REPRESENTATIONS OF SOME RELATIVELY FREE GROUPS IN POWER SERIES RINGS

by

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University of London, Ph.D. Pure Mathematics 1970.

ABSTRACT

W. Magnus represents a free group in a formal power series ring with no relations. We obtain power series representations for certain relatively free groups by putting various relations on the set of variables of the power series. Among those we obtain power series representations for are F/F_m (the free nilpotent groups), F/F'' (the free metabelian group), $F/(F')_3(F_3)'$, $F/(F')_3(F_4)'$, $F/[F'',F]$ (the free centre by metabelian group), $F/[F'',F,F]$ (the free centre by centre by metabelian group) and $F/[F'',F,F,F](F')_3$. In the process it is shown that $F''/[F'',F]$ is free abelian and an explicit basis is given. This basis is used to derive a basis for $[F'',F]/[F'',F,F]$ and various other subgroups of the groups, for which we obtain power series representations, are shown to be free abelian. We prove that all these groups mentioned above are residually torsion free nilpotent using their power series representations.

W. Magnus has also proved that the so-called dimension subgroups and the lower central factors of the free group coincide. In Chapter 5 we present analogues of this result of Magnus for the groups F/F'', $F/(F')_3(F_3)'$ and $F/(F')_3(F_4)'$ and in the process, compute the structure

of the lower central factors of these three groups. We conclude with a contribution to a problem of Fox on the determination of certain ideals in the group ring of the free group.

ACKNOWLEDGEMENT:

I thank my supervisor, Professor K.W. Gruenberg, for his encouragement and advice during the preparation of this thesis and also Miss Valerie Kinsella and Mrs. Lynn Parry for their excellent work in typing. I am also grateful to Royal Holloway College for enabling me to study in London.

Royal Holloway College June 1970

CONTENTS

CHAPTER 1.

Section 1: Basic Definitions: Let a and be be elements of a group G; then the commutator $[a,b] = a^{-1}b^{-1}ab$. The commutator $[a_1,\ldots,a_n]$ is defined for $n > 2$ by putting $[a_1,\ldots,a_n] = [[a_1,\ldots,a_{n-1}],a_n]$. $a_1,\ldots,a_n$ are called the entries. If none of the entries is itself a commutator, then $[a_1,\ldots,a_n]$ is said to be simple and to have weight n (a simple commutator of weight one is just an element a_1). A commutator that is not simple is called complex, and its weight is the sum of the weights of its entries. The conjugate of a by b, $a^b = b^{-1}ab$ and a and b are said to commute if $a^b = a$. The centre of G is the set of all elements x of G such that $[x,g] = 1$ for all g in G.

The upper central series of G

$$Z_o = 1 \leq Z_1(G) \leq Z_2(G) \leq \ldots \leq Z_i(G) \leq Z_{i+1}(G) \leq \ldots$$

is defined by the rule: $Z_{i+1}(G)/Z_i(G)$ is the centre of $G/Z_i(G)$.

If H and K are subgroups of G, then $[H,K]$ is the subgroup generated by all $[h,k]$ with h in H and k in K. In particular the commutator subgroup or derived group of H is $H' = [H,H]$. The lower central series of G

$$G = G_1 \geq G_2 \geq \ldots \geq G_i \geq G_{i+1} \geq \ldots$$

is defined by the rule: $G_{i+1} = [G_i,G]$, and the derived series

$G = G^0 \geq G^1 \geq \ldots \geq G^i \geq G^{i+1} \geq \ldots$

is defined by the rule: $G^{i+1} = [G^i, G^i]$. If $G_{n+1} = 1$ but $G_n \neq 1$, then G is said to be nilpotent of class n, and if $G^{m+1} = 1$ but $G^m \neq 1$ then G is said to be soluble of derived length m. The n-th lower central factor of G is G_n/G_{n+1}.

If P and Q are any properties pertaining to groups then G is said to be P by Q if there exists a normal subgroup N of G such that N has P and G/N has Q. P by P groups are called meta-P groups. G is said to be residually P if given g in G, $g \neq 1$, there exists a normal subgroup N of G, g not in N and G/N has P, or equivalently if all the normal subgroups N of G such that G/N has P intersect in the identity. It is easy to see that if G has P then G is residually P and a residually (residually P) group is just a residually P group.

If H is a subgroup of a group G, then H is said to be fully invariant in G if given any endomorphism θ of G, $H\theta \leq H$. Let F be the free group on a countable set Y. (Countable will mean either finite or denumerable.) G is said to be relatively free in the variety defined by R if G is isomorphic to F/R, where R is a fully invariant subgroup of F. (See Neumann, Hanna [12], for alternative equivalent definitions). If so, then the rank of G is the rank of F, that is, the number of elements in the free generating set Y of F. If F is the free group on $y_1,\ldots,y_r$ then a set of <u>basic commutators</u> in F is a sequence $c_1,c_2,\ldots$ that can be defined as follows. First $c_i = y_i$ $(i = 1,2,\ldots,r)$ are the basic commutators weight one. Next if the basic commutators $c_1,c_2,\ldots,c_t$ of weight less than n have been defined and put in order of non-decreasing weight, then the basic commutators of weight n consist of all commutators $[c_i,c_j]$ such that $t \geq i > j \geq 1$ such that if $c_i = [c_k,c_h]$ then $h \leq j$ and such that the sum of the weights of c_i and c_j is n. The basic commutators of weight n thus defined are put in any order at the end of the sequence. See Hall, M. [7], page 166.

Let r,s be elements of a ring R. Then the <u>additive commutator</u> $(r,s) = rs - sr$. The additive commutator $(r_1,\ldots,r_n)$ is defined for $n > 2$ by $(r_1,\ldots,r_n) = ((r_1,\ldots,r_{n-1}),r_n)$. Let $\mathbb{Z}G$ be the <u>group ring</u> of a group G over the integers. Define the <u>augmentation</u> ε, a ring homomorphism from $\mathbb{Z}G$ to $\mathbb{Z}$ by $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$, $\sum a_g g \mapsto \sum a_g$. The kernel of ε is the <u>augmentation ideal</u>

of of $\mathbb{Z}G$. A <u>right derivation</u> d on $\mathbb{Z}G$ is a mapping from $\mathbb{Z}G$ to $\mathbb{Z}G$ such that for all x and y in $\mathbb{Z}G$,
(i) $(x + y)d = xd + yd$. (ii) $(xy)d = (xd)y + (x\varepsilon)yd$.
A <u>left derivation</u> D on $\mathbb{Z}G$ is a mapping from $\mathbb{Z}G$ to $\mathbb{Z}G$ such that for all x and y in $\mathbb{Z}G$
(i) $D(x + y) = Dx + Dy$
(ii) $D(xy) = x(Dy) + (y\varepsilon)Dx$.
If F is the free group on Y, d_i will denote the <u>right Fox-derivation</u> on $\mathbb{Z}F$ given by $y_j \to \delta_{ij}$, where δ_{ij} is the Kroneker delta, and D_i will denote the <u>left Fox-derivation</u> on $\mathbb{Z}F$ given by $y_j \mapsto \delta_{ij}$ (see Fox [4] and Gruenberg [6] Chapter 4).
If X is a countable set of variables, E will denote the <u>formal power ring</u> in X over $\mathbb{Z}$ (see e.g. Magnus, Karrass and Solitar [11] p.298). A <u>monomial of degree n</u> in E is an expression of the form $p\, x_{i_1} x_{i_2} \ldots x_{i_n}$ with p in $\mathbb{Z}$ and the x_i in X. A_n is the set of monomials of degree n. Every element a of E is an infinite sum $a = a_{(o)} + a_{(1)} + a_{(2)} + \ldots$, where $a_{(r)}$ is the <u>homogeneous component</u> of a of degree r and is a finite sum of monomials of degree r. If $a_{(o)} = a_{(1)} = \ldots = a_{(m-1)} = 0$ but $a_{(m)} \neq 0$ then the <u>order</u> of a is m.
The <u>group of units</u> of E, U(E), is the set of invertible elements of E and consists of elements a in E such that

$a_{(o)} = \pm 1$. W(E) will denote the subgroup of U(E) consisting of elements a in U(E) such that $a_{(o)} = 1$. The leading term of an element a in W(E) is the first non-zero homogeneous component of a - 1.

Section 2: Basic results

F is the free group on Y and Y is in 1 - 1 correspondence with X by $y_i \leftrightarrow x_i$.

Theorem 1.1: (Gruenberg [5] Theorem 2.1(i))

A finitely generated torsion-free nilpotent group is residually a finite p-group for every prime p.

Lemma 1.2: (This is a special case of Lemma 1.9 Gruenberg [5]). Any free group in a variety is residually a finitely generated free group in the same variety.

Theorem 1.3: (Gruenberg [6] Chap.3 Theorem 1)

If $R \triangleleft F$ and R is free on a set Y, then $\mathfrak{r}$ = Ker $(\mathbb{Z}F \to \mathbb{Z}(F/R))$ is free as right (or left) $\mathbb{Z}F$-module on Y - 1.

Lemma 1.4: (Gruenberg [6] Chap.3 Lemmas 3 and 4).

If $\mathfrak{r}$ = Ker $(\mathbb{Z}F \to \mathbb{Z}(F/R))$ and if $\mathfrak{a}$ is a right ideal of $\mathbb{Z}F$, then $\mathfrak{a}/\mathfrak{a}\mathfrak{r}$ is a right F/R module and

(i) If $\mathfrak{a}$ is free as right ideal of $\mathbb{Z}F$ on S, $\mathfrak{a}/\mathfrak{a}\mathfrak{r}$ is F/R - free on S + $\mathfrak{a}\mathfrak{r}$

(ii) If $\mathfrak{a}$ is free as right ideal of $\mathbb{Z}F$ on S, $\mathfrak{b}$ is free as right ideal of $\mathbb{Z}F$ on T and is also two-sided, then $\mathfrak{a}\mathfrak{b}$ is free as right ideal of $\mathbb{Z}F$ on ST.

Corollary: If $\mathfrak{a}$ is free as right ideal of $\mathbb{Z}F$ on S then $\mathfrak{a}/\mathfrak{a}\mathfrak{f}$ is free abelian on $S + \mathfrak{a}\mathfrak{f}$.

(Ditto with right and left interchanged)

Theorem 1.5: (Magnus [10])

$(1 + \mathfrak{f}^n) \cap F = F_n$.

Theorem 1.6: (Gruenberg [6] Chap.4, Proposition 1)

If $\mathfrak{r} = \operatorname{Ker} \mathbb{Z}F \to \mathbb{Z}(F/R)$ then

$(1 + \mathfrak{f}\mathfrak{r}^n) \cap F = (1 + \mathfrak{r}^{n+1}) \cap F = R_{n+1}$.

(Case n = 1 is an old result of Schumann [14]. Cf. also Fox [4].)

We shall be particularly interested in case n = 1 of this theorem when $\mathfrak{r} = \mathfrak{a} = \operatorname{Ker}(\mathbb{Z}F \to \mathbb{Z}(F/F'))$, so that $(1 + \mathfrak{f}\mathfrak{a}) \cap F = F''$.

Theorem 1.7: (Fox [4] (4.5)) Let $\mathfrak{a}$ be any ideal of $\mathbb{Z}F$ that is contained in $\mathfrak{f}$. Then (i) α in $\mathbb{Z}F$ belongs to $\mathfrak{a}\mathfrak{f}^n$ if and only if α belongs to $\mathfrak{f}$ and $D_{i_1} D_{i_2} \ldots D_{i_r} \alpha$ belongs to $\mathfrak{a}\mathfrak{f}^{n-r}$ for all left Fox derivatives $D_{i_1}, D_{i_2}, \ldots, D_{i_r}$ and $0 \leq r \leq n$ (i.e. for any particular r between 0 and n.) (ii) α in $\mathbb{Z}F$ belongs to $\mathfrak{f}^n\mathfrak{a}$ if and only if α belongs to $\mathfrak{f}$ and $\alpha d_{i_1} d_{i_2} \ldots d_{i_r}$ belongs to $\mathfrak{f}^{n-r}\mathfrak{a}$ for all right Fox derivatives

$d_{i_1}, d_{i_2}, \ldots, d_{i_r}$ and $0 \leq r \leq n$.

Theorem 1.8: (See Gruenberg [6] Chap.4, Proposition 4. Also Fox [4]. The original presentation of F in E is due to Magnus [9].)

