FORCES AND ENERGY IN DIELECTRICS.

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A B S T R A C T.

The writer has attempted to give a critical survey of the theory of polarisable deformable media. The subject is treated mainly from the standpoint of classical macroscopic electrostatics. Emphasis has been laid on the principles underlying the subject.

The two main problems are:

(a) To find the body force and couple of electrical origin on polarisable deformable media.

(b) To find the polarisation, field, stress and strain at any point of the medium due to the action of external fields and mechanical surface tractions.

These problems have become subjects of controversy in recent years. The older formulation of the theory was in two parts: (i) Helmholtz's energy method for finding the "body force" of electrical origin; (ii) Voigt's formulae for the relation between stress, strain and polarisation in crystals (piezoelectricity). The recent accounts give a unified version of the theory. Helmholtz's method of deriving body force of electrical origin has been criticised by Smith-White, who proposes the body force formula $\underline{F} = (\underline{P} \cdot \nabla)\underline{E}$

In Section 1 the writer has formally stated the problems to be solved, and also, to clear ideas, has given the solution of an analogous problem in gravitation.

The writer agrees with Smith-White's work (Sections 5 - 14) in general, including his criticisms of the Helmholtz theory,

but disagrees with him over the question of "semi-conservative" systems (see Section 34).

Brown (Sections 15 - 22) finds that the body force depends upon the shape of the volume element considered. This leads to some unusual stress relations with which the writer disagrees (see Section 35).

A mathematically simplified account is given of Toupin's discussion of the stress-strain relations in polarisable elastic media (Sections 23 - 29); this includes a generalisation of Voigt's piezoelectric formulae.

Sections 33 - 36 give the writer's conclusions on the work of Helmholtz, Smith-White, Brown and Toupin.

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INTRODUCTION

A dielectric body placed in an electric field becomes polarised and, as a result, experiences forces causing, in general, a small deformation of the body. This behaviour has long been the subject of study: in crystal physics it reveals itself as the piezoelectric effect; in a more general context as the phenomenon of electrostriction.

The first attempt to obtain formulae for the electric body forces and couples and an associated energy-density was made by Helmholtz. The method is that given by almost all writers of texts on classical mathematical electrostatics: (see e.g. Abraham-Becker (1932), Jeans (1927), Stratton (1941), and Smythe (1950)) an energy-density is obtained and formulae for the body force and couple deduced by considering a virtual deformation of the dielectric.

The subject has aroused considerable controversy, of which a brief account is given in Section 2; in the main it lies within the field of classical macroscopic electrostatics. Earlier criticisms of the Helmholtz method were made by Larmor (1) and Livens (2). More detailed criticisms and alternative theories have been advanced in recent years, notably by Smith-White, Brown, and Toupin; these writers have all <u>postulated</u> formulae for the electric body force and couple, formulae differing, in general, from those obtained by the Helmholtz method.

It is of interest that writers on crystal physics (see e.g. Voigt (18)) have always postulated formulae for body forces and couples, their approach to this problem differing sharply from that of the writers of texts on classical electrostatics; in some respects, however, their mathematical treatment seems incomplete. In a much more elaborate analysis Toupin obtains Voigt's piezoelectric formulae as a special case of a more general result, a contribution forming an important link between the subjects of crystal physics and classical electrostatics.

The following account is divided in four chapters: the first (Section 1) gives a formal statement of the problems and, as a means of clearing ideas about body force and stress-strain relations, a discussion of an analogous problem in gravitation; the second (Sections 2 - 29) gives a statement of the various theories put forward; the third (Sections 30 and 31) deals with the points of disagreement between these theories; the fourth (Sections 33 - 36) gives the writer's attempt to evaluate the several theories and the criticisms made by contending writers.

CHAPTER I

STATEMENT OF PROBLEMS.

1. ANALOGOUS GRAVITATIONAL PROBLEM.

The basic problems to be resolved by any theory of the deformation of a dielectric body in an electrostatic field are stated formally below.

Problem I. Find the spatial displacement, elastic stress, electric field and polarisation at every point of an elastic dielectric body under the action of given mechanical surface tractions and given external fields.

Problem II. Find the spatial displacement, elastic stress, electric field and polarisation at every point of an elastic dielectric body when the displacements of points on the surface of the body are prescribed and the body is under the action of given external fields.

The subject has aroused some controversy concerning formulae for the body force and the stress-strain relation. The points at issue are often far from clear (indeed some writers leave considerable doubt as to whether they are seeking a formula for the stress-strain relation or a formula for the body force); accordingly, it will be as well to clear our ideas by considering a simpler problem analogous to Problem I in which the dielectric body in an electric field is replaced by a gravitating elastic body in a gravitational field. A deformable gravitating body (occupying a region v bounded by the surface S) is subjected to mechanical surface tractions $\underline{\Pi}$ and an external gravitational field $\underline{G}$. The resultant gravitational field is $\underline{G} = \underline{G}_0 + \underline{G}_s$ where $\underline{G}_s$ is the field produced by the gravitating body itself.* We take the initial state of the body to be that which is assumes when $\Pi = 0 = \underline{G}_0$ A typical particle P has coordinates (x,y,z) in the initial state; the effect of introducing the surface tractions and external gravitational field is to give this particle a displacement $\underline{u}$ (x,y,z), where we suppose $\underline{u}$ very small. Our problem is to find $\underline{u}$ and the elastic stress at P as functions of (x,y,z).

The solution is obtained in stages:

1. Body Force. We postulate that the body force per unit volume is

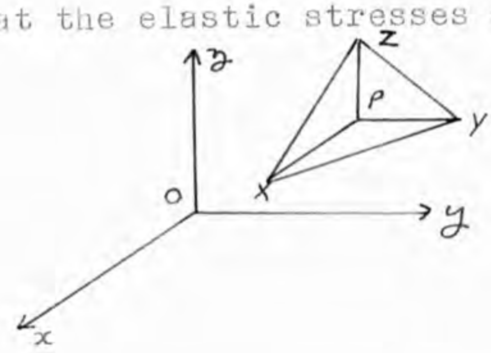
$$\underline{f} = \rho \underline{G}, \tag{1.1}$$

where $\underline{G}$ is the total gravitational field. This postulate is an induction from experience of "mass particles" in gravitational fields (a point emphasised by Kellog "Potential Theory" 1929).

2. Stress Tensor. At a point P in the body we consider an infinitesimal tetrahedron with vertex P and with three faces parallel to the coordinate planes. We assume

* It may be noted that the integral expression for $\underline{G}_s$ at a point inside the gravitating body converges to a unique value. Essentially this is because we are dealing with gravitational forces which vary as $1/r^2$ When we come to deal with dielectrics we find that the forces vary as $1/r^3$ and the corresponding Cauchy-Riemann volume integral is not strictly convergent; thus, for example, excising the usual pill-box and needle cavities about the field point P within the dielectric, different results are in general obtained for the field at P of the remaining dielectric matter.

that the elastic stresses exerted by the rest of the body are



given by $\underline{T}_x$, $\underline{T}_y$, $\underline{T}_z$, $\underline{T}_n$, acting on PYZ, PXZ, PXY, XYZ respectively. These stresses must balance the body force if the tetrahedron is in equilibrium.

We have then

$$\Delta PYZ\,\underline{T}_x + \Delta PXZ\,\underline{T}_y + \Delta PXY\,\underline{T}_n + V\underline{f} = 0\,, \quad \ldots(1.2)$$

where V is the volume of the tetrahedron. It follows immediately that if $\underline{n}$ the outward normal to XYZ has direction cosines (l_1, l_2, l_3) then

$$\underline{T}_n + l_1\underline{T}_x + l_2\underline{T}_y + l_3\underline{T}_z = 0$$

$$\text{or } \underline{T}_n = l_1(t_{11}\underline{i} + t_{21}\underline{j} + t_{31}\underline{k}) + l_2(t_{12}\underline{i} + t_{22}\underline{j} + t_{32}\underline{k}) + l_3(t_{13}\underline{i} + t_{23}\underline{j} + t_{33}\underline{k})\,. \quad \ldots(1.3)$$

*
The absence of $\underline{f}$ from this relation is a result of the limiting process by which it is derived from (1.2).

By considering the equilibrium of an elementary parallelopiped we arrive at the usual equilibrium equation

$$t_{ij,j} + f_i = 0\,, \quad \ldots(1.4)$$

* The outward normal to PZY is in opposite direction to Ox and using the normal convention of elasticity the force is $-t_{11}\underline{i} - t_{21}\underline{j} - t_{31}\underline{k}$

for internal points; at a point on the surface S we obtain

$$t_{ij} n_j = \Pi_i \qquad \ldots(1.5)$$

A consideration of moments shows that

$$t_{ij} = t_{ji} \qquad \ldots(1.6)$$

We note that in the problem of the dielectric body the stress tensor t_{ij} is not symmetric; further, the formula corresponding to (1.5) is rather more complicated. These points are elaborated later.

3. Stress-Strain Relation. To make our problem soluble we need a relation between the stress and the strain at a point.

If $\underline{u}$ is the displacement vector of each point, then a measure of the strain is given by the tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

We can proceed either by the Cauchy method or the Green method. According to the former, the stresses are linear functions of the strains, i.e.

$$t_{ij} = c_{ijkl}\, e_{kl} + c_{ij}(0)$$

where the elastic coefficients c_{ijkl} are independent. Green's energy method leads to equations connecting the elastic coefficients and hence in effect to a reduction in their number. In this method we assume that when the displacement $\underline{u}$ is increased by $\delta\underline{u}$ the work done by the external body forces and surface tractions is equal to the increase in the elastic energy and the self-gravitational energy.

For an arbitrary region τ of the body bounded by a surface Σ

$$\delta\left\{\frac{1}{2}\int_\tau \rho\,\phi_s\,dv + \int_\tau \rho W\,dv\right\} = \int_\Sigma \delta\underline{u}.\underline{\Pi}\,dS + \int_\tau \delta\underline{u}.\underline{f}_0\,dv,$$

where W is the elastic energy per unit mass. Now

$$\underline{f}_s = -\operatorname{grad}\phi_s \;, \quad \underline{f}_0 = -\operatorname{grad}\phi_0,$$

so that

$$\delta\left(\frac{1}{2}\int_\tau \rho\,\phi_s\,dv\right) = -\int_\tau \delta\underline{u}.\underline{f}_s\,dv\,,\ (\text{see } (10\text{-}3)).$$

thus

$$\delta\left(\int_\tau \rho W\,dv\right) = \int_\Sigma \delta\underline{u}.\underline{\Pi}\,dS + \int_\tau \delta\underline{u}.\underline{f}\,dv.$$

As mass is conserved in a virtual displacement $\delta(\rho dv)=0$, also by (1.4)

$$f_i = t_{ij,i},$$

so that using Green's Theorem we obtain

$$\int_\tau \rho\,\delta W\,dv = \int_\Sigma \delta\underline{u}.\underline{\Pi}\,dS - \int_\Sigma \delta\underline{u}.\underline{\Pi}\,dS$$

$$+ \int_\tau t_{ij}\,\delta(u_{i,j})\,dv = \int_\tau t_{ij}\,\delta(u_{i,j})\,dv.$$

Thus

$$\rho \, \delta W = t_{ij} \, \delta(u_{i,j}) .$$

Writing

$$e_{11} = e_1 , \; e_{22} = e_2 , \; e_{33} = e_3 , \; 2e_{23} = e_4 , \; 2e_{31} = e_5 , \; 2e_{12} = e_6 ,$$

$$t_{11} = t_1 , \; t_{22} = \tau_2 , \; t_{33} = \tau_3 , \; t_{23} = \tau_4 , \; t_{31} = \tau_5 , \; t_{12} = \tau_6 ,$$

this becomes

$$\rho \, \delta W = \tau_1 \delta e_1 + \tau_2 \delta e_2 + \tau_3 \delta e_3 + \tau_4 \delta e_4 + \tau_5 \delta e_5 + \tau_6 \delta e_6 , \quad \ldots (1.7)$$

whence $\rho \dfrac{\partial W}{\partial e_i} = \tau_i$.

For infinitesimal displacements

$$\rho = \rho_0 (1 - e_1 - e_2 - e_3) = \rho_0 (1 - \Delta) ,$$

where $\Delta = e_1 + e_2 + e_3$.

If W is a quadratic function in the strains

$$W = c_0' + 2c_i' e_i + c_{ij}' e_i e_j ,$$

without loss of generality we may take $c_{ij}' = c_{ji}'$.

Then by (1.7)

$$\tau_i = 2\rho_0 (1 - \Delta)(c_i' + c_{ij}' e_j)$$

$$= 2\rho_0 c_i' - 2\rho_0 c_i' (e_1 + e_2 + e_3) + 2\rho_0 c_{ij}' e_j .$$

As $c_{ij}' = c_{ji}'$ there is a reduction in the number of elastic coefficients. In a case in which the stress vanished with the strain we would have $c_i' = 0$ and the above equation would become

$$\tau_i = 2\rho_0 c_{ij}' e_j = c_{ij} e_j .$$

As $c'_{ij} = c'_{ji}$ this would imply that $c_{ij} = c_{ji}$ so that the number of elastic coefficients would be reduced from 36 to 21.

4. Conclusion. We have established the relations

$$\underline{f} = \rho \underline{G}, \text{———} \quad (A)$$
$$t_{ij,j} + f_i = 0, \text{———} \quad (B)$$
$$t_{ij} = c_{ij}(0) + c_{ijkl} e_{kl}, \text{———} \quad (C)$$

where the c_{ijkl} are not independent. If we assume that the body is initially homogeneous then the coefficients c_{ijkl} will be spatially constant.

From (C)

$$t_{ij,j} = (c_{ijkl} e_{kl})_{,j} + (c_{ij}(0))_{,j} \quad ,$$

so that (B) becomes

$$[c_{ij}(0) + c_{ijkl} e_{kl}]_{,j} + f_i = 0 \; . \text{———} \quad (D)$$

This represents a system of second-order differential equations in the u_i . These can be solved subject to given boundary conditions. The solution of (D) can then be substituted in (C). Thus, in theory, we have solved Problem 2.

A similar procedure leads to the solution of Problem 1. With a prospect of formulating a similar theory for the elastic dielectric, the following points, although obvious, should be noted:

(a) the body force $\underline{f}$ is postulated to be given by the explicit expression $\underline{f} = \rho \underline{G}$

(b) the deduction of the stress-strain relations from an energy principle is desirable because this method leads to a considerable reduction in the number of the elastic coefficients required.

(c) the equations $f_i = -[c_{ij}(0) + c_{ijkl}e_{kl}]_{,j}$, give the components f_i of the gravitational body force in terms of the strains and elastic coefficients. These elastic coefficients will vary according to the material involved. Our problem is completely solved when, knowing the f_i and the elastic coefficients, we are able to solve these equations to obtain the strains; thus, for the problem to be soluble, it is necessary to have the relation $\underline{f} = \rho \underline{G}$ giving $\underline{f}$ in terms of the gravitational field.

CHAPTER II

SURVEY OF THEORIES.

2. GENERAL HISTORICAL SURVEY.

The usual text-book theory of body force acting in dielectrics is that due to Helmholtz; in this theory the body force is derived from energy considerations (see e.g. Abraham and Becker (1932), Jeans (1927), Stratton (1941) and Smythe (1950)).

Criticisms of Helmholtz's theory have been made by Larmor (1) and Livens (2); however it is still supported by authoritative opinion: for example, in replying to Livens' criticisms, Stratton (3, p. 146) asserts "There appears to be little reason to doubt that the energy method of Korteweg and Helmholtz is fundamentally sound."

Further papers criticising Helmholtz's theory have appeared in recent years; these claim to show that the Helmholtz method is incorrect and put forward alternative theories. The purpose of this dissertation is to give a critical account of these papers, of which the principal are those listed below.

1949. Smith-White (4) Smith-White criticises the Helmholtz method of deriving forces of electrical origin in dielectrics. He proposes the Livens formulae $\underline{F} = (\underline{P} \cdot \nabla)\underline{E}$ and $\underline{G} = \underline{P} \wedge \underline{E}$ to replace the Helmholtz formulae.

1951 - 1952. Cade (5), (6) Cade in turn criticises

Smith-White's theory.

Smith-White (7), (8) replies to these criticisms.

1951. Brown (9) Brown rejects the Helmholtz method and develops his own theory based on the formulae $\underline{F} = (\underline{P} \cdot \nabla)\underline{E}$ and $\underline{G} = \underline{P} \wedge \underline{E}$. He derives general stress-strain relations.

1951. Smith-White (10) publishes a useful account of an elementary non-conservative electrical system. This contains the essentials of Smith-White's work in a particularly simple form.

1951. Smith-White (11) makes no mention of Brown (9). He claims to show that the Helmholtz method is wrong by applying it to a system of discrete charges and dipoles; he points out that for a dielectric body immersed in an incompressible dielectric fluid, the Helmholtz theory gives the same total force and couple as his own. He discusses the formulation of the first law of thermodynamics in a system involving dielectrics.

1952. Cade (12) gives a modified version of the Smith-White theory (he appears to be unaware of the work of Brown (9) at this time) and attempts a comparison of the Helmholtz and Smith-White theories with his own by reference to known experimental data on forces and couples on dielectric bodies in dielectric fluids. He does not include the hydrostatic pressure in his calculations and (in the present writer's opinion) condemns Smith-White wrongly as a result of this.*

<u>1953</u>. <u>Cade</u> (13) This consists of Cade's calculations (using the Helmholtz method) of the total force and couple acting on a dielectric body immersed in a dielectric fluid. He appears to abandon his modified version of Smith-White's theory in favour of that of Brown and repeats his earlier criticism of Smith-White's theory.

<u>1953</u>. <u>Brown</u> (14) gives a condensed version of his 1951 paper (9) and adds to it nothing of basic importance.

<u>1956</u>. <u>Toupin</u> (15) gives an account of the stress-strain relationships in an elastic dielectric. No mention is made of the Helmholtz formulae and the Livens formulae for body force are used without discussion.

* It is clear from Smith-White (11, p. 111) that the Helmholtz and Smith-White theories both give the same result for the resultant force and couple.

3. HELMHOLTZ : THEORY OF BODY FORCE IN FLUID DIELECTRICS.

The Helmholtz theory derives the body force of electrical origin from energy considerations. This is the method given in most textbooks.

As an example of this method we find the body force $\underline{F}^{(h)}$ acting on a fluid dielectric.

The system considered is a <u>fixed</u> distribution of charge in a volume V_1 acting on a dielectric fluid occupying a volume V_2 bounded by a surface S.

We suppose that for the fluid $\underline{D} = K\underline{E}$, where K may be a function of position and density ρ.

Let the liquid in V_2 undergo a small deformation specified by $\delta\underline{u}$, where $\delta\underline{u}$ varies from point to point in V_2. Let us also suppose for simplicity that the normal component of $\delta\underline{u}$ vanishes on the boundary of V_2. The electrical energy of the system is

$$U_{elec} = \frac{1}{2}\int_{V_1} e\phi\, dv .$$

It can be shown (see Appendix A-15) that the variation in U_{elec} following the deformation $\delta\underline{u}$ is

$$\delta U_{elec} = \frac{1}{2}\int_{V_1} e\,\delta\phi\, dv = -\frac{1}{8\pi}\int E^2\,\delta K\, dv .$$

The Helmholtz method rests on the assumption that the work done by the body forces equals the decrease in the energy of the system. We then have

$$\int_{V_2} \underline{F}^{(h)} \cdot \delta \underline{u} \, dv = -\delta \mathcal{U}_{elec} \,. \qquad \ldots (3.1)$$

If ΔK be the increment in K "following the deformation" we have

$$\Delta K = \delta K + \frac{\partial K}{\partial x_\alpha} \delta u_\alpha = \frac{\partial K}{\partial \rho} \Delta \rho \,.$$

From the analysis in (Appendix B-15)

$$\Delta \rho = -\rho \frac{\partial}{\partial x_\alpha} (\delta u_\alpha).$$

Accordingly

$$\delta K = -\frac{\partial K}{\partial x_\alpha} \delta u_\alpha = -\rho \frac{\partial K}{\partial \rho} \frac{\partial (\delta u_\alpha)}{\partial x_\alpha} \,,$$

whence

$$\delta \mathcal{U}_{elec} = \frac{1}{8\pi} \int_{V_2} E^2 \frac{\partial K}{\partial x_\alpha} (\delta u_\alpha) dv + \frac{1}{8\pi} \int_{V_2} E^2 \rho \frac{\partial K}{\partial \rho} \frac{\partial (\delta u_\alpha)}{\partial x_\alpha} dv \,.$$

As the normal component of $\delta \underline{u}$ vanishes on the boundary of V_2, it follows from Green's theorem that

$$\int_{V_2} \frac{\partial}{\partial x_\alpha} \left(E^2 \rho \frac{\partial K}{\partial \rho} \delta u_\alpha \right) dv = 0 \,,$$

so that

$$\int_{V_2} E^2 \rho \frac{\partial K}{\partial \rho} \frac{\partial (\delta u_\alpha)}{\partial x_\alpha} dv = -\int_{V_2} \left\{ \frac{\partial}{\partial x_\alpha} \left(E^2 \rho \frac{\partial K}{\partial \rho} \right) \right\} \delta u_\alpha dv \,.$$

Thus

$$\delta \mathcal{U}_{elec} = \frac{1}{8\pi} \int_{V_2} \left\{ E^2 \frac{\partial K}{\partial x_\alpha} - \frac{\partial}{\partial x_\alpha} \left(E^2 \rho \frac{\partial K}{\partial \rho} \right) \right\} \delta u_\alpha \, dv$$

$$= \frac{1}{8\pi} \int_{V_2} \left\{ E^2 \nabla K + \frac{1}{8\pi} \nabla \left(E^2 \rho \frac{\partial K}{\partial \rho} \right) \right\} \cdot \delta \underline{u} \, dv \,. \qquad \ldots (3.2)$$

Comparing (3.1) and (3.2) it follows that

$$\underline{F}^{(h)} = -\frac{1}{8\pi} E^2 \nabla K + \frac{1}{8\pi} \nabla \left(E^2 \rho \frac{\partial K}{\partial \rho} \right). \qquad \ldots (3.3)$$

Smith-White claims that the body force $\underline{F}$ in the general case is given by

$$\underline{F} = (\underline{P} \cdot \nabla)\underline{E} . \qquad \ldots (3.4)$$

Writing $K = 1 + 4\pi k$ (k being the electric susceptibility), we have $\underline{P} = k\underline{E}$ so that (3.4) becomes

$$\underline{F} = \frac{1}{2} k \nabla (E^2) ,$$

and (3.3) becomes

$$\underline{F}^{(h)} = -\frac{1}{2} E^2 \nabla k + \frac{1}{2} \nabla \left(E^2 \rho \frac{\partial k}{\partial \rho}\right) ,$$

thus

$$\underline{F}^{(h)} - \underline{F} = -\frac{1}{2} \nabla (k E^2) + \frac{1}{2} \nabla \left(E^2 \rho \frac{\partial k}{\partial \rho}\right)$$

$$= \frac{1}{2} \nabla \left(E^2 \rho \frac{\partial k}{\partial \rho} - k E^2\right) = \frac{1}{2} \nabla \left(E^2 \rho^2 \frac{\partial}{\partial \rho}\left(\frac{k}{\rho}\right)\right). \qquad \ldots (3.5)$$

whence it can be seen that, even in the simple case of the general fluid dielectric, the methods of Helmholtz and Smith-White lead to different results for the body force.

4. HELMHOLTZ: THEORY OF BODY FORCE IN SOLID DIELECTRICS.

Stratton (3, 141-146) and Pockels (16) follow the energy method of Helmholtz in obtaining the body force in solid dielectrics.

The configuration considered is that of §3 , the volume V_2 being occupied by a solid elastic dielectric body. It is supposed that in the unstrained state the dielectric body is homogeneous and isotropic as regards its electric and elastic properties. As the change in the dielectric properties depends on the strain components, the body will in general become electrically anisotropic following the deformation.

Stratton takes the electrical energy density u_{elec} in an anisotropic linear medium to be

$$\frac{1}{8\pi}\,\underline{E}\cdot\underline{D} = \frac{1}{8\pi}\,K_{ij}E_iE_j \;.$$

He calculates the variation δu_{elec} (corresponding to a further deformation $\delta\underline{u}$) in two stages: the first contribution $\delta U'$ is that arising from the variation in the parameters K_{ij} ; the second $\delta U''$ is that due simply to the material displacement $\delta\underline{u}$. It is shown (see Appendix A-16) that

$$\delta U' = \frac{1}{8\pi}\int_{V_2}(\delta K_{ij})E_iE_j\,dv\;. \qquad \ldots (4.1)$$

He assumes that the change in the K_{ij} due to strain must be linear functions of the variation in the strain components; thus, for small (secondary) deformations he writes

$$\delta K_{ij} = c_{ijkl}\, \delta e_{kl} \, ,$$

so that

$$c_{ijkl} = \frac{\partial K_{ij}}{\partial e_{kl}} \, .$$

As $K_{ij} = K_{ji}$ and $e_{kl} = e_{lk}$ it follows that

$$c_{ijkl} = c_{jikl} = c_{ijlk} \, . \qquad \ldots (4.2)$$

The dielectric has been assumed to be initially isotropic. This requires that the tensor c_{ijkl} be isotropic; the general isotropic tensor satisfying (4.2) (see Spain: Tensor Analysis p.98) is

$$c_{ijkl} = \alpha\, \delta_{ij}\delta_{kl} + \beta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) ,$$

where α, β are constants.

Putting $\alpha = -a_2/2$, $\beta = a_2 - a_1/4$ we obtain Stratton's formula

$$\delta U_{elec} = -\frac{1}{8\pi}\int_{V_2}\left\{a_2\delta_{ij}\delta_{kl} + \frac{a_1 - a_2}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right\}E^i E^j \delta e_{kl}\, dv \, . \qquad \ldots (4.3)$$

The additional contribution $\delta U''$ due simply to the material displacement $\delta \underline{u}$ is

$$\delta U'' = \frac{1}{8\pi}\int_{V_2} E^2 \nabla K . \delta\underline{u}\, dv \, ,$$

however as the body is initially homogeneous # $\nabla K = 0$ so that this contribution vanishes. A deformation of the dielectric also causes a variation in the elastic energy U_{elas} with the usual notation

$$U_{elas} = \frac{\lambda}{2}(e_{ii})^2 + \mu\left(e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2 + 2e_{23}^2 + 2e_{13}^2\right) ,$$

(See e.g. Sokolnikoff, Mathematical Theory of Elasticity, p. 85.)

Stratton does not assume this in his treatment, but it will make the argument simpler if we do.

then $\delta\mathcal{U}_{elas} = (\lambda_1 e_{ii} + 2\lambda_2 e_{11})\delta e_{11} + (\lambda_1 e_{ii} + 2\lambda_2 e_{22})\delta e_{22}$
$+ (\lambda_1 e_{ii} + 2\lambda_2 e_{33})\delta e_{33} + 4\lambda_2(e_{12}\delta e_{12} + e_{23}\delta e_{23} + e_{31}\delta e_{31})$.

(Note : Stratton defines $e_{ij}\ (i \neq j)$ as $\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$; in the following discussion the more usual notation $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is adopted.)

Stratton then equates the sum $\delta\mathcal{U}_{elec} + \delta\mathcal{U}_{elas}$ with the work done by the body force $\underline{f}$ and surface force $\underline{\Pi}$:

$$\int_{V_2} \underline{f}.\delta\underline{u}\,dv + \int_{S_2} \underline{\Pi}.\delta\underline{u}\,dS = -\delta\int_{V_2}(u_{elec} + u_{elas})\,dv. \qquad \text{... (4.5)}$$

Now, if t_{ij} denotes the stress tensor,

$$\int_{V_2} \underline{f}.\delta\underline{u}\,dv + \int_{S_2} \underline{\Pi}.\delta\underline{u}\,dS = -\int_{V_2} t_{ij,j}\,\delta u_i\,dv +$$
$$+ \int t_{ij}\,\delta u_i n_j\,dS = \int t_{ij}\,\delta(u_{i,j})dv = \int t_{ij}\,\delta e_{ij} dv. \qquad \text{... (4.6)}$$

The variations in the strain components are arbitary; hence, on equating coefficients of corresponding terms in (4.5) and (4.6) we have

$$
\begin{aligned}
t_{11} &= -(\lambda_1 e_{ii} + 2\lambda_2 e_{11}) + \tfrac{1}{2}(a_1 E_1^2 + a_2 E_2^2 + a_2 E_3^2),\\
t_{22} &= -(\lambda_1 e_{ii} + 2\lambda_2 e_{22}) + \tfrac{1}{2}(a_2 E_1^2 + a_1 E_2^2 + a_2 E_3^2),\\
t_{33} &= -(\lambda_1 e_{ii} + 2\lambda_2 e_{33}) + \tfrac{1}{2}(a_2 E_1^2 + a_2 E_2^2 + a_1 E_3^2),\\
t_{12} &= -2\lambda_2 e_{12} + \frac{a_1 - a_2}{2} E_1 E_2,\\
t_{23} &= -2\lambda_2 e_{23} + \frac{a_1 - a_2}{2} E_2 E_3,\\
t_{31} &= -2\lambda_2 e_{13} + \frac{a_1 - a_2}{2} E_1 E_3 . \qquad \text{... (4.7)}
\end{aligned}
$$

Stratton then states that, if the body force is of electrical origin only, the divergence of the elastic stresses vanishes; in this way he obtains the result,

$$f_1 = -\frac{1}{2}\frac{\partial}{\partial x_1}(a_1 E_1^2 + a_2 E_2^2 + a_2 E_3^2) - \frac{1}{2}\frac{\partial}{\partial x_2}\left[(a_1 - a_2)E_1 E_2\right]$$
$$- \frac{1}{2}\frac{\partial}{\partial x_3}\left[(a_1 - a_2)E_1 E_3\right], \qquad \text{... (4.8)}$$

with analogous expressions for f_2 and f_3 .

