TOPICS IN THE MODEL THEORY OF

ABELIAN AND NILPOTENT GROUPS

BY

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A thesis submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

(MATHEMATICAL LOGIC)

at

BEDFORD COLLEGE, UNIVERSITY OF LONDON

1977

Abstract

This thesis falls naturally into two parts, each concerned with the model theory of a different variety of groups. The algebraist will find, in the preliminary chapter, a survey of the necessary model theory.

A classification of abelian groups by their stability properties has been obtained from results of Eklof and Fisher on saturated abelian groups using Shelah's theorem relating saturation and stability. In Chapter two we develop a direct approach to this problem. We obtain a simple formula for calculating the exact cardinality of the Stone space of a given abelian group. We are then able to distinguish between the various stable classes giving new necessary and sufficient conditions for an abelian group to be superstable. Our method generalises easily to modules over Dedekind domains.

The third chapter contains answers to the question of how much saturation or stability is preserved by the free product, $*$, in the variety of all nil-2 groups. First we consider the nil-2 free product of groups, one factor being finite. The elements of such a product are shown to possess a unique normal form which we use to prove, under certain conditions, a "Feferman-Vaught style" theorem for $*$. As a consequence we obtain a condition sufficient for $*$ to preserve both saturation and stability. In the case of saturation, this condition is shown also to be necessary. These results are extended to products of bounded nil-2 groups, the key being a restricted distributive law for $*$ over the direct product. Finally, we classify numerous nil-2 groups by their stability properties. Some questions are left open, the most interesting

of which is whether * preserves ω-stability. Results on the absolutely free product of groups, showing it preserves neither model-theoretic property, are also included in this chapter.

Contents

Chapter 1: Preliminaries

1. Outline

In recent years the ideas and techniques of model theory have been successfully applied to the study of various branches of algebra. We are concerned here with the study of the theory of groups. The thesis falls naturally into two parts, each dealing with a different variety of groups, and for this reason we have departed from the normal practice of introducing results in the first chapter. This section, then, gives only a bare outline of the structure of the thesis, Detailed descriptions of the problems we tackle and brief surveys of the literature on the subject are given in the introductions to each chapter.

Shelah has classified all complete theories by their stability properties (see [37]) and it has been suggested (see [1]) that the notion of stability may be used to classify all groups. Such has already been achieved in the case of abelian groups (see section 2.1), but in an indirect manner using results of Eklof and Fisher [8] on saturated abelian groups. In Chapter 2 we develop a direct approach to this problem, obtaining a simple formula for calculating the exact cardinality of the Stone space of a given abelian group and new necessary and sufficient conditions for an abelian group to be superstable.

In Chapter 3 we investigate the model theory of one of the simplest non-abelian varieties of groups: the variety of all nilpotent groups of class at most 2 (nil-2 groups). We ask questions of the following kind: if P is a given model-theoretic property, under what conditions does the nil-2 free product operation, $*$, preserve P ? Now, if one factor in a given nil-2 free product of groups is finite, the elements possess a convenient unique normal form which can be used to prove a Feferman-Vaught style (see [11]) theorem for the product $*$. As a consequence, we give preservation theorems in restricted situations for each of the following model-theoretic properties: elementary equivalence, saturation and stability. In addition, we classify numerous nil-2 groups according to their stability properties. We do leave some questions on stability open and these are enumerated in section 3.6: the most interesting of these is whether $*$ preserves ω-stability.

We have attempted to make this thesis accessible to both the model theorist and the algebraist with a smattering of logic. Thus, aside from sparing none of the algebraic details in our proofs, we have included in this chapter a section (section 3) summarising the definitions and results on saturation and stability which we need. Our logical terminology is described in the following section. Details of the algebraic notation, definitions and elementary lemmas required are reserved for the second section in each chapter.

2. Logical Terminology

Set-theoretic

We shall assume familiarity with the fundamentals of naive set theory as developed, for example, in Rotman and Kneebone [26]. (For a good summary of all that is required, see Appendix A of Chang and Keisler [7].)

The symbols $\cup$, $\cap$, $-$, $\times$ denote, respectively, the union, intersection, difference and cartesian product of sets and $\subset$ is the subset relation.

An ordinal is conceived of as the set of all smaller ordinals, finite ordinals being identical with the natural numbers. As a rule we shall use lower case Greek letters to denote ordinals; integers are normally denoted by the letters d, e, i, j, k, l, m, n, p, q and r, with p being reserved for prime numbers.

If α , β are ordinals, then $\underline{\alpha < \beta}$ is equivalent to $\underline{\alpha \in \beta}$ and we shall use whichever notation seems more appropriate at the time.

The power, or cardinality, of a set X is denoted by $|X|$. Cardinals are identical with initial ordinals, that is, α is a cardinal if and only if $|\alpha| = \alpha$.

The α-th infinite cardinal is denoted by $\aleph_\alpha$, or alternatively by ω_α ; we use ω in place of $\aleph_0$ or ω_0 .

If κ and λ are cardinals, then $\underline{\kappa+\lambda}$, $\underline{\kappa\cdot\lambda}$ and $\underline{\kappa^\lambda}$ denote, respectively cardinal sum, product and exponentiation. (We use ordinal sum only in the definition of concatenation; see below.) 2^κ is thus the cardinal of the set of all subsets of the cardinal κ . The symbols $\sum$, $\prod$ are used to denote, respectively, the cardinal sum and product of a family of cardinals.

If A and B are sets, then $\underline{f: A \to B}$ denotes a mapping with domain A and range a subset of B . The restriction of f to a subset C of A is denoted by $f \upharpoonright C$. $\underline{{}^A B}$ denotes the set of all mappings $\underline{f: A \to B}$.

If α is an ordinal and X a set, then an α-termed sequence of elements of X (or, from X) is a member of ${}^\alpha X$. Such a sequence will be written, variously, as $\langle x_\beta : \beta < \alpha \rangle$, $\langle x_\beta \rangle_{\beta<\alpha}$ or, when α is clear from the context or is relatively unimportant, $\langle x_0, x_1, \ldots \rangle$ or $\bar{x}$. When $\alpha = 1$, we write x_0 instead of $\langle x_0 \rangle$. If $\bar{x} \in {}^\alpha X$, α is called the length of $\bar{x}$, and we write $\ell(\bar{x}) = \alpha$. The set $\underline{{}^{<\alpha}X}$ denotes the set of all sequences from X having length $< \alpha$. In particular, ${}^{<\alpha}X$ is the set of all finite sequences (that is, sequences of finite length) from X .

Let $\bar{x}$ be an α-termed sequence and $\bar{y}$ a β-termed sequence, then the <u>concatenation</u>, $\bar{x}^\frown\bar{y}$, of $\bar{x}$ and $\bar{y}$ is the $(\alpha+\beta)$-termed (ordinal sum intended) sequence defined by

$$\bar{x}^\frown\bar{y}(\gamma) = x(\gamma) \quad \text{if } \gamma < \alpha$$

$$\bar{x}^\frown\bar{y}(\alpha+\gamma) = \bar{y}(\gamma) \quad \text{if } \gamma < \beta .$$

Thus, intuitively, $\bar{x}^\frown\bar{y}$ is obtained by writing down all the elements of the sequence $\bar{x}$ followed by the elements of $\bar{y}$.

<u>Model-theoretic</u>

We assume the reader is familiar with the elements of first-order predicate logic and the classical definitions and results of its model theory; in other words, the contents of Chapters 1 to 4 of Bell and Slomson [4]. An excellent model-theoretic source for the entire thesis is the book by Chang and Keisler [7].

The letter L always denotes a <u>first-order language</u> with equality which, it is convenient to assume, contains denumerably many variables which we shall denote by u_i, v_i . L may, or may not be denumerable. Most of the time L shall be the language of groups and so it contains a binary function symbol + (or, $\cdot$), for the group operation, and a constant symbol 0 (or, 1) for the group identity. (The symbol - (or, $^{-1}$) for group inverse is then introduced by definition.)

We shall frequently identify L with the set of all formulae in the symbols of L , the Greek letters θ, φ, ψ, χ and σ being used to denote formulae. If φ is a formula with the variables $u_0,u_1,\ldots,u_{n-1}$ free we shall write $\varphi(u_0,\ldots,u_{n-1})$ or $\varphi(\bar{u})$, where $\bar{u} = \langle u_0,\ldots,u_{n-1}\rangle$.

A <u>first-order theory</u> T in a language L is a consistent set of sentences of L . The language of T is sometimes denoted by L(T) , and its cardinality by $|L(T)|$, or $|T|$. Normally, T will be a theory of groups in which case the axioms of T include the group axioms:

(1) $\forall u_1 u_2 u_3 (u_1+(u_2+u_3) = (u_1+u_2)+u_3)$;

(2) $\forall u_1 (u_1+0 = u_1 \wedge 0+u_1 = u_1)$;

(3) $\forall u_1 \exists u_2 (u_1+u_2 = 0 \wedge u_2+u_1 = 0)$.

A theory T is said to be <u>complete</u> if and only if, for each sentence $\sigma \in L(T)$, either σ or $\sim\sigma$ belongs to T .

Structures for L , or <u>L-structures</u> are denoted by capital letters A, B, G, and H, and we shall not distinguish between a structure A and its domain: thus $a \in A$ means that a is an element of the domain of A and $|A|$ denotes the cardinality of the domain of A .

We use the notation $A \simeq B$, $A \subset B$, $A \prec B$ and $A \equiv B$, respectively, for A is isomorphic to B , A is a substructure of B , A is an elementary substructure of B and A is elementarily equivalent to B .

The symbol $\models$ denotes the usual satisfaction predicate. So, if $\varphi(u_0,\ldots,u_{n-1})$ is a formula of L and $\bar{a} = \langle a_0,\ldots,a_{n-1}\rangle$ a sequence from the L-structure A, we write

$$A \models \varphi[a_0,\ldots,a_{n-1}]$$

if $\varphi(\bar{u})$ is satisfied in A by the sequence $\bar{a}$. When the number of free variables in $\varphi(\bar{u})$ is not specified, it is understood that the sequence $\bar{a}$ matches $\bar{u}$.

If T is a theory, we say that A is a <u>model</u> of T , and write $A \models T$, if, for every sentence σ of T , $A \models \sigma$. The <u>theory of A</u> , denoted ThA , is the set of all sentences which hold in A :

$$\text{ThA} = \{\sigma \in L : \sigma \text{ is a sentence and } A \models \sigma\} .$$

Clearly, ThA is a complete theory.

Let A be an L-structure and X a subset of A . Then the expansion of L by X is the new language, denoted $L(X)$, obtained from L by adding one new constant symbol c_a for each $a \in A$:

$$L(X) = L \cup \{c_a : a \in A\} .$$

It is understood that if $a \neq b$, then the symbols c_a and c_b are distinct. Associated with the language $L(X)$ is the expanded structure, $(A , a)_{a\in X}$, obtained by interpreting each new constant c_a by the element a . In fact we shall not normally distinguish between the constant symbol and its intended interpretation, allowing a to denote both constant symbol and element of A . When $X = A$, we shall write A^+ instead of $(A , a)_{a\in A}$ and when X is a sequence $\bar{a}$ from A we shall write $(A , \bar{a})$. The elements of X in such an expansion of the language are called parameters.

The notion of a definable subset of an L-structure A is used throughout the thesis. Let X be a subset of A . Then, X is said to be definable in A if and only if there exists a formula $\varphi(u_0, u_1, \ldots, u_n)$ of L and a sequence $\bar{b} \in {}^nA$ such that

$$X = \{a \in A : A \models \varphi[a,\bar{b}]\} .$$

In such a case we also say that X is definable in A with n parameters from A ; or, the formula $\varphi(u_0,\bar{b})$ of $L(\bar{b})$ defines X in $(A,\bar{b})$. A substructure B of A is said to be definable in A if the domain of B is a definable subset of A .

Finally, a notion needed in Chapter 2. Let T be a first-order theory, L its language and Δ a subset of the formulae of L . Then, T is said to admit elimination of quantifiers modulo Δ , if every formula of L is equivalent, relative to T , to a disjunction of conjunctions of formulae in Δ or their negations. The set Δ is called the set of basic formulae for the elimination of quantifiers in T .

In conclusion, we should point out that, whenever it makes our exposition clearer, we shall use the symbol $\Rightarrow$ for "implies", and the symbol $\Leftrightarrow$ or the standard abbreviation "iff" for "if and only if". All further terminology and notation will be introduced when required.

3. Survey of Saturation and Stability

In this section we survey some of the basic facts about the properties of saturation and stability which we shall use throughout this thesis. Any unexplained algebraic notation may be found in sections 2.2 and 3.2 of the following chapters. The notation and definitions given here are, for the most part, consistent with [37].

Let Δ be a set of formulae in a language L , A an arbitrary L-structure, X a subset of A and m a positive integer. Then, a Δ-m-type over $(A , a)_{a\in X}$ is a set, t , of formulae $\varphi(\bar{u},\bar{x})$ of $L(X)$ such that

(1) $\bar{u} = \langle u_0, u_1, \ldots, u_{m-1}\rangle$, $\bar{x} \in {}^{<\omega}X$ and either $\varphi(\bar{u},\bar{v})$ or its negation belongs to Δ ; and

(2) for every finite subset s of t , $(A,a)_{a\in X} \models \exists\bar{u} \bigwedge_{\varphi\in s} \varphi$.

The elements of X are called the parameters of t . Types will usually be denoted by s , t . When Δ is the set of all formulae of L we shall omit it; when $m=1$, we shall omit it too. If we wish to display the free variables of t we shall write $t(\bar{u})$. When $m=1$, we shall write $t(u)$, u being the free variable in this case.

Let Γ be a subset of Δ , and t a Δ-m-type over $(A , a)_{a\in X}$. Then, the restriction, $t|\Gamma$, of t to Γ is the Γ-m-type over $(A , a)_{a\in X}$ defined by

$$t|\Gamma = \{\varphi(\bar{u},\bar{x}) \in t : \text{either } \varphi(\bar{u},\bar{v}) \text{ or its negation belongs to } \Gamma\} .$$

A Δ-m-type t over $(A , a)_{a\in X}$ is said to be complete if and only if for every formula $\varphi(\bar{u},\bar{v})$ in Δ and sequence $\bar{x}$ from X of suitable length, either $\varphi(\bar{u},\bar{x}) \in t$ or $\sim\varphi(\bar{u},\bar{x}) \in t$. If t is complete, then so is $t|\Gamma$, for every subset, Γ , of Δ . An application of Zorn's Lemma shows that every Δ-m-type can be extended to some complete Δ-m-type. We shall denote the set of all complete Δ-m-types over $(A , a)_{a\in X}$ by $S^m_\Delta(A , a)_{a\in X}$ where, as before, Δ is omitted if it is the set of all formulae of L and m is omitted if m=1 . When X=A we shall write $S^m_\Delta(A)$ instead of $S^m_\Delta(A , a)_{a\in A}$. $S^m(A , a)_{a\in X}$ is the Stone space of the Boolean algebra of all equivalence classes of formulae of L(X) with free variables $u_0,\ldots,u_{m-1}$ under the equivalence relation, $\sim$, defined by

$$\varphi \sim \psi \text{ if and only if } (A,a)_{a\in X} \models \forall\bar{u}(\varphi \leftrightarrow \psi) .$$

By <u>the Stone space of A</u> we mean the set S(A) .

It will often be convenient, from the point of view of notation, to assume that the set X is well-ordered as a sequence $\bar{a}$. We say that a Δ-m-type, t , over $(A , \bar{a})$ is <u>realised in A</u> if there exists a sequence $\bar{b} \in {}^mA$ such that, for every formula $\varphi(\bar{u},\bar{x})$ in t ,

$$(A , \bar{a}) \models \varphi[\bar{b}] .$$

If there is no such sequence in A , then we say that <u>t is omitted by A</u> , or, <u>A omits t</u> . The following definitions are important.

3.1 <u>Definitions</u>

Let $\varkappa$ be a cardinal, finite or infinite. Then, a structure, A , is said to be <u>$\varkappa$-saturated</u> if, for every sequence $\bar{a}$ from A of length $<\varkappa$, every type $t \in S(A , \bar{a})$ is realised in A . A is said to be <u>saturated</u> if it is $|A|$-saturated.

Most of the following Theorems are, by now, part of the folklore of saturated structures. For more details see [4], Chapter 11 or [7], Chapter 5.

3.2 <u>Theorem</u>

Let $\varkappa$ be a cardinal. Then,

(i) every finite structure is $\varkappa$-saturated; and

(ii) every structure has a $\varkappa$-saturated elementary extension.

The next two theorems show that saturated structures also realise complete m-types for certain $m>1$. The proof of Theorem 3.4 is omitted because it is similar to that given for Theorem 3.3.

3.3 Theorem

Let $\varkappa$ be a finite cardinal. Then, a structure A is $\varkappa$-saturated if and only if it realises every complete m-type with r parameters from A, where $m+r$ does not exceed $\varkappa$.

Proof

Let $\varkappa$ be a finite cardinal and A a $\varkappa$-saturated structure. We prove, by induction on m, that A realises every complete m-type with r parameters, provided $m+r \leqq \varkappa$. This follows immediately from our initial assumption if $m=1$. So let $\bar{a}$ be a sequence from A of length r and $t \in S^n(A, \bar{a})$, where $n+r \leqq \varkappa$, and assume, as inductive hypothesis, that the result is proved for all $m<n$ and k such that $m+k \leqq \varkappa$.

First, we define an $(n-1)$-type, t', over $(A, \bar{a})$ by

$$t' = \{\exists u_0 \varphi(u_0, u_1, \ldots, u_{n-1}, a) : \varphi(\bar{u}, \bar{a}) \in t\} .$$

Then, by the inductive hypothesis, since $(n-1)+r = n+r-1 \leqq \varkappa$, A realises every type in $S^{n-1}(A, \bar{a})$ and hence, the type t' is also realised in A, say by $\bar{b} = \langle b_1, \ldots, b_{n-1} \rangle$. Next, set

$$t'' = \{\varphi(u_0, \bar{b}, \bar{a}) : \varphi(\bar{u}, \bar{a}) \in t\} .$$

Then, t'' is a type with $(n-1)+r = n+r-1 < \varkappa$ parameters from A and hence, since A is $\varkappa$-saturated, A realises t''. It is clear that if t'' is realised in A by b_0, then the complete n-type t is realised in A by $\langle b_0, b_1, \ldots, b_{n-1} \rangle$. This completes the proof for $m = n$ and the result follows by induction.

The other direction of the proof is trivial. //

3.4 Theorem

Let $\varkappa$ be an infinite cardinal, then a structure A is

κ-saturated if and only if, for every positive integer m and every sequence $\bar{a}$ from A of length $<\kappa$, every m-type $t \in S^m(A, \bar{a})$ is realised in A .

3.5 Theorem

Let κ be a cardinal, A a κ-saturated structure and B a substructure of A which is definable in A using $r < \kappa$ parameters. Then B is $\kappa \dot{-} r$ saturated, where,

$$\kappa \dot{-} r = \begin{cases} \kappa - r & \kappa < \omega \\ \kappa & \kappa \geqq \omega \end{cases}$$

3.6 Theorem (Waszkiewicz and Weglorz [39], Theorem 1.5)

If A and B are κ-structures for a countable language, then A×B is also κ-saturated.

The concept of a totally transcendental complete theory originated in 1965 in a paper by Morley [23], and was later generalised, by Shelah [35], to the notion of a stable theory. For a thorough analysis of the properties of stable and unstable theories see [37] .

The notion of a stable theory involves the size of the Stone space of any model of the theory. Suppose A is an arbitrary structure. Then, $|S(A)| \geqq |A|$, for every element, a , of A determines a unique type $t \in S(A)$, called the principal type generated by a, and defined by

$$t = \{\varphi(u,\bar{x}) : A^+ \models \varphi[a]\} .$$

If A happens to be finite, then we also have $|S(A)| \leqq |A|$ since each $t \in S(A)$ must include the formula $u = a$ for exactly one a in A and hence each complete type over A^+ is principal. It follows that, for finite structures A , $|S(A)| = |A|$ and so the cardinality of the Stone space is only ever in doubt for infinite structures.

3.7 Definitions

Let κ be an infinite cardinal and T a complete, first-order theory. Then T is said to be $\varkappa$-stable (or, stable in power $\varkappa$) if, for every model A of T, $|A| \leqq \varkappa$ implies $|S(A)| \leqq \varkappa$. If T is not $\varkappa$-stable, then we say that T is $\varkappa$-unstable. T is said to be stable if T is $\varkappa$-stable for some infinite cardinal $\varkappa$. T is unstable if T is $\varkappa$-unstable for every infinite cardinal $\varkappa$. Let A be an arbitrary structure, then A is $\varkappa$-stable (stable, or unstable) if and only if ThA is $\varkappa$-stable (stable, or unstable).

3.8 Theorem

Every finite structure is stable in every infinite power.

3.9 Theorem (Morley, [23])

Let T be a complete first order theory in a countable language. Then, T is ω-stable if and only if T is $\varkappa$-stable for every infinite cardinal $\varkappa$.

The following theorem, due to Shelah, gives a classification of complete theories by the cardinals in which they are stable.

3.10 Theorem (Shelah, [37])

For every complete theory T exactly one of the following alternatives occurs:

(i) T is stable in every cardinal $\varkappa \geqq 2^{|T|}$;

(ii) for every model $A \models T$, $|A|^{\omega} \leqq |S(A)| \leqq |A|^{|T|}$;

(iii) T is unstable.

We make the following distinctions.

3.11 Definitions

Let T be a complete first-order theory. Then, we say that

T is superstable if T is $\varkappa$-stable for every $\varkappa \geqq 2^{|T|}$;

T is strictly-superstable if T is $\varkappa$-stable if and only if $\varkappa \geqq 2^{|T|}$;

T is merely-stable if T is stable but not superstable.

Notice that the merely-stable theories are the ones for which alternative (ii) above holds. In the case of a countable language this is equivalent to being $\varkappa$-stable in all powers $\varkappa$ for which $\varkappa = \varkappa^{\omega}$. By Theorem 3.9, it is evident that a countable ω-stable theory is also superstable. Furthermore, it follows from the proof of that theorem that a countable theory is strictly-superstable if and only if it is superstable but not ω-stable. There are numerous examples, in this thesis, of theories which are unstable, merely-stable, strictly-superstable and ω-stable. The next theorem shows that, for a stable structure A, the cardinality of the Stone space $S^m(A)$, for each $m > 1$, is also bounded by $|A|$.

3.12 <u>Theorem</u> (Shelah [37], Lemma 2.10)

A complete theory T is $\varkappa$-stable if and only if for every $m \geqq 1$ and every model $A \models T$, $|A| \leqq \varkappa$ implies $|S^m(A)| \leqq \varkappa$.

The following characterisation of unstable theories does not involve counting types and we rely heavily upon it, or its Corollary, in Chapter 3.

3.13 <u>Theorem</u> (Shelah [37], Theorem 2.13)

A complete theory T is unstable if and only if there is a formula $\varphi(\bar{u},\bar{v}) \equiv \varphi(u_1,\ldots,u_k,v_1,\ldots,v_k)$, a model $A \models T$ and sequences $\bar{a}_n \in {}^kA$, for each $n \in \omega$, such that

$$A \models \varphi[\bar{a}_m,\bar{a}_n] \text{ if and only if } m < n .$$

3.14 <u>Corollary</u>

A complete theory T is unstable if and only if there is a model $A \models T$, a formula $\varphi(\bar{u},\bar{v}) \equiv \varphi(u_1,\ldots,u_k,v_1,\ldots,v_k)$ which may involve parameters from A and sequences $\bar{a}_n,\bar{b}_n \in {}^kA$ for each $n \in \omega$, such that

(i) the sequences $\bar{a}_n{}^\frown\bar{b}_n$, $n \in \omega$, are all distinct; and

(ii) $A^+ \models \varphi[\bar{a}_m,\bar{b}_n]$ if and only if $m \leqq n$.

<u>Proof</u>

First we assume that T is unstable and let $\varphi(\bar{u},\bar{v})$ be the formula, A the model and $\bar{a}_n$ the sequences, given by Theorem 3.13, satisfying

$$A \models \varphi[a_m,a_n] \text{ if and only if } m < n .$$

It is clear that the sequences $\bar{a}_n$ are all distinct for, if $m < n$, then we have

$$A \models \sim\varphi[\bar{a}_n,\bar{a}_m] \wedge \varphi[\bar{a}_m,\bar{a}_n] .$$

Thus, the sequences $\bar{a}_n^\frown \bar{a}_n$, $n \in \omega$, are all distinct giving (i) with $\bar{b}_n = \bar{a}_n$. Then, defining $\psi(\bar{u},\bar{v})$ by

$$\psi(\bar{u},\bar{v}) \equiv \varphi(\bar{u},\bar{v}) \vee \bigwedge_{1\leq i\leq k} u_i = v_i ,$$

we have

$$\begin{aligned} A \models \psi[\bar{a}_m,\bar{a}_n] &\Leftrightarrow A \models \varphi[\bar{a}_m,\bar{a}_n] \text{, or } \bar{a}_m = \bar{a}_n \\ &\Leftrightarrow m < n \text{ or } m = n \\ &\Leftrightarrow m \leq n , \end{aligned}$$

giving (ii) for $\psi(\bar{u},\bar{v})$ and the sequences $\bar{a}_n,\bar{b}_n$ with $\bar{b}_n = \bar{a}_n$.

Conversely, let $A \models T$ and $\varphi(\bar{u},\bar{v},\bar{x})$ be a formula involving parameters $\bar{x} = \langle x_1,\ldots,x_r\rangle$ from A for which there exist sequences $\bar{a}_n,\bar{b}_n \in {}^kA$ with conditions (i) and (ii) holding. Let $u_{k+1},\ldots,u_{2k+r}$, $v_{k+1},\ldots,v_{2k+r}$ be variables new to $\varphi(\bar{u},\bar{v},\bar{x})$ and $\tilde{\varphi}(u_1,\ldots,u_k,v_{k+1},\ldots,v_{2k+r})$ the formula obtained from φ by replacing each occurrence of the variables $v_1,\ldots,v_k$ by new variables $v_{k+1},\ldots,v_{2k}$ respectively, and each occurrence of the parameters $x_1,\ldots,x_r$ by new variables $v_{2k+1},\ldots,v_{2k+r}$ respectively. Define a formula ψ, with no parameters, by

$$\psi(u_1,\ldots,u_{2k+r},v_1,\ldots,v_{2k+r}) \equiv \tilde{\varphi}(u_1,\ldots,u_k,v_{k+1},\ldots,v_{2k+r}) \wedge \bigvee_{1\leq i\leq 2k} u_i \neq v_i .$$

Then, setting $\bar{c}_n = \bar{a}_n^\frown \bar{b}_n^\frown \bar{x}$, for each $n \in \omega$, we have by (i) and (ii) above,

$$\begin{aligned} A \models \psi[\bar{c}_m,\bar{c}_n] &\Leftrightarrow A \models \tilde{\varphi}[\bar{a}_m,\bar{b}_n,\bar{x}] \text{ and } \bar{a}_m^\frown \bar{b}_m \neq \bar{a}_n^\frown \bar{b}_n \\ &\Leftrightarrow A^+ \models \varphi[a_m,b_n] \text{ and } m \neq n \end{aligned}$$

$$\Leftrightarrow \ m \leqq n \ \text{ and } \ m \neq n$$

$$\Leftrightarrow \ m < n$$

Thus, by Theorem 3.13, T is unstable. //

3.15 <u>Theorem</u> (Sabbagh [27], Proposition 3 bis)

Every definable substructure of a κ-stable structure is κ-stable; the quotient of a κ-stable structure by a definable congruence is κ-stable.

The next theorem is usually attributed to Macintyre [19], who claims it for the case $\kappa = \omega$. A proof, for all $\kappa \geqq |L|$, is given in the introduction to Eklof and Fisher [8]. However, this result is implicit in an earlier paper by Waszkiewicz and Węglorz [39] and it is that proof we give here.

3.16 <u>Theorem</u> (Waszkiewicz and Węglorz)

If A and B are κ-stable structures for a countable language, then $A \times B$ is also κ-stable.

<u>Proof</u>

Let $t \in S(A\times B)$. Then, by [39] Lemma 1.2, there are $t_1 \in S(A)$, $t_2 \in S(B)$ such that if $A_1 \succ A$ and $B_1 \succ B$ with a realising t_1 in A_1 and b realising t_2 in A_2, then (a,b) realises t in $A_1 \times B_1$. Choosing one such pair (t_1,t_2) for each t determines a mapping θ from $S(A\times B)$ into $S(A)\times S(B)$. Furthermore, θ is one-one for suppose $\theta(t) = \theta(t') = (t_1,t_2)$. Now, elementary extensions A_1 of A and B_1 of B exist with $a \in A_1$ realising t_1 and $b \in B_1$ realising t_2. Hence, (a,b) realises both t and t' in $A_1 \times B_1$. Since this is impossible unless $t = t'$ we conclude that θ is one-one. Thus,

$$|S(A\times B)| \leqq |S(A)|\ |S(B)|$$

and from this the result follows immediately. //

We see from the proof above that, whether A and B are stable or not, $|S(A\times B)| \leqq |S(A)|\ |S(B)|$. In [1] (proof of Theorem 1.5), Baldwin and Saxl credit Macintyre with proving $|S(A\times B)| = |S(A)|\ |S(B)|$. Were this the case, then the converse of Theorem 3.16 would follow immediately. However, in a private communication to P. Olin (21, January, 1977), Macintyre denied any claim to this result and stated that he believed it to be an open question. Of course, using Theorem 3.15, it is easy to see that the converse of 3.16 will hold for any pair A, B of structures which are definable in $A \times B$. We also have the following partial converse which is a special case of a result due to Berthier.

3.17 Theorem (Berthier [6], Theorem II. 4)

If A and B are groups and $A \times B$ is stable, then A and B are stable.

In the case of abelian groups we can be more precise.

3.18 Theorem

If A and B are abelian groups, then $A \times B$ is $\varkappa$-stable if and only if A and B are $\varkappa$-stable.

Proof

For the notation and details of the terms used in this proof the reader is referred to Chapter 2 of this thesis.

It is easily verified that, for each prime p and integer $k \geqq 0$,

$$p^k(A \times B) = p^k A \times p^k B ,$$

and, hence

$$p^k(A\times B)/p^{k+1}(A\times B) \simeq p^k A/p^{k+1} A \times p^k B/p^{k+1} B .$$

Thus,

$$tf(p,k;A\times B) = tf(p,k;A)\ tf(p,k;B) .$$

From this and Theorems 2.4.4 and 2.4.7 it follows that $A\times B$ is ω-stable (superstable) if and only if A and B are ω-stable (superstable).

Now it follows from the results in section 2.4 that every abelian group is either ω-stable, strictly-superstable or merely-stable.

Suppose $A \times B$ is $\varkappa$-stable. We show that A and B are both $\varkappa$-stable. There are three possibilities:

(a) <u>$A \times B$ is ω-stable</u>. Then, by what we have proved above A and B are ω-stable and hence, by Theorem 3.9, also $\varkappa$-stable.

(b) <u>$A \times B$ is strictly-superstable</u>. Then, since $A \times B$ is $\varkappa$-stable, we must have $\varkappa \geqq 2^{\omega}$. Thus, since A and B are superstable, they must be $\varkappa$-stable.

(c) <u>$A \times B$ is merely-stable</u>. Then (see the remarks after Definitions 3.11), since $A \times B$ is $\varkappa$-stable, we must have $\varkappa = \varkappa^{\omega}$. But, by Theorem 2.4.1, every abelian group is stable in such powers, and so A and B are $\varkappa$-stable. //

The following theorem has been proved independently by Sabbagh and by Baldwin and Saxl; the proof we give here is our own.

3.19 <u>Theorem</u> (Sabbagh [28]; Baldwin and Saxl [1])

Let $\{G_i\}_{i \in I}$ be a family of groups, infinitely many of which are non-abelian. Then, both the cartesian product $(\overline{\prod} G_i)$ and the direct product $(\prod G_i)$ of the family are unstable.

<u>Proof</u>

Without loss of generality, we shall assume that I is an ordinal and for each $\alpha < \omega$, the group G_α is non-abelian. Choose $a_\alpha, b_\alpha \in G_\alpha, \alpha < \omega$, with $a_\alpha b_\alpha \neq b_\alpha a_\alpha$, and let e denote the identity element for each G_α. For each $n < \omega$, we define elements x_n, y_n belonging to both products $\overline{\prod_{\alpha \in I}} G_\alpha$ and $\prod_{\alpha \in I} G_\alpha$, by:

$$x_n(\alpha) = \begin{cases} a_\alpha & \alpha \leqq n, \\ e & \alpha > n \end{cases}$$

$$y_n(\alpha) = \begin{cases} a_\alpha & \alpha < n. \\ b_n & \alpha = n \\ e & \alpha > n \end{cases}$$

So, with $G = \overline{\prod_{\alpha \in I}} G_\alpha$ or $\prod_{\alpha \in I} G_\alpha$, $\varphi(\bar{u}, \bar{v})$ defined by

$$\varphi(u_1,u_2,v_1,v_2) \equiv u_1v_2 = v_2u_1$$

and $\bar{g}_n = \langle x_n, y_n \rangle$, we have

$$G \models \varphi[\bar{g}_m,\bar{g}_n] \Leftrightarrow G^+ \models x_m y_n = y_n x_m$$
$$\Leftrightarrow m < n .$$

Thus, by Theorem 3.13, G is unstable. //

4. Acknowledgements.

I wish to express my warm thanks to my supervisor, Dr. Wilfrid Hodges, for his continued support and encouragement throughout the period of my research and for his invaluable assistance in overcoming the hurdles of submitting my thesis from another country.

Special thanks go to Professor Philip Olin of York University Toronto for recommending to me the study of nilpotent groups and for kindly supervising my research on Chapter 3 . I am also grateful for his painstaking reading of the entire thesis and his many helpful suggestions for its improvement.

Finally, I thank Mrs. Linda Herskowitz for her excellent typing of the manuscript.

Chapter 2: The Stability of Abelian Groups

1. Introduction

Any investigation of the stability properties of groups would naturally begin with a study of abelian groups. In this section we survey previous work on this instance of the problem and explain our own contribution to its solution.

A partial answer was provided in 1970 by Macintyre [19], who gave a simple algebraic characterisation of the ω-stable abelian groups.