(i) The mapping $\delta: \mathbb{Z}F \to E$ given by $\alpha\delta = \alpha\varepsilon + \sum x_i(\alpha d_i \varepsilon) + \sum x_i x_j(\alpha d_i d_j \varepsilon) + \ldots$ is a ring monomorphism.

(ii) The mapping $p: \mathbb{Z}F \to E$ given by $\alpha p = \alpha\varepsilon + \sum x_i((D_i\alpha)\varepsilon) + \sum x_i x_j((D_i D_j \alpha)\varepsilon) + \ldots$ is a ring monomorphism.

It is easy to see that $p = \delta$ in this theorem. We argue thus. $\delta': F \to U(E)$ given by $y_i\delta' = 1 + x_i$ is a group monomorphism and δ' is the restriction of δ to F. If $\alpha = \sum a_g g$ is in $\mathbb{Z}F$ then $\alpha\delta = \sum a_g(g\delta').\delta'$ is also the restriction of p to F and $\alpha p = \sum a_g(g\delta')$. Hence $p = \delta$.

Let E_n be the ideal of elements in E of order $\geq n$. Then by Theorem 1.8 and (4.1) of Fox [4], $\mathfrak{f}^n\delta = E_n \cap (\mathbb{Z}F)\delta$. Hence by Theorem 1.5 above we get

Theorem 1.9: $(1 + E_n) \cap F\delta = (F_n)\delta$. (See Gruenberg [6] page 61).

Note: Throughout the thesis F will denote the free group on a set of variables Y (countable) and X will always be a set of variables in 1 - 1 correspondence with Y by $x_i \leftrightarrow y_i$. F will also denote the free group on X. In other words X and Y will be interchangeable and the only reason we introduce 2 sets of variables at times is to avoid confusion. We shall reserve $\mathfrak{a}$ for $\ker(\mathbb{Z}F \to \mathbb{Z}(F/F'))$ throughout the thesis. The notation given in Section 1 of this chapter will continue to be used throughout without further reference. The group of units of a power series ring P over the integers should be denoted by U(P) but we shall be more interested in $W(P) = \{a \in P/a_{(o)} = +1\}$ so that when we consider the "group of units" we shall in fact be considering W(P). In other words adopt the convention, group of units $\equiv$ W(P).

Section 3: Summary.

The aim of this thesis is to present analogues of Magnus' representation (Theorem 1.8) of the free group in a formal power series for other relatively free groups. If we put certain relations on the variables in the power series and if these relations are "homogeneous" we expect that the subgroup of the power series with relations generated by 1 + X is isomorphic to some relatively free group. The method of identifying these relatively free groups as given in a power series is usually very difficult. However we note that the power series with relations is isomorphic to the formal power series factored out by the ideal of these relations, call it D, and if we can identify $(1 + D) \cap F\delta$ then we can say what relatively free group we have under consideration.

This "Fox-type" problem can sometimes be as difficult as the original problem but at least it gives us something to get our teeth into. Once we have a group represented in a power series many of its properties are easy consequences.

In Chapter 2 Section 1, we present a power series representation for the free nilpotent groups more for completeness than it actually presents any new properties of these groups. However if anyone wants to go to the trouble, this representation can be used to present a constructive proof of the Theorem of K.W. Gruenberg that these groups are residually finite p-groups for all primes p, (see proof of Theorem 3.7) and it also seems likely that if we let the power series be over the rationals then we get a representation of the free nilpotent D-group (see Baumslag [2] for definitions of free D-groups in a variety). This latter remark also applies to the other power series representations we present in the thesis. In Chapter 2, Section 2, we present a representation of the group ring of the free abelian group in a power series analogous to

Theorem 1.8. This is fundamental for the basic idea developed in Chapter 2, Section 3. In this latter section we present the basic construction which yields Lemma 2.19 Corollary viz. Let $\mathfrak{a} = \mathrm{Ker}(\mathbb{Z}F \to \mathbb{Z}(F/F'))$ and let $P_{n,m}$ denote the power series ring in X over $\mathbb{Z}$ subject to $x_{i_1}x_{i_2}\ldots x_{i_n}(x_{i_{n+1}}x_{i_{n+2}} - x_{i_{n+2}}x_{i_{n+1}})x_{j_1}x_{j_2}\ldots x_{j_m} = 0$, then subgroup G of $W(P_{n,m})$ generated by $1 + X$ is isomorphic to $F/(1 + \mathfrak{f}^n\mathfrak{a}\mathfrak{f}^m) \cap F = C_{n,m}$ say.

In Chapter 3, Section 1, we begin to identify some of these groups $C_{n,m}$.
$C_{1,o} = F/F''$, $C_{2,o} = F/F''$, $C_{3,o} = F/(F')_3(F_3)'$, $C_{4,o} = F/(F')_3(F_4)'$. In Section 2 of this chapter we show how to prove that the group of units of these power series are residually torsion-free nilpotent and when X is finite, residually finite p-groups for all primes p, which imply the corresponding results for the groups embedded in these power series..
We begin Chapter 4 by constructing a set of generators for $F''/[F'',F]$ (which later turn out to be free generators) and use this to prove $C_{1,1} = F/[F'',F]$.
This proves that $F/[F'',F]$ (the free centre by metabelian group) is residually torsion-free nilpotent. Ridley [13] proves this in the case where F has rank two. The basis for $F''/[F'',F]$ is then used to construct a basis

for $[F'',F]/[F'',F,F]$ and hence to show that $C_{1,2} = F/[F'',F,F]$. This proves that $F/[F'',F,F]$ (the free centre by centre by metabelian group) is residually torsion-free nilpotent. We conclude Chapter 4 by showing $C_{2,2} = F/[F'',F,F,F](F')_3$, and hence that this group is residually torsion-free nilpotent.

In Chapter 5 we present a method which computes the structure of the lower central factors of the groups $C_{1,o}$, $C_{3,o}$, $C_{4,o}$ and also prove analogues of Magnus' Theorem 1.5 for these groups. We conclude Chapter 5, and the thesis, with a contribution to a problem of Fox [4] by showing $(1 + \mathfrak{f}^2\mathfrak{r}) \cap F = [R \cap F', R \cap F']R_3$.

CHAPTER 2

Section 1: The free nilpotent groups.

In this section we derive power series representations for the free nilpotent groups F/F_n.

Let G be any group generated by Y. Let $y_i - 1 = x_i$ (in $\mathbb{Z}G$) and let C_n denote the ideal in $\mathbb{Z}G$ generated by $x_{i_1}x_{i_2} \cdots x_{i_n} - x_{i_n}x_{i_1}x_{i_2} \cdots x_{i_{n-1}}$. Then $C_n \leq \mathfrak{g}^n$.

Define $\alpha \equiv \beta \pmod{C_n^*}$ if $\alpha - \beta = \gamma$ with $\gamma \in C_n$ and γ is a finite sum of terms of the form $\tau = \delta\,(x_{i_1}x_{i_2} \cdots x_{i_n} - x_{i_n}x_{i_1}x_{i_2} \cdots x_{i_{n-1}})\eta$ where the x's involved in the expression for α are the only x's involved in the expression for τ, η is either 1 or a product of x's and δ is either (i) 1, (ii) a product of x's (iii) a product of $(1 + x)^{-1}$'s, or (iv) a product of x's and $(1+x)^{-1}$'s.

Lemma 2.1: (a) $x_{i_{n+1}}x_{i_1} \cdots x_{i_n} \equiv x_{i_1}x_{i_2} \cdots x_{i_n}x_{i_{n+1}}$ mod C_n^*, if n is even.

(b) $x_{i_{n+1}}x_{i_1} \cdots x_{i_n} \equiv x_{i_1}x_{i_2} \cdots x_{i_{n-2}}x_{i_n}x_{i_{n-1}}x_{i_{n+1}}$ mod C_n^* if n is odd.

Proof: (a) $x_{i_{n+1}}x_{i_1} \cdots x_{i_n} \equiv x_{i_{n-1}}x_{i_{n+1}}x_{i_1} \cdots x_{i_{n-2}}x_{i_n} \equiv x_{i_{n-1}}x_{i_n}x_{i_{n+1}}x_{i_1} \cdots x_{i_{n-2}}$

$\equiv \cdots$

$\equiv x_{i_1}x_{i_2} \cdots x_{i_{n-1}}x_{i_n}x_{i_{n+1}}$

(b) is similar.

<u>Lemma 2.2:</u> $x_{i_r}{}^2 x_{i_1} \ldots x_{i_{n-1}} \equiv x_{i_r}{}^2 x_{i_{n-1}} x_{i_1} \ldots x_{i_{n-2}}$ mod C_n^*.

<u>Proof:</u> $x_{i_r}{}^2 x_{i_1} \ldots x_{i_{n-1}} \equiv x_{i_r} x_{i_1} \ldots x_{i_{n-2}} x_{i_r} x_{i_{n-1}}$

$\equiv x_{i_1} x_{i_2} \ldots x_{i_{n-2}} x_{i_r} x_{i_r} x_{i_{n-1}}$

$\equiv x_{i_{n-1}} x_{i_1} \ldots x_{i_{n-2}} x_{i_r} x_{i_r}$, by Lemma 2.1.

$\equiv x_{i_r} x_{i_{n-1}} x_{i_1} x_{i_2} \ldots x_{i_{n-2}} x_{i_r}$

$\equiv x_{i_r} x_{i_r} x_{i_{n-1}} x_{i_1} x_{i_2} \ldots x_{i_{n-2}}$

<u>Lemma 2.3:</u> $x_{i_j} x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_j} \ldots x_{i_{n-1}}$

$\equiv x_{i_n} x_{i_j} x_{i_1} x_{i_2} \ldots x_{i_j} \ldots x_{i_{n-1}}$, mod C_n^*.

<u>Proof:</u> $x_{i_j} x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_j} \ldots x_{i_{n-1}}$

$\equiv x_{i_j} x_{i_1} x_{i_2} \ldots x_{i_j} \ldots x_{i_{n-1}} x_{i_n}$

$\equiv x_{i_j} x_{i_2} x_{i_3} \ldots x_{i_j} \ldots x_{i_{n-1}} x_{i_n} x_{i_1}$

$\equiv x_{i_j} x_{i_j} x_{i_{j+1}} \ldots x_{i_{n-1}} x_{i_n} x_{i_1} \ldots x_{i_{j-1}}$

$\equiv x_{i_{j+1}} \ldots x_{i_{n-1}} x_{i_n} x_{i_1} \ldots x_{i_{j-2}} x_{i_j} x_{i_j} x_{i_{j-1}}$

$\equiv x_{i_{j-1}} x_{i_{j+1}} \ldots x_{i_{n-1}} x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_{j-2}} x_{i_j} x_{i_j}$, by Lemma 2.1.

$\equiv x_{i_{j-1}} x_{i_j} x_{i_{j+1}} \ldots x_{i_{n-1}} x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_{j-2}} x_{i_j}$

$\equiv x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_{n-1}} x_{i_j}$

$\equiv x_{i_n} x_{i_j} x_{i_1} x_{i_2} \ldots x_{i_{n-1}}$

Lemma 2.4: If $x_{i_k} = x_{i_j}$ then $x_i x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_j} \ldots$ $x_{i_k} \ldots x_{i_{n-1}} \equiv x_{i_n} x_i x_{i_1} x_{i_2} \ldots x_{i_j} \ldots x_{i_k} \ldots x_{i_{n-1}}$ mod C_n^*.

Proof: $x_i x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_j} \ldots x_{i_k} \ldots x_{i_{n-1}}$

$\equiv x_i x_{i_j} x_{i_{j+1}} \ldots x_{i_k} \ldots x_{i_{n-1}} x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_{j-1}}$

$\equiv x_{i_j} x_{i_{j+1}} \ldots x_{i_k} \ldots x_{i_{n-1}} x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_{j-2}} x_i x_{i_{j-1}}$

$\equiv x_{i_j} x_{i_k} x_{i_{k+1}} \ldots x_{i_{n-1}} x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_{j-2}} x_i x_{i_{j-1}} x_{i_{j+1}}$ $\ldots x_{i_{k-1}}$

$\equiv x_{i_{k+1}} \ldots x_{i_{n-1}} x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_{j-2}} x_i x_{i_{j-1}} x_{i_{j+1}} \ldots$ $x_{i_{k-2}} x_{i_j} x_{i_k} x_{i_{k-1}}$

$\equiv x_{i_{k-1}} x_{i_{k+1}} \ldots x_{i_{n-1}} x_{i_n} x_{i_1} x_{i_2} \ldots x_{i_{j-2}} x_i x_{i_{j-1}} x_{i_{j+1}}$ $\ldots x_{i_{k-2}} x_{i_j} x_{i_k}$, by Lemma 2.1.