If $\operatorname{div} \underline{E} = 0$ and $\operatorname{curl} \underline{E} = 0$ then we can write (4.8) in the form

$$\underline{f} = -\frac{1}{4}(a_1 + a_2) \operatorname{grad} E^2 . \qquad \ldots (4.9)$$

Pockels (16) does not take the elastic energy into account and derives (4.9) from the equation

$$-\delta(u_{elec}) = \int_{V_2} \underline{f} . \delta\underline{u}\, dv + \int_{S_2} \underline{\Pi} . \delta\underline{u}\, dS ,$$

where δu_{elec} is the variation in the <u>electric</u> energy alone. Sratton (3, p.149) discusses the applications of this force formula to the problem of finding the spatial displacement at any point of the dielectric body. He uses the formula

$$\underline{f} + (\lambda_1 + \lambda_2) \nabla \nabla . \underline{u} + \lambda_2 \nabla^2 \underline{u} = 0 . \qquad \ldots (4.10)$$

(This is a standard formula of ordinary elasticity, for the case of an isotropic body.)

Stratton says: "The body forces are given by (4.9) To these must be added when the occasion demands a force $\rho\underline{E}$ to account for a volume charge and a gravitational force ρg ."

Previous to Smith-White it seems that the only published criticisms of the Helmholtz theory were by Larmor (1) and Livens (2). Larmor objected to the integration of the energy density by parts, a process in which an integral over a finite region is replaced by an integral over all space.

Livens proposed that alternative formula for the body force and attempted to show that the energy method was wrong by applying it to the case where the relation between $\underline{D}$ and $\underline{E}$ is non-linear.

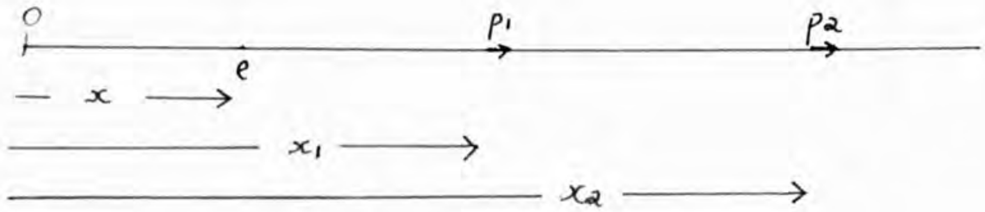
Jeans, Smythe, Abraham-Becker and Stratton have all ignored or dismissed these criticisms. Stratton (1941) states "There appears to be little reason to doubt that the energy method of Korteweg and Helmholtz is fundamentally sound." The developments in this subject since 1949 which are set out in the rest of this dissertation will show that the analyses given by Stratton and Pockels are open to grave doubt.

5. SMITH-WHITE: SIMPLE DIELECTRIC MODEL.

Smith-White (10) points out the error in the Helmholtz theory by considering a simple "non-conservative" electrical system. He shows that for such a system there is a discrepancy between the known force and the force obtained by the Helmholtz method; later he shows that the same discrepancy arises in the analysis for a continuous dielectric medium.

A dielectric body $\mathcal{B}$ in the neighbourhood of a system of charges $\mathcal{E}$ will become polarised and we can regard it as a volume distribution of electric moment. In constructing an analogous elementary system $\mathcal{E}$ is reduced to a single point charge and $\mathcal{B}$ to a pair of elementary dipoles; in order that these dipoles reflect the behaviour of the simplest dielectric materials we suppose them "linearly inductive", i.e., we suppose the strength of each proportional to the strength of the field in which it lies. The coefficients of this proportionality correspond to the dielectric susceptibility in the ordinary case. We also suppose these coefficients to vary with the separation of the dipoles, a dependence which roughly reflects the possible dependence, in the ordinary case, of the dielectric susceptibility k on the material density of the dielectric.

The actual elementary system considered consists of a point charge e and two dipoles p_1 and p_2, all lying along a line OX; the notation will be clear from the figure.



Let ϕ be the electric potential at e and let E_1, E_2 be the electric intensities at p_1, p_2 respectively. Write

$$p_1 = \lambda_1 E_1 , \qquad p_2 = \lambda_2 E_2 , \qquad \ldots(5\text{-}1)$$

$$r_1 = x_1 - x , \quad r_2 = x_2 - x , \quad s = x_2 - x_1 . \qquad \ldots(5\text{-}2)$$

Here we suppose that λ_1, λ_2 are functions of s.

We have

$$\phi = -\frac{p_1}{r_1^2} - \frac{p_2}{r_2^2} ,$$

$$E_1 = \frac{e}{r_1^2} + \frac{2p_2}{s^3} , \qquad \ldots(5\text{-}3)$$

$$E_2 = \frac{e}{r_2^2} + \frac{2p_1}{s^3} .$$

Substituting (5-3) in (5-1), we get

$$p_1 = \frac{\lambda_1 e}{D r_1^2} + \frac{2\lambda_1 \lambda_2 e}{D s^3 r_2^2} , \qquad \ldots(5\text{-}4)$$

$$p_2 = \frac{\lambda_2 e}{D r_2^2} + \frac{2\lambda_1 \lambda_2 e}{D s^3 r_1^2} ,$$

where $D = 1 - \dfrac{4\lambda_1 \lambda_2}{s^6}$. $\qquad \ldots(5\text{-}5)$

If F, F_1, F_2 are the mechanical forces acting on e, p_1, p_2 reckoned positive in the direction from O to X , then

$$F = \frac{2ep_1}{r_1^3} + \frac{2ep_2}{r_2^3},$$

$$F_1 = \frac{-2ep_1}{r_1^3} + \frac{6p_1p_2}{s^4}, \qquad \ldots(5\text{-}6)$$

$$F_2 = \frac{-2ep_2}{r_2^3} - \frac{6p_1p_2}{s^4},$$

so that $F + F_1 + F_2 = 0$.

Substituting from (5-4) in (5-6)

$$F = \frac{2\lambda_1 e^2}{Dr_1^5} + \frac{2\lambda_2 e^2}{Dr_2^5} + \frac{4\lambda_1\lambda_2 e^2}{Ds^3}\,\frac{r_1^2 + r_2}{r_1^3 r_2^3},$$

$$F_1 = \frac{-2\lambda_1 e^2}{Dr_1^5} - \frac{4\lambda_1\lambda_2 e^2}{Ds^3 r_1^3 r_2^2} +$$

$$+ \frac{6\lambda_1\lambda_2 e^2}{D^2 s^4}\left\{\frac{2-D}{r_1^2 r_2^2} + \frac{2\lambda_1}{s^3 r_1^4} + \frac{2\lambda_2}{s^3 r_2^4}\right\}, \qquad \ldots(5\text{-}7)$$

$$F_2 = -\frac{2\lambda_2 e^2}{Dr_2^5} - \frac{4\lambda_1\lambda_2 e^2}{Ds^3 r_1^2 r_2^3} -$$

$$- \frac{6\lambda_1\lambda_2 e^2}{D^2 s^4}\left\{\frac{2-D}{r_1^2 r_2^2} + \frac{2\lambda_1}{s^3 r_1^4} + \frac{2\lambda_2}{s^3 r_2^4}\right\}.$$

It will be shown that the forces in (5-7) cannot in general be derived from an energy principle so that the system considered is, in general, "non-conservative". In a differential displacement of the system along OX the work done by these forces is

$$dW = F\,dx + F_1\,dx_1 + F_2\,dx_2 \qquad \ldots(5\text{-}8)$$

$$= F_1\,dr_1 + F_2\,dr_2$$

as $F + F_1 + F_2 = 0$.

The expression (5-8) in which F_1 and F_2 are given by (5-7), is a perfect differential if, and only if λ_1 and λ_2 are constants. We show this as follows:

Using primes to denote derivatives with respect to s and writing

$$A_1 = \frac{\lambda_1'}{D} + \frac{4\lambda_1(\lambda_1\lambda_2)'}{D^2 s^6},$$

$$A_2 = \frac{\lambda_2'}{D} + \frac{4\lambda_2(\lambda_1\lambda_2)'}{D^2 s^6}, \qquad \ldots (5\text{-}9)$$

$$B = \frac{(\lambda_1\lambda_2)'}{D^2 s^3},$$

a simple calculation shows that

$$\frac{\partial F_2}{\partial r_1} - \frac{\partial F_1}{\partial r_2} = \frac{2e^2 A_1}{r_1^5} + \frac{2e^2 A_2}{r_2^5} + \frac{4e^2 B(r_1+r_2)}{r_1^3 r_2^3}. \qquad \ldots (5\text{-}10)$$

The right hand side of (5-10) vanishes if λ_1, λ_2 are constants. Conversely, if the right hand side of (5-10) vanishes identically in a domain of values of r_1, r_2 and we put $z = r_2 + r_1$, $s = r_2 - r_1$, we see that

$$\frac{2^6 e^2}{(z^2-s^2)^5}\left\{A_1(z+s)^5 + A_2(z-s)^5 + 4Bz(z^2-s^2)^2\right\},$$

is identically zero in z and s. Accordingly, B, A_1, A_2 must vanish identically; by (5-9), as $B=0$, it follows that $(\lambda_1\lambda_2)' \equiv 0$; substituting this result in the expressions for A_1 and A_2 we see that $\lambda_1' \equiv 0$ & $\lambda_2' \equiv 0$

We now apply the Helmholtz method to the above simple system, deriving the forces from the energy function $U = \frac{1}{2}\Sigma e\phi$. We have

$$\mathcal{U} = \frac{e}{2}\left\{-\frac{1}{r_1^2}\left(\frac{\lambda_1 e}{Dr_1^2} + \frac{2\lambda_1\lambda_2 e}{Ds^3 r_2^2}\right) - \frac{1}{r_2^2}\left(\frac{\lambda_2 e}{Dr_2^2} + \frac{2\lambda_1\lambda_2 e}{Ds^3 r_1^2}\right)\right\}$$

$$= \frac{-\lambda_1 e^2}{2D r_1^4} - \frac{\lambda_2 e^2}{2D r_2^4} - \frac{2\lambda_1\lambda_2 e^2}{Ds^3 r_1^2 r_2^2}. \qquad \text{...(5-11)}$$

The Helmholtz forces on the dipoles are given by

$$F_1^{(h)} = -\frac{\partial \mathcal{U}}{\partial r_1}, \quad F_2^{(h)} = -\frac{\partial \mathcal{U}}{\partial r_2}; \qquad \text{...(5-12)}$$

calculating these derivatives and comparing with (5-7) it follows that

$$F_1^{(h)} - F_1 = -\frac{e^2 A_1}{2r_1^4} - \frac{e^2 A_2}{2r_2^4} - \frac{2e^2 B}{r_1^2 r_2^2},$$

$$F_2^{(h)} - F_2 = \frac{e^2 A_1}{2r_1^4} + \frac{e^2 A_2}{2r_2^4} + \frac{2e^2 B}{r_1^2 r_2^2}. \qquad \text{...(5-13)}$$

We see that there is a discrepancy between the actual forces F and the Helmholtz forces $F^{(h)}$ which is of amount

$$\Delta = \frac{e^2 A_1}{2r_1^4} + \frac{e^2 A_2}{2r_2^4} + \frac{2e^2 B}{r_1^2 r_2^2}, \qquad \text{...(5-14)}$$

From (5-3) and (5-9) we see that

$$\Delta = \frac{1}{2} E_1^2 \lambda_1' + \frac{1}{2} E_2^2 \lambda_2'. \qquad \text{...(5-15)}$$

This simple example shows exactly where the Helmholtz theory is wrong: the forces cannot be derived from an energy principle.

We have already shown that the formula given by Helmholtz's energy method for the force per unit volume in a liquid dielectric is

$$F^{(\lambda)} = -\frac{1}{2} E^2 \nabla k + \frac{1}{2} \nabla \left(E^2 \rho \frac{\partial k}{\partial \rho} \right),$$

where $1 + 4\pi k$ is the dielectric constant and ρ is the density of the fluid. The corresponding formula given by Smith-White's theory is

$$\underline{F} = (\underline{p} \cdot \nabla)\underline{E} = \frac{k}{2} \nabla E^2 .$$

We now show that the difference

$$F^{(\lambda)} - F = \frac{1}{2} \nabla \left(E^2 \rho \frac{\partial}{\partial \rho}\left(\frac{k}{\rho}\right)\right), \qquad \ldots (5\text{-}16)$$

corresponds exactly with that given by (5-15).

In a continuous isotropic medium the dipole element is $k\underline{E}dv$ where dv is an element of volume. If to allow for dilation of the medium we use ρ_0, dv_0 to denote, respectively, undeformed or initial density and volume of an element, then $\rho_0 dv_0 = \rho dv$ so that

$$\underline{p} = \frac{k \rho_0 dv_0}{\rho} \underline{E} .$$

We may accordingly suppose $\left(\frac{k}{\rho}\right)\rho_0 dv_0$ to correspond with the coefficient λ of the elementary system: similarly the density corresponds with $\frac{1}{s}$ in the elementary system and $\frac{\rho}{\rho_0}$ corresponds with $\frac{s_0}{s}$ where s_0 is an initial value of s.

Thus

$$s_0 \frac{d\lambda}{ds} = \frac{(d\lambda)}{\left(d \frac{s}{s_0}\right)}$$

corresponds with

$$\frac{\partial \left(\frac{k}{\rho} \rho_0 dv_0 \right)}{\partial \left(\frac{\rho_0}{\rho}\right)} = -dv_0 \rho^2 \frac{\partial}{\partial \rho}\left(\frac{k}{\rho}\right)$$

Substituting (5-15) for the right hand side of (5-13) and noting that

$$\lambda_1' = \frac{d\lambda_1}{ds} = -\frac{\partial \lambda_1}{\partial x_1}, \quad \lambda_2' = \frac{d\lambda_2}{ds} = \frac{\partial \lambda_2}{\partial x_2},$$

we have

$$F_1^{(L)} - F_1 = \frac{1}{2} E_1^2 \frac{\partial \lambda_1}{\partial x_1} - \frac{1}{2} E_2^2 \frac{\partial \lambda_2}{\partial x_2}$$

$$\doteqdot -\frac{1}{2}(x_2 - x_1)\frac{\partial}{\partial x}\left(E^2 \frac{\partial \lambda}{\partial x}\right)$$

$$\doteqdot -\frac{1}{2} s_0 \frac{\partial}{\partial x}\left(E^2 \frac{\partial \lambda}{\partial x}\right)$$

$$= -\frac{1}{2}\frac{\partial}{\partial x}\left(E^2 s_0 \frac{\partial \lambda}{\partial s}\right),$$

and this corresponds with

$$\frac{1}{2}\, dv_0 \frac{\partial}{\partial x}\left(E^2 \rho^2 \frac{\partial}{\partial \rho}\left(\frac{k}{\rho}\right)\right).$$

6. SMITH-WHITE: INTRODUCTION TO GENERAL THEORY OF DISCRETE CHARGES AND DIPOLES.

Smith-White discusses a discrete system of charges and dipoles in showing the Helmholtz theory to be wrong. The theory for such a system is also of value in indicating how to deal with certain difficulties of an analytical nature which occur in the case of continuous distributions.

A dipole of moment $\underline{p} = e\underline{s}$ is conceived as a limiting case of two equal and opposite charges $-e$ and $+e$ separated by a small vector $\underline{s}$. The electric potential at P associated with a dipole $\underline{p}$ at Q is

$$\phi = \phi(P) = \underline{p}\cdot\nabla\left(\frac{1}{r}\right) \qquad \text{where} \quad r = QP$$

and ∇ operates on the coordinates of Q. The corresponding electric field is $\underline{E} = -\nabla\phi$.

If a dipole $\underline{p}$ lies in a field $\underline{E}$ it experiences a mechanical force

$$\underline{F} = (\underline{p}\cdot\nabla)\underline{E}, \qquad \ldots(6\text{-}1)$$

and a mechanical couple

$$\underline{G} = \underline{p}\wedge\underline{E}. \qquad \ldots(6\text{-}2)$$

The functions ϕ and $\underline{E}$ are mere mathematical auxiliaries convenient for the expression of the observed force $\underline{F}$ and couple $\underline{G}$.

One essential difference between a point charge and a point dipole needs emphasis: the charge at a point can be altered only by bringing up additional charge, whereas a dipole $\underline{p}$ can be varied merely by altering the vector separation $\underline{s}$. This non-conservation of dipole moment is of fundamental importance in Smith-White's work; in considering systems of charges and dipoles we shall admit the possibility of varying the dipole strength at any point without bringing up, or removing, actual dipoles.

Consider a system of charges $e_1, \ldots, e_n$ at points $P_1, \ldots, P_n$ and of dipoles $\underline{p}_1, \ldots, \underline{p}_m$ at points $Q_1, \ldots, Q_m$. Write $\overrightarrow{P_i P_j} = \underline{q}_{ij}$, $\overrightarrow{P_i Q_j} = \underline{r}_{ij}$ and $\overrightarrow{Q_i Q_j} = \underline{s}_{ij}$. Let ϕ_i be the electric potential at P_i due to all elements other than e_i and let $\underline{E}_i$ be the electric field at Q_i due to all the elements other than $\underline{p}_i$

For a charge e the potential is $\phi = e/q$ at relative position $\underline{q}$ and the field is $\underline{E} = e\underline{r}/r^3$ at relative position $\underline{r}$; the corresponding quantities for a dipole of strength $\underline{p}$ are

$\phi = -(\underline{p}.\underline{r})/r^3$, at relative position $-\underline{r}$ and

$\underline{E} = \dfrac{-\underline{p}}{s^3} + \dfrac{3(\underline{p}.\underline{s})\underline{s}}{s^5}$, at relative position $\underline{s}$.

For the system of n charges and m dipoles we have

$$\phi_i = \sum_j \frac{e_j}{q_{ij}} - \sum \frac{\underline{p}_j \cdot \underline{r}_{ij}}{r_{ij}^3}, \qquad \ldots(6\text{-}3)$$

$$\underline{E}_i = \sum_j \frac{e_j \underline{r}_{ji}}{r_{ji}^3} - \sum_j{}' \frac{\underline{p}_j}{s_{ji}^3} + 3\sum{}' \frac{(\underline{p}_j \cdot \underline{s}_{ji})\underline{s}_{ji}}{s_{ji}^5} \qquad \ldots(6\text{-}4)$$

In the summations j runs through 1, ..., n or 1, ..., m according as the sum is over the charge or dipoles and

Σ' indicates a sum in which one value $j = i$ is missing.

We now prove Green's Reciprocal Theorem extended to a system of charges and dipoles; i.e., we prove that

$$\sum_i e_i' \phi_i - \sum_i p_i' \cdot \underline{E}_i = \sum_i e_i \phi_i' - \sum_i p_i \cdot \underline{E}_i' , \qquad \ldots(6\text{-}5)$$

where $e_1', \text{------}, e_n'$ and $p_1', \text{------}, p_m'$ are a second set of charges and dipoles situated at the same points $P_1, \text{-------}, P_n; Q_1, \text{------}, Q_m$ and ϕ_i' and $\underline{E}_i'$ are the potential and field corresponding to the ϕ_i and $\underline{E}_i$ of (6-3) and (6-4). By (6-3) and (6-4)

$$\sum_i e_i' \phi_i - \sum_i p_i' \cdot E_i = \sum_P \frac{e_i' e_j}{q_{ij}} - \sum_{i,j} \frac{e_i' p_j + e_i p_j'}{r_{ij}^3}$$

$$+ \sum_P \frac{p_i' \cdot p_j}{s_{ji}^3} - 3 \sum_P \frac{(p_j \cdot \underline{s}_{ji})(p_i' \cdot \underline{s}_{ji})}{s_{ji}^5} , \qquad \ldots(6\text{-}6)$$

where $\sum_{i,j}$ is a summation in which i runs through 1, ..., n and j runs through 1, ..., m and $\sum_P$ is a summation in which the pair i,j runs through all the permutations two at a time of 1, ..., n or 1, ..., m as the case may be. Examining this expression it will be seen that an interchange of the dashed and undashed letters does not alter any one of its four terms. The relation (6-5) then follows.

Supposing that $e_i' = e_i$ (i 1, ..., n) we have

$$\Sigma' e(\phi' - \phi) = \Sigma(\underline{p}.\underline{E}' - \underline{p}'.\underline{E}) . \qquad ...(6\text{-}7)$$

We now obtain certain work formulae as follows: if a charge e in an electric field $\underline{E}$ be given a differential displacement $\Delta\underline{u}$ then the work done by the mechanical force which acts on it is

$$\underline{F}.\Delta\underline{u} = e\underline{E}.\Delta\underline{u} = -e\nabla\phi.\Delta\underline{u} = -e\Delta\phi . \qquad ...(6\text{-}8)$$

Similarly if a dipole $\underline{p}$ in an electrical field $\underline{E}$ be given a differential displacement $\Delta\underline{u}$ the work done by the mechanical force (6-1) which acts on it is

$$\begin{aligned}\underline{F}.\Delta\underline{u} &= [(\underline{p}.\nabla)\underline{E}].\Delta\underline{u} = \underline{p}.\nabla(\underline{E}.\Delta\underline{u}) \\ &= -(\underline{p}.\nabla)\Delta\phi = -\underline{p}.\Delta(\nabla\phi) = \underline{p}.\Delta\underline{E} .\end{aligned} \qquad ...(6\text{-}9)$$

The increments $\Delta\phi$ and $\Delta\underline{E}$ are those corresponding to incrementary changes in the coordinates of e and $\underline{p}$, respectively.

7. SMITH-WHITE: GENERAL WORK FORMULA.

We consider a variation of the electrical system of charges $e_1, \text{------}, e_n$ and dipoles $p_1, \text{---}, p_m$ at points $P_1, \text{---}, P_n$ and $Q_1, \text{-------}, Q_m$ respectively. The variation consists of an infinitesimal displacement of the charges and dipoles and also an infinitesimal change in accordance with the explanation given in §6, in the dipole moments p_i. Thus the independent variables are the 3n cartesian coordinates of the charges e_i, the 3m cartesian coordinates of the dipoles p_i and the 3m rectangular components of the vectors p_i. The following notation for differentials will be used:

Δ denotes a total differential with respect to <u>all</u> independent variables;

Δ' denotes a partial differential with respect to the spatial coordinates of the charges and dipoles;

Δ'' a partial differential with respect to the components of the vectors p_i;

Δ^{e_i} a partial differential with respect to the coordinates of the charge e_i;

Δ^{p_i} a partial differential with respect to the coordinates of p_i.

Then, symbolically,

$$\Delta = \Delta' + \Delta'' , \qquad \text{...(7-1)}$$

and

$$\Delta' = \sum_i \Delta^{e_i} + \sum_i \Delta^{p_i} . \qquad \ldots(7\text{-}2)$$

Consider the function

$$V = \frac{1}{2} \sum_i e_i \phi_i - \frac{1}{2} \sum_i \underline{p}_i \cdot \underline{E}_i . \qquad \ldots(7\text{-}3)$$

If we write $e_i' = e_i$ and $\underline{p}_i' = \underline{p}_i$ in (6-6) and divide by two we have

$$V = \sum_C \frac{e_i e_j}{q_{ij}} - \sum_{i,j} \frac{e_i \underline{p}_j \cdot \underline{r}_{ij}}{r_{ij}^3} + \sum_C \frac{\underline{p}_i \cdot \underline{p}_j}{s_{ji}^3}$$

$$- 3 \sum_C \frac{(\underline{p}_j \cdot \underline{s}_{ji})(\underline{p}_i \cdot \underline{s}_{ji})}{s_{ji}^5} , \qquad \ldots(7\text{-}4)$$

where $\sum_{i,j}$ is a summation in which i runs through i, ..., n and j runs through i, ..., m and $\sum_C$ is a summation in which the pair i,j runs through all combinations, two at a time of 1, ..., n or 1. ..., m as the case may be. In (7-4) the coordinates of e_i are involved only in the terms

$$e_i \sum_j{}' \frac{e_j}{q_{ij}} - e_i \sum \frac{\underline{p}_j \cdot \underline{r}_{ij}}{r_{ij}^3} = e_i \phi_i ,$$

so that $\Delta^{e_i} V = e_i \Delta^{e_i} \phi_i$.

Similarly in (7-4) the coordinates of $\underline{p}_i$ are involved only in the terms

$$- \underline{p}_i \cdot \sum_j \frac{e_j \underline{r}_{ji}}{r_{ji}^3} + \underline{p}_i \cdot \sum_j{}' \frac{\underline{p}_j}{s_{ji}^3} - \underline{p}_i \cdot 3 \frac{\sum (\underline{p}_j \cdot \underline{s}_{ji}) \underline{s}_{ji}}{s_{ji}^5}$$

$$= - \underline{p}_i \cdot \underline{E}_i ,$$

whence $\Delta^{p_i} V = - \underline{p}_i \cdot \Delta^{p_i} \underline{E}_i$.

Thus, by (7-2) $-\Delta' V = - \sum_i e_i \Delta^{e_i} \phi_i + \sum_i \underline{p}_i \cdot \Delta^{p_i} \underline{E}_i ,$

it follows from (6-8) and (6-9) that $-\Delta' V$ is the total work done in the displacement by all the mechanical forces acting on the charge and dipoles in the system. Also,

$$\underline{p}.\underline{E}' - \underline{p}'.\underline{E} = \underline{p}.(\underline{E}'-\underline{E}) - (\underline{p}'-\underline{p}).\underline{E} ;$$

it follows from (6-7) that, for a differential variation of the components of the $\underline{p}_i$,

$$\sum_i e_i \Delta'' \phi_i = \sum_i (\underline{p}_i . \Delta'' \underline{E}_i - \underline{E}_i . \Delta'' \underline{p}_i).$$

Thus

$$\Delta'' V = \frac{1}{2} \sum e_i \Delta'' \phi_i - \frac{1}{2} \sum_i (\underline{p}_i . \Delta'' \underline{E}_i + \underline{E}_i . \Delta'' \underline{p}_i)$$

$$= \frac{1}{2} \sum_i (\underline{p}_i . \Delta'' \underline{E}_i - \underline{E}_i . \Delta'' \underline{p}_i) - \frac{1}{2} \sum_i (\underline{p}_i . \Delta'' \underline{E}_i + \underline{E}_i . \Delta'' \underline{p}_i)$$

$$= - \sum_i \underline{E}_i . \Delta'' \underline{p}_i = - \sum_i \underline{E}_i . \Delta \underline{p}_i ,$$

as $\Delta'' \underline{p}_i = \Delta \underline{p}_i$.

It follows from the above that, if ΔW denotes the work done by the mechanical forces acting on the charges and dipoles,

$$\Delta W = - \Delta V - \sum_i \underline{E}_i . \Delta \underline{p}_i . \qquad \ldots (7\text{-}5)$$

This is the <u>fundamental work formula</u>. Its importande lies in its generality. Smith-White claims that it gives a means of determining whether certain electrical systems are or are not mechanically conservative. We shall be mainly concerned with linear isotropic systems in which case $\underline{p}_i = \lambda_i \underline{E}_i$ and the couple $\underline{G}_i = \underline{p}_i \wedge \underline{E}_i$ will vanish. For such systems then ΔW in (7-5) gives the total work done in the displacement.

Introducing the function

$$u = \frac{1}{2} \sum_i e_i \phi_i , \qquad \dots (7\text{-}6)$$

we find from (7-5)

$$\Delta W = -\Delta u + \frac{1}{2} \sum_i (\underline{p}_i \cdot \Delta \underline{E}_i - \underline{E}_i \cdot \Delta \underline{p}_i), \qquad \dots (7\text{-}7)$$

which is another form of the fundamental work formula.

8. SMITH-WHITE: INDUCTIVE SYSTEMS.

We call a system of point charges and point dipoles an <u>inductive</u> system if the dipole moments p_i depend upon and are determined by the magnitude of the charges $e_1, \ldots, e_n$ and the configuration of the points $P_1, \ldots, P_n, Q_1, \ldots, Q_m$. Systems in which the dipole moments are all constant vectors and systems of charge only may be regarded as special cases.

Thus, for a system of charges only $U = V$; by (7-5) $\Delta W = -\Delta U$, so that the system is mechanically conservative and U is its mechanical potential energy function.

For a system of charges and dipoles in which the components of the dipole moments are fixed $\Delta W = -\Delta V$; accordingly, this system also is mechanically conservative and V is its mechanical potential energy function.

We now consider a <u>linear</u> inductive system in which the dipole moments p_i are determined by

$$p_i = \lambda_i E_i, \qquad \ldots(8\text{-}1)$$

where the λ_i are constants or, more generally, depend upon the configuration of the points $P_1, \ldots, P_n, Q_1, \ldots, Q_m$. Replacing the $\underline{E}_i$ in (8-1) by the expressions in (6-4) we obtain 3m linear equations determining the 3m components of the vectors p_i.

It follows from (8-1) that, in an infinitesimal displacement of the charges and dipoles,

$$\underline{p}.\Delta\underline{E} - \underline{E}.\Delta\underline{p} = \underline{p}.\Delta\underline{E} - \underline{E}.\Delta(\lambda\underline{E})$$

$$= \underline{p}.\Delta\underline{E} - \underline{E}\lambda.\Delta\underline{E} - E^2\Delta\lambda = -E^2\Delta\lambda;$$

by (7-7) the corresponding work done by the mechanical forces acting on the elements of the system is

$$\Delta W = -\Delta U - \frac{1}{2}\sum_i E_i^2 \Delta\lambda_i . \qquad \ldots(8\text{-}2)$$

When the coefficients λ_i are all constants, $\Delta W = -\Delta U$; the system is mechanically conservative and the function U is its <u>mechanical potential energy function.</u>

When the coefficients λ_i are not constants a new situation arises. Consider a linear inductive system in which these coefficients depend on the dipole configuration $Q_1, ---, Q_m$ only, being independent of the positions of $P_1, ---, P_n$. This is the analogue in the discrete case of the dielectric susceptibility varying with the density. We have shown in detail for a special case (See §5) that such a system is mechanically conservative if and only if the coefficients λ_i are all constants. The same result holds more generally: if the coefficients λ_i vary with the configuration of $Q_1, ---, Q_m$ the system is not in general mechanically conservative. However, for a displacement of the charges only, the configuration of the dipoles remaining fixed, (8-2) becomes $\Delta W = -\Delta U$; in this special case, therefore, the system <u>is</u> mechanically conservative, the mechanical potential energy function being U. (Smith-White describes such a system as <u>semi-conservative</u>; he uses the same term to describe analogous continuous systems.)