1.1 Theorem (Macintyre)

An abelian group is ω-stable if and only if it can be written as the direct sum of a divisible group and a bounded group.

However, the proof of this theorem uses two direct sum decompositions for divisible and bounded abelian groups. These decomposition results have, at present, no generalisation covering all abelian groups and thus, there is no obvious way of extending Macintyre's method. The situation for non-ω-stable abelian groups has been handled in a quite different (and more indirect) way.

In 1972, Eklof and Fisher [8] gave new model-theoretic proofs of the results of Szmielew [38] by determining invariants which characterise saturated abelian groups up to isomorphism. In doing so, they were also able to show that every complete theory of abelian groups has a saturated model. Indeed, they established precisely the cardinals in which such models exist. Using this information, together with some highly non-trivial results of Shelah, a complete solution to the problem of stability can be given. (This fact has already been observed by a number of people, among them Baldwin and Saxl [1]). In the next few paragraphs we explain how this is achieved.

Let A be an arbitrary abelian group. Denote by (*) the following pair of conditions:

$$\left.\begin{array}{ll}\text{(a)} & Tf(p;A) < \infty \text{, for all primes } p \text{; and} \\ \text{(b)} & \text{there exist only finitely many primes } p \\ & \text{such that, for some } n\text{, } U(p,n;A) = \infty .\end{array}\right\} \quad (*)$$

Both $Tf(p;A)$ and $U(p,n;A)$ are invariants of A ; we define $Tf(p;A)$ in section 2 and $U(p,n;A)$ is one of the Ulm invariants (see Kaplansky [18],or Eklof and Fisher [8], for details). In fact, from the point of view of understanding the present discussion the definitions of these terms are irrelevant. Conditions (*) are those appearing (under II (b) (i)) in the table on page 146 of Eklof and Fisher's paper. From this table, we extract the following theorem.

1.2 Theorem (Eklof and Fisher)

Let A be an ω-stable abelian group. Then,

(i) if A satisfies (*) , ThA has a saturated model of power $\varkappa$ if and only if $\varkappa \geqq 2^{\omega}$; and

(ii) if A does not satisfy (*) , ThA has a saturated model of power $\varkappa$ if and only if $\varkappa^{\omega} = \varkappa$.

The link between stability and saturation is provided by the next two results.

1.3 Theorem (Shelah, Wierzejewski)

Let T be a countable stable theory and $\varkappa > \omega$. Then, T is $\varkappa$-stable if and only if T has a saturated model of power $\varkappa$.

The "only if" direction of this theorem was stated, without proof, by Shelah in [37]. A proof of both directions is given by Wierzejewski in [40].

1.4 Theorem (Shelah [36])

Let T be a countable unstable theory and $\varkappa > \omega$. Then T has a saturated model of power $\varkappa$ if and only if $\sum_{\lambda<\varkappa} \varkappa^{\lambda} = \varkappa$.

The complete solution to the problem of stability of abelian groups is now easily obtained.

1.5 Theorem (Eklof and Fisher, Macintyre, Shelah, Wierzejewski)

Let A be an arbitrary abelian group.

(i) A has a stable theory.

(ii) A is ω-stable if and only if it is the direct sum of a divisible group and a bounded group.

(iii) A is superstable if and only if it satisfies $(*)$.

Proof

(ii) is of course Macintyre's result; Eklof and Fisher prove it too. (A word of caution here: for Eklof and Fisher "ω_1-stable" is what we mean by "ω-stable".)

(iii) Suppose first that A is superstable. By definition, A is κ-stable for all $\kappa \geqq 2^{\omega}$ and hence, by Theorem 1.3, ThA has saturated models in these powers. Theorem 1.2 and the fact that $(\aleph_{\alpha+\omega})^{\omega} > \aleph_{\alpha+\omega}$ together imply that A satisfies $(*)$.

Conversely, if A satisfies $(*)$, then either A is ω-stable and hence also superstable, or, using Theorem 1.2(i), ThA has saturated models of power κ for all $\kappa \geqq 2^{\omega}$. The rest will follow from Theorem 1.3 provided we can show that A is stable. But this is immediate from Theorem 1.4 since, for any singular cardinal κ, we have $\sum_{\lambda<\kappa} \kappa^{\lambda} > \kappa$.

(i) Let A be an unsuperstable group. Then, by (iii) A does not satisfy $(*)$ and so, by Theorem 1.2 (ii), ThA has a saturated model of power κ if and only if $\kappa^{\omega} = \kappa$. Thus, by Theorem 1.4, to be able to deduce that A is stable we need only produce a cardinal κ for which $\kappa^{\omega} = \kappa$ but $\sum_{\lambda<\kappa} \kappa^{\lambda} > \kappa$. There are plenty of these cardinals: for example, $\beth_{\omega_1}$. //

In this chapter we give a completely new and, we believe, a more natural approach to this whole question. Our method is a direct attack upon the problem. Furthermore, apart from the more elementary facts of algebra and logic, the only deep result we need borrow from elsewhere is Szmielew's elimination of quantifiers theorem (see Theorem 3.1).

Recall, from Chapter 1, that a theory is κ-stable if and only if every model of the theory of power κ has at most κ complete types. Thus, stability is defined in terms of counting types and this is precisely what we do. Indeed, we give an exact formula (Theorem 3.18) for calculating the size of the Stone Space, $S(A)$, of a given infinite abelian group A. This formula is a very simple one involving only the cardinalities of A and certain quotient groups of subgroups of A (see section 2 for the definitions):

$$|S(A)| = |A| \prod_{p \text{ prime}} \prod_{n>0} |p^nA/p^{n+1}A| .$$

Using our formula we deduce immediately that every abelian group has a stable theory (Theorem 4.1). Furthermore, we are able to distinguish between the ω-stable, strictly-superstable, and merely-stable abelian groups (Theorems 4.4, 4.7 and Corollary 4.8). The invariants we give for characterising the superstable groups are, we think, somewhat simpler than those which follow from Eklof and Fisher's work.

This work was carried out independently of Eklof and Fisher [8], and was completed in March 1973 before we realised its connection with that paper. Shortly afterwards, Berthier announced in [5] that the theory of abelian groups is stable. His proof [6] also depends on the elimination of quantifiers result of Szmielew. However, his method is not of the "counting types" variety and does not allow him to separate the stable classes as we do.

We end this section by giving an outline of the structure of

Chapter 1. In section 2 we review the algebraic definitions and results we shall need. Our invariants for characterising the superstable abelian groups are defined there too. Section 3 is devoted to the derivation of our formula for counting types, and, in section 4, we use it to give our version of Theorem 1.5. Finally, in section 5, we discuss how to generalise these results to modules over Dedekind domains.

2. Preliminaries

All groups in this chapter are infinite abelian, and so, for the time being, "group" will mean infinite abelian group. We shall need very few purely algebraic results and it is the purpose of this section to review those definitions and theorems we shall use. Further details of anything left unexplained may be found in Hall [14] or Kaplansky [18].

Let A be an arbitrary group and n an integer.

The subgroup of A consisting of all elements na, with $a \in A$, is denoted by nA.

An integer n is said to divide the element a of A if there exists an element b in A, not necessarily unique, with $a = nb$; we shall write $n|a$ if n divides a.

A is said to be divisible if every non-zero integer divides every element of A; equivalently, A is divisible if $\bigcap_{n>0} nA = A$. It is well-known that a divisible subgroup of A is a direct summand of A. ([18], Theorem 2)

If $nA = 0$, the trivial group, for some integer n, then A is said to be bounded.

Now let B be an arbitrary subgroup of A. Then, A/B denotes the quotient, or factor, group of A by B. The elements of A/B are cosets $a + B$, $a \in A$.

A transversal for B in A is any subset of A containing exactly one element from each coset of B in A .

If C is a subset of A , then B induces an equivalence relation, $\sim$, on C defined by

$$a \sim b \quad \text{if and only if} \quad a-b \in B \ .$$

The set of all equivalence classes of $\sim$ on C is called the quotient set of C modulo $\sim$ and is denoted by $C/\sim$, or by C/B . The dual usage of the quotient symbol should cause no confusion; indeed, when C is also a subgroup of A containing B , then the quotient set C/B coincides with the quotient group of C by B .

In the sequel we shall show that the number of complete types over a group A is completely determined by the cardinalities of A and the successive quotient groups $p^nA/p^{n+1}A$, where n is a non-negative integer and p is prime. For this reason we make the following definition; we shall comment on our notation after proving Lemma 2.2.

2.1 Definition

If A is a group, p a prime and n a non-negative integer, then $tf(p,n;A) = |p^nA/p^{n+1}A|$. We shall not refer to A , writing tf(p,n) instead, whenever it is clear from the context to do so.

The following lemma is straight-forward and so we give only a sketch of its proof.

2.2 Lemma

For any group A , integer $n \geqq 0$ and prime p , we have:

(i) $tf(p,n+1) \leqq tf(p,n)$;

(ii) every coset C of A by p^nA is the union of exactly tf(p,n) cosets of $p^{n+1}A$ in A ; and

(iii) A is the union of $\prod_{k<n} tf(p,k)$ cosets of p^nA .

Proof

Let A be a group, $n \geqq 0$ an integer, p a prime.

(i) If $\{a_\alpha : \alpha < tf(p,n)\}$ is a transversal of $p^{n+1}A$ in p^nA then $|p^{n+1}A/p^{n+2}A| \leqq |\{pa_\alpha + p^{n+2}A : \alpha < tf(p,n)\}|$.

(ii) Let $a \in A$ and C be the coset $a + p^nA$ of p^nA in A . If $\{p^na_\alpha : \alpha < tf(p,n)\}$ is a transversal of $p^{n+1}A$ in p^nA , then the cosets $a + p^na_\alpha + p^{n+1}A$, $\alpha < tf(p,n)$, are all distinct and C is their union.

(iii) Use (ii) and induction on n . //

Now, the conditions we give for characterising the ω-stable and strictly-superstable groups involve the cardinals $tf(p,k)$: for example, (Theorem 4.4) a group A is ω-stable if and only if $tf(p,k;A) = 1$ almost everywhere. In general, the statement $tf(p,k;A) = \kappa$ is not an elementary one. However, we can say elementarily that $tf(p,k;A) = n$, for $0 < n < \omega$; and that $tf(p,k;A) \geqq \omega$. Thus, if $B \equiv A$ then $tf(p,k;B) = tf(p,k;A)$ when this value is finite, and otherwise, both $tf(p,k;A)$ and $tf(p,k;B)$ are infinite. Such expressive power is sufficient for our characterisations.

We have borrowed our notation from the Eklof-Fisher invariant $Tf(p;A)$, [8]. Regarding $p^nA/p^{n+1}A$ as a vector space over the field of integers mod p , $Tf(p;A)$ is defined by:

$$Tf(p;A) = \lim_{n\to\infty} \dim\,(p^nA/p\quad A) .$$

It is clear that $\dim\,(p^nA/p^{n+1}A) = 0$ if and only if $|p^nA/p^{n+1}A| = 1$ and, furthermore, since the field is finite, $\dim\,(p^nA/p^{n+1}A) < \infty$ if and only if $tf(p,n;A) < \omega$. Thus, in particular, $Tf(p;A) < \infty$ if and only if the sequence $tf(p,n;A)$, $n \in \omega$, is eventually finite. The invariants $tf(p,k;A)$ are sufficient for our purposes since we shall not be interested in the exact infinite value of $Tf(p;A)$.

3. Derivation of the Formula

The main theorem proved in this section is that the number of complete types, $|S(A)|$, over a group A is given by

$$|S(A)| = |A| \prod_{p\in P} \prod_{n\in\omega} tf(p,n;A) ,$$

where P is the set of all primes. The basis for our proof is that the theory of abelian groups admits elimination of quantifiers. This result was first proved in 1955, by Szmielew ([38], Theorem 4.22), and a new model-theoretic proof was given in 1972 by Eklof and Fisher ([8], Corollary 4.11). It is Eklof and Fisher's formulation of her result which we give below. The definition of core sentences is given in section 2 of [8]: roughly speaking, they are sentences which define the elementary invariants; thus, two groups are elementarily equivalent if and only if they satisfy the same core sentences. In the following theorem, "$n|u$" is an abbreviation for the formula "$\exists v(u = nv)$".

3.1 Theorem (Szmielew)

Every formula in the language of the theory of abelian groups is equivalent, relative to that theory, to a formula which is a disjunction of conjunctions of core sentences and formulae of the form

$$"p^k| \sum_{1\leq i\leq n} r_i v_i" , \quad "\sum_{1\leq i\leq n} r_i v_i = 0"$$

and their negations. (p,k,n and r_i are all integers, with p a prime and $n,k > 0$.)

Recall, from Chapter 1, that the formulae referred to in an elimination of quantifiers theorem are called basic formulae. Let A be an arbitrary group. Our procedure is to reduce the calculation of the size of the Stone space of A to the enumeration of all complete types restricted to formulae of a particularly simple kind. The first stage in this reduction is to show that a certain subset of the basic formulae will suffice.

3.2 Proposition

Let Δ be the set of all formulae of the form "$p^k|(nu-v)$" and "$nu = v$", where p is a prime, and n and k are positive integers. Then, if A is any group,

$$|S(A)| = |S_\Delta(A)|$$

Proof

Let A be a group. Since each complete type over A contains a complete Δ-type over A, and since distinct complete Δ-types extend to distinct complete types, it suffices to show that the extension of each complete Δ-type is unique. So, let $t_1, t_2 \in S(A)$ and $t_1|\Delta = t_2|\Delta$. We must show that $t_1 = t_2$.

We argue by contradiction. If $t_1 \neq t_2$, then there is a formula $\varphi(u,\bar{v})$ and a sequence $\bar{a}$ from A such that $\varphi(u,\bar{a}) \in t_1$ but $\sim\varphi(u,\bar{a}) \in t_2$. Using Theorem 3.1, there are formulae $\psi_{ij}(u,\bar{v})$, each a basic formula or its negation, such that $\varphi(u,\bar{v})$ may be assumed to have the form

$$\bigvee_i \bigwedge_j \psi_{ij}(u,\bar{v}) .$$

Now, $\varphi(u,\bar{a}) \in t_1$ and so there exists i' such that, for all j, $\psi_{i'j}(u,\bar{a}) \in t_1$. But $\sim\varphi(u,\bar{a}) \in t_2$ and so there is j' such that $\sim\psi_{i'j'}(u,\bar{a}) \in t_2$. Since every complete type over A must contain the same sentences, it is clear that either $\psi_{i'j'}(u,\bar{v})$ or its negation must have the form "$p^k|(r_o u - \sum r_i v_i)$" or "$r_o u = \sum r_i v_i$". Let $\psi(u,v)$ be the formula obtained from $\psi_{i'j'}(u,\bar{v})$ by replacing $\Sigma r_i v_i$ by the single new variable v. Then, either ψ or its negation belongs to Δ and, putting $a = \Sigma r_i a_i$, we have $\psi(u,a) \in t_1$ and $\sim\psi(u,a) \in t_2$. But this is a contradiction, for we assumed $t_1|\Delta = t_2|\Delta$. Thus, $t_1 = t_2$. //

We have stated the above Proposition in the context of abelian groups, but a similar proof yields the corresponding result for any theory with elimination of quantifiers.

3.3 Proposition

If T is a theory with elimination of quantifiers modulo a set F of basic formulae, then there is a subset F' of F such that for all models M of T,

$$|S(M)| = |S_{F'}(M)| .$$

The next lemma enables us to show that the key formulae in the count are those of the form "$p^k | (nu - v)$" .

3.4 Lemma

Let $t \in S_\Delta(A)$ and n be the least positive integer, if one exists, such that $nu = a_o \in t$, for some $a_o \in A$. Then, for all positive integers m and for all $a \in A$, we have $mu = a \in t$ iff $n|m$ and $a = (m/n)a_o$.

Proof

Let $t \in S_\Delta(A)$ and suppose that there exist n and a_o satisfying the hypothesis. Let B be an elementary extension of A with $b \in B$ realising t .

First, suppose that $m = dn$ and $a = da_o$. Since b realises t and $nu = a_o \in t$, we have $nb = a_o$. Thus, $dnb = da_o$ and so, $mb = a$. It follows that $mu = a \in t$.

Conversely, suppose $mu = a \in t$. By the Euclidean algorithm, we can find integers d and r with $m = dn + r$ and $0 \leqq r < n$. Since, $nb = a_o$ and $mb = a$, we have $rb = (m - dn)b = a - da_o$. Thus, $ru = (a - da_o) \in t$. This, contradicts the definition of n , unless $r = 0$. Thus, $r = 0$ and $a = da_o$. So, $n|m$ and $a = (m/n)a_o$. //

3.5 Proposition

Let Π denote the set of all formulae of the form "$p^k | (nu-v)$" , where p is a prime and n and k are positive integers. Then, if A is any infinite group,

$$|S(A)| = |A|\,|S_\Pi(A)|$$

Proof

Clearly, $|S(A)| \geqq |A|$ and $|S(A)| \geqq |S_\Pi(A)|$. Since at least $|A|$ is infinite, it follows that $|S(A)| \geqq |A|\,|S_\Pi(A)|$. Thus, using Proposition 3.3, it suffices to show that each type in $S_\Pi(A)$ extends to at most $|A|$ types in $S_\Delta(A)$.

Consider $t \in S_\Pi(A)$ and let t' be any extension of t to a type in $S_\Delta(A)$. Now, either (a) t' is the unique extension containing $nu \neq a$, for every $n > 0$ and every $a \in A$; or (b) there is an integer $n > 0$ and element a_o of A with $nu = a_o \in t'$. In case (b), let n be the least such integer, Lemma 3.4 shows that the remaining formulae in $t'-t$ are uniquely determined by the pair (n,a_o) : that is to say, $mu = a \in t'$ if $n|m$ and $a = (m/n)a_o$, and $mu \neq a \in t'$, otherwise. Thus, there are at most $\omega|A|$ such extensions of t , and hence, at most $\omega|A| + 1$ extensions altogether. Since A is an infinite group, $\omega|A| + 1 = |A|$, and so we have $|S_\Delta(A)| \leqq (\omega|A| + 1)|S_\Pi(A)| = |A|\,|S_\Pi(A)|$, from which the result now follows. //

Let p be a fixed prime and $\Pi(p)$ denote the set of all formulae of the form "$p^k|(nu-v)$" , where n and k are positive integers . With a slight abuse of notation we shall write $S_p(A)$ instead of $S_{\Pi(p)}(A)$. Let P denote the set of all prime numbers. The proof of the next lemma is obvious; we shall show later on that the inequality may be removed.

3.6 Lemma

$$|S_\Pi(A)| \leqq \prod_{p\in P} |S_p(A)| \; .$$

From now on we focus on an arbitrary, but fixed, prime p and concentrate on determining the cardinality of $S_p(A)$. The following two lemmas are basic to our needs and we shall use them often without necessarily referring to them explicitly.

3.7 Lemma

Let $t \in S_p(A)$ and $n,k > 0$. Then, if $p^k|(nu-a)$ belongs to t for some $a \in A$, we have:

(i) $p^k|(nu-a') \in t$ if and only if $a' \in a + p^kA$;

(ii) $p^i|(nu-a) \in t$ for all i with $0 < i \leqq k$;

(iii) $p^k|(dnu-da) \in t$ for all $d \in \omega$; and

(iv) $p^{k+1}|(pnu-pa) \in t$.

Proof

This is easy and we leave it as an exercise. //

3.8 Lemma

Let $t \in S_p(A)$, $k > 0$, and let n be the least positive integer for which we have $p^k|(nu-a_o) \in t$ for some $a_o \in A$. Then, for all $m > 0$ and $a \in A$,

$$p^k|(mu-a) \in t \text{ iff } n|m \text{ and } a \in (m/n)a_o + p^kA .$$

Proof

First we observe that n, satisfying the hypothesis, must exist since $p^k|(p^ku-p^ka) \in t$, for all $a \in A$. The proof of this Lemma is similar to that of Lemma 3.4, so we omit it. //

3.9 Definitions

(i) Consider the set, S, of all sequences $\langle(n_k,a_k):k > 0\rangle$ of pairs (n_k,a_k), with $n_k > 0$ and $a_k \in A$. Now, associated with each type $t \in S_p(A)$ there are sequences $\sigma = \langle(n_k,a_k):k > 0\rangle \in S$ such that $p^k|(n_ku-a_k) \in t$ for all $k > 0$, and n_k is the least integer $n > 0$ for which there is $a \in A$ with $p^k|(nu-a) \in t$. We shall call any such sequence σ <u>an abbreviation for t</u>. (We may also say that <u>σ abbreviates t</u>, or that <u>t is abbreviated by σ</u>).

(ii) Let $\sigma = \langle(n_k,a_k):k > 0\rangle$ and $\sigma' = \langle(n'_k,a'_k):k > 0\rangle$ be two sequences in S . Then, it is easy to verify that the relation $\sim$ defined on S by

$$\sigma \sim \sigma' \text{ iff for all } k > 0 \text{ , } n_k = n_k' \text{ and } a_k \in a_k' + p^k A \text{ ,}$$

is an equivalence relation.

Now, clearly, every type $t \in S_p(A)$ has many abbreviations. However, it follows from the definitions and Lemma 3.7 that, if σ and σ' both abbreviate the same type t , then $\sigma \sim \sigma'$. Furthermore, Lemma 3.8 shows that if σ is an abbreviation for t , then t is uniquely determined by σ . Indeed, given $\sigma = \langle (n_k, a_k) : k > 0 \rangle$, the formulae in t can be written down:

$$t = \{\sim_{(k,n,a)} p^k | (nu-a) : k > 0 \text{ , } n > 0 \text{ , } a \in A\} \text{ ,}$$

where,

$$\sim_{(k,n,a)} = \begin{cases} \sim\!\sim & \text{if } n_k | n \text{ and } a \in (n/n_k)a_k + p^k A \text{ ;} \\ \\ \sim & \text{otherwise .} \end{cases}$$

Thus, if σ and σ' are abbreviations for t and t' respectively, then $\sigma \sim \sigma'$ if and only if $t = t'$. What this proves is that if T is the subset of S consisting of all abbreviations for types in $S_p(A)$, then there is a one-one correspondence between $S_p(A)$ and the set $T/\sim$ of all equivalence classes of $\sim$ on T . Hence, to enumerate $S_p(A)$ it suffices to count inequivalent abbreviations. The preceding comments prove the following theorem.

3.10 <u>Theorem</u>

$$|S_p(A)| = |T/\sim| \text{ .}$$

Now, using Lemma 2.2, it follows that

$$\begin{aligned} |T/\sim| &\leq \prod_{k>0} (\omega \cdot \prod_{0 \leq n < k} tf(p,n;A)) \\ &\leq 2^{\omega} \prod_{n \in \omega} (tf(p,n))^{\omega} \end{aligned}$$

However, this bound is not fine enough for our purposes, and a closer look at the structure of an abbreviation enables us to lower it.

3.11 Lemma

Let $\langle (n_k,a_k):k > 0\rangle$ be an abbreviation for some type $t \in S_p(A)$ and suppose that $tf(p,k-1;A)$ is finite for some $k > 0$. Then, (i) if $k = 1$, $n_1 = 1$; and (ii) if $k > 1$, then $n_k = n_{k-1}$ and $a_k \in a_{k-1} + p^{k-1}A$.

Proof

Let $tf(p,k-1) = N < \omega$ and $\langle (n_k,a_k):k > 0\rangle$ be an abbreviation for the type $t \in S_p(A)$. Let B be any elementary extension of A with $b \in B$ realising t.

(i) $k = 1$: Since $|A/pA| = tf(p,0) = N$,

$$\mathrm{Th}A \vdash \exists u_1 \ldots u_N \forall u \bigvee_{1 \leq i \leq N} p \mid (u-u_i).$$

Thus, there are elements $b_1,\ldots,b_N$ in A such that

$$\mathrm{Th}A^+ \vdash \forall u \bigvee_{1 \leq i \leq N} p \mid (u-b_i).$$

Since $B \models \mathrm{Th}A^+$, it follows that for some i, $B \models p \mid (b-b_i)$. Hence, $p \mid (u-b_i) \in t$ and so, $n_1 = 1$.

(ii) $k > 1$: By Lemma 2.2, for any $a \in A$, we have $|(a+p^{k-1}A)/p^kA| = tf(p,k-1) = N$. Thus,

$$\mathrm{Th}A \vdash \forall u_o \exists u_1 \ldots u_N \{ \bigwedge_{1 \leq i \leq N} p^{k-1} \mid (u_i - u_o)$$

$$\wedge \forall u (p^{k-1} \mid (u-u_o) \rightarrow \bigvee_{1 \leq i \leq N} p^k \mid (u-u_i)) \}.$$

It follows that there are elements $b_i \in a_{k-1} + p^{k-1}A$, $1 \leq i \leq N$, such that

$$A^+ \models \forall u (p^{k-1} \mid (u-a_{k-1}) \rightarrow \bigvee_{1 \leq i \leq N} p^k \mid (u-b_i) \quad (1)$$

Now, $p^{k-1} \mid (n_{k-1}u-a_{k-1}) \in t$ and so $p^{k-1} \mid (n_{k-1}b-a_{k-1})$. Using the fact that B^+ also satisfies (1) we must have $p^k \mid (n_{k-1}b-b_i)$ for some i, $1 \leq i \leq N$. Thus, $p^k \mid (n_{k-1}u-b_i) \in t$. So by definition of

n_k and Lemma 3.8, $n_k | n_{k-1}$. For the same reasons, since $p^k | (n_k u - a_k) \in t$ implies $p^{k-1} | (n_k u - a_k) \in t$, we also have $n_{k-1} | n_k$. Thus, $n_k = n_{k-1}$. But now, using Lemma 3.7, we must have $a_k \in (a_{k-1} + p^{k-1}A)$. //

3.12 Lemma

Suppose that $\langle (n_k, a_k) : k > 0 \rangle$ is an abbreviation for some type $t \in S_p(A)$. Then, either

(i) $n_k = 1$ and $a_{k+1} \in a_k + p^k A$, for all $k > 0$; or

(ii) there exists $k > 0$ such that $n_k > 1$.

In case (ii), if k is the least such integer, then we have

(a) $$n_r = \begin{cases} 1 & r < k \\ p & r = k \\ p^{\epsilon_r} n_{r-1} & r > k \end{cases}$$

where $\epsilon_r = 0$ or 1, and must be 0 if $tf(p, r-1)$ is finite;

(b) $$a_r \in \begin{cases} a_{r-1} + p^{r-1}A & 1 < r < k \text{ or } r > k \text{ and } \epsilon_r = 0 \\ pa_{r-1} + p^r A & r = k \text{ or } r > k \text{ and } \epsilon_r = 1 . \end{cases}$$

Proof

Let $\langle (n_k, a_k) : k > 0 \rangle$ be an abbreviation for the type $t \in S_p(A)$.

(i) If $n_k = 1$ for all $k > 0$, then since $p^{k+1} | (u - a_{k+1}) \in t$ implies $p^k | (u - a_{k+1}) \in t$, we must have $a_{k+1} \in a_k + p^k A$.

(ii) Let k be the least positive integer for which $n_k > 1$. Then, as in case (i), we have $n_r = 1$ and $a_r \in a_{r-1} + p^{r-1}A$, for $0 < r < k$ (setting $a_o = 0$). We now evaluate n_{k+i}, by induction on $i \in \omega$, pinning down a_{k+i} at the same time.

Consider $i = 0$. If $k = 1$, then $p^k | (pu - pa) \in t$ for all $a \in A$. Otherwise, since $p^{k-1} | (u - a_{k-1}) \in t$, we have $p^k | (pu - pa_{k-1}) \in t$.

Either way, we have $p^k | (pu-pa_{k-1}) \in t$, and hence $n_k | p$. Since $n_k = 1$ is impossible, $n_k = p$. Furthermore, by Lemma 3.7, $a_k \in pa_{k-1} + p^k A$.

Now let $i = r + 1$. Using Lemmas 3.7, 3.8 and the fact that $p^{k+r} | (n_{k+r}u-a_{k+r}), p^{k+r+1} | (n_{k+r+1}u-a_{k+r+1}) \in t$, it follows that $n_{k+r+1} | pn_{k+r}$ and $n_{k+r} | n_{k+r+1}$. Thus, $n_{k+r+1} = p^{\epsilon_{k+r+1}} n_{k+r}$, with $\epsilon_{k+r+1} = 0$ or 1 . By Lemma 3.11, if $tf(p,k + r)$ is finite, then $n_{k+r+1} = n_{k+r}$, so $\epsilon_{k+r+1} = 0$ and $a_{k+r+1} \in a_{k+r} + p^{k+r}A$. If $\epsilon_{k+r+1} = 1$, then $n_{k+r+1} = pn_{k+r}$, and since $p^{k+r+1} | (pn_{k+r}u-pa_{k+r}) \in t$ we have $a_{k+r+1} \in pa_{k+r} + p^{k+r+1}A$. This completes the proof of the Lemma. //

3.13 <u>Lemma</u>

Every sequence $\langle (1,a_k) : k > 0\rangle$ with $a_{k+1} \in a_k + p^k A$ is an abbreviation for some type in $S_p(A)$.

<u>Proof</u>

If $a_{k+1} \in a_k + p^k A$, then the set $\{p^k | (u-a_k) : k > 0\}$ is finitely satisfied in A (by the elements a_k) . //

We now have enough Lemmas to count $S_p(A)$.

3.14 <u>Theorem</u>

$$|S_p(A)| = \prod_{k\in\omega} tf(p,k;A) .$$

<u>Proof</u>

To count $S_p(A)$, remember that it suffices to count inequivalent abbrevations. By the preceding lemma we have immediately that

$$|S_p(A)| \geqq \prod_{k\in\omega} tf(p,k) .$$

Suppose first that $tf(p,0)$ is finite. By Lemma 2.2 (i), $tf(p,k)$ is finite for all $k \in \omega$. Thus, using Lemma 3.11, we see that every abbreviation is of the kind already enumerated. So, for finite $tf(p,0)$ the result follows immediately.

If $tf(p,0)$ is infinite then we have two kinds of abbreviations to consider: those described under (i) and (ii) in Lemma 3.12. Since the number of abbreviations of the former kind is $\prod_{k\in\omega} tf(p,k)$,

and since this cardinal is infinite, it suffices to show that there are at most this number satisfying (ii).

Recall the definition (see 3.9 (ii)) of the equivalence relation $\sim$. Then, using the characterisation of abbreviations satisfying (ii) given under (a) and (b) in Lemma 3.12, we see that their number is at most

$$\prod_{0\leq r<k-1} tf(p,r) \cdot 1 \cdot \prod_{r\geq k} tf(p,r)$$

$$= \varkappa \text{ , say.}$$

(Note that for $r > k$ there are two cases: (1) $tf(p,r-1)$ is finite, in which case $\varepsilon_r = 0$ and so there are $tf(p,r-1)$ possibilities for a_r; or (2) $tf(p,r-1)$ is infinite, in which case $\varepsilon_r = 0$ or 1 and so there are $tf(p,r-1) + 1 = tf(p,r-1)$ possibilities for a_r.)

But now we see immediately that $\varkappa \leq \prod_{r\in\omega} tf(p,r)$ as required. //

The following Corollary now follows using Lemma 3.6.

3.15 <u>Corollary</u>

$$|S_\Pi(A)| \leq \prod_{p\in P} \prod_{k\in\omega} tf(p,k;A) .$$

Before stating Lemma 3.6 we promised that the inequality there could be removed. We shall prove this using Lemma 3.13 and the following version of the Chinese Remainder theorem. Its proof is similar to the one usually given for an integral domain (see, for example, Hardy and Wright [15], page 95, Theorem 121).

3.16 <u>Proposition</u> (Chinese Remainder Theorem for Abelian Groups)

If $q_1 \ldots q_n$ are mutually prime integers and A is an abelian group, then

$$A \models \forall u_1 \ldots u_n \exists u \bigwedge_{1 \leqq i \leqq n} q_i | (u - u_i)$$

3.17 Proposition

$$|S_\Pi(A)| \geqq \prod_{p \in P} \prod_{k \in \omega} tf(p,k,A)$$

Proof

Let p_m , $m \in \omega$, be some enumeration, without repetitions, of the primes, For each $m \in \omega$, let t_m be a type in $S_{p_m}(A)$ abbreviated by some sequence $\langle (1,a_{mk}) : k > 0 \rangle$, where for $k > 0$, $a_{mk+1} \in a_{mk} + p_m^k A$. (Here we are using Lemma 3.13) If we can show that every finite subset of $t = \bigcup_{m \in \omega} t_m$ is satisfied in A^+ , then $t \in S_\Pi(A)$ and we are finished.

Let Γ be any finite subset of t . Then, there is $N \in \omega$ with $\Gamma \subset \bigcup_{m \leqq N} t_m$. In fact, since Γ is finite, there is also an integer $K > 0$ such that

$$\Gamma \subset \{ \sim_{(m,k,n,a)} p_m^k | (nu - a) : a \in A , n > 0 , 0 < k \leqq K , m \leqq N \} ,$$

where, we recall from the comments preceding Theorem 3.10,

$$\sim_{(m,k,n,a)} = \begin{cases} \sim\sim & \text{if } a \in na_{mk} + p_m^k A \text{ ; and} \\ \sim & \text{otherwise .} \end{cases}$$

(Note that $n_{mk} = 1$ for all $m \in \omega$ and $k > 0$.) Call the larger set Γ' . Now, using the Chinese Remainder Theorem, there is $b \in A$ such that

$$A^+ \models \bigwedge_{m \leqq N} p_m^K | (b - a_{mK}) .$$

Suppose $m \leqq N$ and $k \leqq K$. Then, $p_m^k | (b - a_{mK})$. But, it follows from our hypothesis on the elements a_{mi} that, $a_{mK} \in a_{mk} + p_m^k A$. Hence, $p_m^k | (b - a_{mk})$ and thus, for all $n > 0$, $p_m^k | (nb - na_{mk})$. From this it now follows immediately that $p_m^k | (nb - a)$ if and only if $a \in na_{mk} + p_m^k A$. Thus, for all $m \leqq N$, $k \leqq K$, $n > 0$ and $a \in A$, we have

$$p_m^k \mid (nb-a) \text{ iff } a \in na_{mk} + p_m^k A .$$

This shows that b realises Γ' and hence Γ . Thus, t is a Π-type as required. //

The following theorem is now immediate from Propositions 3.5, 3.17 and Corollary 3.15.