$\equiv x_{i_{k-1}} x_{i_k} x_{i_{k+1}} \ldots x_{i_{n-1}} x_{i_n} x_{i_1} \ldots x_{i_{j-2}} x_i x_{i_{j-1}} x_{i_{j+1}}$ $\ldots x_{i_{k-2}} x_{i_j}$

$\equiv x_{i_{j-1}} x_{i_j} x_{i_{j+1}} \ldots x_{i_{k-2}} x_{i_{k-1}} x_{i_k} \ldots x_{i_{n-1}} x_{i_n} x_{i_1} \ldots$

$x_{i_{j-2}} x_i$

$\equiv x_{i_1} \ldots x_{i_{j-2}} x_{i_{j-1}} x_{i_j} x_{i_{j+1}} \ldots x_{i_{k-2}} x_{i_{k-1}} x_{i_k} \ldots x_{i_{n-1}}$

$x_{i_n} x_i$

$\equiv x_{i_n} x_{i_1} \ldots x_{i_{j-2}} x_{i_{j-1}} x_{i_j} \ldots x_{i_{k-1}} x_{i_k} \ldots x_{i_{n-1}} x_i$

$\equiv x_{i_n} x_i x_{i_1} \ldots x_{i_{j-1}} x_{i_j} \ldots x_{i_{k-1}} x_{i_k} \ldots x_{i_{n-1}}$

Lemma 2.5: $\left[y_{i_1}, y_{i_2}, \ldots, y_{i_n}\right] = \left[1 + x_{i_1}, 1 + x_{i_2}, \ldots 1 + x_{i_n}\right] \equiv 1 \bmod C_n{}^*$.

Proof: We use induction on n. Case n = 2 is clear since $\left[1 + x_{i_1}, 1 + x_{i_2}\right] = 1 + (1 + x_{i_1})^{-1}(1 + x_{i_2}{}^{-1})$ $(x_{i_1} x_{i_2} - x_{i_2} x_{i_1})$.

$\left[1 + x_{i_1}, 1 + x_{i_2}, \ldots, 1 + x_{i_n}\right] = 1 + \gamma_o\{ \left[1 + x_{i_1}, 1 + x_{i_2}, \ldots, 1 + x_{i_{n-1}}\right](1 + x_{i_n}) - (1 + x_{i_n}) \left[1 + x_{i_1}, 1 + x_{i_2}, \ldots, 1 + x_{i_{n-1}}\right]\} = b$ say where

$\gamma_o = \left[1 + x_{i_1}, \ldots, 1 + x_{i_{n-1}}\right]^{-1}(1 + x_{i_n})^{-1}$. We see that γ_o is a finite sum $\sum_{i=1}^{n} \gamma_i$ where each γ_i is a product of x's and $(1 + x)^{-1}$'s (or 1).

$b = 1 + \gamma_o\{ \left[1 + x_{i_1}, 1 + x_{i_2}, \ldots, 1 + x_{i_{n-1}}\right] x_{i_n} - x_{i_n}$

$\left[1 + x_{i_1}, 1 + x_{i_2}, \ldots, 1 + x_{i_{n-1}}\right]\}$.

By inductive hypothesis $\left[1 + x_{i_1}, \ldots, 1 + x_{i_{n-1}}\right] = 1 + \alpha$,

with $\alpha \in C_{n-1}$ and α is a finite sum of terms of the form $\gamma\,(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2 \ldots a_{n-2})\delta$ with γ a product of a_i's and $(1 + a_i)^{-1}$'s, δ product of a_i's and the a's are just members of the set $\{x_{i_1}, \ldots, x_{i_{n-1}}\}$.

$\Rightarrow$ $b = 1 + \gamma_o(\alpha\, x_{i_n} - x_{i_n}\, \alpha)$.

Let $x_{i_n} = a_n \Rightarrow b = 1 + \gamma_o(\alpha\, a_n - a_n\, \alpha)$. Hence we see

it is sufficient to prove that $\gamma(a_1a_2\ldots a_{n-1} - a_{n-1} a_1a_2\ldots a_{n-2})\delta\; a_n - a_n\gamma\; (a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})\delta \in C_n^*$. All congruences are mod C_n^*. Suppose $\gamma = \gamma_1(1 + a_i)^{-1}\delta$, where γ_1 is a product like γ and δ_1 is a product like δ. $\gamma(a_1\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})\delta$

$= \gamma_1(1 + a_i)^{-1}\delta_1(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})\delta$

$\equiv \gamma_1(1 + a_i)^{-1}(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})\delta_1\delta$

$= \gamma_1\{(1 - a_i + (1+a_i)^{-1}a_i^{\,2})(a_1a_2\ldots a_{n-1} - a_{n-1}a_1\ldots a_{n-2})\}\delta_1\delta$

$\equiv \gamma_1\{(1 - a_i)(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})\}\;\delta_1\delta$
by Lemma 2.2.

$\equiv \gamma_1(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})(1 - a_i)\;\delta_1\delta$

Hence we see that it is sufficient to show that $\gamma(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})\delta a_n - a_n \gamma (a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})\delta = d \in C_n^*$ where now γ is a product like δ i.e. is a product of a_i.

$d \equiv \gamma\delta(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})a_n - a_n \gamma\delta (a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2}) \equiv 0$ if $\gamma\delta = 1$.

Suppose $\gamma\delta = b_1\ldots b_r$, $b_i \in$ set $\{x_{i_1},x_{i_2},\ldots,x_{i_{n-1}}\}$.

$\Rightarrow d \equiv b_1b_2\ldots b_r(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})a_n - a_n b_1b_2\ldots b_r(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})$

$\equiv (b_1b_2\ldots b_ra_n - a_nb_1b_2\ldots b_r)(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})$.

We have the identity:-

$b_1b_2\ldots b_ra_n - a_nb_1\ldots b_r = (b_1a_n - a_nb_1)b_2\ldots b_r + b_1(b_2a_n - a_nb_2)b_3\ldots b_r + b_1b_2(b_3a_n - a_nb_3)b_4b_5\ldots b_r +\ldots+ b_1b_2\ldots b_{r-1}(b_ra_n - a_nb_r)$. Also $b_1b_2\ldots b_{i-1}(b_ia_n - a_nb_i)b_{i+1}\ldots b_r(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2}) \equiv b_1b_2\ldots b_{i-1}(b_ia_n - a_nb_i)(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2})b_{i+1}\ldots b_r$.

Hence it is sufficient to prove that $p = (b_ia_n - a_nb_i)(a_1a_2\ldots a_{n-1} - a_{n-1}a_1a_2\ldots a_{n-2}) \equiv 0$.

Now $b_i \in$ set $\{x_{i_1},x_{i_2},\ldots,x_{i_{n-1}}\}$ and also $a_1,a_2,\ldots,a_{n-1}$ are contained in this set. Hence amongst $b_i,a_1,a_2,\ldots,a_{n-1}$ there is an equality. If $b_i = a_j$ Lemma 2.3

shows that $p \equiv 0$ and if $a_k = a_r$ for $r \neq k$ then Lemma 2.4 shows that $p \equiv 0$. This completes the proof.

<u>Corollary 1:</u> $G_n \leq (1 + C_n) \cap G$.

<u>Corollary 2:</u> If F is the free group on Y then $(1 + C_n) \cap F = F_n$.

<u>Proof:</u> Follows immediately from Magnus' theorem 1.5.

<u>Corollary 3:</u> Let δ be the mapping of Theorem 1.8, and F the free group on Y. Now let $C_n\delta$ generate the ideal D_n in E. Then $(1 + D_n) \cap F\delta = (F_n)\delta$.

<u>Proof:</u> Follows immediately from Theorem 1.9.

Corollary 3 gives us a power series representation for the free nilpotent group of class n-1, F/F_n, which we state as Theorem 2.6 below.

<u>Theorem 2.6:</u> Let K_n denote the power series ring in X over Z subject to the relations $x_{i_1}x_{i_2}\ldots x_{i_n} - x_{i_n}x_{i_1}x_{i_2}\ldots x_{i_{n-1}} = 0$ (i.e. let the ideal generated by the n-1 homogeneous part be central) then subgroup of $W(K_n)$ generated by $1 + X$ is isomorphic to F/F_n under the mapping $y_i \mapsto 1 + x_i$.

In particular this gives the well known power series representation of F/F' viz. K_2 the power series ring in commuting indeterminates. In the next section we show how this representation of F/F' can be extended to a representation of the group ring $\mathbb{Z}(F/F')$.

Section 2: The free abelian group ring.

Let $G = F/F'$ be the free abelian group on Y and K_2 as in Section 1. $\mathfrak{g}$ is the augmentation ideal of G and ϕ': $G \to W(K_2)$ the embedding of G in K_2.

Lemma 2.7: $\mathfrak{g}^j/\mathfrak{g}^{j+1}$ is freely generated as $\mathbb{Z}$-module by $\{(y_{i_1} - 1)^{\alpha_{i_1}}(y_{i_2} - 1)^{\alpha_{i_2}} \ldots (y_{i_t} - 1)^{\alpha_{i_t}} / \alpha_{i_k} \in \mathbb{Z} - \{0\},\ i_1 < i_2 < \ldots < i_t$ and $\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_t} = j\}$

Proof: Clearly this set generates $\mathfrak{g}^j/\mathfrak{g}^{j+1}$. Suppose $\sum n_{i_1 i_2 \ldots i_t}^{\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_t}} (y_{i_1} - 1)^{\alpha_{i_1}}(y_{i_2} - 1)^{\alpha_{i_2}} \ldots (y_{i_t} - 1)^{\alpha_{i_t}}$ $= w \in \mathfrak{g}^{j+1}$, $n_{i_1 i_2 \ldots i_t}^{\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_t}} \in \mathbb{Z}$. Then $w = \sum m_{j_1 j_2 \ldots j_s}^{\beta_{j_1}\beta_{j_2}\ldots\ldots\beta_{j_s}} (y_{j_1} - 1)^{\beta_{j_1}}(y_{j_2} - 1)^{\beta_{j_2}} \ldots (y_{j_s} - 1)^{\beta_{j_s}}$, with $m_{j_1 j_2 \ldots j_s}^{\beta_{j_1}\beta_{j_2}\ldots\beta_{j_s}} \in \mathbb{Z}G$ and $\beta_{j_1} + \beta_{j_2} + \ldots + \beta_{j_s} = j + 1$. We can extend ϕ' to a ring homomorphism $\phi : \mathbb{Z}G \to K_2$ by $\phi: \sum a_g g \mapsto \sum a_g(g\phi')$. It is clear that an element of $\mathfrak{g}^{j+1}$ will be mapped by ϕ into the ideal of elements in K_2 of order $\geq j + 1$. This implies that $\sum n_{i_1 i_2 \ldots i_t}^{\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_t}} x_{i_1}^{\alpha_{i_1}} x_{i_2}^{\alpha_{i_2}} \ldots x_{i_t}^{\alpha_{i_t}} = 0 \Rightarrow$

$n_{i_1 i_2 \ldots i_t}^{\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_t}} = 0$, completing the proof. As corollaries to this we have the following Lemmas.

Lemma 2.8: ϕ as defined in Lemma 2.7 is mono.

Proof: If $a\phi = 0 \Rightarrow a \in \bigcap_{i=1}^{\infty} \mathfrak{g}^i$. By Hartley [8] Lemma 18, the intersection of the powers of the augmentation ideal of a torsion-free nilpotent group is zero. Therefore $a = 0$.

Proposition 2.9: $\mathcal{G}r\,(\mathbb{Z}G, \mathfrak{g}^k) = \bigoplus_{j\geq 0} \mathfrak{g}^j/\mathfrak{g}^{j+1}$ is isomorphic to the polynomial ring $\mathbb{Z}[X]$ in commuting indeterminates.