9. SMITH-WHITE: INTRODUCTION TO CONTINUOUS THEORY.

There is no difficulty in defining potential ϕ and field $\underline{E}$ at a point outside a dielectric body. For an interior point Q the formula giving the potential is

$$\phi_Q = \int_V \underline{P} . \nabla\left(\frac{1}{r}\right) dv ,$$

the integral being absolutely convergent (for all reasonable distributions). It can be shown that an integral of this type is absolutely convergent if the integrand is of order less than $1/r^{3-\alpha}$ $(\alpha > 0)$ as $r \to +0$. However, if we attempt to calculate the field at Q from the formula

$$\underline{E}_Q = \int_V \left\{ -\frac{\underline{P}}{r^3} + \frac{3(\underline{P}.\underline{r})\underline{r}}{r^5} \right\} dv , \qquad \ldots (9.1)$$

the integral is found to be "semi-convergent", i.e. if $\{V_m\}$ denotes a sequence of similar cavities similarly situated about Q the linear dimensions of V_m tending to zero as $m \to +\infty$ then

$$\lim_{m \to +\infty} \int_{V_m}^{V} \left\{ -\frac{\underline{P}}{r^3} + \frac{3(\underline{P}.\underline{r})\underline{r}}{r^5} \right\} dv ,$$

will, in general, depend on the shape of V_1 .

This difficulty has been discussed by Livens and Smith-White. Smith-White argues as follows:

In electrostatic theory we ordinarily contemplate the following kinds of electrical distributions:

(1) discrete distributions of point charges $q_1, ----, q_n$;

(2) continuous distributions of electricity defined by a charge density e ;

(3) discrete distributions of point electric dipoles;

(4) continuous distributions of electric polarity defined by a moment density $\underline{P}$.

Now the primitive electrostatic law prescribes the mechanical action on each element of type (1) but does not apply directly to give the mechanical action on the elements in any other kind of distribution. It is easy to infer the extensions of Coulomb's Law which enable us to specify this action on the elements of distributions (2) and (3). This however involves an extension of the primitive hypothesis; a similar point is emphasised by Kellog (1929) in connection with the application of Newton's law of gravitation to continuous distributions of matter. On the other hand if we try to write down the mechanical force which acts on an element of the distribution (4) by an integration over the distribution of a contribution from each of the other elements we find that the integral which presents itself is "semi-convergent".

If a theory of electrical distributions of type (4) is to be in line with the theories of the other three types of distribution it must be developed from a hypothesis concerning the mechanical action on the elements of the distribution. What would appear the obvious procedure for finding such a hypothesis fails. There is however another course open to us and this can be regarded as a uniform procedure for all four types of distribution. We begin by noting that the mechanical action on the elements in (1), (2), (3) may be specified in an alternative mathematically equivalent way by first introducing the potential function ϕ, defining the electric field as $\underline{E} = -\nabla\phi$, and then specifying for (1) that the force acting on charge e is $e\underline{E}$; for (2) that the force density acting at a place where the charge density is e is $e\underline{E}$; and for (3) that the force and couple acting on a dipole of strength $\underline{p}$ are respectively $(\underline{p}.\nabla)\underline{E}$ and $\underline{p}\wedge\underline{E}$.

In line with the specifications for distributions (1), (2), (3) we can specify the mechanical action on an element of distribution (4). Suppose the distribution occupies a volume V. Define the potential by

$$\phi = \int_V \underline{P}.\underline{r}/r^3\, dv . \qquad \ldots (9.2)$$

This integral is absolutely convergent inside the distribution. We then define the electric field by $\underline{E} = -\nabla\phi$. We then assert that the force density acting on an element of the distribution is given by

$$\underline{F} = (\underline{P}.\nabla)\underline{E} \qquad \ldots (9.3)$$

and the couple density on it is given by

$$\underline{G} = \underline{P} \wedge \underline{E} . \qquad \ldots (9.4)$$

Also at a surface S of V we can show (see App. A-22) that the limiting form of (9.3) gives a surface traction on S

$$\underline{T} = \tfrac{1}{2} \underline{P} . (\underline{E}_{+} - \underline{E}_{-}) \underline{n} . \qquad \ldots (9.5)$$

Smith-White regards the relations (9.2), (9.3), (9.4) as comprising the appropriate extension of Coulomb's Law required for a mathematical theory of distribution of polarisation. The value of this hypothesis for the physics of dielectrics is to be judged by a comparison of its consequences with the behaviour of actual bodies.

Smith-White argues further that if we develop in detail the theory of point dipole distributions and compare it with the theory of continuous distributions of polarity based on (9.3) and (9.4) we see that the two theories are mathematically similar: that the subtle difficulties in the formulation of the theory of continuous moment distributions are of no real consequence for physics: and that if we refer to such distributions as a matter of mathematical convenience then we must be sure only that our theory is properly parallel to the theory of discrete distributions.

Livens ("Theory of Electricity" p. 66, et seq.) also discusses the semi-convergence of the integral (9.1) at interior points of a dielectric; the argument proceeds as follows.

The integral (9.2) is absolutely convergent; by Green's theorem

$$\phi = \int_V \frac{\underline{P}.\underline{r}}{r^3}\, dv = \int_V \underline{P}.\operatorname{grad}\left(\frac{1}{r}\right) dv$$

(the differentiation being taken with respect to the coordinates of the dipoles)

$$= \int \operatorname{div}\left(\frac{\underline{P}}{r}\right) dv - \int \frac{\operatorname{div} \underline{P}}{r}\, dv$$

$$= \int \frac{\underline{P}.\, d\underline{s}}{r} - \int \frac{\operatorname{div} \underline{P}}{r}\, dv ,$$

where the surface integral is extended over the surface of the dielectric (including the walls of the cavity round the point Q) and the volume integral over the volume of the body.

He then points out that the potential of this polarised body is the same as the potential of the charge distribution specified as

(1) a volume density $e' = -\operatorname{div} \underline{P}$ throughout the body.

(2) a surface density $\sigma' = P_n$ over the surface of the body and the cavity.

The volume distribution e' and the surface distribution σ' over the actual surface of the body give definite contributions to the field at Q but the distribution σ' over the walls of the cavity gives a field depending on the shape of the cavity. Livens argues that this latter component of the field is a purely <u>local</u> part depending on the molecular configuration round the point Q ; since we do not know the molecular configuration, which may be changing rapidly, we cannot know what this local part amounts to.

Livens adopts the arbitrary course of simply neglecting this local molecular part of the field. He says that this is "merely following a usual method in physics and involves but a simple extension of the ideas underlying the Young-Poisson principle of the mutual compensation of molecular forcives employed in their theory of capillary actions. It requires that such local forcives shall set up a purely local physical disturbance of the molecular configuration in the material until it is thereby balanced. Another example of this principle is provided in the ordinary theory of elasticity where in addition to the local strain forces in an elastic medium there are the comparatively very powerful cohesive forces, which are however presumed to form an equilibrating system and not to affect the phenomena as a whole." It is fortunate, remarks Livens, that we can in this way eliminate the influence of the neighbouring elements.

Both Livens and Smith-White effectively say :

(1) the integral for ϕ i.e. $\int \underline{P}\cdot\underline{r}/r^3\,dv$ is absolutely convergent .

(2) define $\underline{E}$ as $-\nabla\phi$.

(3) <u>by hypothesis</u> the force per unit volume is $(\underline{P}\cdot\nabla)\underline{E}$.

10. SMITH-WHITE: THE GENERAL WORK FORMULA.

Consider an electrical system consisting of a continuous charge distribution with density ρ in V_1 and a continuous moment distribution with density $\underline{P}$ in V_2. It is useful to think of the electric moment as "attached" to a medium; the formulae of the prceding section apply to this situation. The force and couple acting on the moment distribution are supposed to be transferred to the medium, this last being the material substance of a continuous dielectric body. For the charge distribution the notion of such a "medium of reference" is unnecessary; or we may say that the medium is the "continuous electricity" itself.

A variation of the electrical system consists of moving the charge in V_1 and deforming the reference medium in V_2. At the same time the electric moment associated with the parts of the medium may alter. An infinitesimal variation would be determined by specifying the displacement vector $\delta\underline{u}$ in V_1 and V_2 and the increment $\Delta\underline{p}$ in V_2. At the same time the medium in V_2 experiences the infinitesimal rotation ~~of~~ $\delta\underline{\Theta}$. (See Appendix B-6.) The work done by the forces and couples acting on the system is

$$\Delta W = \int_{V_1} \underline{F}\delta\underline{u}\, dV + \int_{V_2} \underline{F}\delta\underline{u}\, dV + \int_{V_2} \underline{G}\delta\underline{\Theta}\, dV + \int_{S_2} \underline{T}\delta\underline{u}\, dS, \tag{10.1}$$

where

$$\underline{F} = \rho\underline{E} \text{ in } V_1, \quad \underline{F} = (\underline{P}.\nabla)\underline{E} \text{ in } V_2$$

$$\underline{G} = \underline{P}\wedge\underline{E} \quad \text{"} \quad \text{"}$$

$$\underline{T} = \tfrac{1}{2}\underline{P}.(\underline{E}_+ - \underline{E}_-)\underline{n} \text{ on } S_2.$$

Let n_i be the direction cosines of the outward normal to S_1 or S_2 ; by Green's Theorem

$$\int_{V_1} \underline{F}.\delta\underline{u}\, dV = \int_{V_1} e\,\underline{E}.\delta\underline{u}\, dV = -\int e \frac{\partial \phi}{\partial x_\alpha}\delta u_\alpha\, dv.$$

$$= -\int_{V_1} \frac{\partial}{\partial x_\alpha}(e\phi\delta u_\alpha)\,dv + \int_{V_1} \phi \frac{\partial}{\partial x_\alpha}(e\delta u_\alpha)\,dv$$

$$= -\int_{S_1} e\phi\, n_\alpha \delta u_\alpha\, dS' - \int_{V_1} \phi\, \delta e\, dv,$$ (see App. B-14)

and

$$\int_{V_2} \underline{F}.\delta\underline{u}\, dv = \int_{V_2} [(\underline{P}.\nabla)\underline{E}].\delta\underline{u}\, dv$$

$$= \int_{V_2} P_\beta \frac{\partial E_\alpha}{\partial x_\beta}\delta u_\alpha\, dv = \int_{V_2} P_\beta \frac{\partial E_\beta}{\partial x_\alpha}\delta u_\alpha\, dv$$

Since $\nabla_\wedge \underline{E} = 0$

$$= \int_{V_2} \underline{P}.\frac{\partial \underline{E}}{\partial x_\alpha}\delta u_\alpha\, dv = \int_{V_2} \frac{\partial}{\partial x_\alpha}(\underline{P}.\underline{E}\delta u_\alpha)\,dv - \int_{V_2} \underline{E}.\frac{\partial}{\partial x_\alpha}(\underline{P}\delta u_\alpha)\,dv$$

$$= \int_{S_2} \underline{P}.\underline{E}_-\, n_\alpha \delta u_\alpha\, dS' + \int_{V_2} \underline{E}.\delta\underline{P}\, dv - \int_{V_2} \underline{E}.\Delta\underline{p}\, dv - \int_{V_2} \underline{E}\,\delta\underline{\Theta}_\wedge \underline{P}\, dv.$$

(See App. B-13).

Also
$$\int_{V_2} \underline{G}.\delta\underline{\Theta}\, dv = \int_{V_2} \underline{P}_\wedge \underline{E}\,\delta\underline{\Theta}\, dv = \int_{V_2} \underline{E}\,\delta\underline{\Theta}_\wedge \underline{P}\, dv$$

$$\int_{S_2} \underline{T}.\delta\underline{u}\, dS' = \frac{1}{2}\int_{S_2} \underline{P}.(\underline{E}_+ - \underline{E}_-)\,\underline{n}\delta\underline{u}\, dS'.$$

Write

$$V = \frac{1}{2}\int_{V_1} e\phi\, dv - \frac{1}{2}\int_{V_2} \underline{P}.\underline{E}\, dv .$$

Then

$$\Delta V = \frac{1}{2}\int_{V_1}(e\delta\phi + \phi\,\delta e)\,dv + \frac{1}{2}\int_{S_1} e\phi\, n_\alpha \delta u_\alpha\, dS'$$

$$-\frac{1}{2}\int_{V_2}(\underline{P}.\delta\underline{E} + \underline{E}.\delta\underline{P})\,dv - \frac{1}{2}\int_{S_2}\underline{P}.\underline{E}_-\, n_\alpha \delta u_\alpha\, dS' .$$

Hence, from (10-1),

$$\Delta W + \Delta V + \int_{V_2}\underline{E}.\Delta\underline{p}\,dv$$

$$= \frac{1}{2}\int_{V_1}(e\delta\phi - \phi\,\delta e)\,dv - \frac{1}{2}\int_{S_1} e\phi\, n_\alpha \delta u_\alpha\, dS'$$

$$-\frac{1}{2}\int_{V_2}(\underline{P}.\delta\underline{E} - \underline{E}.\delta\underline{P})\,dv + \frac{1}{2}\int_{S_2}\underline{P}.\underline{E}_+\, n_\alpha \delta u_\alpha\, dS' .$$

As shown in App. A-13', the right hand side of this equation vanishes; accordingly

$$\Delta W = -\Delta V - \int_{V_2}\underline{E}.\Delta\underline{p}\,dv . \qquad \text{... (10.2)}$$

Writing $u = \frac{1}{2}\int_{V_1} e\phi\, dv$, we have

$$\Delta V = \Delta u - \frac{1}{2}\int_{V_2}(\underline{P}.\delta\underline{E} + \underline{E}.\delta\underline{P})\,dv - \frac{1}{2}\int_{S_2}\underline{P}.\underline{E}_- n_\alpha \delta u_\alpha\, dS'.$$

As shown in App. B9, B-13.

$$\underline{P}.\delta\underline{E} + \underline{E}.\delta\underline{P} = \underline{P}.\left[-\frac{\partial \underline{E}}{\partial x_\alpha}u_\alpha + \delta\underline{\Theta}\wedge\underline{E} + \Delta\underline{E}\right]$$

$$+\underline{E}.\left[-\frac{\partial}{\partial x_\alpha}(\underline{P}\delta u_\alpha) + \delta\underline{\Theta}\wedge\underline{P} + \Delta\underline{p}\right] =$$

$$-\frac{\partial}{\partial x_\alpha}(\underline{P}.\underline{E}\delta u_\alpha) + (\underline{P}.\Delta\underline{E} + \underline{E}.\Delta\underline{p}) ,$$

so that

$$\Delta V = \Delta U + \frac{1}{2}\int_{V_2} \frac{\partial}{\partial x_\alpha}(\underline{P}.\underline{E}\delta u_\alpha)dv - \frac{1}{2}\int_{S_2} \underline{P}.\underline{E}\_n_\alpha \delta u_\alpha dS$$

$$-\frac{1}{2}\int_{V_2}(\underline{p}.\Delta\underline{E} + \underline{E}.\Delta\underline{p})\,dv$$

$$= \Delta U - \frac{1}{2}\int_{V_2}(\underline{p}\cdot\Delta\underline{E} + \underline{E}.\Delta\underline{p})dv\,.$$

As $\underline{P} = \underline{p}$ in the initial state, it follows that

$$\Delta W = -\Delta U + \frac{1}{2}\int_{V_2}(\underline{p}.\Delta\underline{E} - \underline{E}.\Delta\underline{p})dv. \qquad \ldots (10.3)$$

11. SMITH-WHITE : APPLICATION OF THE WORK FORMULA.

In the theory of dielectrics we assume that $\underline{P}$ is determined by $\underline{E}$. The relation between $\underline{P}$ and $\underline{E}$ depends upon the physical state of the substance of the dielectric including the condition of strain reckoned from some standard configuration.

The simplest relation is $\underline{P} = k\underline{E}$

If the substance of the dielectric is not homogeneous then k varies from place to place in it; again, if the substance be deformable the value of k for a particular element of the material may depend on its deformation from some standard state. We shall suppose that, in liquid dielectrics, k depends on the density ρ only.

We have $\underline{p} = J\underline{P} = Jk\underline{E}$,

whence $\Delta\underline{p} = Jk\Delta\underline{E} + \underline{E}\,\Delta Jk$,

and $-\underline{E}.\Delta\underline{p} + \underline{p}.\Delta\underline{E} = \underline{p}.\Delta\underline{E} - EJk\Delta\underline{E} - E^2\Delta(Jk) = -E^2\Delta(Jk)$.

Now

$$\Delta(Jk) = J\Delta k + k\Delta J = \frac{\partial k}{\partial \rho}\Delta\rho + k\Delta J ,$$

as $J \doteq 1$ to the first order.

Also $\Delta\rho = -\rho\,\Delta J$ by (B-11),(B - 15),

whence

$$\Delta(Jk) = \left(-\rho\frac{\partial k}{\partial\rho} + k\right)\Delta J$$

$$= -\rho^2\frac{\partial}{\partial\rho}\left(\frac{k}{\rho}\right)\frac{\partial u_\alpha}{\partial x_\alpha} ;$$

Accordingly $\underline{p}\cdot\Delta\underline{E} - \underline{E}\cdot\Delta\underline{p} = E^2\rho^2\frac{\partial}{\partial\rho}\left(\frac{k}{\rho}\right)\frac{\partial u_\alpha}{\partial x_\alpha}$, and formula (10-3) becomes

$$\Delta W = -\Delta U + \frac{1}{2}\int_{V_2} E^2 \rho^2 \frac{\partial}{\partial \rho}\left(\frac{k}{\rho}\right) \frac{\partial u_\alpha}{\partial x_\alpha} dv .$$

... (11.1)

If, in the dielectric, $k = a\rho$ where a is constant throughout V_2 then $\Delta W = -\Delta U$. In this case the system is mechanically conservative and U is its potential energy function. The same result holds if the dielectric is held rigid. In general, however, $\Delta W \neq -\Delta U$ and the system is only "semi-conservative" (cf.Section 8).

Smith-White says: "The recognition of such non-conservative systems is a new feature in electrostatics. To make such systems acceptable from the physical point of view we must show how to fit them into a wider physical scheme in which physical energy is conserved. This offers no difficulty any more than it does in ordinary mechanics, where we are quite familiar with the fact that real mechanical systems are never completely conservative, but are always, to a greater or less degree, dissipative."

12. SMITH-WHITE : THE CONSERVATION OF ENERGY.

Dielectric in a volume V_2 is under the electric influence of a charge distribution in volume V_1 .

It is held in equilibrium by external mechanical surface forces $\underline{\Pi}$ acting on its boundary S_2 . Inside the dielectric the mechanical force and couple of electrical origin i.e. $\underline{F} = (\underline{P}.\nabla)\underline{E}$ and $\underline{G} = \underline{P}\wedge\underline{E}$ is balanced by the mechanical stress in the substance. The statical conditions for this internal equilibrium are written most concisely in tensor notation. Let F_i, G_i be the components of $\underline{F}$ and $\underline{G}$ in directions Ox_i , and if the axes Ox_1, Ox_2, Ox_3 be right handed write $G_1 = G_{23} = -G_{32}$ etc. Then the couple $\underline{G}$ is represented by the anti-symmetric tensor G_{ij} . Let t_{ij} be the mechanical stress tensor at any point in the dielectric. The equilibrium equations are

$$\frac{\partial t_{ij}}{\partial x_j} + F_i = 0 , \qquad \text{... (12.1)}$$

$$t_{ij} - t_{ji} + G_{ij} = 0 . \qquad \text{... (12.2)}$$

Three forces act at the boundary S_2 :

(i) the force $\underline{T}$ given by $\underline{T} = \frac{1}{2}\underline{P}.(\underline{E}_+ - \underline{E}_-)\underline{n}$,

(ii) the external force $\underline{\Pi}$ mentioned above ,

(iii) a force due to the mechanical stress inside V_2 .

If T_i, Π_i are the components of $\underline{T}$, $\underline{\Pi}$ and if n_i are the direction cosines of a normal to S_2 outward from V_2 , the condition of equilibrium of an element dS of S_2 gives

$$-n_j s_{ij} + T_i + \Pi_i = 0 . \qquad \text{... (12.3)}$$

The sole purpose of the charge distribution in V_1, is to provide the source of the external influence in the dielectric. We are not concerned to examine the internal equilibrium of the charge distribution in the way analagous to that for the dielectric. It is sufficient to suppose that the charge is maintained in position by some "external agency" which provides a force, on each volume element dv of the charge, just sufficient to balance the electric force, $e\underline{E}\, dv$.

For the electrical energy of the system we take the quantity

$$V = \frac{1}{8\pi}\int_{E} E^2\, dv \ . \qquad \ldots (12.4)$$

This is not now a mechanical potential energy function but is energy in a physical sense. Only in a special case is V a mechanical potential energy function. This choice for the energy of an electrical system amounts to a definite phyeial [sic] assumption, and is justified by its consequences. There is no other obvious choice for the electrical energy of a system.

We may now formulate the physical equation of energy, expressing the first law of thermodynamics, in a variation of the physical system consisting of a dielectric under the influence of electric charge. We suppose that the variation consists of a movement of the charge, a deformation of the dielectric and an absorption or emission of heat by the dielectric substance. An infinitesimal variation is

specified by the displacement vector $\delta \underline{u}$ defined in V_1 and V_2.
If ΔQ be the heat absorbed by the dielectric, this is accounted for by

(i) an increase ΔI of the "internal energy" of the dielectric ;

(ii) An increase ΔV of the electrical energy of the system ;

(iii) the work $\int_{V_1} \underline{F}.\delta\underline{u}\, dv$ done against the external agency holding the charge ;

(iv) the work $-\int_{S_2} \underline{\Pi}.\delta\underline{u}\, dS'$ done against the external forces holding the dielectric boundary.

So

$$\Delta Q = \Delta I + \Delta V + \int_{V_1} \underline{F}.\delta\underline{u}\, dv - \int_{S_2} \underline{\Pi}.\delta\underline{u}\, dS' . \quad \ldots (12.5)$$

Now set

$$\tilde{\omega}_{ij} = \tfrac{1}{2}(t_{ij} + t_{ji}) , \quad \ldots (12.6)$$

$$a_{ij} = \tfrac{1}{2}(t_{ij} - t_{ji}) . \quad \ldots (12.7)$$

Then by (12-2), (12-6), (12-7),

$$t_{\alpha\beta}\frac{\partial \delta u_\beta}{\partial x_\alpha} = (\tilde{\omega}_{\alpha\beta} + a_{\alpha\beta})(\delta e_{\alpha\beta} + \delta\theta_{\alpha\beta})$$

$$= \tilde{\omega}_{\alpha\beta}\delta e_{\alpha\beta} + \tilde{\omega}_{\alpha\beta}\delta\theta_{\alpha\beta} + a_{\alpha\beta}\delta e_{\alpha\beta} + a_{\alpha\beta}\delta\theta_{\alpha\beta} ,$$

The second and third terms vanish and the last term becomes $-\ \sigma_\beta \delta\theta_\beta$

$$\therefore\ t_{\alpha\beta}\frac{\partial \delta u_\beta}{\partial x_\alpha} = \tilde{\omega}_{\alpha\beta}\delta e_{\alpha\beta} - \sigma_\beta \delta\theta_\beta\ .$$

From (12.3) using Green's theorem and (12.1)

$$-\int_{S_2} \underline{\Pi}.\delta\underline{u}\ dS' = \int_{S_2} T_\beta \delta u_\beta\, dS' - \int_{S_2} n_\beta t_{\alpha\beta}\delta u_\alpha\, dS'$$

$$= \int_{S_2} T_\beta \delta u_\beta\, dS' - \int_{V_2} \frac{\partial}{\partial x_\beta}(t_{\alpha\beta}\delta u_\alpha)\, dv$$

$$= \int_{S_2} T_\beta \delta u_\beta\, dS' - \int_{V_2} \frac{\partial t_{\alpha\beta}}{\partial x_\beta}\delta u_\alpha\, dv - \int_{V_2} t_{\alpha\beta}\frac{\partial \delta u_\alpha}{\partial x_\beta}\, dv$$

$$= \int_{S_2} T_\beta \delta u_\beta\, dS' + \int_{V_2} F_\alpha \delta u_\alpha\, dv + \int_{V_2} \sigma_\alpha \delta\theta_\alpha - \int_{V_2} \tilde{\omega}_{\alpha\beta}\delta e_{\alpha\beta}\, dv$$

$$= \int_{S_2} \underline{T}.\delta\underline{u}\ dS' + \int_{V_2} \underline{F}.\delta\underline{u}\ dv + \int_{V_2} \underline{G}.\delta\underline{\Theta}\ dv - \int_{V_2} \tilde{\omega}_{\alpha\beta}\delta e_{\alpha\beta}\, dv.$$

Hence from (10.1) and (12.5)

$$\Delta Q = \Delta I + \Delta V + \Delta W - \int_{V_2} \tilde{\omega}_{\alpha\beta}\delta e_{\alpha\beta}\, dv$$

$$= \Delta I - \int_{V_2} \tilde{\omega}_{\alpha\beta}\delta e_{\alpha\beta}\, dv - \int_{V_2} \underline{E}.\Delta \underline{p}\, dv\ , \qquad \ldots (12.8)$$

by (10.2). This result applies to any piece of dielectric substance whatever its dimensions. We infer the elementary relation

$$\Delta q = \Delta i - \tilde{\omega}_{\alpha\beta}\delta e_{\alpha\beta} - \underline{E}.\Delta \underline{p} \;, \qquad \ldots (12.9)$$

where i is the internal energy per unit volume and Δq is the heat absorbed per unit volume at any place in the dielectric.

In a fluid $\tilde{\omega}_{ij} = -\tilde{\omega}\,\delta_{ij}$, where $\tilde{\omega}$ is the hydrostatic pressure and $\delta_{ij} = 1$ or 0 as $i = j$ or $i \neq j$

Then

$$\tilde{\omega}_{\alpha\beta}\delta e_{\alpha\beta} = -\tilde{\omega}\,\delta_{\alpha\beta}\delta e_{\alpha\beta} = -\tilde{\omega}\,\Delta J \,,$$

so (12.9) becomes

$$\Delta q = \Delta i + \tilde{\omega}\Delta J - \underline{E}.\Delta \underline{p} \,, \qquad \ldots (12.10)$$

13. SMITH-WHITE: ELECTROSTATIC STRESS.

The body force $\underline{F} = (\underline{P}.\nabla)\underline{E}$ and body couple $\underline{G} = \underline{P}\wedge\underline{E}$ may be expressed in an alternative way by using an electrostatic stress tensor. This is analogous to expressing a vector field $\underline{E}$ by a scalar potential ϕ.

In general if we are given a sec nd order tensor s_{ij} and

$$F_i = \frac{\partial s_{ij}}{\partial x_j}, \quad G_{ij} = s_{ij} - s_{ji}. \qquad \ldots (13.1)$$

Then the F_i are the components of a vector $\underline{F}$ and the G_{ij} are connected with a vector $\underline{G}$ by the relations $G_1 = G_{23} = -G_{32}$ etc. Using a "mechanical" terminology we call s_{ij} a stress tensor, $\underline{F}$ is the body force and $\underline{G}$ the body couple. Let $S_i = s_{ij}n_j$.

We note that the mechanical force on a volume has components

$$\int_V F_i\,dV = \int_V \frac{\partial s_{i\alpha}}{\partial x_\alpha}\,dV = \int_S n_\alpha s_{i\alpha}\,dS = \int_S S_i\,dS.$$

Also the moment with respect to the origin has components

$$\int_V (x_i F_j - x_j F_i + G_{ij})\,dV = \int_V \left(x_i \frac{\partial s_{j\alpha}}{\partial x_\alpha} - x_j \frac{\partial s_{i\alpha}}{\partial x_\alpha} + G_{ij}\right)dV$$

$$= \int_V \left\{ \frac{\partial}{\partial x_\alpha}(x_i s_{j\alpha} - x_j s_{i\alpha}) + s_{ji} - s_{ij} + G_{ij} \right\} dV$$

$$= \int_S n_\alpha (x_i s_{j\alpha} - x_j s_{i\alpha})\,dS = \int_S (x_i S_j - x_j S_i)\,dS.$$

For a charged dielectric we have that the body force is

$$\underline{F} = {}^e\rho\,\underline{E} + (\underline{P}.\nabla)\underline{E}, \qquad \ldots (13.2)$$

and the body couple is

$$\underline{G} = \underline{P} \wedge \underline{E} .$$
... (13.3)

We may express this $\underline{F}$ by means of anelectrostatic stress tensor as follows

$$F_i = e E_i + P_\alpha \frac{\partial E_i}{\partial x_\alpha} .$$

But $\nabla . \underline{D} = 4\pi e$

$$\therefore \quad F_i = \frac{E_i}{4\pi} \frac{\partial D_\alpha}{\partial x_\alpha} + P_\alpha \frac{\partial E_i}{\partial x_\alpha} = \frac{E_i}{4\pi} \frac{\partial D_\alpha}{\partial x_\alpha} + \frac{D_\alpha - E_\alpha}{4\pi} \frac{\partial E_i}{\partial x_\alpha}$$

$$= \frac{1}{4\pi} \frac{\partial}{\partial x_\alpha} (D_\alpha E_i) - \frac{E_\alpha}{4\pi} \frac{\partial E_\alpha}{\partial x_i} .$$

$$= \frac{1}{4\pi} \frac{\partial}{\partial x_\alpha} \left(D_\alpha E_i - \frac{1}{2} E^2 \delta_{\alpha i} \right) = \frac{\partial s_{i\alpha}}{\partial x_\alpha} ,$$

where

$$s_{ij} = \frac{\left(E_i D_j - \frac{E^2}{2} \delta_{ij} \right)}{4\pi} .$$
... (13.4)

Also

$$S_{ji} - s_{ij} = (D_i E_j - D_j E_i)/_{4\pi} = P_i E_j - P_j E_i$$

Thus the mechanical field given by (13.2) and (13.3) may be derived from the tensor

$$s_{ij} = \frac{E_i D_j - \frac{1}{2} E^2 \delta_{ij}}{4\pi} .$$

This is the Maxwell stress tensor as given by Livens (2. 1926)

For the case in which $\underline{D} = K \underline{E}$ the stress (13.4) is

$$s_{ij} = \frac{1}{4\pi} \left(K E_i E_j - \frac{E^2}{2} \delta_{ij} \right) .$$
... (13.5)

In the literature the discussion is usually restricted to this case, [redacted] instead of (13.5) the electrostatic stress is found to be either

$$M^{\circ}_{ij} = \frac{K}{4\pi}\left(E_i E_j - \frac{E^2}{2}\delta_{ij}\right), \qquad \ldots (13.6)$$

which corresponds to the body force

$$\underline{F}_{0}^{(h)} = e\underline{E} - \frac{E^2}{8\pi}\nabla K; \qquad \ldots (13.7)$$

or

$$M_{ij} = \frac{K}{4\pi}\left(E_i E_j - \frac{1}{2}E^2\delta_{ij}\right) + \frac{1}{8\pi}E^2\rho\frac{\partial K}{\partial\rho}\delta_{ij}, \qquad \ldots (13.8)$$

which corresponds to the body force

$$\underline{F}^{(h)} = e\underline{E} - \frac{E^2}{8\pi}\nabla K + \frac{1}{8\pi}\nabla\left(E^2\rho\frac{\partial K}{\partial\rho}\right). \qquad \ldots (13.9)$$

14. SMITH-WHITE: FLUID DIELECTRIC.

Consider a fluid dielectric in equilibrium in an electric field.

