THE EMBEDDING OF RELATIVISTIC SPACE-TIMES IN FLAT SPACES

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NOTATION

The summation convention is used throughout, unless specifically suspended.

E_n - a pseudo-Euclidean n dimensional space with metric form,

$$-ds^2 = \sum_{\alpha} \epsilon_\alpha (dz^\alpha)^2$$

R_n - a Riemannian (usually relativistic) n dimensional space with metric form,

$$-ds^2 = g_{ij}\, dx^i\, dx^j$$

$d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$ where θ, ϕ are spherical polar angular coordinates.

ABSTRACT

This dissertation is divided into the following five chapters:

CHAPTER 1

The definition of a flat space is discussed from the point of view of the metric form in the space being reducable to one with constant coefficients for some coordinate system, and this is shown to be equivalent to the vanishing of the ~~Ricci~~ Riemann tensor. Some general results concerning the isometric embedding of Riemannian metrics in flat spaces are derived.

CHAPTER 2

In this chapter, Kasner's proof that an embedding of any solution of Einstein's vacuum equations in a five dimensional flat space is impossible, is given in detail, with a short note regarding the signature of the embedding space. Kasner's minimal embedding of the Schwarzschild Exterior Solution is derived. The contents of this chapter are further discussed in chapter 4.

CHAPTER 3

Fronsdal's embedding of the Schwarzschild Exterior Solution in E_6 is treated in detail as an illustration of the use of the embedding method in general relativity.

CHAPTER 4

The contents of this chapter are wholly based upon a systematic approach to the problem of finding the embedding transformations and minimal embedding spaces for certain relativistic metrics, by the use of sets of partial differential equations. The embeddings of the Schwarzschild Exterior Solution by Kasner and Fronsdal are slightly generalised. Kasner's result of Chapter 2 is criticized on the grounds that its initial assumptions are too restrictive, and give rise to misleading results. In view of this, it is proved that the Schwarzschild Exterior Solution is not embeddable in E_5 by an independent method. Other embeddings of relativistic metrics are derived.

CHAPTER 5

By way of motivation for the present reawakening of interest in the embedding problem, a proposed relation of the embedding space to the elementary particle symmetries is discussed in general terms.

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INTRODUCTION

In his book on "Riemannian Geometry", L. P. Eisenhart has devoted much space to the study of Riemannian surfaces embedded in higher dimensional flat spaces. In the case of general Riemannian surfaces, Eisenhart found this to be a convenient method for studying their properties. However, in relativistic spaces, it has been found that, for the most part, the general curvilinear coordinates in the spaces themselves are not too difficult to deal with, and the embedding flat spaces are of little use in the understanding of gravitation and cosmology. The use of the technique for studying relativistic spaces is well illustrated by C. Fronsdal's completion of the Schwarzschild Exterior Solution by an embedding in a flat space of six dimensions.[1] However, the same results were later obtained by M. D. Kruskal without using an embedding.[1] Thus, interest in the method subsided again.

The embedding method had been considerably used earlier by Schrodinger in his book "Expanding Universes", where he conducts a lengthy discussion of the de Sitter Universe consdiered as a pseudo-hypersphere in a pseudo-Euclidean five dimensional space. Eddington also uses it as a simple illustration of the transition from a Newtonian to a General Relativistic theory of the solar gravitational field in his introduction to relativity theory - "Space, Time and Gravitation", (pages 95, 96).

The real revival of interest in the embedding problem is due to the possibility that the elementary particle symmetries may in some way arise from the space-time geometry. New symmetries may be assodiated with some space orthogonal to the four dimensional space in which we exist, in which case, the extra dimensions of the embedding space may provide us with such an orthogonal space. R. Penrose[5] has said, "The number (and signature) of these new dimensions is to be determined, in this approach, by the condition that the embedding space be minimal and pseudo-Euclidean. It is, thus, of some relevance go try to determine the number and nature of the extra dimensions required for the isometric embedding of a general, physically interesting, general-relativistic space-time." A more detailed discussion of the motivation for the study of this problem has been postponed to the last chapter.

()A minimal embedding is one in which the pseudo-Euclidean embedding space has the minimum number of extra dimensions over the original Riemannian space).

With regard to the general relativistic part of the problem, a very extensive list of embeddings has been published by J. Rosen.[4] J. Plebanski[9] has published a comprehensive account of the embedding of spherically symmetric space-times in flat spaces. The more theoretical aspects of the problem have recently been discussed by A. Friedman[3] following on from the earlier classical works by Janet,[11] Cartan[12] and Burstin,[13] on positive definite metrics, which Friedman has generalised to indefinite metrics. For the main part, only the 'physically interesting' relativistic space-times are discussed in this dissertaiion. The case for a connection with elementary particle physics has been fully discussed in three papers which formed part of a 'Seminar on the Embedding Problem'[4,5,6,7,8] by C. Fronsdal,[6] D. Joseph[7] and Y, Ne'eman,[8] and by E. T. Newman.[14]

The penultimate chapter of this dissertation is concerned with a more systematic approach to the problem of finding the embedding transformations. Kasner,[16] Rosen and Plebanski have used a 'guesswork' approach to this problem, based upon expressing the metric in terms of squares of perfect differentials. This method is equivalent to that of solving the partial differential equations which express the equivalence of the metrics in the curved and flat spaces (these equations will here be called the 'isometry equations', not to be confused with Killing's equations, which refer to preservation of the metric form by coordinate transformations within the curved space).

The advantage of using these equations can be seen from the derivation of two slightly more general classes of embeddings of the Schwarzschild Exterior Solution which reduce to the known embeddings, by Kasner and Fronsdal, in particular cases. A criticism of Kasner's proof that no solution of Einstein's vacuum equations can be embedded in E_5 is found and an existence theorem proved for the embedding of certain types of metric spaces in E_5. J. Plebanski's embedding of a spherically symmetric space-time is also generalised in a way similar to that of the Schwarzschild Exterior Solution.

CHAPTER 1

FLAT SPACES AND GENERAL RESULTS CONCERNING THE EMBEDDING OF RIEMANNIAN SPACES IN FLAT SPACES

FLAT SPACES AND GENERAL RESULTS CONCERNING THE EMBEDDING OF RIEMANNIAN SPACES IN FLAT SPACES

1. Flat Spaces

A pseudo-Euclidean flat space is one in which the metric tensor takes the form,

$$-ds^2 = \sum_\alpha \epsilon_\alpha (dx^\alpha)^2 \,, \quad \epsilon_\alpha = \pm 1 \qquad (1)$$

in some coordinate system - called pseudo-cartesian coordinates. The excess of the number of positive terms over the number of negative terms is an invariant - the signature. If the signature is n, then the space of which (1) is the positive definite metric is said to be a Euclidean space of n dimensions, In the following, positive definite metrics will be rarely used and the notation E_n will be used for any space having a pseudo-Euclidean metric of n dimensions.

A necessary and sufficient condition for there to exist a coordinate system in which the metric takes the above form in a Riemannian space R_n, is that the Riemann tensor is zero. It is necessary to define flatness in this way since a coordinate transformation may take the metric into another form,which it is not immediately obvious is equivalent to the above form.

We know that if two symmetric quadratic differential forms are equivalent,

i.e. $\quad g'_{\mu\nu}\, dx'^{\mu} dx'^{\nu} = g_{ij}\, dx^{\mu} dx^{\nu}$

then it is necessary that,

wh

$$g'^{\mu\nu} = g_{ij} \frac{\partial x^i}{\partial x'^{\mu}} \cdot \frac{\partial x^j}{\partial x'^{\nu}} \qquad (2)$$

where

$$x^i = x^i(x'^1, \dots x'^n) \qquad (3)$$

are the n independent equations of the transformation from x coordinates to x' coordinates.

Now

$$\frac{\partial^2 x^l}{\partial x'^{\mu} \partial x'^{\sigma}} = \begin{Bmatrix} \lambda \\ \mu\sigma \end{Bmatrix}' \frac{\partial x^l}{\partial x'^{\lambda}} - \begin{Bmatrix} l \\ i\ j \end{Bmatrix} \frac{\partial x^i}{\partial x'^{\mu}} \cdot \frac{\partial x^j}{\partial x'^{\sigma}} \qquad (4)$$

where

$$\begin{Bmatrix} l \\ i\ j \end{Bmatrix} = \frac{1}{2} g^{lk} \left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right] \qquad (5)$$

- the Christoffel symbols of the second kind.

Thus, if the quadratic forms are equivalent, the equations (4) admit n(n+1) solutions, which are the functions x^i and $\frac{\partial x^i}{\partial x'^{\mu}}$, which must also satisfy the $\frac{1}{2}$n(n+1) equations (2).

Differentiating (4) with respect to x'^{ν} and differentiating similar equations for $\frac{\partial^2 x^l}{\partial x'^{\mu} \partial x'^{\nu}}$ w.r.t. x'^{σ}, the equation,

$$R'^{\lambda}_{\ \mu\sigma\nu}\ \frac{\partial x^l}{\partial x'^\lambda} = R^l_{\ ijk}\ \frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu} \qquad (6)$$

can be derived by eliminating third order derivatives, where,

$$R^l_{\ ijk} = \frac{\partial}{\partial x^j}\left\{{l \atop ik}\right\} - \frac{\partial}{\partial x^k}\left\{{l \atop ij}\right\} + \left\{{m \atop ik}\right\}\left\{{l \atop mj}\right\} - \left\{{m \atop ij}\right\}\left\{{l \atop mk}\right\}$$

is the Riemann tensor with respect to g_{ij}. Equation (6) is equivalent to,

$$R'_{\tau\mu\sigma\nu} = R_{hijk}\ \frac{\partial x^h}{\partial x'^\tau}\cdot\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu} \qquad (7)$$

which is the integrability condition on equations (4), by virtue of the elimination of the third order derivatives in the derivation of (6). Since equations (4) were obtained from (2) by differentiation with respect to x', any solutions of (4) must make

$$g'_{\mu\nu} - g_{ij}\ \frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} = \text{constant}$$

and if the initial values of the solutions of (4) are choosen to make the constants zero, the solutions will satisfy (2) also. This imposes $\frac{1}{2}n(n+1)$ constraints on the constants of integration of (4), leaving at most $\frac{1}{2}n(n+1)$ arbitrary constants, which may be further reduced in number by the integrability conditions on (4).

In particular if the $g'_{\mu\nu}$ are constants, i.e. they represent a flat space according to the original definition, then $R'_{\tau\mu\sigma\nu} = 0$ and hence $R_{hijk} = 0$ by virtue of (7). Conversely, if the Riemann tensors vanish for both sets of g_{ij}, $g'_{\mu\nu}$, equations (7) are identically true and the two sets of quadratic differential forms are transformable into each other, by a transformation involving $\frac{1}{2}n(n+1)$ arbitrary constants.

Hence we have proved the relationship between flatness and the vanishing of the Riemann tensor as required, and we also note that the transformation necessarily involves $\frac{1}{2}n(n+1)$ arbitrary constants.

2. The Isometric Embedding of Riemannian Spaces in Flat Spaces

Consider a Riemannian m-dimensional space R_m with metric form,

$$-ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta, \quad \alpha,\beta = 1,\cdots m \qquad (1)$$

and a flat space E_n, with metric form,

$$-ds^2 = \sum_{i=1}^{n} \epsilon_i (dz^i)^2 \qquad (2)$$

referred to cartesian coordinates z^i.

A necessary and sufficient condition that R_m can be isometrically embedded in E_n is that there exist n independent real analytic functions,

$$z^i = z^i(x^1, \cdots x^m) \qquad (3)$$

such that,

$$\sum_i \epsilon_i \frac{\partial z^i}{\partial x^\alpha} \cdot \frac{\partial z^i}{\partial x^\beta} = g_{\alpha\beta} \qquad (4)$$

An R_m in E_n is defined by a set of n independent equations (3). Differentiating (3) gives,

$$dz^i = \frac{\partial z^i}{\partial x^\alpha} \cdot dx^\alpha$$

Hence from (2), on the surface defined by (3) it is necessary that,

$$-d\sigma^2 = \sum_i \epsilon_i \left(\frac{\partial z^i}{\partial x^\alpha} dx^\alpha\right)\left(\frac{\partial z^i}{\partial x^\beta} \cdot dx^\beta\right)$$

equating this to (1) gives (4). Conversely, if such a set of functions exist satisfying equation (4), they define an R_m with metric tensor $g_{\alpha\beta}$.