Proof: Note that $\mathbb{Z}[X]$ is just the direct sum of the homogeneous components of K_2. Hence an isomorphism $\psi: \mathcal{G}r(\mathbb{Z}G, \mathfrak{g}^k) \to \mathbb{Z}\,X$ is given by $\psi: (y_{i_1} - 1)^{\alpha_{i_1}}(y_{i_2} - 1)^{\alpha_{i_2}} \ldots (y_{i_t} - 1)^{\alpha_{i_t}} + \mathfrak{g}^{j+1} \mapsto x_{i_1}^{\alpha_{i_1}} x_{i_2}^{\alpha_{i_2}} \ldots x_{i_t}^{\alpha_{i_t}}$, by Lemma 2.7.

Proposition 2.10: $\varprojlim \mathbb{Z}G/\mathfrak{g}^k = K_2$

Proof: Comes directly from definition of the inverse limit and Lemma 2.7.

(For definition of the inverse limit $\varprojlim$, see e.g. Eilenberg and Steenrod [3]).

Section 3: Derivations in E.

Define a linear mapping $\bar{d}_j$ of E into E by

$1\,\bar{d}_j = 0,$

$(x_{i_1}x_{i_2}\ldots x_{i_n})\bar{d}_j = \delta_{i_1 j}x_{i_2}\ldots x_{i_n}$. Then $\bar{d}_j : A_n \to A_{n-1}$.

Lemma 2.11: For a and b in E,

$(ab)\bar{d}_j = (a\bar{d}_j)b + (b\bar{d}_j)a_{(o)}$

Proof: $((ab)\bar{d}_j)_{(r)} = (ab)_{(r+1)}\bar{d}_j$

$$= \left(\sum_{i=0}^{r+1} a_{(i)}b_{(r+1-i)}\right)\bar{d}_j$$

$$= a_{(o)}(b_{(r+1)}\bar{d}_j) + \left(\sum_{i=1}^{r+1} a_{(i)}b_{(r+1-i)}\right)\bar{d}_j$$

$$= a_{(o)}(b_{(r+1)}\bar{d}_j) + \sum_{i=1}^{r+1} (a_i\bar{d}_j)b_{(r+1-i)}$$

$$= a_{(o)}(b\bar{d}_j)_{(r)} + \sum_{i=1}^{r+1} (a\bar{d}_j)_{(i-1)}b_{(r+1-i)}$$

$$= a_{(o)}(b\bar{d}_j)_{(r)} + \sum_{i=0}^{r} (a\bar{d}_j)_{(i)}b_{(r-i)}$$

$$= a_{(o)}(b\bar{d}_j)_{(r)} + (a\bar{d}_j b)_{(r)}$$

$$= \{a_{(o)}(b\bar{d}_j) + (a\bar{d}_j)b\}_{(r)}$$

$$\Rightarrow (ab)\bar{d}_j = (a\bar{d}_j)b + (b\bar{d}_j)a_{(o)}.$$

Lemma 2.12: For all α in $\mathbb{Z}F$, $(\alpha d_j)\delta = (\alpha\ \delta)\bar{d}_j$ (δ is the δ of Theorem 1.8).

Proof: $\{(\alpha d_j)\delta\}_{(r)}$

$$= \sum_{i_1, i_2, \ldots i_r} x_{i_1} x_{i_2} \ldots x_{i_r} \, (\alpha d_j d_{i_1} d_{i_2} \ldots d_{i_r} \varepsilon)$$

$$= \{ \sum_{i_1, i_2, \ldots i_r} x_{i_1} x_{i_2} \ldots x_{i_r} \, (\alpha d_j d_{i_1} d_{i_2} \ldots d_{i_r} \varepsilon) \} \bar{d}_j$$

$$= \{ \sum_{i, i_1, i_2, \ldots i_r} x_i x_{i_1} x_{i_2} \ldots x_{i_r} (\alpha d_i d_{i_1} d_{i_2} \ldots d_{i_r} \varepsilon) \} \bar{d}_j$$

$$= \{(\alpha\delta)\bar{d}_j\}_{(r)}$$

$\Rightarrow (\alpha d_j)\delta = (\alpha\delta)\bar{d}_j$.

Lemma 2.13: Let $\mathcal{B}$ be an ideal of $\mathbb{Z}F$ that is contained in $\mathcal{F}$ and let $(\mathcal{B})\delta$ generate the ideal D_o in E such that $D_o \cap (\mathbb{Z}F)\delta = (\mathcal{B})\delta$. If $(\mathcal{F}^n\mathcal{B})\delta$ generates the ideal D_n in E, then $D_n \cap (\mathbb{Z}F)\delta = (\mathcal{F}^n\mathcal{B})\delta$.

Proof: By induction on n. Case n = 0 is part of hypothesis. $(\mathcal{F}^n\mathcal{B})\delta \leq D_n \cap (\mathbb{Z}F)\delta$ is clear. Suppose $a \in \mathbb{Z}F$ and $a\delta \in D_n$

$\Rightarrow a\delta = \sum \underline{\alpha}_i \; \beta_i \; \gamma_i \; \underline{\delta}_i$, $\underline{\alpha}_i$ and $\underline{\delta}_i \in E$,
$\beta_i \in (\mathcal{F})\delta$, $\gamma_i \in (\mathcal{F}^{n-1}\mathcal{B})\delta$.

$(ad_j)\delta = (a\delta)\bar{d}_j$ by Lemma 2.12

$= (\sum \underline{\alpha}_i \; \beta_i \; \gamma_i \; \underline{\delta}_i)\bar{d}_j$

$= \sum\{(\underline{\alpha}_i\beta_i)\bar{d}_j\gamma_i \; \underline{\delta}_i + \gamma_i \; \underline{\delta}_i\bar{d}_j(\underline{\alpha}_i\beta_i)_{(o)}\}$

$= \sum(\underline{\alpha}_i \beta_i)\bar{d}_j \gamma_i \underline{\delta}_i$ which is in D_{n-1}. Hence by induction

$(ad_j)\delta \in (\mathfrak{f}^{n-1}\mathfrak{G})\delta$, for all d_j.

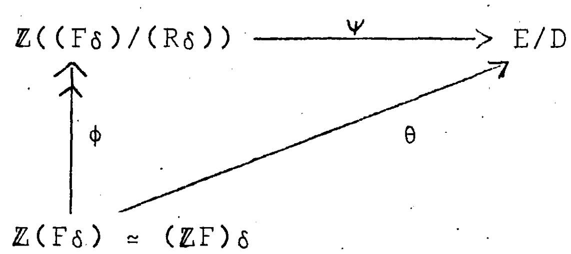
$\Rightarrow ad_j \in \mathfrak{f}^{n-1}\mathfrak{G}$, for all d_j.

$\Rightarrow a \in \mathfrak{f}^{n}\mathfrak{G}$. (By Theorem 1.7).

$\Rightarrow a\delta \in (\mathfrak{f}^{n}\mathfrak{G})\delta$

<u>Lemma 2.14:</u> Let $\mathfrak{r} = \operatorname{Ker} \mathbb{Z}F \to \mathbb{Z}(F/R)$ and let $\mathfrak{r}\delta$ generate the ideal D in E. Then $\mathbb{Z}F\delta \cap D = \mathfrak{r}\delta$ if and only if the homomorphism ψ': $(F\delta)/(R\delta) \to E/D$ given by $1 + x \mapsto 1 + x + D$ extends to a ring monomorphism ψ: $\mathbb{Z}((F\delta)/(R\delta)) \to E/D$ (ψ: $\sum a_g g \mapsto \sum a_g(g\psi')$).

<u>Proof:</u>



θ is induced by the natural map $E \to E/D$. This diagram commutes. (ϕ is onto). That is, $\phi\psi = \theta$. Suppose $(\mathbb{Z}F)\delta \cap D = \mathfrak{r}\delta$ and let $a\phi \in \operatorname{Ker} \psi$, $a \in \mathbb{Z}F\delta$.

$\Rightarrow a\phi\psi = 0 \Rightarrow a\theta = 0 \Rightarrow a \in D$

$\Rightarrow a \in D \cap \mathbb{Z}F\delta = \mathfrak{r}\delta \Rightarrow a\phi = 0$

$\Rightarrow \psi$ is a monomorphism.

Suppose ψ is a monomorphism. Clearly $\mathfrak{r}\delta \leq \mathbb{Z}F\delta \cap D$.

Let $a\delta \in \mathbb{Z}F\delta \cap D$, $a \in \mathbb{Z}F$, $\Rightarrow (a\delta)\phi\ \psi = (a\delta)\theta = 0$

$\Rightarrow (a\delta)\phi = 0 \quad \Rightarrow a\delta \in \mathfrak{f}\delta$

<u>Lemma 2.15</u>: Let $\mathfrak{a} = \text{Ker}\ \mathbb{Z}F \to \mathbb{Z}(F/F')$ and suppose $(\mathfrak{f}^n\mathfrak{a})\delta$ generates the ideal D_n in E. Then $\mathbb{Z}F\delta \cap D_n = (\mathfrak{f}^n\mathfrak{a})\delta$.

<u>Proof</u>: By Lemmas 2.8, 2.14 and 2.13.

<u>Corollary</u>: Let P_n be the power series ring in X over $\mathbb{Z}$ subject to the relations $x_{i_1}x_{i_2}\ldots x_{i_n}(x_{i_{n+1}}x_{i_{n+2}} - x_{i_{n+2}}x_{i_{n+1}}) = 0$, then the subgroup G of $W(P_n)$ generated by $1 + X$ is isomorphic to $F/(1 + \mathfrak{f}^n\mathfrak{a}) \cap F$.

<u>Proof</u>: Is clear from Lemma since $\mathfrak{f}^n\mathfrak{a}$ is the ideal on $(y_{i_1} - 1)(y_{i_2} - 1)\ldots(y_{i_n} - 1)\{(y_{i_{n+1}} - 1)(y_{i_{n+2}} - 1) - (y_{i_{n+2}} - 1)(y_{i_{n+1}} - 1)\}$.

We shall show below how this enables us to prove that these groups are residually torsion free nilpotent. Note that these groups are relatively free since $(1 + \mathfrak{f}^n\mathfrak{a}) \cap F$ is a fully invariant subgroup of F.

Define a linear mapping $\overline{D}_j$ of E into E by $\overline{D}_j(x_{i_1}x_{i_2}\ldots x_{i_n}) = \delta_{i_n j}\, x_{i_1}\ldots x_{i_{n-1}}$. $\overline{D}_j 1 = 0$.

<u>Lemma 2.16</u>: For a and b in E, $\overline{D}_j(ab) = b_{(o)}\overline{D}_j(a) + a\overline{D}_j(b)$.

Proof: $\{\overline{D}_j(ab)\}_{(r)} = \overline{D}_j\{(ab)_{(r+1)}\}$

$$= \overline{D}_j\left(\sum_{i=0}^{r+1} a_{(i)}b_{(r+1-i)}\right)$$

$$= \overline{D}_j\left(\sum_{i=0}^{r} a_{(i)}b_{(r+1-i)} + a_{(r+1)}b_{(o)}\right)$$

$$= \sum_{i=0}^{r} a_{(i)}\overline{D}_j(b_{(r+1-i)}) + b_{(o)}\overline{D}_j(a_{(r+1)})$$

$$= \sum_{i=0}^{r} a_{(i)}(\overline{D}_j(b))_{(r-i)} + b_{(o)}(\overline{D}_j a)_{(r)}$$

$$= \{a\,\overline{D}_j(b)\}_{(r)} + b_{(o)}(\overline{D}_j a)_{(r)}$$

$$= \{a\,\overline{D}_j(b) + b_{(o)}\overline{D}_j(a)\}_{(r)}$$

$$\Rightarrow \overline{D}_j(ab) = b_{(o)}\overline{D}_j(a) + a\,\overline{D}_j(b)$$

Lemma 2.17: For all α in $\mathbb{Z}F$, $(D_j\alpha)\delta = \overline{D}_j(\alpha\delta)$

Proof: Similar to Lemma 2.12.