3.18 Theorem

For every infinite abelian group A ,

$$|S(A)| = |A| \prod_{p\in P} \prod_{k\in K} tf(p,k;A)$$

In the next section we exploit this formula, and classify abelian groups by their stability properties.

4. Exploitation of the Formula

It is an immediate corollary of the main theorem of the previous section that every abelian group has a stable theory (Theorem 4.1, below). By a careful analysis of the exact values of the various invariants involved in the formula, we are able to distinguish between those groups which are ω-stable, strictly-superstable and merely-stable (Theorems 4.4, 4.7 and Corollary 4.8, below).

4.1 Theorem (Berthier [6], Theorem III. 1.1)

Every abelian group has a stable theory.

Proof

Let A be an arbitrary group. Since all finite structures are stable we shall assume that A is infinite. For all $k \in \omega$ and $p \in P$, $tf(p,k) \leq |A|$ and hence, using the formula of Theorem 3.18, we have

$$|S(A)| \leq |A| \prod_{p\in P} \prod_{k\in\omega} |A| = |A|^{\omega} .$$

Thus, ThA is stable in power $\varkappa$, where $\varkappa^{\omega} = \varkappa$. //

Before stating the next theorem we make the following convention.

4.2 Convention

The phrase "almost everywhere" means "for all except, possibly, finitely many pairs (p,k), $p \in P$, $k \in \omega$" .

4.3 Theorem

For every infinite abelian group A we have

(i) $tf(p,k) = 1$ almost everywhere if and only if A is ω-stable; and

(ii) if $tf(p,k) < \omega$ almost everywhere then A is superstable.

Proof

Let A be an infinite abelian group and define

$$P_1 = \{p \in P : \text{for some } k \in \omega,\ tf(p,k) \geqq \omega\},$$

$$P_2 = \{p \in P-P_1 : \text{for some } k \in \omega,\ tf(p,k) > 1\}.$$

It is clear that for $p \notin P_1 \cup P_2$, $\prod_{k\in\omega} tf(p,k) = 1$. Thus, setting

$$\prod_{k\in\omega} tf(p,k) = \begin{cases} \varkappa_p & \text{for } p \in P_1 ; \\ \lambda_p & \text{for } p \in P_2 , \end{cases}$$

we have, from Theorem 3.18,

$$|S(A)| = |A| \prod_{p\in P_1} \varkappa_p \prod_{p\in P_2} \lambda_p . \tag{1}$$

Observe that $\varkappa_p \geqq \omega$ and $\lambda_p < 1$. (2)

(i) First assume that $tf(p,k) = 1$ almost everywhere, so that both P_1, and P_2 are finite. Since we also have

$$\varkappa_p \leqq |A| \text{ and } \lambda_p < \omega$$

in this case, it follows from (1) that $|S(A)| \leqq |A|^{|P_1|+1} \cdot \omega = |A|$. Thus, A is ω-stable.

Now suppose that the condition on the $tf(p,k)$ fails. Then, either some $\varkappa_p \geqq 2^\omega$; some $\lambda_p \geqq 2^\omega$; P_1 is infinite, or P_2 is infinite. Using (1) and (2), any one of these four possibilities leads to $|S(A)| \geqq |A| \cdot 2^\omega$. Consequently, A is unstable in any power $\varkappa$ for which $\omega \leqq \varkappa < 2^\omega$. This proves that A is not ω-stable.

(ii) If $tf(p,k) < \omega$ almost everywhere, then P_1 is finite and $\varkappa_p \leqq |A| \cdot 2^\omega$, $\lambda_p \leqq 2^\omega$. So, from (1) we have

$$|S(A)| \leqq |A| \cdot (|A| \cdot 2^\omega)^{|P_1|} \cdot (2^\omega)^{|P_2|} \leqq |A| \cdot 2^\omega .$$

Thus, A is $\varkappa$-stable for all $\varkappa \geqq 2^\omega$. In other words A is superstable. //

For ease of reference we isolate (i) of the theorem above.

4.4 Theorem

An infinite abelian group A is ω-stable if and only if $tf(p,k;A) = 1$ almost everywhere.

The hypothesis of (ii) in Theorem 4.3 is also necessary for a superstable group. To prove this we shall use the following simple criterion under which a group is not superstable.

4.5 Theorem (Baldwin and Saxl [1], Theorem 1.2)

If there is a sequence of definable subgroups B_n , $n \in \omega$, of a group A with $B_n \supset B_{n+1}$, such that $|B_n/B_{n+1}|$ is infinite, then A is not superstable.

4.6 Corollary

If A is superstable, then $tf(p,k;A) < \omega$ almost everywhere.

Proof

Let A be an abelian group for which $tf(p,k) \geqq \omega$ for infinitely many pairs (p,k) . There are two ways in which this may happen: (i) for some $p \in P$, $tf(p,k) \geqq \omega$ for all $k \in \omega$; or (ii) there are infinitely many distinct primes p_n , $n \in \omega$,with $tf(p_n,k_n) \geqq \omega$ for some $k_n \in \omega$. In each case we exhibit a descending chain of definable subgroups of A satisfying the hypothesis of Theorem 4.5. It will then follow that A is not superstable.

(i) The subgroups p^kA , $k \in \omega$, form a sequence of the desired kind.

(ii) By Lemma 2.2 it follows that $tf(p_n,0) \geq \omega$, for all $n \in \omega$. The sequence B_n, $n \in \omega$, defined by $B_0 = A$, $B_{n+1} = p_n B_n$ is a descending chain of definable subgroups of A. It remains to check that $|B_n/B_{n+1}|$ is infinite.

Now $|B_0/B_1| = |A/p_0A| = tf(p_0,0)$ which is infinite. For $n > 0$, let $\{a_\alpha : \alpha < tf(p_n,0)\}$ be a transversal for p_nA in A. Let $\alpha \neq \beta$ and $p_0 \dots p_{n-1}a_\alpha - p_0 \dots p_{n-1}a_\beta \in p_0 \dots p_nA$, with a view to finding a contradiction. There exists $a \in A$ with $p_0 \dots p_{n-1}(a_\alpha - a_\beta) = p_0 \dots p_n a$. Now, $p_0 \dots p_{n-1}$ and p_n are co-prime and so there are integers r and s such that $rp_0 \dots p_{n-1} + sp_n = 1$ Thus,

$$a_\alpha - a_\beta = rp_0 \dots p_n a + sp_n(a_\alpha - a_\beta)$$
$$= p_n(rp_0 \dots p_{n-1}a + sa_\alpha - sa_\beta) .$$

Hence, $a_\alpha - a_\beta \in p_nA$ which contradicts the definition of a transversal. So, the elements $p_0 \dots p_{n-1}a_\alpha$, $\alpha < tf(p_n,0)$ must all lie in distinct cosets of B_{n+1} in B_n and hence $|B_n/B_{n+1}| \geq tf(p_n,0) \geq \omega$. //

Combining this Corollary with (ii) of Theorem 4.3 we have the following result.

4.7 Theorem

An infinite abelian group A is superstable if and only if $tf(p,k;A) < \omega$ almost everywhere.

4.8 Corollary (of Theorems 4.3 and 4.7)

An infinite abelian group A is strictly superstable if and only if

(i) $tf(p,k;A) < \omega$ almost everywhere; and

(ii) $tf(p,k;A) > 1$ for infinitely many pairs (p,k).

Proof

This follows immediately from the fact that an abelian group is strictly-superstable if and only if it is superstable but not ω-stable. //

We conclude this section by commenting on the equivalence between our criteria for ω-stability and superstability and those (obtained by other means) which we have discussed in the introduction to this chapter.

The necessary and sufficient conditions for superstability which may be obtained from Eklof and Fisher's work, [8], are

(i) $Tf(p;A) < \infty$ for all $p \in P$; and

(ii) there exist only finitely many $p \in P$ such that for some n , $U(p,n;A) = \infty$.

We have already discussed $Tf(p;A)$ (in section 2) and remarked that $Tf(p;A) < \infty$ if and only if the sequence $tf(p,k;A)$, $k \in \omega$, is eventually finite. It is an elementary algebraic exercise to show directly that $U(p,n;A) < \infty$ for all $n \in \omega$ if and only if $tf(p,0;A) < \omega$.

The more familiar necessary and sufficient conditions for ω-stability are those discovered by Macintyre [19]. We indicate, briefly, the direct proof of their equivalence to our conditions.

4.9 Theorem (Macintyre)

A is ω-stable if and only if A is the direct sum of a divisible group and a bounded group.

Proof

If A is ω-stable, then by Corollary 4.4 there are an enumeration p_i, $i \in \omega$, of P and integers $N \in \omega$, $k_i > 0$ for $i < N$ such that for $i \geq N$, $tf(p_i,0) = 1$ and for $i < N$, k_i is the least positive integer with $tf(p_i,k_i) = 1$. Setting $n = \prod_{i<N} p_i^{k_i}$,

it can be shown that nA is a divisible subgroup of A . Thus, nA is a direct summand of A . Writing $A = nA + B$, it follows that $nA = n^2A + nB = nA + nB$, since nA is divisible. Hence $nB = 0$, showing that B is bounded.

Conversely, if $A = D + B$, where D is divisible and B is bounded, and if n is the least positive integer such that $nB = 0$, it follows that $nA = nD + nB = D$. From this it can be shown that if $n = p_0^{k_0} \dots p_{N-1}^{k_{N-1}}$, then $tf(p,0) = 1$ for all $p \neq p_i$, $i < N$ and $tf(p_i,k_i) = 1$, for all $i < N$. From this we conclude, via Corollary 4.4, that A is ω-stable. //

5. Modules over Dedekind Domains

The theory of abelian groups is a special case of the theory of modules over a principal ideal domain. So, the natural question to ask is whether our results generalise. To attempt such a generalisation we shall need an elimination of quantifiers theorem and Eklof and Fisher have proved one (see [8], Theorem 5.5) for the case of modules over a Dedekind domain. Hence, all the results of the previous sections have analogues for such modules and we propose to give here a brief sketch of how this may be achieved, pointing out the main differences with the case of Z-modules. We should, of course, remark that, as for abelian groups, all our results may be obtained from Eklof and Fisher's work via Shelah's theorems relating stability and saturation.

A good source for the definitions and properties of Dedekind domains is Zariski and Samuel [41]. For our part, the main facts we use are the following:

(i) every ideal is a product of prime ideals;

(ii) a non-zero ideal is prime if and only if it is maximal;

(iii) every ideal has a finite basis: for each ideal E in a Dedekind domain D there are elements $\alpha_1 \dots \alpha_n \in E$, for some $n \in \omega$, such that $E = D\alpha_1 + \dots + D\alpha_n$.

Let D be a fixed Dedekind domain, T the theory of mudules over D (for a list of the axioms of T see [8], page 161) and

M an arbitrary, but fixed, model of T. If E is an ideal in D, then "$E|u$" abbreviates the formula $u \in EM$ (Notice, that since E has a finite basis this can be said in $L(T)$.) Now, using Eklof and Fisher's elimination of quantifiers theorem, we obtain the analogue of Theorem 3.2. In this situation, Δ is now the set of all formulae of the form

$$"P^k|(du-v)" \text{ and } "du = v",$$

where P is a prime ideal in D, $d \in D$ and $k > 0$. From now on we fix $\lambda = |D| + \omega = |L(T)|$. We shall give the proof of the analogue of Theorem 3.5 since it shows clearly the part played by the cardinal λ.

5.1 <u>Theorem</u>

Let Π denote the set of all formulae of the form "$P^k|(du-v)$", where P is a prime ideal, $d \in D$ and $k > 0$. Then if M is an infinite D-module,

$$|M|\ |S_\Pi(M)| \leq |S(M)| \leq \lambda|M|\ |S_\Pi(M)| .$$

Thus, for M with $|M| \geq \lambda$, we have $|S(M)| = |M|\ |S_\Pi(M)|$. //

<u>Proof</u>

Clearly it suffices to show that each $t \in S_\Pi(M)$ has at most $\lambda|M|$ extensions in $S_\Delta(M)$. Fix $t \in S_\Pi(M)$ and let $t' \in S_\Delta(M)$ extend t.

First we observe that the set $E = \{d \in D : du = m \in t', \text{ some } m \in M\}$ is an ideal in D. Remembering that ideals in D are finitely generated, it follows that those pairs $(d,m) \in D \times M$ for which $du = m \in t'$ are completely determined by the pairs (α_i, m_i), $1 \leq i \leq n$, where $\alpha_1 \ldots \alpha_n$ is some basis for E and $\alpha_i u = m_i \in t'$. Since there are at most λ possibilities for the ideal E and at most $|M|$ possibilities for m_i once a basis for E is fixed, it follows that there are at most $\lambda|M|$ possibilities for t'. Thus, $|S(M)| = |S_\Delta(M)| \leq \lambda|M|\ |S_\Pi(M)|$. //

Let $\mathcal{P}$ denote the set of all prime ideals in D . Then, with obvious notation, it is clear that $|S_{\Pi}(M)| \leq \prod_{P\in\mathcal{P}} |S_P(M)|$. Thus, as before, we concentrate on determining $|S_P(M)|$ for an arbitrary, but fixed, prime ideal P . Defining $tf(P,k;M) = |P^kM/P^{k+1}M|$, all the definitions and lemmas numbered 2.2, 3.7-3.17 go through with appropriate notational changes and a minimum of difficulty. To aid the reader interested in working through the details we shall give the statements, without proof, of the major changes.

5.2 Lemma (compare with 3.8)

Let $t \in S_P(M)$ and $k > 0$. Then the set

$$E = \{d \in D : P^k|(du\text{-}m) \in t \text{ , for some } m \in M\}$$

is an ideal. Let $\alpha_1,\ldots,\alpha_n$ be any basis for E and $m_i \in M$ satisfy $P^k|(\alpha_i u - m_i) \in t$, $1 \leq i \leq n$. Then,

$$P^k|(du\text{-}m) \in t \text{ iff } \begin{cases} d \in E \text{ , so } d = \sum_{1\leq i\leq n} d_i\alpha_i \text{ , some } d_i \in D \text{ ; and} \\ m \in \sum_{1\leq i\leq n} d_i m_i + P^kM \text{ .} \end{cases}$$

5.3 Definitions (compare with 3.9)

(i) Let S be the set of all sequences $\sigma = \langle\sigma_k : k > 0\rangle$, where for $k > 0$, $\sigma_k = (\bar{\alpha}_k,\bar{m}_k)$ and for some $n_k > 0$, $\bar{\alpha}_k = \langle\alpha_{k1},\ldots,\alpha_{kn_k}\rangle$ and $\bar{m}_k = \langle m_{k1},\ldots,m_{kn_k}\rangle$ are sequences from D and M respectively.

An abbreviation for $t \in S_{\Pi}(M)$ is any sequence $\sigma \in S$ such that for each $k > 0$,

(a) $P^k|(\alpha_{ki}u - m_{ki}) \in t$, $1 \leq i \leq n_k$; and

(b) $\alpha_{k1},\ldots,\alpha_{kn_k}$ is a basis for the ideal $(\bar{\alpha}_k)$ associated with t in the sense of Lemma 5.2, above.

(ii) If σ , σ' are two sequences from S then, using Lemma 5.2, it can be verified that the relation, $\sim$, defined on S by

$\sigma \sim \sigma'$ if and only if, for all $k > 0$,

$$\text{(a)} \quad (\bar{\alpha}_k) = (\bar{\alpha}'_k) ;$$

$$\text{(b) if } \alpha_{ki} = \sum_j d_{kij}\, \alpha'_{kj} \text{ then}$$

$$m_{ki} \in \sum_j d_{kij}\, m'_{kj} + P^k M ; \text{ and}$$

$$\text{(c) if } \alpha'_{ki} = \sum_j d'_{kij}\, \alpha_{kj} \text{ then}$$

$$m'_{ki} \in \sum_j d'_{kij}\, m_{kj} + P^k M ,$$

is an equivalence relation.

It is now easily shown that, if σ and σ' abbreviate t and t' respectively, then $\sigma \sim \sigma'$ if and only if $t = t'$. The following Lemmas are the analogues of Lemmas 3.11 and 3.12.

5.4 Lemma

Let $t \in S_P(M)$ and suppose that $tf(P,k-1;M)$ is finite for some $k > 0$. Then, t is abbreviated by a sequence $\sigma = \langle(\bar{\alpha}_k, \bar{m}_k) : k > 0\rangle$ in S where

(i) if $k = 1$, then $(\bar{\alpha}_k) = D$ and so we may take $\bar{\alpha}_k = 1$, the unit of D; and

(ii) if $k > 1$, then $\bar{\alpha}_k = \bar{\alpha}_{k-1}$ and $m_{ki} \in m_{k-1,i} + P^{k-1}M$.

5.5 Lemma

Let $t \in S_P(M)$ and $\beta_1,\ldots,\beta_r$ be a fixed basis for P. Then t is abbreviated by a sequence $\sigma = \langle(\bar{\alpha}_k, m_k) : k > 0\rangle$ in S of one of the following forms:

(i) $\sigma = \langle(1, m_k) : k > 0\rangle$ where $m_{k+1} \in m_k + P^k M$;

(ii) there exists $k > 0$ such that $\bar{\alpha}_k \neq 1$.

In case (ii), if k is the least positive integer such that $\bar{\alpha}_k \neq 1$, then, with $\bar{\beta} = (\beta_1,\ldots,\beta_r)$,

(a)
$$\bar{\alpha}_r = \begin{cases} 1 & \text{for } r < k \\ \bar{\beta} & \text{for } r = k \\ \beta^{\epsilon_r}.\bar{\alpha}_{r-1} & \text{for } r > k \end{cases}$$

where $\epsilon_r = 0$ or 1, and must be 0 if $tf(P,r-1)$ is finite, and $\bar{\beta}.\bar{\alpha}_n$ is defined to be the sequence $\langle \beta_i \alpha_{nj} \rangle_{ij}$;

(b)
$$\begin{array}{ll} m_r \in m_{r-1} + P^{r-1}M & 1 < r < k \\ m_{ki} \in \beta_i m_{k-1} + P^k M & r = k \\ m_{ri} \in m_{r-1i} + P^{r-1}M & r > k\ ,\ \epsilon_r = 0 \\ m_{rij} \in \beta_i m_{r-1j} + P^r M & r > k\ ,\ \epsilon_r = 1\ . \end{array}$$

The Chinese Remainder theorem (Proposition 3.16) can also be generalised to the present situation.

5.6 Proposition (Chinese Remainder Theorem for D-Modules)

If $Q_1, \ldots, Q_n$ are comaximal ideals in a Dedekind domain D and if M is an arbitrary D-module, then

$$M \models \forall u_1, \ldots u_n \exists u \bigwedge_{1 \leq i \leq n} Q_i \mid (u - u_i)$$

We are now able to state the analogue of the main theorem of section 3.

5.7 Theorem

For every D-module M,

$$|M| \prod_{P \in \mathcal{P}} \prod_{k \in \omega} tf(P,k;M) \leq |S(M)| \leq \lambda . |M| \prod_{P \in \mathcal{P}} \prod_{k \in \omega} tf(P,k;M)\ .$$

Thus, for $|M| \geq \lambda$,

$$|S(M)| = |M| \prod_{P \in \mathcal{P}} \prod_{k \in \omega} tf(P,k;M)\ .$$

It follows immediately from this that every D-module is stable. (But see also W. Baur's Theorem 2.1 in [3]; or, the abstract of E.R. Fisher [12], in conjunction with Theorems 1.3, 1.4.) A closer look at the values assumed by $tf(P,k)$ gives more precise information about the cardinals $\varkappa \geqq \gamma$ in which a given D-module is $\varkappa$-stable. First some notation:

$$\mathcal{P}_1 = \{P \in \mathcal{P} : \text{for some } k \in \omega , \ tf(P,k) \geqq \omega\} .$$

$$\mathcal{P}_2 = \{P \in \mathcal{P} - \mathcal{P}_1 : \text{for some } k \in \omega , \ tf(P,k) > 1\} .$$

$$\mu_1 = |\mathcal{P}_1| .$$

$$\mu_2 = |\mathcal{P}_2| .$$

5.8 <u>Theorem</u>

Let M be an infinite D-module. Then,

(i) M is $\varkappa$-stable for all $\varkappa \geqq \lambda \cdot 2^{\omega + \mu_2}$ for which $\varkappa^{\omega + \mu_1} = \varkappa$;

(ii) if $tf(P,k) = 1$ almost everywhere, then M is $\varkappa$-stable for all $\varkappa \geqq \lambda$;

(iii) if the hypothesis of (ii) fails, then M is unstable in all powers $\varkappa$ for which $\omega \leqq \varkappa < 2^{\omega + \mu_1 + \mu_2}$; and

(iv) if $tf(P,k) < \omega$ almost everywhere, then M is $\varkappa$-stable for all $\varkappa \geqq \lambda \cdot 2^{\mu_2 + \omega}$.

<u>Proof</u>

First set

$$\prod_{k \in \omega} tf(P,k) = \begin{cases} \varkappa_P & \text{for } P \in \mathcal{P}_1 \text{ ; and} \\ \lambda_P & \text{for } P \in \mathcal{P}_2 . \end{cases}$$

Then, from Theorem 5.7, we have

$$|M| \prod_{P \in \mathcal{P}_1} \varkappa_P \prod_{P \in \mathcal{P}_2} \lambda_P \leqq |S(M)| \leqq \lambda \cdot |M| \prod_{P \in \mathcal{P}_1} \varkappa_P \prod_{P \in \mathcal{P}_2} \lambda_P , \qquad (1)$$

where $\prod_{P \in \mathcal{P}_i} = 1$ if $\mathcal{P}_i = \phi$, $i = 1 , 2$.

Furthermore, $\varkappa_P \geqq \omega$ and $\lambda_P > 1$. (2)

(i) Since $\varkappa_P \leqq |M|^\omega$ and $\lambda_P \leqq 2^\omega$, we have from (1),

$$|S(M)| \leqq \lambda \cdot |M| \cdot (|M|^{\omega})^{\mu_1} \cdot (2^{\omega})^{\mu_2} \leqq \lambda \cdot |M|^{\omega+\mu_1} \cdot 2^{\omega+\mu_2} ,$$

from which the result follows.

(ii) If $tf(P,k) = 1$ almost everywhere, then $\mu_1, \mu_2 < \omega$ and $\kappa_P \leqq |M|$, $\lambda_P < \omega$. Thus from (1) and (2),

$$|M| \leqq |S(M)| \leqq \lambda \cdot |M|^{\mu_1+1} \cdot \omega \leqq \lambda |M| ,$$

so, M is κ-stable for all $\kappa \geqq \lambda$.

(iii) If the hypothesis of (ii) fails then either some $\kappa_P \geqq 2^{\omega}$; or some $\lambda_P \geqq 2^{\omega}$; $\mu_1 \geqq \omega$, or $\mu_2 \geqq \omega$. Using (1) and (2) any one of these possibilities implies that $|S(M)| \geqq |M| \cdot 2^{\omega+\mu_1+\mu_2}$ and hence, M is unstable in power κ where $\omega \leqq \kappa < 2^{\omega+\mu_1+\mu_2}$.

(iv) If $tf(P,k) < \omega$ almost everywhere, then $\mu_1 < \omega$ and $\kappa_P \leqq |M| \cdot 2^{\omega}$, $\lambda_P \leqq 2^{\omega}$. From (1) again, we have

$$|S(M)| \leqq \lambda \cdot |M| \cdot (|M| \cdot 2^{\omega})^{\mu_1} \cdot (2^{\omega})^{\mu_2} \leqq \lambda \cdot |M| \cdot 2^{\omega+\mu_2} .$$

Thus, M is κ-stable for all $\kappa \geqq \lambda \cdot 2^{\omega+\mu_2}$. //

We conclude this section by making two observations on the consequences of Theorem 5.8.

First, if λ , μ_1 and μ_2 satisfy $\lambda < 2^{\omega+\mu_1+\mu_2}$ (in particular, this will be the case if $\lambda = \omega$), then by (iii), if the hypothesis of (ii) fails, M is λ-unstable. Thus, for such modules the converse of (ii) holds. This proves the following corollary.

5.9 <u>Corollary</u>

If M is any infinite module for which $\lambda < 2^{\omega+\mu_1+\mu_2}$ then, M is κ-stable for all $\kappa \geqq \lambda$ if and only if $tf(P,k;M) = 1$ almost everywhere.

Our second observation is that since $\mu_2 + \omega \leqq \lambda = |L(T)|$, (iv) of Theorem 5.8 shows that if $tf(P,k) < \omega$ almost everywhere, then M is superstable (κ-stable for all $\kappa \geqq 2^{|L(T)|}$). Using the obvious generalisation of Theorem 4.5 for modules, or Shelah's Corollary 6.10 in [37], it is possible to prove the converse. Thus, we have also obtained necessary and sufficient conditions for superstable D-modules.

5.10 Corollary

An infinite D-module M is superstable if and only if $tf(P,k;M) < \omega$ almost everywhere.

Chapter 3: The Model Theory of Nil-2 Groups

1. Introduction

In Chapter 2, we saw that either of the concepts of saturation or stability may be applied to give a complete classification of abelian groups. The study of the classification for non-abelian groups has been initiated by Baldwin and Saxl in [1], and by Sabbagh in [28].

Quoting results of Mal'cev (from [22], pages 244-6), Sabbagh claims that each of the linear groups $GL_n(K)$, $SL_n(K)$ and $PSL_n(K)$, where K is a field and n a positive integer, is stable if K is stable and ω-stable if K is algebraically closed. Both papers provide examples of unstable non-abelian groups, notably, any group which contains an isomorphic copy of each finite group. And, as a consequence, the following groups are unstable: any group containing the restricted symmetric group on an infinite set, and every existentially closed (hence, algebraically closed) group.

One theorem, common to both papers, asserts that every non-abelian variety of groups contains an unstable group. A variety of groups is a class of groups defined by a set of equations. Yet very little is known about the model theory of one of the simplest non-abelian varieties, namely, the variety N_2 of all nilpotent groups of class at most 2 . In the remainder of this thesis we shall be concerned with this topic. Before outlining the structure of this chapter we shall briefly review the existing literature on the subject. Definitions of the terminology we use here will be given in section 2 or may be found in the references.

The nil-2 variety is important for another reason, it is well-known (see, for example, [10] Proposition 1) that every variety containing a finite non-abelian group includes as a sub-variety either the variety generated by some Frobenius group (see [10], page 261) or one of the following nil-2 varieties: the variety, N_2^p , of all nil-2 and exponent p groups where p is an odd prime; the variety $N_2^{4,2}$ of all nil-2 and exponent 4 groups with derived group of exponent 2. In [10], Eršov shows that each of these nil-2 varieties, and hence also the variety N_2 , has an undecidable theory. In contrast, he also proves that the class of free groups in N_2^p has a decidable theory.

Questions, of decidability had been considered earlier in a paper by Mal'cev (see [21]). Mal'cev showed that, for each $n \geqq 2$, every free nilpotent group of class at most n has an undecidable theory. Furthermore, as Sabbagh has observed, in [29], the techniques developed in that paper may also be used to show that these groups are unstable. We have generalised this result (Theorem 6.13) obtaining as a special case that the nil-2 free product of torsion-free groups, one of which has a basis (see Definition 3.1), is unstable.

Recently, progress has been made in the study of model companions and existentially closed structures for nilpotent groups. In a talk given at the Abraham Robinson Memorial Conference (see [30] and [31]) and in a later paper (see [32] and [33]) Saracino has shown that, for $n \geqq 2$, neither of the theories K_n , of all nil-n groups, or K_n^+ , of all torsion-free nil-n groups has a model companion. When $n = 2$, he gives a stronger result showing that, for each theory, the classes of existentially closed finitely generic and infinitely generic models are distinct. Of course, for $n = 1$ it is already known (see [9]) that K_1 , the theory of abelian groups, has a model companion.

Problems of a different flavour, for algebraically closed torsion-free nil-2 groups, are discussed in [2] by B. Baumslag and

F. Levin. For example: a torsion-free nil-2 group (T_2-group) can be embedded in an algebraically closed T_2-group of the same cardinality without increasing the rank (as a vector space over the rationals) of the derived group; a T_2-group G is algebraically closed if and only if it is one-unknown closed and G/G' has infinite rank; countable algebraically closed T_2-groups are isomorphic if and only if their centres are isomorphic.

The only other paper we know of which deals with the model theory of nil-2 groups is that of P. Olin [25]. As part of a wider investigation of free products and elementary equivalence, Olin has considered the case of groups. He attributes to B. Jónsson the suggestion of the variety N_2^3 , of nil-2 and exponent 3 groups as the place to begin this investigation for groups. One result, of particular interest to us, is the following.

1.1 Theorem (Olin [25], Theorem 2)

Let V be any variety of groups containing the variety of all nil-2 and exponent 3 groups and contained in the variety of all nil-2 groups, and let $*$ denote the V-free product. Then, there exist denumerable groups A , B in V , with $A \succ B$, such that if C is the cyclic group of order 3, then $C*A \not\equiv C*B$.

Consequently, the nil-2 free product does not preserve elementary equivalence. By contrast, our results (see Theorem 4.4 and its corollaries) show that a certain degree of preservation does take place. In particular, if A and B are both either abelian, free nil-2 or existentially closed nil-2 groups and if $A \equiv B$, then $C*A \equiv C*B$, for all finite nil-2 groups C .

All our results or nil-2 groups were obtained after seeing a preliminary version of Olin's paper, and, indeed, as we have already mentioned in the acknowledgements, we are indebted to him for having recommended to us such a study.

These are the questions we shall attempt to answer in this chapter of the thesis:

(1) How much saturation is preserved by the nil-2 free product in the variety N_2 ?

(2) How much stability is preserved by the nil-2 free product in the variety N_2 ?

Theorems relating to question (1) are to be found in section 5. As a consequence of our answers, in section 6, to question (2) we have classified numerous nil-2 groups according to their stability properties. While we hope that the theorems of both these sections may, ultimately, be useful in classifying nil-2 groups, nevertheless, we believe that each of the questions raised above is interesting for its own sake.

The problem of how much saturation or stability is preserved by the various products in group theory has received some attention already. It is well-known that the direct product preserves both saturation (see Waszkiewicz and Węglorz [39]) and stability (see Waszkiewicz and Węglorz [39]; Macintyre [19]; Eklof and Fisher [8]). On the other hand, in [1], Baldwin and Saxl give an example of an unstable group which is the semi-direct product of two ω-stable abelian groups.

Now, the V-free product is associated with a given variety V in the same way as the direct product is associated with the variety of abelian groups. The results of section 7 show that the full free product preserves neither saturation nor stability. In the case of saturation this failure is extremely bad, for we show (Theorem 7.2) that the free product of every pair of non-trivial groups is 2-unsaturated.

Our results show that the nil-2 free product falls between the two extremes of absolute preservation and the total lack of it. We conclude by giving a brief outline of the remaining sections of this chapter.

Section 2 contains definitions of most of the algebraic terms we shall use, proofs of basic lemmas on commutators, and, for the model-theorist, a survey of the combinatorial group-theoretic notions required. Most nil-2 products we shall form will involve at least one factor having a basis. This concept is defined, in section 3, in terms of the corresponding notion for abelian groups. It turns out that the elements of a nil-2 free product of groups, at least one of which has a basis, have a unique normal form (Theorems 3.7 and 3.9). The existence of these normal forms is the key to many of our theorems.

Another key result is Theorem 4.1, a restricted analogue, for the nil-2 free product, of the well-known Feferman-Vaught Theorem for generalised products (see [11]). As an application we prove the preservation result for elementary equivalence (Theorem 4.4) which should be compared with Theorem 1.1 above.

There are two main theorems in section 5, each giving a necessary and sufficient condition for the preservation of saturation in a restricted situation. In the first (Theorem 5.16) one factor in the nil-2 free product is assumed finite, while in the second (Theorem 5.23) both factors are bounded. In each case the hard part is showing that the condition is necessary. The proof of sufficiency in the first theorem follows from an application of the "Feferman-Vaught" theorem of section 4. We show (Theorem 5.1) that each type over $C*G$, where C is finite, determines another type over G whose satisfaction in G entails the satisfaction of the original type in $C*G$. (Here $*$ denotes the nil-2 free product.) For Theorem 5.23, the proof of sufficiency is then a corollary of Theorem 5.16 and a restricted distributive law for the nil-2 free product over the direct product (Theorem 5.19).

Section 6 contains a number of positive and negative results in answer to the question on preservation of stability. On the positive side, we show that the conditions of Theorems 5.16 and 5.23 are sufficient, in each case, to give preservation of stability (Theorem 6.1 and Corollary 6.4). In fact we are able to strengthen the latter result, under certain conditions to torsion factors (Theorem 6.3). On the negative side, we show that the nil-2 free product of groups is unstable in certain cases where one or both factors are non-torsion (Theorems 6.10, 6.13, Corollaries 6.14, 6.15). These results enable us to produce examples which show that the nil-2 free product can fail to preserve mere-stability and strict-superstability. As far as ω-stability is concerned, although we can produce numerous instances of its preservation (Corollaries 6.2, 6.5, 6.7 and 6.8), the answer to the question of whether ω-stability is always preserved eludes us. Corollary 6.11 and Proposition 6.12 provide examples of unstable nil-2 free products where one factor is ω-stable and the other is superstable.

We have already discussed the content of section 7. We include these results in this chapter, although they have nothing to do with nil-2 groups, because they do relate to the general questions raised here.

2. Preliminaries

Most groups studied in this chapter of the thesis are nilpotent of class 2 (See Definition 2.2, below). However, when we know that a particular result holds in general we shall prove it for all groups. Thus, we shall normally qualify the term "group" unless it is very clear from the context not to do so. All groups, including abelian groups, shall normally be written multiplicatively. The following is an account of some of the notation, definitions and elementary algebraic results we shall need; anything we leave unexplained may be found in the books of M. Hall [14], H. Neumann [24], or Magnus, Karass and Solitar [20] .

Let G be an arbitrary group and X a subset of G . Then $gp(X)$ denotes the subgroup of G consisting of all finite products of elements of X and their inverses; we say that <u>$gp(X)$ is generated by X</u> . G is said to be <u>finitely generated</u> if, for some finite subset X of G , $gp(X) = G$; otherwise G is <u>infinitely generated</u>.

