $T^\perp$ is obtained from $\pi_0$ by the automorphism

$$\alpha_\zeta : W(\varphi) \to e^{iB(\zeta,\varphi)} W(\varphi) \qquad (4.2.29)$$

and therefore the representation spaces $\mathcal{H}_\zeta$ for the $\pi_\zeta$ may be chosen to be identical.

A calculation analogous to (4.2.27) shows that the automorphism (4.2.29) is essentially a displacement

$$A^\mu \to A^\mu - \zeta^\mu.$$

Thus we have representations of $\Delta_c(N_0)$ which are related by displacements of the potential.

The Hilbert space of the Fock representation can be written

$$\mathcal{H} = \int_{\zeta \in T^\perp} \mathcal{H}_\zeta \, d\mu(\zeta) \qquad (4.2.30)$$

where $\mu$ is specified by its characteristic function

$$\psi \to \exp\left(-\tfrac{1}{4}B(\psi, J_F \psi)\right).$$

The functions $F$ on $T^\perp$ with $F(\zeta) \in \mathcal{H}_\zeta$ and such that

$$\int \|F(\zeta)\|_\zeta^2 d\mu(\zeta) < \infty$$

are dense in $\mathcal{H}$. For such an $F \in \mathcal{H}$, and for $\varphi \in N_0$,

$$(\pi_F(W(\varphi))F)(\zeta) = \pi_\zeta(W(\varphi)).F(\zeta) \ . \qquad (4.2.31)$$

For $\varphi \in T_0$ , this becomes

$$(\pi_F(W(\varphi))F)(\zeta) = e^{-i\,B(\,\zeta,\,\varphi)}F(\zeta) \ . \qquad (4.2.32)$$

For $\psi \in T^\perp \cap M_0$, $W(\psi)$ is represented by

$$(\pi_F(W(\psi))F)(\zeta) = e^{-\frac{1}{2}B(\,\zeta,\,J\cdot F\,\psi)\,-\frac{1}{4}B(\,\psi,\,J\,F\,\psi)}F(\zeta+\psi) \qquad (4.2.33)$$

The generating functional for the representation $\pi_0$,

$$\rho_0(W(\varphi)) = \exp(-\int \overline{\hat{\varphi}^\mu(\underline{k})}\,\hat{\varphi}_\mu(\underline{k})\,\frac{d^3k}{|\underline{k}|}) \qquad (4.2.34)$$

is Poincaré invariant, so that Poincaré transformations are implemented in $\pi_0$. We use the representation $\pi_0$ to find a physical representation of the free field. $\pi_0$ is a representation of $\Delta_c(N_0)$ in which the degrees of freedom are just the physical ones, since the supplementary condition operators vanish. In fact, the 2-sided ideal of the supplementary condition operators in $\Delta_c(N_0)$ is just $\ker \pi_0$, so that the representation determines a representation $\tilde{\pi}_0$ of

$\Delta_c(N_0)/\ker \pi_0$ in the obvious way. This quotient
algebra is the algebra of observables for the free
electromagnetic field. It does not carry a unitary
representation of the Lorentz boosts, but it is $\mathbb{R} \times \mathbb{E}^3$
covariant.

We wish to describe inequivalent representations
which may be interpreted as sectors of non-zero charge,
and thereby to define charge transfer operators. We
therefore endeavour to define displaced representations
which represent the introduction of an external classical
electron field. The displaced representation of $\Delta_c(M_0)$
are easier to analyse, and we consider these first.

We choose (following Carey and Hurst) the
displacement as the Fourier transform of the current
corresponding to a classical point electron moving with
velocity

$$v^\mu = p^\mu/m_e ;$$

namely,
$$\hat{j}^\mu(k) = \frac{ep^\mu}{p.k} \qquad (4.2.35)$$

(cf. also Jauch and Rohrlich 1976, Kulish and Faddeev
1971, Zwanziger 1975.)
However $\hat{j}$ as defined above is not in the appropriate
set $\bar{D}^+(\mathbb{R}^{3+1})$ to give rise to a translation covariant
representation. This is a result of its ultraviolet
behaviour, and we introduce a cut-off, defining

$$\hat{j}_f^\mu(k) = \frac{ep^\mu}{p.k} f(p.k), \qquad (4.2.36)$$

where f is a real function which is one on a
neighbourhood of the origin and zero on a neighbourhood

of infinity.

