Weakly Hamiltonian-connected Ordinary multipartite Tournaments

Jørgen Bang–Jensen*Gregory Gutin†and Jing Huang‡

Abstract

We characterize weakly Hamiltonian-connected ordinary multipartite tournaments. Our result generalizes such a characterization for tournaments by Thomassen and implies a polynomial algorithm to decide the existence of a Hamiltonian path connecting two given vertices in an ordinary multipartite tournament and find one, if it exists.

1 Introduction

It is well-known that tournaments have a very rich structure. Recently it has been shown that there are several classes of much more general digraphs, containing the tournaments, which share a lot of this structure with the tournaments, see for example [2, 3, 4, 6, 12, 13].

Multipartite tournaments are another generalization of tournaments. A digraph is a *multipartite tournament* if it can be obtained from a complete $k-$partite graph, for some $k \geq 2$, by orienting the edges. For a survey on results on multipartite tournaments see [9]. An *ordinary multipartite tournament* is a special kind of multipartite tournament in which all arcs between two classes in the partition have the same direction. One can also view it as a digraph obtained from a tournament by replacing each vertex with a set of independent vertices. Obviously all tournaments are ordinary multipartite tournaments. Ordinary-multipartite tournaments are also quasi-transitive. A digraph D is *quasi-transitive* if, whenever $x{\rightarrow}y$ and $y{\rightarrow}z$ are arcs of D, then x and z are adjacent. It has been shown that these digraphs share many properties with tournaments, [2, 6], hence, so do ordinary multipartite tournaments.

One notable difference between tournaments and multipartite tournaments is that while every tournament has a Hamiltonian path, there is no degree of strong connectivity that guarantees the existence of a Hamiltonian path in a multipartite tournament, cf. [5]. In fact, here the existence of an *almost factor*, i.e., a disjoint collection of cycles and one path, covering all the vertices of the digraph, is important. A Hamiltonian path is itself an almost factor, so clearly the existence of an almost factor is required in any digraph with a Hamiltonian path. The second author showed that this is also sufficient for multipartite tournaments [11].

*Dept. of Math. and Compt. Sci., Odense University, DK-5230, Odense, Denmark

†Dept. of Math. Tel Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel. Financial support by Department of Mathematics and Computer Science, Odense University is greatfully acknowledged.

‡Dept. of Math. and Compt. Sci., Odense University, DK-5230, Odense, Denmark. Financial support by the Danish Research Counsil under grant no. 11-9743-1 is greatfully acknowledged.

Thomassen [14] completely characterized those tournaments which are *weakly Hamiltonian-connected* that is, for any two vertices x and y, there is a Hamiltonian path connecting x and y. In the present paper, we give a complete characterization of weakly Hamiltonian-connected ordinary multipartite tournaments. Our result generalizes Thomassen's characterization for tournaments. From our characterization, it follows that Hamiltonian-connected ordinary multipartite tournaments share the same 'forbidden pattern' with Hamiltonian-connected tournaments, except for minor necessary differences, namely, the existence of an appropriate almost factor.

A polynomial algorithm to decide the existence of a Hamiltonian path connecting two given vertices and find one if there exists follows from our result.

Another recent result on weakly Hamiltonian-connectedness is [7] which gives a characterization of bipartite tournaments (a *bipartite tournament* is an orientation of a complete bipartite graph, and hence a special case of multipartite tournaments) which have a Hamiltonian path connecting two given vertices and gives a polynomial algorithm to find one if it exists.