A group is said to be <u>torsion</u>, or <u>periodic</u>, if all its elements have finite order. A group is <u>non-torsion</u> if it possesses at least one element of infinite order, and <u>torsion-free</u> if all its elements have infinite order. A torsion group G is <u>bounded</u> if there is a positive integer n for which $g^n = 1$, for all $g \in G$. The least positive integer with this property is called the <u>exponent</u> of G and we write $\exp G = n$.

The <u>centre</u> of a group G is denoted by $Z(G)$; the <u>centraliser</u> of an element $g \in G$ is the set of all elements of G which commute with g .

If N is a normal subgroup of a group G , then we shall write <u>$N \lhd G$</u> . G/N denotes the <u>quotient</u>, or <u>factor</u>, group of G by N .

Let G be a group and X a subset of G . Then, the <u>normal subgroup of G generated by X</u> (or, <u>normal closure in G of $gp(X)$</u>) is defined to be the intersection of all the normal subgroups of G which contain X . Equivalently, the normal closure in G of $gp(X)$ is the subgroup of G generated by all elements $g^{-1} x g$, where $g \in G$, $x \in X$.

If H is a subgroup of G , then for all a , $b \in G$, we shall write $a = b$ modulo H in case $ab^{-1} \in H$. If K is another subgroup of G, then we shall write $g \in K$ modulo H in case $g = kh$, for some $k \in K$, $h \in H$.

For integers m, n we write $m|n$ if m is a divisor of n. The congruence relation, $\equiv$ mod k, for a positive integer k, is defined as usual by $m \equiv n \bmod k$ if and only if $k|(m-n)$. $n \bmod k$ will be used to denote the unique integer m such that $0 \leqq m < k$ and $m \equiv n \bmod k$. We shall often have occasion to interpret $\equiv \bmod \infty$ as equality.

The following standard abbreviations for abelian groups will be used (in these special cases we shall depart from our stated intention and write them additively):

Z	the additive group of integers;
Q	the additive group of rational numbers;
Z_n	the group of integers under addition mod n, where n is a positive integer;
$Z(p^\infty)$, p a prime	the group of all rationals r such that $0 \leqq r < 1$ and $r = i/p^n$ for some integers $i \geqq 0$, $n \geqq 0$, under addition mod 1.

The commutators $[g_0, g_1, \ldots g_n]$ of elements g_i of a group G are defined recursively by:

$$[g_0] = g_0 ;$$

$$[g_0, g_1] = g_0^{-1} g_1^{-1} g_0 g_1 ;$$

$$[g_0, g_1, \ldots, g_{n+1}] = [[g_0, g_1, \ldots, g_n], g_{n+1}] .$$

If $H_0, H_1, \ldots, H_n$ are subgroups of G, then $[H_0, H_1, \ldots, H_n]$ denotes the subgroup of G generated by all commutators $[h_0, h_1, \ldots, h_n]$ with $h_i \in H_i$, $i \leqq n$. In particular, $[G,G]$ is the commutator subgroup, or derived group, of G and is usually denoted by G'.

The following lemma can be derived immediately from the definitions.

2.1 Lemma

For any elements a, b, c of a group G :

(i) $ba = ab\,[b, a]$;

(ii) $[b,a] = [a, b]^{-1}$;

(iii) $[ab, c] = [a, c][a,c,b][b,c]$;

(iv) $[a, bc] = [a, c][a,b][a,b,c]$.

2.2 Definition

A group G is said to be nilpotent of class $\leq n$, or nil-n , if $G \models \forall u_0 u_1 \ldots u_n\, [u\ ,u_1,\ldots,u_n] = 1$.

We are interested in the nil-2 groups: those groups G for which $[a,b,c] = 1$, for all $a,b,c \in G$. Thus, G is nil-2 if and only if $G' \subset Z(G)$. The following lemma is easily deduced from Lemma 2.1. We use it constantly throughout this chapter and are sure to do so often without explicit reference.

2.3 Lemma

If G is nil-2, then for all $a,b,c \in G$,

(i) $[ab,c] = [a,c][b,c] = [b,c][a,c] = [ba,c]$;

(ii) $[a,bc] = [a,b][a,c] = [a,c][a,b] = [a,cb]$;

(iii) if $g \in G'$ then $[a,g] = [g,a] = 1$;

(iv) $[a,b]^{-1} = [a^{-1},b] = [a,b^{-1}]$;

(v) for any integer n , $[a,b]^n = [a^n,b] = [a,b^n]$;

(vi) for any integer n , $a^n b^n = (ab)^n [a,b]^{\frac{1}{2}n(n-1)}$.

Proof

(i) and (ii) follow immediately from Lemma 2.1 (iii) and (iv).

(iii) follows from (i) and (ii) and the definition of nil-2.

(iv) Using (i),

$$1 = [1,b] = [aa^{-1},b] = [a,b][a^{-1},b] ,$$

and hence, $[a,b]^{-1} = [a^{-1},b]$. Similarly using (ii), $[a,b] = [a,b^{-1}]$.

(v) We prove $[a,b]^n = [a^n,b]$; the other equality will then follow using Lemma 2.1 (ii). First assume that $n \geqq 0$ and use induction. The result is obvious for $n = 0$ and, assuming it holds for $n = k$, it follows from (i) that

$$[a,b]^{k+1} = [a,b]^k[a,b] = [a^k,b][a,b] = [a^{k+1},b] .$$

Hence, by induction, $[a,b]^n = [a^n,b]$ for all $n \geqq 0$. The result for negative n follows from this using (iv).

(vi) First assume that $n \geqq 0$. Clearly the result holds for $n = 0$. Assume it holds for $n = k$. Then, by Lemma 2.1 (i),

$$a^{k+1}b^{k+1} = a^k(ab^k)b = a^kb^ka[a,b^k]b = a^kb^kab[a,b^k] .$$

Thus, by our assumption and (v),

$$a^{k+1}b^{k+1} = (ab)^k[a,b]^{\frac{1}{2}k(k-1)}ab[a,b]^k = (ab)^{k+1}[a,b]^{\frac{1}{2}(k+1)k} .$$

Thus, by induction, (vi) holds for all $n > 0$. Again, the result for negative n follows from (iv) .

The next lemma is also used often, frequently in conjunction with the second normal form theorem (Theorem 3.9). But first we need a definition.

2.4 Definition

The pseudo-order, $o(g)$, of an element g in a group G is the least positive integer n , if one exists, such that $g^n \in G'$. If no such integer exists, then we write $o(g) = \infty$.

2.5 Lemma

Let G be a nil-2 group and a , b elements of G with finite pseudo-orders m , n respectively. Then,

(i) for all integers k , $a^k \in G'$ if and only if $m|k$;

(ii) $o(ab) \mid o(a)o(b)$;

(iii) for all $g \in G$, $[a,g]^m = 1$;

(iv) if $d = \gcd(m,n)$, the greatest common divisor of m and n, then $[a,b]^d = 1$; in particular, if $o(a)$ and $o(b)$ are coprime, then a and b commute;

(v) if $d = \gcd(m,n)$, then for all integers k , there exists an integer r with $0 \leqq r < d$, such that $[a,b]^k = [a,b]^r$.

<u>Proof</u>

First we observe that the pseudo-order of $a \in G$ is equal to the order, in the usual sense, of the element aG' of the abelian group G/G' . Thus (i) and (ii) follow from the corresponding results in abelian group theory. For (i), $a^k \in G'$ iff $(aG')^k = G'$ iff $\text{order}(aG') | k$ iff $m | k$. For (ii) , $o(ab) = \text{order}(abG')$, and since $abG' = (aG')(bG')$, it follows that $o(ab) | \text{order}(aG') \cdot \text{order}(bG')$ and hence, $o(ab) | o(a)o(b)$.

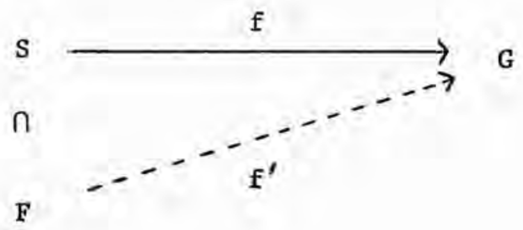
(iii) Since $a^m \in G'$, Lemmas 2.3 (iii) and (v) show that $[a,g]^m = [a^m,g] = 1$.

(iv) If $d = \gcd(m,n)$, then there exist integers α , β with $\alpha m + \beta n = d$. So $[a,b]^d = [a,b]^{\alpha m}[a,b]^{\beta n} = 1$, by (iii).

(v) Let k be an integer. By the Euclidean algorithm there exist integers q and r with $k = qd + r$ and $0 \leqq r < d$. Thus, by (iv), $[a,b]^k = [a,b]^{qd}[a,b]^r = [a,b]^r$. //

For the remainder of this section we shall survey the basic ideas from combinatorial group theory which are essential to an understanding of the material in this chapter.

A group F is said to be <u>free</u> on the set $S \subset F$, if S generates F and for every group G and mapping $f: S \to G$ there exists a unique homomorphism $f': F \to G$ extending f . We have the following picture:



Given such an F, the cardinality of S is known to be unique and is called the rank of F. It can be shown that free groups of each rank exist and are unique up to isomorphism.

Perhaps a more intuitive idea of what a free group is may be obtained from its construction. Let X be a non-empty (finite or infinite) set of symbols x. We shall denote these symbols also by x^{+1} and for each $x \in X$ construct another symbol x^{-1}. Then, any finite sequence

$$w = x_1^{\epsilon_1} \ldots x_n^{\epsilon_n} \tag{1}$$

where $\epsilon_i = \pm 1$, $1 \leqq i \leqq n$, and repetitions of the symbols involved are allowed, is called a word on X. If, in (1), no symbol x_i^{+1} is adjacent to its associated symbol x_i^{-1}, then w is called a reduced word. It can be shown that the set of all reduced words on X forms a group: the group operation is concatenation followed by the successive deletion of all adjacent pairs $x_i^{\epsilon_i}, x_i^{-\epsilon_i}$; the group identity is the empty word with no symbols. It is known that the free group F on the set $S \subset F$ is (isomorphic to) the group of all reduced words on S constructed in this manner.

In general, when speaking of words we shall normally mean reduced words on a fixed countable set $X = \{x_n : n \in w\}$. If w is a reduced word which involves at most the symbols $x_o, \ldots, x_n$ then we shall write $w(x_o, \ldots, x_n)$. Let F_ω denote the free group on X.

Let G be an arbitrary group and w a reduced word $(w \in F_\omega)$. Then, w is said to be a law in G if and only if for every homomorphism $f: F_\omega \to G$, $f(w) = 1$. Equivalently, w is a law in G if and only if, with the obvious notation, $w(\bar{g}) = 1$, for every sequence $\bar{g}$ from G of the appropriate length.

Let W be a set of (reduced) words. Then, the verbal subgroup, $W(G)$, of a group G associated with W is defined by

$$W(G) = gp(\{w(\bar{g}) : w \in W , \bar{g} \in {}^{<\omega}G\}) .$$

As an illustration, the derived group G' of G is the verbal subgroup of G associated with the set $\{[x_0,x_1]\}$ of words.

Let V be a set of words. Then the <u>variety</u> of groups defined by V is the class of all groups for which each word in V is a law. We shall often use V to denote both the set of words and the variety it defines. Furthermore, if V is a variety and v a word in V, then we shall call "$v = 1$" a <u>law of the variety</u>. The empty set defines the variety of all groups and the variety of all abelian groups is defined by the law $[x_0,x_1] = 1$. The variety of all nil-n groups is defined by the law $[x_0,\ldots,x_n]= 1$ and we denote it by N_n. We are primarily interested in the variety N_2. (A characterisation of all the distinct sub-varieties of N_2 has been given by B. Jonsson in [17]. These are the varieties $B(m,n)$ defined by the laws $x_0^m = 1$, $[x_0,x_1]^n = 1$ where $n \cdot \gcd(2,m)|m$.)

Clearly, a variety is closed under the taking of subgroups, quotient groups and cartesian product groups. By a classical result of Birkhoff (see [24], section 1.5, for the proof in the case of groups) the converse is also valid, and this provides an alternative definition of a variety.

Let V be an arbitrary variety of groups, F a group in V and S a subset of F. Then, F is said to be <u>free in the variety V on S</u>, or <u>V-free on S</u>, if and only if S generates F and, for every group G in V and mapping $f: S \to G$, there exists a unique homomorphism $f': F \to G$ extending f. If V is the variety of all groups then this definition coincides with the one, previously given, of a free group. Given such an F in V, the cardinality of S is known to be unique and is called the rank of F. V-free groups of each rank exist and are unique up to isomorphism. Indeed, if $\tilde{F}$ is the (absolutely) free group on S, then the V-free group on S is (isomorphic to) the group $\tilde{F}/V(\tilde{F})$.

Finally, we describe the V-free product of groups in a given variety V. This product may be seen as one way to generalise the direct product operation in the variety of abelian groups. First we should mention that we reserve the notation $\prod_{i\in I} A_i$ for the direct product of groups A_i, $i \in I$. Elements of this group will be denoted by $\prod_{i\in I} a_i$ (or $\prod_{i\in I} a(i)$), where for each $i \in I$, $a_i \in A_i$ ($a(i) \in A_i$) and it is understood that at most finitely many a_i ($a(i)$) are different from 1. The full cartesian product of the groups A_i, $i \in I$, will be denoted by $\overline{\prod_{i\in I}} A_i$ and its elements by $\overline{\prod_{i\in I}} a_i$ or, $\overline{\prod_{i\in I}} a(i)$.

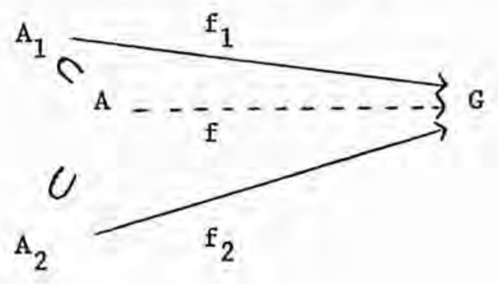
Of course, the direct product is a subgroup of the cartesian product and when the index set is finite, they coincide. (We shall choose the direct product notation or $A_1 \times \ldots \times A_n$, for both products in this case.) If $A_i = A$, for every $i \in I$, then we shall denote by $A^{(I)}$ (A^I) the direct (cartesian) power of A.

2.6 Definition

A group A is called the free product of its subgroups A_1, A_2 ($A_1 \cap A_2 = 1$) if and only if

(i) A is generated by $A_1 \cup A_2$; and

(ii) for every pair of homomorphisms, $f_i\colon A_i \to G$, of A_i into a given group G, $i = 1, 2$, there exists a unique homomorphism $f\colon A \to G$ whose restriction to A_i is f_i.



It can be shown that free-products exist and are unique up to isomorphism, so we denote the free product of A_1 and A_2 by $A_1 * A_2$. In a similar way, we may define the free product of a collection $\{A_i : i \in I\}$ of groups which we denote $\prod^*_{i\in I} A_i$. The following result gives a normal form for the elements of $\prod^*_{i\in I} A_i$.

2.7 Lemma (Normal form for the elements of a free product)

Every element $a \neq 1$ of $\prod^*_{i\in I} A_i$ may be written, uniquely, in the form

$$a_1 a_2 \dots a_n$$

with each $a_k \neq 1$ belonging to some group A_{i_k} and $i_k \neq i_{k+1}$.

Let V be an arbitrary variety of groups. Then the V-free product is defined in the obvious way as follows.

2.8 Definition

The group A in V is called the V-free product (or, the free product in the variety V; verbal product; varietal product) of its subgroups A_1 and A_2 ($A_1 \cap A_2 = 1$) if and only if

(i) A is generated by $A_1 \cup A_2$; and

(ii) for every pair of homomorphisms, $f_i : A_i \to G$, of A_i into a group $G \in V$, $i = 1, 2$, there exists a unique homomorphism $f: A \to G$ whose restriction to A_i is f_i.

Again, V-free products can be shown to exist in every variety and are unique up to isomorphism. We denote the V-free product of $A_1, A_2 \in V$ by $A_1 *_V A_2$ and shall drop the V when it is clear from the context to do so. Indeed, for most of this chapter $A_1 * A_2$ denotes the N_2, or nil-2, free product of groups $A_1, A_2 \in N_2$. (We deviate from this only in Theorem 5.19 and in section 7.) As for the free product, $\prod^{*V}_{i\in I} A_i$ denotes the V-free product of groups

$A_i \in V$, $i \in I$, and is defined in the obvious way.

The actual construction of the V-free product $A_1 *_V A_2$ shows that, within isomorphism, $A_1 *_V A_2 = A_1 * A_2 / V(A_1 * A_2)$. Hence, the elements of $A_1 *_V A_2$ may be construed as words on the symbols of A_1 and A_2 (where we may assume that $A_1 \cap A_2 = 1$) with concatenation for multiplication and with words identified only if the group axioms and the laws of V permit it.

3. The Normal Form Theorems

Consider two groups G and H belonging to the variety, N_2 , of all nilpotent groups of class at most 2. The aim of this section is to show that, in certain cases, the elements of the N_2-free product G*H may be represented, uniquely, in a particularly simple form. As we have already mentioned in the introduction to this chapter, the results here were obtained after seeing a preliminary version of [25].

Repeated applications of the laws $vu = uv[v,u]$, $[u,v,w] = 1$ and $[v,u] = [u,v^{-1}]$ holding in N_2 , enables one to show that an arbitrary element $x \in G*H$ may be written in the form $x = ghc$, where $g \in G$, $h \in H$, and c is a product of mixed commutators $[g',h']$ with $g' \in G$ and $h' \in H$. We can be more precise about the composition of the element c in such a representation of x , if we fix a set of generators for one of the factors G or H . Suppose that G is generated by elements g_i , $i \in I$. Then, using the additional law $[uv,w] = [u,w][v,w]$, we may write $x \in G*H$ in the form

$$x = gh \prod_{i \in I} [g_i, h_i] \qquad (1)$$

where $g \in G$, h , $h_i \in H$ and only finitely many h_i are different from 1 .

In general, such a representation of $x \in G*H$ is not unique. For example, if some generator, say g_o , has finite order n , then the elements $[g_0,h]$ and $[g_0,h^{n+1}]$ are equal but have different representations in the form (1). Perhaps a more subtle way in which uniqueness might be destroyed is if there are relations holding amongst the generators. Consider, for example, any nil-2 groups G generated by two elements g_1 and g_2. Then G is also generated by elements g_1,g_2,g_3 where $g_3 = g_1g_2$. Thus, for all $H \in N_2$, the elements $[g_1,h][g_2,h]$ and $[g_3,h]$ are equal.

These considerations lead us to restrict our attention to those $G \in N_2$ which possess, what we shall call, a basis modulo G'. This idea is defined for all groups in terms of the corresponding notion for abelian groups (see Hall [14], page 37). We recall that definition first.

A set of elements $\{a_i : i \in I\}$ in an abelian group A is said to be <u>independent</u> if and only if a finite product $a_{i_1}^{n_1}\ldots a_{i_k}^{n_k} = 1$ only only when $a_{i_j}^{n_j} = 1$, for all $j = 1,\ldots,k$. (Remember that we are now writing abelian groups multiplicatively.) The set $\{a_i : i \in I\}$ is a <u>basis</u> for A if, in addition to being independent, it also generates A. Clearly, a set $\{a_i : i \in I\}$ is a basis for A if and only if A is the direct product of the cyclic groups $gp(a_i)$ generated by the a_i, $i \in I$. Finitely generated, hence also finite, abelian groups (see [14], Theorem 3.2.2) and bounded abelian groups (see [18], Theorem 6) are all examples of abelian groups with a basis. Of course, we do not claim that such a basis is unique.

3.1 <u>Definition</u>

A set $\{a_i : i \in I\}$ of elements in an arbitrary group G is said to form <u>a basis for G modulo G'</u> if and only if the set $\{a_iG' : i \in I\}$ forms a basis for the abelian group G/G'. (In future, we shall suppress the "modulo G'" and refer, simply, to a basis for G.)

Clearly, all finitely generated and all bounded groups possess a basis. Our first lemma shows that the elements of a group with a basis may be written, uniquely modulo G', as a finite product of the generators. First we recall the following definition from section 2.

3.2 Definition

The pseudo-order, $o(g)$, of an element g in a group G is the least positive integer n, if one exists, such that $g^n \in G'$. If no such integer exists we write $o(g) = \infty$.

Clearly, the pseudo-order of an element $g \in G$ is equal to the order, in the usual sense, of the element gG' of G/G'.

3.3 Lemma

Let $\{a_i : i \in I\}$ be a basis for a group G. Then, every element of G can be written, uniquely, in the form

$$\prod_{i \in I} a_i^{\alpha_i} . a' \qquad (2)$$

where $a' \in G'$, the α_i are integers, only finitely many of which are non-zero, and $0 \leq \alpha_i < o(a_i)$ when $o(a_i) < \infty$. Furthermore, if $G \in N_2$, then G' is generated by the elements of the basis.

Proof

Let G be a group with a basis $\{a_i : i \in I\}$. Then, $\{a_i G' : i \in I\}$ is a basis for G/G', and so, for each $g \in G$, there exist unique integers α_i, only finitely many of which are non-zero, with $0 \leq \alpha_i < o(a_i)$ when $o(a_i)$ is finite, such that

$$gG' = \prod_{i \in I} (a_i G')^{\alpha_i} = \prod_{i \in I} a_i^{\alpha_i} . G' .$$

It follows that there is $a' \in G'$ with $g = \prod_{i \in I} a_i^{\alpha_i} . a'$. The uniqueness of a' follows from the uniqueness of the α_i.

Now assume that $G \in N_2$. To prove that G' is generated by the elements of the basis, it suffices to show that each commutator of G is so generated. Let $g_1, g_2 \in G$. Since each of g_1, g_2 can be written in the form (2), it follows, with obvious notation, that

$$[g_1,g_2] = [\prod_{i\in I} a_i^{\alpha_i}.a' , \prod_{i\in I} a_i^{\beta_i}.a''] = \prod_{i,j\in I} [a_i,a_j]^{\alpha_i\beta_j} .$$

This shows that the basis generates $[g_1,g_2]$, for all $g_1,g_2 \in G$. //

It follows from this lemma that, for a nil-2 group G, a set $\{a_i : i \in I\}$ is a basis for G if and only if it generates G and is independent modulo G' (that is, a finite product $\prod a_i^{n_i}$ belongs to G' only when each $a_i^{n_i}$ belongs to G'). There is a temptation to carry the analogy with abelian groups even further and hope to prove that $\{a_i : i \in I\}$ is a basis for G if and only if G is the N_2-free product of the cyclic subgroups generated by the a_i.

This temptation is even greater when we compare the conclusion of Lemma 3.3 and the form the commutators take for $G \in N_2$ with the expression (4) in the second normal form theorem (Theorem 3.9, below). Indeed, using this theorem, it is easy to see that one direction of the assertion does hold: if G is the N_2-free product of cyclic groups $gp(a_i)$, $i \in I$, then $\{a_i : i \in I\}$ forms a basis for G. However, the converse is false as the following example shows.

Let G be the nil-2 group $Z_2*(Z_2\times Z_2)$. Using the second normal form theorem (Theorem 3.9, below), or by direct calculation, it can be shown that the set $\{a,b,c\}$, where a generates Z_2 and $\{b,c\}$ is a basis for $Z_2\times Z_2$, is a basis for G. The elements of G can then be written, uniquely, in the form

$$a^i b^j c^k [a,b]^l [a,c]^m ,$$

where i, j, k, l and $m \in \{0,1\}$. However, $G \not\cong gp(a)*gp(b)*gp(c)$ for, in G, the elements b and c commute whilst in the latter group they do not. Observe that G has cardinality 2^5 and its elements all have order 1, 2 or 4 . Using the second normal form theorem and considering all the possibilities, it can be seen that there is no nil-2 group of cardinality 2^5 which is the N_2-free product of cyclic groups of order 2 or 4 . Thus, G has no basis for which it is the N_2-free product of the corresponding cyclic subgroups.

In the normal form theorems and, indeed, throughout this chapter, an important role is played by certain subgroups of $G \in N_2$. For this reason, we single out their definition.

3.4 Definition

For each group G , the subsets $H_\infty(G)$ and $H_n(G)$, $n \in \omega$, of G are defined by $H_\infty(G) = H_o(G) = G'$ and

$$H_n(G) = \{g^n g' : g \in G \text{ , } g' \in G'\}.$$

3.5 Lemma

For every group G and each $n \geqq 0$, $H_n(G)$ is a subgroup of G .

Proof

A straightforward argument by induction using the law $xy = yx[x,y]$ yields: $x^n y^n = (xy)^n$ modulo G' , for all x , $y \in G$ and $n \in \omega$. From this the Lemma follows immediately. //

We shall often have occasion to use the following simple property of these subgroups.

3.6 Lemma

Let G , $K \in N_2$ with G a subgroup of K . Then, for all $k \in K$ and for all $g \in H_{o(k)}(G)$, we have $[k,g] = 1$.

Proof

Let G , $K \in N_2$, G be a subgroup of K and let $k \in K$ with $o(k) = n \leqq \infty$. If $o(k) = \infty$, then $H_{o(k)}(G) = G'$ and so, since K

is nil-2, $[k,g] = 1$ for all $g \in H_{o(k)}(G)$. Suppose that $n < \infty$. If $g \in H_n(G)$, then $g = g_1^n g_2$, for some $g_1 \in G$, $g_2 \in G'$. Thus, $[k,g] = [k,g_1^n g_2] = [k^n,g_1]$, by Lemma 2.3. But $o(k) = n$ and so $k^n \in K'$. Hence $[k,g] = [k^n,g] = 1$. //

3.7 <u>Theorem</u> (The first normal form theorem)

Let C , $G \in N_2$ and C have a basis $\{a_i : i \in I\}$, where $o(a_i) = m_i (\leqq \infty)$. (For notational convenience it is assumed that $0 \notin I$.) Then, the elements of $C*G$ can be written in the form

$$cg_0 \prod_{i\in I} [a_i,g_i] \qquad (3)$$

where $c \in C$, $g_0, g_i \in G$ and only finitely many $g_i \notin H_{m_i}(G)$.

Furthermore,

(i) $c_1 g_{10} \prod_{i\in I} [a_i,g_{1i}] = c_2 g_{20} \prod_{i\in I} [a_i,g_{2i}]$ if and only if $c_1 = c_2$, $g_{10} = g_{20}$ and, for each $i \in I$, $g_{1i}\, g_{2i}^{-1} \in H_{m_i}(G)$;

(ii) $c_1 g_{10} \prod_{i\in I} [a_i,g_{1i}] \cdot c_2 g_{20} \prod_{i\in I} [a_i,g_{2i}]$
$= c_1 c_2 g_{10} g_{20} \prod_{i\in I} [a_i, g_{10}^{-\alpha_{2i}} g_{1i} g_{2i}]$

where, for $j = 1$, 2 , $c_j = \prod_{i\in I} a_i^{\alpha_{ji}}$ modulo C' ;

(iii) $cg_0 \prod_{i\in I} [a_i,g_i] = 1$ if and only if $c = 1$, $g_0 = 1$ and $g_i \in H_{m_i}(G)$, for every $i \in I$;

(iv) $(cg_0 \prod_{i\in I} [a_i,g_i])^{-1} = c^{-1} g_0^{-1} \prod_{i\in I} [a_i, g_0^{-\alpha_i} g_i^{-1}]$,

where $c = \prod_{i\in I} a_i^{\alpha_i}$ modulo C' .

<u>Proof</u>

Let W be the set of all words in the symbols of the set $C \cup G$ (see section 2 for the definition of a word) and $\hat{W}$ the subset of W of all words of the form (3) above. We define two words $c_k g_{k0} \prod_{i \in I} [a_i, g_{ki}]$, $k = 1$, 2 , to be equivalent if and only if $c_1 = c_2$, $g_{10} = g_{20}$ and, for each $i \in I$, $g_{1i} g_{2i}^{-1} \in H_{m_i}(G)$. It is easy to see that this defines an equivalence relation, $\sim$, on $\hat{W}$. Let $\{w\}$ denote the equivalence class of $w \in \hat{W}$ and let K be the set of all such equivalence classes. The bulk of the proof of this theorem is the verification that K forms a nil-2 group under a multiplication given by $\{w_1\} \circ \{w_2\} = \{w_1 \cdot w_2\}$, with $w_1 \cdot w_2$ defined by (ii); identity $\{1 \cdot 1 \cdot \prod_{i \in I} [a_i, 1]\}$, and $\{w\}^{-1} = \{w^{-1}\}$, with w^{-1} given by (iv) .

Let us assume, for the moment, that K does form a nil-2 group. We show that $C*G$ is isomorphic to K . Now, the mappings $f_1: C \to K$, $f_2: G \to K$ defined by $f_1(c) = \{c \cdot 1 \circ \prod_{i \in I} [a_i, 1]\}$, $f_2(g) = \{1 \cdot g \cdot \prod_{i \in I} [a_i, 1]\}$ are group isomorphisms, and it is clear that their images generate K . Thus, the homomorphism, $f: C*G \to K$, extending f_1 and f_2 , given by the definition (see Definition 2.8) of $*$, is onto K . From the preceeding discussion (see (1)), we know that the elements of $C*G$ can be written in the form $cg_0 \prod_{i \in I} [a_i, g_i]$ with only finitely many g_i different from 1 . Suppose that

$$f(cg_0 \prod_{i \in I} [a_i, g_i]) = \{1 \cdot 1 \cdot \prod_{i \in I} [a_i, 1]\} ,$$

the identity in K . Then,

$$\{cg_0 \prod_{i \in I} [a_i, g_i]\} = \{1 \cdot 1 \cdot \prod_{i \in I} [a_i, 1]\}$$

because f extends f_1 and f_2 . Hence, by the definition of $\sim$, $c = 1$, $g_0 = 1$ and $g_i \in H_{m_i}(G)$. But then, using Lemma 3.6, we have, $cg_0 \prod_{i \in I} [a_i, g_i] = 1$. This shows that f is an isomorphism and hence, $C*G \simeq K$. From this, and the definition of $\sim$, it follows that (i)-(iv) hold in $C*G$.

Finally we give the verification that K is a nil-2 group. First we check that multiplication and inverses are well-defined. For each $k = 1 , 2$, let

$$w_k = c_k g_{k0} \prod_{i\in I} [a_i, g_{ki}] \sim \hat{c}_k \hat{g}_k \prod_{i\in I} [a_i, \hat{g}_{ki}] = \hat{w}_k$$

Then,

$$c_k = \hat{c}_k \ , \quad g_{k0} = \hat{g}_{k0} \text{ and } g_{ki}\hat{g}_{ki}^{-1} \in H_{m_i}(G) \ , \quad k = 1 , 2 .$$

Since $c_k = \hat{c}_k$, it follows from Lemma 3.3, that there exist unique α_{ki} , such that

$$c_k = \hat{c}_k = \prod_{i\in I} a_i^{\alpha_{ki}} \text{ modulo } C' .$$

So,

(a) $c_1 c_2 = \hat{c}_1\hat{c}_2$, $g_{10}g_{20} = \hat{g}_{10}\hat{g}_{20}$ and $(g_{10}^{-\alpha_{2i}} g_{1i} g_{2i})(\hat{g}_{10}^{-\alpha_{2i}}\hat{g}_{1i}\hat{g}_{2i})^{-1} \in H_{m_i}(G)$;

(b) $c_1^{-1} = \hat{c}_1^{-1}$, $g_{10}^{-1} = \hat{g}_{10}^{-1}$ and $(g_{10}^{-\alpha_{1i}} g_{1i}^{-1})(\hat{g}_{10}^{-\alpha_{1i}}\hat{g}_{1i}^{-1})^{-1} \in H_{m_i}(G)$.

Now, (a) implies that $w_1 w_2 \sim \hat{w}_1\hat{w}_2$ and hence multiplication is well-defined; (b) implies that $w_1^{-1} \sim \hat{w}_1^{-1}$ and hence inverses are well defined.

The verifications that $\{1\cdot 1\cdot \prod_{i\in I} [a_i,1]\}$ acts as identity, and that $\{w\}^{-1} = \{w^{-1}\}$ for $w \in K$, are easy and so we omit them.

For the associative law: let $w_k = c_k g_{k0} \prod_{i\in I} [a_i, g_{ki}]$, $k = 1, 2, 3$ and $c_k = \prod_{i\in I} a_i^{\alpha_{ki}}$ modulo C' . In the following we use the definition of $\sim$ and Lemma 2.3 constantly.

$$w_1(w_2 w_3) \sim c_1 g_{10} \prod_{i\in I}[a_i, g_{1i}]\cdot c_2 c_3 g_{20} g_{30} \prod_{i\in I}[a_i, g_{20}^{-\alpha_{3i}} g_{2i} g_{3i}]$$

$$\sim c_1(c_2c_3)g_{10}(g_{20}g_{30})\prod_{i\in I}[a_i,g_{10}^{-(\alpha_{2i}+\alpha_{3i})\bmod m_i}g_{20}^{-\alpha_{3i}}g_{1i}g_{2i}g_{3i}]\ ,$$

since $c_2c_3 = \prod_{i\in I} a_i^{(\alpha_{2i}+\alpha_{3i})\bmod m_i}$ modulo C'. (Note that $\equiv$ mod ∞ must be interpreted as equality.) Thus, using the definition of $\sim$ and the associative law in C and G,

$$w_1(w_2w_3) \sim (c_1c_2)c_3(g_{10}g_{20})g_{30}\prod_{i\in I}[a_i,(g_{10}g_{20})^{-\alpha_{3i}}g_{10}^{-\alpha_{2i}}g_{1i}g_{2i}g_{3i}]$$

$$\sim c_1c_2g_{10}g_{20}\prod_{i\in I}[a_i,g_{10}^{-\alpha_{2i}}g_{1i}g_{2i}]\cdot c_3g_{30}\prod_{i\in I}[a_i,g_{3i}]$$

$$\sim (w_1w_2)w_3\ .$$

Hence, $\{w_1\}(\{w_2\}\{w_3\}) = (\{w_1\}\{w_2\})\{w_3\}$, proving the associative law.