Smith-White's equation of equilibrium is

$$\nabla \tilde{\omega} = \frac{k}{2} \nabla E^2 , \qquad \ldots (14.1)$$

where $\tilde{\omega}$ is the hydrostatic pressure developed in the fluid.

The Helmholtz equation of equilibrium will be

$$\nabla \tilde{\omega}^{(h)} = \frac{1}{2} E^2 \nabla k + \frac{1}{2} \nabla \left(E^2 \rho \frac{\partial k}{\partial \rho} \right) ; \qquad \ldots (14.2)$$

in these formulae k and ρ refer to the actual deformed state of the dielectric in the existing field.

For a fluid which is originally homogeneous and in which k and ρ are functions of the pressure only the equations (14.1) and (14.2) may be integrated, giving

$$\int \frac{d\tilde{\omega}}{k} = \frac{E^2}{2} ,$$

$$\int \frac{d\tilde{\omega}^{(h)}}{\rho} = \frac{E^2}{2} \frac{\partial k}{\partial \rho} ,$$

respectively . If, in addition, the fluid be effectively incompressible, we have

$$\tilde{\omega} = \frac{k}{2} E^2 + \tilde{\omega}_0 , \qquad \ldots (14.3)$$

and

$$\tilde{\omega}^{(h)} = \frac{1}{2} E^2 \rho \frac{\partial k}{\partial \rho} + \tilde{\omega}_0 . \qquad \ldots (14.4)$$

Thus $$\tilde{\omega}^{(h)} - \tilde{\omega} = \frac{1}{2} E^2 \rho^2 \frac{\partial}{\partial \rho} \left(\frac{k}{\rho} \right) .$$

If K is not proportional to ρ the Helmholtz theory and the Smith-White theory give <u>different values for the pressure in the dielectric fluid</u>

Suppose we have a solid body immersed in a homogeneous fluid, the system being held in equilibrium by a suitable mechanical constraining force and couple applied to the body. The fluid develops the pressure ϖ . The force exerted directly by the electric field on the body has components

$$\int_S n_\alpha s_{\alpha i} \, dS \quad ,$$

where s_{ij} is the tensor (13.5) and the integration is over the surface of the body. The reaction of the pressure in the fluid has components

$$\int_S \varpi n_i \, dS = \int_S \varpi \delta_{\alpha i} n_\alpha \, dS \ .$$

The resultant force acting on the body, which must be balanced by the constraint, has components

$$\int_S (s_{\alpha i} - \varpi \delta_{\alpha i}) n_\alpha \, dS \ . \qquad \ldots (14.5)$$

On the usual Helmholtz theory this force has components

$$\int_S (M_{\alpha i} - \varpi^{(h)} \delta_{\alpha i}) n_\alpha \, dS \ . \qquad \ldots (14.6)$$

Now from (13.5) and (14.3) or (13.8) and (14.4)

$$s_{ij} - \varpi \delta_{ij} = M^0_{ij} - \varpi_0 \delta_{ij} \ ,$$
$$M_{ij} - \varpi^{(h)} \delta_{ij} = M^0_{ij} - \varpi_0 \delta_{ij} \ ,$$

so that (14.5) and (14.6) reduce to the same expression

$$\int_S M^{(0)}_{\alpha i} n_\alpha \, dS' . \qquad \ldots (14.7)$$

Thus the two theories give the same net force acting on the solid. In a homogeneous fluid the stress M^{0}_{ij} is self-equilibrating i.e. it corresponds to no body force. Then the integral (14.7) may be taken over any surface S in the fluid which encloses the body.

From (13.6) the force components (14.7) are those of the vector

$$\frac{K}{4\pi} \int_S \left(\underline{E} (\underline{E} . \underline{n}) - \frac{1}{2} E^2 \underline{n} \right) dS' , \qquad \ldots (14.8)$$

where $\underline{n}$ is the unit outward normal on S . This is the usual expression for the force acting on a body immersed in a fluid.

15. <u>BROWN: OUTLINE OF BROWN'S METHOD OF CALCULATING BODY FORCES</u>.

Suppose we wish to calculate the force on a body of magnetically polarised matter τ which is part of a larger physical body B and which is therefore in direct contact with τ' (the part of the body not included in τ). In this case a "force" can be formally calculated by integrating over τ and over τ' the formula for the magnetic force exerted by a dipole of moment $\underline{M}'d\tau'$ on a dipole of moment $\underline{M}d\tau$ with $\underline{M}$ the macroscopic magnetisation (magnetic moment per unit volume) and $d\tau$ the element of volume. <u>The "long range force" exerted by τ' on τ is defined as the force formally calculated by this method</u>.

In calculating the force on a magnetised body he first calculates the force and torque exerted on a whole magnetised body (the "test body") by sources of field entirely outside it.

He next evaluates the long range part of the magnetic force and torque exerted on matter in an arbitrary volume τ of a magnetized body by all sources outside including the rest of the body. To do this he first calculates the magnetic force and torque exerted by all sources outside a closed surface S_1 on all the polarised matter inside a closed surface S_2 completely surrounded by S_1 and ~~then~~ separated from it by a distance molecularly large (Fig.1); he then finds the limiting values of this

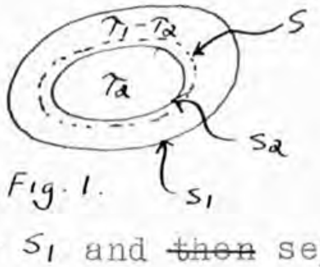


Fig. 1.

force and torque as S_1 and S_2 are brought together to the limiting surface S .

Brown says: "The force and torque calculated in this manner are reliable estimates of physically significant quantities only before the limiting process is carried out. When the distance between S_1 and S_2 becomes of a molecular or atomic order of magnitude the continuous dipole densities cease to give reliable results. The errors correspond to forces which fall off with distance faster than do dipole forces; because of this short range character they can be taken into account macroscopically as stresses. In addition there are other short range forces that act across the surface S . In a macroscopic theory there is no possibility of separating these two contributions to the stress. However, the resultant stress components must obey certain laws that can be deduced by macroscopic theory alone."

16. BROWN: THE FORCES ON A WHOLE BODY.

The magnetic moment of the matter in a volume element $d\tau$ of the test body is $\underline{M}\,d\tau$ where $\underline{M}$ is the magnetisation in $d\tau$. Let the magnetic field of the sources external to the body be $\underline{H}_0$. Then the forces exerted on the volume element by the sources of the external field is $(\underline{M}.\nabla)\underline{H}_0\,d\tau$ and there is also a couple $\underline{M}\wedge\underline{H}_0 d\tau$ so that the torque about an arbitrary origin O with respect to which the position vector of $d\tau$ is $\underline{r}$ is

$$\left\{ \underline{r}\wedge\left[(\underline{M}.\nabla)H_0\right] + \underline{M}\wedge\underline{H}_0\right\} d\tau.$$

The total force and torque exerted by external sources are found by integrating these expressions over the volume τ occupied by the body. It is known that the force exerted by external sources on a small body is $(\underline{M}\cdot\nabla)\underline{H}_0\,d\tau$ but when $d\tau$ is part of a larger body this is not capable of demonstration. The ultimate test for these formulae is their agreement with experiment.

Brown gives two useful general formulae not usually given in textbooks :

$$\int(\underline{n}.\underline{u})\,\underline{v}\,dS - \int(\nabla.\underline{u})\underline{v}\,d\tau = \int(\underline{u}.\nabla)\underline{v}\,d\tau\,, \qquad (16\text{-}1)$$

and

$$\int(\underline{n}.\underline{u})\,\underline{r}\wedge\underline{v}\,dS - \int(\nabla.\underline{u})\underline{r}\wedge\underline{v}\,d\tau$$

$$= \int\left[\underline{r}\wedge[(\underline{u}.\nabla)\underline{v}] + \underline{u}\wedge\underline{v}\right] d\tau\;. \qquad (16\text{-}2)$$

Applying these formulae to the magnetic case we obtain

$$\int(M.\nabla)\underline{H}\,d\tau = \int(-\nabla.\underline{M})\,\underline{H}\,d\tau + \int(\underline{M}.\underline{n})\underline{H}\,dS\,, \qquad (16.1')$$

$$\int \left[\underline{r} \wedge [(\underline{M}.\nabla)\underline{H}] + \underline{M} \wedge \underline{H} \right] d\tau = \int M_n \underline{r} \wedge \underline{H} \, dS - \int (\nabla.\underline{M}) \underline{r} \wedge \underline{H} \, d\tau. \tag{16.2'}$$

The first of these relations implies that the total force acting on a moment distribution is equivalent to the total force acting on the equivalent Poisson charge distribution of volume density $-\nabla.\underline{M}$ and surface density $\underline{M}\cdot\underline{n}$ (It may be noted that Marziani (21) devotes two papers to proving this result by a roundabout energy method and under the restrictive assumption that the dielectric is rigid. This seems rather pointless in view of the fact that a simple application of (16.1) gives the required result.)

17. BROWN: THE LONG RANGE FORCES ON PART OF A BODY.

Let S be a surface entirely inside a polarised body enclosing a volume τ. Let S_1 and S_2 be surfaces drawn outside and inside S enclosing volumes τ_1 and τ_2 which lie entirely within the polarised body. To find the magnetic force $\underline{F}$ exerted on τ by all sources outside S Brown first calculates the magnetic force $\underline{F}_{12}$ on τ_2 due to all sources outside S_1 ; $\underline{F}$ will then be the limit of $\underline{F}_{12}$ as $S_1, S_2 \to S$ He divides the magnetic field in two parts:

(1) $\underline{H}_0$ is the field due to sources outside S_1 ,

(2) $\underline{H}_{12}$ is the field (calculated inside S_2) arising from all sources inside S_1 .

The total field inside S_2 is

$$\underline{H} = \underline{H}_0 + \underline{H}_{12} . \tag{17.1}$$

Then

$$\underline{F} = \lim_{S_1, S_2 \to S} \int_{\tau_2} (\underline{M} . \nabla) \underline{H}_0 \, d\tau . \tag{17.2}$$

Substituting for $\underline{H}_0$ from (17.1) in (17.2),

$$\underline{F} = \lim_{S_1, S_2 \to S} \int_{\tau_2} (\underline{M} . \nabla) \underline{H} \, d\tau + \underline{F}_1 , \tag{17.3}$$

where

$$\underline{F}_1 = - \lim_{S_1, S_2 \to S} \int_{\tau_2} (\underline{M} . \nabla) \underline{H}_{12} \, d\tau ,$$

and

$$\lim_{S_1, S_2 \to S} \int_{\tau_2} (\underline{M} . \nabla) \underline{H} \, d\tau = \int_{\tau} (\underline{M} . \nabla) \underline{H} \, d\tau .$$

Brown evaluates $\underline{F}_1$ in the following way. By (16-1'),

$$\lim_{S_1,S_2\to S}\left\{\int_{\tau_2}(\underline{M}\cdot\nabla)\underline{H}_{12}\,d\tau_2\right\} = \lim_{S_1,S_2\to S}\left\{\int_{S_2}(\underline{n}_2\cdot\underline{M}_2)\underline{H}_{12}\,dS_2 + \int_{\tau_2}(-\nabla_2\cdot\underline{M}_2)\,\underline{H}_{12}\,d\tau_2\right\}, \tag{17.4}$$

where $-\nabla_2\cdot\underline{M}_2$ refers to a differentiation with respect to coordinates of points in τ_2 .

The field inside S_2 due to sources inside S_1 will be the same as the field due to the equivalent Poisson volume and surface charges of densities $-\nabla_1\cdot\underline{M}_1$ and $\underline{M}.\underline{n}$ and is given by

$$\underline{H}_{12} = \left\{\int_{S_1}(\underline{n}_1\cdot\underline{M}_1)\frac{\underline{l}_{12}}{r_{12}^2}\,dS_1 + \int_{\tau_1}(-\nabla_1\cdot\underline{M}_1)\frac{\underline{l}_{12}}{r_{12}^2}\,d\tau_1\right\}, \tag{17.5}$$

where $\underline{l}_{12}$ is a unit vector pointing from surface or volume element 1 to the relevant field point in τ_2 and r_{12} is the corresponding mutual distance. Substituting for $\underline{H}_{12}$ from (17-5) in (17-4) we obtain the limit of a double integral which, since $\underline{l}_{12} = -\underline{l}_{21}$, can be written as

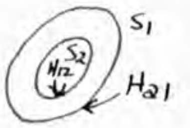
$$-\lim_{S_1,S_2\to S}\left\{\int_{S_1}(\underline{n}_1\cdot\underline{M}_1)\underline{H}_{21}\,dS_1 + \int_{\tau_1}(-\nabla_1\cdot\underline{M}_1)\,\underline{H}_{21}\,d\tau_1\right\}, \tag{17.6}$$

(i.e. the Coulomb forces between the pole distributions 1 and 2 obey the law of action and reaction.)

If the two expressions (17-4) and (17-6) are added and divided by 2, the volume terms vanish in the limit but the surface terms do not for $\underline{H}_{12}$ is always evaluated <u>inside</u> S_1 whereas $\underline{H}_{21}$ is evaluated <u>outside</u> S_2 . The limit is

$$\underline{F}_1 = \frac{1}{2} \int_S \underline{n}.\underline{M} \left(\underline{H}_1^+ - \underline{H}_1^- \right) dS' \, , \tag{17.7}$$

where $\underline{H}_1$ is the field due to sources contained in τ, the symbols $+$ and $-$ indicating values just outside and just inside S. The discontinuity in $\underline{H}_1$ across the surface S



affects the normal component only and is of amount $4\pi \, \underline{n}.\underline{M} = 4\pi M_n$.

Therefore

$$\underline{H}_1^+ - \underline{H}_1^- = 4\pi \, \underline{n} \, M_n \, , \tag{17.8}$$

and

$$\underline{F}_1 = 2\pi \int \underline{n} \, M_n^2 \, dS' \, . \tag{17.9}$$

The total long range force exerted on the matter in τ (i.e. the force exerted by all sources outside τ) is thus given by

$$\underline{F} = \int (\underline{M}.\nabla) \underline{H} \, d\tau + 2\pi \int \underline{n} \, M_n^2 \, dS' \, . \tag{17.10}$$

This may be expressed in terms of $\underline{B}$ by setting $\underline{H} = \underline{B} - 4\pi \underline{M}$ in (17-10): we obtain

$$\underline{F} = \int \left[(\underline{M}.\nabla) \underline{B} + \underline{M} \wedge (\nabla \wedge \underline{B}) \right] d\tau - 2\pi \int \underline{n} \, M_t^2 \, dS' \, . \tag{17.11}$$

Because of the nature of the last term in (17-10) the long range force "cannot be expressed as simply so much per unit volume; it depends also on the shape of the volume considered".

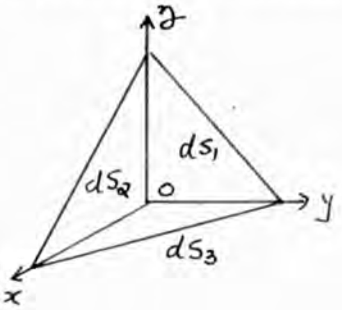
18. BROWN: STRESSES.

Brown has to modify the procedures of standard elasticity theory to accommodate his force formula (17-10) In the standard theory of elasticity (see e.g. Love (22)) it is assumed that the mechanical stress tensor is symmetrical. This is not true in the magnetic case where we have a couple density $\underline{M}\wedge\underline{H}$. Both Smith-White and Brown agree that the corresponding mechanical stress tensor is not symmetrical. The usual relation $t_{23} = t_{32}$ must be replaced by

$$t_{23} - t_{32} = \left(\underline{M}\wedge\underline{H}\right)_1 = M_2 H_3 - M_3 H_2 . \qquad \ldots(18.1)$$

Brown also has to make a further modification of the stress tensor on account of the surface forces in his force formula.

We consider the equilibrium of a small tetrahedron with three of its faces as the coordinate planes of the rectangular axes Ox, oy, oz the fourth face having a normal with direction cosines l, m, n. In the ordinary case we



merely have external forces (e.g. gravitational forces) which can be expressed as so much per unit volume. Resolving along Ox we have, with a standard notation,

$$t_{11}\, dS_1 + t_{12}\, dS_2 + t_{13}\, dS_3 + F_1\, dV = t_1(\nu)\, d\Sigma ,$$

where $dV = \frac{1}{3} p\, d\Sigma$; it follows from the usual limiting process that $t_1(\nu) = t_{11} l + t_{12} m + t_{13} n$,

since, in the limit, $p F_1$ tends to zero. Thus in the ordinary case the body force does not appear in the stress tensor because it is associated with the volume of the element. Brown has surface forces as well as body forces; in this case, resolving along Ox we have

$$\left(t_1(\nu) + 2\pi(M_n^2)\right) d\Sigma =$$

$$\left(t_{11} + 2\pi M_1^2\right) dS_1 + t_{12}\, dS_2 + t_{13}\, dS_3 + F_1\, dV \ .$$

The usual limiting process yields the relation

$$t_1(\nu) = \left[t_{11} + 2\pi(M_1^2 - M_n^2)\right] l + t_{12} m + t_{13} n \ . \qquad \ldots(18.2)$$

This gives the traction across a plane with normal $\underline{n} = (l, m, n)$ in terms of the traction across coordinate planes. In the same way it can be shown that

$$t_2(\nu) = t_{21} l + \left[t_{22} + 2\pi(M_2^2 - M_n^2)\right] m + t_{23} n \ ,$$

$$t_3(\nu) = t_{31} l + t_{32} m + \left[t_{33} + 2\pi(M_3^2 - M_n^2)\right] n \ ,$$

where $t_i(\nu)$ refers to the i'th component of the traction. Substituting $\underline{H} = \underline{B} - 4\pi \underline{M}$ in $(18\text{-}1)$ gives

$$t_{23} - t_{32} = (\underline{M} \wedge \underline{H})_1 = (\underline{M} \wedge \underline{B})_1$$

since $\quad \underline{M} \wedge \underline{M} = 0$.

The $\underline{B}$ form of equation $(18\text{-}2)$ is

$$t_1(\nu) = \left[t_{11} - 2\pi(M_2^2 + M_3^2 - M_t^2)\right] l + \qquad \ldots(18.2')$$
$$+ t_{12} m + t_{13} n \ .$$

Brown then deduces his equation of motion. He considers a rectangular volume element with sides parallel to the coordinate planes; by the usual procedure he shows that

$$(t_{11} + 2\pi M_1^2)_{,1} + t_{12,2} + t_{13,3} + \qquad \ldots(18.3)$$
$$+ \left[(\underline{M} \cdot \nabla)\underline{H}\right]_1 + \rho F_1 = \rho f_1 \ .$$

Substituting $\underline{H} = \underline{B} - 4\pi\underline{M}$ in (18-3) we have

$$\left(t_{11} - 2\pi(M_2^2 + M_3^2)\right)_{,1} + t_{12,2} + t_{13,3}$$
$$+ \left[(M\cdot\nabla)\underline{B} + \underline{M}\wedge(\nabla\wedge\underline{B})\right]_1 + \rho F_1 = \rho f_1 . \quad \ldots(18\text{-}3')$$

Here (F_1, F_2, F_3) is the mechanical body force per unit mass (e.g. gravity), ($f_1\ f_2\ f_3$) is the acceleration and ρ is the mass density. At a free surface, $X_\nu = Y_\nu = Z_\nu = 0$; more generally, the values of these quantities at a bounding surface of a body are equal to the tractions applied to the surface from outside.

Brown also discusses what happens if the long range forces are computed by associating with each volume element the Poisson equivalent charge $-\nabla.\underline{M}\,d\tau$ and with each surface element the equivalent charge $\underline{n}.\underline{M}\,dS$. (See Section 16.) Later, when discussing the work done in a displacement, Brown says "When only forces and torques are being computed, the alternative interpretation in terms of polarisation charge density was treated with considerable tolerance because it led to results equivalent to those obtained by the dipole method. Here it is necessary to be less tolerant."

Brown's basic point of view is that the forces act on the dipoles and not on the equivalent charges; accordingly only the equations corresponding to the force acting on the dipoles are set down here.

19. <u>BROWN: FLUIDS</u>.

Brown defines a fluid as a material incapable in equilibrium of exerting tangential tractions on a <u>solid surface</u> with which it is in contact or of having such tractions exerted on itself.

Considering a surface normal to a coordinate axis, Brown deduces that $t_{23}=0, t_{32}=0$, etc., so that by (18-1), $\underline{M}\wedge\underline{H}=0$, i.e. $\underline{M}$ must be parallel to $\underline{H}$. For an arbitrarily oriented surface it follows from (18-2) that, as $(t_1(\nu),t_2(\nu),t_3(\nu))$ must be along (l,m,n)

$$t_1(\nu) = [t_{11}+2\pi(M_1^2-M_n^2)]l + t_{12}m + t_{13}n = l\xi,$$
$$t_2(\nu) = t_{21}l + [t_{22}+2\pi(M_2^2-M_n^2)]m + t_{23}n = m\xi,$$
$$t_3(\nu) = t_{31}l + t_{32}m + [t_{33}+2\pi(M_3^2-M_n^2)] = n\xi .$$

These relationshold for an arbitrary direction (l,m,n) ; consequently

$$t_{11}+2\pi(M_1^2-M_n^2) = t_{22}+2\pi(M_2^2-M_n^2) = t_{33}+2\pi(M_3^2-M_n^2)$$
$$t_{12} - t_{23} = t_{31} = 0$$
$$t_{11}+2\pi M_1^2 - t_{22}+2\pi M_2^2 = t_{33}+2\pi M_3^2 = \qquad \ldots(19.1)$$
$$= -p' \ (\text{say}).$$

Thus the vector force per unit area exerted across dS

$$\underline{t}(\nu) = -(p'+2\pi M_n^2)\underline{n} = -p_\nu \underline{n} . \qquad \ldots(19.2)$$

The equation of motion (18-3) reduces to

$$-\nabla p' + (\underline{M}.\nabla)\underline{H} + \rho\underline{E} = \rho\underline{f} , \qquad ;\ldots(19.3)$$

where $\underline{F} = (F_1, F_2, F_3)$ is the body force per unit mass and $\underline{f} = (f_1, f_2, f_3)$ is the acceleration.

As $\underline{M}$ is in the same direction as $\underline{H}$ we have $\underline{M} = \alpha \underline{H}$. Also

$$(\underline{M}.\nabla)\underline{H} = \alpha(\underline{H}.\nabla)\underline{H} = \frac{\alpha}{2} \,\mathrm{grad}\, H^2 ,$$

as $\nabla \wedge \underline{H} = 0$, further $\mathrm{grad}\, H^2 = 2H \,\mathrm{grad}\, H$,

so that $(\underline{M}\cdot\nabla)\underline{H} = M \,\mathrm{grad}\, H$.

Accordingly, the equation of motion is

$$-\nabla p' + M(\nabla H) + \rho \underline{F} = \rho \underline{f} . \qquad \ldots(19\text{-}4)$$

20. BROWN: EQUILIBRIUM RELATIONS IN FLUIDS.

Brown assumes the fluid to be in equilibrium in a conservative field of force so that

$$\underline{F} = -\operatorname{grad} V \ , \quad \underline{f} = 0 \ , \qquad \ldots(20.1)$$

where V is the potential of the mechanical body forces.

By (19-4) the equilibrium equation is

$$-\operatorname{grad} p' + M \operatorname{grad} H - \rho \operatorname{grad} V = 0 \ . \qquad \ldots(20.2)$$

Putting $\quad Z = -p' - \rho V \ , \qquad \ldots(20.3)$

this becomes

$$\operatorname{grad} Z = - M \operatorname{grad} H - V \operatorname{grad} \rho \ . \qquad \ldots(20.4)$$

Vector multiplication by $\operatorname{grad} H$ gives

$$\operatorname{grad} H \wedge \operatorname{grad} Z = - V \operatorname{grad} \rho \wedge \operatorname{grad} H \ .$$

Scalar multiplication by $\operatorname{grad} Z$ then gives

$$\operatorname{grad} Z \cdot \{ \operatorname{grad} \rho \wedge \operatorname{grad} H \} = 0 \ .$$

Thus

$$\begin{vmatrix} \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \\ \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial \rho}{\partial z} \end{vmatrix} = 0 \ , \qquad \ldots(20.5)$$

i.e. $$\frac{\partial (Z, H, \rho)}{\partial (x, y, z)} = 0 \ .$$

Thus there must be a functional relation connecting Z, H and ρ. If this relation be

$$Z = Z(H, \rho) \ , \qquad \ldots(20.6)$$

then

$$dZ = \frac{\partial Z}{\partial H} dH + \frac{\partial Z}{\partial \rho} d\rho$$

However by (20-4), $dZ = -M\,dH - V\,d\rho$, ...(20.7)
(where d represents the change on moving from $\underline{r}$ to $\underline{r} + d\underline{r}$). Comparing these expressions for dZ

$$M = -\frac{\partial Z}{\partial H} \quad , \quad V = -\frac{\partial Z}{\partial \rho} . \qquad \ldots(20.8)$$

It follows from (19-2) and (20-3) that the normal pressure across a surface with normal $\underline{n}$ is

$$p_\nu = p' + 2\pi M_n^2 \ , \qquad \ldots(20.9)$$

where

$$p' = -Z - \rho V = -Z + \rho \frac{\partial Z}{\partial \rho} .$$

Brown points out the similarity of these equations to those obtained by thermodynamic reasoning.

He also remarks that the presence of polarisation necessitates some care in the treatment of pressure. In ordinary hydrodynamics a fluid, defined as a substance capable of exerting only normal tractions across a bounding surface is found to exert the <u>same</u> normal traction across differently oriented surfaces; in the presence of polarisation this is no longer true. By (20-9) the pressure across a surface normal to the polarisation vector exceeds that across a surface tangential to it by $2\pi M^2$.

21. BROWN: ELECTROSTRICTION.

(1) Compressible Fluid.

In this section we give Brown's discussion of the behaviour of a dielectric fluid under the influence of an electric field. The general fluid equilibrium equations deduced in Section 20 will be used with $\underline{H}$ and $\underline{M}$ replaced by $\underline{E}$ and $\underline{P}$ respectively.

Brown assumes that the polarisation is proportional to the field intensity, the susceptibility being a function of the density ρ :

$$\underline{P} = k(\rho)\underline{E} = \frac{1}{4\pi}[K(\rho)-1]\underline{E} . \qquad \ldots(21.1)$$

By (20-8)

$$P = -\frac{\partial Z(E,\rho)}{\partial E} ; \qquad \ldots(21.2)$$

integrating (20-1) it follows that

$$-Z = \frac{1}{2}k(\rho)E^2 + \psi(\rho) \qquad \ldots(21.3)$$

Then,

$$V = -\frac{\partial Z}{\partial \rho} = \frac{1}{2}k'(\rho)E^2 + \psi'(\rho) , \qquad \ldots(21.4)$$

and

$$p' = -Z - \rho V = \frac{1}{2}[k(\rho) - \rho k'(\rho)]E^2 + \psi(\rho) - \rho\psi'(\rho) . \qquad \ldots(21.5)$$

The pressure at zero field intensity and at density ρ is

$$p(\rho) = \psi(\rho) - \rho\psi'(\rho) ; \qquad \ldots(21.6)$$

accordingly $\psi(\rho)$ can be evaluated from the pressure-density relation at zero field.

We now suppose that V is constant: $V = V_0$ say;

this will be the case in the absence of non-electrical forces such as gravity. (21-4) then yields the following relation between the densities at two points where the field intensities are E_0 and E_1 :

$$\frac{1}{2}\left[E_1^2 k'(\varrho_1) - E_0^2 k'(\varrho_0)\right] = -\left[\psi'(\varrho_1) - \psi'(\varrho_0)\right]. \quad \ldots(21.7)$$

Differentiating (21-6)

$$p'(\varrho) = -\varrho\,\psi''(\varrho), \quad \ldots(21.8)$$

whence

$$\psi'(\varrho_1) - \psi'(\varrho_0) = -\int_{\varrho_0}^{\varrho_1} \frac{dp(\varrho)}{\varrho}, \quad \ldots(21.9)$$

so that (21-7) becomes

$$\begin{aligned}\int_{\varrho_0}^{\varrho_1} \frac{dp(\varrho)}{\varrho} &= \frac{1}{2}\left[E_1^2 k'(\varrho_1) - E_0^2 k'(\varrho_0)\right] \\ &= \frac{1}{8\pi}\left[E_1^2 K'(\varrho_1) - E_0^2 K'(\varrho_0)\right].\end{aligned} \quad \ldots(21.10)$$

This is equation (14 c) of Abraham-Becker (17) if $p(\varrho)$ is identified with the quantity p appearing in their analysis.

The pressure across a surface with normal $\underline{n}$ at a point where the field intensity is $\underline{E}$ differs from the pressure which would obtain with the same density distribution but with zero field by an amount

$$\begin{aligned} p_\nu - p(\varrho) &= p' + 2\pi P_n^2 - p(\varrho) \\ &= \frac{1}{2}\left[k(\varrho) - \varrho k'(\varrho)\right]E^2 + 2\pi k^2(\varrho) E_n^2 .\end{aligned} \quad \ldots(21.11)$$

As

$$k = \frac{K-1}{4\pi},$$

we have

$$p_v - p(\rho) = \frac{1}{8\pi}\Big\{ (K(\rho) - 1 - \rho K'(\rho)) E^2 + (K(\rho)-1)^2 E_n^2 \Big\} . \qquad \ldots (21.12)$$

This is equation (15 a) of Abraham-Becker (17), p.101, with the same interpretation of p as before: the Abraham-Becker "hydrostatic pressure" is the pressure that would obtain for the same density distribution if the field intensity were zero.