Note first of all that $\delta = p$ in Theorem 1.8.

$$\{(D_j\alpha)\delta\}_{(r)} = \sum_{i_1 i_2,\ldots,i_r} x_{i_1}x_{i_2}\cdots x_{i_r}(D_{i_1}D_{i_2}\cdots D_{i_r}D_j\alpha)\varepsilon$$

$$= \overline{D}_j \sum_{i_1,i_2,\ldots i_r} x_{i_1}x_{i_2}\cdots x_{i_r}x_{i_j}(D_{i_1}D_{i_2}\cdots D_{i_r}D_j\alpha)\varepsilon$$

$$= \overline{D}_j \sum_{i_1,i_2,\ldots,i_r,i} x_{i_1}x_{i_2}\cdots x_{i_r}x_i(D_{i_1}D_{i_2}\cdots D_{i_r}D_i\alpha)\varepsilon$$

$$= \overline{D}_j\,(\alpha\delta)_{(r+1)}$$

$= \{\overline{D}_j(\alpha\delta)\}_{(r)}$

$\Rightarrow (D_j\,\alpha)\,\delta = \overline{D}_j(\alpha\delta).$

Lemma 2.18: Let $\mathcal{B}$ be an ideal of $\mathbb{Z}F$ that is contained in $\mathfrak{f}$ and let $(\mathcal{B})\delta$ generate the ideal B_o in E such that $B_o \cap \mathbb{Z}F\delta = \mathcal{B}\,\delta$. Then if $(\mathcal{B}\mathfrak{f}^n)\delta$ generates the ideal B_n in E,

$$B_n \cap \mathbb{Z}F\delta = (\mathcal{B}\mathfrak{f}^n)\delta.$$

Proof: Induction on n.

Case n=0 is part of hypothesis.

$(\mathcal{B}\mathfrak{f}^n)\delta \leq B_n \cap \mathbb{Z}F\delta$ is clear. Suppose $a \in \mathbb{Z}F$ and $a\delta \in B_n \Rightarrow a\delta = \sum \underline{\alpha}_i\beta_i\gamma_i\underline{\delta}_i$, where $\underline{\alpha}_i, \underline{\delta}_i$ are in E, $\beta_i \in (\mathcal{B}\mathfrak{f}^{n-1})\delta$ and $\gamma_i \in \mathfrak{f}\,\delta$.

$$\begin{aligned}(D_j a)\delta &= \overline{D}_j(a\delta), \text{ by Lemma 2.17}\\ &= \overline{D}_j(\textstyle\sum \underline{\alpha}_i\beta_i\gamma_i\underline{\delta}_i)\\ &= \textstyle\sum \overline{D}_j(\underline{\alpha}_i\beta_i\gamma_i\underline{\delta}_i)\\ &= \textstyle\sum \underline{\alpha}_i\beta_i\overline{D}_j(\gamma_i\underline{\delta}_i)+(\gamma_i\underline{\delta}_i)_{(o)}\overline{D}_j(\underline{\alpha}_i\beta)\\ &= \textstyle\sum \underline{\alpha}_i\beta_i\overline{D}_j(\gamma_i\underline{\delta}_i), \text{ which is in } B_{n-1}.\end{aligned}$$

Hence by induction

$(D_j a)\delta \in (\mathcal{B}\mathfrak{f}^{n-1})\delta$, for all D_j.

$\Rightarrow D_j a \in \mathcal{B}\mathfrak{f}^{n-1}$, for all D_j

$\Rightarrow a \in \mathcal{B}\mathfrak{f}^n$, by Theorem 1.7.

$\Rightarrow a\delta \in (\mathcal{B}\mathfrak{f}^n)\delta$

Lemma 2.19: Let $\mathfrak{a} = \operatorname{Ker} \mathbb{Z}F \to \mathbb{Z}(F/F')$. If $(\mathfrak{f}^n\mathfrak{a}\mathfrak{f}^m)_\delta$ generates the ideal $D_{n,m}$ in E then $(\mathfrak{f}^n\mathfrak{a}\mathfrak{f}^m)_\delta = D_{n,m} \cap \mathbb{Z}F\delta$.

Proof: From lemmas 2.15 and 2.18.

Corollary: Let $P_{n,m}$ be the power series ring in X over $\mathbb{Z}$ subject to

$$x_{i_1}x_{i_2}\dots x_{i_n}(x_{i_{n+1}}x_{i_{n+2}} - x_{i_{n+2}}x_{i_{n+1}})x_{j_1}x_{j_2}\dots x_{j_m} = 0$$

then subgroup G of $W(P_{n,m})$ generates by 1 + X is isomorphic to $F/(1+\mathfrak{f}^n\mathfrak{a}\mathfrak{f}^m)\cap F$.

Proof: Is clear since $\mathfrak{f}^n\mathfrak{a}\mathfrak{f}^m$ is the ideal on

$$(y_{i_1}-1)(y_{i_2}-1)\dots(y_{i_n}-1)(y_{i_{n+1}}y_{i_{n+2}} - y_{i_{n+2}}y_{i_{n+1}}).$$

$$(y_{j_1}-1)(y_{j_2}-1)\dots(y_{j_m}-1).$$ /

Note that $F/(1+\mathfrak{f}^n\mathfrak{a}\mathfrak{f}^m)\cap F$ is a relatively free group since $(1+\mathfrak{f}^n\mathfrak{a}\mathfrak{f}^m)\cap F$ is a fully invariant subgroup of F.

The problem now will be to identify the groups $(1+\mathfrak{f}^n\mathfrak{a}\mathfrak{f}^m)\cap F$. Let $C_{n,m} = F/(1+\mathfrak{f}^n\mathfrak{a}\mathfrak{f}^m)\cap F$ and we shall continue to use this notation in the following chapters.

CHAPTER 3

Section 1: $C_{1,0}, C_{2,0}, C_{3,0}$ and $C_{4,0}$.

Lemma 3.1: $(1 + \mathfrak{f}^2\mathfrak{a}) \cap F = (1 + \mathfrak{f}\mathfrak{a}) \cap F = F''$

Proof: $F'' \leq (1 + \mathfrak{a}^2) \cap F \leq (1 + \mathfrak{f}^2\mathfrak{a}) \cap F$

$\leq (1 + \mathfrak{f}\mathfrak{a}) \cap F = F''$ by Theorem 1.6.

Corollary: Let Q denote the power series ring in X over Z subject to $x_{i_1}(x_{i_2}x_{i_3} - x_{i_3}x_{i_2}) = 0$ then subgroup of W(Q) generated by 1 + X is isomorphic to F/F".

Let F be free on Y. Before proceeding we introduce some well known commutator identities. If a,b and c are any elements of a group G then

1. $[a,bc] = [a,c]\,[a,b]^c = [a,c]\,[a,b]\,[a,b,c]$.
2. $[ab,c] = [a,c]^b\,[b,c] = [a,c]\,[a,c,b]\,[b,c]$.
3. $[a^{-1},b] = [a,b]^{-a^{-1}} = [a,b]^{-1}\,[[a,b]^{-1},a^{-1}]$.
4. $[a,b^{-1}] = [a,b]^{-b^{-1}} = [a,b]^{-1}\,[[a,b]^{-1},b^{-1}]$.
5. $ab = ba[a,b]$.

We shall refer to these as (R).

Lemma 3.2: $(1 + \mathfrak{f}^3\mathfrak{a}) \cap F = (F')_3(F_3)'$.

Proof: $(F')_3 \leq (1 + \mathfrak{a}^3) \cap F \leq (1 + \mathfrak{f}^3\mathfrak{a}) \cap F$.

Let $f_1, f_2 \in F_3$ then $f_1 - 1 \in \mathfrak{f}^3 \cap \mathfrak{a}$ and $f_2 - 1 \in \mathfrak{f}^3 \cap \mathfrak{a}$.

$[f_1,f_2] = 1 + f_1^{-1}f_2^{-1}\{(f_1-1)(f_2-1) - (f_2-1)(f_1-1)\} \in 1 + \mathfrak{f}^3\mathfrak{a}$.

Hence $(F')_3(F_3)' \leq (1 + \mathfrak{f}^3\mathfrak{a}) \cap F$.

Suppose $a \in (1 + \mathfrak{f}^3\mathfrak{a}) \cap F$. Then $a \in F''$ by theorem 1.6 (all congruences are mod $(F')_3(F_3)'$ unless otherwise stated).

$$\Rightarrow a = \Pi \left[[y_{i_1}, y_{i_2}]^{\alpha_i}, [y_{i_3}, y_{i_4}]^{\beta_i} \right]^{\gamma_i} \text{ with } \alpha_i, \beta_i \in F,$$

$\gamma_i \in F'$.

$$\Rightarrow a \equiv \Pi \left[[y_{i_1}, y_{i_2}]^{\alpha_i}, [y_{i_3}, y_{i_4}]^{\beta_i} \right]$$

$$= \Pi \left[[y_{i_1}, y_{i_2}][y_{i_1}, y_{i_2}, \alpha_i], [y_{i_3}, y_{i_4}][y_{i_3}, y_{i_4}, \beta_i] \right]$$

$$\equiv \Pi \left[[y_{i_1}, y_{i_2}], [y_{i_3}, y_{i_4}] \right] \left[[y_{i_1}, y_{i_2}], [y_{i_3}, y_{i_4}, \beta_i] \right]$$

$$\left[[y_{i_1}, y_{i_2}, \alpha_i], [y_{i_3}, y_{i_4}] \right], \text{ by (R).}$$

$$\equiv \Pi \left[[y_{i_1}, y_{i_2}], [y_{i_3}, y_{i_4}] \right] \left[[y_{i_1}, y_{i_2}], [y_{i_3}, y_{i_4}, \beta_i] \right]$$

$$\left[[y_{i_3}, y_{i_4}], [y_{i_1}, y_{i_2}, \alpha_i]^{-1} \right], \text{by (R).}$$

Call this (A). Cancel inverse pairs. By (R) we see that a is congruent to a product type (A) (where now we allow the double commutators in (A) to have negative sign) in which the 2-commutators are basic ($i_1 > i_2$, $i_3 > i_4$). Cancel inverse pairs after this reduction and call the new product obtained (B). We proceed by induction on the number of distinct (basic) 2-commutators in (B) to show $a \equiv 1$. If no 2-commutator is left after cancellation we are through. Let $[y_{i_1}, y_{i_2}]$ be a particular 2-commutator in (B). We may now collect in one commutator all the commutators in (B) involving $[y_{i_1}, y_{i_2}]$ (modulo $(F')_3(F_3)'$) using (R). Thus

$$a \equiv \left[[y_{i_1}, y_{i_2}],\ \Pi[y_{i_3}, y_{i_4}]^{\varepsilon}\ \Pi[y_{j_1}, y_{j_2}, \alpha_j]^{\eta}\right]$$

Π (type (B) $[y_{i_1}, y_{i_2}]$ not a 2-commutator), with $\varepsilon = \pm 1$, $\eta = \pm 1$, $[y_{i_3}, y_{i_4}] \neq [y_{i_1}, y_{i_2}]$. Now for f_1, f_2 and $f_3 \in F$, $[f_1, f_2] \equiv 1 + (f_1 f_2 - f_2 f_1) \bmod \mathfrak{f}^3$ and $[f_1, f_2, f_3] \equiv 1 \bmod \mathfrak{f}^3$. Hence since $a - 1 \in \mathfrak{f}^3\mathfrak{a}$ and $(F')_3(F_3)' \leq 1 + \mathfrak{f}^3\mathfrak{a}$,

$$\{(y_{i_1}-1)(y_{i_2}-1) - (y_{i_2}-1)(y_{i_1}-1)\}\{\Pi[y_{i_3}, y_{i_4}]^{\varepsilon}$$

$$\Pi[y_{j_1}, y_{j_2}, \alpha_j]^{\eta} - 1\} - \sum \varepsilon\{(y_{i_3}-1)(y_{i_4}-1) - (y_{i_4}-1)(y_{i_3}-1)\}$$

$$\{([y_{i_1}, y_{i_2}] - 1)\} + \sum\{(y_{k_1}-1)(y_{k_2}-1) - (y_{k_2}-1)(y_{k_1}-1)\}(\gamma_k - 1)$$

$= \alpha$ say, is contained in $\mathfrak{f}^3\mathfrak{a}$, where $[y_{k_1}, y_{k_2}] \neq [y_{i_1}, y_{i_2}]$, and $\gamma_k \in F$.