Finally, with w_1, w_2, w_3 as above, we verify the nil-2 law. Now, $[w_1,w_2,w_3] = [[w_1,w_2], w_3]$ by definition. But,

$$[w_1,w_2] = w_1^{-1}w_2^{-1}w_1w_2$$

$$\sim c_1^{-1}g_{10}^{-1}\prod_{i\in I}[a_i,g_{10}^{-\alpha_{1i}}g_{1i}^{-1}]\cdot c_2^{-1}g_{20}^{-1}\prod_{i\in I}[a_i,g_{20}^{-\alpha_{2i}}g_{2i}^{-1}]$$

$$\cdot\ c_1c_2g_{10}g_{20}\prod_{i\in I}[a_i,g_{10}^{-\alpha_{2i}}g_{1i}g_{2i}]$$

$$\sim c_1^{-1}c_2^{-1}g_{10}^{-1}g_{20}^{-1}\prod_{i\in I}[a_i,g_{10}^{-\alpha_{2i}}g_{10}^{-\alpha_{1i}}g_{20}^{-\alpha_{2i}}g_{1i}^{-1}g_{2i}^{-1}]$$

$$\cdot\ c_1c_2g_{10}g_{20}\prod_{i\in I}[a_i,g_{10}^{-\alpha_{2i}}g_{1i}g_{2i}]$$

$$\sim [c_1,c_2][g_{10},g_{20}]\prod_{i\in I}[a_i,g_{10}^{\alpha_{1i}+\alpha_{2i}}g_{20}^{\alpha_{1i}+\alpha_{2i}}g_{10}^{-\alpha_{1i}-2\alpha_{2i}}g_{20}^{-\alpha_{2i}}]$$

$$\sim [c_1,c_2][g_{10},g_{20}]\prod_{i\in I}[a_i,g_{10}^{-\alpha_{2i}}g_{20}^{\alpha_{1i}}]\ .$$

So, using this equivalence with $[w_1,w_2]$ in place of w_1 and w_3

in place of w_2, we have

$$[w_1,w_2,w_3] \sim [c_1,c_2,c_3][g_{10},g_{20},g_{30}] \prod_{i\in I} [a_i,[g_{10},g_{20}]^{-\alpha_{3i}}]$$

$$\sim [c_1,c_2,c_3][g_{10},g_{20},g_{30}] \prod_{i\in I} [a_i,1]$$

$$\sim 1 ,$$

using the nil-2 law for C and G . Hence, $[\{w_1\}, \{w_2\}, \{w_3\}]$ is the identity in K and so K is nil-2 . This completes the proof. //

Our second normal form theorem treats the case where both factors in the free product have a basis. Let A and B be nil-2 groups with basis $\{x_i : i \in I\}$, $\{y_j : j \in J\}$ respectively. We show that every element $g \in A*B$ can be written uniquely in the form

$$ab \prod_{(i,j)\in I\times J} [x_i,y_j]^{\gamma_{ij}}$$

where $a \in A$, $b \in B$ and the γ_{ij} are integers. In the case where $o(x_i) = m_i$ and $o(y_j) = n_j$ are both finite, the γ_{ij} will be unique modulo the greatest common divisor of m_i and n_j . First we make a definition extending the concept of greatest common divisor to include infinity.

3.8 Definition

Let m and n be positive integers or ∞ . Then, we define $\gcd(m,n) \leqq \infty$ by:

$$\gcd(m,n) = \begin{cases} \text{the greatest common divisor of M and n if } m , n < \infty ; \\ \text{the minimum of } m , n \text{ otherwise .} \end{cases}$$

3.9 Theorem (The second normal form theorem)

Let A and B be nil-2 groups such that

(i) A has a basis $\{x_i : i \in I\}$, with $o(x_i) = m_i (\leqq\infty)$; and

(ii) B has a basis $\{y_j : j \in J\}$, with $o(y_j) = n_j (\leqq \infty)$.

Then, for each element g of $A*B$ there are unique $a \in A$, $b \in B$ and integers γ_{ij} , only finitely many of which are non-zero, with $0 \leqq \gamma_{ij} < \gcd(m_i, n_j) = d_{ij}$ when d_{ij} is finite, such that

$$g = ab \prod_{(i,j) \in I \times J} [x_i, y_j]^{\gamma_{ij}} \tag{4}$$

<u>Proof</u>

The fact that every $g \in A*B$ can be represented in the form (4) follows from (i) and (ii), the first normal form theorem (Theorem 3.7), Lemma 3.3 and Lemmas 2.3 and 2.5. To show uniqueness we first show that $g = 1$ if and only if $a = b = 1$ and $\gamma_{ij} = 0$ for all $i \in I$, $j \in J$. (5)

Let g be an arbitrary element of $A*B$ written in the form (4): $g = ab \prod_{i,j} [x_i, y_i]^{\gamma_{ij}}$, with $a \in A$, $b \in B$, γ_{ij} integers with only finitely many $\gamma_{ij} \neq 0$ and $0 \leqq \gamma_{ij} < d_{ij}$ if $d_{ij} < \infty$. Clearly, if $a = b = 1$ and $\gamma_{ij} = 0$ for all $i \in I$, $j \in J$ then $g = 1$. For the converse, we first observe that

$$\prod_{i,j} [x_i, y_j]^{\gamma_{ij}} = \prod_{i,j} [x_i, y_j^{\gamma_{ij}}] = \prod_{i \in I} [x_i, \prod_{j \in J} y_j^{\gamma_{ij}}] .$$

Thus, using Theorem 3.7 (iii) with A in place of C and B in place of G we have that if $g = 1$, then

$$a = b = 1 \text{ and } \prod_{j \in J} y_j^{\gamma_{ij}} \in H_{m_i}(B) .$$

Hence, by Lemma 3.6,

$$g = 1 \Rightarrow a = b = 1 \text{ and } [x_i, \prod_{j \in J} y_j^{\gamma_{ij}}] = 1 \text{ , for all } i \in I .$$

But, for each $i \in I$,

$$[x_i, \prod_{j\in J} y_j^{\gamma_{ij}}] = \prod_{j\in J} [x_i, y_j]^{\gamma_{ij}} = \prod_{j\in J} [y_j, x_i^{-\gamma_{ij}}]$$

and so, using 3.7 (iii) with the roles of A and B reversed we have,

$$[x_i, \prod_{j\in J} y_j^{\gamma_{ij}}] = 1 \Rightarrow \prod_{j\in J} [y_j, x_i^{-\gamma_{ij}}] = 1$$

$$\Rightarrow \text{for each } j \in J\ ,\ x_i^{-\gamma_{ij}} \in H_{n_j}(A)\ .$$

To summarise, since $H_{n_j}(A)$ is a subgroup of A, we have shown $(*)$:

$$g = 1 \Rightarrow a = b = 1 \text{ and } x_i^{\gamma_{ij}} \in H_{n_j}(A)\ , \text{ for every } i \in I\ ,\ j \in J\ .$$

Consider a fixed pair $(i,j) \in I{\times}J$, and let $\gamma = \gamma_{ij}$. We have two cases to consider: $n_j = \infty$, $n_j < \infty$. In each case we show that $x_i^{\gamma} \in H_{n_j}(A)$ implies that $\gamma = 0$.

<u>Case 1</u>: $n_j = \infty$. In this case $H_{n_j}(A) = A'$ and so, by Lemma 2.5 and the definition of pseudo-order, $x_i^{\gamma} \in A'$ implies that $m_i | \gamma$ if $m_i < \infty$ and $\gamma = 0$ if $m_i = \infty$. But, if $m_i < \infty$, then $0 \leqq \gamma < \gcd(m_i,n_j) = m_i$ and so, in either case we have $\gamma = 0$.

<u>Case 2</u>: $n_j < \infty$. In this case $\gcd(m_i,n_j) \leqq n_j < \infty$ and so $0 \leqq \gamma < \gcd(m_i,n_j)$. Suppose $x_i^{\gamma} \in H_{n_j}(A)$. Then, there are elements $a \in A$, $a' \in A'$ such that $x_i^{\gamma} = a^{n_j}a'$. But A has a basis and so by Lemma 3.3, there are integers α_k for each $k \in I$, and $a'' \in A'$ with $a = \prod_{k\in I} x_k^{\alpha_k}\cdot a''$. Thus, using Lemma 2.3,

$$\begin{aligned} x_i^{\gamma} &= (\prod_{k\in I} x_k^{\alpha_k}.a'')^{n_j}.a' \\ &= \prod_{k\in I} x_k^{\alpha_k n_j}\cdot a''' , \end{aligned}$$

for some $a''' \in A'$. For each $k \in I$ such that $m_k < \infty$, let β_k be the integer such that $\beta_k \equiv \alpha_k n_j \mod m_k$ and $0 \leqq \beta_k < m_k$. For $m_k = \infty$, set $\beta_k = \alpha_k n_j$. Then,

$$x_i^{\gamma} = \prod_{k \in I} x_k^{\beta_k} . a'''' , \qquad (**)$$

with $a'''' \in A'$ and $0 \leqq \beta_k < m_k$ if $m_k < \infty$. Hence, by Lemma 3.3, since (**) yields two representations in the form (2) of the same element of A, we must have, in particular, $\beta_i = \gamma$. If $m_i = \infty$, this means that $\gamma = \alpha_i n_j$. But since $0 \leqq \gamma < \gcd(m_i, n_j) \leqq n_j$, it follows that $\alpha_i = 0$ and hence $\gamma = 0$. If $m_i < \infty$, then since $\gamma = \beta_i \equiv \alpha_i n_j \mod m_i$ it follows that $\gcd(m_i, n_j) | \lambda$. But $0 \leqq \gamma < \gcd(m_i, n_j)$ and so, again, $\gamma = 0$. So either way $\delta = 0$.

Now combining what we have just proved in cases 1 and 2 with (*) above, we have shown that

$$g = 1 \Rightarrow a = b = 1 \text{ and } \gamma_{ij} = 0 \text{ for all } i \in I, \ j \in J .$$

Finally, we show that (5) implies uniqueness. Now, $a_1 b_1 \prod_{i,j} [x_i, y_j]^{\gamma_{ij}} = a_2 b_2 \prod_{i,j} [x_i, y_j]^{\delta_{ij}}$

if and only if

$$g = a_1 b_1 b_2^{-1} a_2^{-1} \prod_{i,j} [x_i y_j]^{\gamma_{ij} - \delta_{ij}} = 1 .$$

But,

$$g = a_1 a_2^{-1} b_1 b_2^{-1} [b_1 b_2^{-1}, a_2^{-1}] \prod_{i,j} [x_i, y_j]^{\gamma_{ij} - \delta_{ij}}$$

$$= a_1 a_2^{-1} b_1 b_2^{-1} \prod_{i,j} [x_i, y_j]^{\gamma_{ij} - \delta_{ij} - \alpha_{2i}(\beta_{1j} - \beta_{2j})}$$

where for each $k = 1$, 2 , using Lemma 3.8, α_{ki} , β_{kj} are the unique integers such that

$$a_k = \prod_i x_i^{\alpha_{ki}} \text{ modulo } A' \quad \text{and} \quad b_k = \prod_j y_j^{\beta_{kj}} \text{ modulo } B' .$$

Thus, by (5),

$$g = 1 \text{ iff } a_1 a_2^{-1} = 1 , \ b_1 b_2^{-1} = 1 \text{ and}$$

$$\gamma_{ij} - \delta_{ij} = \alpha_{2i}(\beta_{1j} - \beta_{2j}) , \ i \in I , \ j \in J .$$

But, $b_1 = b_2$ implies that $\beta_{1j} = \beta_{2j}$ for all j , and hence,

$g = 1$ iff $a_1 = a_2$, $b_1 = b_2$ and $\gamma_{ij} = \delta_{ij}$, for all $i \in I$, $j \in J$.

Thus we have shown that

$$a_1 b_1 \prod_{i,j} [x_i, y_j]^{\gamma_{ij}} = a_2 b_2 \prod_{i,j} [x_i, y_i]^{\delta_{ij}}$$

iff $a_1 = a_2$, $b_1 = b_2$ and $\gamma_{ij} = \delta_{ij}$, for all $i \in I$, $j \in J$.

This completes the proof of the theorem. //

4. A "Feferman-Vaught Style" Theorem

It is known, from the work of Feferman and Vaught [11], that the truth of a formula in the direct product $A \times B$, of structures A , B is effectively determined by the truth of a finite set of formulae in the individual factors A , B . As a consequence of this, the direct product operation preserves elementary equivalence. Indeed, it can also be deduced that $\times$ preserves saturation (see Waszkiewicz and Węglorz [39]) and stability (see Wierzejewski [40]; Macintyre [19]; Eklof and Fisher [8]).

In this section we use the first normal form theorem to give our own version of a Feferman-Vaught style theorem for the N_2-free product, $C*G$, of a finite $C \in N_2$, and an arbitrary $G \in N_2$ subject to certain conditions imposed on G by C (see Theorem 4.1, below). We also give an application of this result to the problem of the preservation by $*$, of elementary equivalence (see Theorem 4.4). In later sections we apply it to the problems of preservation of saturation and of stability.

Before stating Theorem 4.1, we make certain notational conventions which we shall generally adhere to throughout this section. Suppose that C is a fixed, finite, nil-2 group and G an arbitrary nil-2 group. Since C is finite, it has a basis, say $\{a_i : 1 \leqq i \leqq n\}$ where $o(a_i) = m_i < \infty$. Now, by the first normal form theorem (Theorem 3.7), an element $h \in C*G$ can be written in the form

$$h = cg_0 \prod_{1 \leqq i \leqq n} [a_i, g_i] ,$$

where $c \in C$ and $g_0 \in G$ are uniquely determined by h , and for each $i = 1, \ldots, n$, the elements g_i of G are unique modulo the subgroup $H_{m_i}(G)$ of G . With every element $h \in C*G$, written in the above way, we shall associate the sequence $\bar{g} = \langle g_0, g_1, \ldots, g_n \rangle \in {}^{n+1}G$. Correspondingly, with every variable u_j of L , the language of groups we associate a sequence of variables, $\bar{u}_j = \langle u_{j0}, u_{j1}, \ldots, u_{jn} \rangle$, from L . Let L' be the language obtained from L by adding a set, $\{x_n : n \in \omega\}$, of new constant symbols, and $\bar{x} = \langle x_0, x_1, \ldots, x_{r-1} \rangle$ for some $r \in \omega$.

4.1 <u>Theorem</u>

Let $\psi_i(u, \bar{x})$ be formulae of L' for each $i = 1, \ldots, n$ and call a group G <u>pertinent</u> if there exists a sequence $\bar{d} \in {}^{r}G$ with $\psi_i(u, \bar{x})$ defining $H_{m_i}(G)$ in $(G, \bar{d})$.

Then, for every formula $\varphi(u_0, \ldots, u_{k-1}) \in L$ there exist formulae $\theta_{\bar{c}}(\bar{u}_0, \ldots, \bar{u}_{k-1}, \bar{x}) \in L'$ for each $\bar{c} \in {}^{k}C$, such that for all pertinent $G \in N_2$ and all $c_j \in C$, $g_{ji} \in G$ $(0 \leqq j \leqq k-1 , 0 \leqq i \leqq n)$,

$$C*G \models \varphi[\bar{h}] \text{ if and only if } (G, \bar{d}) \models \theta_{\bar{c}}[\bar{g}] ,$$

where,

$$\begin{cases} \bar{h} = \langle h_0,\ldots,h_{k-1}\rangle \ , \quad h_j = c_j g_{j0} \prod_{1\leq i\leq n} [a_i,g_{ji}] \ ; \\ \bar{g} = \bar{g}_0{}^\frown \bar{g}_1{}^\frown \cdots {}^\frown \bar{g}_{k-1} \ , \quad \bar{g}_j = \langle g_{j0},\ldots,g_{jn}\rangle \ ; \text{ and} \\ \bar{c} = \langle c_0,\ldots,c_{k-1}\rangle \end{cases}$$

Proof

Let $\psi_i(u,\bar{x})$ be given formulae of L' , $i = 1,\ldots,n$. If $\varphi(\bar{u})$ is a formula of L for which the theorem holds with a set $\{\theta_{\bar{c}} : \bar{c} \in {}^kC\}$, then we shall say that φ is determined by the set $\{\theta_{\bar{c}} : \bar{c} \in {}^kC\}$. Let D be the set of all determined formulae of L . We show, by induction on the complexity of φ , that every formula $\varphi \in L$ belongs to D .

φ atomic: Consider $\varphi(u_1,u_2,u_3) \equiv u_1u_2 = u_3$.

Let $\bar{c} = \langle c_1,c_2,c_3\rangle \in {}^3C$. By Lemma 3.3, for each $j = 1, 2, 3$, there exist unique integers α_{ji} , $0 \leq \alpha_{ji} < m_i$, $i = 1,\ldots,n$, such that $c_j = \prod_{1\leq i\leq n} a_i^{\alpha_{ji}}$ modulo C' . For each $\bar{c} \in {}^3C$ we define

$$\theta_{\bar{c}}(\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{x}) \equiv \begin{cases} u_{10}u_{20} = u_{30} \\ \wedge \bigwedge_{1\leq i\leq n} \exists v(v = u_{10}^{-\alpha_{2i}} u_{1i}u_{2i}u_{3i}^{-1} \wedge \psi_i(v,\bar{x})) \quad \text{if } c_1c_2 = c_3 \ ; \\ u_{10} \neq u_{10} \quad \text{otherwise} . \end{cases}$$

Then, with appropriate $G \in N_2$, $\bar{d}$ in ${}^rG$, $\bar{c}$ in ${}^3C$, $\bar{g}_1$, $\bar{g}_2$ and $\bar{g}_3$ in ${}^{n+1}G$ and appropriate $\bar{h}$ in ${}^3(C{*}G)$, we have, using the first normal form theorem, the following sequence of equivalent statements:

(i) $C{*}G \models \varphi[\bar{h}]$;

(ii) $c_1g_{10} \prod_{1\leq i\leq n} [a_i,g_{1i}]\cdot c_2g_{20} \prod_{1\leq i\leq n} [a_i,g_{2i}]$

$$= c_3 g_{30} \prod_{1 \leq i \leq n} [a_i, g_{3i}] \ ;$$

(iii) $c_1 c_2 g_{10} g_{20} \prod_{1 \leq i \leq n} [a_i, g_{10}^{-\alpha_{2i}} g_{1i} g_{2i}] = c_3 g_{30} \prod_{1 \leq i \leq n} [a_i, g_{3i}]$;

(iv) $c_1 c_2 = c_3$, $g_{10} g_{20} = g_{30}$ and $g_{10}^{-\alpha_{2i}} g_{1i} g_{2i} g_{3i}^{-1} \in H_{m_i}(G)$,

for each $i = 1,\ldots,n$;

(v) $c_1 c_2 = c_3$, $g_{10} g_{20} = g_{30}$ and

$(G,\bar{d}) \models \psi_i[g_{10}^{-\alpha_{2i}} g_{1i} g_{2i} g_{3i}^{-1}]$, for each $i = 1,\ldots,n$;

(vi) $(G,\bar{d}) \models \theta_{\bar{c}}[\bar{g}_1, \bar{g}_2, \bar{g}_3]$.

Thus,

$$C^*G \models \varphi[\bar{h}] \text{ if and only if } (G,\bar{d}) \models \theta_{\bar{c}}[\bar{g}] \ .$$

Hence, all atomic formulae are determined.

$\varphi \equiv \sim\varphi'$ and $\varphi' \in D$: Assume $\varphi \equiv \sim\varphi'$ and that φ' is determined by the set $\{\theta'_{\bar{c}} : c \in {}^k C\}$. Then, it is clear that φ is determined by the set $\{\theta_{\bar{c}} : \bar{c} \in {}^k C\}$, where $\theta_{\bar{c}} \equiv \sim\theta'_{\bar{c}}$. Thus, $\varphi \in D$.

$\varphi \equiv \varphi' \wedge \varphi''$ and $\varphi', \varphi'' \in D$: Assume that $\varphi \equiv \varphi' \wedge \varphi''$, and that both φ' and φ'' are determined. By adding redundant variables, if necessary, we may assume that both φ' and φ'' have free variables $u_0,\ldots,u_{k-1}$. Let φ', φ'' be determined by the sets $\{\theta'_{\bar{c}} : c \in {}^k C\}$, $\{\theta''_{\bar{c}} : c \in {}^k C\}$, respectively. Then if $\theta_{\bar{c}} \equiv \theta'_{\bar{c}} \wedge \theta''_{\bar{c}}$, it is clear that φ is determined by the set $\{\theta_{\bar{c}} : \bar{c} \in {}^k C\}$ and hence $\varphi \in D$.

$\varphi \equiv \exists u \varphi'$ and $\varphi' \in D$: By relabelling the variables, if necessary, we may assume that $\varphi \equiv \exists u_k \varphi'(u_0,\ldots,u_{k-1},u_k)$ and that $\varphi'(u_0,\ldots,u_k)$ is determined by the set $\{\theta'_{\bar{c}}(\bar{u}_0,\ldots,\bar{u}_k,\bar{x}) : \bar{c} \in {}^{k+1} C\}$. For each $c \in {}^k C$, we define

$$\theta_{\bar{c}}(\bar{u}_0,\ldots,\bar{u}_{k-1},\bar{x}) \equiv \exists u_{k0}\ldots u_{kn} \bigvee_{c\in C} \theta'_{\bar{c}\frown c}(\bar{u}_0,\ldots,\bar{u}_k,\bar{x}) \; .$$

Then, with appropriate $G \in N_2$, $\bar{d}$ in ${}^{r}G$, $\bar{c} \in {}^{k}C$, $\bar{g}_0,\ldots,\bar{g}_{k-1}$ in ${}^{n+1}G$ and $\bar{h}$ in ${}^{k}(C{*}G)$, we have the following sequence of equivalent statements:

(i) $C{*}G \models \varphi[\bar{h}]$;

(ii) there exists $h = cg_0 \prod_{1\le i\le n} [a_i,g_i] \in C{*}G$, such that $C{*}G \models \varphi'[\bar{h},h]$;

(iii) there exist $c \in C$ and $\bar{g} = (g_0,\ldots,g_n)$ in G such that $(G,\bar{d}) \models \theta'_{\bar{c}\frown c}[\bar{g}_0,\ldots,\bar{g}_{k-1},\bar{g}]$;

(iv) $(G,\bar{d}) \models \exists u_{k0}\ldots u_{kn} \bigvee_{c\in C} \theta'_{\bar{c}\frown c}[\bar{g}_0,\ldots,\bar{g}_{k-1}]$;

(v) $(G,\bar{d}) \models \theta_{\bar{c}}[\bar{g}_0,\ldots,\bar{g}_{k-1}]$.

(The equivalence (ii)$\leftrightarrow$(iii) is proved using the inductive hypothesis.) Thus,

$$C{*}G \models \varphi[\bar{h}] \text{ if and only if } (G,\bar{d}) \models \theta_{\bar{c}}[\bar{g}] \; ,$$

and hence, $\varphi \in D$.

The theorem now follows by induction. //

We conclude this section by making some observations on the theorem above and deducing some immediate corollaries.

First we remark that the full strength of the hypothesis that C be finite is used. The formulae determining an existential formulae $\exists u\varphi$ are defined by forming a disjunction over the determining formulae for φ . Hence, we require that the determining set for a given formula be finite and we can see from the atomic case in the proof above that this will not be so if C is infinite.

The theorem also holds under the stronger, but simpler, hypothesis that G' is definable. For suppose that $\psi(u,\bar{x})$ defines G' in $(G,\bar{a})$, with $\bar{a}$ in G. Then, for each $n > 0$, $H_n(G)$ is defined by the formula

$$\psi_n(u,\bar{x}) \equiv \exists vw(u = v^n w \wedge \psi(w,\bar{x})) .$$

This remark leads to the following corollary.

4.2 Corollary

Theorem 4.1 also holds for every nil-2 G with a definable derived group.

There are numerous examples of nil-2 groups G satisfying the hypothesis of the preceding corollary: obvious ones are the abelian groups. Recently, (see [31], Proposition 11) Saracino has proved that for every existentially complete, nil-2 group G, $Z(G) = G'$, and hence the derived group of such a G is definable. Other examples are afforded by the free nil-2 groups. Let F be free nil-2 on $\{x_i : i \in I\}$. Then, using the second normal form theorem, the elements of F have the form $\prod_i x_i^{\alpha_i} \prod_{i,j} [x_i,x_j]^{\gamma_{ij}}$, where the α_i, γ_{ij} are integers. Lemma 5.20 below yields a characterisation of the elements of the centre of F and it is easy to see that $Z(F) = F'$, so F' is definable.

The following corollary is used in the proof of the promised preservation result for elementary equivalence.

4.3 Corollary

For each sentence σ of L, the determining set for σ obtained in Theorem 4.1 consists of a single sentence of L'. Thus, given formulae $\psi_i(u,\bar{x}) \in L'$, $1 \leq i \leq n$, for every sentence $\sigma \in L$, there exists a sentence $\theta \in L'$ such that for all appropriate $G \in N_2$ and $\bar{a}$ in ${}^{<\omega}G$,

$$C^*G \models \sigma \text{ if and only if } (G,\bar{a}) \models \theta .$$

Proof

The following observation suffices to prove the corollary: if φ is a formula of L having $k \geqq 0$ free variables, then the determining set for φ consists of at most $|C|^k$ formulae of L' each having at most $k(n+1)$ free variables. //

4.4 Theorem

Let G_1, G_2 be nil-2 groups for which there exist formulae $\psi_i(u,\bar{x})$, $1 \leqq i \leqq n$, of L' and sequences $\bar{a}_k$ from G_k, $k = 1, 2$, with $\psi_i(u,\bar{x})$ defining $H_{m_i}(G_k)$ in $(G_k,\bar{a})$. Then,

$$(G_1,\bar{a}_1) \equiv (G_2,\bar{a}_2) \text{ implies } C^*G_1 \equiv C^*G_2 .$$

Proof

Assume that G_1, G_2 are given nil-2 groups satisfying the hypotheses of the theorem. For each sentence $\sigma \in L$, let θ_σ be the sentence of L' obtained above in Corollary 4.3. Then, if $(G_1,\bar{a}_1) \equiv (G_2,\bar{a}_2)$ we have, for each sentence σ of L:

$$\begin{aligned} C^*G_1 \models \sigma &\Leftrightarrow (G_1,\bar{a}_1) \models \theta_\sigma \\ &\Leftrightarrow (G_2,\bar{a}_2) \models \theta_\sigma \\ &\Leftrightarrow C^*G \models \sigma . \end{aligned}$$

Hence, $C^*G_1 \equiv C^*G_2$, proving the theorem. //

4.5 Corollary

If G_1, $G_2 \in N_2$ and if there is a formula $\psi(u,\bar{x}) \in L'$ and sequences $\bar{a}_i$ from G_i with $\psi(u,\bar{x})$ defining G_i' in $(G_i,\bar{a}_i)$, $i = 1, 2$, then for every finite $C \in N_2$,

$$(G_1,\bar{a}_1) \equiv (G_2,\bar{a}_2) \text{ implies } C^*G_1 \equiv C^*G_2 .$$

Proof

This is immediate from Corollary 4.2 and Theorem 4.4 . //

P. Olin (see Theorem 1.1) has an example which shows that Corollary 4.5 fails when the definability hypothesis is removed. His example comprises denumerable nil-2 (and exponent 3) groups A , B such that $B \prec A$ but $C*B \neq C*A$, where C is the cyclic group of order 3. The fact that B' is the centre of B but A' is not the centre of A plays an important part in the proof of his result.

The comments preceding Corollary 4.3 yield the following special case of 4.5.

4.6 Corollary

Let C be a finite nil-2 group. Then, for all nil-2 groups A , B which are i) abelian; ii) free; or iii) existentially complete, $A \equiv B$ implies that $C*A \equiv C*B$.

5. Preservation of Saturation

In this section we shall exploit the "Feferman-Vaught" theorem of the previous section to show that saturation is preserved by the N_2-free product operation when one factor is finite and the other obeys certain definability conditions (Theorem 5.2). We also show that this result is best possible for products with one finite factor (Theorem 5.15). Finally, in Theorem 5.23, we extend both results to the class of all bounded nil-2 groups.

One first theorem shows that each type t over $C*G$, where C is finite and G has the appropriate definable subgroups (C and G belonging to N_2), determines a type τ over G such that, whenever τ is realised in G , so is t realised in $C*G$. Theorem 5.2 is an immediate corollary of this result. With a little more work we are also able to show that, subject to the same definability conditions, stability, too, is preserved, but we shall reserve this for the next section.

We shall continue to observe the notational conventions noted in the third paragraph of section 4 .

5.1 Theorem

Let C be a finite nil-2 group with basis $\{a_i : 1 \leq i \leq n\}$ and $o(a_i) = m_i (< \infty)$, and G an arbitrary nil-2 group for which there exist formulae $\psi_i(u,\bar{x}) \in L'$ and a sequence $\bar{d}$ from G with $\psi_i(u,\bar{x})$ defining $H_{m_i}(G)$ in $(G,\bar{d})$, $1 \leq i \leq n$. Then, for every k-type $t(u_1,\ldots,u_k)$ over $(C*G, z_\nu)_{\nu<\lambda}$, $z_\nu = b_\nu y_{\nu 0} \prod_{1\leq i\leq n} [a_i, y_{\nu i}]$ with $b_\nu \in C$ and $y_{\nu 0}, y_{\nu i} \in G$, there exist $\bar{c} = \langle c_1,\ldots,c_k\rangle \in {}^kC$ and a $k(n+1)$-type $\tau_{\bar{c}}(\bar{u}_1,\ldots,\bar{u}_k)$ over $(G,\bar{d},\bar{y}_\nu)_{\nu<\lambda}$, where $\bar{y}_\nu = \langle y_{\nu 0},\ldots,y_{\nu n}\rangle$, such that for all $\bar{g} = \bar{g}_1 {}^\frown \ldots {}^\frown \bar{g}_k$, with $\bar{g}_j = \langle g_{j0},\ldots,g_{jn}\rangle$, $1 \leq j \leq k$, $\bar{g}$ realises $\tau_{\bar{c}}$ in G if and only if $\bar{h}$ realises t in $C*G$, where $\bar{h} = \langle h_1,\ldots,h_k\rangle$, with $h_j = c_j g_{j0} \prod_{1\leq i\leq n} [a_i, g_{ji}]$, $1 \leq j \leq k$.

Proof

Although we have stated this theorem in all generality, we shall prove it only for the case $k=1$. We hope it shall be clear from the proof we give that the generalisation to $k > 1$ is straightforward.

Let C and G satisfy the hypothesis and let $t(u)$ be a 1-type over $(C*G, z_\nu)_{\nu<\lambda}$, with $z_\nu = b_\nu y_{\nu 0} \prod_{1\leq i\leq n} [a_i, y_{\nu i}]$ and $b_\nu \in C$, $y_{\nu 0}$, $y_{\nu i} \in G$.

Consider an arbitrary formula $\varphi(u)$ of t . Then, $\varphi(u) \in L(C*G)$, so we let $z_1,\ldots,z_m$ be a list of the constants from $C*G$ appearing in φ . Let $u_1,\ldots,u_m$ be variables new to

$\varphi(u)$ and denote by $\widetilde{\varphi}(u,u_1,\ldots,u_m)$ the formula of L obtained from φ by replacing each occurrence of a constant z_j by the variable u_j, $1 \leqq j \leqq m$. Now let $\{\widetilde{\theta}_{\bar{c}}(\bar{u},\bar{u}_1,\ldots,\bar{u}_m,\bar{x}) : \bar{c} \in {}^{m+1}C\}$ be a determining set for $\widetilde{\varphi}$ obtained using Theorem 4.1. Then, for all $c \in C$ and $\bar{g} = \langle g_0,g_1,\ldots,g_n\rangle$ from G, we have

$$C*G \models \widetilde{\varphi}[h,z_1,\ldots,z_m] \text{ iff } (G,\bar{d}) \models \theta_{c^\frown \bar{b}}[\bar{g},\bar{y}_1,\ldots,\bar{y}_m]$$

where $h = cg_0 \overline{\prod_{1\leqq i\leqq n}} [a_i,g_i]$ and the $\bar{b}$ and $\bar{y}_j$ are appropriate for z_j, $1 \leqq j \leqq m$.

Let $\theta_c(\bar{u},\bar{x})$, $c \in C$, be the formula of $L'(G)$ obtained from $\widetilde{\theta}_{c^\frown \bar{b}}(\bar{u},\bar{u}_1,\ldots,\bar{u}_m,\bar{x})$ by replacing each occurrence of the sequence of variables $\bar{u}_j$ by the sequence $\bar{y}_j$ of constants from G, $1 \leqq j \leqq m$. (Strictly speaking these formulae should be indexed by $c^\frown \bar{b}$, $c \in C$. But, since $\bar{b}$ is the same for each formula and, in any case, it is fixed by the constants in $\varphi(u)$, we shall suppress it.) Thus, from (1), for all $c \in C$ and $\bar{g} = \langle g_0,\ldots,g_n\rangle$ from G, we have

$$(C*G,z_\nu)_{\nu<\lambda} \models \varphi[h] \text{ iff } (G,\bar{d},\bar{y}_\nu)_{\nu<\lambda} \models \theta_c[\bar{g}],$$

where $h = cg_0 \overline{\prod_{1\leqq i\leqq n}} [a_i,g_i]$. (2)

For each $\varphi(u) \in t(u)$, denote the formulae obtained in this way by $\theta^{\varphi}_c(\bar{u},\bar{x})$, $c \in C$. Then, for each $c \in C$ let D_c be the set defined by:

$$D_c = \{\theta^{\varphi}_c(\bar{u},\bar{x}) : \varphi \in t(u)\}.$$

<u>Claim:</u> there exists $c \in C$ such that D_c is finitely satisfiable in $(G,\bar{d},\bar{y}_\nu)_{\nu<\lambda}$.

<u>Proof of Claim:</u> Suppose the claim were false. Then for all $c \in C$, there exists a finite subset Δ_c of D_c such that

$$(G,\bar{d},\bar{y}_\nu)_{\nu<\lambda} \models \sim\exists\bar{u} \bigwedge_{\theta\in\Delta_c} \theta . \tag{3}$$

Set $\Gamma = \bigcup_{c\in C} \{\varphi : \theta_c^\varphi \in \Delta_c\}$. Note Γ is finite because C is. Then Γ is a finite subset of the type $t(u)$ and so is realised in $(C{*}G,z_\nu)_{\nu\in\lambda}$, say by $h_0 = c_0 g_{00} \prod_{1\leq i\leq n} [a_i, g_{0i}]$. Thus,

$$(C{*}G,z_\nu)_{\nu<\lambda} \models \bigwedge_{\varphi\in\Gamma} \varphi[h_0] .$$

Hence, by (2),

$$(G,\bar{d},\bar{y}_\nu)_{\nu<\lambda} \models \bigwedge_{\varphi\in\Gamma} \theta_{c_0}^\varphi[\bar{g}_0] ,$$

with $\bar{g}_0 = \langle g_{00},\ldots,g_{0n}\rangle$. So, in particular, by the definition of Γ , we have

$$(G,\bar{d},\bar{y}_\nu)_{\nu<\lambda} \models \bigwedge_{\theta\in\Delta_{c_0}} \theta[\bar{g}_0] .$$

But this contradicts (3), thus proving the claim.