(2) <u>Incompressible Fluid</u>.

In Brown's discussion of an incompressible fluid the mechanical body-force potential appears as the independent variable. Considering the electric case, it follows from (20-2) that p' is a function of E and V such that

$$\frac{\partial p'}{\partial E} = P , \quad \frac{\partial p'}{\partial V} = \rho .$$

For an incompressible fluid, $\rho = const. = \rho_0$, so that

$$p' = \rho_0 V + f(E) \quad , \text{ where } \quad f'(E) = P \ ;$$

~~and~~

thus P depends on E only and not on the mechanical forces.

For a linear incompressible fluid $P = kE$ where k is independent of E and of the mechanical forces; in this case

$$p' = \rho_0 V + \frac{kE^2}{2} + const.,$$

so that in the absence of mechanical body forces

$$p' = p_0 + \frac{k}{2} E^2 ,$$ where p_0 is the pressure at a point when $E = 0$.

Thus, for such a fluid,

$$
\begin{aligned}
p_\nu &= p_0 + \frac{k}{2} E^2 + 2\pi k^2 E_n^2 \\
&= p_0 + \frac{1}{8\pi}(K-1)\left[E^2 + (K-1)E_n^2\right] \qquad \ldots(21.13) \\
&= p_0 + \frac{1}{8\pi}(K-1)\left[E_t^2 + K E_n^2\right] .
\end{aligned}
$$

22. BROWN: STRESS-STRAIN RELATIONS FOR A POLARISABLE ELASTIC SOLID.

Brown derives the stress-strain relations for a polarisable elastic solid by an energy method. This is a generalisation of the usual procedure followed in classical elasticity. Brown points out that he now appeals to an energy argument to derive simple and plausible formulae describing the properties of special classes of matter; in the Helmholtz theory, on the other hand, (see Sections 2 and 3) an energy method was used to derive the body force.

Brown computes the work done by long-range forces and tractions for a small deformation of an elastic dielectric body which at the instant under consideration occupies a volume τ; certain terms can be identified as representing the increase in a potential energy of known form; for isothermal or adiabatic changes it can be shown, by thermodynamic arguments applied to reversible processes, that the remaining terms represent the increment in a thermodynamic potential. The use of this thermodynamic potential will simplify the magnetoelastic relations, in the same way as the use of the elastic energy simplifies the stress-strain relations in classical elasticity.

Consider a magnet of moment $\underline{m}$ in an external field $\underline{H}_0$. If the magnet is given a small displacement $\delta\underline{u}$ and rotated through a small angle $\delta\underline{\Theta}$ then

$$d\underline{m} = \delta\underline{\Theta} \wedge \underline{m} + D\underline{m} , \qquad \ldots (22.1)$$

where d is the change of $\underline{m}$ with respect to axes that translate but do not rotate with $\underline{m}$ and D is the change in $\underline{m}$ with respect to axes that translate and rotate with $\underline{m}$. Then,

$$-d(-\underline{m}.\underline{H}_0) = \underline{m}.d\underline{H}_0 + d\underline{m}.\underline{H}_0$$

$$= \underline{m}.\left[(\delta\underline{u}.\nabla)\underline{H}_0\right] + \delta\underline{\Theta}\wedge\underline{m}.\underline{H}_0 + D\underline{m}.\underline{H}_0 \qquad \ldots (22.2)$$

$$= \delta\underline{u}.\left[(\underline{m}.\nabla)\underline{H}_0\right] + \delta\underline{\Theta}\wedge\underline{m}.\underline{H}_0 + \underline{H}_0.D\underline{m} .$$

(The interchange of $\underline{m}$ and $\delta\underline{u}$ in the first term is permissible because $\mathrm{curl}\,\underline{H}_0 = 0$.) The first two terms give the work done by the force $(\underline{m}.\nabla)\underline{H}_0$ and the couple $\underline{m}\wedge\underline{H}_0$. Besides these there is a term $\underline{H}_0.D\underline{m}$ due to internal changes in the magnet. If the magnet is to be treated as a conservative dynamical system it is necessary to introduce, besides the external coordinates necessary for a permanent magnet, additional internal coordinates - the components of $\underline{m}$ in axes attached to the magnet - and to include a term $\underline{H}_0.D\underline{m}$ in the differential expression for work done.

For the magnetic body under consideration, occupying the volume τ , $\underline{m}$ is to be replaced by $\underline{M}\,d\tau = \underline{M}'dm$ where dm is the mass of the element of volume $d\tau$.

We assume that the only body force is of magnetic origin; thus, for example, we exclude a gravitational body force. The work done by long-range magnetic forces exerted by sources outside τ is

$$\delta W_m = \int \left\{ \delta \underline{u}.\left[(\underline{M}'.\nabla)\underline{H}_0\right] + \delta \underline{\Theta}.\left[\underline{M}' \wedge \underline{H}_0\right] + \underline{H}_0 . D\underline{M}' \right\} dm$$

$$= \int \underline{\delta u}.\left[(\underline{M}.\nabla)\underline{H}_0\right] d\tau + \int \delta\underline{\Theta}.\left[\underline{M}\wedge\underline{H}_0\right] d\tau + \int \underline{H}_0 . D\underline{M}' dm . \quad \ldots (22.3)$$

The work done by the surface tractions is

$$\delta W_t = \int \left\{ t_1 \delta u_1 + t_2 \delta u_2 + t_3 \delta u_3 \right\} dS$$

$$= \int \left\{ (t_{11} + 2\pi M_1^2 - 2\pi M_n^2) \delta u_1 l + t_{12} \delta u_1 m + t_{13} \delta u_1 n \cdots \right\} dS$$

$$= \int \left\{ \delta u_1 \left[(t_{11} + 2\pi M_1^2)_{,1} + t_{12,2} + t_{13,3} \right] + \cdots\cdots \right\} d\tau \quad \ldots (22.4)$$

$$- 2\pi \int M_n^2 \, \underline{n} . \underline{\delta u} \, dS$$

$$+ \int \left\{ (t_{11} + 2\pi M_1^2)(\delta u_1)_{,1} + t_{12} (\delta u_1)_{,2} + t_{13} (\delta u_1)_{,3} + \cdots\cdots \right\} d\tau .$$

Thus, by (18.3)

$$\delta W_t = \int - \underline{\delta u} . \left[(\underline{M}.\nabla) \underline{H} \right] d\tau - 2\pi \int M_n^2 \underline{n} . \underline{\delta u} \, dS \quad \ldots (22.4')$$

$$+ \int \left\{ (t_{11} + 2\pi M_1^2)(\delta u_1)_{,1} + t_{12} (\delta u_1)_{,2} + t_{13} (\delta u_1)_{,3} + \cdots\cdots \right\} d\tau .$$

Also,

$$\delta e_{11} = (\delta u_1)_{,1} \qquad \delta e_{12} = \tfrac{1}{2}\left((\delta u_1)_{,2} + (\delta u_2)_{,1}\right).$$

$$\delta \underline{\Theta} = \tfrac{1}{2}\,\mathrm{curl}\,\delta\underline{u} = \tfrac{1}{2}\Big((\delta u_3)_{,2} - (\delta u_2)_{,3},\ (\delta u_1)_{,3} - (\delta u_3)_{,1},$$
$$(\delta u_2)_{,1} - (\delta u_1)_{,2} \Big)$$

$$\therefore \begin{cases} (\delta u_2)_{,1} = \delta e_{12} + \delta \Theta_3 \\ (\delta u_1)_{,2} = \delta e_{12} - \delta \Theta_3 \end{cases},$$

and from (18.1)

$$t_{23} - t_{32} = (\underline{M} \wedge \underline{H})_1 \quad .$$

In terms of these quantities the last integral in (22.4) becomes

$$\int \left\{ (t_{11} + 2\pi M_1^2)\delta e_{11} + (t_{12} + t_{21})\delta e_{12} + \cdots - \underline{M}\wedge\underline{H}\cdot\delta\underline{\Theta} \right\} d\tau \,. \qquad (22.5)$$

Thus,

$$\delta W_m + \delta W_t = \int \left\{ \delta\underline{u}\cdot[(\underline{M}.\nabla)\underline{H}_0] + \delta\underline{\Theta}.(\underline{M}\wedge\underline{H}_0)\right\} d\tau$$

$$+ \int \underline{H}_0 . D\underline{M}'\, dm - \int \left\{ \delta\underline{u}.[(\underline{M}.\nabla)\underline{H}] + \delta\underline{\Theta}.(\underline{M}\wedge\underline{H})\right\} d\tau$$

$$- 2\pi \int M_n^2\, \underline{n}.\delta\underline{u}\, dS + \int \left\{ (t_{11} + 2\pi M_1^2)\delta e_{11} + (t_{12} + t_{21})\delta e_{12} + \cdots \right\} d\tau$$

... (22.6)

$$= \int \underline{H}_0 . D\underline{M}'\, dm - \int \left\{ \delta\underline{u}.[(\underline{M}.\nabla)\underline{H}_1] + \delta\underline{\Theta}.(\underline{M}\wedge\underline{H}_1)\right\} d\tau$$

$$- 2\pi \int M_n^2\, \underline{n}.\delta\underline{u}\, dS + \int \Big\{ (t_{11} + 2\pi M_1^2)\delta e_{11} + \cdots$$

$$+ (t_{12} + t_{21})\delta e_{12} + \cdots \Big\} d\tau \quad ,$$

where $\underline{H}_1 = \underline{H} - \underline{H}_0$ is the part of the magnetic intensity due to the magnetisation in τ.

Set

$$U_m = -\frac{1}{2}\int_\tau \underline{M}.\underline{H}_1 d\tau ,$$

then ~~(see App. A)~~ by (10-2) (with e = 0),

$$\delta U_m = -\int \underline{H}_1 . D\underline{M}' dm - 2\pi \int M_n^2 \underline{n}.\underline{\delta u}\, dS$$

$$- \int \left\{ \delta\underline{\Theta}.(\underline{M} \wedge \underline{H}_1) + \underline{\delta u}.[(\underline{M}.\nabla)\underline{H}_1] \right\} d\tau . \qquad \ldots (22.7)$$

Finally,

$$\delta W_m + \delta W_t = \delta U_m + \int \underline{H}.D\underline{M}' dm$$

$$+ \int \left\{ (t_{11} + 2\pi M_1^2)\delta e_{11} + \cdots (t_{12} + t_{21})\delta e_{12} + \cdots \right\} d\tau . \qquad \ldots (22.8)$$

A similar procedure based on the $\underline{B}$ equations leads to

$$\delta W_m + \delta W_t = \delta U_m' + \int \underline{B}.D\underline{M}' dm$$

$$+ \int \left\{ [t_{11} - 2\pi(M_2^2 + M_3^2)]\delta e_{11} + \cdots (t_{12} + t_{21})\delta e_{12} + \cdots \right\} d\tau , \qquad (22.8')$$

where $$U_m' = -\frac{1}{2}\int_\tau \underline{M}.\underline{B}_1 d\tau .$$

Set

$$dF \equiv \underline{H}.d\underline{M}' + V\left\{ (t_{11} + 2\pi M_1^2)de_{11} + \cdots + (t_{12} + t_{21})de_{12} + \cdots \right\} , \qquad (22.9)$$

$$dF' \equiv \underline{B}.d\underline{M}' + V\left\{ [t_{11} - 2\pi(M_2^2 + M_3^2)]de_{11} + \right. \qquad \ldots (22.9')$$

$$\left. + \cdots + (t_{12} + t_{21})de_{12} + \cdots \right\} ,$$

where $V = (1/\rho)$ is the specific volume. In a reversible isothermal or adiabatic change dF and dF' must be perfect differentials of quantities F, F' determined by the magnetisation and strains in the body. The relation between these functions is

$$F' - F = 2\pi M^2 V = \frac{2\pi M'^2}{V} . \qquad \ldots (22.10)$$

If $\underline{H}$ (or $\underline{B}$) is used in place of $\underline{M}$ as independent variable the function

$$Z = F - \underline{H}.\underline{M}' , \quad (\text{or } Z' = F' - \underline{B}.\underline{M}'), \qquad \ldots (22.11)$$

is more convenient. In the formulae for dZ and dZ' the first terms in (22.9) and (22.9') are replaced by $-\underline{M}'.D\underline{H}$ and $-\underline{M}'.D\underline{B}$ respectively. (Note that $D(\underline{M}'.\underline{H})$ and $d(\underline{M}'.\underline{H})$ are equal, as $\underline{M}'.\underline{H}$ is a scalar.) For a fluid the function Z reduces to that of Section 18: in this case

$$t_{12} = t_{21} = 0 ,$$

$$t_{11} + 2\pi M_1^2 = -p' ;$$

and $\underline{M}'$ is along $\underline{H}$ so that

$$\underline{M}'.D\underline{H} = \frac{M'}{H} \underline{H}.D\underline{H} ,$$

whence

$$\begin{aligned} dZ &= -M'dH - p'V(de_{11} + de_{22} + de_{33}) \\ &= -M'dH - p'dV . \end{aligned} \qquad \ldots (22.12)$$

Brown points out that the choice of $\underline{H}$, $\underline{B}$ or $\alpha \underline{H} + (1-\alpha)\underline{B}$ as the variable to be used is a matter of convenience only. No interpretation of any of these as an effective field intensity is either justified by the analysis or necessary for the derivation of useful results.

Brown then applies these formulae to the case of an elastic solid. For reversible behaviour the thermodynamic potential F may be taken to be a single valued function of the nine variables

$$M_1', M_2', M_3', e_{11}, e_{22}, e_{33}, e_{12}, e_{23}, e_{13} .$$

If it is so expressed the other nine variables are given by

$$H_1 = \frac{\partial F}{\partial M_1'} , \; ---- , \; V(t_{11} + 2\pi M_1^2) = \frac{\partial F}{\partial e_{11}} ,$$

$$V\,(t_{12} + t_{21}) = \frac{\partial F}{\partial e_{12}} . \qquad \ldots (22.13)$$

For many materials it is sufficient to express F as a power series in all nine variables and to neglect terms of higher order than the second. This procedure leads to formulae for $H_1, H_2, H_3, t_{11} + 2\pi M_1^2, t_{12} + t_{21}$ etc.

In the case of crystals, relations among the coefficients of the power series are determined by crystal symmetry. The relations are those worked out in detail by Voigt (18). Voigt himself developed his theory by a calculation of electric work in the absence of strain and of strain work in the absence of polarisation; he assumed that the incremental work was the sum of these expressions and that the interaction could be sufficiently taken into account by merely inserting product terms in the thermodynamic potential. In the light of Brown's work this procedure seems to be incorrect. The form of the relevant equations suggests that in many cases however the differences affect only second order terms in the final formulae and are therefore not significant over the range to which Voigt's theory applies. Nevertheless the difference of interpretation is important in principle and there may be cases in which the discrepancies are significant.

23. TOUPIN: MATHEMATICS OF FINITE DEFORMATIONS.

Toupin (15) uses general curvilinear coordinates throughout. In this account of Toupin's work only rectangular cartesian coordinates will be used; this will simplify the mathematics considerably.

Toupin uses two coordinate systems, one to describe the initial state and one to describe the final state of the medium. We shall use only one rectangular cartesian system but will denote initial state values by X^A and final state values by x^i .

The deformation will be specified by the relation

$$x^i = x^i(X^A) \; . \qquad \ldots (23.1)$$

Set

$$\frac{\partial x^i}{\partial X^A} \equiv x^i{}_{;A} \; ,$$

$$\frac{\partial X^A}{\partial x^i} \equiv X^A{}_{;i} \; .$$

If $T^{A\ldots}_{i\ldots}(X,x)$ is a tensor (i.e. a quantity subject to the transformation law

$$\bar{T}^{AB\ldots}_{i\ldots} = \frac{\partial \bar{X}^A}{\partial X^z} \frac{\partial x^a}{\partial \bar{x}^i} \cdots\cdots T^{z\ldots}_{a\ldots}$$

)

and if X and x are related by (23.1) then the derivative of $T^{A\ldots}_{i\ldots}$ with respect to X^B is defined as

$$T^{A\cdots}_{i\cdots;B} = \frac{\partial T^{A\cdots}_{i\cdots}}{\partial X^B} + \frac{\partial T^{A\cdots}_{i\cdots}}{\partial x^j} x^j_{;B} \quad . \qquad \ldots (23.2)$$

The derivative with respect to x^j is defined as

$$T^{A\cdots}_{i\cdots;j} = \frac{\partial T^{A\cdots}_{i\cdots}}{\partial x^j} + \frac{\partial T^{A\cdots}_{i\cdots}}{\partial X^B} X^B_{;j} \quad . \qquad \ldots (23.3)$$

Multiplying (23.3) by $x^j_{;C}$, we have

$$x^j_{;C} T^{A\cdots}_{i\cdots;j} = T^{A\cdots}_{i\cdots;j} x^j_{;C} + T^{A\cdots}_{i\cdots;B} \frac{\partial X^B}{\partial x^j} \frac{\partial x^j}{\partial X^C}$$

$$= T^{A\cdots}_{i\cdots;j} x^j_{;C} + T^{A\cdots}_{i\cdots;C} = T^{A\cdots}_{i\cdots;C} \quad ,$$

whence

$$T^{A\cdots}_{i\cdots;B} = T^{A\cdots}_{i\cdots;j} x^j_{;B} \quad ;$$

similarly

$$T^{A\cdots}_{i\cdots;j} = T^{A\cdots}_{i\cdots;B} X^B_{;j} \quad .$$

Consider the two points X^A and $X^A + dX^A$ of the medium in the initial state C_0. In the final state C the same two particles will have coordinates x^i and $x^i + dx^i$.

Since we refer to the same particles the coordinates will be related by (23.1). Thus

$$dx^i = dX^A x^i_{;A} \quad .$$

The square of the initial distance between the two particles is given by

$$ds_0^2 = \delta_{AB}\, dX^A dX^B = c_{ij}\, dx^i dx^j \quad , \qquad \ldots (23.4)$$

where

$$c_{ij} = \delta_{AB} X^A_{;i} X^B_{;j} \quad . \qquad \ldots (23.5)$$

The square of the distance between the particles in the final configuration is

$$ds^2 = \delta_{ij}\, dx^i dx^j = C_{AB}\, dX^A dX^B \, , \qquad \ldots (23.6)$$

where

$$C_{AB} = \delta_{ij}\, x^i_{;A}\, x^j_{;B} \, . \qquad \ldots (23.7)$$

The displacement u^A is given by

$$u^A = x^A - X^A \, ,$$

thus

$$\frac{\partial u^A}{\partial X^B} = \frac{\partial x^A}{\partial X^B} - \frac{\partial X^A}{\partial X^B}$$

$$\therefore \quad \frac{\partial x^A}{\partial X^B} = \frac{\partial u^A}{\partial X^B} + \delta_{AB}$$

$$x^i_{;B} = \frac{\partial u^i}{\partial X^B} + \delta_{iB} \quad ; \qquad \ldots (23.8)$$

also

$$\frac{\partial u^A}{\partial x^i} = \frac{\partial x^A}{\partial x^i} - \frac{\partial X^A}{\partial x^i} \, ,$$

and

$$X^A_{;i} = \delta_{Ai} - \frac{\partial u^A}{\partial x^i} \, . \qquad \ldots (23.9)$$

From (23.5) and (23.7)

$$C_{AB} = \delta_{ij} \left(u^i_{;A} + \delta_{Ai} \right)\left(u^j_{;B} + \delta_{jB} \right)$$

$$= \delta_{AB} + \left(u_{B;A} + u_{A;B} \right) + \delta_{ij}\, u^i_{;A}\, u^j_{;B} \, ,$$

and

$$c_{ij} = \delta_{ij} - \left(u_{i;j} + u_{j;i} \right) + \delta_{KL}\, u^K_{;i}\, u^L_{;j} \, . \qquad \ldots (23.10)$$

If $ds_0^2 = ds^2$ for all curves in C_0 then the distance between neighbouring points in C_0 is unchanged by the deformation. The difference $ds^2 - ds_0^2$ can be taken as a measure of strain in finite deformations. We will show that it is directly related to the usual strain measure for infinitesimal deformations.

The difference $ds^2 - ds_0^2$ can be expressed in terms of either initial coordinates X^A or final coordinates x^i :

$$ds^2 - ds_0^2 = (C_{AB} - \delta_{AB})\,dX^A dX^B$$

$$= (\delta_{ij} - c_{ij})\,dx^i dx^j \ .$$

Write

$$C_{AB} - \delta_{AB} \equiv 2\eta_{AB} \ ,$$

$$\delta_{ij} - c_{ij} \equiv 2e_{ij} \ ,$$

then,

$$2\eta_{AB} = C_{AB} - \delta_{AB} = u_{B;A} + u_{A;B} + \delta_{ij}\, u_{i;A}\, u_{j;B} \ ,$$

$$2\epsilon_{ij} = \delta_{ij} - c_{ij} = u_{i;j} + u_{j;i} - \delta_{KL}\, u_{K;i}\, u_{L;j} \ .$$

Neglecting products of "displacement gradients" we obtain the "infinitesimal" strains of classical theory

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right) \ .$$

24. TOUPIN: STATIC MECHANICAL EQUILIBRIUM OF CONTINUOUS MEDIA.

Consider a dielectric in a region V with boundary B. We assume that the medium is in static equilibrium with a set of mechanical surface tractions Π^i and an external field E_0^i. The external field will give rise to an extrinsic body force density F_0^i and an extrinsic moment density G_{ij}^0.

The components of the resultant force are given by

$$F^A_{ext} = \int_V F_0^A \, dV + \int_B \Pi^A \, dS \; . \qquad \ldots (24\text{-}1)$$

The particles of the medium also exert forces on each other: for example the cohesive forces which bind the medium into an elastic solid are of this type. Further if the dielectric is polarised the self-field of the dielectric will exert a force on a polarised particle.

Toupin makes the following stress hypothesis:

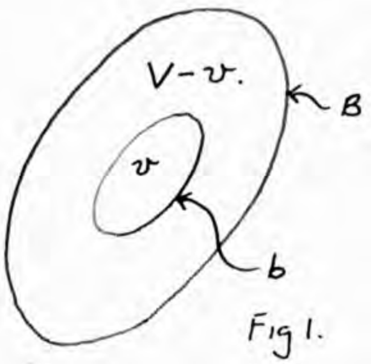


Fig 1.

Let v be an arbitrary regular region of space. This region may be entirely or partially contained in V or its intersection with V may be zero. The forces of interaction between particles contained in v and $V - v$ are equipollent to a system of stress vectors distributed over the surface of the region v. Let b denote the surface of v.

The stress vector field is a function depending only on the position on the surface of b and on the direction of the normal to b . Let t^i denote the field of stress vectors. Then $t^i = t^i(\underline{x}, \underline{n})$ where n^i are the components of the unit outward normal to the surface b at the point x^i . The resultant force exerted by the particles in the region $V - v$ on the particles in v is given by

$$F^A_{int} = \int_b t^A(x^i, n_i)\, dS . \qquad \ldots(24\text{-}2)$$

The total force on the particles contained in the region v is the sum of the resultant extrinsic force (24-1) and the resultant interparticle force (24-2). The total on the arbitrary region v (See Fig 2).

$$F^A = \int_b t^A dS + \int_{v \cap B} \Pi^A dS + \int_{v \cap V} F_0^A dV . \qquad \ldots(24\text{-}3)$$

If the medium is in static mechanical equilibrium this total force vanishes for an arbitrarily chosen region v . Applying this condition of equilibrium to a tetrahedron with a vertex which is not a point of B and proceeding in the usual way we can demonstrate the existence of a <u>stress tensor</u> field $t^{ij}(x)$ such that

Fig 2

$$t^i(\underline{x}, \underline{n}) = t^{ij}(x)\, n_j . \qquad \ldots(24\text{-}4)$$

We may also apply the condition of vanishing total force to a pill box region which contains points of the surface B. In the usual way we obtain

$$[\![t^{ij}]\!] n_j + \Pi^i = 0 \, , \qquad \ldots(24\text{-}5)$$

where $[\![t^{ij}]\!] = t^{+ij} - t^{-ij}$; where t^{+ij} and t^{-ij} are the limiting values of the stress tensor as the surface B is approached from the exterior and interior of the dielectric respectively. The resultant moment about the origin exerted on the dielectric by the surface tractions and external electric field is given by

$$G^{AB}_{ext} = \int_V G^{AB}_{o}\, dV + \int_B (r^A \Pi^B - r^B \Pi^A)\, dS$$
$$+ \int_V (r^A F^B_o - r^B F^A_o)\, dV \, . \qquad \ldots(24\text{-}6)$$

The interparticle interaction gives rise to a resultant moment on the region v which is given by

$$G^{AB}_{int} = \int_b (r^A t^B - r^B t^A)\, dS \, . \qquad \ldots(24\text{-}7)$$

The total moment on the region v is given by

$$G^{AB} = \int_{v \cap B} (r^A \Pi^B - r^B \Pi^A)\, dS + \int_{v \cap V} (r^A F^B_o - r^B F^A_o)\, dV$$
$$+ \int_{v \cap V} G^{AB}_{o}\, dV + \int_b (r^A t^B - r^B t^A)\, dS \, . \qquad \ldots(24\text{-}8)$$

If the medium is in static mechanical equilibrium this total moment on the arbitrary region v must vanish. From these integral forms of the conditions of equilibrium it follows that

$$t^{ij}_{\ ,j} + F_0^{\ i} = 0 \qquad x \in V - B$$

$$t^{ij}_{\ ,j} = 0 \qquad x \in E - V - B \qquad \ldots(24\text{-}9)$$

$$t^{ji} - t^{ij} + G_0^{\ ij} = 0 \qquad x \in V - B$$

$$t^{ij} - t^{ji} = 0 \qquad x \in E - V - B \qquad \ldots(24\text{-}10)$$

where E denotes all Euclidean space.

25. TOUPIN: THE FIELD IN A DIELECTRIC.

If we consider a volume distribution of dielectric with polarisation density $\underline{P}$ in a volume v_2 and a volume distribution of charge, density e, in a volume v_1, then the electric potential ϕ is given by

$$\phi = \int_{v_1} \frac{e\,dv}{r} + \int_{v_2} \underline{P}\cdot\nabla\left(\frac{1}{r}\right) dv. \qquad \ldots(25\text{-}1)$$

Toupin defines $\underline{E}_M$ as the negative gradient of ϕ. Toupin's $\underline{E}_M$ is the same as Smith-White's $\underline{E}$ (see Section 9)

$$\underline{E}_M = -\nabla\phi . \qquad \ldots(25\text{-}1')$$

He splits up this field into two parts as follows:

set

$$\phi_{MS} = \int_{v_2} \underline{P}\cdot\nabla\left(\frac{1}{r}\right) dv,$$

$$\phi_0 = \int_{v_1} \frac{e\,dv}{r.} \quad ;$$

then $\underline{E}_M = \underline{E}_{MS} + \underline{E}_0$,

where $\underline{E}_{MS} = -\nabla\phi_{MS}$,

$$\underline{E}_0 = -\nabla\phi_0 .$$

$\underline{E}_0$ is the field arising from sources outside the dielectric v_2 and is called the ~~extreme~~ extrinsic field. $\underline{E}_{MS}$ is the field arising from the dielectric itself and is called the Maxwell self field.

Lorentz has calculated the electrostatic field of an array of point dipoles arranged on a uniform space lattice (23 pp. 305 - 308); he has shown that the self electric field of a lattice having cubic symmetry when evaluated at a lattice point is given by

$$\underline{E}_S = \frac{4\pi}{3}\underline{P} + \underline{E}_{MS} . \qquad \ldots(25\text{-}2)$$

Whatever the symmetry we shall write E^i_S in the form

$$E^i_S = E^i_L + E^i_{MS} \qquad \ldots(25\text{-}3)$$

where E^i_L is termed the Lorentz local field. From (25-2) we see that when the lattice is cubically symmetrical then $E^i_L = \frac{4\pi}{3} P^i$.

If a cubic lattice is deformed homogeneously and the deformed lattice has lower symmetry than cubic, then the local field will have a value which differs from $\frac{4\pi}{3} P^i$ Hence the local field is a function of the lattice deformation. This result lends support to the following hypothesis.

In the continuum theory Toupin sets down as a primitive assumption that

$$E^i_S = E^i_{MS} + E^i_L(x^i_{;A}, P^i) . \qquad \ldots(25\text{-}4)$$

That is the electrostatic self field of a polarised and deformed continuous elastic dielectric is the sum of the Maxwell electrostatic self field and a local field which is a state function of the displacement gradients and polarisation density.

The lattice model described makes this assumption very plausible. Toupin assumes that the relation (25-4) holds not only in dielectrics having a crystal structure but also in elastic dielectrics such as rubber or plastic.

Toupin takes a volume distribution of dipoles as a model of an elastic dielectric. Each dipole consists of two equal electric charges of opposite sign separated along a line parallel to the polarisation vector. If this dipole is in static mechanical equilibrium, the forces which act on either charge must have a zero resultant. The electrical force which acts on a charge e placed in an electrostatic field $\underline{E}$ is $e\underline{E}$. The electrostatic field which acts on the charge in a dipole in an elastic dielectric has three distinct components:

(1) the Maxwell self electric field $\underline{E}_{MS}$,

(2) the Lorentz local field $\underline{E}_L$,

(3) the external field $\underline{E}_0$.