In theory, it should be possible to determine whether or not an embedding of a given R_m is possible in a given E_n, and the form of the embedding transformation, from these equations. All embeddings at the present appear to be based upon more intuitive trial and error procedures, since the partial differential equitions (4) are difficult to solve in general. They have been solved in a later chapter for certain more familiar general relativistic space times whefe the equations can be simplified considerably.

A result which is of use in determining the possible signatures of the embedding space for physically interesting space times is the following. Since the coordinates in R_m can be choosen so that at any particular point of R_m the metric can be reduced to one with constant coefficients, which has, say, p positive and q negative terms, the metric of the embedding space must have at least p positive and at least q negative terms, since m of the coordinates in the E_n can be identified with coordinates in the R_m at the point.

A result originally proved by Janet and Cartan in 1926/27 is that any R_m with positive definite metric can be embedded in E_n where $n \leq \frac{1}{2}m(m+1)$. If (1) is a positive definite form and all the ϵ_i are plus one in (2) then (4) gives $\frac{1}{2}m(m+1)$ equations for the determination of the functions z^i. Hence from the theory of partial differential equations there are n real solutions to (4) if $n = \frac{1}{2}m(m+1)$, so we have an upper bound on the number of dimensions necessary for the embedding flat space.

Recently Friedman[10] has extended this result to metrics which are not positive definite, in the following form. Any R_m with metric tensor having p positive and q negative eigen-values can be isometrically embedded in an E_n with r positive and s negative metric coefficients where $n \leq \frac{1}{2}m(m+1)$ and $r \geq p$, $s \geq q$.

In the case of four dimensional relativistic space times therefore, we can always embed them in Euclidean spaces of ten or less dimensions.

CHAPTER 2

THE EMBEDDING OF EMPTY SPACE SOLUTIONS OF EINSTEIN'S EQUATIONS IN 5D FLAT SPACES AND KASNER'S EMBEDDING OF THE SCHWARZSCHILD EXTERIOR SOLUTION IN E_6

THE EMBEDDING OF EMPTY SPACE SOLUTIONS OF EINSTEIN'S EQUATIONS IN FIVE DIMENSIONAL FLAT SPACES

There are very few general results available in the theory of the embedding method, One of the earliest to be derived, and one of the most useful as far as relativistic metrics are concerned, is that, it is impossible to perform the local embedding of a vacuum solution of Einstein's equations in a five dimensional flat space. This result is due to Kasner[15] and has been esqecially useful in connection with the Schwarzschild exterior solution.

Kasner uses an E_5 with a set of rectangular cartesian coordinates (x^1, x^2, x^3, x^4, w) as the embedding space. In this space an R_4 is defined by an equation of the form,

$$\omega = \omega(x^1, x^2, x^3, x^4)$$

so that the metric form in R_4 is,

$$-ds^2 = dx^{1^2} + dx^{2^2} + dx^{3^2} + dx^{4^2} + d\omega^2$$

where

$$d\omega = \frac{\partial\omega}{\partial x^i}dx^i, \quad i = 1, 2, 3, 4$$

ie.

$$-ds^2 = g_{ik}\, dx^i\, dx^k$$

where

$$g_{ii} = 1 + \omega_i^2 \quad , \quad \omega_i = \frac{\partial \omega}{\partial x^i}$$

and

$$g_{ik} = \omega_i \omega_k \quad , \quad i \neq k$$

Writing

$$g = | g_{ik} |$$

we have

$$g = \begin{vmatrix} 1+\omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 & \omega_1\omega_4 \\ \omega_2\omega_1 & 1+\omega_2^2 & & \\ & & \text{etc} & \end{vmatrix}$$

$$= 1 + \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 .$$

Thus far there are no restrictions on the form of $w(x^1, x^2, x^3, x^4)$, but it is required that the R_4 which w represents be a solution of the Einstein equations in empty space. Hence we wish to find a w such that the Einstein tensor, G_{ik}, is zero. To do this, the contracted Riemann-Christoffel tensor, and hence the Christoffel symbols of the second kind must be calculated in terms of w.

Calculation of the Christoffel Symbols of the Second Kind

In terms of the g_{ik};-

$$\left\{ {\gamma \atop \alpha\beta} \right\} = \frac{1}{2} g^{\gamma\epsilon} \left[g_{\alpha\epsilon,\beta} + g_{\beta\epsilon,\alpha} - g_{\alpha\beta,\epsilon} \right]$$

From the equations for the g_{ik} we have,

$$\frac{\partial g_{ij}}{\partial x^k} = g_{ij,k} = (\omega_i \omega_j)_{,k} = \omega_i \omega_{jk} + \omega_{ik} \omega_j$$

Hence

$$\left\{ {\gamma \atop \alpha\beta} \right\} = \omega_{\alpha\beta} \omega_\epsilon g^{\gamma\epsilon}$$

Now

$$g^{\mu\nu} = \frac{\text{minor of } g_{\mu\nu} \text{ in } |g_{\mu\nu}|}{g}$$

Hence $g^{\gamma\gamma} = \dfrac{1 + \Sigma' \omega_{\epsilon'} \omega_{\epsilon'}}{g}$ where $\sum\limits_{\epsilon'}'$ means summation over $\epsilon' \neq \gamma$.

Therefore, suspending the summation convention on the left hand side,

$$\omega_\gamma g^{\gamma\gamma} = \omega_\gamma \frac{1 + \Sigma' \omega_{\epsilon'} \omega_{\epsilon'}}{g}$$

and

$$\Sigma' \omega_{\epsilon'} g^{\gamma\epsilon'} = \Sigma' \omega_{\epsilon'} \frac{\omega_\gamma \omega_{\epsilon'}}{g} = \frac{\omega_\gamma \Sigma' \omega_{\epsilon'} \omega_{\epsilon'}}{g}$$

So that,

$$\omega_\epsilon g^{\gamma\epsilon} = \omega_\gamma \frac{1 + 2\Sigma' \omega_{\epsilon'} \omega_{\epsilon'}}{g}$$

Writing,

$$P_\gamma = 1 + 2\Sigma' \omega_{\epsilon'} \omega_{\epsilon'} = 2g - 1 - 2\omega_\gamma^2$$

So that

$$\left\{ {\gamma \atop \alpha\beta} \right\} = \frac{\omega_{\alpha\beta}\, \omega_\gamma P_\gamma}{g} .$$

Calculation of the Ricci Tensor

The condition $G_{ik} = 0$ gives,

$$R_{ik} - \tfrac{1}{2} g_{ik} R = 0$$

hence

$$g^{ik}\left[R^{ik} - \tfrac{1}{2} g^{ik} R\right] = 0$$

i.e. $R - 2R = 0$

so that $G_{ik} = 0 \Leftrightarrow R_{ik} = 0$.

The Ricci tensor is given by,

$$R_{\mu\nu} = \begin{Bmatrix} \alpha \\ \mu\sigma \end{Bmatrix}\begin{Bmatrix} \sigma \\ \alpha\nu \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}\begin{Bmatrix} \sigma \\ \alpha\sigma \end{Bmatrix} + \frac{\partial}{\partial x^\nu}\begin{Bmatrix} \sigma \\ \mu\sigma \end{Bmatrix} - \frac{\partial}{\partial x^\sigma}\begin{Bmatrix} \sigma \\ \mu\nu \end{Bmatrix}$$

using

$$\begin{Bmatrix} \sigma \\ \mu\sigma \end{Bmatrix} = \frac{\partial}{\partial x^\mu}\left(\log \sqrt{g}\right)$$

and altering some of the dummy suffices $R_{\mu\nu}$ can be written,

$$R_{\mu\nu} = -\frac{\partial}{\partial x^\alpha}\begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} + \begin{Bmatrix} \beta \\ \mu\alpha \end{Bmatrix}\begin{Bmatrix} \alpha \\ \nu\beta \end{Bmatrix} + \frac{\partial^2}{\partial x^\mu \partial x^\nu}\left(\log\sqrt{g}\right) - \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}\frac{\partial}{\partial x^\alpha}\left(\log\sqrt{g}\right).$$

To calculate the Ricci tensor at this stage would clearly be quite difficult. However, the process can be considerably simplified; since the origin,(and orientation) of the Cartesian coordinates, is quite arbitrary, it may be choosen as a point on the surface with the fifth coordinate in the direction of the normal to the surface. If the origin is a regular point, we can expand $\omega(x^1, x^2, x^3, x^4)$

in terms of the xs. The deviation from the tangent plane is of second order compared with distances in the plane and hence the expansion does not contain linear terms in the xs.

Accordingly

$$\omega = \tfrac{1}{2} a_{\mu\nu} x^{\mu} x^{\nu} + \text{higher terms}$$

near the origin. This quadric is known as the indicatrix for a fixed w.[19]

The radius of curvature of any normal section of the surface, is given by,

$$\rho = \frac{t^2}{2\omega} = \frac{1}{a_{\mu\nu} l_{\mu} l_{\nu}}$$

where t is the radius of curvature of the indicatrix in the direction, with direction cosines (l^1, l^2, l^3, l^4), of the normal section.

If the principal axes of the indicatrix are made to coincide with the cartesian axes, w can be written in the form,

$$2\omega = k_1 x^{1^2} + k_2 x^{2^2} + k_3 x_3^2 + k_4 x^{4^2},$$

where the principal radii of curvature of the surface $\rho_i = \frac{1}{k_i}$, since the direction cosines are now those of the cartesian axes.

Hence, at the origin and with this special orientation;- $\omega_i = 0$, $\omega_{ii} = k_i$, $\omega_{ij} = 0$, $i \neq j$.

so that $g = \begin{vmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{vmatrix} = 1$

and $P_\gamma = 1$ at the origin.

Therefore

$$\left\{ {\gamma \atop \alpha\beta} \right\} = \frac{\omega_{\alpha\beta}\,\omega_\gamma P_\gamma}{g} = 0$$

Hence at the origin, we have,

$$R_{\mu\nu} = -\frac{\partial}{\partial x^\alpha}\left\{ {\alpha \atop \mu\nu} \right\} + \frac{\partial^2}{\partial x^\mu \partial x^\nu}\left(\log\sqrt{g}\right)$$

$$= -\frac{\partial}{\partial x^\alpha}\left(\frac{\omega_{\mu\nu}\,\omega_\alpha P_\alpha}{g}\right) + \frac{\partial}{\partial x^\mu}\left(\frac{\omega_\nu\,\omega_{\nu\nu}}{2g}\right)$$

(Since $g = 1 + \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2$).

$$= -\omega_{\alpha\alpha}\frac{\omega_{\mu\nu} P_\alpha}{g} - \omega_\alpha \frac{\partial}{\partial x^\alpha}\left(\frac{\omega_{\mu\nu} P_\alpha}{g}\right) + \frac{\partial}{\partial x^\mu}\left(\frac{\omega_\nu\,\omega_{\nu\nu}}{g}\right)$$

$$= -\omega_{\mu\nu}\left(\omega_{11} + \omega_{22} + \omega_{33} + \omega_{44}\right) + \frac{\omega_{\mu\nu}\,\omega_{\nu\nu}}{g}$$

$$= -\omega_{\mu\nu}\left(\omega_{11} + \omega_{22} + \omega_{33} + \omega_{44} - \omega_{\nu\nu}\right).$$

Remembering throughout that at the origin $g = 1$, $P_\gamma = 1$, $\omega_\nu = 0$.

Hence

$$R_{\mu\nu} = 0 \qquad \mu \neq \nu$$

$$= -\omega_{\nu\nu}\left(\sum_{\alpha=1}^{4}\omega_{\alpha\alpha} - \omega_{\nu\nu}\right), \quad \mu = \nu.$$

The conditions that $w = w^{\alpha}(x^1, x^2, x^3, x^4)$ represents a four dimensional solution of Einstein's empty space equations are thus,

$$R_{11} = -\omega_{11}(\omega_{22} + \omega_{33} + \omega_{44}) = 0 \quad , \text{etc}$$

i.e.

$$k_1(k_2 + k_3 + k_4) = 0$$
$$k_2(k_1 + k_3 + k_4) = 0$$
$$k_3(k_1 + k_2 + k_4) = 0$$
$$k_4(k_1 + k_2 + k_3) = 0$$

i.e.

$$k_1 = k_2 = k_3 = k_4 = 0$$

corresponding to the trivial case when R_4 is itself an E_4. Thus it is not possible to represent a natural gravitational field satisfying $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$, $R_{\mu\nu\sigma\tau} \neq 0$, in five Euclidean dimensions.