Using the claim, let c be an element of C for which D_c is finitely satisfiable in $(G,\bar{d},\bar{y}_\nu)_{\nu<\lambda}$. Then, $\tau_c(\bar{u}) = D_c$ is an $(n+1)$-type over $(G,\bar{d},\bar{y}_\nu)_{\nu<\lambda}$ and the rest follows from (2). //

Now consider two nil-2 groups C and G for which Theorem 5.1 holds, and let r be the total number of constants needed to define the subgroups $H_{m_i}(G)$, $1 \leq i \leq n$. Then, if t is a type over $C{*}G$ with μ parameters, it is clear that the associated type $\tau_{\bar{c}}$ over G has at most $r+(n+1)\mu$ parameters. The following theorem now follows immediately from 5.1. (Remember that, by Theorem 1.3.2, every finite structure is $\varkappa$-saturated, for all cardinals $\varkappa$.)

5.2 Theorem

Let G be any nil-2 group and C a finite nil-2 group with a basis $\{a_i : 1\leq i\leq n\}$ for which the subgroups $H_{o(a_i)}(G)$ are definable in G . Let r be the total number of constants needed in the definition of these subgroups. Then, for all cardinals $\lambda > r$, if G is λ-saturated it follows that $C*G$ is $\varkappa$-saturated, where $\varkappa = \lambda$ if λ is infinite and where, if λ is finite, $\varkappa$ is any cardinal satisfying $r+(n+1)\varkappa \leqq \lambda$.

Proof

Let C and G satisfy the hypotheses, $\lambda > r$ be a cardinal such that G is λ-saturated and let $\varkappa$ be a cardinal satisfying the given conditions with respect to λ . Suppose that t is any 1-type over $C*G$ having $\mu < \varkappa$ parameters from $C*G$. Then, appealing to Theorem 5.1, there is a sequence $\bar{c}$ from C and an $(n+1)$-type $\tau_{\bar{c}}$ over G such that any sequence from G realising $\tau_{\bar{c}}$ can be combined with $\bar{c}$ in the correct way to give a member from $C*G$ realising t . By our previous results, $\tau_{\bar{c}}$ has at most $r+(n+1)\mu$ parameters from G .

If λ is infinite, then $\varkappa = \lambda$ and so $r+(n+1)\mu < \lambda$. Thus, by Theorem 1.3.4, G realises $\tau_{\bar{c}}$.

If λ is finite, then

$$(n+1)+r+(n+1)\mu = r+(n+1)(\mu+1) \leqq r+(n+1)\varkappa \leqq \lambda .$$

Thus, by Theorem 1.3.3, it follows that $\tau_{\bar{c}}$ is realised in G .

Consequently, t is realised in $C*G$ and, since t was arbitrary, it follows that $C*G$ is $\varkappa$-saturated. //

In Theorem 5.15 we shall show that the definability hypothesis of the above theorem is essential to its validity. Indeed, we show that when this hypotheses is relaxed, $C*G$ may not even be 1-saturated, regardless of the degree of saturation possessed by G . (See the remarks following Theorem 5.16.) The proof of this fact emerges after a series of technical lemmas, but first we make a definition capturing the definability hypothesis.

5.3 Definition

Let C and G be nil-2 groups, C having a basis. Then, a basis $\{a_i : i \in I\}$ for C is said to be G-suitable if the subgroups $H_{o(a_i)}(G)$ of G are all definable in G.

We shall show that if C, $G \in N_2$, C is finite but has no G-suitable basis, then C*G is not 1-saturated.

Let G be an arbitrary group and m a positive integer. Our first lemma gives a condition which, provided G is 1-saturated, is equivalent to $H_m(G)$ being definable in G. We shall use only one direction of this result, and the proof for it does not use the fact that G is 1-saturated. Roughly speaking, for a 1-saturated group G, $H_m(G)$ is definable if and only if there is a bound on the number of commutators required to represent each of its elements. First we define, for each $k \in \omega$, a formula of L which, if satisfied by an element $g \in H_m(G)$, expresses the fact that g cannot be written down using less than k commutators.

5.4 Definition

For each m, $k \in \omega$, the formula $\chi_k^m(u)$ is defined by

$$\chi_k^m(u) \equiv \sim\exists u_0 u_1 \ldots u_k v_1 \ldots v_k (u = u_0^m [u_1,v_1] \ldots [u_k,v_k]) .$$

5.5 Lemma

If G is a 1-saturated group and $m \in \omega$, then the following are equivalent:

(i) $H_m(G)$ is definable (without parameters) in G;

(ii) there is $K \in \omega$ such that for all $g \in H_m(G)$,

$$G \models \sim\chi_K^m[g] .$$

Proof

(i) ⇒ (ii) : Suppose that $\psi(u)$ defines $H_m(G)$ in G, but that for all $k \in \omega$, there exist $g_k \in H_m(G)$ with $G \models \chi_k^m[g_k]$. Define a set of formulae, $t(u)$, of L by

$$t(u) = \{\psi(u)\} \cup \{\chi_k^m(u) : k \in \omega\} .$$

Then $t(u)$ is a type over G being finitely satisfied in G by the g_k, $k \in \omega$. However, $t(u)$ is not realised in G since every element g which satisfies $\psi(u)$ belongs to $H_m(G)$ and hence can be written in the form $g_1^m g_2$ with $g_1 \in G$, $g_2 \in G'$; if k is the number of commutators comprising g_2, then g will satisfy the formula $\sim\chi_k^m(u)$. This contradicts the fact that G is 1-saturated.

(ii) ⇒ (i) : If there exists such a $K \in \omega$, then clearly $H_m(G)$ is defined in G by $\sim\chi_K^m(u)$. //

The following is a corollary of the proof of Lemma 5.5.

5.6 Corollary

If G is a group and m a non-negative integer for which $H_m(G)$ is not definable in G (with or without parameters), then for all $k \in \omega$, there exist $g_k \in H_m(G)$ such that

$$G \models \chi_k^m[g_k] .$$

5.7 Remark

If we strengthen the hypothesis of Lemma 5.5 to read that G is $(r+1)$-saturated, then the result still holds when we allow r parameters to be used in defining $H_m(G)$.

The idea behind the proof of the converse of Theorem 5.2 (Theorem 5.15, below) is now quite easy to describe. If G, $C \in N_2$

and C is finite but has no G-suitable basis, then no matter what basis we choose for C, there will always be at least one generator, say a, such that $H_{o(a)}(G)$ is not definable in G. Suppose $o(a) = m$. Then, we exhibit elements $h_k \in Z(C*G)$, $k \in \omega$, such that $C*G \models \chi_k^m[h_k]$. Of course, it would be very convenient if we could take $h_k = g_k$, the elements obtained using Corollary 5.6. But these elements do not necessarily belong to the centre of $C*G$. So first we show that g_k may be chosen to lie in G' (and, therefore also in $Z(C*G)$). This is the content of Lemma 5.10. Next, in Lemma 5.11, we show that not only does g_k satisfy $\chi_k^m(u)$ in G, it also satisfies this formula in $C*G$. Then we are able to define a type over $C*G$, with no parameters, in the same way that we did in the proof of Lemma 5.5, (i) ⇒ (ii). Finally, and this is the hard part, we show that $C*G$ omits this type. On the way, we shall require some information about the precise nature of the elements of $Z(C*G)$ and $(C*G)'$. This we shall obtain first.

For the moment, let C be an arbitrary, but fixed, finite, nil-2 group with a basis $\{a_i : 1 \leq i \leq n\}$, $o(a_i) = m_i$, and let G be any nil-2 group.

5.8 <u>Lemma</u>

$h \in Z(C*G)$ if and only if $h = cg_0 \prod_{1\leq i\leq n} [a_i, g_i]$, for some $c \in C$, $g_0, g_i \in G$, where $c = \prod_{1\leq i\leq n} a_i^{\alpha_i}$ modulo C', $0 \leq \alpha_i < m_i$ and

(i) $c \in Z(C)$;

(ii) $g_0 \in \bigcap_{1\leq i\leq n} H_{m_i}(G) \cap Z(G)$; and

(iii) $H_{\alpha_i}(G) \subset H_{m_i}(G)$, $1 \leq i \leq n$.

Proof

Let $h \in C*G$. Then, by the first normal form theorem (Theorem 3.7) there exist $\alpha_i < m_i$, and $g_0, g_i \in G$, $1 \leq i \leq n$, such that $h = cg_0 \prod_{1\leq i\leq n} [a_i, g_i]$, with $c = \prod_{1\leq i\leq n} a_i^{\alpha_i}$ modulo C' .

Since $C*G$ is generated by the set $\{a_i : i = 1,\ldots,n\} \cup G$,

$$h \in Z(C*G) \Leftrightarrow \text{a) } [h,a_i] = 1 \text{ , for each } i = 1,\ldots,n \text{ ; and}$$
$$\text{b) } [h,g] = 1 \text{ , for all } g \in G \text{ .}$$

Now, using Lemma 2.3 and Theorem 3.7,

$$\begin{aligned} \text{a) holds } &\Leftrightarrow [c,a_i][g_0,a_i] = 1 \text{ , } i = 1,\ldots,n \\ &\Leftrightarrow [c,a_i] = 1 \text{ and } g_0 \in H_{m_i}(G) \text{ , } i = 1,\ldots,n \\ &\Leftrightarrow c \in Z(C) \text{ and } g_0 \in \bigcap_{1\leq i\leq n} H_{m_i}(G) \text{ .} \end{aligned}$$

Similarly,

$$\begin{aligned} \text{b) holds } &\Leftrightarrow [c,g][g_0,g] = 1 \text{ , for all } g \in G \\ &\Leftrightarrow [g_0,g] \prod_{1\leq i\leq n} [a_i, g^{\alpha_i}] = 1 \text{ , for all } g \in G \\ &\Leftrightarrow [g_0,g] = 1 \text{ and } g^{\alpha_i} \in H_{m_i}(G) \text{ , for each } i = 1,\ldots,n, \\ &\quad \text{and all } g \in G \\ &\Leftrightarrow g_0 \in Z(G) \text{ and } H_{\alpha_i}(G) \subset H_{m_i}(G) \text{ , } i = 1,\ldots,n \text{ .} \end{aligned}$$

Thus,

$$\begin{aligned} h \in Z(C*G) \Leftrightarrow\ & c \in Z(C) \text{ , } g_0 \in \bigcap_{1\leq i\leq n} H_{m_i}(G) \cap Z(G) \text{ and} \\ & \text{for each } i = 1,\ldots,n \text{ , } H_{\alpha_i}(G) \subset H_{m_i}(G) \text{ .} \end{aligned}$$ //

5.9 Lemma

$h \in (C*G)'$ if and only if $h = cg_0 \prod_{1\leq i\leq n} [a_i, g_i]$, where $c \in C'$ and $g_0 \in G'$. Furthermore, if h is a commutator, so are c and g_0.

Proof

Let $h = cg_0 \prod_{1\leq i\leq n} [a_i,g_i]$ be an element of $C*G$. Clearly, if $c \in C'$ and $g_0 \in G'$, then $h \in (C*G)'$. The rest of the Lemma will follow if we can show that, if h is a commutator, then so are c and g_0.

Consider two elements $h_j = c_j g_{j0} \prod_{1\leq i\leq n} [a_i,g_{ji}]$, $j = 1, 2$, of $C*G$, where $c_j = \prod_{1\leq i\leq n} a_i^{\alpha_{ji}}$ modulo C'. Then, by Lemma 2.3

$$\begin{aligned}[h_1,h_2] &= [c_1g_{10},c_2g_{20}] \\ &= [c_1,c_2][g_{10},g_{20}][c_1,g_{20}][c_2,g_{10}^{-1}] .\end{aligned}$$

Thus, substituting for c_1, c_2 in the third and fourth commutators,

$$[h_1,h_2] = [c_1,c_2][g_{10},g_{20}] \prod_{1\leq i\leq n} [a_i,g_{20}^{\alpha_{1i}}g_{10}^{-\alpha_{2i}}]$$

showing that $[h_1,h_2]$ can be written in the required form. //

5.10 Lemma

If $G \in N_2$, $m \in \omega$ and $H_m(G)$ is not definable in G then for all $k \in \omega$, there exist $g_k \in G'$ such that $G \models \chi_k^m[g_k]$.

Proof

Suppose there is $K \in \omega$ such that for all $g \in G'$, $G \models \sim\chi_K^m[g]$. Then, for all $g \in G'$, there exist $g_0,g_1,\ldots,g_K,h_1,\ldots,h_K \in G$ such that

$$g = g_0^m[g_1,h_1]\ldots[g_K,h_K] \tag{1}$$

Now let x be an arbitrary element of $H_m(G)$. Then, there exists an integer r and elements $x_0, x_1, \ldots, x_r, y_1, \ldots, y_r \in G$ such that

$$x = x_0^m [x_1, y_1] \ldots [x_r, y_r] .$$

But setting $g' = [x_1, y_1] \ldots [x_r, y_r]$ we have $g' \in G'$ and hence we can write g' in the form (1) above. So,

$$x = x_0^m g' = x_0^m g_0^m [g_1, h_1] \ldots [g_K, h_K] .$$

By Lemma 2.3, $x_0^m g_0^m = (x_0 g_0)^m [x_0, g_0]^{\frac{1}{2}m(m-1)}$ and so

$$x = (x_0 g_0)^m [x_0, g_0^{\frac{1}{2}m(m-1)}][g_1, h_1] \ldots [g_K, h_K] ,$$

showing that $G \models \sim\chi^m_{K+1}[x]$. Thus, by Corollary 5.6, $H_m(G)$ is definable. //

5.11 <u>Lemma</u>

Let $G \in N_2$ and C be a finite nil-2 group. Then, for all $g \in G$ and $m, k \in \omega$

$$G \models \chi^m_k[g] \text{ implies } C{*}G \models \chi^m_k[g] .$$

<u>Proof</u>

Let $C, G \in N_2$ with C finite, and let $\{a_i : 1 \leq i \leq n\}$ be a basis for C . Let $g \in G$ and $m, k \in \omega$. We show that if $C{*}G \models \sim\chi^m_k[g]$, then $G \models \sim\chi^m_k[g]$.

Now, if $C{*}G \models \sim\chi^m_k[g]$, then there exist $h_0 \in C{*}G$ and $h_1, \ldots, h_k$, each a commutator of $C{*}G$ such that

$$g = h_0^m h_1 \ldots h_k .$$

Suppose that $h_0 = c_0 g_0$ modulo $(C{*}G)'$. Then, there exist $g_i \in G$ such that $h_0^m = c_0^m g_0^m \prod_{1 \leq i \leq n} [a_i, g_i]$. By Lemma 5.9 above, for each $j = 1, \ldots, k$, there exist $g_{j1}, \ldots, g_{jn} \in G$ and commutators

c_j in C and g_{j0} in G such that $h_j = c_j g_{j0} \prod_{1 \leq i \leq n} [a_i, g_{ji}]$.

Substituting in the equation above for g and using the fact that c_j and g_{j0} are central in $C{*}G$ we have

$$g = c_0^m g_0^m \prod_{1 \leq i \leq n} [a_i, g_i] \prod_{1 \leq i \leq k} (c_j g_{j0} \prod_{1 \leq i \leq n} [a_i, g_{ji}])$$

$$= c_0^m c_1 \ldots c_k g_0^m g_{10} \ldots g_{k0} \prod_{1 \leq i \leq n} [a_i, g_i g_{1i} \ldots g_{ki}] \; .$$

But $g \in G$ and hence by the first normal form theorem we have, in particular,

$$g = g_0^m g_{10} \ldots g_{k0} \; .$$

But, each g_{j0} is a commutator in G. So,

$$G \models \sim \chi_k^m [g] \; ,$$

completing the proof of the theorem. //

We are now almost ready to prove the promised converse of Theorem 5.2. Defining the type we need now poses no problems but there are still difficulties in showing that it is omitted in $C{*}G$. The following lemmas are required to remove them.

5.12 Lemma

If C is a finite group, then a basis for C modulo C' may always be chosen so that the pseudo-order of each basis element is a power of a prime.

Proof

This follows from the definition (see 3.1) of a basis and the well-known fact that such a basis may always be chosen for a finite abelian group (See, for example, Hall [14], page 40, Theorem 3.3.1.) //

5.13 Lemma

Let $G \in N_2$ and r, s be positive integers.

(i) If $r | s$ then $H_s(G) \subset H_r(G)$.

(ii) If $H_s(G) \subset H_r(G)$ and $H_s(G)$ is definable in G , then $H_r(G)$ is also definable in G .

Proof

Let $G \in N_2$ and r, s positive integers with $r | s$. Then for some integer k , $s = kr$, and so, if $g \in H_s(G)$, there exist $g_1 \in G$, $g_2 \in G'$ such that $g = g_1^s g_2 = (g_1^k)^r g_2$. This shows that $g \in H_r(G)$ and hence $H_s(G) \subset H_r(G)$. Furthermore, if $\psi(u,\bar{x})$ defines $H_s(G)$ and $H_s(G) \subset H_r(G)$, then it is clear that $H_r(G)$ is defined by the formula

$$\varphi(u,\bar{x}) \equiv \exists vw(u = v^r w \wedge \psi(w,\bar{x})) \; . \; //$$

5.14 Lemma

Let $G \in N_2$, p be a prime, e a positive integer and let M_{p^e} be the set $\{\alpha < p^e : H_\alpha(G) \subset H_{p^e}(G)\}$. Then,

(i) M_{p^e} is a subgroup of the integers mod p^e generated by some integer p^d , with $0 < d \leqq e$;

(ii) if $e > e > 0$, then $M_{p^{e_1}} \subset M_{p^{e_1}}$ and $\alpha \in M_{p^{e_2}}$ implies $\alpha \bmod p^{e_1} \in M_{p^{e_1}}$ (that is, $M_{p^{e_1}} = M_{p^{e_2}} \bmod p^{e_1}$) .

Proof

(i) With G, p, e and M_{p^e} as defined in the Lemma, we assume that $\alpha, \beta \in M_{p^e}$, so that $H_\alpha(G), H_\beta(G) \subset H_{p^e}(G)$. Let $\alpha-\beta \bmod p^e = k$. Then for some integer λ, $k = \alpha-\beta+\lambda p^e$ and so, for all $g \in G$, we have $g^k = g^\alpha g^{-\beta} g^{\lambda p^e} \in H_{p^e}(G)$. Thus,

$k \in M_{p^e}$ and so M_{p^e} is a subgroup. The rest follows by elementary group theory.

(ii) Assume that $G, p, e_1, e_2, M_{p^{e_1}}, M_{p^{e_2}}$ satisfy the hypotheses of the lemma.

First assume that $\alpha \in M_{p^{e_2}}$. Then, $H_\alpha(G) \subset H_{p^{e_2}}(G)$. But by Lemma 5.13 (i), $H_{p^{e_2}}(G) \subset H_{p^{e_1}}(G)$ and so $H_\alpha(G) \subset H_{p^{e_1}}(G)$. Suppose that $k = \alpha \bmod p^{e_1}$. Then, for some integer λ, $k = \alpha + \lambda p^{e_1}$. Thus, for all $g \in G$, $g^k = g^\alpha g^{\lambda p^{e_1}} \in H_{p^{e_1}}(G)$. This shows that $k = \alpha \bmod p^{e_1} \in M_{p^{e_1}}$. Hence, $\alpha \in M_{p^{e_2}}$ implies $\alpha \bmod p^{e_1} \in M_{p^{e_1}}$.

Next we show that $M_{p^{e_1}} \subset M_{p^{e_2}}$. Using (i) of this lemma we assume that $M_{p^{e_1}} = gp(p^d)$, with $0 < d \leqq e_1$. If $d = e_1$ then $M_{p^{e_1}}$ is the trivial group and hence $M_{p^{e_1}} \subset M_{p^{e_2}}$ is immediate. So we assume that $d < e_1$. Let $\alpha \in M_{p^{e_1}}$, then there exist integers λ, μ with $\alpha = \lambda p^d + \mu p^{e_1}$ and so, for all $g \in G$, $g^\alpha = (g^{\lambda + \mu p^{e_1 - d}})^{p^d}$. Hence, to prove $M_{p^{e_1}} \subset M_{p^{e_2}}$, it suffices to prove that $g^{p^d} \in H_{p^{e_2}}(G)$, for all $g \in G$. Now, $p^d \in M_{p^{e_1}}$ and so, for all $g \in G$, $g^{p^d} = g_1^{p^{e_1}} g_2 = (g_1^{p^{e_1 - 1}})^p g_2$, for some $g_1 \in G$, $g_2 \in G'$. But $d \leqq e_1 - 1$ and so $p^{e_1 - 1} \in gp(p^d) = M_{p^{e_1}}$. Thus, $H_{p^{e_1 - 1}}(G) \subset H_{p^d}(G)$. Hence, there exist $g_3 \in G$, $g_4 \in G'$ with $g_1^{p^{e_1 - 1}} = g_3^{p^{e_1}} g_4$, and so, for some

$g_5 \in G'$, $g^{p^d} = g_3^{p^{e_1+1}} g_5 \in H_{p^{e_1+1}}(G)$. Now, writing $g^{p^d} = (g_3^{p^{e_1-1}})^{p^2} g_5$

and repeating the above argument, we have $g^{p^d} \in H_{p^{e_1+2}}(G)$. A further e_2-e_1-2 repetitions yields $g^{p^d} \in H_{p^{e_2}}(G)$ as required, proving that $M_{p^{e_1}} \subset M_{p^{e_2}}$. //

5.15 Theorem

Let G be a nil-2 group and C a finite nil-2 group having no G-suitable basis. Thus, $C*G$ is 1-unsaturated.

Proof

Let C and G satisfy the hypotheses of the theorem. Then, if $\{a_i : 1 \leq i \leq n\}$ is a basis for C, it is not G-suitable. So, for some i_o, $H_{o(a_{i_o})}(G)$ is not definable in G. Set $o(a_{i_o}) = m$.

Now, by Lemma 5.13 (ii), for $\alpha \leq m$ with $H_\alpha(G) \subset H_m(G)$ it follows that $H_\alpha(G)$ is not definable in G. But then, using Lemmas 5.10 and 5.11, there exist, for each $k \in \omega$, $g_k \in G'$ with $C*G \models \chi_k^\alpha[g_k]$. Using the nil-2 law we have $G' \subset Z(C*G)$ and hence, the existence of these g_k shows that the following subset of L is a type, with no parameters, over $C*G$:

$$t_\alpha^m(u) = \{\forall v[u,v] = 1\} \cup \{\chi_k^\alpha(u) : k \in \omega\} .$$

We shall show that there exists $\alpha \leq m$ with $H_\alpha(G) \subset H_m(G)$ such that $C*G$ omits $t_\alpha^m(u)$.

Now, any $h \in C*G$ which realises $t_\alpha^m(u)$ in $C*G$ must be central. So suppose $h \in Z(C*G)$. Then, by Lemma 5.8, with

$o(a_i) = m_i$, we have $h = cg_0 \prod_{1\leq i\leq n} [a_i, g_i]$ where, in particular,

$g_0 \in H_{m_{i_o}}(G)$, and $c = \prod_{1\leq i\leq n} a_i^{\alpha_i}$ modulo C' with $0 \leqq \alpha_i < m_i$

and $H_{\alpha_i}(G) \subset H_{m_i}(G)$. Thus, for some $b \in G$, and remembering that $m = m_{i_o}$, we have

$$h = \prod_{1\leq i\leq n} a_i^{\alpha_i} \cdot b^m \text{ modulo}(C*G)' . \tag{1}$$

We shall have finished if we can show how to express h as an α-th power modulo$(C*G)'$ for some $\alpha \leqq m$ with $H_\alpha(G) \subseteq H_m(G)$; for then it would follow that $C*G \models \sim\chi_k^\alpha[h]$, for some $k \in \omega$.

Using Lemma 5.12, we may assume that $m_i = p_i^{e_i}$, where p_i is a prime and $e_i > 0$. Now the primes p_i are not necessarily distinct. However, by Lemma 5.13, we may assume that if $m_{i_o} = m (= p^e$, say) and if $p_i = p$, then $e_i \leqq e$. Thus we may make the assumptions:

$$\begin{cases} \text{(i)} \;\; m_i = p_i^{e_i} \text{ and } m_{i_o} = m = p^e \text{ ; and} \\ \text{(ii)} \;\; \text{if } p_i = p \text{ then } e_i \leqq e . \end{cases} \tag{2}$$

Consider an arbitrary i , $1 \leqq i \leqq n$. Let $M_i = M_{p_i^{e_i}}$ be the set defined in Lemma 5.14 and suppose that it is generated by $p_i^{d_i}$. ($M_{i_o} = M$ and $d_{i_o} = d$) With h in the form (1), above, we have $\alpha_i \in M_i$. There are two possibilities.

<u>Case 1</u>: $p_i = p$. In this case, by (2), we have $M_i \subset M$. So $\alpha_i \in M$ and hence for some integer λ_i, $\alpha_i \equiv \lambda_i p^d \text{mod } p^e$.

Case 2: $p_i \neq p$. Then, since p_i, p are distinct primes, $\gcd(p_i^{e_i}, p^d) = 1$ and so for some integer λ, $1 \equiv \lambda p^d \bmod p_i^{e_i}$. Thus, $\alpha_i \equiv \alpha_i \lambda p^d \bmod p_i^{e_i}$.

So in either case, there exists an integer λ_i such that, since $o(a_i) = p_i^{e_i}$, $a_i^{\alpha_i} = a_i^{\lambda_i p^d}$ modulo C'. Hence, from (1) we have, working modulo $(C{*}G)'$:

$$h = \prod_{1\le i\le n} a_i^{\lambda_i p^d} \cdot b^{p^e}$$

$$= \Big(\prod_{1\le i\le n} a_i^{\lambda_i} \cdot b^{p^{e-d}}\Big)^{p^d} .$$

Thus, for all $h \in Z(C{*}G)$, $\exists\, h_1 \in C{*}G$, $h_2 \in (C{*}G)'$ such that $h = h_1^{p^d} h_2$. Let k be the number of commutators comprising h_2. Then, $C{*}G \models \sim \chi_k^{p^d}[h]$. It follows that the type $t_\alpha^m(u)$, with $m = p^e$ and $\alpha = p^d$ is omitted by $C{*}G$, proving that $C{*}G$ is 1-unsaturated. //

Combining Theorems 5.2 and 5.15 and forfeiting some of the strength of the latter, we obtain the following result.

5.16 Theorem

Let C and G be nil-2 groups with C finite and G $\varkappa$-saturated, where $\varkappa$ is an infinite cardinal. Then, $C{*}G$ is $\varkappa$-saturated if and only if C has a G-suitable basis.

The following remarks show that the above Theorem does not hold vacuously. In [8], Eklof and Fisher prove the existence of $\varkappa$-saturated abelian groups, for every infinite cardinal $\varkappa$. Furthermore, for an abelian group G, we have $G' = 1$ and hence every subgroup $H_m(G)$, $m \geq 0$, is definable in G . So there exist κ-saturated nil-2 groups G for which every finite nil-2 group has a G-suitable basis.

For the other direction we need a κ-saturated nil-2 group G for which some finite nil-2 group has no G-suitable basis. The following example arose from a conversation with A. Mekler. There certainly exist nil-2 groups G such that for any $m \in \omega$, $H_m(G)$ possesses elements g_k , $k \in \omega$, with $G \models \chi_k^m[g_k]$. (For instance, let G be the free nil-2 group of countable rank on the set $\{x_n : n \in \omega\}$. Set $g_k = [x_1,x_2]\ldots[x_{2k+1},x_{2k+2}]$. Then, an argument similar to that given in the proof of Claim (a), Theorem 5.23 below, shows that $G \models \chi_k^m[g_k]$.) Let G be such a group, then if $\tilde{G}$ is any elementary extension of G we also have $g_k \in H_m(\tilde{G})$ and $\tilde{G} \models \chi_k^m[g_k]$. Thus, choosing a κ-saturated $\tilde{G}$ we also have, by Remark 5.7, that $H_m(\tilde{G})$ is not definable in $\tilde{G}$.

Before discussing the possibility of extending Theorem 5.16 we consider the problem of the existence of saturated models in the variety N_2 . For a given theory T , saturated models of T need not always exist and, in general, we can be sure of their existence only by assuming the GCH or the existence of inaccessible cardinals. (See, for example Chang and Keisler [7], Proposition 5.1.5.) The results of Eklof and Fisher [8] show that the theory of abelian groups has saturated models in all infinite cardinals. We now use this fact to show that the same is true of the theory of all non-abelian, nil-2 groups.

5.17 <u>Theorem</u>

The theory of nil-2 groups has non-abelian saturated models in every infinite cardinality.

Proof

Let C be any non-abelian, finite, nil-2 group. (The smallest one has order 8 and is $Z_2 * Z_2$, where Z_2 is the cyclic group of order 2.) Let A be a saturated abelian group of cardinality $\kappa \geqq \omega$. Since C is finite and A is infinite, the nil-2 free product preserves the cardinality of A: to see this, consult the first normal form theorem and count. So, $|C*A| = |A| = \kappa$. But, since $A' = 1$, every basis for C is A-suitable and hence, by Theorem 5.16, $C*A$ is κ-saturated. Thus, $C*A$ is a saturated, non-abelian (since C is non-abelian), nil-2 group of cardinality κ. //

The following proposition shows that there is no point in seeking to extend Theorem 5.16 to the case where C is infinite but still finitely generated.

5.18 Proposition

No infinite, finitely generated group is ω-saturated. Indeed, if an infinite group G can be generated by n elements then G is $(n+1)$-unsaturated.

Proof

Let G be infinite and $G = gp\{a_0,\ldots,a_{n-1}\}$. Then, the elements of G are all finite products $g_0\ldots g_{k-1}$ where each g_i is either a generator or the inverse of some generator. The type omitted by G is, roughly speaking, the set $\{u \neq g : g \in G\}$ and we need use only the generators to describe it.

For each $s \in {}^{<\omega}n$ and $t \in {}^{<\omega}\{-1,+1\}$, where $\ell(s) = \ell(t) = k$ and $k > 0$, define

$$\varphi_{s,t}(u,v_0,\ldots,v_{n-1}) \equiv u \neq v_{s(0)}^{t(0)} v_{s(1)}^{t(1)} \ldots v_{s(k-1)}^{t(k-1)} .$$

Then, with $\bar{a} = \langle a_0,\ldots,a_{n-1}\rangle$,

$$\Gamma(u) = \{\varphi_{s,t}(u,\bar{a}) : s \in {}^{<\omega}n \ , \ t \in {}^{<\omega}\{-1,+1\} \ , \ \ell(s) = \ell(t)\}$$

is a type over G (since G is infinite) with n parameters, which is omitted by G. Hence, G is (n+1)-unsaturated. //

Now consider an infinitely generated, torsion group C. (Recall that C is torsion if every element of C has finite order. This restriction is important if we wish to use our previous results.) If C is unbounded, then it possesses elements of increasing finite order and hence the set $\{u^n \neq 1 : n > 0\}$ is a type, with no parameters, which C omits. It follows that no unbounded torsion group is 1-saturated. On the other hand, from the work of Eklof and Fisher [8], we know that there are κ-saturated, bounded, infinite abelian groups for every infinite cardinal κ. We conclude this section with a preservation result for the nil-2 free product restricted to the class, $\mathfrak{B}$, of all bounded nil-2 groups.

We have already observed, in section 3, that every group G belonging to $\mathfrak{B}$ possesses a basis. We shall prove (see Theorem 5.23, below) that a necessary and sufficient condition for the nil-2 free product of groups A, B $\in \mathfrak{B}$ to preserve saturation is that A and B possess "suitable" bases. The proof of the sufficiency of this condition relies heavily on the following distributive law for the V-free product operation, in an arbitrary variety V, over the direct product operation. We do not know whether this result holds for the cartesian product.

5.19 <u>Theorem</u> (Distributive law for $*_V$ over $\prod$)

Let V be an arbitrary variety of groups and $*_V$ denote the V-free product operation. Let A_i, B_i be groups in V for each $i \in I$ which, when identified as subgroups of

$(\overline{\prod_{i\in I}} A_i) *_V (\overline{\prod_{i\in I}} B_i)$ in the obvious way, satisfy $[A_i, B_j] = 1$ for $i \neq j$. Then,

$$(\overline{\prod_{i\in I}} A_i) *_V (\overline{\prod_{i\in I}} B_i) \simeq \overline{\prod_{i\in I}} (A_i *_V B_i) .$$

Proof

Let V be an arbitrary variety of groups, $*$ the full free product operation and $*_V$ the V-free product operation. Let A_i, B_i be groups in V for each $i \in I$ satisfying the hypothesis of the theorem. We make the following definitions and identifications:

$A = \overline{\prod_{i\in I}} A_i$, with A_i identified as a subgroup of A ;

$B = \overline{\prod_{i\in I}} B_i$, with B_i identified as a subgroup of B ;

$G = A*B$, with A and B identified as subgroups of G ;

$G_i = gp(A_i \cup B_i)$, $i \in I$;

K = normal closure (in G) of $gp(\bigcup_{i\neq j} [A_i, B_j])$;

$H_i = A_i * B_i$, $i \in I$;

$H = \overline{\prod_{i\in I}} H_i$, with H_i identified as a subgroup of H .

Now, we recall from section 2 that, up to isomorphism, the V-free product $M_1 *_V M_2$, of groups M_1 and M_2 is $M_1 * M_2 / V(M_1 * M_2)$. Thus, within the present framework, the hypothesis on the groups $[A_i, B_j]$, $i \neq j$, becomes $K \subset V(A*B)$. Indeed, since K is normal in $A*B$, we also have

$$K \triangleleft V(A*B) . \tag{1}$$

The result we wish to prove is:

$$G/V(G) \simeq \overline{\prod_{i\in I}} H_i/V(H_i) . \tag{2}$$

The proof of (2) is quickly deduced from the following claims.