By Theorem 1.7, $\alpha d_{i_1} d_{i_2} \in \mathfrak{f}\mathfrak{a}$.

$$\Rightarrow \Pi[y_{i_3}, y_{i_4}]\ \Pi[y_{j_1}, y_{j_2}, \alpha_j]^{\eta} - 1 \in \mathfrak{f}\mathfrak{a}.$$

$\Rightarrow \Pi[y_{i_3}, y_{i_4}] \; \Pi[y_{j_1}, y_{j_2}, \alpha_j]^{n} \; \varepsilon F''$ by Theorem 1.6.

$\Rightarrow a \equiv \Pi$ (type (B) with one less distinct 2-commutator),

$\Rightarrow$ by induction, $a \equiv 1$.

Corollary: Let P be the power series ring in X over $\mathbb{Z}$ subject to $x_{i_1}x_{i_2}x_{i_3}(x_{i_4}x_{i_5} - x_{i_5}x_{i_4}) = 0$ then subgroup G of $W(P)$ generated by $1 + X$ is isomorphic to $F/(F')_3(F_3)'$.

Lemma 3.3: $(1 + \mathfrak{f}^4\mathfrak{a}) \cap F = (F')_3(F_4)'$.

Proof: $(F')_3 \leq (1 + \mathfrak{a}^3) \cap F \leq (1 + \mathfrak{f}^4\mathfrak{a}) \cap F$ since $\mathfrak{a} \leq \mathfrak{f}^2$.

Let f_1, f_2 be in F_4 then

$[f_1, f_2] = 1 + f_1^{-1}f_2^{-1}\{(f_1-1)(f_2-1)-(f_2-1)(f_1-1)\}$ and

hence $(F_4)' \leq (1 + \mathfrak{f}^4\mathfrak{a}) \cap F$

$\Rightarrow (F')_3(F_4)' \leq (1 + \mathfrak{f}^4\mathfrak{a}) \cap F$.

Suppose $a - 1 \; \varepsilon \; \mathfrak{f}^4\mathfrak{a}$ with $a \; \varepsilon \; F \Rightarrow a \; \varepsilon \; (F')_3(F_3)'$ by Lemma 3.2. (All congruences will be mod $(F')_3(F_4)'$ unless otherwise stated).

$\Rightarrow a \equiv f$ with $f \; \varepsilon \; (F_3)'$

$\Rightarrow a \equiv \Pi[\alpha_{i_1}^{\eta_{i_1}} \cdots \alpha_{i_n}^{\eta_{i_n}} f_i, \; \beta_{j_1}^{\varepsilon_{j_1}} \cdots \beta_{j_m}^{\varepsilon_{j_m}} g_j]^{\ell_{i,j}}$,

α_{i_ℓ}, β_{j_k} are basic commutators weight 3, $i_1 < i_2 < \ldots < i_n$ and $j_1 < j_2 < \ldots < j_m$ in the ordering of the basic commutators weight 3, η_{i_ℓ}, ε_{j_k} are in $\mathbb{Z}$, f_i, g_j are in F_4 and the $\ell_{i,j}$ are in F_3.

$$\Rightarrow a \equiv \Pi \left[\alpha_{i_1}^{\eta_{i_1}} \alpha_{i_2}^{\eta_{i_2}} \cdots \alpha_{i_n}^{\eta_{i_n}} f_i, \beta_{j_1}^{\varepsilon_{j_1}} \beta_{j_2}^{\varepsilon_{j_2}} \cdots \beta_{j_m}^{\varepsilon_{j_m}} g_j\right]$$

$$\equiv \Pi \left[\alpha_{i_1}^{\eta_{i_1}} \alpha_{i_2}^{\eta_{i_2}} \cdots \alpha_{i_n}^{\eta_{i_n}}, \beta_{j_1}^{\varepsilon_{j_1}} \beta_{j_2}^{\varepsilon_{j_2}} \cdots \beta_{j_n}^{\varepsilon_{j_n}} g_j\right]$$

$$\left[f_i, \beta_{j_1}^{\varepsilon_{j_1}} \beta_{j_2}^{\varepsilon_{j_2}} \cdots \beta_{j_m}^{\varepsilon_{j_m}} g_j\right], \text{by (R)}$$

$$\equiv \Pi \left[\alpha_{i_1}^{\eta_{i_1}} \alpha_{i_2}^{\eta_{i_2}} \cdots \alpha_{i_n}^{\eta_{i_n}}, \beta_{j_1}^{\varepsilon_{j_1}} \beta_{j_2}^{\varepsilon_{j_2}} \cdots \beta_{j_m}^{\varepsilon_{j_m}} g_j\right]$$

$$\left[f_i, \beta_{j_1}^{\varepsilon j_1} \beta_{j_2}^{\varepsilon j_2} \cdots \beta_{j_m}^{\varepsilon j_m}\right] - (B), (\text{by (R)})$$

Cancel inverse pairs and after this we proceed by induction on the number of distinct basic commutators weight 3 remaining. If no basic commutator weight 3 is left we are through. We collect using (R) all terms of the product involving a particular basic α_j say, noting that α_j^{-1} is also collected using $[\alpha_j^{-1}, \beta] \equiv [\alpha_j, \beta^{-1}]$ for β in F'.

Thus $a \equiv \Pi[\alpha_j, pq]\ \Pi$ (type (B) not involving the basic α_j), where p is a product of basics weight 3, not involving α_j and $q \in F_4$.

For x, y, z and $w \in F$

$[x,y,z] \equiv 1 + (xy-yx)z-z(xy-yx) \bmod \mathfrak{f}^4$

$[x,y,z,w] \equiv 1 \bmod \mathfrak{f}^4$

Since $a-1 \in \mathfrak{f}^4\mathfrak{a}$ and $(F')_3(F_4)' \leq (1 + \mathfrak{f}^4\mathfrak{a})$

$\Rightarrow (\alpha_j-1)(pq-1)-(p-1)(\alpha_j-1) + \sum\pm(\gamma_k-1)(\delta_k-1) \in \mathfrak{f}^4\mathfrak{a}$,
where the γ_i are basics $\neq \alpha_j$, and $\delta_k \in F_3$. Also
$p-1 \equiv \sum\pm(w_k-1) \bmod \mathfrak{f}^4$, where w_k is a basic $\neq \alpha_j$.
$\Rightarrow (\alpha_j-1)(pq-1) + \sum\pm(\gamma_k-1)(\delta_k-1)$ is in $\mathfrak{f}^4\mathfrak{a}$ - (C),
where γ_i is a basic $\neq \alpha_j$, $\delta_k \in F_3$.
Let $\alpha_j = [y_{i_1}, y_{i_2}, y_{i_3}]$, $i_1 > i_2$, $i_2 \leq i_3$.

The only other basic commutator weight 3 that involves each of y_{i_1}, y_{i_2} and y_{i_3} is $[y_{i_3}, y_{i_2}, y_{i_1}]$, $y_{i_3} \neq y_{i_2}$.

$\alpha_j-1 \equiv (y_{i_1}y_{i_2} - y_{i_2}y_{i_1})y_{i_3} - y_{i_3}(y_{i_1}y_{i_2} - y_{i_2}y_{i_1}) \bmod \mathfrak{f}^4$.

$[y_{i_3}, y_{i_2}, y_{i_1}] -1 \equiv (y_{i_3}y_{i_2} - y_{i_2}y_{i_3})\, y_{i_1} - y_{i_1}$
$(y_{i_3}y_{i_2} - y_{i_2}y_{i_3}) \bmod \mathfrak{f}^4$.

$(y_{i_3}, y_{i_2}, y_{i_1})$ does not involve y_{i_3}, y_{i_2} and y_{i_1} in the sequence i_2, i_1, i_3 for $i_3 \neq i_1$.
From (C) we get

$(y_{i_1}, y_{i_2}, y_{i_3})(pq-1) + \sum\pm(y_{k_1}, y_{k_2}, y_{k_3})(\delta_k-1) = \gamma, \quad \in \mathfrak{f}^4\mathfrak{a}$.

$\Rightarrow \gamma d_{i_2} d_{i_1} d_{i_3} \in \mathfrak{f}\mathfrak{a}$ (if $i_2 = i_3$ take $\gamma d_{i_1} d_{i_2} d_{i_3} \in \mathfrak{f}\mathfrak{a}$)

$\Rightarrow pq-1 \in \mathfrak{f}\mathfrak{a}$

$\Rightarrow pq \in F''$ by Theorem 1.6.

$\Rightarrow a \equiv \Pi$ (type B with one less distinct basic commutator weight 3).

$\Rightarrow$ by induction, $a \equiv 1$.

Corollary: Let S be the power series ring in X over $\mathbb{Z}$ subject to $x_{i_1}x_{i_2}x_{i_3}x_{i_4}(x_{i_5}x_{i_6}-x_{i_6}x_{i_5}) = 0$ then subgroup G of W(S) generated by 1 + X is isomorphic to $F/(F')_3'(F_4)'$ under the mapping $y_i \mapsto 1 + x_i$.

Section 2: Residual Properties

This section is devoted to proving that the groups of units of the $P_{n,m}$ (as constructed in Lemma 2.19 Corollary) are residually torsion free nilpotent and when the set of variables is finite, are residually finite p-groups for all primes p. This will prove that the groups $F/(1 + \mathfrak{f}^n \mathfrak{a} \mathfrak{f}^m) \cap F$ embedded in these power series are residually torsion free nilpotent and residually finite p-groups for all primes p, (without any restriction to finite generation by Lemma 1.2). We shall confine our attention to Q, the power series ring in X over $\mathbb{Z}$ subject to $x_{i_1}(x_{i_2}x_{i_3} - x_{i_3}x_{i_2}) = 0$ but it is easy to see how these results can be generalised to $P_{n,m}$ (with probably a little notational difficulty!)

In Q every element s in the multiplicative semigroup of Q generated by X can be written uniquely in the form

$$s = x_{i_1}x_{i_2}x_{i_3}\cdots x_{i_n}, \quad i_2 \leq i_3 \leq \cdots \leq i_n \tag{1}$$

Term ms with $m \in \mathbb{Z}$, a monomial in Q. Let K_i be the ideal of elements in Q of order $\geq$ i. Then $\cap K_i = 0$.

Lemma 3.4: Let $a \in W(Q)$ and let the order of $a-1$ be i. Then the order of a^m-1 is also i, for $m \in Z - \{0\}$.

Proof: Let $a = 1 + a_{(i)} + g(x)$, $a_{(i)} \neq 0$

$g(x) \in \underset{j>i}{\pi} K_j$ (the Cartesian product.)

$a^m = \{1 + a_{(i)} + g(x)\}^m \Rightarrow (a^m)_{(i)} = ma_{(i)}$ and $(a^m)_{(k)} = 0$ for $0 < k < i$ and $(a^m)_{(o)}=1$. By (1) if $ma_{(i)} = 0 \Rightarrow a_{(i)} = 0$. Hence the order of a^m-1 is i.

Theorem 3.5: W(Q) is residually torsion free nilpotent.

Proof: $\cap(1 + K_i) = 1$ and $(1 + K_i) \lhd W(Q)$. Clearly $\{W(Q)\}_i \leq 1 + K_i$. (Note $\{W(Q)\}_i$ is the ith term of the lower central series of W(Q)). Hence $W(Q)/(1 + K_i)$ is nilpotent and is torsion free by Lemma 3.4. $\Rightarrow$ W(Q) is residually torsion free nilpotent.

As a corollary to this we get the well known theorem:

Theorem 3.6: The free metabelian group is residually torsion free nilpotent.

Proof: By Lemma 3.1, Corollary and Theorem 3.5. /

We can now use Lemma 1.2 and Theorems 1.1 and 3.6 to prove that the free metabelian group is residually a finite p-group for all primes p. However this result is a corollary of the following constructive Theorem.

<u>Theorem 3.7</u>: If X is finite then W(Q) is residually a finite p-group for all primes p.

<u>Proof</u>: Let $X = x_1, x_2, \ldots, x_{r+1}$. Define

$$R_{i,n} = \{1 + p^i g(x) + f(x)/f(x) \in \prod_{i>n} K_i,\ g(x) \in \bigoplus_{i=1}^{n} K_i\}$$

(Π denotes the Cartesian product and $\oplus$ the direct sum).