Kasner's proof, as it stands, is only true for an embedding space with a positive definite metric form. A formal demonstration that is also true for indefinite metrics is obtained by considering the proof for

$$-d\sigma^2 = dx^{1^2} + dx^2 + dx^{3^2} + dX^{4^2} + dw^2$$

where $X^4 = ix^4$. Then algebraically the proof proceeds as before and the condition,

$$R_{ik} = 0 \quad , \quad i \neq k$$
$$R_{ii} = -\omega_{ii}\left(\sum_{\alpha=1}^{4} \omega_{\alpha\alpha} - \omega_{ii}\right)$$

is derived for w to represent an R_4 in E_5.

If at this stage we write ix^4 for X^4 then,

$$\omega_4 = \frac{\partial\omega}{\partial X^4} = -i\,\frac{\partial\omega}{\partial x^4}$$

$$\omega_{44} = -\frac{\partial^2\omega}{\partial x^{4^2}}$$

and the indicatrix becomes,

$$2\omega = k_1 x^{1^2} + k_2 x^{2^2} + k_3 x^{3^2} - k_4 x^{4^2}$$

So that ,

$$\frac{\partial^2\omega}{\partial x^{4^2}} = -k_4$$

and

$$\omega_{44} = k_4$$

as before.

Thus we derive the same conditions on the ks.

(If the conditions on the ks had become of the form

$$k_1(k_2 + k_3 - k_4) = 0$$

etc.

it would not have necessarily followed that all the ks were zero).

KASNER'S EMBEDDING OF THE SCHWARZSCHILD EXTERIOR SOLUTION IN E_6 [16]

The results of the preceeding section show that the Schwarzschild Exterior Solution space-time cannot be embedded in an E_5. Thus an embedding in E_6 would be minimal. Kasner first obtained such a solution by considering the metric form,[20]

$$-ds^2 = \left(1-\frac{2m}{r}\right)dt^2 - \left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \qquad (1)$$

and introducing cartesian coordinates to give the equivalent form,

$$-ds^2 = -dz'^2 - dz^{2\,2} - dz^{3\,2} - \frac{2m}{r-2m}dr^2 + \frac{r-2m}{2m}dt^2 \qquad (2)$$

with

$$r^2 = z'^2 + z^{2\,2} + z^{3\,2} \qquad (2')$$

Introducing a new variable, R, such that,

$$dR = \left(\frac{2m}{r-2m}\right)^{\frac{1}{2}} dr$$

i.e.

$$R = \left(8m(r-2m)\right)^{\frac{1}{2}}, \qquad (3)$$

reduces the dr^2 term to a perfect differential and the metric form then becomes,

$$-ds^2 = -dz'^2 - dz^{2\,2} - dz^{3\,2} - dR^2 + \frac{R^2}{R^2+16m^2}dt^2 \qquad (4)$$

Considering now only the last two terms, which involve R and t only, they can be replaced by the sum of three unit squares

$$dz^{4^2} + dz^{5^2} - dz^{6^2} \qquad (5)$$

where

$$\left.\begin{aligned} z^4_4 &= R(R^2+16m^2)^{-\frac{1}{2}} \sin t \\ z^5 &= R(R^2+16m^2)^{-\frac{1}{2}} \cos t \\ z^6 &= \int \left(1+\frac{256m^4}{(R^2+16m^2)^3}\right)^{\frac{1}{2}} dR \end{aligned}\right\} \qquad (6)$$

Hence the metric form of the embedding E_6 is,

$$-d\sigma^2 = -dz^{1^2} - dz^{2^2} - dz^{3^2} + dz^{4^2} + dz^{5^2} - dz^{6^2}$$

and the R_4 is represented by equations 2', 3, and 6.

There are two interesting subspaces of this R_4. Firstly, the cross-section by the hyperplanes $t = 0$, $z^6 = 0$, illustrating the inter-relation of spatial measurements for a given time, in the surface, and secondly, that represented by equations 6, illustrating the inter-relation of space and time measurements.

With $t = 0$,

$$z^4 = 0, \quad z^5 = \frac{R}{(R^2+16m^2)^{\frac{1}{2}}}, \quad z_6 = \int (\quad)^{\frac{1}{2}} dR \qquad (7)$$

and 4 becomes,

$$-d\sigma^2 = -dz^{1^2} - dz^{2^2} - dz^{3^2} - dR^2 \qquad (8)$$

Equations 7, 2' and 3, represent an R_3 in the E_4 with metric form 8, (the R_3 is also in the E_5 with metric form

$$-d\sigma^2 = -dz^{1^2} - dz^{2^2} - dz^{3^2} + dz^{5^2} - dz^{6^2}).$$

Using the representation in E_4 and putting $z^6 = 0$

we obtain Flamm's paraboloid,[18] by rotating the parabola $R = (2m(r-2m))^{\frac{1}{2}}$ about its directrix.

The surface represented by equations 6 is also a surface of revolution,

$$Z^{4^2} + Z^{5^2} = R^2(R^2+16m^2)$$
$$Z^6 = \int \left(1 + \frac{256m^4}{(R^2+16m^2)^3}\right)^{\frac{1}{2}} dR$$

The generating curve is transcendental, since the integral involves hyper-elliptic functions. The parameter t represents the angle of rotation to the point (R,t). These two subspaces characterize the Schwarzschild Exterior Solution.

Writing the embedding in terms of r,

$$Z^4 = \left(\frac{r-2m}{r}\right)^{\frac{1}{2}} \sin t$$
$$Z^5 = \left(\frac{r-2m}{r}\right)^{\frac{1}{2}} \cos t$$
$$Z^6 = \int \left(\frac{2m + \frac{m^2}{r^3}}{r-2m}\right)^{\frac{1}{2}} dr$$

In obtaining the equations of the surface that these equations represent, we must determine r as a function of Z_6. This can only be done, in this embedding, if $r > 2m$, thus, as Fronsdal has pointed out, Kasner's embedding cannot be used to continue the Schwarzschild Solution into the region $r < 2m$. The embedding also identifies points at time t with points at time $t + 2n\pi$, this prompted Fronsdal to replace the trigonemetric functions with hyperbolic functions, and thus obtain an embedding which could be extended into the region $r < 2m$.

CHAPTER 3

FRONSDAL'S COMPLETION OF THE SCHWARZSCHILD SOLUTION BY AN EMBEDDING IN A PSEUDO-EUCLIDEAN SIX SPACE

FRONSDAL'S COMPLETION OF THE SCHWARZSCHILD SOLUTION BY AN EMBEDDING IN A PSEUDO-EUCLIDEAN SIX SPACE [1]

We consider a manifold as a set of geometrical objects (points), then if we can define a 1:1 relationship between the points of the n-dimentional arithmetic space $\{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n$ are ordered real numbers, we have defined a coordinate system on that manifold. The coordinate system may not cover the whole manifold in which case it is referred to as a coordinate patch - a familiar example of a coordinate patch is the spherical polar coordinates on the surface of a sphere, where the pole of the sphere does not admit a 1:1 identification with a point in the arithmetic set.

The coordinate system in which the Schwarzschild Exterior Solution takes the form, [20]

$$-ds^2 = \left(1-\tfrac{1}{r}\right)dt^2 - \left(1-\tfrac{1}{r}\right)^{-1}dr^2 - r^2\, d\Omega^2$$

is incomplete, in so far as, the Schwarzschild manifold exists for all r in the range $0<r<\infty$, whereas the coordinate patch extends over the interval $1<r<\infty$.

The fact that the singularity at $r=1$ is due to the coordinates used, can best be illustrated by considering the equations of the geodesics in the plane $\theta = \pi/2$.

In general, the equations of the geodesics are given by the four equations,

$$\frac{d^2x^i}{ds^2} + \left\{ {i \atop \alpha\beta} \right\} \frac{dx^\alpha}{ds} \cdot \frac{dx^\beta}{ds} = 0 .$$

In the case of the Schwarzschild Exterior Solution, these equations become,[20]

$$\frac{d^2r}{ds^2} + \frac{1}{2}\frac{d\lambda}{dr}\left(\frac{dr}{ds}\right)^2 - re^{-\lambda}\left(\frac{d\phi}{ds}\right)^2 + e^{\nu-\lambda}\frac{d\nu}{d\lambda}\left(\frac{dt}{ds}\right)^2 = 0 \qquad \underline{A}$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\cdot\frac{d\phi}{ds} = 0 \qquad \underline{B}$$

$$\frac{d^2t}{ds^2} + \frac{d\nu}{ds}\cdot\frac{dt}{ds} = 0 \qquad \underline{C}$$

in the plane $\theta = \pi/2$, where $e^\nu = e^{-\lambda} = \left(1 - \frac{1}{r}\right)$.

A first integral of A is provided by the metric form itself, and the other two may be integrated by inspection. Thus we have,

$$1 = -e^\lambda\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2 + e^\nu\left(\frac{dt}{ds}\right)^2 \qquad (1)$$

$$\frac{d\phi}{ds} = h/r^2 \qquad (2)$$

$$\frac{dt}{ds} = ke^{-\nu} \qquad (3)$$

in the plane $\Theta = \pi/2$, with h,k constants of integration.
(1) and (3) give ;-

$$\left(\frac{dr}{ds}\right)^2 + r^2\left(1-\frac{1}{r}\right)\left(\frac{d\phi}{ds}\right)^2 - k^2 = -1\left(1-\frac{1}{r}\right)$$

i.e.

$$\left(\frac{dr}{ds}\right)^2 + \left(r^2 - r\right)\frac{h^2}{r^4} = k^2 - 1 + \frac{1}{r}$$

i.e.

$$\left(\frac{dr}{ds}\right)^2 = k^2 - 1 + \frac{1}{r} - \frac{h^2}{r^2} + h^2/r^3 .$$

By considering the analogous Newtonian equations for the motion of a small test body in a spherically symmetric gravitational field, we say that h, k^2 are essentially equivalent to angular momentum and energy, respectively, in this solution.

Now, if $0 \leq r < 1$ then $\dot{r}^2 > 0$, ($\dot{r} = \frac{dr}{d\tau}$) where τ is the proper time (defined below) and if $r = 1 + \alpha$, where α is small, $(1+\alpha)^{-n} \sim 1 - \alpha n$, gives,

$$\dot{r}^2 \approx k^2 - (h^2+1)\alpha$$

hence $\dot{r}^2 = 0$ when $\alpha = \frac{k^2}{h^2+1}$, so that the smallest value of α for which $\dot{r}^2 = 0$ is $\alpha = 0$ corresponding to $r = 1$.

Hence, the functional behaviour of $r(\tau)$ in the region $0 < r < 1+$ is either (a), (b) or (c) in Fig.1,