In addition to the resultant electrostatic force other forces act on the charged particle. These are the molecular forces which are made up of the Coulomb attraction between the charges of the particle, dynamical forces and other non classical or quantum forces. Let $e\underline{F}$ denote the resultant of these molecular forces. Then at static equilibrium applying the Newtonian law of force balance

$$e\,(F^i + E^i_L + E^i_{MS} + E^i_0) = 0 . \qquad \ldots(25\text{-}5)$$

Set $F^i + E^i_L = \bar{E}^i_L$ where $\bar{E}^i_L$ will be termed the effective local field. Toupin assumes that the effective local field at a point in an elastic dielectric is a function of the displacement gradients $x^i_{;A}$ and polarisation density at that point. Thus we have

$$\bar{E}^i(x^i_{;A}\ P^i) + E_0^i + E^i_{MS} = 0 \ . \qquad \ldots(25\text{-}6)$$

Toupin calls (25-6) the equation of <u>intramolecular force</u> balance. The total Maxwell field is just

$\underline{E}_M = \underline{E}_{MS} + \underline{E}_0$, so that (25-6) can be written in the form

$$\bar{E}^i_L(x^i_{;A}, P^i) + E^i_M = 0 \ . \qquad \ldots(25\text{-}7)$$

26. TOUPIN: EQUILIBRIUM RELATIONS.

Toupin assumes that the stress tensor in a polarised elastic dielectric has the form

$$t^{ij} = t_L^{ij}(x^i_{;A}, P^i) + S_{MS}^{ij} ; \qquad \ldots(26.1)$$

where S_{MS}^{ij} is the Livens-Smith-White tensor

$$S_{MS}^{ij} = E_{MS}^{i} D_{MS}^{j} - \frac{\delta_{ij}}{8\pi} E_{MS}^{2} , \qquad \ldots(26.2)$$

(this corresponds to a force density $(\underline{P}.\nabla)\underline{E}_{MS}$) and where t_L^{ij} is a state function called the local stress. We note that the stress tensor S_{MS}^{ij} does not vanish at points outside the dielectric. The resultant electrostatic force on any region which lies entirely outside the dielectric is zero. The local stress is assigned the value zero outside the dielectric.

Toupin uses the Livens-Smith-White formulae for the force and couple density due to the external field $\underline{E}_0$

$$\underline{F}_0 = (\underline{P}.\nabla)\underline{E}_0 , \qquad \ldots(26.3)$$

$$\underline{G}_0 = \underline{P}\wedge\underline{E}_0 . \qquad \ldots(26.4)$$

We have $S_{MS\,,j}^{ij} = E^{i}_{MS\,,j} P^{j}$ (see § 13)

Also, (see App. A-29)

$$[S_{MS}^{ij}]\, n_j = \frac{1}{8\pi}(P_j n_j)^2 n_i . \qquad \ldots(26.5)$$

Substituting these values in (24-9), (24-10) we obtain the following relations

$$t^{ij}_{L,j} + s^{ij}_{MS,j} + E^{i}_{o,j}\, P^{j} = 0 , \qquad \ldots(26.6)$$

$$t^{ji}_{L} - t^{ij}_{L} + s^{ij}_{MS} - s^{ji}_{MS} + $$
$$+ E^{j}_{o} P^{i} - E^{i}_{o} P^{j} = 0 , \qquad \ldots(26.7)$$

$$\bar{E}^{i}_{L} + E_{o}^{i} + E^{i}_{MS} = 0 . \qquad \ldots(26.8)$$

We may write these in the form

$$t^{ij}_{L,j} + E^{i}_{M,j}\, P^{j} = 0 , \qquad \ldots(26.9)$$

$$t^{ji}_{L} - t^{ij}_{L} + E^{j}_{M} P^{i} - E^{i}_{M} P^{j} = 0 , \qquad \ldots(26.10)$$

$$\bar{E}^{i}_{L} + E^{i}_{M} = 0 . \qquad \ldots(26.11)$$

Substituting (26-5) in the boundary condition (24-5) we have

$$- t^{-ij}_{L} n_j + \frac{1}{8\pi}(P_j n_j)^2 n^i + \Pi^i = 0 , \qquad \ldots(26.12)$$

where t^{-ij}_{L} is the limiting value of the local stress as the boundary of the dielectric is approached from the interior.

We also have the following equations

$$\nabla \cdot \underline{D}_{MS} = E^{i}_{MS,i} + 4\pi P^{i}_{,i} , \qquad \ldots(26.13)$$

$$\left[D^i_{MS} \right] n^i = 0 = \left[E^i_{MS}\, n_i \right] + 4\pi \left[P^i n_i \right]. \qquad \ldots(26.14)$$

Finally in addition to the above equations (26-9);-(26-12) two sets of constitutive relations characteristic of the material must be given. These are

$$t_L^{ij} = t_L^{ij}\left(x^i_{;A}, P^i \right), \qquad \ldots(26.15)$$

$$E^i_L = E^i_L\left(x^i_{;A}, P^i \right). \qquad \ldots(26.16)$$

The form of these constitutive relations for the local stress and effective local field is restricted by certain symmetry properties of the material.

27. TOUPIN: VIRTUAL WORK FOR THE ELASTIC DIELECTRIC.

In classical infinitesimal elasticity theory there are two methods of deriving stress-strain relations;

(1) Cauchy's method. We assume that the components of the stress tensor are some general functions of the strains e.g. $t_{ij} = c_{ijkl}\, e_{kl}$ where c_{ijkl} is some fourth order tensor.

(2) Green's method. We use the method of virtual work and derive the stresses in terms of the derivatives of an energy function W with respect to the strains:

$$\frac{\partial W}{\partial e_{ij}} = t_{ij} .$$

Green's method is a special case of Cauchy's method. Toupin generalises Green's method to include the case of an elastic dielectric. The natural state of an elastic dielectric is the equilibrium state which the material assumes in the absence of applied surface tractions and external electric field. If X^A and x^i are the rectangular cartesian coordinates of the same particle in the natural and deformed states, then

$$x^i = x^i(X^A) . \qquad \ldots(27\text{-}1)$$

As X^A ranges over the region V_0 occupied by the body in its natural state then the correspondence (27-1) constitutes a continuous mapping of the region V_0 onto the region V occupied by the body in its deformed and polarised state.

Let B_0 and B denote the boundary of the dielectric in the natural and deformed states respectively. The total mass of the body is given by

$$M = \int_V \rho \, dV . \qquad \ldots(27\text{-}2)$$

Let

$$J = \det x^i{}_{;A} . \qquad \ldots(27\text{-}3)$$

The law of conservation of mass may be stated in the form

$$\rho_0 (X^A) = J \rho (x^i(X^A)) , \qquad \ldots(27\text{-}4)$$

where ρ_0 is the density in the natural state.

If $\underline{P}'$ is the polarisation <u>per unit mass</u> then

$$P^i = \rho P'^i , \qquad \ldots(27\text{-}5)$$

where $\underline{P}$ is the polarisation <u>per unit volume</u>. Let Σ denote the stored energy function of deformation and polarisation (i.e. the stored energy for unit mass). Then Σ is a state function

$$\Sigma = \Sigma (x^i{}_{;A} , P'^i) . \qquad \ldots(27\text{-}6)$$

Toupin's equation of virtual work for the elastic dielectric is

$$\delta \left[- \int_V \rho \, \Sigma (x^i{}_{;A} , P'^i) + \frac{1}{8\pi} \int_E \phi_{;i} \phi_{;i} \, dV \stackrel{*}{-} \int_V \phi_{;i} P^i \, dV \right]$$

$$\ldots(27\text{-}7)$$

$$+ \int_B \Pi^i \delta' x^i \, dS + \int_V F_0{}^i \delta' x^i \, dV + \int_V \rho E_0^i \delta'' P'^i \, dV = 0 .$$

* There is a misprint in Toupin's original paper; his $+$ sign should be $-$ if his equation (27-11) is to hold.

Toupin's explanation of this equation is as follows:

"The last three integrals in this variational expression represent respectively the work done by the applied surface tractions if the general point on the boundary of the dielectric is displaced from its equilibrium position by a small amount $\delta' x^i$, the work done by the body force if the general point in the dielectric is displaced from its equilibrium position by a small amount $\delta' x^i$ and the work done by the external field in increasing the polarisation at the general point by a small amount $\delta'' P'^i$ from its equilibrium value. The sum of these three virtual work terms is set equal to the variation in potential energy of the elastic dielectric. This potential energy is written as the sum of three parts which are enclosed in the large brackets in (27-7). The first of these terms represents the variation in the stored energy of deformation and polarisation. This term is quite analogous to the stored elastic energy of elasticity theory. The second term is the variation in the potential energy of the self electric field. The third term in the bracket represents an interaction energy between the self field and a polarised particle of the dielectric. The independent variations of the field variables are now listed.

$$x^i \rightarrow x^i + \delta' x^i \,, \qquad \ldots(27\text{-}8)$$

$$p^{/i} \rightarrow p^{/i} + \delta'' p^{/i} \,, \qquad \ldots(27\text{-}9)$$

$$\phi \rightarrow \phi + \delta''' \phi \;; \qquad \ldots(27\text{-}10)$$

where $\underline{E}_{MS} = -\nabla \phi$.

The total variation in large brackets (27-7) means the resultant first-order change in the values of these integrals under the replacements (27-8) (27-9) (27-10)." Toupin performs the variation and obtains (27- 7) in the form

$$\int_{B_0} \Big\{ \Big[-\varrho_0 \frac{\partial \Sigma}{\partial x^i_{;A}} + J(S^{+ij}_{MS} - S^{-ij}_{MS}) X^A_{;j} + \pi^i N^A \Big(\frac{dS}{dS_0}\Big) \Big] \delta' x^i$$

$$+ \Big[\frac{(E^{+j}_{MS} - E^{-j}_{MS} - P^j) X^A_{;j}}{4\pi} \Big] J \delta''' \phi \Big\} N_A \, dS_0$$

$$+ \int_{V_0} \Big\{ \varrho_0 \Big[-\frac{\partial \Sigma}{\partial p^{/j}} + E_{MS\,j} + E_{0j} \Big] \delta'' p^{/j}$$

$$+ \Big[\Big(\varrho_0 \frac{\partial \Sigma}{\partial x^i_{;A}}\Big)_{;A} + J S^{ij}_{MS;j} + J F_0^{\,i} \Big] \delta' x^i \qquad \ldots(27\text{-}11)$$

$$+ \frac{J}{4\pi} \Big[-\nabla^2 \phi + 4\pi \nabla \cdot \underline{P} \Big] \delta''' \phi \, dV_0$$

$$+ \int_{E-V_0} \Big\{ \overset{*}{-} \frac{1}{4\pi} \nabla^2 \phi \, \delta''' \phi + S^{ij}_{MS;j} \, \delta' x^i \Big\} J \, dV_0 = 0 \,,$$

where N^A are the components of the outward unit normal to the surface of the undeformed dielectric. The ratio of

* There is a misprint in the original Toupin paper; a — sign is missing.

the magnitudes of the surface elements dS/dS_0 is given by

$$dS/dS_0 = J\sqrt{(C^{-1})^{AB} N_A N_B} = J\Big/\sqrt{(c^{-1})^{ij} n_i n_j}\ . \qquad \ldots(27\text{-}12)$$

Toupin obtains the following field equations

$$J^{-1}\left(\rho_0 \frac{\partial \Sigma}{\partial x^i_{;A}}\right)_{;A} + s^{ij}_{MS;j} + E^i_{0;j}\, P^j = 0, \qquad \ldots(27\text{-}13)$$

$$-\frac{\partial \Sigma}{\partial P^i} + E^i_{MS} + E^i_0 = 0, \qquad \ldots(27\text{-}14)$$

$$-\nabla^2\phi + 4\pi\,\nabla\cdot\underline{P} = 0\ . \qquad \ldots(27\text{-}15)$$

The boundary condition is given by

$$-\rho\, \frac{\partial \Sigma}{\partial x^i_{;A}} N_A - \left(s^{-ij}_{MS} - s^{+ij}_{MS}\right) X^A_{;j} N_A \qquad \ldots(27\text{-}16)$$

$$+\ \Pi^i\sqrt{(C^{-1})^{AB} N_A N_B} = 0\ .$$

In Section 36 the writer has given a simpler method of deriving these expressions.

Using the identities

$$(J X^A_{;i})_{;A} = 0\ , \quad (J^{-1} x^i_{;A})_{;i} = 0\ ;$$

and the relation

$$n_i = J N_A X^A_{;i}$$

between the components of the unit normals to the deformed and undeformed dielectric, the following equations can be obtained:

$$\left(\rho \frac{\partial \Sigma}{\partial x^i_{;A}} x^j_{;A} \right)_{;j} + s^{ij}_{MS;j} + E_{0i;j} P_j = 0 , \qquad \ldots(27\text{-}17)$$

$$- \frac{\partial \Sigma}{\partial P'^i} + E^i_{MS} + E_0^{\,i} = 0 , \qquad \ldots(27\text{-}18)$$

$$- \nabla^2 \phi + 4\pi \nabla . \underline{P} = 0 , \qquad \ldots(27\text{-}19)$$

$$- \rho \frac{\partial \Sigma}{\partial x^i_{;A}} x^j_{;A} n_j + [s^{ij}_{MS}] n_j + \Pi^i = 0 . \qquad \ldots(27\text{-}20)$$

Toupin identifies the local stress and effective local field as the expressions

$$t^{ij}_L = \rho \frac{\partial \Sigma}{\partial x^i_{;A}} x^j_{;A} , \qquad \ldots(27\text{-}21)$$

$$\bar{E}^i_L = - \frac{\partial \Sigma}{\partial P'^i} . \qquad \ldots(27\text{-}22)$$

The set of equilibrium conditions obtained above then becomes identical with the set obtained in Section 26.

Toupin shows that the moment equation (26-10) is satisfied identically if the stored energy function is invariant under a rigid rotation of the deformed and polarised dielectric. His proof is based on a result from the theory of invariant functions of several vectors.

Let $F\,(V_1^i, V_2^i, \cdots, V_n^i)$ be a function of the components $V_r^i \; (r = 1, 2, \cdots, n)$ which is invariant under the substitutions

$$V_r^i \rightarrow R^i_j \, V_r^j$$

where R^i_j is an arbitrary rotation, i.e.

$$R^i_k R^i_\ell = \delta_{k\ell} \quad \text{and} \quad \det R^i_j = 1 .$$

An infinitesimal rotation has the form

$R^i_j = \delta^i_j + \epsilon^i_j$ where ϵ^{ij} is an arbitrary antisymmetric tensor. Since

$$F(V_1^i, \ldots, V_n^i) = F(R^i_j V_1^j, \ldots, R^i_j V_n^j)$$

$$= F(\bar{V}_1^i, \ldots, \bar{V}_n^i) .$$

we have as necessary conditions

$$dF = \frac{\partial F}{\partial \bar{V}^i} \frac{\partial \bar{V}^i}{\partial R^k_\ell} dR^k_\ell = 0 .$$

Hence for differentials ϵ^k_ℓ about the values $R^k_\ell = \delta^k_\ell$ the above condition is

$$\frac{\partial F}{\partial V^i} V^j \epsilon^i_j = 0 \quad ;$$

where ϵ_{ij} is an arbitrary antisymmetric tensor. This condition implies that the coefficients of ϵ_{ij} in this expression are the components of a symmetric tensor. We use the notation $T^{[i \cdots k]}$ to denote the antisymmetric part of a tensor. Thus we obtain the necessary conditions

$$\sum \frac{\partial F}{\partial V_{r[i}} V_r^{j]} = 0 \quad . \qquad \ldots(27\text{-}23)$$

It can be shown that the conditions (27-23) are sufficient to ensure the invariance of F under finite rotations. If the deformed and polarised state of the elastic dielectric is rotated rigidly in space the displacement derivatives and polarisation vector change to new values given by

$$x^i{}_{;A} \rightarrow R_{ij}\, x^j{}_{;A} \;,$$

$$p'^i \rightarrow R_{ij}\, p'^j \;.$$

Hence if we assume that the stored energy function of deformation and polarisation is invariant under a rigid rotation of the deformed and polarised state it follows from the above theorem that

$$\frac{\partial \Sigma}{\partial x^{[i}{}_{;A}}\, x^{j]}{}_{;A} + \frac{\partial \Sigma}{\partial p'^{[i}}\, p'^{j]} = 0 \;.$$

Multiplying this equation by ρ and using (27-21) and (27-22) it may be put in the form

$$t_L^{[ij]} - \bar{E}_L^{[i} p^{j]} = 0 \;. \qquad \ldots(27\text{-}24)$$

This equation embodies the physical idea that the moment exerted by the local stresses is balanced by the moment exerted by the local field $\underline{E}_L$ acting on the polarised particle. (27-24) is identically satisfied whether we are in an equilibrium state or not. If we use the equilibrium condition (26-11) $\bar{E}^i_L + E^i_M = 0$ then the moment equation (26-10) is satisfied identically.

Toupin has thus shown that the variational principle (27-7) gives the same field equations and boundary conditions as were obtained from equilibrium considerations. The importance of this energy function method is that it imposes considerable restrictions on the constitutive relations for effective local field and local stress. Toupin concludes:

"As in elasticity theory, it is probable that for many elastic dielectrics the restrictions imposed on the constitutive relations by using (27-21) and (27-22) instead of the Cauchy forms (26-15) and (26-16) are desirable and are actually borne out by experiment."

28. TOUPIN: POLYNOMIAL APPROXIMATIONS FOR THE STORED ENERGY FUNCTION.

Toupin considers the following polynomial form for the energy function

$$\rho_0 \Sigma = H_0^{A} \bar{P}_A + H_1^{AB} \bar{P}_A \bar{P}_B + H_2^{AB} E_{AB}$$

$$+ H_3^{ABCD} E_{AB} E_{CD} + H_4^{ABC} E_{AB} \bar{P}_C + H_5^{ABCD} E_{AB} \bar{P}_C \bar{P}_D \qquad \ldots (28.1)$$

$$+ H_6^{ABCDE} E_{AB} E_{CD} \bar{P}_E + H_7^{ABCDEF} E_{AB} E_{CD} \bar{P}_E \bar{P}_F ,$$

where $E_{AB} = C_{AB} - \delta_{AB}$ is the tensor measure of strain, $\bar{P}_A = x^i_{;A} P^{\prime}_L$ and $H_\Gamma^{AB\cdots}$ $(\Gamma = 1, 2, \cdots, 7)$ are independent of the E_{AB} and $\bar{P}_A$. Toupin calls the tensors $H_\Gamma^{AB\cdots}$ "material descriptors"; in several important cases, many of these vanish identically owing to the material symmetry. For example, if the natural state is isotropic then the tensors H_Γ must be isotropic so that the terms $H_4^{ABC} E_{AB} \bar{P}_C$ and $H_6^{ABCDE} E_{AB} E_{CD} \bar{P}_E$ vanish.

Toupin's general expressions for local stress and local field have been shown to be

$$t_L^{ij} = \rho \frac{\partial \Sigma}{\partial x^i_{;A}} x^j_{;A} , \qquad \ldots (28.2)$$

$$E_L^{i} = - \frac{\partial \Sigma}{\partial P^{\prime}_L} . \qquad \ldots (28.3)$$

Set

$$x^j_{;C} \frac{\partial E_{AB}}{\partial x^i_{;C}} = x^i_{;B} x^j_{;A} + x^i_{;A} x^j_{;B} \equiv M^{ij}_{AB},$$

$$x^j_{;B} \frac{\partial \overline{P}_A}{\partial x^i_{;B}} = x^j_{;A} P'_i \equiv N^{ij}_A ,$$

$$\frac{\partial \overline{P}_A}{\partial P'_i} = x^i_{;A} .$$

Using (28.1) in (28.2) and (28.3) we have

$$\begin{aligned} \frac{\rho_0}{\rho} t^{ij}_L &= H_0^{A} N^{ij}_A + 2H_1^{AB} N^{ij}_A \overline{P}_B + H_2^{AB} M^{ij}_{AB} \\ &+ 2H_3^{ABCD} M^{ij}_{AB} E_{CD} + H_4^{ABC} M^{ij}_{AB} \overline{P}_C + H_4^{ABC} E_{AB} N^{ij}_C \\ &+ H_5^{ABCD} M^{ij}_{AB} \overline{P}_C \overline{P}_D + 2H_5^{ABCD} E_{AB} \overline{P}_C N^{ij}_D \\ &+ 2H_6^{ABCDE} E_{AB} M^{ij}_{CD} \overline{P}_E + H_6^{ABCDE} E_{AB} E_{CD} N^{ij}_E \\ &+ 2H_7^{ABCDEF} E_{AB} \overline{P}_E \overline{P}_F M^{ij}_{CD} + 2H_7^{ABCDEF} E_{AB} E_{CD} \overline{P}_E N^{ij}_F , \end{aligned} \quad \ldots (28.4)$$

$$\begin{aligned} \rho_0 \overline{E}^i_L = -\Big[& H_0^{A} x^i_{;A} + 2H_1^{AB} x^i_{;A} \overline{P}_B + H_4^{ABC} E_{AB} x^i_{;C} \\ &+ 2H_5^{ABCD} E_{AB} \overline{P}_C x^i_{;D} + H_6^{ABCDE} E_{AB} E_{CD} x^i_{;E} \\ &+ 2H_7^{ABCDEF} E_{AB} E_{CD} \overline{P}_E x^i_{;F} \Big]. \end{aligned} \quad \ldots (28.5)$$

It follows from (28.4) that if the local stress is to vanish in the natural state we must have $H_2^{AB} = 0$. Similarly from (28.5) we must have $H_0^{A} = 0$ if the effective local field is to vanish in the natural state.

The relations (28.4) and (28.5) are quite general and hold for finite as well as infinitesimal deformations. In the rest of this section we will deal only with infinitesimal deformations. It will be easier in this case to work with the $u^i_{;j}$ rather than the $x^i_{;A}$ as strain variables. The expressions (28.4) and (28.5) will now be expressed in terms of the $u^i_{;j}$ and P'_i where the variables $u^i_{;j}$ are regarded as infinitesimals.

We have the following relations

$$M^{ij}_{AB} \doteq (\delta_{iA}\delta_{jB} + \delta_{iB}\delta_{jA}) + (\delta_{iA}\delta_{KB} + \delta_{iB}\delta_{KA})u^j_{;K}$$

$$+ (\delta_{iA}\delta_{KB} + \delta_{jB}\delta_{KA})u^i_{;K} \, ,$$

$$N^{ij}_{A} \doteq \delta_{jA}P'_j + \delta_{KA}u^j_{;K}P'_i \, ,$$

$$x^i_{;A} \doteq \delta_{iA} + \delta_{KA}u^i_{;K} \, .$$

Substituting these approximate expressions for M^{ij}_{AB}, N^{ij}_{A} and $x^i_{;A}$ in (28.4) and (28.5) we have

$$\frac{\rho_0}{\rho} t^{ij}_L = 2H_1^{jK}P'_iP'_K + 2H_1^{jK}u^l_{;K}P'_iP'_l$$

$$+ 2H_1^{Kl}u^j_{;K}P'_iP'_l + 4H_3^{ijKl}e_{Kl} + 2H_4^{ijK}P'_K$$

$$+ 2H_4^{ijk} u^l_{;k} P'_l + 2H_4^{ikl} u^j_{;k} P'_l + 2H_4^{jkl} u^i_{;k} P'_l$$

$$+ H_4^{klj} e_{kl} P'_i + 2H_5^{ijkl} P'_k P'_l$$

$$+ 2H_5^{imkl} u^j_{;m} P'_k P'_l + 2H_5^{jmkl} u^i_{;m} P'_k P'_l$$

$$+ 4H_5^{ijkl} u^m_{;l} P'_k P'_m + 2H_5^{klmj} e_{kl} P'_m P'_i \quad \ldots (28.6)$$

$$+ 2H_6^{klijm} e_{kl} P'_m + 2H_7^{klijmn} e_{kl} P'_m P'_n$$

$$\rho_0 \bar{E}^i_L \doteq -\left[2H_1^{ik} P'_k + 2H_1^{ik} u^l_{;k} P'_l + 2H_1^{kl} u^i_{;k} P'_l\right.$$

$$\left.+ H_4^{kli} e_{kl} + 2H_5^{klmi} e_{kl} P'_m\right] , \quad \ldots (28.7)$$

where $2e_{ij} = u_{i;j} + u_{j;i}$.

For infinitesimal displacements we also have

$\rho_0/\rho = 1 + e_{kk}$; hence this factor may be cleared from the expression for the stress by multiplying each term on the right which does not already contain a displacement gradient by the factor $(1 - e_{kk})$. Toupin points out that even for infinitesimal displacement gradients the components of the local stress and effective local field do not in general reduce to polynomials in the symmetric part of the displacement gradients only, as is sometimes assumed.

If the equations are completely linearised by dropping all terms involving squares of the polarisation or a product of a polarisation component and a displacement gradient we obtain the linear relations

$$t_L^{ij} \doteqdot 4H_3{}^{ijkl}e_{kl} + 2H_4{}^{ijk}p'_k , \qquad \ldots (28.8)$$

$$\rho_0 \bar{E}_L^i \doteqdot -\left[2H_1{}^{ik}p'_k + H_4{}^{kli}e_{kl}\right]. \qquad \ldots (28.9)$$

At static equilibrium we have from (25.3)

$$\bar{\underline{E}}_L + \underline{E}_M = 0 ,$$

where $\underline{E}_M$ is the total Maxwell field at a point inside the dielectric. This is the field which occurs in Voigt's relations (18). Also the total stress t^{ij} which is always symmetric if we neglect the Maxwell stress tensor (a legitimate approximation in this linearised theory since it always involves the square of the field or polarisation) must be assumed to be the stress tensor referred to in Voigt's theory since the concept of a local stress is not introduced. From (28.8) and (28.9) it follows that, at <u>static</u> equilibrium,

$$t^{ij} \doteqdot c^{ijkl}e_{kl} + q^{ijk}p_k , \qquad \ldots (28.10)$$

$$E_M^i \doteqdot (\chi^{-1})^{ik}p_k + p^{kli}e_{kl} ; \qquad \ldots (28.11)$$

where

$$c_{ijkl} = 4H_3{}^{ijkl} , \quad q^{ijk} = \left(\frac{2}{\rho_0}\right)H_4{}^{ijk}$$

$$(\chi^{-1})^{ij} = \left(\frac{2}{\rho_0^2}\right)H_1{}^{ij} , \quad p^{kli} = \frac{1}{\rho_0}H_4{}^{kli} .$$

The linear relations (28.10) and (28.11) are identical in form with the piezoelectric relations proposed by Voigt.

For a dielectric whose natural state is isotropic the material descriptors $H_{\Gamma}^{AB\cdots}$ must be isotropic tensors. The most general forms allowed by isotropic symmetry are as follows

$$
\begin{aligned}
2H_1^{ij} &= a_1 \delta_{ij} , \\
4H_3^{ijkl} &= \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) , \\
2H_5^{ijkl} &= a_2 \delta_{ij}\delta_{kl} + a_3(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) , \\
2H_7^{ijklmn} &= a_4 \delta_{ij}\delta_{kl}\delta_{mn} \\
&\quad + a_5(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})\delta_{mn} \qquad \ldots (28.12) \\
&\quad + a_6\left[(\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in})\delta_{kl} + (\delta_{km}\delta_{ln} + \delta_{lm}\delta_{kn})\delta_{ij}\right] \\
&\quad + a_7\left[\delta_{jk}(\delta_{im}\delta_{ln} + \delta_{in}\delta_{lm}) + \delta_{ik}(\delta_{jm}\delta_{ln} + \delta_{jn}\delta_{lm})\right. \\
&\quad \left. + \delta_{jl}(\delta_{im}\delta_{kn} + \delta_{in}\delta_{km}) + \delta_{il}(\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km})\right] .
\end{aligned}
$$

It can be shown from (28.12) that nine scalar material descriptors ($\lambda, \mu, a_1, \cdots, a_7$) are necessary to determine the special form of the stored energy function (28.1) in the isotropic case. If the dielectric is also homogeneous these nine descriptors will be spatially constant. It is appropriate to call these scalars <u>material constants</u>. Substituting (28.12) in (28.6) and (28.7) we obtain

$$t^{ij}_{L} \doteq \Big[\lambda e_{KK} + a_2 (P')^2 + 2(a_2 + a_6)\, \underline{P}'.\underline{e}.\underline{P}'$$

$$+ (a_4 - a_2)\, e_{KK} (P')^2 \Big] \delta_{ij}$$

$$+ \left[2\mu + 2(a_2 + a_5)(P')^2\right] e_{ij} + \qquad \ldots (28.13)$$

$$+ \left[(a_1 + 2a_3) + (a_2 + 2a_6 - a_1 - 2a_3) e_{KK}\right] P'_i P'_j$$

$$+ 2(a_1 + 3a_3 + 2a_7)\, e_{jK} P'_i P'_K + 4(a_3 + a_7)\, e_{iK} P'_j P'_K,$$

$$-\rho_0 \bar{E}^{\,i}_{L} \doteq (a_1 + a_2 e_{KK}) P'_i + 2(a_1 + a_3)\, e_{iK} P'_K. \qquad \ldots (28.14)$$

The constitutive relations for isotropic materials may be further specialised by dropping all the terms which contain a product of a displacement gradient and a component of polarisation. We then have

$$t^{ij}_{L} \doteq \lambda e_{KK} \delta_{ij} + 2\mu e_{ij} + a_2 (P')^2 \delta_{ij} + (a_1 + 2a_3) P'_i P'_j ,$$

$$\rho_0 \bar{E}^{\,i}_{L} \doteq -a_1 P'_i . \qquad \ldots (28.15)$$

Then, using the equilibrium condition $E_M + \bar{E}_L = 0$, we can write the local stress in this approximation in the form

$$t^{ij}_{L} \doteq e_{KK} \delta_{ij} + 2\mu e_{ij} + A_1 E_M^2 \delta_{ij} + A_2 E^i_M E^j_M ,$$

where the A's are material constants. This formula is identical with the stress-strain field relation derived by Stratton (3, pp. 140-146).

29. TOUPIN: NOTE ON PRESENTATION.

In this treatment of Toupin's work (Sections 23 - 28) the writer has tried to give as simplified a presentation as is consistent with explaining the basic ideas. The main emphasis has been laid on two features in Toupin's work: firstly on the derivation of constitutive relations between stress, strain, field and polarisation by use of an energy function, and secondly on polynomial approximations to this energy function.

The following simplifications have been made:

(1) In Section 3 of his paper Toupin discusses two point tensor fields: this discussion is not particularly relevant to the rest of his paper and has been omitted. (See also Section 36.)

(2) Rectangular cartesian coordinates have been used throughout; this greatly simplifies the presentation. In this way the theory has been presented without using parallel displacement tensors and Christoffel symbols, and the amount of tensor analysis has been reduced to a minimum.