Then $R_{i,n} \lhd W(Q)$. $\left[W(Q): R_{i,n}\right] = \prod_{j=1}^{n} \left[R_{i,j-1}:R_{i,j}\right]$

and we let $R_{i,0} = W(Q)$. The number of distinct elements of degree j in the multiplicative semigroup of Q, (generated by X) is by (1)

$(r+1)\binom{r-1+j}{j}$. Then $\left[R_{i,j-1}:R_{i,j}\right] = p^{i(r+1)\binom{r-1+j}{j}}$

since $\{1 + \sum C_{t_1 \ldots t_j} x_{t_1} \ldots x_{t_j} /$ with $0 \leq C_{t_1 \ldots t_j} < p^i$, $t_2 \leq t_3 \leq \ldots \leq t_j\}$ gives a transversal of $R_{i,j}$ in $R_{i,j-1}$. Hence $W(Q)/R_{i,n}$ is a p-group. Suppose $a \neq 1$ is contained in W(Q) and let the order of a-1 be ℓ. $a_{(\ell)} \neq 0$ and $a_{(\ell)}$ is a finite sum of monomials of length ℓ. Suppose $a_{(\ell)} = p^{i-1}g(x)$ where a monomial of g(x) is not divisible by p. Then $a \notin R_{i,\ell}$.

Using the methods derived in this section we prove in a similar manner as Theorems 3.6 and 3.7.

<u>Theorem 3.8</u>: $F/(F')_3(F_3)'$ is residually torsion-free nilpotent.

Proof: By Lemma 3.2 Corollary $F/(F')_3(F_3)'$ is embedded in $P_{3,0}$.

Theorem 3.9: $F/(F')_3(F_3)'$ is residually a finite p-group for all primes p.

Theorem 3.10: $F/(F')_3(F_4)'$ is residually torsion-free nilpotent.

Proof: By Lemma 3.3 Corollary $F/(F')_3(F_4)'$ is embedded in $P_{4,0}$.

Theorem 3.11: $F/(F')_3(F_4)'$ is residually a finite p-group for all primes p.

CHAPTER 4

Section 1: $F/[F'', F]$; The free centre by metabelian group.

In this section we show that $F''/[F'', F]$ is free abelian and an explicit basis is given. We also show $C_{1,1} = F/[F'', F]$ and hence that $F/[F'', F]$ is residually torsion free nilpotent. We use lengthy computations with commutators and the reader is assumed to be very familiar with commutator identities. Lemmas 4.1 - 4.6 below are an attempt to familiarise the reader with the identities we shall frequently use.

We collect in Lemma 4.1 some well-known results to which we shall make frequent reference later on.

Lemma 4.1: G any group.

(i) If $a_1 \in G'$, $a_2, \ldots, a_n \in G$ then

$$[a_1, a_2, \ldots, a_n]^{-1} \equiv [a_1^{-1}, a_2, \ldots, a_n] \text{ mod } G''$$

(ii) If a, b and c $\in$ G then

$$\begin{aligned}[a, b, c] &\equiv [b, c, a]^{-1}[c, a, b]^{-1} \\ &\equiv [b, c, a]^{-1}[a, c, b] \\ &\equiv [c, b, a][c, a, b]^{-1} \\ &\equiv [c, b, a][a, c, b] \text{ mod } G''.\end{aligned}$$

(These are just restatements of the Jacobi Identity.)

(iii) If $a_1 \in G'$, $a_2, \ldots, a_n \in G$ then

$[a_1, a_2, \ldots, a_n] \equiv [a_1, a_{i_2}, \ldots, a_{i_n}]$

mod G", where $i_2, \ldots, i_n$ is any permutation of $2, \ldots, n$.

(iv) If a and b ε G', c ε G" and $a_1, a_2, \ldots, a_n \varepsilon G$ then

$[a\, b\, c, a_1, a_2, \ldots, a_n]$

$\equiv [a, a_1, a_2, \ldots, a_n][b, a_1, a_2, \ldots, a_n]$ mod G" .

(v) If $a, b, c, a_1, a_n, \ldots, a_n \varepsilon G$ then

$[a, b, c, a_1, a_2, \ldots, a_n] \equiv [b, c, a, a_1, a_2, \ldots, a_n]^{-1}$.

$[c, a, b, a_1, a_2, \ldots, a_n]^{-1}$

$\equiv [c, b, a, a_1, a_2, \ldots, a_n][c, a, b, a_1, a_2, \ldots, a_n]^{-1}$

$\equiv [b, c, a, a_1, a_2, \ldots, a_n]^{-1}[a, c, b, a_1, a_2, \ldots, a_n]$

$\equiv [c, b, a, a_1, a_2, \ldots, a_n][a, c, b, a_1, a_2, \ldots, a_n]$

mod G".

(vi) If $a_1, a_2, \ldots, a_n \varepsilon G$ then

(α) $[a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_n]$

$\equiv [a_1, a_2, \ldots, a_i^{-1}, a_{i+1}, \ldots, a_n]^{-1}$.

$[a_1, a_2, \ldots, a_i^{-1}, a_i, a_{i+1}, \ldots, a_n]^{-1}$ mod G"

for $2 \leq i \leq n$.

(β) $[a_1, a_2, \ldots, a_n] \equiv [a_1^{-1}, a_2, \ldots, a_n]^{-1}$.

$[a_1^{-1}, a_2, a_1, a_3, \ldots, a_n]^{-1}$ mod G" .

Proof: (i), (ii) and (iii) are well known.

(iv) is easy by induction on n.

(v) is just a combination of (i), (ii) and (iv).

(vi) comes from the congruences

$$[x, y] \equiv [x^{-1}, y]^{-1}[x^{-1}, y, x]^{-1} \mod G''$$

$$[x, y] \equiv [x, y^{-1}]^{-1}[x, y^{-1}, y]^{-1} \mod G''$$

Lemma 4.2 below is due to Ridley [13].

Lemma 4.2: G any group, a, b and c ε G' e, f ε G, then

(i) $[a^{-1}, b] \equiv [a, b]^{-1} \equiv [a, b^{-1}], \mod (G')_3$.

(ii) $[a^e, b^f] \equiv [a^{ef^{-1}}, b], \mod [G'', G]$.

(iii) $[ab, c] \equiv [a, c][b, c], \mod (G')_3$.

(iv) $[a^b, c] \equiv [a, c], \mod (G')_3$.

(v) $[a, e, b] \equiv [a, [b, e^{-1}]] \mod [G'', G]$.

Proof: (i), (iii) and (iv) are clear.

For (ii):- $[a^e, b^f] = [a^{ef^{-1}}, b]^f$

$$= [a^{ef^{-1}}, b][a^{ef^{-1}}, b, f] \equiv [a^{ef^{-1}}, b] \mod [G'', G]$$

For (v):- $[[a, e], b] = [a^{-1}a^e, b]$

$\equiv [a^{-1}, b][a^e, b]$ from (iii)

$\equiv [a, b^{-1}][a, b^{e^{-1}}]$ from (i) and (ii)

$\equiv [a, b^{-1} b^{e^{-1}}]$ from (iii)

$\equiv [a, [b, e^{-1}]]$.

Let G be any group generated by X (countable). Let $a = [x_{i_1}^{\varepsilon_{i_1}}, x_{i_2}^{\varepsilon_{i_2}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}]$ be a commutator in G, then we say the sign of x_{i_j} <u>tallies</u> in a *at* x_{i_k} if $x_{i_j} \neq x_{i_k}$ or if $x_{i_j} = x_{i_k}$ and $\varepsilon_{i_j} = \varepsilon_{i_k}$ for $1 \leq k \leq n$. Otherwise we say the sign of x_{i_j} does not tally.

<u>Lemma 4.3</u>: Let $a = [x_{i_1}^{\varepsilon_{i_1}}, x_{i_2}^{\varepsilon_{i_2}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}] \neq 1$ where the signs of $x_{i_1}, \ldots, x_{i_{n-1}}$ tally, and $i_3 \leq i_4 \leq \cdots \leq i_{n-1}$. Then a is a product modulo G" of commutators of the form $b = [x_{j_1}^{\eta_{j_1}}, x_{j_2}^{\eta_{j_2}}, \ldots, x_{j_m}^{\eta_{j_m}}]$, where $j_1 = i_1$, $j_2 = i_2$, $j_3 \leq j_4 \leq \cdots \leq j_m$ and the sign of x_{j_k} tallies for $1 \leq k \leq m$, and also for every s, $1 \leq s \leq n$, $i_s = j_k$ for some k, $1 \leq k \leq m$.

<u>Proof</u>: If $i_n \neq i_k$ for any k, $1 \leq k \leq n-1$ or if $i_n = i_k$ and $\varepsilon_{i_m} = \varepsilon_{i_k}$ for any k $1 \leq k \leq n$ then by Lemma 4.1 we are through. If $i_n = i_k$ and $\varepsilon_{i_n} = -\varepsilon_{i_k}$ for some k, $1 \leq k \leq n-1$ then we proceed by induction on the number to of times i_n occurs amongst $i_1, i_2, \ldots, i_{n-1}$. If t = 1 Lemma 4.1 (iii) and (vi) gives the result. If t > 1 then Lemma 4.1 (iii) and (vi) shows that a is a product mod G" of a commutator of the required type and one of the same form as a but where i_n occurs less than t times amongst the indices $i_1, i_2, \ldots, i_{n-1}$.

<u>Lemma 4.4</u>: Let $a = [x_{i_1}^{\varepsilon_{i_1}}, x_{i_2}^{\varepsilon_{i_2}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}]$, with $n \geq 4$, $i_3 \leq i_4 \leq \cdots \leq i_{n-1}$ and the sign tallies of x_{i_k}

for $3 \leq k \leq n-1$. Then a is a product modulo G" of commutators of the form

$b = [x_{i_1}, x_{i_2}, x_{j_3}^{\eta_{j_3}}, \ldots, x_{j_m}^{\eta_{j_m}}]$ with $j_3 \leq j_4 \leq \cdots \leq j_m$, the sign of x_{j_t} tallies for $3 \leq t \leq m$ and for every s, $3 \leq s \leq m$, $i_s = j_k$ for some k, $3 \leq k \leq m$.

Proof: The proof is similar to the proof of the previous Lemma.

Lemma 4.5: Let $a = [x_{i_1}^{\varepsilon_{i_1}}, x_{i_2}^{\varepsilon_{i_2}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}] \neq 1$. Then a is a product modulo G" of commutators of the form

$b = [x_{j_1}^{\eta_{j_1}}, x_{j_2}^{\eta_{j_2}}, \ldots, x_{j_m}^{\eta_{j_m}}]$, with $j_1 = i_1$, $j_2 = i_2$, $j_3 \leq j_4 \leq \cdots \leq j_m$, the sign of x_{j_t} tallies for $1 \leq t \leq m$, and for every i_s, $1 \leq s \leq n$, $i_s = j_k$ for some k, $3 \leq k \leq m$.

Proof: We use induction on n. Case n = 2 is clear. Suppose $n \geq 3$. By the induction hypothesis $[x_{i_1}^{\varepsilon_{i_1}}, x_{i_2}^{\varepsilon_{i_2}},\ldots,x_{i_{n-1}}^{\varepsilon_{i_{n-1}}}]$ is a product modulo G" of commutators of the required form. Hence by Lemma 4.1 we need only show that $[x_{j_1}^{\eta_{j_1}}, x_{j_2}^{\eta_{j_2}}, \ldots, x_{j_m}^{\eta_{j_m}}, x_{i_n}^{\varepsilon_{i_n}}]$ with $[x_{j_1}^{\eta_{j_1}}, x_{j_2}^{\eta_{j_2}}, \ldots, x_{j_m}^{\eta_{j_m}}]$ a commutator of the required form, is a product of commutators of the required form. Lemma 4.3 does this for us.

Lemma 4.6: Let $a = [[x_{i_1}^{\varepsilon_{i_1}}, x_{i_2}^{\varepsilon_{i_2}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i^{\eta_i}, x_j^{\eta_j}]]$. Then a is a product modulo [G", G] of commutators of the form $b = [[x_{i_1}, x_{i_2}, x_{j_3}^{\eta_{j_3}}, \ldots, x_{j_m}^{\eta_{j_m}}], [x_i, x_j]]$ where (i) $j_3 \leq \ldots \leq j_m$

(ii) if $j_k = j_t$ for $3 \leq k, t \leq m$ then $\eta_{j_k} = \eta_{j_t}$

(iii) for every s, $3 \leq s \leq n$, $i_s = j_t$ for some t, $3 \leq t \leq m$.