Now define a displaced Fock representation of $\Delta_c(M_0)$ by

$$\pi^F_{j_f}(W(\varphi)) = e^{i\,B(j_f,\,\varphi)}\,\pi_F(W(\varphi)) \qquad (4.2.37)$$

In the new representation, translations will be implemented by the operator

$$V_{j_f}(x) = \Gamma_{j_f}(x)V_F(x) \qquad (4.2.38)$$

where $V_F(x)$ implements translations in the Fock representation, and

$$\Gamma_{j_f}(x) = W_F(j_f - U^x(x)j_f). \qquad (4.2.39)$$

By considering the behaviour of $j$ for small $|\underline{k}|$, we find that $j_f$ is not in the one-particle space (i.e. its norm corresponding to the inner product (4.2.17) is infinite) and hence $\pi^F_{j_f}$ is inequivalent to $\pi_F$. But for each $x$,

$$(1 - U^x(x))j_f$$

does have finite norm. Hence $\Gamma_{j_f}$ is well-defined.

Since we are dealing with a case of non-zero spin in 3+1 dimensions, we cannot expect all rotations and Lorentz boosts to be implemented in the displaced representation.

The cocycle $\Gamma_{j_f}$ defined here is an operator

on the representation space of the field algebra of the free electromagnetic field. Hence, although it is similar in structure to the cases we have discussed previously, it is not strictly analogous. For comparison with other models, it will be necessary to put the displaced sectors in the context of an algebraic structure involving representations of the observable algebra. This is the context in which charge transfer cocycles have usually been discussed. (Doplicher, Haag and Roberts 1971, Roberts 1976, Fröhlich 1977, 1979.)

As a first step, we therefore determine what effect the $j_f$-automorphism (4.2.37) of $\pi_F(\Delta_c(M_0))$ has on the component representations $\pi_\zeta(\Delta_c(N_0))$ in the direct integral decomposition of $\pi_F$.

We have, for $F = F(\zeta)$ in the dense set of cylinder functions which generate $\mathcal{H} = \int_{T^\perp} d\mu(\zeta)\mathcal{H}$ , and for $\varphi \in N_0$,

$$(\pi_F(W(\varphi))F)(\zeta) = \pi(W(\varphi)).F(\zeta), \qquad (4.2.40)$$

and hence

$$(\pi^F_{j_f}(W(\varphi))F)(\zeta) = (e^{i\,B(j_f\cdot\varphi)}\pi_F(W(\varphi))F)(\zeta)$$

$$= e^{i\,B(j_f\cdot\varphi)}\pi_\zeta(W(\varphi)).F(\zeta). \qquad (4.2.41)$$

That is, the displaced representation of $\Delta_c(M_0)$ restricted to $\Delta(N_0)$ is given by

$$\pi^F_{j_f}(W(\varphi)) = e^{i\,B(j_f\cdot\varphi)}\int_{\oplus T^\perp} \pi_\zeta(W(\varphi))d\mu(\zeta). \qquad (4.2.42)$$

In particular, the component of the displaced

representation corresponding to $\pi_0$ is defined by the following action on $W(\varphi)$ for $\varphi \in N_0$:

$$\pi_{j_f}(W(\varphi)) = e^{i B(j_f, \varphi)} \pi_0(W(\varphi)) \qquad (4.2.43)$$

The kernel of the new representation, $J = \ker \pi_{j_f}$ is different from $I = \ker \pi_0$. $\pi_{j_f}$ again determines a representation $\tilde{\pi}_{j_f}$ of an algebra representing the degrees of freedom of the electromagnetic field, but the algebra is now $\Delta_c(N_0)/J$ rather than $\Delta_c(N_0)/I$. This corresponds to a new choice of supplementary condition in the heuristic description. For $\hat{\psi} \in \mathcal{T}_0$, $\hat{\psi}^\mu(x) = \partial^\mu \chi(x)$, say, we have

$$\pi_{j_f}(W(\psi)) = e^{i B(j_f, \psi)} \pi_0(W(\psi))$$

$$= e^{i B(j_f, \psi)} \qquad (4.2.44)$$

so that the condition that $W(\varphi)$ should be in the kernel of $\pi_{j_f}$ corresponds heuristically not to

$$\partial^\mu A_\mu = 0$$

but to $\qquad (\partial^\mu A_\mu)(x) = g(x)$

where $g$ depends on $j_f$.

If we ignore the ultraviolet cut-off, we find that $W(\varphi) \in \ker \pi_j$ corresponds to

$$(\partial^\mu A_\mu)(x) = -D(x) \qquad (4.2.45)$$

which is the appropriate supplementary condition for the 1-particle space of the interacting electromagnetic field.