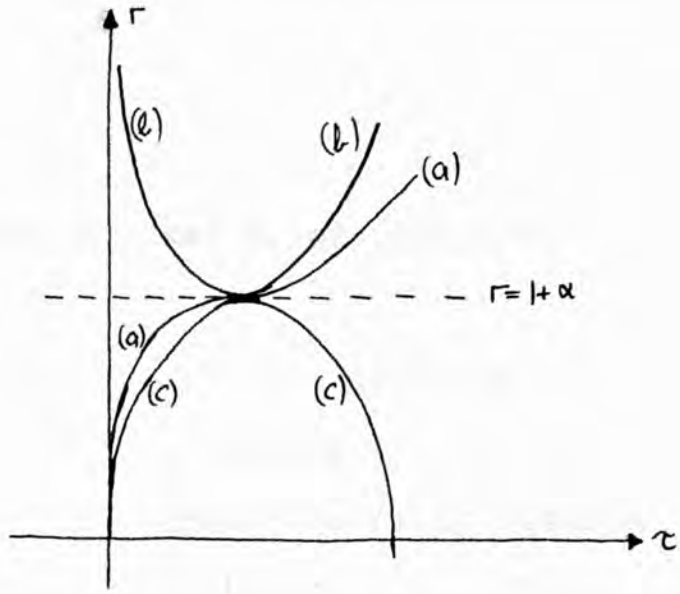

Fig 1

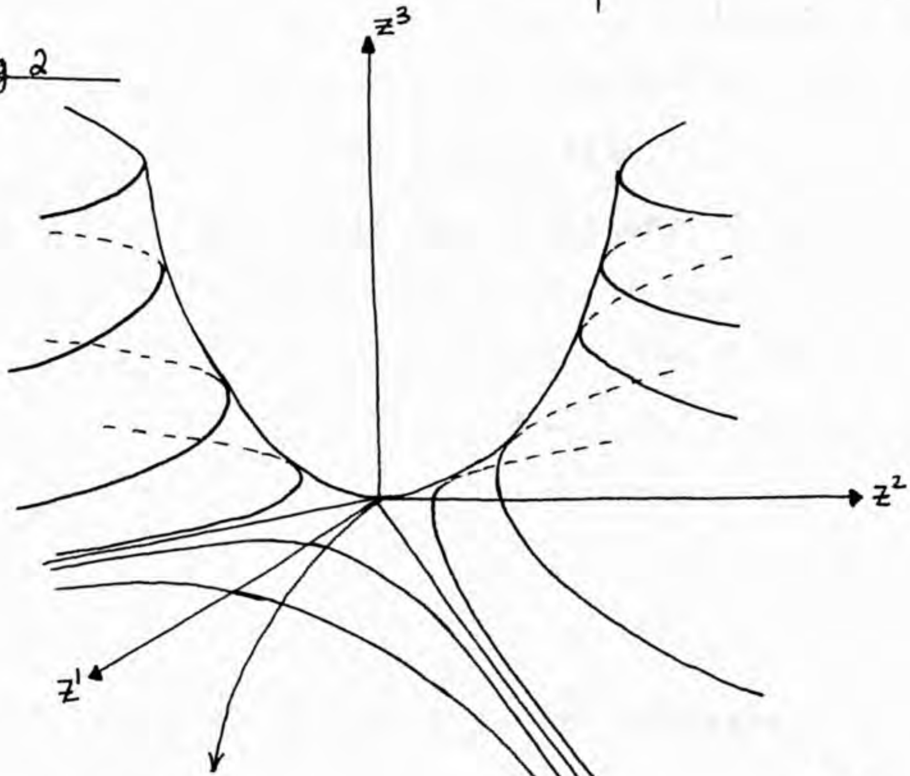

Fig 2

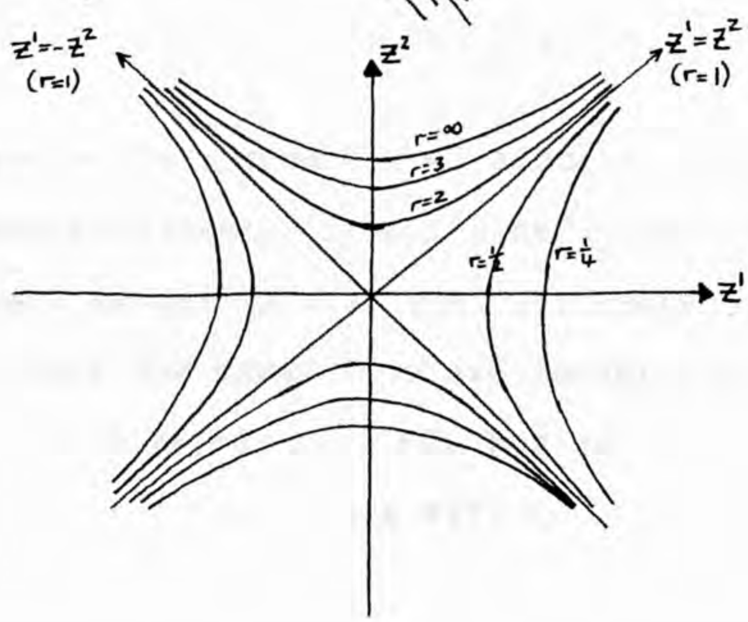

Fig 3

depending upon the values of h and k, assuming $r(\tau)$ is continuous.

Corresponding to (a), we have geodesics starting or ending at the origin passing through $r=1$. Corresponding to (b), we have geodesics fully outside $r=1$, and to (c) geodesics fully inside $r=1$. Clearly a test body reaching the origin from a finite distance does so in a finite proper time.

We obtain the relationship between proper time τ and Schwarzschild time t from,

$$\frac{ds^2}{d\tau^2} = \epsilon = \begin{cases} +1 & \text{for a time-like geodesic} \\ -1 & \text{for a space-like geodesic} \\ 0 & \text{for a null geodesic} \end{cases}$$

i.e.

$$\epsilon = -\left(1-\tfrac{1}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 + \left(1-\tfrac{1}{r}\right)\dot{t}^2$$

and using the equation for $\dot{r}^2$, we have

$$\left(1-\tfrac{1}{r}\right)\dot{t}^2 = \left(1-\tfrac{1}{r}\right)^{-1}\left(k^2-1+\tfrac{1}{r}-\tfrac{h^2}{r^2}+\tfrac{h^2}{r^3}\right)+r^2\dot{\phi}^2+\epsilon$$

and hence as we approach $r=1$ along a geodesic $t \to \pm\infty$. This results directly from the definition of t and is equivalent to saying that the coordinates in which the metric takes the usual form are incomplete, since geodesics can penetrate $r=1$, but we cannot describe them in terms of functions $r(\tau)$ and $t(\tau)$, except in the region $1 < r \infty$, whereas the Schwarzschild manifold exists for $0<r<\infty$.

By representing the Schwarzschild exterior manifold as an R_4 in E_6, we can, if desired, remove the necessity for finding coordinates in the R_4 by writing its equations in terms of the pseudo-rectangular cartesian coordinates of the E_6. If we still require to define coordinates in the R_4 then they may be choosen in terms of these rectangular coordinates or the original Schwarzschild coordinates.

Kasner's embedding has already been mentioned, and for reasons given there, it cannot be used to extend the representation of the Schwarzschild manifold to the region $r<1$. However, by replacing trigonometric functions with the hyperbolic functions, we can obtain another embedding, this time in the E_6 with,

$$-ds^2 = dz^{1^2} - dz^{2^2} - dz^{3^2} - dz^{4^2} - dz^{5^2} - dz^{6^2}$$

$$z^1 = 2\left(1-\tfrac{1}{r}\right)^{\frac{1}{2}} \sinh t/2$$

$$z^2 = 2\left(1-\tfrac{1}{r}\right)^{\frac{1}{2}} \cosh t/2$$

$$z^3 = g(r)$$

$$z^4 = r\sin\theta\sin\phi, \quad z^5 = r\sin\theta\cos\phi, \quad z^6 = r\cos\theta,$$

where

$$g(r) = \int\left(\frac{r^2+r+1}{r^3}\right)^{\frac{1}{2}} dr .$$

Eliminating the coordinates θ, ϕ and t we get,

$$Z^{2^2} - Z^{1^2} = 4\left(1 - \frac{1}{r}\right)$$

$$Z^3 = \int \left(\frac{r^2 + r + 1}{r^3}\right)^{\frac{1}{2}} dr \qquad \text{(A)}$$

$$Z^{4^2} + Z^{5^2} + Z^{6^2} = r^2$$

and we could eliminate r also from these equations if we so wished, to completely free the representation of the Schwarzschild coordinates, but it is simpler to retain r as a parameter over the surface, besides, the term $r^2 d\Omega^2$ in the metric, identifies the r with the physical radius, and difficulties in the description of the geodesics were due to a poor choice of the time coordinate. Hence the surface represented by the set of equations (A) is the Schwarzschild surface R_4, for $r > 1$, but the surface so defined is analytic for the interval $0 < r < \infty$, and hence, it is an analytic continuation of the Schwarzschild exterior solution into the region $0 < r \leq 1$.

A Discussion of the Properties of the Surface R_4 Defined by Equations (A)

1. Vizualization of the surface

We cannot vizualize the surface R_4 in E_6 directly, but we can obtain an, R_2 in E_3, reduced representation of the surface, by considering the section of the R_4 by hyperplanes θ = constant, ϕ = constant.

The metric in the R_2 is then,

$$-ds^2 = \left(1-\tfrac{1}{r}\right)dt^2 - \left(1-\tfrac{1}{r}\right)^{-1}dr^2$$

and the metric in the embedding E_3 is,

$$-ds^2 = dz^{1^2} - dz^{2^2} - dz^{3^2}$$

and the surface is defined by the first two of equations (A). The surface is drawn diagramatically in Fig. 2. The function $z_3(r)$ is monotonically increasing and the two quadrants in which $r > 1$ represent the usual Schwarzschild exterior solution twice over, whilst the regions $r < 1$ represent the analytic continuation of of the solution. Clearly, any plane $z_3(r_0)$ = constant cuts the surface in two hyperbolic branches, given by

$$z^{2^2} - z^{1^2} = 4\left(1-\tfrac{1}{r_0}\right)$$

For $\quad r$ large $\quad \frac{dz^3}{dr} \sim 1$

and for $\quad r$ small $\quad \frac{dz^3}{dr} \sim r^{-3/2}$

and for $\quad r \sim 1 \quad \frac{dz^3}{dr} \sim 2$

and hence, the general shape, as in Fig. 2.

On projecting the surface down onto the z^1, z^2 plane, Fig. 3 is obtained, which consists of a set of hyperbolae of constant r. For $r = 1$, the hyperbolae degenerate into the two straight lines $z^2 = \pm z^1$ and for large r they approach the hyperbola $z^{2^2} - z^{1^2} = 4$.

2. Coordinates in the surface

The rectangular coordinate z^1 is timelike, whilst z^2 and z^3 are spacelike, as can be seen from the metric form in the E_3, thus we could choose z^1 as a time coordinate in Fig. 3 and z^2 as a space coordinate. However, we would like to retain r as a space coordinate as mentioned before, and certain considerations enter into the choosing of a time coordinate. Firstly, for large values of r we require the metric to approach the Minkowski metric, and secondly, the metric tensor should be independent of the time coordinate. The original Schwarzschild time is, of course, such a coordinate, and it is defined in the top quadrant of Fig. 3 by

$$t = \log\left(\frac{1+ Z^1/Z^2}{1- Z^1/Z^2}\right)$$

but, as we have seen, it tends to plus or minus infinity near the lines $Z^1 = Z^2$ and $Z^1 = -Z^2$ respectively, and hence is not a global definition. The only alternative definition which will not introduce time into the metric tensor is

$$T = t + f(r)$$

(which gives $dt = dT - f'(r)\,dr$

and hence a metric tensor of form,

$$-ds^2 = \left[f'(r)^2(1-\tfrac{1}{r}) - (1-\tfrac{1}{r})^{-1}\right]dr^2 + 2(1-\tfrac{1}{r})f'(r)\,dr\,dT + (1-\tfrac{1}{r})dT^2 - r^2 d\Omega^2).$$

However, $f(r)$ must have the same value both at $Z^1 = Z^2$ and at $Z^1 = -Z^2$ and hence no choice of $f(r)$ can make T finite at both of $Z^1 = \pm Z^2$. If we choose $f(r) = -\log(r-1)$ (17) then T is finite on $Z^1 = -Z^2$ but not on $Z^1 = +Z^2$. Thus, here we have a time coordinate for one half of the completed manifold, and in fact this is the largest portion of the manifold on which we can define a time coordinate satisfying our requirements, of course, a time coordinate can be similarly defined over any of the halves of the completed manifold, bounded by lines $Z^1 = +Z^2$ or $Z^1 = -Z^2$.

3. Geodesics

It can be easily seen that, in the z^1, z^2, z^3 space, geodesics can be drawn on the R_2 surface which pass through the line r=1, when the unnecessary complications of the r, t coordinates are removed. The lines corresponding to r=1 still retain some distinction from other r=constant curves in that they are lines of zero length, and from the form of the metric in the subspace $\theta=\phi=$constant, this can be seen to be inherently connected with the signature of the metric, which is invariant under any coordinate transformation.

CHAPTER 4

THE EMBEDDING OF RELATIVISTIC SPACE-TIMES BY THE USE OF PARTIAL DIFFERENTIAL EQUATIONS

THE EMBEDDING OF RELATIVISTIC SPACE-TIMES BY THE USE OF PARTIAL DIFFERENTIAL EQUATIONS

INTRODUCTION

Many authors have found embeddings of various relativistic space-times, notably Kasner,[16] Fronsdal,[1] Rosen,[4] and Plebanski[9] but no systematic method has been used in deriving them. Merely guessing functions which will reduce metrics to the sums of squares of perfect differentials seems unsatisfactory as far as obtaining more complete solutions is concerned. Rosen states that his very comprehensive list of embeddings was obtained without the use of partial differential equations. Earlier we obtained a necessary and sufficient condition for an embedding to be possible, and an obvious approach is to use this condition to derive the embedding transformations and as a criterion as to whether or not they can exist in any particular case. This is the basis of Kasner's results.

It may be argued that the solutions of the partial differential equations are so difficult to obtain as to prohibit the use of this method, even though the equations themselves are simple in form. However, in cases of the simpler relativistic metrics, (Schwarzschild Interior and Exterior Solutions, Reissner-Weyl charged particle solution, Einstein and Einstein de-Sitter Universes) the solutions can be readily obtained by making certain simplifying assumptions, and the solutions so obtained even though they may not be complete solutions do contain all the embeddings found by previous methods. Furthermore, the forms of the differential equations show why the metric of the embedding space can have certain signatures and no others.