Claim 1 $$G/K \simeq H$$

Claim 2 $$H/V(H) \simeq \prod_{i\in I} H_i/V(H_i)$$

We shall assume, for the moment, that the claims are justified and show how to derive (2).

It is easily verified (or see [20], page 79, Theorem 2.5) that for all groups M_1, M_2,

$$M_1 \simeq M_2 \Rightarrow M_1/V(M_1) \simeq M_2/V(M_2) .$$

Thus, by Claim 1 ,

$$(G/K)/V(G/K) \simeq H/V(H) \tag{3}$$

Suppose that $w(x_1,\ldots,x_n)$ is a word in V and $g_1,\ldots,g_n \in G$. Then, since K is normal in G , it follows that

$$w(g_1K,\ldots,g_nK) = w(g_1,\ldots,g_n)K .$$

Hence,

$$V(G/K) = \{gK : g \in V(G)\}$$

But, since by (1), $K \lhd V(G)$, the set on the right is the group $V(G)/K$. Thus,

$$V(G/K) = V(G)/K . \tag{4}$$

So, (3) and (4) together with some elementary group theory yield

$$G/V(G) \simeq (G/K)/(V(G)/K) = (G/K)/V(G/K) \simeq H/V(H) .$$

The result we require, namely (2), now follows immediately from Claim 2. It remains only to prove the claims.

Proof of Claim 1

Let $\alpha_i : A_i \to A_i * B_i$, $\beta_i : B_i \to A_i * B_i$ be the natural embeddings from A_i, B_i respectively into $A_i * B_i$, for each $i \in I$. Set $\alpha = \prod_{i\in I} \alpha_i$, $\beta = \prod_{i\in I} \beta_i$, so that α, β are embeddings of

A, B respectively into $\overline{\prod_{i\in I}} A_i * B_i = H$. Let $\theta : A*B \to H$ be the homomorphism extending α and β given by the definition (see 2.6) of $*$. Since H is generated by $\alpha(A) \cup \beta(B)$ it is clear that θ is onto H . The claim follows at once if we can prove that $\mathrm{Ker}\theta = K$.

First we verify that $K \subset \mathrm{Ker}\theta$. Now, K is generated by all elements of the form $g^{-1}[a_j,b_k]g$, where $g \in G$, $a_j \in A_j$, $b_k \in B_k$ and $j \neq k$. It suffices to prove that these generators lie in the kernel of θ . Suppose $\theta(g) = \overline{\prod_{i\in I}} h_i$ and $\theta(g^{-1}[a_j,b_k]g) = \overline{\prod_{i\in I}} c_i$. Then,

$$c_i = \begin{cases} h_i^{-1}h_i & i \notin \{j,k\} ; \\ h_j^{-1}\alpha_j(a_j^{-1})\alpha_j(a_j)h_j & i = j ; \\ h_k^{-1}\beta_k(b_k^{-1})\beta_k(b_k)h_k & i = k . \end{cases}$$

In all cases, $c_i = 1$ and hence $\theta(g^{-1}[a_j,b_k]g) = 1$. Thus, $K \subset \mathrm{Ker}\theta$.

To prove that $\mathrm{Ker}\theta \subset K$ we first show that each member $g \in A*B$ may be expressed in the form

$$g_1g_2\ldots g_nk \tag{5}$$

where $k \in K$ and, for each $j = 1,\ldots,n$, $g_j \in G_{i_j}$ for some $i_j \in I$ where $i_1,i_2,\ldots,i_n$ are pairwise distinct.

Now, G is generated by $\bigcup_{i\in I} G_i$ and hence each $g \in G$ is a product of finitely many elements from the subgroups G_i . Using the commutator laws

$$[a,bc] = [a,c]c^{-1}[a,b]c ,$$
$$[ab,c] = b^{-1}[a,c]b[b,c]$$

and the fact that $[A_i,A_j] = [B_i,B_j] = 1$ for all $i \neq j$, it is easy to see that each commutator in $[G_i,G_j]$, $i \neq j$, can be written as a product of finitely many elements from K. Thus, modulo K, the elements of G_i and G_j commute:

$$[G_i,G_j] \subset K \text{, for all } i \neq j \text{ .} \tag{6}$$

We may now use (6) to rearrange the component parts of an element of G. Each application of (6) produces an element of K. However, K is normal in G and so for all $k \in K$, $g \in G$ there is $k' \in K$ such that $kg = gk'$. Thus, in such a rearrangement of an element $g \in G$, the elements of K so produced may be collected on the right yielding an expression for g in the required form.

Now, let $g \in \mathrm{Ker}\theta$ and write g in the form (5) above. Then, since $K \subset \mathrm{Ker}\theta$,

$$\theta(g) = \theta(g_1 \ldots g_n k) = \theta(g_1) \ldots \theta(g_n) \text{ .}$$

Since $\theta(g_j)$ all belong to different groups H_{i_j} and $\theta(g) = 1$, it follows that $\theta(g_j) = 1$ for all $j = 1,\ldots,n$. Notice that $\theta(G_i) = A_i * B_i$. But, (see [20], page 186, Corollary 4.1.2) $G_i \simeq A_i * B_i$. Hence, $\theta \restriction G_i$ is an isomorphism. Thus, $g_j = 1$, for all $j = 1,\ldots,n$ and so $g = k$. This shows that $\mathrm{Ker}\theta \subset K$ and completes the proof of Claim 1.

Proof of Claim 2

This claim holds for any group H which is the direct product of subgroups H_i, $i \in I$. By considering the obvious mapping it is easily verified that

$$H/(\prod_{i \in I} V(H_i)) \simeq \prod_{i \in I} (H_i/V(H_i)) \tag{7}$$

But $\prod_{i\in I} V(H_i)$ is generated by the subgroups $V(H_i)$, each of which is a subgroup of $V(H)$. Hence, $\prod_{i\in I} V(H_i) \subset V(H)$. Furthermore, if $w(x_1,\ldots,x_n) \in V$ and $h_1,\ldots,h_n \in H$, then it is clear that

$$w(h_1,\ldots,h_n) = \prod_{i\in I} w(h_1(i),\ldots,h_n(i)) .$$

Thus, $V(H) \subset \prod_{i\in I} V(H_i)$. Consequently, $V(H) = \prod_{i\in I} V(H_i)$ and the claim follows immediately from (7). This completes the proof of the theorem. //

Before stating and proving Theorem 5.23, we need two Lemmas. The first characterises the elements of the centre of the nil-2 free product of the groups in which we are interested. For the notation we refer the reader to Definition 3.8 and the second normal form theorem (Theorem 3.9). * denotes, as usual, the nil-2 free product.

5.20 Lemma

Let A and B be nil-2 groups with bases $\{x_i : i \in I\}$, $o(x_i) = m_i$, and $\{y_j : j \in J\}$, $o(y_j) = n_j$ respectively. Then an element $ab \prod_{i,j\in I\times J} [x_i,y_j]^{\gamma_{ij}}$ belongs to the centre $Z(A*B)$, of $A*B$ if and only if

(i) $a \in Z(A)$ and $b \in Z(B)$; and

(ii) if $a = \prod_i x_i^{\alpha_i}$ modulo A' , $b = \prod_j y_j^{\beta_j}$ modulo B' , with $0 \leqq \alpha_i < m_i$ and $0 \leqq \beta_j < n_j$ when m_i and n_j are finite, then $\alpha_i \equiv \beta_j \equiv 0 \bmod \gcd(m_i,n_j)$, for all $i \in I$, $j \in J$. (As usual, $\equiv \bmod \infty$ is interpreted as equality.)

Proof

Let A and B satisfy the hypotheses and $g = ab$ modulo $(A*B)'$ be an arbitrary element of $A*B$. Using Lemma 3.3 we may write $a = \prod_i x_i^{\alpha_i}$ modulo A' and $b = \prod_i y_j^{\beta_j}$ modulo B' , where if m_i and n_j are finite, $0 \leq \alpha_i < m_i$ and $0 \leq \beta_j < n_j$.

First assume that (i) and (ii) hold. To prove that $g \in Z(A*B)$ it suffices to show that g commutes with each of the generators x_i, y_j . Let $i \in I$. Then, since $a \in Z(A)$ and $A*B$ is nil-2, $[x_i,g] = [x_i,b]$. Thus, using Lemma 2.5, since $\beta_j \equiv 0 \bmod \gcd(m_i,n_j)$, $[x_i,g] = \prod_j [x_i,y_j]^{\beta_j} = 1$. Similarly, $[y_j,g] = 1$ for all $j \in J$ and hence, $g \in Z(A*B)$.

Conversely, suppose $g \in Z(A*B)$. Then, for all $i \in I$, $1 = [x_i,g] = [x_i,a] \prod_{j\in J} [x_i,y_j]^{\beta_j}$. So, by the second normal form theorem and Lemma 2.5, for each $i \in I$, we have $[x_i,a] = 1$ and $\beta_j \equiv 0 \bmod \gcd(m_i,n_j)$, for all $j \in J$. Thus, $a \in Z(A)$ and $\beta_j \equiv 0 \bmod \gcd(m_i,n_j)$, for all $i \in I$, $j \in J$. The remaining parts of (i) and (ii) follow from the fact that $[y_j,g] = 1$, for all $j \in J$. //

Let G be an arbitrary group, and p a prime. Then G is said to be a p-group if the order of every element of G is a power of p . A p-subgroup of G is a subgroup of G which is also a p-group. A Sylow p-subgroup is a maximal p-subgroup of G . The following lemma is well-known (see, for example, Scott [34], page 144, 6.4.13).

5.21 Lemma

Every torsion nilpotent group is the direct product of its Sylow subgroups.

Proof

Let A be a torsion nilpotent group. Since A is nilpotent, each Sylow subgroup is a normal subgroup. These subgroups form a direct product in A consisting precisely of all elements of finite order. Since A is torsion, this direct product must coincide with A . //

5.22 Definitions

Let A and B be arbitrary nil-2 groups and let A_p , B_p denote the Sylow p-subgroups of A and B respectively.

We say that A is inclined to B if and only if, for each prime p ,

(i) whenever A_p is infinite, B_p is finite; and

(ii) whenever A_p is finite, A_p has a B_p-suitable basis (see definition 5.3).

A and B are said to be compatible if and only if A is inclined to B and B is inclined to A .

5.23 Theorem

Let A and B be bounded, nil-2 groups and $\varkappa$ an infinite cardinal such that A and B are both $\varkappa$-saturated. Then, A*B is $\varkappa$-saturated if and only if A and B are compatible.

Proof

Let A and B be bounded, nil-2 groups. Thus, with the notation of 5.22 and using Lemma 5.21, there is a positive integer k and distinct primes $p_1,\dots,p_k$ such that

$$A \simeq A_{p_1} \times \dots \times A_{p_k} \quad \text{and} \quad B \simeq B_{p_1} \times \dots \times B_{p_k} , \qquad (1)$$

where A_{p_i} and B_{p_i} are the Sylow p_i-subgroups (possibly degenerate) of A and B respectively. Now, since elements of coprime (pseudo-)orders in a nil-2 group commute (Lemma 2.5) we have, for $i \neq j$, $[A_{p_i}, B_{p_j}] = 1$, where these groups are thought of as subgroups of $A*B$. Thus, by the Distributive law (Theorem 5.19),

$$A*B \simeq \prod_{1\leq i\leq k} A_{p_i} * B_{p_i} . \tag{2}$$

Let $\varkappa$ be an infinite cardinal and A and B be $\varkappa$-saturated.

First we suppose that A and B are compatible with the intent of showing that $A*B$ is $\varkappa$-saturated.

Since the direct product operation preserves saturation (Theorem 1.3.6) we are finished if we can show that each summand, $A_{p_i} * B_{p_i}$, in the expression (2) above is $\varkappa$-saturated.

Consider a fixed i, $1 \leq i \leq k$. We show first that both A_{p_i} and B_{p_i} are $\varkappa$-saturated. Since A is bounded, so is A_{p_i} and hence if $m = \exp A_{p_i}$ we have $x \in A_{p_i}$ if and only if $x^m = 1$ (Note that if x does not belong to A_{p_i} then the order of x will be of the form $p_i^r \cdot q$ where $r \geq 0$ and $q > 1$ with $\gcd(p_i, q) = 1$. So, $x^m \neq 1$ since $p_i^r \cdot q \nmid m$.) Thus, A_{p_i} is a definable subgroup of A and hence by Theorem 1.3.5, A_{p_i} is $\varkappa$-saturated. Similarly, B_{p_i} is $\varkappa$-saturated.

Now if both A_{p_i} and B_{p_i} are finite, $A_{p_i} * B_{p_i}$ is also finite (consult the second normal form theorem and count). Hence, $A_{p_i} * B_{p_i}$ is $\varkappa$-saturated for all $\varkappa$. Otherwise, since A and B are compatible, it follows from (i) of the definition of inclination that one of A_{p_i} and B_{p_i} must be finite.

Suppose A_{p_i} is finite. Then, by (ii) of the definition of inclination, A_{p_i} has a B_{p_i}-suitable basis, and thus, since B_{p_i} is $\varkappa$-saturated, it follows, from Theorem 3.16, that $A_{p_i} * B_{p_i}$ is $\varkappa$-saturated. This proves that $A*B$ is $\varkappa$-saturated as explained above.

For the converse assume that A and B are not compatible; suppose A is not inclined to B. There are two possibilities:
1) for some i_o, both $A_{p_{i_o}}$ and $B_{p_{i_o}}$ are infinite; or
2) for some i_o, $A_{p_{i_o}}$ is finite, but has no $B_{p_{i_o}}$-suitable basis.

<u>Case 1</u>: Suppose both $A_{p_{i_o}}$ and $B_{p_{i_o}}$ are infinite and set $p_{i_o} = p$. Use the expression (1) above to choose a basis $\{x_i : i \in I\}$ for A with $o(x_i)$ a power of one of the primes $p_1, \ldots, p_k$ for each $i \in I$. Since A_p is infinite, there are infinitely many $i \in I$ for which $o(x_i)$ is a power of p. But A_p is bounded and so, by the pigeon-hole principle, there is an $\alpha > 0$ such that $o(x_i) = p^\alpha$ for infinitely many values of $i \in I$. So, without loss of generality, we shall assume that $I = \eta_1 - \{0\}$, for some cardinal η_1 and that for each n, $0 < n < \omega$, $o(x_n) = p^\alpha = r$, say. In the same way, we may choose a basis $\{y_j : j \in J\}$ for B, with $J = \eta_2 - \{0\}$, for some cardinal η_2, such that for each n, $0 < n < \omega$, $o(y_n) = p^\beta = s$, say. Assume further that $\alpha \leqq \beta$ and so $r | s$. (The proof for $\beta < \alpha$ is similar.) Our objective is to show that the element

$[x_1,y_1]\dots[x_{2n},y_{2n}]$ cannot be written in a certain form (see (3), below) using $<n$ commutators from $A*B$. This will enable us to define a type omitted by $A*B$.

Let $m_i = o(x_i)$ and $n_j = o(y_j)$ as usual, and $g \in Z(A*B)$. Then, by Lemma 5.20, working modulo $(A*B)'$,

$$g = \prod_{i\in I} x_i^{\alpha_i} \prod_{j\in J} y_j^{\beta_j}$$

where $\alpha_i \equiv \beta_j \equiv 0 \bmod \gcd(m_i,n_j)$ for all $i \in I$, $j \in J$. Thus, in particular, $\alpha_i \equiv 0 \bmod \gcd(m_i,s)$ and $\beta_j \equiv 0 \bmod \gcd(r,n_j)$. So there exist integers λ_i, μ_j with $\alpha_i \equiv \lambda_i s \bmod m_i$ and $\beta_j \equiv \mu_j r \bmod n_j$. So, still working modulo $(A*B)'$:

$$g = \prod_{i\in I} x_i^{\lambda_i s} \prod_{j\in J} y_j^{\mu_j r}$$

$$= (\prod_{i\in I} x_i^{\lambda_i (s/r)} \prod_{j\in J} y_j^{\mu_j})^r,$$

since $r|s$. Thus, for all $g \in Z(A*B)$, there is $h \in A*B$ with

$$g = h^r \text{ modulo } (A*B)'. \qquad (3)$$

Now, for each n, $0 < n < \omega$, let $c_n = [x_1,y_1]\dots[x_{2n},y_{2n}]$.

<u>Claim (a)</u>: c_n cannot be written in the from (3) above using $<n$ commutators.

<u>Proof of Claim (a)</u>: Suppose that c_n can be written in this way, so that, for some $h,g_1,\dots,g_{n-1},h_1,\dots,h_{n-1} \in A*B$,

$$c_n = h^r[g_1,h_1]\dots[g_{n-1},h_{n-1}]. \qquad (4)$$

Consider h first and suppose, using the second normal form theorem, that $h = ab \prod_{i,j} [x_i,y_j]^{\epsilon_{ij}}$, for some $a \in A$, $b \in B$ and suitable

$\varepsilon_{ij} < \infty$. Then, using Lemma 2.3

$$h^r = a^r b^r [a,b^{\frac{1}{2}r(r-1)}] \prod_{i \geq \omega, j \in J} [x_i,y_j]^{r\varepsilon_{ij}} , \tag{5}$$

since if $0 < i < \omega$, $o(x_i) = r$ and hence $[x_i,y_j]^{r\varepsilon_{ij}} = 1$.

Now consider $[g_\ell,h_\ell]$, $1 \leq \ell \leq n-1$, letting $g_\ell = a_\ell b_\ell$, and $h_\ell = \tilde{a}_\ell \tilde{b}_\ell$ modulo $(A*B)'$. Then,

$$\begin{aligned} [g_\ell,h_\ell] &= [a_\ell b_\ell,\tilde{a}_\ell \tilde{b}_\ell] \\ &= [a_\ell,\tilde{a}_\ell][b_\ell,\tilde{b}_\ell][a_\ell,\tilde{b}_\ell][\tilde{a}_\ell,b_\ell^{-1}] \\ &= a'_\ell b'_\ell \, [a_\ell,\tilde{b}_\ell][\tilde{a}_\ell,b_\ell^{-1}] , \end{aligned} \tag{6}$$

where a'_ℓ and b'_ℓ are commutators of A and B respectively. Now substituting from (5) and (6) in (4) above we have that c_n can be written in the form

$(a^r b^r a'_1 \ldots a'_{n-1} b'_1 \ldots b'_{n-1})$

$\cdot$ (2n-1) commutators of the form $[\tilde{a},\tilde{b}]$

$\cdot$ commutators of the form $[x_i,y_j]$, $i \geq \omega$, $j \in J$.

Hence we may write

$$c_n = a^r a' b^r b' \prod_{1 \leq \ell \leq 2n-1} [\hat{a}_\ell,\hat{b}_\ell] \cdot c , \tag{7}$$

where, $a' \in A'$, $b' \in B'$, $\hat{a}_\ell \in A$, $\hat{b}_\ell \in B$ and c is a product of commutators of the form $[x_i,y_j]$ involving x_i <u>only for $i \geq \omega$</u> . Suppose that $\hat{a}_\ell = \prod_{i \in I} x_i^{\alpha_{i\ell}}$ modulo A' and $\hat{b}_\ell = \prod_{j \in J} y_j^{\beta_{\ell j}}$ modulo B' (using Lemma 3.3), then

$$[\hat{a}_\ell,\hat{b}_\ell] = \prod_{i,j} [x_i,y_j]^{\alpha_{i\ell}\beta_{\ell j}} .$$

Thus, from (7) we have

$$c_n = a^r a' b^r b' \prod_{1 \leq \ell \leq 2n-1} \left(\prod_{i,j} [x_i,y_j]^{\alpha_{i\ell}\beta_{\ell j}}\right) \cdot c$$

$$= a^r a' b^r b' c \prod_{i,j} [x_i,y_j]^{\sum_{1 \leq \ell \leq 2n-1} \alpha_{i\ell}\beta_{\ell j}} .$$

Thus, we have two expressions, in the correct form, for the same element $c_n = \prod_{1 \leq i \leq 2n} [x_i,y_i]$ of $A*B$. Hence using Lemma 2.5 and the second normal form theorem, and remembering that c involves $[x_i,y_j]$ only if $i \geq \omega$ we have, in particular, since $\gcd(r,s) = r$, that

$$\sum_{1 \leq \ell \leq 2n-1} \alpha_{i\ell}\beta_{\ell j} \equiv \bmod r \begin{cases} 1 & \text{if } i = j = 1,\ldots,2n ; \\ \\ 0 & \text{if } i \neq j ,\ 1 \leq i,j \leq 2n . \end{cases} \tag{8}$$

Now let $\underline{X}$ denote the $2n \times (2n-1)$ matrix $(\alpha_{i\ell})$ and $\underline{Y}$ the $(2n-1) \times 2n$ matrix $(\beta_{\ell j})$, both over the ring of integers modulo r. Then (8) expresses the fact that

$$\underline{X} \cdot \underline{Y} = \underline{I}$$

where $\underline{I}$ is the identity matrix of order $2n$ over the given ring. Now let $r(\underline{M})$ denote the rank of a matrix $\underline{M}$. Then, since $r(\underline{X} \cdot \underline{Y}) \leq \min \{r(\underline{X}), r(\underline{Y})\}$ (see Herstein [16], page 222, Lemma 6.3) it follows that $r(\underline{X} \cdot \underline{Y}) \leq 2n-1$ and hence $r(\underline{I}) < 2n$. This, of course, is a contradiction and hence we have proved the claim.

Now, using claim (a) and the fact that since $c_n \in (A*B)'$ we also have $c_n \in Z(A*B)$, the following set is a type over $A*B$:

$$t(u) = \{\forall v[u,v] = 1\} \cup \{\chi_n^r(u) : n \in \omega\} .$$

(For the definition of $\chi_n^r(u)$, see 5.4.) But (3) shows that $A*B$ omits $t(u)$. It follows that $A*B$ is 1-unsaturated. (We actually prove a stronger result than we need, see Corollary 5.24 below.) This completes the proof in Case 1 .

Case 2: Assume that $A_{p_{i_o}}$ is finite but has no $B_{p_{i_o}}$-suitable basis. Set $p_{i_o} = p$.

Using (1) above, choose a basis $\{x_i : i \in I\}$ for A and $\{y_j : j \in J\}$ for B , where, for each $i, j, o(x_i), o(y_j)$ is a power of one of the primes $p_1,\ldots,p_k$. Now, $\{x_i : x_i \in A_p\}$ is a basis for A_p and hence is not B_p-suitable. Thus, for some $x_i \in A_p$ we have that $H_{o(x_i)}(B_p)$ is not definable in B_p . There are two consequences of this. The first is that, if $p^e = \max\{o(x_i) : x_i \in A\}$, then, by Lemma 5.13, $H_{p^e}(B_p)$ is not definable in B_p . The second is that, if $p^d = \max\{o(y_j) : y_j \in B_p\}$, then $H_{p^d}(B_p) = B'_p$ and so, $H_{p^d}(B_p)$ is not definable in B_p . (Otherwise, by the comments preceding Corollary 4.2, we could define $H_{p^e}(B_p)$ in B_p .)

Now if $H_m(B_p)$ is not definable in B_p , then since A_p is finite, we may use Lemmas 5.10 and 5.11 obtaining, for each $n \in \omega$, $g_n \in B'_p$ such that $A_p*B_p \models \chi_n^m[g_n]$. But $\sim\chi_n^m(u)$ is a positive existential formula and hence, from (2), $A*B \models \chi_n^m[g_n]$. Thus, for each $m = p^e, p^d$, the following set is a type over $A*B$:

$$t_m(u) = \{\forall v[u,v] = 1\} \cup \{\chi_n^m(u) : n \in \omega\} .$$

It will follow that $A*B$ is 1-saturated once we have proved the next claim.

Claim (b): for some $m = p^e$ or p^d, $A*B$ omits $t_m(u)$.

Proof of Claim (b): Let $g \in Z(A*B)$. Then, by Lemma 5.20, we may write

$$g = \prod_{i\in I} x_i^{\alpha_i} \prod_{j\in J} y_j^{\beta_j} \text{ modulo } (A*B)' ,$$

where, $\alpha_i \equiv \beta_j \equiv 0 \bmod \gcd(m_i,n_j)$ and $m_i = o(x_i)$ and $n_j = o(y_j)$ as usual. Thus, in particular

$\alpha_i \equiv 0 \bmod \gcd(m_i,p^d)$ and $\beta_j \equiv 0 \bmod \gcd(p^e,n_j)$. There are two possibilities: $e < d$, $e \geqq d$.

Assume $e < d$. Then,

$$\gcd(m_i,p^d) = \begin{cases} 1 & \text{if } \gcd(m_i,p) = 1 ; \\ m_i & \text{otherwise, by the definition of } p^e . \end{cases}$$

If $\gcd(m_i,p^d) = 1$, then there exist integers s, t with $sm_i+tp^e = 1$, and so, $\alpha_i = \alpha_i sm_i+\alpha_i tp^e$.

Thus, $x_i^{\alpha_i} = x_i^{\alpha_i tp^e}$ modulo A'. If $\gcd(m_i,p^d) \neq 1$, then $\alpha_i \equiv 0 \bmod m_i$, and $m_i | p^e$. Thus, in this case

$x_i^{\alpha_i} = x_i^{p^e} = 1$ modulo A'. Hence for some λ_i,

$$x_i^{\alpha_i} \underset{\text{modulo } A'}{=} \begin{cases} x_i^{\lambda_i p^e} & \text{if } \gcd(m_i,p) = 1 ; \\ x_i^{p^e} = 1 & \text{otherwise} . \end{cases}$$

Similarly, since

$$\gcd(p^e, n_j) = \begin{cases} 1 & \text{if } \gcd(p,n_j) = 1 \ ; \\ n_j & \text{if } \gcd(p,n_j) \neq 1 \text{ and } n_j < p^e \ ; \\ p^e & \text{if } \gcd(p,n_j) \neq 1 \text{ and } n_j \geqq p^e \ , \end{cases}$$

we have, for some μ_j ,

$$y_j^{\beta_j} =_{\text{modulo } B'} \begin{cases} y_j^{\mu_j p^e} & \text{if } \gcd(p,n_j) = 1 \ ; \\ y_j^{p^e} = 1 & \text{if } \gcd(p,n_j) \neq 1 \text{ and } n_j < p^e \ ; \\ y_j^{\mu_j p^e} & \text{if } \gcd(p,n_j) \neq 1 \text{ and } n_j \geqq p^e \ . \end{cases}$$

Thus, in all cases, there are integers λ_i, μ_j such that

$$g = \prod_i x_i^{\lambda_i p^e} \prod_j y_j^{\mu_j p^e} \quad \text{modulo}(A*B)'$$

$$= \left(\prod_i x_i^{\lambda_i} \prod_j y_j^{\mu_j}\right)^{p^e} \quad \text{modulo}(A*B)' \ .$$

Such a g cannot realise $t_{p^e}(u)$ and hence $A*B$ omits $t_{p^e}(u)$.

<u>Now assume $e \geqq d$</u> . Then, in exactly the same way we are able to show that every $g \in Z(A*B)$ has the form h^{p^d} modulo $(A*B)'$ with $h \in A*B$, and hence, g cannot realise $t_{p^d}(u)$. Thus, in this case, $A*B$ omits $t_{p^d}(u)$.

This completes the proof in Case 2, and hence also the proof of the theorem. //

In the course of proving Theorem 5.23 we actually proved a stronger result than stated.

5.24 <u>Corollary</u>

If A and B are incompatible, bounded, nil-2 groups then $A*B$ is 1-unsaturated.

5.25 Examples

The following examples show that Theorem 5.23 does not hold vacuously. It is well-known (see, for example, [7], page 101, Theorem 2.3.13) that the unique denumerable model of an ω-categorical, complete theory is saturated. Hence, since every bounded, abelian group has an ω-categorical theory (see [8], page 146), it follows that every denumerable, bounded abelian group is saturated. We shall use this fact frequently to construct our examples.

Example (i)

Let $A = Z_2 \times Z_3^{(\omega)}$ and $B = Z_3 \times Z_2^{(\omega)}$. Then, A and B are compatible, bounded, denumerable, saturated, abelian groups. Indeed, there are pairs of compatible, bounded, κ-saturated, abelian groups for every cardinal κ. (See [8], page 146).

Example (ii)

Clearly, if $A = B = Z_2^{(\omega)}$, then A and B are both ω-saturated, but they are also incompatible since they fail to satisfy clause (i) of the definition of inclination.

Example (iii)

Let $A = Z_2 \times Z_3^{(\omega)}$ and $B = \prod_{n \in \omega}^{*} Z$, (the nil-2 free product of ω copies of Z_2). A is ω-saturated but B is 1-unsaturated. To see the latter observe first, using the second normal form theorem, that $Z(B) = B'$. Hence B' is definable in B. However, since there also exist in B products of commutators of unbounded length whose length cannot be "reduced", it follows, from Lemma 5.5, that B must be 1-unsaturated. So consider instead an ω-saturated elementary extension $\widetilde{B}$ of B. Since B has exponent 4, the same is true of $\widetilde{B}$. So, $\widetilde{B}$ is a bounded,

ω-saturated, nil-2 group. Furthermore, clause (i) of the definition of inclination holds for the pair $A, \widetilde{B}$. But $H_2(B) = B'$ and, as we have already remarked, it can be shown that there exist $b_k \in B'$, for $k \in \omega$, such that $B \models \chi_k^2[b_k]$. Hence, $\widetilde{B} \models \chi_k^2[b_k]$ and so, by Lemma 5.5 again, $H_2(\widetilde{B})$ is not definable in $\widetilde{B}$. Thus, A and $\widetilde{B}$ are incompatible through failure to satisfy clause (ii) of the definition of inclination.

6. Preservation of Stability

The first theorem in this section is the analogue for stability of the corresponding result for saturation in section 5 (see Theorem 5.2). Before reading the proof the following should be recalled: the definition of a G-suitable basis (Definition 5.3), the construction of a determining set given in the proof of Theorem 4.1 and the notational conventions outlined in the paragraph preceding that theorem.

6.1 Theorem

Let κ be an infinite cardinal, G a nil-2 group and C any finite nil-2 group possessing a G-suitable basis. Then, if G is κ-stable so is $C*G$.

Proof

Let C, G and κ satisfy the hypotheses and assume that G is κ-stable. Of course, if G is finite then so is $C*G$ and hence there is nothing to prove since every finite structure is stable in every infinite power. Thus we may also assume that G is infinite. Let $\{a_i : 1 \leq i \leq n\}$ be any G-suitable basis for C . Then each subgroup $H_{o(a_i)}(G)$ of G is definable in G , possibly with parameters: let $\bar{d}$ be a (finite) sequence consisting of all the parameters used in the defining formulae of these subgroups. Let $\bar{z} = \langle z_\nu \rangle_{\nu<\lambda}$ be a sequence from $C*G$ of length $\lambda \leq \varkappa$ and $\bar{y} = \langle \bar{y}_\nu \rangle_{\nu<\lambda}$ the corresponding sequence of length $(n+1)\lambda \leq \varkappa$ from G. (In other words, if $z_\nu = b_\nu y_{\nu 0} \prod_{1\leq i\leq n} [a_i, y_{\nu i}]$, then $\bar{y}_\nu = \langle y_{\nu 0}, \ldots, y_{\nu n} \rangle$). Then, since G is $\varkappa$-stable we have $|S(G,\bar{d},\bar{y})| \leq \varkappa$. We wish to prove that $|S(C*G,\bar{z})| \leq \varkappa$.

For each type $t(u) \in S(C*G,\bar{z})$ let (c_t, τ_t), where $c_t \in C$ and $\tau_t(\bar{u})$ is an $(n+1)$-type over $(G,\bar{d},\bar{y})$, be some pair associated with t in the sense of Theorem 5.1. For each $c \in C$ define

$$T_c(C*G,\bar{z}) = \{t \in S(C*G,\bar{z}) : c_t = c\} .$$

Then, since C is finite and $|S(C*G,\bar{z})|$ is infinite, there must be some $c_o \in C$ with

$$|T_{c_o}(C*G,\bar{z})| = |S(C*G,\bar{z})| \qquad (1)$$

We show that distinct types $t_1, t_2 \in T_{c_o}(C*G,\bar{z})$ are associated with distinct types τ_{t_1}, τ_{t_2} over $(G,\bar{d},\bar{y})$.

Suppose that $t_1, t_2 \in T_{c_o}(C*G,\bar{z})$ and $t_1 \neq t_2$. Then, there is some formula $\varphi(u) \in t_1$ such that $\sim\varphi(u) \in t_2$. Now, if $\{\theta_c(\bar{u}) : c \in C\}$ is a determining set for $\varphi(u)$, it follows from the proof of Theorem 4.1 that $\{\sim\theta_c(\bar{u}) : c \in C\}$ is a determining set for $\sim\varphi(u)$. Hence, by the construction of the type $\tau(\bar{u})$ corresponding to $t(u)$ given in the proof of Theorem 5.1, we have

$$\theta_{c_o}(\bar{u}) \in \tau_{t_1} \quad \text{and} \quad \sim\theta_{c_o}(\bar{u}) \in \tau_{t_2} .$$

Hence, $\tau_{t_1} \neq \tau_{t_2}$. Thus, the complete types over $(G,\bar{d},\bar{y})$ determined by τ_{t_1} and τ_{t_2} will also differ, giving

$$|S(G,\bar{d},\bar{y})| \geq |T_{c_0}(C*G,\bar{z})| .$$

Thus, from (1),

$$|S(C*G,\bar{z})| \leq |S(G,\bar{d},\bar{y})| \leq \varkappa ,$$

proving that $C*G$ is $\varkappa$-stable . //

The following corollary is a consequence of the above theorem and the results of Chapter 2 of this thesis. It also provides a contrast with Baldwin and Saxl's Theorem 4.1 in [1] and Sabbagh's Remarques 2 (iii) in [28] .

6.2 Corollary

(i) Every nil-2 group of the form $C*A$, where C is finite and A is abelian is stable.

(ii) There exist ω-stable, strictly-superstable and also merely-stable, non-abelian nil-2 groups.

Proof

(i) follows from the fact that every abelian group is stable and that C will possess an A-suitable basis.

For (ii) we first observe that $C*A$ will be at least as stable as A is. Thus, examples of ω-stable, non-abelian groups in N_2 are easily constructed by forming the nil-2 free product of a finite, non-abelian, nil-2 group with an ω-stable, abelian group.