The translation covariant displaced representations of $\Delta_c(M_0)$ can be classified using the cohomological techniques developed by Araki (1970), and discussed in chapter 2 in relation to displaced Fock representations. This classification is in terms of additive cocycles in the one-particle space. In chapter 2, we saw that the classification of displaced Fock representations could be viewed essentially as a special case of the structure associated with inequivalent representations of a more general type where the cocycles are charge transfer operators. This general structure needs to be modified for the case of the electromagnetic field. Fröhlich (1979) has pointed out that, due to the mildness of the localization requirement, a representation-dependent analysis of cohomology classes is needed, exactly as in the 1+1-dimensional case. This is essentially because the cocycle associated with a given path can no longer be localized purely near the endpoints. Thus for the same reason as in 1+1 dimensions, the equivalence and triviality analysis given in chapter 2 breaks down, and hence the representation-independent cohomology classification no longer applies.

In our discussion of the Doplicher-Haag-Roberts description of field theories, we saw that their specification of "representations interesting for particle physics" could be re-formulated in terms of the properties which the associated charge transfer operators had to have. To discuss the present case in a similar way, the description of what constitutes an

interesting representation of the observable algebra
must be modified in a number of ways. Consequently the
properties required of the charge transfer operators
will also be modified. The first change is the milder
localization requirement already mentioned. This was
also used in the scalar model in section 4.1.

Secondly, full covariance cannot be required.
For the case of displaced representations, this follows
from the theorem proved at the beginning of the chapter.
There are also some more general results along these lines.
(e.g. Fröhlich, Morchio and Strocchi 1979). In the
general case, it is possible to obtain sectors with
rotation covariance but not with Lorentz boost covariance.
As a consequence, an inducing construction of the type
used in section 4.1 will be needed to regain Lorentz
boost covariance.

Finally, we wish to give the description of the
representation structure in terms of the observable
algebra rather than the field algebra. We shall see
that, largely because of the supplementary condition,
the description of the representation structure of the
electromagnetic field in terms of the observable algebra
is quite complicated.

To see what happens to the concept of charge
transfer when the description is given in terms of the
observable algebra, we return to the case of displaced
representations, and investigate more closely the
automorphism of $\pi_F(\Delta_c(M_0))$ implemented by $\Gamma_{j_f}(x)$.

In particular, what action does it determine on component representations of $\Delta_c(N_0)$ in the direct integral decomposition?

First we use the formulae (4.2.31)-(4.2.33) to determine how $\Gamma_{j_f}$ acts on component Hilbert spaces $\mathcal{H}_\zeta$. $(j_f - U^x(x)j_f)$ can be approximated arbitrarily closely by $\psi^\mu + \partial^\mu\chi$ where $(\psi^\mu) \in T^\perp \cap M_0$ and $(\partial^\mu\chi) = \lambda \in T_0$. Then we have

$$[W_F(\psi+\lambda)F](\zeta) = [W_F(\psi)W_F(\lambda)F](\zeta)$$

$$= e^{-\frac{1}{2}B(\zeta, J_F\psi) - \frac{1}{4}B(\psi, J_F\psi)}(W_F(\lambda)F)(\zeta+\psi)$$

$$= e^{j B(\zeta+\psi, \lambda)} e^{-\frac{1}{2}B(\zeta, J_F\psi) - \frac{1}{4}B(\psi, J_F\psi)} F(\zeta+\psi)$$

If we were dealing with a conserved current, the element $\psi$ of $T^\perp$ would be zero, and the sectors would not be mixed. But in the model we have set up, the description of charge transfer requires a rather large representation of $\Delta_c(N_0)$. Each component representation has a slightly different kernel, $K_\zeta$ say, and therefore determines a representation of a different algebra, $\Delta_c(N_0)/K_\zeta$, describing the observable degrees of freedom. In the limit, as $x$ tends to infinity in a spacelike direction, the representation $\pi_{j_f}$ of $\Delta_c(N_0)$ is obtained. This representation does not appear in the direct integral of representations $\pi_\zeta$ and we obtain the new observable algebra corresponding to the supplementary condition (4.2.45).

The representation (4.2.42) is still not large

enough to describe Lorentz boosts. Its representation space is large enough to carry the field algebra of the free electromagnetic field, but not surprisingly, it will have to be enlarged to describe any interacting theory, even the first approximation which we have set up.