2 Terminology and preliminaries

All digraphs in this paper are oriented, i.e., they contain no cycles of length 2. Let D be a digraph. If there is an arc from a vertex x to a vertex y in D, then we say that x *dominates* y and denote it by $x \to y$. If A and B are two subsets of $V(D)$ and every vertex of A dominates each vertex of B, then we say that A *dominates* B and denote it by $A \to B$. If x and y are two vertices of D and P is a directed path from x to y, then we say that P is an (x, y)-path. An (x, y)-*Hamiltonian path* is an (x, y)-path which contains all vertices of D. If u and v are two vertices of a path P, we use $P[u, v]$ denote the subpath of P from u to v. If C is a cycle and u, v are two vertices of C, then we use $C[u, v]$ to denote the subpath of C from u to v. If H is a subgraph of a digraph D and x is a vertex of D then we denote by $O_H(x)$ $(I_H(x))$ the set of vertices of H dominated by (respectively dominates) the vertex x.

A digraph D is *strong* if there is an (x, y)-path and a (y, x)-path for any two distinct vertices x and y of D. A *factor* of a digraph D is a spanning subgraph such that every vertex has precisely one in-neighbour and precisely one out-neighbour, i.e., it consists of disjoint cycles covering the vertices of D.

The following result was obtained by the second author, cf. [10, 11].

Theorem 2.1 *An ordinary multipartite tournament D has a Hamiltonian path if and only if D has a path P such that $D - P$ has a factor. A strong ordinary multipartite tournament has a Hamiltonian cycle if and only if it has a factor.* □

Lemma 2.2 *Let x, y and z be three vertices of an ordinary multipartite tournament. If x dominates (resp. is dominated by) y which is not is not adjacent to z, then x dominates z (resp. is dominated by) z.* □

So an ordinary multipartite tournament satisfies the following *quasi-transitive* property, cf. [6, 8], i.e., $x \rightarrow y \rightarrow z$ implies that x and z are adjacent. This fact will be frequently used in the sequel.

Lemma 2.3 *Let S and S' be two strong components in an ordinary multipartite tournament such that at least one of them is non-trivial. Then either $S \rightarrow S'$ or $S' \rightarrow S$.*

Proof: Since one of S and S' is non-trivial, there is at least one arc between S and S'. Suppose that $x \rightarrow y$ is an arc where $x \in S$ and $y \in S'$. For each vertex $z \in S'$, there is a (y, z)-path in S' and x must dominate each vertex on this (y, z)-path. In particular, x dominates z. Hence $x \rightarrow S'$. Similarly, $S \rightarrow y$. Therefore $S \rightarrow S'$. $\square$

Let D be an ordinary multipartite tournament. It is not difficult to see that if D contains an almost factor, then the strong components of D can be uniquely ordered $S_1, S_2, \ldots, S_k$ so that $S_i \rightarrow S_j$ for any $i < j$. We shall call S_1 the *initial component* and S_k the *terminal component* of D. This fact will be frequently used in the sequel without further emphasis.

We shall use x^+ (x^-) to denote the successor (predecessor) of x on a path or a cycle. Whenever we use this terminology it will be clear which is the relevant path (cycle).

Lemma 2.4 *Let C and C' be two disjoint cycles in an ordinary multipartite tournament D. If C and C' are not completely adjacent, then there is a cycle in D containing all vertices of $V(C) \cup V(C')$.*

Proof: If $x \in V(C)$ is not adjacent to $y \in V(C')$, then by Lemma 2.2 $x \rightarrow y^+$ and $y \rightarrow x^+$. Hence $xC'[y^+, y]C[x^+, x]$ is a Hamiltonian cycle in $V(C) \cup V(C')$. $\square$

Let x be a vertex of the digraph D and $C : u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_k \rightarrow u_1$ be a cycle in $D - x$. A *partner* of x on C is an arc $u_i \rightarrow u_{i+1}$ such that $u_i \rightarrow x \rightarrow u_{i+1}$. Note that a vertex which has a partner can be inserted into C to obtain a longer cycle.