(3) In Sections 11 and 13 of his paper Toupin deduces that the stored energy function is a function of a certain number of variables. In the case of a homogeneous isotropic dielectric the energy function is a function of six scalar invariants; in the anisotropic case, the

number of invariants depends upon the material symmetry. Discussion of these problems has been omitted because these sections are not necessary for obtaining Toupin's results, using polynomial approximations to the stored energy function.

Toupin's results for these polynomial approximations have been given in full and form one of the main features of his work.

CHAPTER III

C R I T I C I S M S.

30: SMITH-WHITE: CRITICISMS OF THE KORTEWEG-HELMHOLTZ THEORY.

The authors of many standard texts do not give a satisfactory derivation of a formula for the energy in a dielectric system.

For example, Jeans and Abraham-Becker assume that the results which are obtained by considering a parallel plate condenser hold quite generally. In this way it is found that the expression $\frac{1}{8\pi}\underline{D}.\underline{E}$ represents the energy density of the system.

Stratton (3) does attempt a rigorous derivation of a formula for the energy. He deduces two expressions for the energy density of a dielectric system. These two expressions are not the same in general, but no comment is made on this fact.

The first expression is obtained by the following argument.* Consider a dielectric occupying a volume ~~v_2~~ v_2 and suppose we gradually accumulate a charge distribution of final density e, in a volume v_1. To increase the charge density by δe in the element dv the work done is $\phi\,\delta e\,dv$ and the total work done in increasing the charge density by δe in v_1 is

$$\int_{v_1} \phi\,\delta e\,dv\,. \qquad \ldots(30.1)$$

The increase in charge inside v_1 will increase the electrical potential by $\delta\phi$ with the consequent increase

* For simplicity, certain details have been omitted from the argument.

of energy of the charge $e\,dv$ in dv by an amount $e\,\delta\phi\,dv$. Thus the total increment of energy is

$$\int_{V_1} e\,\delta\phi\,dv \ . \qquad \ldots(30.2)$$

It is now argued that the work done is equal to the increment of energy, so that each is equal to

$$\frac{1}{2}\int_{V_1}(e\,\delta\phi + \phi\,\delta e)\,dv \ .$$

Integrating, it follows that the energy of the system is

$$u = \frac{1}{2}\int_{V_1} e\,\phi\,dv \ . \qquad \ldots(30.3)$$

The error in the above argument is obvious if we refer to (App. A-12). It is not true that

$$\int_{V_1} e\,\delta\phi\,dv = \int_{V_1}\phi\,\delta e\,dv \ ,$$

in general.

In the next paragraph (3, p. 107), Stratton derives another formula for the energy density. He argues as follows. Corresponding to the increase of charge density δe in V_1, the work done is given by (30.1). This is taken to be the increment of electrostatic energy; transforming this by (App. A-9) the increment of energy is obtained as $\frac{1}{4\pi}\int_E \underline{E}\cdot\delta\underline{D}\,dv$;

thus the total energy is

$$\frac{1}{4\pi}\int_E dv \int_0^{D} \underline{E}.\,\delta\underline{D} \quad , \qquad \ldots(30.4)$$

if we assume that at each point of the dielectric $\underline{D}$ depends on $\underline{E}$ in such a way that $\underline{E}.\delta\underline{D}$ is a perfect differential.

The two formulae (30.3) and (30.4) are not the same in general. This follows from the fact that $\int_{V_1} e\,\delta\phi\,dv$ is not, in general, equal to $\int_{V_1} \phi\,\delta e\,dv$.

It will be seen that there is a lack of precision in the derivation of energy formulae in many standard authors. However, be this as it may, the main feature of Smith-White's criticism of the Helmholtz method is implicit in his derivation of the general work formulae

(10.2) $$\Delta W = -\Delta V - \int_{V_2} \underline{E}.\,\Delta\underline{p}\,dv \quad ,$$

and

(10.3) $$\Delta W = -\Delta U + \frac{1}{2}\int_{V_2}(\underline{p}.\,\Delta\underline{E} - \underline{E}.\,\Delta\underline{p})\,dv \ .$$

Where

$$V = \frac{1}{2}\int_{V_1} e\phi\,dv - \frac{1}{2}\int_{V_2} \underline{P}.\underline{E}\,dv \, ,$$

and

$$U = \frac{1}{2}\int_{V_1} e\phi\,dv \quad .$$

It has been shown in Section 11 that U and V are only suitable as mechanical potential energy functions in certain special cases; in general they cannot be used as such. This is an important criticism which cannot be ignored. It indicates that a reformulation of the theory of polarisable bodies is necessary.

31. CADE: CRITICISM OF SMITH-WHITE'S THEORY.

Cade (5) claims to have discovered a fundamental error in Smith-White's theory.

He holds that Smith-White's fundamental force formula

$$\underline{F} = (\underline{P}.\nabla)\underline{E}\,, \qquad \dots (31.1)$$

where

$$\underline{E} = -\operatorname{grad}\phi\,, \qquad \dots (31.2)$$

does not take into account the fact that a volume element at the field point considered provides a contribution to the field of the same order as $\underline{E}$ so that, in finding the force on the element, this contribution must be subtracted.

He proceeds to investigate the matter as follows: "We consider a small element of dielectric and regard this and the rest of the dielectric as two polarised bodies. Their polarisations may be replaced by Poisson's equivalent charge distributions. It follows that the field strength acting on the element differs from $\underline{E}$ through the removal of a volume charge of density $-\operatorname{div}\underline{P}$ belonging to the element and the addition of a surface distribution of density P_n on the wall of the cavity occupied by the element, the suffix denoting the component of $\underline{P}$ normal to the wall and directed into the cavity. In addition to the force given by (31.1) one therefore has

(a) The force due to the charges on the wall of the cavity acting upon the charges on the surface of the element.

(b) The forces due to the same charges acting upon the volume charge of the element.

(c) The effect of the additional electric intensity due to the removed charge $-div\,\underline{P}$ upon the surface charge of the element.

(d) The effect of this electric intensity upon the volume charge of the element .

It will easily be seen that (d) is zero and that (b) and (c) cancel so that all that remains is (a) which is in general not zero.

The acceptance of (31-1) then according to Cade is wrong. Cade remarks that the Livens-Smith-White couple density in a dielectric i.e. $\underline{G} = \underline{P} \wedge \underline{E}$ <u>is unaffected by the present criticism</u> for it can be readily seen that the additional field strength acting on a volume element will be a multiple of $\underline{P}$.

Smith-White's energy theory is unaffected by these criticisms. The effective extra forces are due to surface charges existing as equal and opposite adjacent layers and so the work done by them is zero.

Cade then goes on to say that the force (31-1) will not in general yield the correct result for the mechanical force on a region of dielectric; although it will in the special case where the region contains the whole of the dielectric and the dielectric is rigid, for then the additional forces cancel. He points out that in this case the Livens-Smith-White theory gives the same result as the Maxwell-Helmholtz theory (see

Section 10) so that our former ideas of the mechanical action are unaffected.

Smith-White (7) in a reply emphasises that the equations $(31-1)$ and $(31-2)$ are hypotheses which will ultimately be tested by experiment. Also that they are in line with the theory of discrete distributions of dipole moment. (Smith-White deals with this point more fully in (11); for an account of this see Section 9)

Cade (6) in a reply thinks that this hypothesis of Smith-White's conflicts with known electrical principles. He points out that when we write down the electric intensity at a point in a dielectric in terms of an integral over the doublet distribution the integral from the strict mathematical viewpoint is non convergent but the difficulty is overcome from a physical standpoint by specifying that a cavity surrounding the point which is in due course allowed to become increasingly small shall have a certain shape. It is well known that for tubular and disc shaped cavities we obtain field intensities of $\underline{E}$ and $\underline{E} + 4\pi \underline{P}$. He goes on to say that the difficulty of non convergence therefore does not prevent us from finding the force on an element of dielectric but merely demands that we specify the shape of the element. He points out that this situation, except for the analogous one in magnetism, is probably unique in theoretical physics.

In Cade's opinion, a calculation of the force density will lead to the term $(\underline{P}.\nabla)\underline{E}$ together with an extra term, so that Smith-White's hypothesis that $(\underline{P}.\nabla)\underline{E}$ alone is the force, conflicts with established electrical principles.

Smith-White (8) in reply considers that the action of a mechanical force on a body is either specified by the force acting at a single point of the body, or it may be distributed throughout the volume of the body. Though it would be possible to contemplate more general kinds of distributions it is usually sufficient to consider only distributions which can be specified by a force density function. The function is then defined throughout the region occupied by the body. Such a specification of continuously distributed force is possible only if the force acting on an element of the body is proportional to the volume being independent of the shape of the element. The notion of force density is meaningless in any other case.

CHAPTER IV

C O N C L U S I O N S.

32. GENERAL DISCUSSION.

The writer thinks that the treatment of Problems I, II, of Section 1 should follow that of the analogous problem in gravitation given in the same section.

The essential features of such a treatment are:

A. We postulate that

(1) The body force density is given by $\underline{F} = (\underline{P}\cdot\nabla)\underline{E}$.

(2) The surface force is given by $\underline{T} = 2\pi P_n^2 \underline{n}$.

(It is shown in Appendix A·22 that this is a limiting form of the formula $\underline{F} = (\underline{P}\cdot\nabla)\underline{E}$.)

(3) The body couple density is given by $\underline{P} \wedge \underline{E}$

B. An energy method involving both electric and elastic energy is used to obtain the stress-strain relations. (Toupin (see Sections 27, 28) gives a general derivation of these relations. Brown (see Section 22) shows that they will involve the field and polarisarion, so that it is not permissible to take over the stress-strain relations that apply in the absence of an electric field.)

It will be seen that an argument based on energy considerations is omitted from (A) and finds its proper place in (B).

33. HELMHOLTZ: CONCLUSION.

When discussing a system of charges, most textbooks give the explicit formula for the law of force between charges and then proceed to deduce the mutual potential energy of the charges.

The Helmholtz theory for polarisable bodies reverses this order and deduces the law of force from an expression supposed to represent the energy of the system. It is not clear why Korteweg and Helmholtz should have used this procedure, particularly when Maxwell himself had suggested that the solution of an analogous problem in magnetism was to be obtained by assuming the expression $\underline{F} = (\underline{M}.\nabla)\underline{H}$ for the force density.

It is difficult to decide from Stratton's treatment of the Helmholtz theory (see Sections 3, 4) whether he is attempting to derive a formula for the (electric) body-force or a relation between the stress and strain in a polarised medium. The main steps in his discussion are as follows:

(1) He states that he intends to derive a formula for the body-force in deformable solid dielectrics.

(2) For an infinitesimal deformation of the medium he writes

δ (electric energy + elastic energy) = work done by body-force.

(3) This equality is used to obtain the formula (4 - 7).

(4) He then states "If the body-force is of electrical

origin only, the divergence of the elastic stresses vanishes".

(5) Under this last assumption he deduces the formula (4 - 9) for the body-force.

(6) Later (Stratton 3, p.149), when discussing electro-striction, he uses the formula

$$(4 - 10) \qquad \underline{f} + (\lambda_1 + \lambda_2)\nabla\nabla.\underline{u} + \lambda_2\nabla^2\underline{u} = 0 ,$$

where $\underline{f}$ is given by (4 - 8). (Here he has implicitly assumed that the stress-strain relations are unaltered by the polarisation of the medium).

The writer's view (see Section 32,B) is that, in using the equation in (2), Stratton is deriving a stress-strain-field relation (and not a formula for body force); indeed, Toupin shows that, if certain approximations are made, the relation (4 - 7) can be obtained as a special case of his own more general results.

Stratton's argument is an elaboration of the original Helmholtz theory; a typical account of the latter is that given by Pockels (16), who proceeds as follows:

(1) He states that he intends to derive a formula for the body force in deformable solid dielectrics.

(2) For an infinitesimal deformation of the system he writes

δ (electric energy) = work done by body force.

(3) This equality is used to obtain the formula (4 - 8) and its special case (4 - 9).

(4) In applying this theory to isotropic solid bodies (16, P.362) he uses an equation similar to (4 - 10); he assumes that the stress-strain relations are unaltered by polarisation.

It is interesting to note that, in his account of Voigt's empirical theory of piezo-electricity (16, p.374), the dependence of the stress-strain relations on the field appears explicitly; however, Pockels offers no comment on this difference. In the same connection it may be noted that the Helmholtz theory lacks any clear experimental support: most of the experimenters mentioned in the article 'Electrostriction" in The International Critical Tables (19) have based their work on the Helmholtz theory; it is evident from this article that there is no general agreement between theory and experiment. On the other hand, Voigt's theory of piezo-electricity (obtained by Toupin as a special case of his own more general theory) has met with complete experimental verification; indeed, the main results have now found an important place in electrical-engineering practice. However, it is fair to point out that, in electrostriction, the effects involved are extremely small and correspondingly difficult of measurement.

Smith-White's criticisms given in Section 30, lodge theoretical objections against the Helmholtz theory.

The above facts lend support to Toupin's work: in the first place it gives a unified view of the problem of deformable dielectrics, whereas the Helmholtz theory - as it stands - does not cover piezo-electric phenomena; in the second place, it includes Voigt's results and in this respect is amply verified by experimental evidence.

34. SMITH-WHITE: CONCLUSION

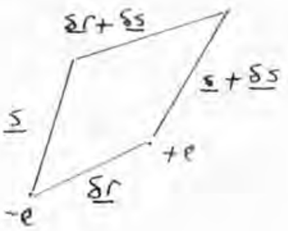
The early criticisms of Larmor and Livens seem to have made little impression on the authors of mathematical texts on Electricity. Stratton (3) is the only author to mention Larmor and Livens in this connection: he says "These criticisms however do not appear to be well founded" and "There appears to be little reason to doubt that the energy method of Korteweg and Helmholtz is fundamentally sound"

Smith-White's criticism of the Helmholtz theory has been given in Section 30. It was there shown that neither of the functions U nor V is suitable in general as a mechanical potential energy function. This shows that the Helmholtz method of deriving the body force is fundamentally unsound.

The writer agrees with Smith-White's conclusions in the main but thinks that his use of the term "semi-conservative" to describe a system containing polarisable matter causes confusion. As Brown remarks in connection with (22.2) if a magnet is to be treated as a conservative dynamical system it is necessary to introduce, besides the external coordinates necessary for a permanent magnet, additional internal coordinates, the components of $\underline{m}$ in axes attached to the magnet, and to include a term, $\underline{H}_0 . D\underline{m}$ in the differential expression for work done.

In fine, work is done in deforming the dipole as well as in displacing it. This can be illustrated by the following "dumbell" model.

We consider two charges $-e$ and $+e$ whose vector separation is $\underline{\delta r}$ and write $\underline{p} = e\underline{\delta r}$. Suppose that these charges are given the respective infinitesimal displacements $\underline{s}$ and $\underline{s} + \underline{\delta s}$. The corresponding work



done by the field is $-e\underline{E}.\underline{s}$ on $-e$ and $e(\underline{E} + \underline{\delta E}).(\underline{s} + \underline{\delta s})$ on $+e$ so that the net work done by the field is $e\underline{E}.\underline{\delta s} + e\underline{s}.\underline{\delta E}$

$$= \underline{E}.\delta\underline{p} + [(\underline{p}.\nabla)\underline{E}].\underline{s} \qquad \ldots (34.1)$$

since $\underline{\delta E} = (\underline{\delta r}.\nabla)\underline{E}$ and $e\underline{\delta s} = \delta\underline{p}$.

We see that in addition to the work done by the body force $(\underline{p}.\nabla)\underline{E}$ there is a contribution $\underline{E}.\Delta\underline{p}$ which arises from the deformation of the dipole.

If we have a system of such dipoles the work done by this field $\underline{E}$ is

$$\Sigma \left\{ \underline{E}.\Delta\underline{p} + [(\underline{p}.\nabla)\underline{E}].\underline{s} \right\} .$$

For the system of charges and dipoles of Section 7 the total work done in a virtual displacement is

$$\Sigma\, \underline{E}.\Delta\underline{p} + \Sigma\, [(\underline{p}.\nabla)\underline{E}].\underline{s} + \Sigma e\underline{E}.\underline{s} ,$$

(7.5) shows that this sum is equal to $-\Delta V$

Smith-White states that the importance of (7.5) "springs from the generality of the circumstances to which it applies".

In Section 7 however Smith-White appears to include only terms of the type $[(\underline{p}.\nabla)\underline{E}].\underline{S}$ when considering the work done on the dipoles by the field. It is clear from (34.1) that it is necessary to include terms of the type $\underline{E}.\delta\underline{p}$ to show that the system is conservative. This is no more than saying that, in general, a deformable body has an energy due to deformation as well as an energy due to position in a field of force.

From this point of view the significance of the example given in Section 5 becomes more clear. Smith-White showed that, in general, the expression

$$F\,dx + F_1\,dx_1 + F_2\,dx_2 = F_1\,dr_1 + F_2\,dr_2\,,$$

is not a perfect differential. In the light of (34.1) this is only to be expected: we obtain a perfect differential

$$F\,dx + F_1\,dx_1 + F_2\,dx_2 + E_1\,dp_1 + E_2\,dp_2\,,$$

(see 7.5)) by adding the work done in the internal deformation of the dipoles, so that the system <u>is</u> conservative. This criticism does not invalidate the usefulness of Smith-White's fundamental work formulae (10.2) and (10.3) in showing that neither $\mathcal{U}$ nor V is suitable as a potential function from which the body force $(\underline{P}.\nabla)\underline{E}$ and body couple $\underline{P}\wedge\underline{E}$ can be derived.

Smith-White's useful discussion of the "effective field" in a dielectric has been given in Section 9. He insists that the theory of continuous distributions should run parallel to that of discrete distributions. For a discrete distribution of dipoles he finds that there are two alternative procedures for obtaining $\underline{E}$: either $\underline{E}$ can be defined directly or it can be defined as the gradient of $-\phi$. In the case of a continuous distribution the first procedure breaks down but the second method is still available as ϕ can be expressed as a convergent integral.

Smith-White has shown (see Section 14) that, for a solid body immersed in an incompressible fluid dielectric, his own theory gives the same force and couple as the Maxwell theory. Cade (12) has tried to compare the Helmholtz and Smith-White theories with known experimental measurements of the couples on ellipsoids in dielectric fluids. He has left out of account the effect of the hydrostatic pressure in his treatment of Smith-White's theory and thereby erroneously concludes that calculations based on the Smith-White theory are in sharp disagreement with the experimental evidence.

It must be repeated that in the special case of an incompressible fluid dielectric the formulae obtained by Smith-White's theory and those given by Helmholtz reduce to the formulae obtained earlier by Maxwell.

35. BROWN: CONCLUSION.

Consider a dielectric body V_2 under the influence of charge contained in a separate volume V_1.

Brown (see Section 17) obtains the "long-range" force on a part τ of the volume V_2 as

(17.9) $$\underline{F} = \int_\tau (\underline{M}.\nabla)\underline{H}\,d\tau + 2\pi\int_S \underline{n}\,M_n^2\,dS.$$

On the other hand, Smith-White postulates that the body force density at an interior point of a dielectric is

$$\underline{F} = (\underline{P}.\nabla)\underline{E} \quad ,$$

and that the surface force at an actual surface of the dielectric is

$$\underline{T} = 2\pi\,\underline{n}\,P_n^2 \quad .$$

(Smith-White regards the surface force $\underline{T} = 2\pi P_n^2\,\underline{n}$ as being a limiting case of the volume force $\underline{F} = (\underline{P}.\nabla)\underline{E}$. (See App A - 22)

Thus the force given by Brown's formula (17.9) is the same as that given by the Smith-White formulae provided the volume is the whole of the dielectric and not just part of it. Disagreement arises when τ represents only part of the volume V_2 . The writer is of the opinion that in this case the reasoning used by Brown in obtaining

(17.9) contains a flaw: he conceives of the volume τ as being isolated in empty space (e.g., he says "the discontinuity in $\underline{H}_1$ across the surface S affects the normal components only and is of amount $4\pi M_n$") and thereby introduces what we might call a conceptual discontinuity in the dielectric and, in consequence, in the vector $\underline{P}$.

This conceptual discontinuity gives rise to the surface integral $2\pi \int M_n^2 \underline{n}\, dS$ in (17.9). If the surfaces S_1 and S_2 appearing in Section 17 are entirely inside the dielectric body then in reality $\underline{H}_1^+ = \underline{H}_1^-$ in (17.6) and the surface term $F_1 = \frac{1}{2}\int \underline{n}\cdot\underline{M}\,(\underline{H}_1^+ - \underline{H}_1^-)\,dS$ $= \int 2\pi M_n^2 \underline{n}\, dS$ vanishes. Thus the body force is $(\underline{P}.\nabla)\underline{E}$ per unit volume at an interior point of a dielectric; Brown's formula (17-9) should only be applied to the whole volume V_2 .

Using (17.9) Brown cannot use a body-force density alone but must include in the force of electrical origin on an element of dielectric a term involving a surface integral over the boundary surface of the element. This is the reason for the unusual stress-relations

(18.2)
$$\begin{aligned} t_1(\nu) &= [t_{11} + 2\pi(M_1^2 - M_n^2)]l + t_{12}m + t_{13}n, \\ t_2(\nu) &= t_{21}l + [t_{22} + 2\pi(M_2^2 - M_n^2)]m + t_{23}n, \\ t_3(\nu) &= t_{31}l + t_{32}m + [t_{33} + 2\pi(M_3^2 - M_n^2)]n. \end{aligned}$$

The contributions $2\pi(M_1^2 - M_n^2)$, $2\pi(M_2^2 - M_n^2)$, $2\pi(M_3^2 - M_n^2)$ arise from the surface force $2\pi M_n^2 \underline{n}$.

As pointed out above, this "surface force" vanishes in the interior of the dielectric; making this correction (18.2) reduce to the usual stress-relations

$$t_1(\nu) = t_{11}l + t_{12}m + t_{13}n \text{ , etc.} \qquad \ldots (35.1)$$

the corresponding equations of motion being

$$t_{11,1} + t_{12,2} + t_{13,3} + [(\underline{M}\cdot\nabla)\underline{H}]_1 + \rho F_1 = \rho f_1 . \qquad \ldots (35.2)$$

For a fluid dielectric the pressure will be the same in all directions at an interior point; but at any point on the actual surface of the fluid the force exerted will depend on the local orientation of this surface.

(19-2) is correct at a <u>real</u> surface of the fluid but is not correct at an interior point of the fluid; in the latter case it must be replaced by the relation

$$p\nu = -p'\underline{n}$$

Brown obtains what the writer opines to be the correct equilibrium relation

(20.2) $\operatorname{grad} p' + M \operatorname{grad} H - \rho \operatorname{grad} V = 0$,

for a fluid dielectric; putting $V = 0$ this is formally the same as the equation

(14-1) $\operatorname{grad} \tilde{\omega} = \frac{k}{2} \operatorname{grad} E^2$,

obtained by Smith-White.

Brown's deductions from (20-2) are useful: he shows that there exists a functional relation between Z , H

and ϱ and proves that

$$M = -\frac{\partial Z}{\partial H}, \quad V = -\frac{\partial Z}{\partial \varrho}, \qquad \text{... (35.3)}$$

(Later he shows that these results are special cases of the formulae (21 - 13).)

Brown applies the relations (35.3) to the case of a compressible fluid under the action of an electric field and is thereby a le to throw considerable light on the Helmholtz theory of electrostriction. From the equations

$$\underline{P} = k(\varrho)\,\underline{E},$$

$$P = \frac{-\partial Z(E,\varrho)}{\partial E},$$

$$V = -\frac{\partial Z(E,\varrho)}{\partial \varrho},$$

he deduces that

$$\begin{aligned} p' &= \tfrac{1}{2}\left[k(\varrho) - \varrho k'(\varrho)\right]E^2 + \Psi(\varrho) - \varrho\Psi'(\varrho) \\ &= \tfrac{1}{2}\left[k(\varrho) - \varrho k'(\varrho)\right]E^2 + p(\varrho), \end{aligned} \qquad \text{... (35.4)}$$

where $p(\varrho)$ is the pressure at zero field.

Further results obtained by Brown (see Section 21) make it clear that the pressure-density relation appearing in the account of electrostriction by Abraham-Becker (17) is that corresponding to zero field.

From (35.4)

$$\begin{aligned} \nabla p' = {} & \tfrac{1}{2}\left[\nabla\varrho\left(\frac{\partial k}{\partial\varrho}\right) - \nabla\varrho\left(k'(\varrho)\right) - \varrho\nabla\frac{\partial k}{\partial\varrho}\right]E^2 \\ & + \tfrac{1}{2}\left[k - \varrho k'(\varrho)\right]\nabla E^2 + \nabla p(\varrho), \end{aligned}$$

whence

$$\operatorname{grad} p' - \frac{k}{2}\operatorname{grad} E^2 = \operatorname{grad} p(\rho) -$$
$$-\frac{E^2}{2}\rho \operatorname{grad}\left(\frac{\partial k}{\partial \rho}\right) - \frac{\rho}{2}\frac{\partial k}{\partial \rho}\operatorname{grad} E^2$$
$$= \operatorname{grad} p(\rho) - \frac{1}{2}\left[\operatorname{grad}\left(E^2\rho\frac{\partial k}{\partial \rho}\right) - E^2\frac{\partial k}{\partial \rho}\operatorname{grad}\rho\right] \quad \ldots (35.5)$$
$$= \operatorname{grad} p(\rho) - \frac{1}{2}\left[\operatorname{grad}\left(E^2\rho\frac{\partial k}{\partial \rho}\right) - E^2\operatorname{grad} k\right].$$

According to Smith-White the equilibrium equation is

$$\operatorname{grad} p' - \frac{k}{2}\operatorname{grad} E^2 = 0. \quad \ldots (35.6)$$

In this case it would follow from (35.5) that

$$\operatorname{grad} p(\rho) - \frac{1}{2}\left[\operatorname{grad}\left(E^2\rho\frac{\partial k}{\partial \rho}\right) - E^2\operatorname{grad} k\right] \ldots (35.7)$$

This is the Helmholtz equation of equilibrium. Thus it is seen that, for a fluid dielectric, the Helmholtz formula (35.7) is equivalent to the Smith-White formula (35.6); the term $\frac{1}{2}\operatorname{grad}\left(E^2\rho\frac{\partial k}{\partial \rho}\right) - \frac{E^2}{2}\operatorname{grad} k$ in the former is the body force obtained by Helmholtz.

Brown's proof of the existence of a thermodynamic potential function has been given in Section 22. Earlier in the present section it has been pointed out that the stress relations (18-2) used by Brown should be replaced by the simpler relations (35.1); making the appropriate changes in the argument in Section 22, the final result obtained (cf. (22-13)) is

$$H_1 = \frac{\partial F}{\partial M_1'}, \quad ----, \quad V(t_{11}) = \frac{\partial F}{\partial e_{11}},$$

$$V\,(t_{12} + t_{21}) = \frac{\partial F}{\partial e_{12}}.$$

36. TOUPIN: CONCLUSION.

That Toupin has given the most general account to date of the stress-strain relations in deformable dielectrics will be clear from the following comments:

(1) It includes finite deformations, infinitesimal deformations being considered as special cases. (Various earlier attempts have been made to generalise Voigt's theory of piezoelectricity; these, like the original theory, are based on a consideration of infinitesimal deformations. Toupin's generalisation of Voigt's theory represents the most significant advance since it includes the case of finite deformations.)

(2) The fact that the polarisation of a dielectric depends on the strain as well as on the field (e.g., a quartz crystal can be polarised in the absence of an external field by subjecting it to mechanical pressure) is a fundamental postulate in the theory.

(3) Initial and final states of the medium are expressed in terms of arbitrary sets of curvilinear coordinates; in consequence, application of Toupin's results to particular configurations is often facilitated. (As noted in § 23, rectangular cartesian coordinates have been used throughout the writer's presentation of Toupin's work; the consequent loss of generality is outweighed by the simplification of the argument.)

Toupin introduces two-point tensor fields at the beginning of his work (15, pp. 854-858). The writer is of the opinion that this analytical tool is of interest but is not necessary for a complete discussion of the deformation of an elastic dielectric: two-point tensors may be of use when they are functions of two <u>independent</u> sets of points; however, in the present case, described by Toupin (15, p.858) as one in which, ".... the argument points X^A and x^i are not independent but are functionally related by the mapping $x^i = x^i(X^A)$ "; the presentation is simplified by avoiding their use.

The Livens formulae for body force and couple are used without discussion by Toupin. (No mention is made of the Helmholtz formula for body force.) Toupin follows the same approach as Smith-White and Brown in postulating the body force directly and then using an energy method to derive the stress-strain relations.

The main feature of Toupin's work is his derivation of constitutive relations between stress, strain, field and polarisation by use of an energy principle. This is an extension of the method used in ordinary elasticity, but here the elastic energy depends on the polarisation as well as the strain.

It is possible to simplify Toupin's derivation of (27-11) from (27-7) as follows:

Firstly, two of the integrals in (27-7) can be combined:

$$\frac{1}{8\pi}\int_E E_{MS}^2\,dv + \int_V \underline{E}_{MS}\cdot\underline{P}\,dv$$

$$= \left[\frac{1}{8\pi}\int_E E_{MS}^2\,dv + \frac{1}{2}\int_E \underline{E}_{MS}\cdot\underline{P}\,dv\right] + \frac{1}{2}\int_V \underline{E}_{MS}\cdot\underline{P}\,dv$$

$$= \frac{1}{8\pi}\int_E \underline{E}_{MS}\cdot\underline{D}\,dv + \frac{1}{2}\int_V \underline{E}\cdot\underline{P}\,dv \quad .$$

By Appendix A7 we have that $\int_E \underline{E}_{MS}\cdot\underline{D}\,dv = 0$, (since $e=0$)

$$\therefore \frac{1}{8\pi}\int_E \underline{E}_{MS}^2\,dv + \int_E \underline{E}_{MS}\cdot\underline{P}\,dv = \frac{1}{2}\int_E \underline{E}_{MS}\cdot\underline{P}\,dv. \qquad \ldots(36\text{-}1)$$

It is now necessary to calculate the variation of one integral only, instead of two as in Toupin's treatment. Secondly, Toupin considers three independent variations

(27-8) $\quad x^i \to x^i + \delta' x^i$,

(27-9) $\quad P_i' \to P_i' + \delta'' P_i'$,

(27-10) $\quad \phi \to \phi + \delta''' \phi$,

and deduces Maxwell's law $\operatorname{div}(\underline{E}_{MS} + 4\pi\underline{P}) = 0$ as a side result of his argument. (The method is similar to Palatini's derivation (20) of Einstein's field equations.) A considerable simplification is attained if we follow the usual course of <u>assuming</u> Maxwell's law. In this way, as shown by Smith-White (see §10), a variation of (37-1) yields the result

$$\Delta W = -\Delta V - \int_{V_2} \underline{E} . \Delta \underline{p} \, dv .$$

Applying this formula to a system of dipoles only, we obtain (in Smith-White's notation) the relation

$$\int_{V_2} \underline{F} . \underline{u} \, dv + \int_{S_2} \underline{T} . \underline{u} \, dS =$$

$$= \Delta \frac{1}{2} \int_{V_2} \underline{P} . \underline{E} \, dV - \int_{V_2} \underline{E} . \Delta \underline{p} \, dv ,$$

where $\underline{F} = (\underline{P} . \nabla) \underline{E}$ and $\underline{T} = \frac{1}{2} \underline{P} . (\underline{E}_{+} - \underline{E}_{-}) \underline{n}$ and $\Delta \underline{p}$ is the charge in $\underline{p}$ following the displacement but referred to axes that have the same orientation throughout and do not partake of the rotation $\delta \underline{\Theta}$.