We recall that the embedding condition was first derived by representing the R_4 surface by a set of restrictions on the coordinates of the E_n. These restrictions in general are given by the equations

$z^\alpha = z^\alpha (r, \theta, \phi, t)$ where z^α are the coordinates in the E_n and r, θ, ϕ, t are parameters on the surface R_4. However, when dealing with relativistic metrics, we may wish to identify certain of the r, θ, ϕ, t parameters with coordinates z^α. For example, when an $r^2 d\Omega^2$ term appears in the relativistic metric, we identify $z^1 = r \sin\theta \sin\phi$, $z^2 = r \sin\theta \cos\phi$, $z^3 = r \cos\theta$

with the rectangular cartesian x, y, z, since r has an interpretation as the radius of the sphere defined by the θ, ϕ coordinates. However, there is no particular reason why we should identify the t coordinate with the flat space t. When this identification is not valid it manifests itself by making the equations of the embedding unsolvable for the particular number of embedding dimensions under consideration. This point will be clarified with respect to the Schwarzschild Interior Solution, in particular.

As an example, consider the R_4 defined by Kasner, in E_5, by equations $z_1 = x$, $z_2 = y$, $z_3 = z$, $z_4 = t$, $w = w(x, y, z, t)$ here all four parameters in the surface are defined by identification with coordinates in E_5. Another way of representing an R_4 in E_5 is by equations $z_1 = x$, $z_2 = y$, $z_3 = z$, $z_4 = z_4(x, y, z, t)$, $z_5 = z_5(x, y, z, t)$ in which t is now a parameter in the R_4 and not a coordinate in the E_5. There are an infinite number of sets of rectangular cartesian coordinates in the E_5 all related by linear transformations and if the parameter t cannot be identified with a linear transformation of $z_4(t)$ it has to be treated as a parameter on R_4 and not a coordinate in E_5.

Bearing this in mind, Kasner's result will be reworked for the Schwarzschild Exterior Solution and a spurious result obtained for the Schwarzschild Interior Solution.

A minimal embedding of the Schwarzschild Interior Solution is found quite straghtforwardly from the isometry equations.

In this section, the Euclidean spatial part of the metric will be represented in spherical polar coordinates.

A REAPPRAISAL OF KASNER'S RESULT IN THE SPECIAL CASE OF THE SCHWARZSCHILD EXTERIOR SOLUTION

Representing an R_4 by the surface $\omega = \omega(r, \theta, \phi, t)$ (with r, θ, ϕ spherical polar coordinates), in an E_5, we have that the metric tensor induced on the R_4 by the metric tensor of E_5 $\left(-ds^2 = dr^2 + r^2 d\Omega^2 - dt^2 + \epsilon\, d\omega^2,\ \epsilon = \pm 1\right)$,

is

$$a_{\alpha\beta} = \begin{pmatrix} 1+\epsilon\left(\frac{\partial\omega}{\partial r}\right)^2 & \epsilon\frac{\partial\omega}{\partial r}\cdot\frac{\partial\omega}{\partial\theta} & \epsilon\frac{\partial\omega}{\partial r}\cdot\frac{\partial\omega}{\partial\phi} & \epsilon\frac{\partial\omega}{\partial r}\cdot\frac{\partial\omega}{\partial t} \\ \epsilon\frac{\partial\omega}{\partial\theta}\cdot\frac{\partial\omega}{\partial r} & r^2+\epsilon\left(\frac{\partial\omega}{\partial\theta}\right)^2 & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & r^2\sin^2\theta+\epsilon\left(\frac{\partial\omega}{\partial\phi}\right)^2 & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & -1+\epsilon\left(\frac{\partial\omega}{\partial t}\right)^2 \end{pmatrix}$$

Hence with the usual Schwarzschild Exterior metric, we require,

$$\left.\begin{aligned} 1+\epsilon\left(\frac{\partial\omega}{\partial r}\right)^2 &= \left(1-\frac{1}{r}\right)^{-1} \\ \epsilon\left(\frac{\partial\omega}{\partial\theta}\right)^2 &= 0 \\ \epsilon\left(\frac{\partial\omega}{\partial\phi}\right)^2 &= 0 \\ -1+\epsilon\left(\frac{\partial\omega}{\partial t}\right)^2 &= -\left(1-\frac{1}{r}\right) \end{aligned}\right\} \underline{A} \quad \& \quad \left.\begin{aligned} \frac{\partial\omega}{\partial r}\cdot\frac{\partial\omega}{\partial\theta} &= 0 \\ \frac{\partial\omega}{\partial r}\cdot\frac{\partial\omega}{\partial\phi} &= 0 \\ &\vdots \\ \frac{\partial\omega}{\partial\phi}\cdot\frac{\partial\omega}{\partial t} &= 0 \end{aligned}\right\} \underline{B}$$

Either set of equations show immediately that the embedding is impossible. Equations A by $\frac{\partial^2 \omega}{\partial r \partial t} \neq \frac{\partial^2 \omega}{\partial t \partial r}$ and equations B show that only one of $\frac{\partial \omega}{\partial r}, \frac{\partial \omega}{\partial \theta}, \frac{\partial \omega}{\partial \phi}, \frac{\partial \omega}{\partial t}$ is non-zero.

However by replacing the right hand side of $\underline{A}$ by the corresponding terms for the Schwarzschild Interior Solution,[21]

$$-ds^2 = \left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 d\Omega^2 - \frac{1}{4}\left(3(A) - \left(1-\frac{r^2}{R^2}\right)^{\frac{1}{2}}\right)^2 dt^2$$

we see that again a solution is impossible, when the surface R_4 is represented by a single equation. The Schwarzschild Interior Solution can, however, be embedded in an E_5 as is illustrated in the next section.

Hence, we may ask if Kasner's result really does prove that the Schwarzschild Exterior Solution is not embeddable in E_5, perhaps it only appears so, due to Kasner's restrictive choice of embedding functions; z_1, z_2, z_3 as spherical polar coordinates, $z_4 = t$, $z_5 = w(r, \theta, \phi, t)$, which were so chosen to make the calculation of the Ricci tensor simple. In view of this, the embedding of the Schwarzschild Exterior Solution is investigated when it is represented by $r, \theta, \phi, u(r,\theta,\phi,t), v(r,\theta,\phi,t)$ in the E_5 with metric $-ds^2 = dr^2 + r^2 d\Omega^2 + \epsilon_1 du^2 + \epsilon_2 dv^2$

In this case, the isometry equations are;

$$\epsilon_1\left(\frac{\partial u}{\partial r}\right)^2 + \epsilon_2\left(\frac{\partial v}{\partial r}\right)^2 = \frac{1}{r-1} = \ell(r) > 0 \qquad (1)$$

$$\epsilon_1\left(\frac{\partial u}{\partial t}\right)^2 + \epsilon_2\left(\frac{\partial v}{\partial t}\right)^2 = -\left(\frac{r-1}{r}\right) = m(r) < 0 \qquad (2)$$

$$\left.\right\} r > 1$$

$$\epsilon_1 \frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial t} + \epsilon_2 \frac{\partial v}{\partial r}\cdot\frac{\partial v}{\partial t} = 0 \qquad (3)$$

where it has been assumed that,

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial \phi} = \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial \phi} = 0$$

owing to the spherical symmetry of the metric, which must be preserved in the embedded surface. This assumption will be used throughout this section.

From equations (1) and (2), $\epsilon_1 = -\epsilon_2 = \pm 1$

With $\epsilon_1 = -\epsilon_2 = 1$ equation (3) gives,

$$\frac{\frac{\partial u}{\partial r}}{\frac{\partial v}{\partial r}} = \frac{\frac{\partial v}{\partial t}}{\partial u/\partial t} \qquad (4)$$

The functions u(r,t), v(r,t) must have variables separable if they are to represent a stationary solution with respect to t, otherwise curves r = constant would have varying forms for varying t. Let

$$u = p(r)\,f(t) \;, \quad v = q(r)\,g(t)$$

then (4) $\Rightarrow$

$$\frac{p'f}{q'g} = \frac{qg'}{pf'} \qquad p' = \frac{\partial p}{\partial r}\,,\; f' = \frac{\partial f}{\partial t}$$

thus $\frac{p'}{q'} = \frac{q}{p}$ + $\frac{f}{g} = \frac{g'}{f'}$

Hence $p^2 = q^2 + A$, $f^2 = g^2 + B$ where A, B are arbitrary constants. When $A \neq 0$ we may take $f^2 = g^2 + 1$ without loss of generality. In this case $f' = gg'/f$ and equation (2) gives,

$$(f')^2 \left(p^2 - q^2 \frac{f^2}{g^2}\right) = \left(\frac{r}{r-1}\right)^{-1}$$

which has no solution unless $p^2 = q^2$ and $f'^2\left(1 - \frac{f^2}{g^2}\right) = \text{const.}$ i.e. $A = 0$.

With $A = 0$, (1) and (2) become,

$$(p')^2 \left(f^2 - g^2\right) = \frac{1}{r-1} = l(r) \qquad (5)$$

$$p^2 \left(f'^2 - g'^2\right) = \frac{r-1}{r} = -m(r) \qquad (6)$$

These equations only have a solution when the functions $l(r)$, $m(r)$ are related by $\frac{d}{dr}\left[(m(r))^{\frac{1}{2}}\right] = l(r)^{\frac{1}{2}}$ to within a multiplicative constant. (Hence the Schwarzschild Interior Solution has an embedding.) In such a case, the embedding is given by $p(r) = (-m(r))^{\frac{1}{2}}$ and $f^2 - g^2 = 1$ i.e. $f = \cosh k(t)$, $g = \sinh k(t)$

The condition $f'^2 - g'^2 =$ constant from equation (6) gives $k(t) = kt + h$.

Thus, we may deduce that the Schwarzschild Exterior Solution is not embeddable in an E_5 whatever the form of the embedding equations, since in this case the functions l(r), m(r) are not related as above.

THE EMBEDDING OF THE SCHWARTZSCHILD INTERIOR SOLUTION[21] IN E_5

Let the solution be represented by an R_4 defined by equations,

$$u = u(r,t), \quad v = v(r,t), \ r, \theta, \phi$$

in E_5 with metric

$$-ds^2 = dr^2 + r^2 d\Omega^2 + \epsilon_1 du^2 + \epsilon_2 dv^2$$

Then the isometry equations are,

$$\left.\begin{aligned} 1 + \epsilon_1 \left(\frac{\partial u}{\partial r}\right)^2 + \epsilon_2 \left(\frac{\partial v}{\partial r}\right)^2 &= \left(1 - r^2/R^2\right)^{-1} && > 0 \\ \epsilon_1 \left(\frac{\partial u}{\partial t}\right)^2 + \epsilon_2 \left(\frac{\partial v}{\partial t}\right)^2 &= -\frac{1}{4}\left[3A - \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}}\right]^2 && < 0 \end{aligned}\right\} r < R$$

$$\epsilon_1 \frac{\partial u}{\partial r} \cdot \frac{\partial u}{\partial t} + \epsilon_2 \frac{\partial v}{\partial r} \cdot \frac{\partial v}{\partial t} = 0$$

with $\epsilon_1 = 1, \ \epsilon_2 = -1$,

$$\left(\frac{\partial u}{\partial r}\right)^2 - \left(\frac{\partial v}{\partial r}\right)^2 = \frac{r^2/R^2}{1 - r^2/R^2} \tag{7}$$

$$\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial v}{\partial t}\right)^2 = -\frac{1}{4}\left[3A - \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}}\right]^2 \tag{8}$$

$$\frac{\partial u}{\partial r} \cdot \frac{\partial u}{\partial t} - \frac{\partial v}{\partial r} \cdot \frac{\partial v}{\partial t} = 0 \tag{9}$$

let $u = p(r) \cosh kt, \quad V = p(r) \sinh kt$

then by (7)

$$\left(\frac{\partial p}{\partial r}\right)^2 = \frac{r^2/R^2}{1 - r^2/R^2}$$

i.e. $$p(r) = \left[C - R\left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}}\right]$$

Equation (8) implies $$p^2(r)\, k^2 = \frac{1}{4}\left[3A - \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}}\right]^2$$

i.e. $k^2 = \frac{1}{4R^2}, \quad C = 3AR$

Equations (9) are satisfied automatically.