Lemma 2.5 *Let D be a digraph which contains a cycle C and a path P in $D - C$ such that each vertex of P has a partner on C. Then D has a cycle containing all vertices of C and P.*

Proof: Denote P by $u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_r$ and C by $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_s \rightarrow v_1$. Let $v_i \rightarrow v_{i+1}$ be a partner of u_1, namely, $v_i \rightarrow u_1 \rightarrow v_{i+1}$. Let u_j be the vertex of P with the greatest subscript j such that $u_j \rightarrow v_2$. Then we have a new cycle C' : $v_i P[u_1, u_j] C[v_{i+1}, v_i]$, which contains all arcs of C except $v_i \rightarrow v_{i+1}$. If $j = r$, then we are done; otherwise note that each vertex of $P[u_{i+1}, u_r]$ has a partner on C' and we can continue as above to insert vertices of $P[u_{i+1}, u_r]$ into C'. $\square$

Lemma 2.6 *Suppose that a strong ordinary multipartite tournament D has an (x, y)-path P such that $D - P$ has a factor. Then D has a Hamiltonian path starting at x and a Hamiltonian path ending at y.*

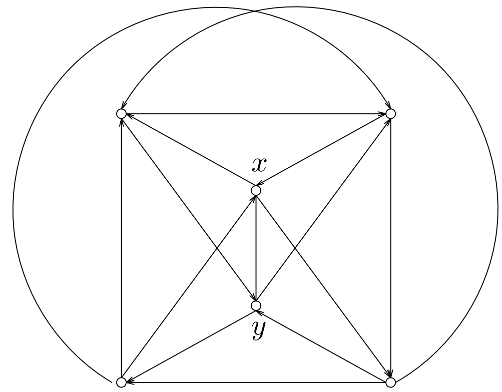

Figure 1: The exceptional tournaments where the edge between x and y can be oriented arbitrarily

Proof: Choose a path P' starting at x as long as possible so that $D-P'$ has a factor which consists of minimal number of cycles $C_1, C_2, \ldots, C_q$. Then by Theorem 2.1 and Lemma 2.4 we may assume that $C_i \rightarrow C_j$ when $i<j$. Let $x=u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_r$ be the path P'. If $q \neq 0$, then u_r is completely dominated by C_1. Since D is strong, there is an arc from P' to C_1. Let u_i be the vertex of P' with largest i such that there is an arc from u_i to C_1. Hence C_1 can be inserted between u_i and u_{i+1}, contradicting the choice of P'. So $q=0$ and P' is a Hamiltonian path starting at x. A similar argument can be applied to show that D has a Hamiltonian path ending at y. □

In [14], Thomassen proved the following.

Lemma 2.7 *Suppose that T is a tournament which contains two vertices x and y such that $T, T-x$ and $T-y$ are strong. Then there is no Hamiltonian path connecting x and y if and only if T is isomorphic to one of the tournaments shown in Figure 1.* □

Lemma 2.8 *Suppose that D is a strong ordinary multipartite tournament containing two adjacent vertices x and y such that $D-\{x,y\}$ has a Hamiltonian cycle C but D has no Hamiltonian path connecting x and y. Then C is an even cycle, $O_C(x)=I_C(y)$, $I_C(x)=O_C(y)$, and x,y dominate alternating vertices along the cycle C.*

Proof: Since x and y are adjacent, every vertex of C is adjacent to at least one of x and y. If some $u \in V(C)$ is adjacent to exactly one of x and y, say x, then $u^- \rightarrow y \rightarrow u^+$. There is an (x,y)-Hamiltonian path $xC[u,u^-]y$ if $x \rightarrow u$ and there is a (y,x)-Hamiltonian path $yC[u^+,u]x$ if $u \rightarrow x$, contradicting our hypothesis. So each vertex is adjacent to both x and y. If some vertex of C dominates both x and y, then there is a vertex $v \in V(C)$ such that v dominates both x and y and there is an arc from $\{x,y\}$ to v^+. It is easy to see that there is a Hamiltonian path connecting x and y, a contradiction. Similarly no vertex of C is dominated by both x and y. If one of x and y, say x, dominates two consecutive vertices w and w^+ of C, then y is dominated by w and w^+ and hence there is an (x, y)-Hamiltonian path, a contradiction. Therefore we have proved the claims in the lemma. □

We shall see from Theorem 3.1 in the next section that the only possible case when D may fail to have a Hamiltonian path connecting x and y is when the length of C is four, in which case D is isomorphic to one of the tournaments shown in Figure 1.