We apply the inducing construction to the representation (4.2.42) by taking a direct integral over representations $\pi_{j_f(p)}$ where $j_f(p)$ is given by

$$j_f(p) = \frac{ep^\mu}{p.k} f(p.k)$$

where $p = (p^\mu)$ satisfies $p_0 = \sqrt{\underline{p}^2+m^2}$, $m = $ mass of electron;

i.e. the values of $p$ range over the electron mass shell, $V$ say. Denote the Hilbert space on which the direct integral representation acts by

$$\mathcal{H}^c = \int_{\oplus V} d\mu(p)\,\mathcal{H}_p$$

where each $\pi_{j_f(p)}$ acts on $\mathcal{H}_p$. A dense set of functions in $\mathcal{H}^c$ is spanned by classes of functions $F$ on $V$ such that for $p \in V$,

$$F(p) \in \mathcal{H}_p\ ,$$

and

$$\int \|F(p)\|^2\ \frac{d^3p}{\sqrt{\underline{p}^2+m^2}} < \infty$$

It is encouraging that $\mathcal{H}^c$ is exactly the one-particle space of the Hilbert space for asymptotic

interactions obtained by Carey and Hurst via a slightly different approach.

Spaces of states containing arbitrary numbers of particles could be built up by successive use of the limiting charge transfer operation, and with an associated inducing construction, this gives rise to the full asymptotic Hilbert space as described in Carey and Hurst (1979).

As in the previous section, this construction leads to the possibility of certain gauge transformations not included in the original space, so that the gauge group is extended for the interacting theory.

CHAPTER 5

CONCLUSIONS

The models presented here indicate certain
directions in which the concept of charge transfer and
hence superselection structure must be generalized in
order to describe more realistic theories.  The analyses
of superselection structure and charge transfer given by
Doplicher, Haag and Roberts (1969a,b) and Roberts (1976)
describe idealized field theories for elementary
particles, in which the charges are strictly localized,
only gauge theories of the first kind are allowed, and
all irreducible representations considered are Poincaré
covariant.  These assumptions allow a detailed description
of the structure expected of a field theory.

Roberts also deals with the 1+1-dimensional case
under similar assumptions, and the structure here is
rather different.  Because the charge transfer operator
cannot be localized about the endpoints of the charge
transfer path, the cohomological classification takes
a more complicated form, since it becomes representation
dependent.

The problem of constructing models in 1+1 dimensions
has proved considerably more feasible than in higher
dimensions.  This is primarily the result of special
topological properties of 1+1 dimensions.  Unfortunately,
these same properties must also limit the value of 1+1

dimensional models in drawing conclusions about the
expected structure of field theories in higher dimensions.
An obvious example is the separation of spacelike left
and right infinity. It would seem desirable, therefore,
to attempt to construct non-vacuum sectors in 1+1
dimensions which did not rely for their very existence
on this feature. We have seen that this becomes a
realistic proposition if strict localization is relaxed.
Admittedly, many of the factors which make the
construction easier depend on topological solitons, but
even without this, 1+1 dimensional models should be
easier to handle.

Meanwhile, the only examples we have at present
of non-linear quantum soliton sectors are of the type
which rely on multiple vacua, and the separation of
spacelike left and right infinity. Fröhlich (1977)
exploits the detailed properties of cocycles in 1+1
dimensions to demonstrate their construction for the
$P(\varphi)_2$ model.

The algebraic structure associated with these
soliton sectors is largely similar to that of the
Streater-Wilde model. This encourages us to regard
displaced Fock representations as playing a rôle similar
to that of non-vacuum sectors of non-linear theories.
In fact, the displacement $\phi^x$ can be identified with
a weak solution of the wave equation and dual subsidiary
conditions, if it is interpreted as a generalized
function. That is, if we have a Weyl algebra corresponding

to a space $M$ of solutions of the free field equations
(2.1.7), and $M$ is parametrized by the Cauchy data
$(\varphi_\alpha, \pi_\alpha)$ of its elements, then $\phi^x$ can be associated
with the generalized functions $\phi^x_\alpha, \Pi^x_\alpha$ in $S^*(\mathbb{R}^3)$
given by the formula

$$\text{Im}\langle \phi^x, \phi \rangle = \langle \phi^x_\alpha(\cdot, t), \pi_\alpha(\cdot, t) \rangle_{L^2} - \langle \Pi^x_\alpha(\cdot, t), \varphi_\alpha(\cdot, t) \rangle_{L^2} .$$

Then each $(\phi^x_\alpha(\underline{x}, t))$ will be the solution to the wave
equation with Cauchy data $(\phi^x_\alpha(0, t), \Pi^x_\alpha(0, t))$, and
will satisfy the dual of the subsidiary conditions given
in (2.1.7).