So that an embedding of the Schwarzschild Interior Solution is given by,

$$Z_1 = r \sin\Theta \sin\phi, \quad Z_2 = r\sin\Theta\cos\phi, \quad Z_3 = r\cos\Theta$$

$$Z_4 = R\left[3A - \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}}\right] \cosh\frac{t}{2R}$$

$$Z_5 = R\left[3A - \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}}\right] \sinh\frac{t}{2R}$$

in $$-ds^2 = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 - dz_5^2$$

The previously known embedding of the Schwarzschild Interior Solution was given by Rosen[4] and was in E_6. Thus, Rosen's embedding was not minimal as the one given above must be.

We note that the signature of the E_5 metric is the only one possible from the form of the isometry equations, and that the constant k is determined and not arbitrary.

Hence, although it first appears that the

Schwarzschild Interior Solution is not embeddable in E_5,, this is not so, the spurious result being due to the fact that we cannot identify the time coordinate of the solution with a flat space coordinate.

THE EMBEDDING OF THE SCHWARZSCHILD EXTERIOR SOLUTION IN E_6

The metric form in E_6 is taken as,

$$-ds^2 = dr^2 + r^2\, d\Omega^2 + \epsilon_4\, du^2 + \epsilon_5\, dv^2 + \epsilon_6\, dw^2$$

where r, θ, ϕ are spherical polar coordinates and an R_4 is defined by

$$u(r,\theta,\phi,t)\,,\quad v(r,\theta,\phi,t)\,,\quad r,\theta,\phi\,,\quad w(r,\theta,\phi,t)$$

in E_6, where t is as yet an unspecified parameter on the surface.

In this case,

$$a_{\alpha\beta} = \begin{pmatrix} 1+\epsilon_4\left(\frac{\partial u}{\partial r}\right)^2+\epsilon_5\left(\frac{\partial v}{\partial r}\right)^2+\epsilon_6\left(\frac{\partial w}{\partial r}\right)^2 & \cdot\ \cdot \\ \epsilon_4\frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial \theta}+\epsilon_5\frac{\partial v}{\partial r}\frac{\partial v}{\partial \theta}+\epsilon_6\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta} & \text{etc} \\ \cdot\ \cdot & \cdot\ \cdot \end{pmatrix}$$

Hence, on identifying $a_{\alpha\beta}$ with $g_{\alpha\beta}$ of Schwarzschild Solution, and t with the Schwarzschild t, we have

$$\left.\begin{aligned}1+\epsilon_4\left(\frac{\partial u}{\partial r}\right)^2+\epsilon_5\left(\frac{\partial v}{\partial r}\right)^2+\epsilon_6\left(\frac{\partial w}{\partial r}\right)^2 &= \left(1-\frac{1}{r}\right)^{-1}\\ \epsilon_4\left(\frac{\partial u}{\partial \theta}\right)^2+\epsilon_5\left(\frac{\partial v}{\partial \theta}\right)^2+\epsilon_6\left(\frac{\partial w}{\partial \theta}\right)^2 &= 0\\ \epsilon_4\left(\frac{\partial u}{\partial \phi}\right)^2+\epsilon_5\left(\frac{\partial v}{\partial \phi}\right)^2+\epsilon_6\left(\frac{\partial w}{\partial \phi}\right)^2 &= 0\\ \epsilon_4\left(\frac{\partial u}{\partial t}\right)^2+\epsilon_5\left(\frac{\partial v}{\partial t}\right)^2+\epsilon_6\left(\frac{\partial w}{\partial t}\right)^2 &= -\left(1-\frac{1}{r}\right)\end{aligned}\right\} \quad \underline{\alpha}$$

and

$$\left.\begin{aligned}\epsilon_4\frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial \theta}+\epsilon_5\frac{\partial v}{\partial r}\cdot\frac{\partial v}{\partial \theta}+\epsilon_6\frac{\partial w}{\partial r}\cdot\frac{\partial w}{\partial \theta} = 0\\ \text{etc.}\end{aligned}\right\} \quad \underline{\beta}$$

With the assumption,

$$\frac{\partial u}{\partial \theta}=\frac{\partial v}{\partial \theta}=\frac{\partial w}{\partial \theta}=\frac{\partial u}{\partial \phi}=\frac{\partial v}{\partial \phi}=\frac{\partial w}{\partial \phi}=0$$

equations α and β reduce to the three equations :-

$$\epsilon_4\left(\frac{\partial u}{\partial r}\right)^2+\epsilon_5\left(\frac{\partial v}{\partial r}\right)^2+\epsilon_6\left(\frac{\partial w}{\partial r}\right)^2 = \frac{1}{r-1} \qquad (10)$$

$$\epsilon_4\left(\frac{\partial u}{\partial t}\right)^2+\epsilon_5\left(\frac{\partial v}{\partial t}\right)^2+\epsilon_6\left(\frac{\partial w}{\partial t}\right)^2 = \frac{1-r}{r} \qquad (11)$$

$$\epsilon_4\frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial t}+\epsilon_5\frac{\partial v}{\partial r}\cdot\frac{\partial v}{\partial t}+\epsilon_6\frac{\partial w}{\partial r}\cdot\frac{\partial w}{\partial t} = 0 \qquad (12)$$

We have to solve equations (10), (11), (12) simultaneously to obtain an embedding. The general solution of such equations may be difficult to obtain.

However, a set of solutions may be obtained by assuming $\frac{\partial \omega}{\partial t} = 0$. This gives the simplified set of equations consisting of equation (10) with,

$$\epsilon_4 \left(\frac{\partial u}{\partial t}\right)^2 + \epsilon_5 \left(\frac{\partial v}{\partial t}\right)^2 = \frac{1-r}{r} \qquad (13)$$

$$\epsilon_4 \frac{\partial u}{\partial r}\frac{\partial u}{\partial t} + \epsilon_5 \frac{\partial v}{\partial r}\frac{\partial v}{\partial t} = 0 \qquad (14)$$

These three equations determine the possible signatures for the embedding space.

Firstly for r > 1, equation (10) demands that at least one of the ϵs is +1, and equation (13) demands that at least one of ϵ_4 , ϵ_5 is -1 so that we have 4 cases,

$\underset{\sim}{a}$ $\quad \epsilon_6 = +1 \qquad \epsilon_4 = \epsilon_5 = -1$

$\qquad\qquad\qquad$ or $\epsilon_4 = +1, \epsilon_5 = -1$

$\underset{\sim}{b}$ $\quad \epsilon_6 = -1 \qquad \epsilon_4 = \epsilon_5 = +1$

$\qquad\qquad\qquad$ or $\epsilon_4 = +1, \epsilon_5 = -1$

Secondly for r < 1, equation (10) demands at least one of ϵs is -1, and equation (13) demands that at least one of ϵ_4 , ϵ_5 is +1. Thus again we have 4 cases,

$\underset{\sim}{a}$ $\quad \epsilon_6 = +1 \qquad \epsilon_4 = \epsilon_5 = -1$

$\qquad\qquad\qquad$ or $\epsilon_4 = +1, \epsilon_5 = -1$

$\underset{\sim}{b}$ $\quad \epsilon_6 = -1 \qquad \epsilon_4 = \epsilon_5 = +1$

$\qquad\qquad\qquad$ or $\epsilon_4 = +1, \epsilon_5 = -1$

These are all distinct since ϵ_6 is distinguished from ϵ_4, ϵ_5 by $\frac{\partial \omega}{\partial t} = 0$.

Hence the signature of the final embedding space must be either $++++--$ or $+++++-$, it cannot be $+++---$. Since the original metric has signature $+++-$, there are no other possibilites, for the embedding space must have at least three positive and one negative coefficient.

We see that for an embedding to exist in the region $r>1$ and $r<1$ we must have

$\epsilon_6 = \pm 1,\ \epsilon_4 = 1,\ \epsilon_5 = -1$ (cf Fronsdal's embedding).

Consider the cases when (i) $\epsilon_4 = -\epsilon_5 = 1$

(ii) $\epsilon_4 = \epsilon_5 = -1$

(i) Here we have,

$$\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial v}{\partial t}\right)^2 = \frac{1-r}{r}$$

To demonstrate that this has a solution with the corresponding equations (10) and (14), let

$$\frac{\partial u}{\partial t} = p(r)\,\mathrm{Sinh}\,kt\,, \quad \frac{\partial v}{\partial t} = p(r)\,\mathrm{Cosh}\,kt$$

then

$$p(r) = \left(\frac{r-1}{r}\right)^{\frac{1}{2}}, \quad u = \frac{1}{k}\,p(r)\,\mathrm{Cosh}\,kt, \quad v = \frac{1}{k}\,p(r)\,\mathrm{Sinh}\,kt$$

putting u and v in (10) gives,

$$\frac{1}{k^2}\left(\frac{\partial p}{\partial r}\right)^2 + \epsilon_6\left(\frac{\partial w}{\partial r}\right)^2 = \frac{1}{r-1}$$

With $\epsilon_6 = +1$,

$$\frac{\partial w}{\partial r} = \left\{\frac{1 - \frac{1}{4k^2r^3}}{r-1}\right\}^{\frac{1}{2}}$$

i.e.

$$w(r) = \int \left(\frac{4k^2r^3-1}{r-1}\right)^{\frac{1}{2}} \left(4k^2r^3\right)^{-\frac{1}{2}}\,dr$$

With $\epsilon_6 = -1$

$$\omega(r) = \int \left(\frac{1-4k^2r^3}{r-1}\right)^{\frac{1}{2}} (4k^2r^3)^{-\frac{1}{2}} dr$$

Hence we have a set of embeddings given by,

$$u = \frac{1}{k}\left(\frac{r-1}{r}\right)^{\frac{1}{2}} \cosh kt$$
$$v = \frac{1}{k}\left(\frac{r-1}{r}\right)^{\frac{1}{2}} \sinh kt$$
$$\omega = \int \left\{\pm\left(\frac{4k^2r^3-1}{r-1}\right)\right\}^{\frac{1}{2}} (4k^2r^3)^{-\frac{1}{2}} dr$$

With $k = \frac{1}{2}$, $\omega = \int \left(\pm \frac{r^2+r+1}{r^3}\right)^{\frac{1}{2}} dr$

and corresponding u and v. This is Fronsdal's solution in the case $\epsilon_6 = +1$,

$$-ds^2 = dr^2 + r^2 d\Omega^2 + du^2 - dv^2 + dw^2$$

(ii)

$\epsilon_4 = \epsilon_5 = -1$

In this case, we let $\frac{\partial u}{\partial t} = p(r)\sin kt$, $\frac{\partial v}{\partial t} = p(r)\cos kt$ and by similar reasoning, this gives $p(r) = \left(\frac{r-1}{r}\right)^{\frac{1}{2}}$, and two cases, $\epsilon_6 = \pm 1$, arise for w(r).

$$\epsilon_6 = +1 \Rightarrow \omega(r) = \int \left\{\frac{1+4k^2r^3}{r-1}\right\}^{\frac{1}{2}} (4k^2r^3)^{-\frac{1}{2}} dr$$

Putting k = 1 gives Kasner's embedding in the units in which the coordinate singularity is at r=1.

The equations (10), (11) and (12) may have many other solutions, but the subset of all possible solutions obtained by putting $\frac{\partial\omega}{\partial t} = 0$ can already be seen to include the previously known solutions. The assumption that the variables are separable follows naturally from the static nature of the original solution with respect to t.

THE EMBEDDING OF THE HOMOGENEOUS COSMOLOGICAL MODEL[24] IN E_5

The metric of this model is[24]

$$-ds^2 = dt^2 - f^2(r,t)\left(dr^2 + r^2 d\Omega^2\right)$$

where $f^2(r,t) = e^{g(t)}\left(1+\frac{r^2}{4R^2}\right)^{-2}$.

The isometry equations are in this case, (where the R_4 is represented by equations , $z^\alpha = z^\alpha(r,\theta,\phi,t)$, $\alpha = 1, \cdots 5$ in an E_5 with $-ds^2 = \sum_\alpha \epsilon_\alpha (dz^\alpha)^2$;-

$$\sum_\alpha \epsilon_\alpha (z^\alpha_r)^2 = -f^2(r,t) \quad , \quad z^\alpha_r = \frac{\partial z^\alpha}{\partial r} \quad \text{etc} \tag{15}$$

$$\sum_\alpha \epsilon_\alpha (z^\alpha_\theta)^2 = -r^2 f^2(r,t) \tag{16}$$

$$\sum_\alpha \epsilon_\alpha (z^\alpha_\phi)^2 = -r^2 \sin^2\theta\, f^2(r,t) \tag{17}$$

$$\sum_\alpha \epsilon_\alpha (z^\alpha_t)^2 = 1 \tag{18}$$

and the set of equations $\sum_\alpha \epsilon_\alpha z^\alpha_r z^\alpha_\theta = 0$ with similar equations in $z^\alpha_r, z^\alpha_\theta, z^\alpha_\phi, z^\alpha_t$ taken in pairs.