3 Weakly Hamiltonian-connected vertices

Here is the main result.

Theorem 3.1 *Let D be an ordinary multipartite tournament and x, y be distinct vertices of D. Then D has a Hamiltonian path connecting x and y (from x to y or from y to x), if and only if D has a path P connecting x and y such that $D - P$ has a factor and D does not satisfy any of conditions* $(1) - (4)$ *below.*

(1) D is not strong and either the initial or the terminal component of D (or both) contains none of x and y;

(2) D is strong, $D - x$ is not strong and either y belongs to neither the initial nor the terminal component of $D - x$, or y belongs to the initial (terminal) component of $D - x$ and there is no (y, x)-path ((x, y)-path) P' such that $D - P'$ has a factor.

(3) D is strong, $D - y$ is not strong and either x belongs to neither the initial nor the terminal component of $D - y$, or x belongs to the initial (terminal) component of $D - y$ and there is no (x, y)-path ((y, x)-path) P' such that $D - P'$ has a factor.

(4) $D, D - x$, and $D - y$ are all strong and D is isomorphic to one of the tournaments shown in Figure 1.

Proof: First observe that the existence of a path P connecting x and y such that $D - P$ has a factor is necessary, since any Hamiltonian path connecting x and y has that property. Now, if one of conditions $(1) - (4)$ holds, then it is easy to see that D has no Hamiltonian path connecting x and y. We prove by induction on n, the number of vertices of D, that the converse is true as well.

First we claim that if D has no Hamiltonian path connecting x and y and $D, D - x$, or $D - y$ is not strong, then $(1), (2)$, or (3) must hold. Indeed, if D is not strong and if the initial component S_1 of D contains x and the terminal component S_t of D contains y, then it is clear that our condition implies that there is an (x, y)-path P such that $D - P$ has a factor. Therefore, by Lemma 2.6 the initial component S_1 contains a Hamiltonian path starting at x, the terminal component S_t has a Hamiltonian path ending at y and any other component S_i $(1 < i < t)$ of D contains an almost factor. By Theorem 2.1, any S_i $(1 < i < t)$ has a Hamiltonian path. Hence, by Lemma 2.3 D has an (x, y)-Hamiltonian path, a contradiction.

Now we consider what happens when D is strong and $D - x$ is not strong. Suppose that the initial component of $D - x$ contains y and there is a (y, x)-path P' such that $D - P'$ has a factor. By Lemma 2.6 the initial component of $D - x$ has a Hamiltonian path starting at y. If the terminal component of $D - x$ intersects P', then again by Lemma 2.6 it has a Hamiltonian path ending at the predecessor of x on P'. If the terminal component does not intersect P', then it has a Hamiltonian cycle and hence it has a Hamiltonian path ending at a vertex which has an arc to x. Hence it is clear that D has a (y, x)-Hamiltonian path, a contradiction. For the case when $D - y$ is not strong, we can show that condition (3) holds.

So we assume that $D, D - x$, and $D - y$ are all strong and we shall prove that either D has a Hamiltonian path connecting x and y or D is isomorphic to one of the tournaments in Figure 1. By Lemma 2.8, $n \geq 6$.

Suppose first that x is adjacent to y and that $D - \{x, y\}$ has a Hamiltonian cycle $u_1 \to u_2 \to \ldots \to u_t \to u_1$. By Lemma 2.8, either D has a Hamiltonian path connecting x and y or t is even, say $t = 2k$ for some k, and we may choose the labeling of the cycle such that, $O_C(x) = I_C(y) = \{u_1, u_3, \ldots, u_{2k-1}\}$, and $I_C(x) = O_C(y) = \{u_2, u_4, \ldots, u_{2k}\}$. If $k = 2$, i.e., $n = 6$, then it is clear that D is a tournament. By Lemma 2.7, either D has a Hamiltonian path connecting x and y or D is isomorphic to one of the tournaments in Figure 1. So $k \geq 3$.