Consider the group $G = Z_2*Z$. Since Z is superstable it follows that G is superstable; we shall show that G is strictly-superstable. By Theorem 3.9, if a generates Z_2 and b

generates Z , then the elements of G can be written, uniquely, in the form

$$a^{\alpha} b^{k} [a,b]^{\beta} ,$$

where $0 \leq \alpha , \beta < 2$ and k is an integer. From this it follows that G is a non-abelian group in which the elements of order 2 , namely those for which $k = 0$, form a definable normal subgroup H . Thus, appealing to Theorem 1.3.15, G/H is $\varkappa$-stable whenever G is $\varkappa$-stable. But,

$$G/H = \{b^k H : k \text{ is an integer}\} \simeq Z$$

and hence G/H is strictly-superstable. This means that G is $\varkappa$-unstable for all $\varkappa < 2^{\omega}$ and completes the proof that G is a strictly-superstable, non-abelian, nil-2 group. A similar argument will show that the non-abelian, nil-2 group $Z_2 *_{n \in \omega} Z_{2^n}$ is merely-stable. //

We may now use Theorem 6.1 to give the analogue for stability of the positive direction of Theorem 5.23. We obtain this as a corollary of a more general result concerning stable, compatible (see Definition 5.22), torsion, nil-2 groups.

Consider two torsion groups $A, B \in N_2$. Using Lemma 5.21 we may write $A = \prod_p A_p$, $B = \prod_p B_p$, where A_p , B_p denote the Sylow p-subgroups of A , B respectively. Then, by the distributive law (Theorem 5.19), $A*B = \prod_p A_p * B_p$. If $A_p * B_p$ is non-abelian for infinitely many values of p , it follows from Theorem 1.3.19 that $A*B$ is unstable. Of course, if $A_p * B_p$ is abelian, then so are A_p and B_p and, furthermore, $A_p * B_p = A_p \times B_p$. So, on the other hand, if only finitely many factors $A_p * B_p$ are non-abelian, we may write

$$A*B = \prod_{1\le i\le n} (A_{p_i} * B_{p_i}) \times \prod_{p\ne p_i} (A_p \times B_p)$$

$$= \prod_{1\le i\le n} (A_{p_i} * B_{p_i}) \times \mathfrak{A} \times \mathfrak{B} , \qquad (1)$$

where $\mathfrak{A} = \prod_{p\ne p_i} A_p$ and $\mathfrak{B} = \prod_{p\ne p_i} B_p$ are both abelian. Since $\mathfrak{A} \times \mathfrak{B}$ is an abelian group it is also stable. If we assume that A is stable, then since $A = \prod_{1\le i\le n} A_{p_i} \times \mathfrak{A}$, Theorem 1.3.17 implies that each subgroup A_{p_i} is stable. Similarly, if B is stable, then so are the subgroups B_{p_i} . Thus, on the further assumption that $\prod_{1\le i\le n} A_{p_i}$ and $\prod_{1\le i\le n} B_{p_i}$ are compatible, Theorem 6.1 yields that each factor $A_{p_i} * B_{p_i}$ is stable. Since by Theorem 1.3.16, $\times$ preserves stability, we see at once from (1) that, under the above assumptions, $A*B$ is stable. Suppose that A and B are $\varkappa$-stable. Then, we can be more precise about the stability of $A*B$ if we assume that the groups A_{p_i}, $\mathfrak{A}$ (respectively, B_{p_i} , $\mathfrak{B}$) are definable subgroups of A (respectively, B). For then, by Theorem 1.3.15, it follows that these subgroups are also $\varkappa$-stable and hence so is $A*B$.

These remarks prove the following Theorem.

6.3 <u>Theorem</u>

Let A and B be nil-2 torsion groups and, for each prime p , let A_p, B_p denote their Sylow p-subgroups.

(i) If $A*B$ is stable, then A_p*B_p is abelian for all but, possibly, finitely many values of p .

(ii) If A and B are stable, A_p*B_p is abelian for all

$p \neq p_1,\ldots,p_n$ and the groups $\prod_{1\leq i\leq n} A_{p_i}$, $\prod_{1\leq i\leq n} B_{p_i}$ are compatible, then $A*B$ is stable; if, in addition, the subgroups A_{p_i} and $\prod_{p\neq p_i} A_p$ (respectively, B_{p_i}, $\prod_{p\neq p_i} B_p$) are definable in A (respectively, B), then $A*B$ is $\varkappa$-stable whenever both A and B are $\varkappa$-stable.

6.4 Corollary

Let A and B be bounded, compatible, nil-2 groups. Then if A and B are $\varkappa$-stable, so is $A*B$.

Proof

Under the hypotheses of the Corollary, there exist $p_1 \ldots p_n$ such that $A = \prod_{1\leq i\leq n} A_{p_i}$, $B = \prod_{1\leq i\leq n} B_{p_i}$, and each A_{p_i}, B_{p_i} is bounded. Furthermore, as we proved in Theorem 5.23, each subgroup A_{p_i} (respectively, B_{p_i}) is definable in A (respectively, B). The Corollary now follows immediately from Theorem 6.3(ii). //

It is now a simple matter to produce examples which show that the nil-2 free product operation fails to preserve either strict-superstability or mere-stability even when the free factors involved are also abelian. For instance, consider the groups $A = \prod_p Z_p$ and $\prod_p \prod_n Z_{p^n}$. From the characterisations given in Chapter 2 (see Theorems 2.4.4 and 2.4.7) it is easy to check that A is strictly-superstable and B is merely-stable. However, for each prime p , both $Z_p * Z_p$ and $\prod_n Z_{p^n} * \prod_n Z_{p^n}$ are non-abelian and hence, by Theorem 6.3 (i), neither $A*A$ nor $B*B$ is stable.

The situation for ω-stability is not so clear even if we restrict attention to abelian free factors. Of course we do have some preservation of ω-stability. Corollary 6.5, below, may be obtained from the above Corollary using the fact that every bounded abelian group is ω-stable. The hypothesis on the Sylow subgroups ensures that the free factors are compatible.

6.5 Corollary

If A and B are bounded abelian groups and there is no prime p for which the Sylow p-subgroups of A and B are both infinite, then $A*B$ is ω-stable.

The simplest example of a nil-2 free product of bounded abelian groups not covered by the Corollary above is $Z_2^{(\omega)}*Z_2^{(\omega)}$. Proposition 6.9, below, gives a necessary and sufficient condition for $*$, restricted to abelian groups, to preserve ω-stability. It also shows that if we are to resolve this problem at all it is essential that we solve it first for groups like the one just mentioned. The next lemma is used in the proof of this proposition and as a by-product gives us one more preservation result.

6.6 Lemma

If D, T are subgroups of some nil-2 group with D divisible and T torsion, then $[D,T] = 1$.

Proof

Clearly it suffices to show that $[d,t] = 1$ for all $d \in D$, $t \in T$. So let $d \in D$, $t \in T$. Since T is torsion, $t^n = 1$ for some integer n; since D is divisible, $d = d_1^n$, for some $d_1 \in D$. Thus, by Lemma 2.3, $[d,t] = [d_1^n,t] = [d_1,t]^n = [d_1,t^n] = 1$. //

6.7 Corollary

If $D, T \in N_2$ with D divisible and T torsion, then $D*T$ is $\varkappa$-stable whenever both D and T are $\varkappa$-stable.

Proof

By the lemma above, $D*T = D\times T$ and so the result follows from

Theorem 1.3.16. //

Since the abelian group $Z(p^\infty)$ is both divisible and torsion, we have the following corollary of 6.7.

6.8 Corollary

Let p and q denote distinct primes. Then the following nil-2 groups are ω-stable:

$$Z(p^\infty)*Z(p^\infty),\ Z(p^\infty)*Z(q^\infty),\ Z(p^\infty)*Q\ .$$

6.9 Proposition

The nil-2 free product of ω-stable abelian groups is ω-stable if and only if

(i) $Q^{(\varkappa)}*Q^{(\lambda)}$ is ω-stable, for all cardinals $\varkappa, \lambda$; and

(ii) P_1*P_2 is ω-stable, where P_1 and P_2 are both bounded abelian p-groups for the same prime p .

Proof

The necessity of conditions (i) and (ii) is obvious.

For the sufficiency, let G_1, G_2 be two ω-stable abelian groups and assume that both (i) and (ii) hold. By Theorem 2.4.8, there exist bounded groups B_i and divisible groups D_i with $G_i = B_i \times D_i$, $i = 1, 2$. Using the lemma above and the distributive law (Theorem 5.19) we have

$$G_1*G_2 = (B_1*B_2) \times (D_1*D_2) \tag{1}$$

We consider each direct factor separately.

By Lemma 5.21, there exist primes $p_1,\ldots,p_k$ such that $B_i = P_{i1}\times\ldots\times P_{ik}$, where each P_{ij} is the Sylow p_j-subgroup of B_i , $i = 1, 2$. Now, using the fact that elements of coprime order in a nil-2 group commute (Lemma 2.5), a further use of the distributive law yields

$$B_1 * B_2 = \prod_{1 \leq i \leq k} P_{1i} * P_{2i} \ . \tag{2}$$

It is well-known (see [18], Theorem 4) that a divisible abelian group D can be written in the form

$$D = Q^{(\varkappa)} \times \prod_p Z(p^\infty)^{(\varkappa_p)}$$

for some cardinals $\varkappa$, $\varkappa_p$. Suppose that for each i = 1, 2 ,

$$D = Q^{(\varkappa_i)} \times \prod_p Z(p^\infty)^{(\varkappa_{ip})}$$

then, by Lemma 6.6 and the distributive law,

$$D_1 * D_2 = Q^{(\varkappa_1)} * Q^{(\varkappa_2)} \times \prod_p Z(p^\infty)^{(\varkappa_{1p} + \varkappa_{2p})} \tag{3}$$

Writing $\lambda_p = \varkappa_{1p} + \varkappa_{2p}$, (1), (2) and (3) give

$$G_1 * G_2 = \prod_{1 \leq i \leq k} (P_{1i} * P_{2i}) \times (Q^{(\varkappa_1)} * Q^{(\varkappa_2)}) \times \prod_p Z(p^\infty)^{(\lambda_p)} \ .$$

Now, by hypothesis $P_{1i} * P_{2i}$ and $Q^{(\varkappa_1)} * Q^{(\varkappa_2)}$ are ω-stable and since $\prod_p Z(p^\infty)^{(\lambda_p)}$ is a divisible abelian group, it, too, is ω-stable. Hence, since the direct product operation preserves stability (Theorem 1.3.16), $G_1 * G_2$ is ω-stable. //

Although we cannot give an example where * fails to preserve ω-stability, our next result yields an instance of an unstable, nil-2 free product with one ω-stable factor and one strictly-superstable factor: the group $Z * Z(p^\infty)$. Further examples of such a product are afforded by Proposition 6.12 and Theorem 6.13. For the definition of pseudo-order see Definitions 2.4.

6.10 <u>Theorem</u>

If A is a non-torsion nil-2 group with a basis and B is a nil-2 group possessing elements of increasing and unbounded finite pseudo-order, then A*B is unstable.

Proof

Let A be a nil-2 group with a basis. By Lemmas 2.5 and 3.3 it follows easily that if every generator of A has finite pseudo-order, then A must be torsion. Thus, if A is non-torsion, the basis for A must include at least one generator, say a, with $o(a) = \infty$. Let B be a nil-2 group possessing elements of increasing, unbounded finite pseudo-order. Then, there exist b_n in B , with $o(b_n) = k_n < \infty$ and $k_0 \ldots k_n < k_{n+1}$, for each $n \in \omega$. Set $\ell_n = k_0 \ldots k_n$. We shall show that $[a^{\ell_m}, b_n] = 1$ if and only if $n \leqq m$ and hence, by Corollary 1.3.14 it will follow that A*B is unstable.

Now, if $[a^{\ell_m}, b_n] = 1$ then $[a, b_n^{\ell_m}] = 1$ and hence, by the first normal form theorem (Theorem 3.7), $b_n^{\ell_m} \in B'$, since $o(a) = \infty$. But, by Lemma 2.5, this implies that $o(b_n) | \ell_m$, or equivalently, $k_n | k_0 \ldots k_m$. Since for $n > m$ we have $k_n \geqq k_{m+1} > \ell_m$ it follows that $n \leqq m$. Thus, $[a^{\ell_m}, b_n] = 1$ implies $n \leqq m$. Conversely, if $n \leqq m$, then $o(b_n) | \ell_m$ and hence by Lemmas 2.3 and 2.5, $[a^{\ell_m}, b_n] = [a, b_n^{\ell_m}] = 1$. //

6.11 Corollary

The nil-2 free product of an ω-stable and a strictly-superstable group can be unstable.

Proof

Let $A = Z$ and $B = Z(p^\infty)$ in Theorem 6.10 . //

Theorem 6.10 above shows that * can fail to preserve stability when one factor is not torsion. We next consider what happens when both factors possess elements of infinite order. We shall show, in Theorem 6.13, that when one factor in such a free product also possesses a basis and satisfies a simple condition, then the group is unstable. The proof of this result was motivated by the following considerations.

By a classical result of Mal'cev, the elementary theory of the free nil-2 group, $Z*Z$, of rank 2 is undecidable. A proof of this is given in [21], section 15.3. Mal'cev shows how to introduce operations of addition and multiplication on the centre of $Z*Z$ thereby obtaining an interpretation in $Z*Z$ of the ring of integers. G. Sabbagh has observed ([29]) that this enables one to show that $Z*Z$ is unstable and we thank him for bringing this to our attention.

To make the key idea in our generalisation of this result as clear as possible we propose to prove first that the group $Z*Q$ is unstable. This group is also of independent interest since it is yet another instance of Corollary 6.11, this time with both free factors being torsion-free.

Let a be a fixed generator of Z and b any fixed element of $Q-\{0\}$. The first normal form theorem (Theorem 3.7) yields a unique representation of the elements of $Z*Q$ in the form $a^k q_1[a,q_2]$, with k an integer and q_1,q_2 belonging to Q. Furthermore, it is easy to verify that for $g \in Z*Q$:

$$[a,g] = 1 \text{ if and only if } g \in Z \text{ modulo}(Z*Q)' \qquad (1)$$

Let L deonte the language of groups and define formulae $\varphi(u_1,u_2,u_3,a,b)$, $\psi(u,v,a,b) \in L(Z*Q)$ by :

$$\varphi(u_1,u_2,u_3,a,b) \equiv \bigwedge_{1\le i\le 3} \exists v_i(u_i=[v_i,b] \wedge [a,v_i] = 1)$$

$$\wedge\ \forall v_1v_2((u_1=[a,v_1] \wedge u_2=[v_2,b]\wedge[a,v_2]=1) \rightarrow u_3=[v_2,v_1]) \ ;$$

$$\psi(u,v,a,b) \equiv \exists w\varphi(u,w,v,a,b) \ .$$

Roughly speaking, φ defines a multiplication $\cdot$ on the set of all elements of $Z*Q$ of the form $[a,b]^n$ by

$$[a,b]^m \cdot [a,b]^n = [a^n,b^m] = [a,b]^{mn} ;$$

and $\psi(u,v)$ says that "u divides v" : that is,

$$\psi(u,v) \equiv \exists w(u \cdot w = v) .$$

For each $n \in \omega$, set $c_n = [a,b]^{2^n}$. Then, we shall show that

$$(Z*Q)^+ \models \psi[c_m,c_n] \text{ if and only if } m \leqq n .$$

This, together with Corollary 1.3.14, then yields that $Z*Q$ is unstable.

First assume that $(Z*Q)^+ \models \psi[c_m,c_n]$. Then, by (1) above, there exists $h = [a^k,b]$ in $Z*Q$ such that for all x, y satisfying

(a) $c_m = [a,x]$; and

(b) $h = [y,b]$ with $[a,y] = 1$,

we have $c_n = [y,x]$. But $x = b^{2^m}$ and $y = a^k$ satisfy (a) and (b) and so, $c_n = [a^k,b^{2^m}] = [a,b]^{k\cdot 2^m}$. Thus, $[a,b]^{2^n} = [a,b]^{k\cdot 2^m}$ and so, by the uniqueness of the representation of the elements of $Z*Q$, we have $2^n = k\cdot 2^m$. It follows that $m \leqq n$.

Conversely, if $m \leqq n$, then setting $k = 2^{n-m}$ and $h = [a,b]^k$ it is easily verified, using (1) above that $(Z*Q)^+ \models \varphi[c_m,h,c_n]$. Hence, $(Z*Q)^+ \models \psi[c_m,c_n]$.

This proves the following special case of our result.

6.12 <u>Proposition</u>

The nil-2 group $Z*Q$ is unstable.

6.13 <u>Theorem</u>

Let A and B be non-torsion nil-2 groups such that A has a basis and an element of infinite pseudo-order with an abelian centraliser. Then, $A*B$ is unstable.

Proof

Let A and B satisfy the hypotheses. Then we may assume that A has a basis $\{a_i : i \in I\}$ with $o(a_i) = m_i$ and $m_0 = \infty$, such that the centraliser $\{x \in A : [x,a_0] = 1\}$ of a_0 is abelian. We may also assume, as we did for A in the proof of Theorem 6.10, that B possesses at least one element, call it b_0, with $o(b_0) = \infty$.

Using the first normal form theorem, the elements of A*B may be written in the form $ab \prod_{i\in I} [a_i,b_i]$, with $a \in A$, and $b, b_i \in B$ satisfying

$$ab \prod_{i\in I} [a_i,b_i] = 1 \text{ iff } a = b = 1 \text{ and } b_i \in H_{m_i}(B) . \qquad (1)$$

First we obtain more information about the elements of A*B commuting with a_0.

Let $g = ab$ modulo$(A*B)'$. If $[a_0,g] = 1$, then $[a_0,a][a_0,b] = 1$ and so, by (1), $[a_0,a] = 1$ and $b \in H_\infty(B)$. Thus, since $H_\infty(B) = B'$ we have $b \in B'$. It follows that $g = a$ modulo$(A*B)'$ with a commuting with a_0. Since every g of this form clearly commutes with a_0 we have

$$[a_0,g] = 1 \text{ iff } g = a \text{ modulo}(A*B)' \text{ with } [a,a_0] = 1 . \qquad (2)$$

Now we define the formulae which show that A*B is unstable:

$$\varphi(u_1,u_2,u_3,a_0,b_0) \equiv \bigwedge_{1\le i\le 3} \exists v_i (u_i = [v_i,b_0] \wedge [a_0,v_i] = 1)$$

$$\wedge\ \forall v_1 v_2 ((u_1 = [a_0,v_1] \wedge u_2 = [v_2,b_0] \wedge [a_0,v_2] = 1) \rightarrow u_3 = [v_2,v_1]) ;$$

$$\psi(u,v,a_0,b_0) \equiv \exists w \varphi(u,w,v,a_0,b_0) .$$

For each $n \in \omega$, set

$$c_n = [a_0,b_0]^{2^n} .$$

From the definitions and (1) and (2) above, we have that $(A*B)^+ \models \psi[c_m,c_n]$ if and only if there exists $a \in A$ commuting with a_0, such that for $g = [a,b_0]$ any pair $x, y \in A*B$ satisfying

(i) $c_m = [a_0,x]$; and

(ii) $g = [y,b_0]$ with $[a_0,y] = 1$,

also satisfies $c_n = [y,x]$.

First assume that $(A*B)^+ \models \psi[c_m,c_n]$. Then, there exists $g = [a,b_0]$ in $A*B$, for some $a \in A$ with $[a_0,a] = 1$ such that any pair x, y satisfying (i) and (ii) also satisfies $c_n = [y,x]$. But $x = b_0^{2^m}$ and $y = a$ satisfy (i) and (ii). Hence, $c_n = [a,b_0^{2^m}]$, or, $[a_0,b_0^{2^m}] = [a,b_0^{2^m}]$. Then, if $a = \prod_i a_i^{\alpha_i}$ modulo A' we have,

$$[a_0,b_0^{2^m\alpha_0-2^n}] \prod_{i\neq 0} [a_i,b_0^{2^m\alpha_i}] = 1 .$$

So, in particular, using (1) above, $b_0^{2^m\alpha_0-2^n} \in H_\infty(B)$. But $H_\infty(B) = B'$ and $o(b_0) = \infty$, so $2^m\alpha_0 = 2^n$ from which it follows that $m \leqq n$. So $(A*B)^+ \models \psi[c_m,c_n]$ implies that $m \leqq n$.

For the converse, assume that $m \leqq n$ and set $k = 2^{n-m}$. It is obvious that a_0^k commutes with a_0 so let $g = [a_0^k,b_0]$. Assume that $x, y \in A*B$ satisfy conditions (i) and (ii).

If $x = ab$ modulo$(A*B)'$ then (i) implies that $[a_0,b_0^{2^m}] = [a_0,a][a_0,b]$ and so, $[a_0,a][a_0,bb_0^{-2^m}] = 1$. Thus, by (1), $[a_0,a] = 1$ and $bb_0^{-2^m} \in B'$. The latter implies that $b = b_0^{2^m}$ modulo$(A*B)'$ and so

$$x = ab_0^{2^m} \text{ modulo}(A*B)' \text{ , where } [a,a_0] = 1 . \qquad (3)$$

Using (2), (ii) implies that $[a_0^k, b_0] = [y, b_0]$,

where $y = \prod_i a_i^{\alpha_i}$ modulo$(A*B)'$ and $[\prod_i a_i^{\alpha_i}, a_0] = 1$.

So, $[a_0, b_0^{\alpha_0 - k}] \prod_{i\neq 0} [a_i, b_0^{\alpha_i}] = 1$ and hence by (1), $k = \alpha_0$ and

$b_0^{\alpha_i} \in H_{m_i}(B)$, for each $i \neq 0$. Thus,

$$y = a_0^k \prod_{i\neq 0} a_i^{\alpha_i} \text{ modulo}(A*B)', \qquad (4)$$

where $b_0^{\alpha_i} \in H_{m_i}(B)$ and $[\prod_{i\neq 0} a_i^{\alpha_i}, a_0] = 1$.

Now, (3) and (4) together with (1) and the fact that $k = 2^{n-m}$ imply:

$$[y,x] = [a_0^k \prod_{i\neq 0} a_i^{\alpha_i}, ab_0^{2^m}]$$

$$= [a_0, a]^k [\prod_{i\neq 0} a_i^{\alpha_i}, a][a_0, b_0]^{k\cdot 2^m} \prod_{i\neq 0}[a_i, b_0^{2^m \alpha_i}]$$

$$= [\prod_{i\neq 0} a_i^{\alpha_i}, a][a_0, b_0]^{2^n} .$$

But both a and $\prod_{i\neq 0} a_i^{\alpha_i}$ belong to the centraliser of a_0 which, by hypothesis, is abelian. Hence $[\prod_{i\neq 0} a_i^{\alpha_i}, a] = 1$ and so, $[y,x] = [a_0, b_0]^{2^n} = c_n$, as required.

Thus, we have shown that $(A*B)^+ \models \psi[c_m, c_n]$ if and only if $m \leqq n$. By Corollary 1.3.14, it follows that $A*B$ is unstable. //

6.14 <u>Corollary</u> (of the proof of Theorem 6.13)

If A is a torsion-free nil-2 group with a basis and B is a non-torsion nil-2 group, then $A*B$ is unstable.

Proof

Let A and B satisfy the hypothesis and $\{a_i : i \in I\}$ be a basis for A with $o(a_i) = m_i$. Now, since A is torsion-free, if $m_i < \infty$ then a_i must belong to the centre of A ; otherwise, picking $a \in A$ with $[a,a_i] \neq 1$, we have $[a,a_i]^{m_i} = 1$ which is a contradiction. Furthermore, if every m_i were finite then it would follow that A was torsion. Thus, we may assume that $m_0 = \infty$, and for $i \neq 0$, either $m_i = \infty$ or if not, $a_i \in Z(A)$.
As before, we choose $b_0 \in B$ with $o(b_0) = \infty$.

The additional hypothesis (namely, that a_0 has an abelian centraliser) in the statement of the theorem is used only in the proof that $m \leqq n$ implies $(A*B)^+ \models \psi[c_m,c_n]$. Examining the equation for y (see (4), above) we see that we now have $y = a_0^k \prod_{i\neq 0} a_i^{\alpha_i}$ modulo$(A*B)'$, where if $m_i = \infty$, then $b_0^{\alpha_i} \in B'$ and and so $\alpha_i = 0$ and if $m_i < \infty$, then $b_0^{\alpha_i} \in H_{m_i}(B)$ and $a_i \in Z(A)$. Thus, with x given by equation (3) we have,

$$[y,x] = [a_0^k \prod_{a_i \in Z(A)} a_i^{\alpha_i}, ab_0^{2^m}]$$

$$= [a_0,a]^k \prod_{a_i \in Z(A)} [a_i,a]^{\alpha_i} [a_0,b_0]^{k\cdot 2^m} \prod_{a_i \in Z(A)} [a_i,b_0^{\alpha_i 2^m}]$$

$$= [a_0,b_0]^{2^n}$$

$$= c_n ,$$

as required. //

6.15 Corollary (of 6.14)

The nil-2 free product of two torsion-free nil-2 groups where one group has a basis is unstable.

6.16 Corollary (Sabbagh [29])

Every free nil-2 group of rank greater than 1 is unstable.

6.17 Corollary (of 6.16) (Sabbagh [29])

For each $n \geqq 2$, any free nilpotent group of class n and rank greater than 1 is unstable.

Proof

Let F be a free nil-n group of rank $\geqq 2$ with $n \geqq 2$. We use induction on n. For $n = 2$, this is just Corollary 6.16. For $n > 2$, set $G = F/Z(F)$ and assume the Corollary is valid for all k with $2 \leqq k < n$. Since $Z(F)$ is a definable subgroup of F, if F is κ-stable then so is G (see Theorem 1.3.15). But G is the nil-$(n-1)$ group of rank = rank F and so by the inductive hypothesis, G is unstable. Thus, F is unstable. //

This concludes the results which we have on the power of the nil-2 free product operation to preserve stability. A list of the questions we leave open is given at the end of this section.

The theorems we have proved here provide numerous examples of both stable and unstable nil-2 groups. Of course, our ultimate goal is to give a complete classification of all the stable nil-2 groups as we have already accomplished for abelian groups. The final proposition in this section is a first step in that direction.

Consider an arbitrary nil-2 group G and let A denote its associated abelian group G/G'. We recall that the subgroup $H_n(G)$, for $n \in \omega$ is defined by

$$H_n(G) = \{g^n g' : g \in G, g' \in G'\},$$

and it is easily verified that $H_n(G)$ is a normal subgroup of G. Thus, we may form the quotient group $G/H_n(G)$. Indeed it is evident that for $m, n \in \omega$ with $m | n$, $H_n(G)$ is a normal subgroup of $H_m(G)$ and so the group $H_m(G)/H_n(G)$ may also be formed. Since A is abelian, $H_n(A)$ is just the group of all a^n, with $a \in A$. To maintain consistency with Chapter 2 we shall denote this group by nA. Then, the following lemma is easily verified by elementary group theoretic means. For the definition of $tf(p,n;A)$ consult Definition 2.2.1.

6.18 Lemma

Let G be an arbitrary nil-2 group and A the abelian group G/G'. Then, for each prime p and integer $n \geqq 0$,

$$|H_{p^n}(G)/H_{p^{n+1}}(G)| = tf(p,n;A) .$$

Proof (Sketch)

For each $g \in G$ we denote the corresponding element gG' of A by $\tilde{g}$. Then, the elements of p^nA have the form $\tilde{g}^{p^n}$, with $g \in G$. Setting $H_{p^{n+1}}(G) = X$ and $p^{n+1}A = Y$ we define an isomorphism θ of $H_{p^n}(G)/X$ onto p^nA/Y by

$$\theta(g^{p^n}X) = \tilde{g}^{p^n}Y .$$

The result now follows from the definition of $tf(p,n;A)$. //

The definition following gives the natural generalisation of the term $tf(p,n;A)$ to the variety of all nil-2 groups; the lemma above justifies it.

6.19 Definition

If G is a nil-2 group and n a non-negative integer, then $tf(p,n;G) = tf(p,n;G/G')$.

Now, under the hypothesis that G' be definable, Theorem 1.3.15 shows that any stability possessed by G is passed on to the abelian group G/G'. So, using Theorem 2.4.4 and Corollary 2.4.6 we obtain the following necessary conditions for G to be (i) ω-stable, and (ii) superstable.

6.20 Proposition

If G is a nil-2 group with a definable derived group then

(i) G is ω-stable only if $tf(p,n;G) = 1$ almost everywhere;

(ii) G is superstable only if $tf(p,n;G)$ is finite almost everywhere.

6.21 Counterexample

The example we give here shows that the converse of part (ii) above is false. Let $G = Z*Q$. Then, by Proposition 6.12, G is unstable. From Lemma 5.8, it is easy to see that $G' = Z(G)$ and hence G' is definable. But for each prime p and integer $n \geqq 0$ we have $|H_{p^n}(G)/H_{p^{n+1}}(G)| = p$ and so, $tf(p,n;G)$ is finite everywhere.

6.22 Open Questions

I : Does Theorem 6.1 remain valid when we remove the hypothesis that C posess a G-suitable basis?

II : What happens in Theorem 6.3 (ii) if we do not make the assumption on compatibility?

We can give a partial answer to this question in the case where the failure to be compatible results from both A and B possessing infinite Sylow p-subgroups for the same prime p.

6.23 Proposition

If A and B are nil-2 groups both possessing a basis and, for some prime p, elements of increasing and unbounded finite pseudo-order equal to a power of p, then $A*B$ is unstable.

Proof

Let A and B satisfy the hypotheses. Then, using Lemma 2.5 (ii), we may assume that the bases for A and B include generators of increasing and unbounded finite pseudo-order equal to a power of p. Choose generators a_n, b_n, $n \in \omega$, from these bases with $o(a_n) = p^{\alpha_n}$, $o(b_n) = p^{\beta_n}$ and

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_n < \beta_n < \alpha_{n+1} < \beta_{n+1} < \ldots .$$

Clearly, this may be done. Set $x_n = a_n$ and $y_n = b_n p^{\beta_n - 1}$.

Then, using the second normal form theorem, we have

$$[x_m, y_n] = 1 \Leftrightarrow [a_m, b_n]^{p^{\beta_n - 1}} = 1$$

$$\Leftrightarrow \gcd(p^{\alpha_m}, p^{\beta_n}) \mid p^{\beta_n - 1} .$$

Now, if $m \leqq n$, then $\alpha_m < \beta_n$ and so $\gcd(p^{\alpha_m}, p^{\beta_n}) = p^{\alpha_m}$ which divides $p^{\beta_n - 1}$, since $\alpha_m \leqq \beta_n - 1$. Conversely, if $m > n$ then $\alpha_m > \beta_n$ and so $\gcd(p^{\alpha_m}, p^{\beta_n}) = p^{\beta_n}$ which clearly does not divide $p^{\beta_n - 1}$. Hence, $[x_m, y_n] = 1$ if and only if $m \leqq n$, which fact, together with Corollary 1.3.14, shows that $A*B$ is unstable. //

III : Does the nil-2 free product operation preserve ω-stability? In particular, are $Q*Q$ and $Z_2^{(\omega)} * Z_2^{(\omega)}$ stable? (See Proposition 6.9)

IV : Is the converse of Proposition 6.20 (i) valid?

If open question III has a negative answer in the case of the group $Z_2^{(\omega)} * Z_2^{(\omega)}$ then we are able to show that this question also has a negative answer. Let G be the nil-2 group $A*B$, where $A = B = Z_2^{(\omega)}$ and let $\{a_n : n \in \omega\}$, $\{b_n : n \in \omega\}$ be bases for A, B respectively. Then, by Lemma 3.3 and Theorem 3.9, the elements of G can be written in the form

$$\prod_n a_n^{\alpha_n} \prod_n b_n^{\beta_n} \prod_{m,n} [a_m, b_n]^{\gamma_{mn}}$$

where $0 \leq \alpha_n, \beta_n, \gamma_{m,n} < 2$. Now, using Lemma 5.20, one easily shows that $G' = Z(G)$ and hence is definable.

Furthermore, since

$$H_n(G) = \begin{cases} G' & n \text{ even ;} \\ G & \text{is odd ,} \end{cases}$$

it is clear that $tf(p,n;G) = 1$ almost everywhere. Thus, if the group $Z_2^{(\omega)} * Z_2^{(\omega)}$ is not ω-stable it provides a counterexample to the converse of Proposition 6.20 (i).

7. The Full Free Product

In this section we prove two theorems we obtained whilst working on the nil-2 free products and of relevance to the central questions raised in this chapter. Our first theorem shows that, unlike the direct product and the nil-2 free product, the full free product never preserves saturation; our second gives an analogous result for stability. As a consequence of the latter we prove that no free group is ω-stable, a result since superseded by Gibone [13], who has shown that no non-abelian free group is superstable. The question of whether they are stable at all remains open.

The proofs of both results are similar and derive from the observation that the abelian group Z is definable in every free product, $A*B$, with non-trivial factors. To see this we need the following well-known result from combinatorial group theory (see [20], page 187, Corollary 4.1.6). Throughout this section, $*$ denotes the full free product.

7.1 Lemma

Let A and B be non-trivial groups with $a \in A$, $b \in B$ and a, b different from 1 . Then, an element x in A*B commutes with ab if and only if $x \in gp\{ab\}$.

7.2 Theorem

If A and B are non-trivial groups, then A*B is not even 2-saturated.

Proof

Let A and B be non-trivial and $a \in A$, $b \in B$ any elements different from 1 . Then, by the lemma, $[x,ab] = 1$ if and only if $x = (ab)^n$, for some integer n . This shows that the following type with one parameter is not realised in A*B :

$$p(u) = \{u \neq (ab)^n : n \in Z\} \cup \{[u,ab] = 1\} \text{ . //}$$

7.3 Theorem

If A and B are non-trivial groups, then A*B is ω-unstable.

Proof

Let A and B be non-trivial and $a \in A$, $b \in B$ be different from 1 . Then, using the lemma, we see that the subgroup $gp\{ab\}$ is defined in A*B by the formula $[u,ab] = 1$. Thus, A*B contains a definable subgroup isomorphic to the integers which, as we know from Chapter 2, is strictly-superstable. It follows, from Theorem 1.3.15, that A*B is ω-unstable. //

7.4 Corollary (see also Gibone [13])

Every free group is ω-unstable.

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