In Toupin's notation this formula becomes

$$\delta \left\{ \frac{1}{2} \int_V \underline{E}_{MS} . \underline{P} \, dv \right\} = \int_V E^{MS}_i \, \rho \, \delta'' P_i' dv$$

$$+ \int_V S^{MS}_{ij,j} \, \delta' x_i \, dv + \int_B (s^{MS}_{ij} - s^{MS}_{ij}) n_j \, \delta' x_i \, dS .$$

Now $$\delta \left\{ - \int_V \rho \, \Sigma (x^i_{;A} , P_i') dv \right\}$$

$$= - \int_V \rho \frac{\partial \Sigma}{\partial x^i_{;A}} \delta x^i_{,A} \, dv - \int \rho \frac{\partial \Sigma}{\partial P_i'} \delta P_i' \, dv$$

$$= -\int \varrho \frac{\partial \Sigma}{\partial x^i_{;A}} (\delta' x^i)_{;j}\, x^j_{;A}\, dv - \int \varrho \frac{\partial \Sigma}{\partial P_i'} \delta'' P_i'\, dv$$

$$= -\int_B \varrho \frac{\partial \Sigma}{\partial x^i_{;A}} x^j_{;A}\, n_j\, dS' + \int_V \left(\varrho \frac{\partial \Sigma}{\partial x^i_{;A}} x^j_{;A}\right)_{;j} \delta' x^i$$

$$-\int_V \varrho \frac{\partial \Sigma}{\partial P_i'} \delta'' P_i'\, dv ,$$

so that (27-11) simplifies to

$$\int_V \left\{ \left(\varrho \frac{\partial \Sigma}{\partial x^i_{;A}} x^j_{;A}\right)_{;j} + S^{MS}_{ij;j} + E^0_{i;j} P_j \right\} \delta' x^i\, dv$$

$$+ \int_V \left\{ -\frac{\partial \Sigma}{\partial P_i'} + E^i_{MS} + E_0^i \right\} \varrho\, \delta'' P_i'\, dv$$

$$+ \int_B \left\{ -\varrho \frac{\partial \Sigma}{\partial x^i_{;A}} x^j_{;A}\, n_j + [S^{MS}_{ij}]\, n_j + \Pi_i \right\} \delta' x_i\, dS' = 0.$$

Accordingly,

(27-17) $$\left(\varrho \frac{\partial \Sigma}{\partial x^i_{;A}} x^j_{;A}\right)_{;j} + S^{MS}_{ij;j} + E_{0i;j} P_j = 0$$

(27-18) $$-\frac{\partial \Sigma}{\partial P_i'} + E^i_{MS} + E_0^i = 0 ,$$

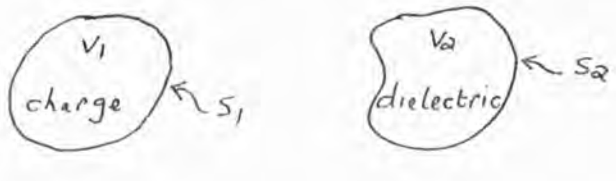
(27-20) $$-\rho \frac{\partial \Sigma}{\partial x^i_{;A}} x^j_{;A} n_j + [\![s^{MS}_{ij}]\!] n_j + \Pi_i = 0 .$$

Toupin's other equation (27-19) is of course the Maxwell law implicitly assumed in the above derivation.

Toupin (see §28) has examined polynomial approximations to the stored energy function. By approaching the subject from the <u>finite</u> deformation point of view he is able to give general expressions (28-6) and (28-7) for local stress and local field respectively. (28-8), (28-9) are obtained as special cases of these results and represent a generalisation of Voigt's formulae. It should be noted that even for infinitesimal displacements, t^{ij}_L and E^i_L are not functions of polarisation and the symmetric part of the strain tensor alone but may also depend on the antisymmetric part of the strain tensor.

APPENDIX A: ELECTROSTATICS.

1. FORMAL TRANSFORMATIONS



Consider a charge distribution density e in a volume V_1 with surface S_1 and a dipole moment distribution of density $\underline{P}$ in volume V_2 with surface S_2. The electrostatic potential ϕ is given by

$$\phi = \int_{V_1} \frac{e\,dv}{r} + \int_{V_2} \underline{P}.\nabla\left(\frac{1}{r}\right) dv \quad . \qquad \ldots (A - 1)$$

Transforming (A - 1) by Green's theorem

$$\phi = \int_{V_1} e\frac{dv}{r} + \int_{V_2} \frac{-\operatorname{div}\underline{P}}{r}\,dv + \int_{S_2} \frac{P_n\,dS}{r} \;, \qquad \ldots (A - 2)$$

We have

$$\underline{D} = \underline{E} + 4\pi\underline{P} \;, \qquad \ldots (A - 3)$$

and

$$\operatorname{div}\underline{D} = \begin{cases} 4\pi e & \text{in } V_1 \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots (A - 4)$$

For any volume V bounded by surface S

$$\int_S D_n\,dS = \int_V \operatorname{div}\underline{D}\,dv = 4\pi\int_V e\,dv = 4\pi Q \;, \qquad \ldots (A - 5)$$

where Q is the total charge contained in V.

Consider a second distribution of charge with density

e' in V_1 and moment with density $\underline{P}'$ in V_2. For this distribution, let $\phi', \underline{E}', \underline{D}'$ be the functions corresponding to the $\phi, \underline{E}, \underline{D}$ of the first distribution. The identity

$$\phi \operatorname{div} \underline{D}' = \operatorname{div}(\phi \underline{D}') - \underline{D}' . \nabla\phi ,$$

gives

$$4\pi e' \phi = \operatorname{div}(\phi \underline{D}') + \underline{D}' . \underline{E} .$$

Applying Green's theorem to each of the three components of space separated by the surfaces S_1 and S_2 and adding, we have

$$\int_{V_1} e' \phi \, dv = \frac{1}{4\pi} \int_E \underline{D}' . \underline{E} \, dv , \qquad \ldots (A - 6)$$

where $\int_E$ signifies an integration over the whole of space.

In (A - 6) write $e' = e$, $\underline{D}' = \underline{D}$ then

$$\int_{V_1} e \, \phi \, dv = \frac{1}{4\pi} \int_E \underline{D} . \underline{E} \, dv . \qquad \ldots (A - 7)$$

Subtracting (A-7) from (A - 6)

$$\int_{V_1} (e' - e) \phi \, dv = \frac{1}{4\pi} \int_E (\underline{D}' - \underline{D}) . \underline{E} \, dv . \qquad \ldots (A - 8)$$

For a differential variation $e \rightarrow e + \delta e$, $\underline{D} \rightarrow \underline{D} + \delta\underline{D}$, this gives

$$\int_{V_1} \phi \, \delta e \, dv = \frac{1}{4\pi} \int_E \underline{E} . \delta \underline{D} \, dv . \qquad \ldots (A - 9)$$

Interchange dashed and undashed letters in (A - 6) and subtract the new result from it. So

$$\int_{V_1} (e'\phi - \phi e')\, dv = \frac{1}{4\pi}\int_E (\underline{D}'.\underline{E} - \underline{D}.\underline{E}')\, dv$$

... (A - 10)

$$= \frac{1}{4\pi}\int_{V_2} (\underline{\dot{D}}'.\underline{E} - \underline{D}.\underline{E}')\, dv$$

$$= \int_{V_2} (\underline{P}'.\underline{E} - \underline{P}.\underline{E}')\, dv \qquad \text{... (A - 11)}$$

For a differential variation $e \to e + \delta e$, $\underline{P} \to \underline{P} + \delta \underline{P}$, and $\underline{D} \to \underline{D} + \delta \underline{D}$ this gives

$$\int_{V_1} (\phi\, \delta e - e\, \delta\phi)\, dv = \frac{1}{4\pi}\int_{V_2} (\underline{E}.\delta\underline{D} - \underline{D}.\delta\underline{E})\, dv \qquad \text{... (A - 12)}$$

$$= \int_{V_2} (\underline{E}.\delta\underline{P} - \underline{P}.\delta\underline{E})\, dv\, .$$

... (A - 13)

If the boundaries S_1 and S_2 move during the deformation the more general result

$$\int_{V_1} (e\,\delta\phi - \phi\,\delta e)\, dv - \int_{S_1} e\phi\, n_\alpha \delta u_\alpha\, dS'$$

$$= \int_{V_2} (\underline{P}.\delta\underline{E} - \underline{E}.\delta\underline{P})\, dv - \int_{S_2} \underline{P}.\underline{E}_+\, n_\alpha \delta u_\alpha\, dS', \qquad \text{... (A - 13')}$$

can be proved as follows.

Suppose for simplicity of argument that the boundaries of S_1 and S_2 are given an infinitesimal displacement $\delta \underline{u}$ and move uniformly outwards to S_1' and S_2' enclosing volumes V_1' and V_2'.

Let the new distribution be of charge density e' in V_1' and dipole density $\underline{P}$ in V_2'. Applying Green's theorem to each of the five components of space separated by S_1, S_2, S_1', S_2' we have

$$\int_{V_1} e\phi' dv = \frac{1}{4\pi}\int_E \underline{D}.\underline{E}' dv , \qquad \ldots (A - 6')$$

and

$$\int_{V_1'} e'\phi\, dv = \frac{1}{4\pi}\int_E \underline{D}'.\underline{E}\, dv . \qquad \ldots (A - 6'')$$

Subtract (A - 6') from (A - 6"), then

$$\int_{V_1'} e'\phi dv - \int_{V_1} e\phi' dv = \frac{1}{4\pi}\int_E \underline{D}'.\underline{E}\, dv - \frac{1}{4\pi}\int \underline{D}.\underline{E}' dv$$

$$= \int_E \underline{P}'.\underline{E}\, dv - \int_E \underline{E}'.\underline{P}\, dv = \int_{V_2'} \underline{P}'.\underline{E}\, dv - \int_{V_2} \underline{P}.\underline{E}' dv .$$

This gives

$$\int_{V_1} (\phi\,\delta e - e\,\delta\phi)\, dv + \int_{V_1'-V_1} e'\phi\, dv$$

$$= \int_{V_2} (\delta\underline{P}.\underline{E} - \underline{P}.\delta\underline{E})\, dv + \int_{V_2'-V_2} \underline{P}'.\underline{E}\, dv .$$

Since $\delta\underline{u}$ is an infinitesimal displacement this can be written in the form

$$\int_{V_1} (\phi\,\delta e - e\,\delta\phi)\, dv + \int_{S_1} e\phi\, n_\alpha \delta u_\alpha\, dS$$

$$= \int_{V_2} (\underline{E}.\delta\underline{P} - \underline{P}.\delta\underline{E})\, dv + \int_{S_2} \underline{P}.\underline{E} + n_\alpha\delta u_\alpha\, dS . \qquad \ldots (A - 13')$$

If in (A - 10) we take $e' = e$ and write $\underline{D} = K\underline{E}$, $\underline{D}' = K'\underline{E}'$ we have

$$\int_{V_1} e(\phi - \phi')\,dv = \frac{1}{4\pi}\int (K' - K)\underline{E}'.\underline{E}\,dv. \qquad \ldots \text{(A - 14)}$$

For a differential variation $K \to K + \delta K$ this gives

$$\int_{V_1} e\,\delta\phi\,dv = -\frac{1}{4\pi}\int_{V_2} E^2\,\delta K\,dv \qquad \ldots \text{(A - 15)}$$

More generally if $D_i = K_{ij}E_j$ where K_{ij} is symmetric then

$$\int_{V_1} e\,\delta\phi = -\frac{1}{4\pi}\int E_i E_j\,\delta K_{ij}\,dv \qquad \ldots \text{(A - 16)}$$

The identity

$$\underline{P}.\nabla\phi = \operatorname{div}(\phi\underline{P}) - \phi\operatorname{div}\underline{P},$$

gives

$$-\underline{P}.\underline{E} = \operatorname{div}(\phi\underline{P}) - \phi\operatorname{div}\underline{P}.$$

Integrating over V_2 and applying Green's theorem

$$-\int_{V_2}\underline{P}.\underline{E}\,dv = \int_{S_2}\phi P_n\,dS - \int_{V_2}\phi\operatorname{div}\underline{P}\,dv.$$

Now define functions U and V by

$$U = \frac{1}{2}\int_{V_1} e\phi\,dv, \qquad \ldots \text{(A - 17)}$$

$$V = \frac{1}{2}\int_{V_1} e\phi\,dv - \frac{1}{2}\int_{V_2}\underline{P}.\underline{E}\,dv.$$

Then by (A - 7)

$$u = \frac{1}{8\pi}\int_E \underline{D}.\underline{E}\,dv$$

$$= \frac{1}{8\pi}\int_E (\underline{E} + 4\pi\underline{P}).\underline{E}\,dv \qquad \ldots (A - 18)$$

$$= \frac{1}{8\pi}\int_E E^2 dv + \frac{1}{2}\int \underline{P}.\underline{E}\,dv \;;$$

and

$$V = \frac{1}{2}\int_{V_1} e\phi\,dv + \frac{1}{2}\int_{S_2} \phi P_n\,dS' - \frac{1}{2}\int_{V_2}\phi\,\mathrm{div}\,\underline{P}\,dv \qquad \ldots (A - 19)$$

$$= \frac{1}{8\pi}\int_E E^2 dv\,. \qquad \ldots (A - 20)$$

2. SURFACE FORCE AT THE BOUNDARY OF A DIELECTRIC.

At the surface S_2 of V_2 we show that the limiting form of $\underline{F} = (\underline{P} \cdot \nabla)\underline{E}$ gives a surface traction on S_2

$$\underline{T} = \frac{1}{2} \underline{P} \cdot (\underline{E}_+ - \underline{E}_-) \underline{n},$$

where $\underline{n}$ is the unit normal outward from V_2 and $\underline{E}_+, \underline{E}_-$ are the electric fields just outside and just inside V_2.

Consider a dielectric medium of semi-infinite extent bounded by a plane surface, take this surface to be $x_3 = 0$ and suppose the dielectric occupies the space $x_3 < 0$. The tangential components of $\underline{E}$ are continuous across the bounding surface, but the normal component is discontinuous. We imagine the surface discontinuity in $\underline{P}$ to be replaced by a thin layer in which $\underline{P}$ changes rapidly but continuously and regard the discontinuity as being generated when this layer becomes infinitely thin. As the limit is approached $\frac{\partial E_3}{\partial x_3}$ becomes infinite but the other derivatives remain finite. The traction $\underline{T}$ acting on the surface discontinuity may now be obtained by integrating the volume force throughout the layer covering unit area on the plane $x_3 = 0$. The components of force are

$$F_\alpha = P_\beta \frac{\partial E_\alpha}{\partial x_\beta},$$

and the volume of integration tends to zero as we approach the limit so that the only contribution to the traction is due to F_3 . Thus $T_1 = T_2 = 0$ and

$$T_3 = \int P_3 \frac{\partial E_3}{\partial x_3} dx_3 .$$

The relation $\operatorname{div} \underline{E} = -4\pi \operatorname{div} \underline{P}$, reduces effectively to

$$\frac{\partial E_3}{\partial x_3} = -4\pi \frac{\partial P_3}{\partial x_3} .$$

Hence

$$T_3 = -4\pi \int P_3 \frac{\partial P_3}{\partial x_3} dx_3 = \left[-2\pi P_3^2 \right]_{P_3}^{0} = 2\pi P_3^2 .$$

As the normal component of the vector $\underline{D} = \underline{E} + 4\pi \underline{P}$ is continuous it follows that if we distinguish the electric field in $x_3 < 0$ and $x_3 > 0$ by the signs $-$ and $+$ respectively then $2\pi P_3 = \frac{1}{2} \left(E_{3+} - E_{3-} \right)$,

and so $T_3 = \frac{1}{2} P_3 \left(E_{3+} - E_{3-} \right)$.

Finally since $\underline{E}_+ - \underline{E}_-$ is a vector perpendicular to the plane $x_3 = 0$ we may express this result in the form

$$\underline{T} = \frac{1}{2} \left\{ \underline{P} . \left(\underline{E}_+ - \underline{E}_- \right) \right\} \underline{n} \qquad \text{... (A - 21)}$$

$$= 2\pi P_n^2 \, \underline{n} \, , \qquad \text{... (A - 22)}$$

where $\underline{n}$ is the unit normal from the $-$ to the $+$ side. This formula applies generally to any free surface bounding a dielectric medium if the $-$ sign refers to the medium and the $+$ sign to outside the medium.

Ultimately of course this formula rests on the same footing as $\underline{F} = (\underline{P}.\nabla)\underline{E}$ and $\underline{G} = \underline{P}\wedge\underline{E}$ i.e. it is a hypothesis, the extension of Coulomb's law appropriate to the surface discontinuity of the polarisation across S_2

It has been shown in Section 13 that the force $\underline{F} = (\underline{P}\cdot\nabla)\underline{E}$ can be represented by the stress tensor s_{ij} given in (13 - 4); it will be of interest to show that the discontinuity in $s_{ij}\,n_j$ across the boundary of the dielectric is the surface force (A - 22), i.e.

$$[\![s_{ij}]\!]\, n_j = \left(2\pi\, P_n^2\right) n_i\ .$$

Taking the same system of coordinates as before we have

$$\begin{cases} n_1 = 0 \\ n_2 = 0 \\ n_3 = 1 \end{cases}$$

We have

$$[\![s_{1j}]\!]\, n_j = [\![s_{13}]\!]\, n_3 \qquad \ldots \text{(A - 23)}$$

$$[\![s_{2j}]\!]\, n_j = [\![s_{23}]\!]\, n_3 \qquad \ldots \text{(A - 24)}$$

$$[\![s_{3j}]\!]\, n_j = [\![s_{33}]\!]\, n_3 \qquad \ldots \text{(A - 25)}$$

But

$$s_{ij} = \frac{E_i D_j - \frac{1}{2}\delta_{ij} E^2}{4\pi}$$

$$\therefore \quad s_{13}^{+} - s_{13}^{-} = 0 \qquad \ldots \text{(A - 26)}$$

$$s_{23}^{+} - s_{23}^{-} = 0 \qquad \ldots \text{(A - 27)}$$

$$S_{33}^{+} - S_{33}^{-} = \frac{E_3^{+} D_3^{+} - \frac{1}{2} E^{+2} - E_3^{-} D_3^{-} + \frac{1}{2} E^{-2}}{4\pi}$$

$$= \frac{D_3^{+}\left(E_3^{+} - E_3^{-}\right) - \frac{1}{2}\left(E_3^{+2} - E_3^{-2}\right)}{4\pi}$$

$$= \frac{\left(E_3^{+} - \frac{1}{2}\left(E_3^{+} + E_3^{-}\right)\right)\left(E_3^{+} - E_3^{-}\right)}{4\pi} \qquad \ldots \text{(A - 28)}$$

$$= \frac{\left(E_3^{+} - E_3^{-}\right)^2}{8\pi} = 2\pi P_3^2 .$$

From (A - 26), (A - 27) and (A - 28) in (A - 23), (A - 24) and (A - 25)

$$[\![S_{1j}]\!] n_j = 0 ,$$
$$[\![S_{2j}]\!] n_j = 0 ,$$
$$[\![S_{3j}]\!] n_j = 2\pi P_3^2 .$$

Since $n_j = (0,0,1)$ we can write this as

$$[\![S_{ij}]\!] n_j = 2\pi P_3^2 n_i = 2\pi \left(P_j n_j\right)^2 n_i \quad \therefore \text{(A-29)}$$

This is a tensor equation, therefore if true for one, then true for all coordinate systems.

APPENDIX B: ELASTICITY

The results given here are for infinitesimal displacements from the undeformed state.

Taking a fixed rectangular system of axes, if x_1, x_2, x_3 are the coordinates of a particle which originally occupied the position X_1, X_2, X_3 . then the deformation is specified by the relation

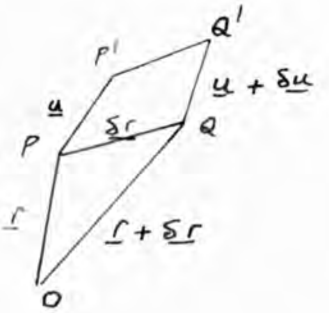
$$x_i = x_i(X_A) \qquad \text{... (B - 1)}$$

Let

$$u_i = x_i - X_i \qquad \text{... (B - 2)}$$

The vector $\underline{u}$ gives the displacement of each point.

Consider two neighbouring points P and Q of a material body with position vectors $\underline{r}$ and $\underline{r} + \underline{\delta r}$ which are given the infinitesimal displacements $\underline{u}$ and $\underline{u} + \underline{\delta u}$ respectively. We have



$$\underline{\delta u} = (\underline{\delta r} . \nabla)\, \underline{u} .$$

Then

$$\delta u_i = \partial x_j \frac{\partial u_i}{\partial x_j} ,$$

$$\therefore \delta u_i = \delta x_j \left(\frac{1}{2} \overline{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}} + \frac{1}{2} \overline{\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}} \right) . \qquad \text{(B - 3)}$$

$$= e_{ij}\, \delta x_j + [\underline{\Theta} \wedge \underline{\delta r}]_i ,$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \ldots (B - 4)$$

$$-\theta_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \qquad \ldots (B - 5)$$

and

$$\underline{\Theta} = \frac{1}{2} \operatorname{curl} \underline{u} \qquad \ldots (B - 6)$$

$\underline{\Theta}$ has components θ_i and

$$\theta_1 = +\theta_{23} = -\theta_{32} \text{ etc.} \ldots (B - 7)$$

The second term in (B - 3) represents a rigid body rotation, which does not alter the relative positions of each point; the first term gives a measure of the relative movement of nearby points. The functions e_{ij} are known as the strains and give a measure of the actual stretching and twisting of the material.

Consider a secondary deformation $\delta \underline{u}$ of the material. Let ϕ be a function of position in the medium and suppose that ϕ depends also on the deformation. Denote by $\delta\phi$ the differential of ϕ at a fixed point in space and by $\Delta\phi$ the differential of ϕ "following the displacement".

Then

$$\delta\phi = -\frac{\partial\phi}{\partial x_\alpha}\delta u_\alpha + \Delta\phi \qquad \ldots (B - 8)$$

For a vector function $\underline{E}$ let $\Delta\underline{E}$ denote its differential "following the displacement" and referred to axes which partake of the infinitesimal rotation $\delta\underline{\Theta}$. Then applying (B - 8) to each of the components of $\underline{E}$ and allowing for the rotation we find

$$\delta \underline{E} = -\frac{\partial \underline{E}}{\partial x_\alpha} \delta u_\alpha + \delta \underline{\Theta} \wedge \underline{E} + \Delta \underline{E} \qquad \ldots (B - 9)$$

Consider now a distribution of a quantity Q (e.g. charge) throughout a deformable medium. Let P be the density of the distribution and let p be the "density" reckoned per unit volume of the undeformed state. Then

$$p = J P,$$

where

$$J = \frac{\partial (x_1, x_2, x_3)}{\partial (X_1, X_2, X_3)} . \qquad \ldots (B - 10)$$

For deformations from the undeformed state $J \doteqdot 1 \quad p = P$ and

$$\Delta J = \frac{\partial \delta u_\alpha}{\partial x_\alpha} \qquad \ldots (B - 11)$$

So

$$\Delta p = J \Delta P + P \Delta J = \Delta P + P \frac{\partial \delta u_\alpha}{\partial x_\alpha}$$

$$\therefore \delta P = -\frac{\partial P}{\partial x_\alpha} \delta u_\alpha - P \frac{\partial \delta u_\alpha}{\partial x_\alpha} + \Delta p$$

$$\ldots (B - 12)$$

$$= -\frac{\partial}{\partial x_\alpha} (P \delta u_\alpha) + \Delta p \quad .$$

For the density $\underline{P}$ of distribution of a vector quantity let $\Delta \underline{p}$ denote a differential referred to axes which partake of the rotation $\delta \underline{\Theta}$. Then corresponding to (B - 12) we have

$$\delta \underline{P} = -\frac{\partial}{\partial x_\alpha}(\underline{P}\,\delta u_\alpha) + \delta\underline{\Theta} \wedge \underline{P} + \Delta \underline{P} \qquad \ldots (B - 13)$$

For a quantity e whose density per unit mass does not change (e.g. charge) (B - 12) gives

$$\delta e = -\frac{\partial}{\partial x_\alpha}(e\,\delta u_\alpha). \qquad \ldots (B - 14)$$

and (B - 8) gives

$$\Delta e = -e\,\frac{\partial(\delta u_\alpha)}{\partial x_\alpha} \qquad \ldots (B - 15)$$

NOTATION.

Quantity	Smith-White	Brown	Toupin	Used in this Dissertation
Displacement	$\underline{u}$	$\underline{u}$	$\underline{u}$	$\underline{u}$
Force (of electrical origin)	$\underline{F} = (\underline{P}\cdot\nabla)\underline{E}$		$f^i = P^j E^i_{;j}$	$\underline{F}$
Boundary Force (of electrical origin)	$\underline{T} = 2\pi P_n^2 \underline{n}$		$[t^{ij}_{MS}]$ $= \frac{\epsilon_0}{2}(P_j n_j)^2 n_i$	$\underline{T}$
Mechanical Boundary Force	$\underline{\Pi}$		T^i	$\underline{\Pi}$
Mass Density	τ	ρ	ρ	ρ
Polarisation per unit mass	$p (= \nu \underline{P})$	$\underline{M}'$ where $\underline{M}\,d\tau = \underline{M}'\,dm$ $\therefore \underline{M} = \underline{M}'\rho$ but $\rho = \rho_0/J$ $\therefore \underline{M}' = J\underline{M}/\rho_0$	$\underline{\pi}$ where $\underline{P} = \rho\underline{\pi}$ $\underline{\pi} = \frac{J\underline{P}}{\rho_0}$	$\underline{M}'$ or $\underline{P}'$ for Brown or Toupin; p for Smith-White. Note: $\underline{M}'$ (Brown) $= \underline{\pi}$ (Toupin) $= p/\rho_0$ (Smith-White)

N O T A T I O N.

Quantity	Smith-White	Brown	Toupin	Used in this Dissertation.
Element of Area	dF	ds	ds	ds
Charge Density	ρ			e
Couple (of electrical origin)	$\underline{G} = \underline{P} \wedge \underline{E}$ $G_3 = P_1 E_2 - E_1 P_2$	$\underline{M} \wedge \underline{H}$ No symbol	$m_{ij} = p^i E^j - p^j E^i$ $(m_{ij} = -m_{ji})$	Use $\underline{G}$ for vector G_{ij} for tensor Note: m_{12} (Toupin) $= G_3$ (Smith-White)
Deformation determinant	$\nu = \dfrac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}.$		$J = \lvert \det x^i_{;A} \rvert$	J Note: ν (Smith-White) $= J$ (Toupin)
Dielectric constant	K	K_e		K

N O T A T I O N.

Quantity	Smith-White	Brown	Toupin	Used in this Dissertation.
Electric Potential	$\phi = \begin{cases} \text{charge} \\ + \\ \text{dipoles} \end{cases}$ charge dipoles.		$\phi_M = \phi_0 + \phi_{MS}$ where ϕ_0 - charge ϕ_{MS} - dipoles	ϕ Note: ϕ_M (Toupin) $= \phi$ (Smith-White)
Strain	$v_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)$	$e_{xx}, e_{xy} \cdots$ etc	$\tilde{e}_{ij}$	e_{ij}
Anti-symmetric part of Strain Tensor	$\vartheta_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right)$			ϑ_{ij}
Electrical Stress Tensor	"Livens' version of Maxwell's tensor" $s_{ij} = \frac{D_i E_j - \frac{1}{2} E^2 \delta_{ij}}{4\pi}$		"Maxwell stress tensor" $t_{ijMS} =$ $E_{iMS}(e_0 E_{jMS} + P_j)$ $- \frac{e_0}{2} E_{MS}^2$.	s_{ijMS} for Toupin s_{ij} for Smith-White Note: inversion of subscripts in Smith-White.

NOTATION.

Quantity	Smith-White	Brown	Toupin	Used in this Dissertation.
Mechanical Stress Tensor	s_{ij} plane direction	X_y direction plane	t_{ij} direction plane	t_{ij} direction plane Note: Smith-White uses subscripts the other way round.
Symmetric part of Mechanical Stress Tensor	$\tilde{\omega}_{ij} = \frac{1}{2}(s_{ij} + s_{ji})$			$\tilde{\omega}_{ij}$
Anti-symmetric part of Mechanical Stress Tensor	$a_{ij} = \frac{1}{2}(s_{ij} - s_{ji})$			a_{ij}
Susceptibility	k	χ_e		k
Element of Volume	dv	$d\tau$	dV	dv ($d\tau$ for Brown)
$\underline{D}/\underline{E}$	$\underline{D} = \underline{E} + 4\pi\underline{P}$	$\underline{D} = \underline{E} + \gamma\underline{P}$	$\underline{D} = \epsilon_0\underline{E} + \underline{P}$	$\underline{D} = \underline{E} + 4\pi\underline{P}$

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