Clearly, the general solutions are very difficult to obtain and we have to resort to some simplification. As a first step, taking a hint from the previossly derived embeddings, the $r^2 d\Omega^2$ term can be accounted for by putting:-

$$Z^1 = f(r,t)\ r \cos\theta$$
$$Z^2 = f(r,t)\ r \sin\theta \cos\phi$$
$$Z^3 = f(r,t)\ r \sin\theta \sin\phi$$

$\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$ and Z^4, Z^5 independent of θ and ϕ. The equations then reduce to:-

$$\epsilon_4 {Z^4_r}^2 + \epsilon_5 {Z^5_r}^2 = \left(\frac{\partial}{\partial r}(rf)\right)^2 - f^2 \qquad (19)$$

$$\epsilon_4 {Z^4_t}^2 + \epsilon_5 {Z^5_t}^2 = 1 + r^2\left(\frac{\partial f}{\partial t}\right)^2 \qquad (20)$$

$$\epsilon_4 Z^4_r Z^4_t + \epsilon_5 Z^5_r Z^5_t = \tfrac{1}{2}\dot{g} f^2 r \frac{(1-\frac{r^2}{4R^2})}{(1+\frac{r^2}{4R^2})} \qquad (21)$$

all the other equations being satisfied automatically.

But $\frac{\partial f}{\partial t} = \frac{1}{2}\dot{g} f$, $\frac{\partial}{\partial r}(rf) = f \frac{(1-\frac{r^2}{4R^2})}{(1+\frac{r^2}{4R^2})}$, $\left(\dot{g} = \frac{dg}{dt}\right)$,

so that equation (19) becomes $\epsilon_4 {Z^4_r}^2 + \epsilon^5 {Z^5_r}^2 = f^2 \frac{(-\frac{r^2}{R^2})}{(1+\frac{r^2}{4R^2})^2}$

which has an obvious solution $\epsilon_4 = -1$, $Z^5_r = 0$, $Z^4_r = \frac{rf}{R}\left(1+\frac{r^2}{4R^2}\right)^{-1}$

i.e.
$$Z^4 = \frac{2Re^{g/2}}{(1+\frac{r^2}{4R^2})} + A(t) \qquad (22)$$

putting this into equation (21):-

$$\frac{-rf}{R(1+\frac{r^2}{4R^2})}\left(\dot{g} R \frac{e^{g/2}}{1+\frac{r^2}{4R^2}} + \dot{A}(t)\right) = \tfrac{1}{2}\dot{g} f^2 r \frac{(1-\frac{r^2}{4R^2})}{(1+\frac{r^2}{4R^2})}$$

i.e.
$$2\left[1 + \frac{\dot{A}(t)e^{-g/2}}{\dot{g}R}\left(1+\frac{r^2}{4R^2}\right)\right] = 1 - \frac{r^2}{4R^2}$$

hence $\dot{A}(t) = R\dot{g}e^{g/2}\left(-\tfrac{1}{2}\right)$, $+$ $A(t) = -Re^{g/2}$

so that $Z^4 = Re^{g/2}\left(1-\frac{r^2}{4R^2}\right)\left(1+\frac{r^2}{4R^2}\right)^{-1} = Rf(r,t)\left(1-\frac{r^2}{4R^2}\right) \qquad (23)$

We still have to satisfy (20) with $Z^5_r = 0$. Using (23), (20) becomes:-

$$\epsilon_5 Z^{5\,2}_t = 1 + \frac{\dot{g}^2 f^2}{4}\left(1+\frac{r^2}{4R^2}\right)^2 R^2$$

$$= 1 + \tfrac{1}{4}\dot{g}^2 e^{g(t)} R^2.$$

Hence, with $\epsilon_5 = +1$, $Z^5 = \int (1 + \frac{1}{4}\dot{g}^2 e^g R^2)^{\frac{1}{2}} dt$, $Z^5_r = 0$.
Thus an embedding is defined by,

$$Z^1 = f(r,t)\, r \cos\theta$$
$$Z^2 = f(r,t)\, r \sin\theta \cos\phi$$
$$Z^3 = f(r,t)\, r \sin\theta \sin\phi$$
$$Z^4 = R\, f(r,t)\left(1 - \frac{r^2}{4R^2}\right)$$
$$Z^5 = \int \left(1 + \tfrac{1}{4}\dot{g}^2 e^g R^2\right)^{\frac{1}{2}} dt$$

in an E_5 with $-d\sigma^2 = -dz^{1\,2} - dz^{2\,2} - dz^{3\,2} - dz^{4\,2} + dz^{5\,2}$.
This embedding was originally given by Rosen[4] although no mention was made of how it was derived. In this case, the method above cannot be claimed to have any advantages over 'intuitive' methods since it requires close scrutiny of the equatinns to notice that $Z^5_r = 0$ is the most promising start to obtaining a soltion, and then it is not obvious that it will, in fact, give a complete solution.

AN EMBEDDING OF THE DE SITTER COSMOLOGICAL MODEL[23] IN E_5

The metric of the model[23] is taken in the form

$$-ds^2 = \left(1-\frac{r^2}{R^2}\right)dt^2 - \left(1-\frac{r^2}{R^2}\right)dr^2 - r^2 d\Omega^2$$

With the usual $z^1 = r\cos\theta$, $z^2 = r\sin\theta\cos\phi$, $z^3 = r\sin\theta\sin\phi$ and z^4, z^5 independent of θ and ϕ, the equations of the embedding take the form:

$$\epsilon_4 {z^4_r}^2 + \epsilon_5 {z^5_r}^2 = l(r) = \frac{-r^2/R^2}{1-\frac{r^2}{R^2}} \qquad (24)$$

$$\epsilon_4 {z^4_t}^2 + \epsilon_5 {z^5_t}^2 = m(r) = 1-\frac{r^2}{R^2} \qquad (25)$$

$$\epsilon_4 z^4_r z^4_t + \epsilon_5 z^4_r z^5_t = 0 \qquad (26)$$

in an embedding E_5 with metric,

$$-ds^2 = -dr^2 - r^2 d\Omega^2 + \epsilon_4 {dz^4}^2 + \epsilon_5 {dz^5}^2.$$

It will be recalled that, in the discussion of the Schwarzschild Exterior Solution, equations, (5) and (6) were used to derive a condition on l(r) and m(r) for a separation of variables solution to be possible.
The condition being that,

$$\frac{d}{dr}\left(m(r)\right)^{\frac{1}{2}} = l(r)^{\frac{1}{2}}$$

The right hand sides of (24) and (25) obey this condition to within a multiplicative constant, hence, a solution of the form,

$$z^4 = p(r)\sinh kt$$

$$z^5 = p(r)\cosh kt$$

is possible.

We require $\epsilon_4 = -\epsilon_5 = -1$ from (24) and (25), and that $p(r) = \frac{1}{k}\left(1-\frac{r^2}{R^2}\right)^{\frac{1}{2}}$ from (25), and $k = \frac{1}{R}$ from (24).

Thus, an embedding is given by; Z^1, Z^2, Z^3 as above,

$$Z^4 = (R^2 - r^2)^{\frac{1}{2}} \sinh \frac{t}{R}$$

$$Z^5 = (R^2 - r^2)^{\frac{1}{2}} \cosh \frac{t}{R}$$

in the E_5 with

$$-ds^2 = -dz^{1^2} - dz^{2^2} - dz^{3^2} - dz^{4^2} + dz^{5^2}$$

For completeness, an embedding of the Einstein model(22), $-ds^2 = dt^2 - \left(1-\frac{r^2}{R^2}\right)^{-1} dr^2 - r^2 d\Omega^2$ is given, although it is trivial. By inspection,

$$Z^1 = t,\ Z^2 = R\left(1-\frac{r^2}{R^2}\right)^{\frac{1}{2}},\ Z^3 = r\cos\theta,\ Z^4 = r\sin\theta\cos\phi,\ Z^5 = r\sin\theta\sin\phi$$

and embedding E_5 is $-ds^2 = dz^{1^2} - dz^{2^2} - dz^{3^2} - dz^{4^2} - dz^{5^2}$.

WHITTAKER'S QUASI-UNIFORM GRAVITATIONAL FIELD

$$-ds^2 = \left(1+\frac{2x_1}{a}\right)dx_4^2 - \left(1+\frac{2x_1}{a}\right)^{-1}dx_1^2 - dx_2^2 - dx_3^2$$

We attempt an embedding in E_4 with

$$-ds^2 = dz_1^2 - dz_2^2 - dz_3^2 - dz_4^2$$

Putting $z_2 = x_2,\ z_3 = x_3$ the isometry equations are:-

$$\left(\frac{\partial z_1}{\partial x_1}\right)^2 - \left(\frac{\partial z_4}{\partial x_1}\right)^2 = \left(1+\frac{2x_1}{a}\right)^{-1}$$

$$\left(\frac{\partial z_1}{\partial x_4}\right)^2 - \left(\frac{\partial z_4}{\partial x_4}\right)^2 = -\left(1+\frac{2x_1}{a}\right)$$

$$\frac{\partial z_1}{\partial x_1}\cdot\frac{\partial z_1}{\partial x_4} - \frac{\partial z_4}{\partial x_1}\,\frac{\partial z_4}{\partial x_4} = 0$$

The signature has to be as above since the original R_4 has signature (+ ---).

We note the similarity of these equations to (1), (2) and (3) in the attempted embedding of the Schwarzschild Exterior Solution in E_5. In that case we derived the condition $\frac{d}{dr}(m(r))^{\frac{1}{2}} = \ell(r)^{\frac{1}{2}}$ for a solution by separation of variables to exist, when the solution is stationary with respect to time. The condition is satisfied here to within a multiplicative constant when we put $r = x_1$ and $t = x_4$. Hence, we try a solution of the form,

$$z_1^* = A\left(1+\frac{2x_1}{a}\right)^{\frac{1}{2}}\cosh kx_4$$

$$z_2^* = A\left(1+\frac{2x_1}{a}\right)^{\frac{1}{2}}\sinh kx_4$$

from equations (5) and (6).

The following conditions on A, k are necessary;-

$$A^2 k^2 = 1$$

and $A^2/a^2 = 1$ from the isometry equations,

i.e. $A^2 = a^2 = \frac{1}{k^2}$

Hence, a unique soltuion (to within±1) exists by separation of variables , and is

$$z_1 = a'\left(1 + \frac{2x_1}{a}\right)^{\frac{1}{2}} \cosh \frac{x_4}{a}$$

$$z_2 = x_2 \ , \quad z_3 = x_3$$

$$z_4 = a\left(1 + \frac{2x_1}{a}\right)^{\frac{1}{2}} \sinh \frac{x_4}{a}$$

Thus, the R_4 is embeddable in the Minkowski E_4! This can only be true if the R_4 actually is the Minkowski E_4, but with an unusual coordinate system defined on it. Of course, this is a well known result, as are the functions z_1, z_2, z_3, z_4 , expressing the coordinate transformations, but it does illustrate the use of the embedding method in this case.

AN EMBEDDING OF A SPHERICALLY SYMMETRIC R_4 IN AN E_6

Using the line element in the form (9)

$$-ds^2 = f(r)\,dt^2 - g(r)\,dr^2 - r^2\,d\Omega^2$$

and with the first three flat space coordinates as usual when metric with $r^2 d\Omega^2$ term is present, and u, v, ω independent of θ, ϕ in the E_6, the embedding equations are:-

$$\epsilon_4\left(\frac{\partial u}{\partial r}\right)^2 + \epsilon_5\left(\frac{\partial v}{\partial r}\right)^2 + \epsilon_6\left(\frac{\partial \omega}{\partial r}\right)^2 = 1 - g(r) \qquad (27)$$

$$\epsilon_4\left(\frac{\partial u}{\partial t}\right)^2 + \epsilon_5\left(\frac{\partial v}{\partial t}\right)^2 + \epsilon_6\left(\frac{\partial \omega}{\partial t}\right)^2 = f(r) \qquad (28)$$

$$\epsilon_4\frac{\partial u}{\partial r}\frac{\partial u}{\partial t} + \epsilon_5\frac{\partial v}{\partial r}\cdot\frac{\partial v}{\partial t} + \epsilon_6\frac{\partial \omega}{\partial r}\cdot\frac{\partial \omega}{\partial t} = 0 \qquad (29)$$

As in the case of the Schwarzschild Exterior Solution, let $\frac{\partial \omega}{\partial t} = 0$, and consider the case $\epsilon_4 = -\epsilon_5 = -1$, to allow for different signs of $1 - g(r)$ and $f(r)$, if they occur. Solving (28) by separation of variables,

$$u = p(r)\sinh kt \qquad (30)$$

$$v = p(r)\cosh kt \qquad (31)$$

gives $\quad p(r) = \frac{1}{k} f(r) \qquad (32)$

Hence, (27) becomes

$$p'(r)^2 + \epsilon_6\left(\frac{\partial \omega}{\partial r}\right)^2 = 1 - g(r)$$

i.e. $$\omega(r) = \int\left[\epsilon_6\left(1 - g(r) - \frac{1}{k^2} f'(r)^2\right)\right]^{\frac{1}{2}} dr, \qquad (33)$$

ϵ_6 being chosen to make the integrand real.