Claim 1. Either D has a Hamiltonian path connecting x and y, or there exist different a, b such that $u_{a-1} \to u_{b+1}$ where u_a, u_b are in $O_C(x) = I_C(y)$ or in $I_C(x) = O_C(y)$. Furthermore every Hamiltonian path of $D - \{x, y\}$ connecting u_a and u_b can be extended to a Hamiltonian path of D connecting x and y.

Proof of Claim 1: The second statement of Claim 1 follows from the fact that u_a, u_b are in $O_C(x) = I_C(y)$ or in $I_C(x) = O_C(y)$. Note that, for every i, u_i is adjacent to u_{i+2}. If $u_{j+2} \to u_j$ for some j, then let $a = j + 3$, $b = j - 1$ and the claim follows. So assume that $u_i \to u_{i+2}$ for every i. Then u_i is adjacent to u_{i+4}. In particular, u_1 is adjacent to u_5. If $k = 3$, then D is a tournament which is not isomorphic to the one in Figure 1, and by Lemma 2.7 there is a Hamiltonian path connecting x and y. If $k \geq 4$, then let $a = 2$, $b = 4$ if $u_1 \to u_5$ and let $a = 6$, $b = t$ if $u_5 \to u_1$, and the claim follows. □

By Claim 1, we may assume, without loss of generality, that $D - \{x, y\}$ has a (u_a, u_b)-path $P' : u_a \to u_{a+1} \to \ldots \to u_b$ and $D - \{x, y\} - P'$ has a Hamiltonian cycle $C' : u_{a-1} \to u_{b+1} \to u_{b+2} \to \ldots \to u_{a-1}$, and u_a, u_b are in $O_C(x) = I_C(y)$ or in $I_C(x) = O_C(y)$. According to the second statement of Claim 1, we may assume that there is no Hamiltonian path connecting u_a and u_b in $D - \{x, y\}$. By induction, $D - \{x, y\}$ satisfies (2), (3) or (4) of Theorem 3.1, with D replaced by $D - \{x, y\}$, x by u_a, and y by u_b. Suppose that condition (2) holds, namely, $D - \{x, y, u_a\}$ is not strong. Note that, because of the existence of the path P', u_b can not appear in the terminal component T of $D - \{x, y, u_a\}$ and, since $u_{a-1} \in T$, T contains all the vertices of C'. Hence u_b completely dominates all vertices of C'. Therefore D has a Hamiltonian path connecting x and y (note that C' contains in-neighbours of x and y). Similarly there is a Hamiltonian path connecting x and y when $D - \{x, y, u_b\}$ is not strong. Finally suppose that condition (4) holds, i.e., $D - \{x, y\}$ is isomorphic to the tournament in Fig. 1. In this case, D is a tournament since x and y are adjacent to all vertices of $D - \{x, y\}$. By Lemma 2.7, D has a Hamiltonian path connecting x and y.

So we assume, from now on, that x and y are not adjacent whenever $D - \{x, y\}$ has a Hamiltonian cycle.

Suppose that D has a path $P : v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_t$ connecting x and y and $D - P$ has a factor which consists of cycles $C_1, C_2, \ldots, C_q$. Choose such P as long as possible and subject to that q is minimal. Suppose first $q \geq 2$. Without loss of generality assume that P is an (x, y)-path. By Theorem 2.1 and Lemma 2.3, we may also assume that $C_i \rightarrow C_j$ when $i < j$. Since $D - y$ is strong, there is an arc from some v_i to C_1 where $i \leq t - 1$. Then there must be arc from v_{t-1} to C_1 since otherwise all vertices of C_1 can be inserted between two consecutive vertices of P, contradicting the choice of P. Hence $y \rightarrow C_q$. Applying a similar argument, we can show that there is an arc from C_q to v_2 and $C_1 \rightarrow x$. This implies that the length of P is at least two. Let u be a vertex of C_1 dominated by v_{t-1} and u' be a vertex of C_q which dominates v_2. Then $yC_q[u'^+, u']P[v_2, v_{t-1}]C_1[u, u^-]x$ is a (y, x)-path of length greater than the length of P, contradicting the choice of P. Hence $q \leq 1$.