Thus the displacement provides a solution of the
wave equation which lies outside the one-particle space,
and which might play the same rôle as a soliton
solution in a non-linear theory. Therefore, displaced
Fock representations provide well-defined structures
involving inequivalent representations of the observable
algebra, which, it can be argued, should play the same
rôle as the soliton sectors of non-linear theories. So
it might be hoped that discoveries about the structure of
free theories will carry over to the soliton sectors of
non-linear models.

A second reason why models involving different
representations of the free field are of interest is
that they can be used in approximate calculations for
interacting theories; an example is our use (section
4.2) of displaced Fock representations to describe the
interaction of the electromagnetic field with an external
classical current. This application, together with the

analogy between soliton sectors and displaced Fock
representations, encourages us to suggest that true
soliton sectors should also describe certain known
particles of field theory.

We have related the displaced Fock representations,
and their cohomological classification, to a more general
analysis of the superselection structure of field
theories. The additive cocycles, $\psi_x = \Psi^x - U^x(x)\Psi^x$,
whose equivalence classes give a classification of
displaced Fock representations can be seen to correspond
to charge transfer operators

$$\Gamma(x) = W_F(\psi(-x)).$$

These charge transfer operators have been shown under
fairly general conditions to be local cocycles. (In
the case where the space-time translation group has a
multiplier representation, the cocycle identity must be
modified, (2.3.6).) For the 1+1-dimensional displaced
Fock representations which occur in the Streater-Wilde
model, the correspondence between the displacements and
strictly local cocycles as defined by Roberts (1976) is
perfect. For non-Fock representations in higher
dimensions, the displacements correspond to cocycles
which are asymptotically rather than strictly localized.
Such a relaxation of the localization requirement would
in any case be necessary for the discussion of many
realistic theories. The classification of displaced
representations is just the same with either localization

requirement, but the analysis of more general representations is more difficult in the case of asymptotic localization; the cohomology structure becomes representation-dependent, just as in 1+1 dimensions.

Other modifications to the algebraic structure result from a consideration of covariance requirements. We have looked at these in examples built up from displaced Fock representations. In a case such as the Streater-Wilde model (or the $P(\varphi)_2$ model), where each soliton sector is fully covariant, the charge transfer cocycles may be extended to cocycles over the Poincaré group. But it is not possible to obtain Poincaré covariant sectors of the displaced Fock type in 3+1 dimensions. Representations which are associated with charge transfer cocycles but lack full covariance are of interest because, for example, it has been shown that full covariance is not possible for non-Fock representations of the electromagnetic field. A natural way to restore covariance is by an inducing construction. This construction, which was described in section 4.1, acts on a space which is suitable for the description of the field in inter-action with an external particle. It also leads to the possibility of certain gauge transformations. If the original representation of the translation group on which the inducing construction was performed is a multiplier representation, we have the possibility that in order to achieve full covarience, a Lorentz transformation may need to be accompanied by one of these gauge transformations.

This situation is familiar from studies of the interacting electromagnetic field.

The structure of displaced Fock sectors of the electromagnetic field looks rather similar to the scalar case if we consider the field algebra, but the description in terms of the observable algebra is complicated by the need to deal with the supplementary condition.

We have also identified a 2-cohomology structure for induced representations built up from multiplier representations. The 2-cocycles form an abelian set of operators, so we have a rather degenerate example of the non-abelian 2-cohomology which Roberts (1977) suggests should be associated with second kind gauge groups. But even in this case, the 2-cohomology structure is associated with the superselection structure. Clearly examples involving different kinds of representations will be needed in order to illustrate more complicated structures of gauge theories. True non-linear theories are very hard to develop in a rigorous way in higher dimensions, but some interesting possibilities might arise if other non-Fock representations of the free field are used. For example, for the electromagnetic field, Kraus, Polley and Reents (1977) have considered different representations of the free field. It would be interesting to study the charge transfer operation in this case.

There is also a need for further study of the algebraic rôle of the supplementary condition in a

fully interacting theory of quantum electrodynamics.

Another direction for development is in the detailed study of the structure of solution spaces for classical non-linear relativistic field equations. Parenti, Strocchi and Velo (1977) have developed a framework for such a study, in which conserved dynamical charges are identified to label the different solution sectors. It would be interesting to develop the relationship of this structure to a corresponding quantum theory.

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