This embedding defined by (30), (31), (32), (33) and the spherical polar coordinates, includes the Schwarzschild Exterior Solution (with or without cosmological term) and the Reissner-Weyl, charged particle solution, and is essentially a treatment of the embedding of a spherically symmetric space-time in E_6 as was first performed by J. Plebanski[9].

CONCLUSION

It is hoped that the usefulness of the partial differential equations approach to the problem has been demonstrated in the case of the simpler relativistic metrics. Even though assumptions have to be made in order to solve the equations, it is clear what these assumptions are, and an attempt has been made to justify them. It is not claimed that the method is a great step forward, indeed, the whole use of embedding of R_4s in E_ns may turn out to be a red-herring, but the structure of the possible solutions, especially with regard to the signature of the embedding space, and the role of the r, θ, ϕ, t coordinates, in the R_4, is emphasized. Certain minor results may be proved by this method simply because it is a systematic approach to the problem.

CHAPTER 5

THE RELATION OF THE EMBEDDING PROBLEM TO ELEMENTARY PARTICLE PHYSICS

THE RELATION OF THE EMBEDDING PROBLEM TO ELEMENTARY PARTICLE PHYSICS

In this chapter we propose to discuss in a general way recent suggestions that the extra dimensions introduced in the embedding may lead to symmetry transformations in the Euclidean embedding space, which may give rise to invariance principles corresponding to the so called internal symmetries of elementary particles. This suggestion has in fact been almost wholly responsible for the revival of interest in the embedding problem as applied to 'physical' space times.

A detailed treatment of existing theory in this field is beyond the scope of this dissertation and only a brief outline of motivating ideas is given here. A list of references is given which it is hoped will prove useful for further investigation.

It may be argued that General Relativity can play no part in the theory of elementary particles since the gravitational interactions are so weak compared with other interactions, and from the point of view of the dynamics of the interactions this is almost certainly true.

However, when we discuss the symmetries of elementary particles a different situation may arise.

A symmetry (or invariance) of the world exists whenever the description of the laws of physics is unaffected by a change in the frame of reference. E.g. the position of the origin of a space coordinate system is quite arbitrary, and changing it makes no difference in the description of the motion of bodies, because forces between them depend only on relative positions and not on any absolute position. This symmetry of space to translations implies the law of conservation of linear momentum.

Particle symmetries such as conservation of linear and angular momentum and of energy can be derived from the properties of the Lorentz Group of allowable symmetry transformations in Minkowski space. Conservation of linear and angular momentum from translations and rotations in space, and conservation of energy from translations in time.

But of course the space time in which we make observations is not actually flat and the Minkowski space time is only a local approximation to the laboratory space-time and the Lorentz Group is applicable only in that it is some kind of approximation to the group of allowable symmetry preserving transformations.

The use of the Lorentz Group in deriving external symmetries is well understood and is usually based upon the fact that it arises as a solution of Killing's equations

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$$

for symmetry transformations, in flat space-time. Killing's equations give us a lead to the finding of exact symmetry preserving transformations in any Riemannian space, and at this point it is instructive to show their derivation and in particular the derivation of the Lorentz Group for flat space-time.

We formulate analytically the idea of a transformation of the points of an R_4 into themselves by introducing a set of coordinates $\{x^{\mu}\}$ on the R_4 at a point P, and a set $\{x'^{\mu}\}$ at a point P', and express the association of point P' with point P under a transformation of the R_4 into itself by the set of functions,

$$x'^{\mu} = x'^{\mu}\left(\{x^{\nu}\}\right)$$

(The coordinatization of the R_4 is assumed to be at least patchwise continuous, so that we can express the entire transformation in terms of such a set of functions.)

We shall consider only infinitesimal transformations, as any finite transformation can be built up from them, and we represent the transformation by a vector field $\xi^{\mu}(x)$ on the R_4 which takes the point P $\{x^{\mu}\}$ into the point P' $\{x^{\mu} + \xi^{\mu}(x)\}$.

Now we require that the vector field $\xi^{\mu}(x)$ leave the metric invariant, in transforming the point P to P'. The metric is taken to be a tensor,

$$g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \cdot \frac{\partial x^{\sigma}}{\partial x'^{\nu}}$$

and the transformation is,

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

the inverse transformation is,

$$x^{\mu} = x'^{\mu} - \xi^{\mu}(x')$$

since the ξs are infinitesimal here, and in the following.

Thus

$$g'_{\mu\nu} = g_{\rho\sigma} \{ \delta^{\rho}_{\mu} - \xi^{\rho}_{,\mu} \} \{ \delta^{\sigma}_{\nu} - \xi^{\sigma}_{,\nu} \}$$

$$\simeq g_{\mu\nu} - g_{\mu\sigma} \xi^{\sigma}_{,\nu} - g_{\rho\nu} \xi^{\rho}_{,\mu} \qquad (1)$$

(evaluated at P)

However

$$g'_{\mu\nu} = g_{\mu\nu}(x^{\mu} + dx^{\mu})$$

$$\simeq g_{\mu\nu}(x^{\mu}) + dx^{\alpha} \frac{\partial g_{\mu\nu}(x^{\mu})}{\partial x^{\alpha}}$$

(using a Taylor expansion).

$$= g_{\mu\nu}(x^{\mu}) + \xi^{\alpha} \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \qquad (2)$$

Hence a necessary and sufficient condition that the metric be left invariant by the transformation is that (1) = (2). The difference $\delta g_{\mu\nu}$ between (1) and (2) is,

$$\xi^{\alpha} \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} + g_{\mu\sigma} \xi^{\sigma}{}_{,\nu} + g_{\rho\nu} \xi^{\rho}{}_{,\mu}$$

and we require this to be zero. This condition constitutes Killing's equations, it is usually written in the form,

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$$

using the definition of covariant derivative.

If the R_4 is the Minkowski E_4 with cartesian coordinates,

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$$

and Killing's equations become,

$$g_{\mu\sigma} \xi^{\sigma}{}_{,\nu} + g_{\rho\nu} \xi^{\rho}{}_{,\mu} = 0$$

$$\text{i.e. } \xi_{\mu,\nu} + \xi_{\nu,\mu} = 0$$

hence for $\mu = \nu = \alpha$,

$$\xi_{\alpha,\alpha} = 0 \Rightarrow \xi_{\alpha} = a_{\alpha} + f_{\alpha}(x^{\beta})$$

a_{α} = constant, $\beta \neq \alpha$.

$$\mu \neq \nu \Rightarrow \xi_{\mu,\nu} = - \xi_{\nu,\mu}$$

but $\xi_{\mu} = a_{\mu} + f_{\mu}(x^{k})$, $k \neq \mu$, $\rightarrow \frac{\partial \xi_{\mu}}{\partial x^{k}} = \frac{\partial f_{\mu}}{\partial x^{k}}$

$\xi_{\nu} = a_{\nu} + f_{\nu}(x^{l})$, $l \neq \nu$, $\rightarrow \frac{\partial \xi_{\nu}}{\partial x^{l}} = \frac{\partial f_{\nu}}{\partial x^{l}}$

hence since $\frac{\partial t_\mu}{\partial x^k}$ is independent of x^μ , and $\frac{\partial t_\nu}{\partial x^l}$ is independent of x^ν for all μ, ν; $\frac{\partial t_\mu}{\partial x^k}$ is independent of x^ν, and $\frac{\partial t_\nu}{\partial x^l}$ is independent of x^μ for all μ, ν.

Therefore we have $\frac{\partial t_\mu}{\partial x^k} = b_{\mu k}$

where $b_{\mu k}$ is on antisymetric constant matrix.

So
$$\frac{\partial \xi_\mu}{\partial x^k} = b_{\mu k}$$

And
$$\xi_\mu = b_{\mu k} x^k + a_\mu$$

i.e. the admitted transformations preserving the Minkowski E_4 are the ten parameter (6 from $b_{\mu k}$,4 from a_μ) group of transformations,which we know as the full Lorentz Group.

It is clear that in a general R_4 the solution to Killings equations may not even exist, since the gravitational field destroys the symmetry of the space-time, and the use of the Lorentz Group is no longer justified except as an approximation.

The Lorentz Group seems to be a good approximation in that external symmetries are well established, linear and angular momentum are conserved to highest degrees of observational approximation, but what of the so called "internal" symmetries? Here we may have a case in which the Lorentz Group is not even an allowable approximation.

(Internal symmetries are for example conservation of charge, strangeness, and isotopic spin. The invariance principle corresponding to the conservation of isotopic spin is the invariance of the system under rotations in the "isotopic-spin space", rather than in ordinary space and as far as is known the isotopic spin space has no proved relation to any other space ordinary or not).

As D. Joseph[7] has shown, that the source of the difficulties lies in the use of general curvilinear coordinates when we give up the Lorentz Group, In the traditional approach using general curvilinear coordinates the group of coordinate transformations has an infinite number of parameters, whereas the Lorentz Group has only ten. Of course this difficulty would have arisen in the flat space Lorentz Group approximation had we used curvilinear coordinates there. But we choose to use the preferred set of pseudo-Euclidean coordinates in flat space time, which gave the Minkowski metric. In the same way a set of preferred coordinates exist for the curved space-time, where we define pseudo-Euclidean coordinates by

$$ds^2 = \sum_{i=1}^{10} \epsilon_i (dx_i)^2 , \quad \epsilon_i = \pm 1$$

(the fact that ten coordinates are required for a 4 dimensional curved space-time has been mentioned earlier).

The Lorentz Group preserving the Minkowski metric is now replaced by the group of pseudo-rotations and translations in E_{10} preserving

$$ds^2 = \sum_{i=1}^{10} \epsilon_i (dx_i)^2.$$

Although the space dimensions have been increased from 4 to 10 the group parameters have been decreased from to 55. Laboratory experiments are conducted in the embedded Riemannian 4-space, hence we would expect observed symmetries to be obtained only by transformations in E_{10} which leave R_4 invariant, of course for a general R_4 such transformations do not necessarily exist. However, we can approximate to the R_4 at a point by using the tangent E_4 at that point. There exist two types of transformations preserving this E_4, firstly transformations of E_4 into E_4 (analogous to rotations of a plane about a line perpendicular to it in E_3), and secondly transformations in E_{10} which leave E_4 fixed. The first set of transformations of points in E_4 into points in E_4 are analogous to Lorentz Transformations and correspond to the use of a Minkowski frame at the point as an approximation.

It has been suggested that in a 'corrugated' space-time the best approximation might be the associated

E_4 to which the corrugated R_4 best fits on a larger scale. Such considerations seem a little premature at this stage in the investigation of this problem and a full investigation of the tangent plane approximation should be the first step.

However, in both of these cases the pseudo-rotations in the approximated E_4 would yield 'external' conservation laws, and those in the normal flat space would generate additional local symmetries, which we hope would coincide with the so called internal symmetries.

In this connection it may be legitimate to say that the totality of possible particle symmetries in an R_4 may be obtained by considering a Minkowski E_4 embedded in an E_{10}. That is, regarding the Minkowski E_4 as the limiting case in which the R_4 becomes an E_4, far away from any sources of curvature, where we may expect a maximum number of particle symmetries, since the presence of any curvature sources must reduce the number of invariance transformations possible in the space. We take E_{10} for the embedding space as it has the maximum dimensions necessary for the embedding of an R_4, as an extreme case. Hence, we would say that the maximum number of exact external symmetries can be derived from the pseudo-rotations in the E_4, i.e. the Lorentz Group and that these are already well known. The maximum number of internal symmetries would then be derived from the number of transformations in E_{10} which preserve E_4.

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