So $D - P$ has a Hamiltonian cycle and we use $C : u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_s \rightarrow u_1$ to denote such a cycle.

Claim 2. There exists an arc from v_2 to C and an arc from C to v_{t-1}, or D has a Hamiltonian path connecting x and y.

Proof of Claim 2: Suppose that $C \rightarrow v_2$. Then $C \rightarrow x$ or we can insert all vertices of C between x and v_2 to obtain an (x, y)-Hamiltonian path. If there is an arc from v_{t-1} to C, then $D - \{x, y\}$ has a Hamiltonian cycle. Hence x and y are not adjacent. Then $yP[v_2, v_{t-1}]C[u, u^-]x$ is a Hamiltonian path where u is a vertex of C dominated by v_{t-1}. If there is no arc from v_{t-1} to C, i.e., $C \rightarrow v_{t-1}$, then all vertices of C can be inserted between two consecutive vertices of P to obtain an (x, y)-Hamiltonian path, because $D - y$ is strong. By a similar argument, it can be shown that there is an arc from C to v_{t-1} or D has a Hamiltonian path connecting x and y. □

Suppose D does not have a Hamiltonian path connecting x and y. By Claim 2, there is an arc from v_2 and an arc from C to v_{t-1}. Since D has no Hamiltonian path connecting x and y, each v_i with $2 \leq i \leq t - 1$ has arcs in both directions to C. Each of such vertices must completely adjacent to C. Indeed, if v_i is not adjacent to some u_j, then $v_i \rightarrow u_{j+1}$, $u_j \rightarrow v_{i+1}$, and hence there is (x, y)-Hamiltonian path $P[x, v_i]C[u_{j+1}, u_j]P[v_{i+1}, y]$. So each vertex v_i with $2 \leq i \leq t - 1$ has a partner on C. Hence by Lemma 2.5 $D - \{x, y\}$ has a Hamiltonian cycle. Therefore x and y are not adjacent and we have $v_{t-1} \rightarrow x$ and $y \rightarrow v_2$.

By a similar discussion as above, we can show that both x and y are completely adjacent to C. Since D has no Hamiltonian path connecting x and y, there is at least one arc from C to x and there is at least one arc from y to C. Since x and y are not adjacent, there is an arc from x to C. Now there is a Hamiltonian cycle C' in $D - y$ because each vertex v_i with $1 \leq i \leq t - 1$ has a partner on C. Let x^- denote the predecessor of x on C'. Then x^- dominates x which is not adjacent to y. By Lemma 2.2, x^- dominates y. Therefore there is an (x, y)-Hamiltonian path $C'[x, x^-]y$, a contradiction. This completes the proof of Theorem 3.1. □

The following immediate consequence of Theorem 3.1 characterizes weakly Hamiltonian-connected ordinary multipartite tournaments.

Corollary 3.2 *An ordinary multipartite tournament D is weakly Hamiltonian-connected if and only if it satisfies* $(1), (2), (3)$, *and* (4) *below.*

(1) D is strong.

(2) For every pair of distinct vertices x and y of D, there is a path P connecting x and y so that $D - P$ has a factor.

(3) For each vertex x of D, $D - x$ has at most two components and, for each vertex y in the initial (terminal) component, there is a (y, x) $((x, y))$-path P' such that $D - P'$ has a factor.

(4) D is not isomorphic to the tournament in Fig. 1. □

The proof [14] of Thomassen's theorem, mentioned above, is based on the following result: For any three distinct vertices of a strong tournament T, there is a Hamiltonian path connecting two of them. This theorem has also an independent significance. We shall formulate and prove an analogous result for strong ordinary multipartite tournaments.

Corollary 3.3 *Let x, y and z be three vertices of a strong ordinary multipartite tournament D. Suppose that, for each pair of x, y and z, there is a path P connecting them so that $D - P$ has a factor. Then there is a Hamiltonian path connecting two of them.*

Proof: If both $D-x$ and $D-y$ are strong, then, by Theorem 3.1, either D has a Hamiltonian path connecting x and y, or D is isomorphic to one of the tournaments in Figure 1, in which case there is a Hamiltonian path connecting x and z. Similarly, if both $D - x$ and $D - z$, or $D - y$ and $D - z$ are strong. So without loss of generality assume that neither $D - x$ nor $D - y$ is strong. Let P be a path connecting x and y in D such that $D - P$ has a factor F. Obviously, all the components of $D - x$, except possibly one, contain cycles of F and hence all of them, except possibly one, are non-trivial. Let $S_1, S_2, \ldots, S_t$ be the strong components of $D - x$ where $S_i \rightarrow S_j$ for all $1 \leq i < j \leq t$. Note that S_t has an arc to x, since D is strong.

If neither S_1 nor S_t contains y, then $D - y$ is strong, contradicting our assumption. So assume that the initial component contains y. (The discussion is similar in the other possible case). In the connection with condition (2) of Theorem 3.1, we may also assume that there is an (x, y)-path $P' : x \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_r = y$ such that $D - P'$ has a factor. Since $P' - x$ is contained in S_1, each S_i, $i > 1$, has a factor and hence contains a Hamiltonian cycle C_i, by Theorem 2.1.

If $r = 1$ then, by Lemma 2.6, S_1 contains a Hamiltonian path starting at y. This path can easily be extended to a (y, x)-Hamiltonian path in D, since each S_i, $i > 1$, is Hamiltonian.

Assume now that $r \geq 2$. By Lemma 2.6, S_1 has a Hamiltonian path P'' starting at v_1. If $z \in S_j$, $1 < j < t$, then let $v \in S_t$ be any vertex which dominates x and D has the (y, z)-Hamiltonian path

$$P''[y, a]C_{j+1}[b, b^-] \ldots C_t[v^+, v]xP''[v_1, y^-]C_2[c, c^-] \ldots\ldots C_j[z^+, z]$$

where a is the last vertex of P'' and b, c are any vertices in C_{j+1}, C_2 respectively. If $z \in S_t$, then D has the (x, z)-Hamiltonian path

$$xP''C_2[c, c^-]...C_t[z^+, z].$$

So assume that $z \in S_1$.

If $S_1 - y$ is strong, then $D - y$ is strong contradicting our assumption above. Let $T_1, T_2, \ldots, T_s$, $s \geq 2$, be the strong components of $S_1 - y$ where $T_i \rightarrow T_j$ for every $i < j$. Note that each T_i either consists of some cycles from the factor of $D - P'$ and hence has a Hamiltonian cycle, or consists of a portion of $P'[v_1, v_{r-1}]$ and some cycles from the factor of $D - P'$ and hence has a Hamiltonian path. Note also that there is at least one arc from y to T_1 and at least one arc from T_s to y. If T_1 consists of a portion of $P'[v_1, v_{r-1}]$ and some cycles from the factor of $D - P'$, then it is clear that T_1 contains v_1, hence $D - y$ is strong, contradicting our hypothesis. So T_1 contains no vertices of $P'[v_1, v_{r-1}]$ and hence it has a Hamiltonian cycle to which there is at least one arc from y. Therefore it is easy to see that D has a (y, x)-Hamiltonian path. $\Box$

4 A polynomial algorithm

We show that there is a polynomial algorithm to decide the existence of a Hamiltonian path connecting two given vertices of an ordinary multipartite tournament and to find one, if it exists. Instead of giving the algorithm in detail, we shall argue that such an algorithm is inherent in our proof. Let x and y be two vertices of an ordinary multipartite tournament D. First, according to Theorem 3.1, we need to check whether there is an (x, y) ((y, x))-path P such that $D - P$ has a factor. Let D' the digraph obtained from D by deleting all arcs from y (x) and all arcs into x (y) and then adding the arc $y \rightarrow x$ ($x \rightarrow y$). It is clear that there is an (x, y) ((y, x))-path P in D such that $D - P$ has a factor if and only if D' has a factor. For the purpose of deciding whether D' has a factor, we construct a bipartite graph B from D' as follows: The vertex set of B consists of two copies v, v' of every vertex v of D'. The edge set of B consists of all edges vu' where $v \rightarrow u$ in D'. Then it is easy to see that D' has a factor if and only if B has a perfect matching. The existence of a perfect matching in a bipartite graph can be checked in time $O(n^{1.5}\sqrt{m/\log n})$ [1]. So in time $O(n^{1.5}\sqrt{m/\log n})$ we can decide whether there is an (x, y) or a (y, x)-path P such that $D - P$ has a factor.

It can be checked in time $O(n^{1.5}\sqrt{m/\log n})$ whether D satisfies any of conditions $(1) - (3)$ of theorem 3.1. If D does not satisfy any of conditions $(1) - (3)$ and either D or $D - x$ or $D - y$ is not strong, then we know the desired path exists and it can be found by turning the proof of Lemma 2.6 into an $O(n^2)$ time algorithm.

Suppose that $D, D - x$, and $D - y$ are all strong. It can be checked in constant time whether D is isomorphic to the tournament in Fig. 1. If not, then, according to Theorem 3.1, D has the desired path. To find such a path, we follow the proof in section 3. Most of the steps constructively find the desired path and rest of steps can also easily converted into an $O(n^2)$ time algorithm.

Hence we have the following

Theorem 4.1 *There exists an $O(n^{1.5}\sqrt{m/\log n})$ algorithm to decide if a given ordinary multipartite tournament has a Hamiltonian path connecting two specified vertices x and y. Furthermore, within the same time bound such a path can be found if it exists.* □

5 Concluding remarks

In this paper we have shown that ordinary multipartite tournaments have a structure which is closely related to that of tournaments, when we consider weakly Hamiltonian-connectedness. We point out that Theorem 3.1 does not extend to general multipartite tournaments. To see this consider the multipartite tournament D obtained from a Hamiltonian bipartite tournament B with classes X and Y, by adding two new vertices x and y along with the following arcs: all arcs from x to X and from Y to x, respectively all arcs from y to Y and X to y and an arc between x and y in any direction. It is easy to see that D satifies none of the conditions $(1)-(4)$ in Theorem 3.1, yet there can be no Hamiltonian path with endvertices x and y in D, because any such path would contain a Hamiltonian path of B starting and ending in X or starting and ending in Y. Such a path cannot exist for parity reasons. Note also that we can choose B so that the resulting multipartite tournament is highly connected. However, it can be shown that if D is a strong multipartite tournament containing an (x, y)-path P such that $D - P$ has a factor with t cycles and $D - x$, $D - y$ are strong, then D contains a path with endvertices x and y of length at least $n - t - 2$.

In [14] Thomassen went a lot further than just weakly Hamiltonian-connectedness. He also showed that every 4-connected tournament is strongly Hamiltonian-connected (i.e. we can specify the starting and the ending vertex of a Hamiltonian path). We conjecture that a similar result holds for ordinary multipartite tournaments, namely, we conjecture that if D is a 4-connected ordinary multipartite tournament with an $(x, y)-$path P such that $D-P$ has a factor, then D has an $(x, y)-$Hamiltonian path.

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