<u>Proof</u>: By Lemma 4.1 (iii) and (vi), and Lemma 4.4 we can assume $\varepsilon_{i_1} = +1$, $\varepsilon_{i_2} = +1$, $i_3 \leq i_4 \leq \ldots \leq i_n$ and if $i_t = i_s$ for $1 \leq t, s \leq n$ then $\varepsilon_{i_t} = \varepsilon_{i_s}$. If $\eta_i = -1$ then $[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i^{-1}, x_j^{\eta_j}]]$

$$\equiv [[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j^{\eta_j}]]^{-1}$$

$$[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j^{\eta_j}, x_i^{-1}]^{-1}$$

= pq say

$$q = [[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}, x_i], [x_i, x_j^{\eta_j}]]$$

by Lemma 4.2.

By Lemma 4.4 we can now assume $\eta_i = +1$. Similarly we can deal with $\eta_j = -1$.

The reader is advised to be very familiar with the last six Lemmas before proceeding. We also introduce some further terminology. We say an <u>amalgamation</u> of x_{i_k} is necessary in the commutator $[x_{i_1}^{\varepsilon_{i_1}}, x_{i_2}^{\varepsilon_{i_2}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}]$ if the sign of

x_{i_k} does not tally in this commutator (we have to apply Lemma 4.5 in order to express the commutator modulo G" as a product of commutators in which the sign of x_{i_k} does tally). From now on F is the free group on X and as usual $\mathfrak{f} = \text{Ker}(\mathbb{Z}F \to \mathbb{Z})$, and $\mathfrak{a} = \text{Ker}\, \mathbb{Z}F \to \mathbb{Z}(F/F')$. Free generators of F' are derived in Gruenberg [5] Theorem 5.2 namely the set, W consisting of commutators of the form $[x_{i_1}^{\varepsilon_{i1}}, x_{i_2}^{\varepsilon_{i2}}, \ldots, x_{i_n}^{\varepsilon_{in}}]$ with $i_1 > i_2, i_2 \leq i_3 \leq \cdots \leq i_n$ and the sign of x_{i_k} tallies for all k, $1 \leq k \leq n$. Hence by Theorem 1.3, $\mathfrak{a}$ is free as right (or left) $\mathbb{Z}F$-module on W-1, $\Rightarrow$ by Lemma 1.4 $\mathfrak{f}\mathfrak{a}$ is free as right (or left) $\mathbb{Z}F$-module on (X - 1)(W - 1), $\Rightarrow \mathfrak{f}\mathfrak{a}/\mathfrak{f}\mathfrak{a}\mathfrak{f}$ is free abelian on (X-1)W-1) by Lemma 1.4 Corollary. This latter fact is crucial for what is to follow. We shall also say that an amalgamation of x_{i_k} is necessary in $a = (x_i - 1)([x_{i_1}^{\varepsilon_{i1}}, x_{i_2}^{\varepsilon_{i2}}, \ldots, x_{i_n}^{\varepsilon_{in}}] - 1)$ if the sign of x_{i_k} does not tally and we have to apply Lemma 4.5 in order to express the commutator of a as a product, modulo F", of commutators in which the sign of x_{i_k} tallies and hence to express a as a sum modulo $\mathfrak{f}\mathfrak{a}\mathfrak{f}$ of terms of the form $b = (x_i - 1)([x_{j_1}^{n_{j1}}, x_{j_2}^{n_{j2}}, \ldots, x_{j_m}^{n_{jm}}] - 1)$ where now the sign of x_{i_k} in b tallies.

We introduce an ordering on the basic 2-commutators by

$[x_i, x_j] < [x_k, x_\ell]$ if $j < \ell$ and

$[x_i, x_j] < [x_k, x_j]$ if $i < k$.

(This ordering is valid in any group for which

$[x_i, x_j] = [x_k, x_\ell] \Rightarrow i = k$ and $j = \ell$.)

The following proposition derives generators for $F''/[F'', F]$ which later turn out to be free generators.

Note that frequent use of Lemmas 4.1 and 4.2 will be made and we shall at times use these Lemmas without reference.

<u>Proposition 4.7</u>: $F''/[F'', F]$ is generated by the double commutators of the form

$[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]]$ with $\varepsilon_{i_k} = \pm 1$

subject to the following conditions:

(i) $i_1 > i_2$ $\quad i_2 \leq i_3 \leq \cdots \leq i_n$

$i > j$ $\quad j \leq i_3 \leq \cdots \leq i_n$

(ii) If $i_\alpha = i_\beta$ for $3 \leq \alpha, \beta \leq n$ then $\varepsilon_{i_\alpha} = \varepsilon_{i_\beta}$.

(iii) $[x_{i_1}, x_{i_2}] \leq [x_i, x_j]$ in the ordering of the basic 2-commutators (ordered as shown above) and if

$[x_{i_1}, x_{i_2}] = [x_i, x_j]$ then $\varepsilon_{i_3} = +1$.

(iv) (α) If $i_2 = j \neq i_3$ then either $i_1 \leq i_3$ or else

$i_3 < i_1 \leq i \leq i_4$, $\varepsilon_{i_3} = +1$.

(β) (If $i_2 = j \neq i_3$) and $i_1 = i_3 < i = i_4$ then $\varepsilon_{i_3} = +1$. (For this condition (iv) if an index is not applicable to the double commutator just omit it from the condition.)

Proof: Let $G = F/[F'', F]$.

Then G' is generated by $\{[x_i, x_j]^\alpha / i > j, \alpha \in G\}$. Hence G'' is generated by $\{[[x_{i_1}, x_{i_2}]^\alpha, [x_i, x_j]^\beta] / i_1 > i_2, i > j, \alpha$ and $\beta \in G\}$. By Lemma 4.2

$$[[x_{i_1}, x_{i_2}]^\alpha, [x_i, x_j]^\beta] = [[x_{i_1}, x_{i_2}]^{\alpha\beta^{-1}}, [x_i, x_j]]$$

$$= [[x_{i_1}, x_{i_2}]^{x_{i_3}^{\alpha_3} \cdots x_{i_m}^{\alpha_m} a}, [x_i, x_j]], \text{ where}$$

$\alpha\beta^{-1} = x_{i_3}^{\alpha_3} \cdots x_{i_m}^{\alpha_m} a$, $i_3 < \cdots < i_m$ and $a \in G'$

$$= [[x_{i_1}, x_{i_2}]^{x_{i_3}^{\alpha_3} \cdots x_{i_m}^{\alpha_m}}, [x_i, x_j]] \text{ by Lemma 4.2.}$$

It is easy to see using $[x, y\mathbf{z}] = [x, z][x, y][x, y, z]$ that $[x_{i_1}, x_{i_2}]^{x_{i_3}^{\alpha_3} \cdots x_{i_m}^{\alpha_m}}$ is a product of commutators of the form

$$[x_{i_1}, x_{i_2}, x_{j_3}^{\varepsilon_{j_3}}, \ldots, x_{j_n}^{\varepsilon_{j_n}}], \; j_3 \leq j_4 \leq \cdots \leq j_n$$

and if $j_k = j_s$ then $\varepsilon_{j_k} = \varepsilon_{j_s}$. Hence G'' is generated by

$\{[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j] / i_1 > i_2,$

$i > j$, $i_3 \leq \dots \leq i_n$ and if $i_k = i_s$ then $\varepsilon_{i_k} = \varepsilon_{i_s}$ for $3 \leq k \leq n$ and $3 \leq s \leq n\}$. Suppose $i_2 > i_3$ then

$$[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \dots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]]$$

$$= [[x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_1}, x_{i_4}^{\varepsilon_{i_4}}, \dots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]] \cdot$$

$$[[x_{i_1}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \dots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]]$$

$= pq$ say.

Note that $i_2 > i_3$ and hence $i_1 > i_3$. Now apply Lemma 4.6 to write p and q as a product of commutators of the form

$$[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \dots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]] \ , \ i_1 > i_2 \ ,$$

$i_2 \leq i_3 \leq \dots \leq i_n$, $i > j$ and if $i_k = i_s$, $3 \leq k, s \leq n$ then $\varepsilon_{i_k} = \varepsilon_{i_s}$. Hence G" is generated by commutators of this form. If $j > i_3$ then

$$[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \dots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]]$$

$$= [[x_{i_1}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \dots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j, x_{i_3}^{-\varepsilon_{i_3}}]]$$

by Lemma 4.2.

$$= [[x_{i_1}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_j, x_{i_3}^{-\varepsilon_{i_3}}, x_i]] \,.$$

$$[[x_{i_1}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_3}^{-\varepsilon_{i_3}}, x_j]]$$

$$= [[x_{i_1}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}, x_i^{-1}], [x_j, x_{i_3}^{-\varepsilon_{i_3}}]] \,.$$

$$[[x_{i_1}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}, x_j^{-1}], [x_i, x_{i_3}^{-\varepsilon_{i_3}}]]$$

$= pq$ say. Apply Lemma 4.6 to p and q to show that G'' is generated by $\{[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]] /$ $i_1 > i_2$, $i_2 \leq i_3 \leq \cdots \leq i_n$, $i > j$, $j \leq i_3$ and if $i_k = i_s$ for $3 \leq k, s \leq n$ then $\varepsilon_{i_k} = \varepsilon_{i_s}\}$.

By Lemma 4.2 $[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]]$

$$= [[x_{i_1}, x_{i_2}], [x_i, x_j, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}]]$$

$$= [[x_i, x_j, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}], [x_{i_1}, x_{i_2}]]^{-1} \qquad \text{(A)}$$

Hence we can assume condition (iii).

We are left to show condition (iv). First of all we show condition (iv) (α). Let $a =$

$$[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]] \,, \quad i_2 \neq i_3$$

and suppose $i_1 > i_3$. Suppose further i_3 is a repeated entry of a. Then clearly for $n = 3$, $i_1 \le i_3$. (Note that $i_1 \le i$ from condition (iii).) So we consider $i_1 > i_3 = i_4$. Then

$$a = [[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]$$

$$= [[x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_1}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]$$

$$[[x_{i_1}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]$$

$= bc$ say.

b is a product of commutators of the correct form by Lemma 4.6.

$$c = [[x_{i_1}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}, x_{i_4}^{-\varepsilon_{i_4}}]]$$

$$= [[x_{i_1}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_{i_4}^{-\varepsilon_{i_4}}, x_{i_2}, x_i]]$$

$$[[x_{i_1}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_4}^{-\varepsilon_{i_4}}, x_{i_2}]]$$

$= de$ say.

$$d = [[x_{i_1}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}, x_i^{-1}], [x_{i_4}^{-\varepsilon_{i_4}}, x_{i_2}]]$$

$$= [[x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_1}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}, x_i^{-1}], [x_{i_4}^{-\varepsilon_{i_4}}, x_{i_2}]]^{-1}$$

$$[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}, x_i^{-1}], [x_{i_4}^{-\varepsilon_{i_4}}, x_{i_2}]]$$

$= q^{-1}p$ say.

Apply (A) and Lemma 4.6 to p and q to express them as products of commutators satisfying (i), (ii), (iii) and (iv).

$$e = [[x_{i_1}, x_{i_3}{}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_2}{}^{-1}, x_{i_5}{}^{\varepsilon_{i_5}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_{i_4}{}^{-\varepsilon_{i_4}}]]$$

$$= [[x_{i_1}, x_{i_3}{}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_5}{}^{\varepsilon_{i_5}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_{i_4}{}^{-\varepsilon_{i_4}}]]^{-1}$$

$$[[x_{i_1}, x_{i_3}{}^{\varepsilon_{i_3}}, x_{i_2}{}^{-1}, x_{i_5}{}^{\varepsilon_{i_4}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_{i_4}{}^{-\varepsilon_{i_4}}]]^{-1}$$

$= s^{-1}t^{-1}$ say.

$$s^{-1} = [[x_{i_1}, x_{i_2}, x_{i_3}{}^{\varepsilon_{i_3}}, x_{i_5}{}^{\varepsilon_{i_5}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_{i_4}{}^{-\varepsilon_{i_4}}]]^{-1}$$

$$[[x_{i_3}{}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_1}, x_{i_5}{}^{\varepsilon_{i_5}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_{i_4}{}^{-\varepsilon_{i_4}}]]$$

and we apply Lemma 4.6 again to these to express them as products of commutators satisfying (i), (ii), (iii) and (iv) since $i_2 \neq i_3$.

$$t^{-1} = [[x_i, x_{i_4}{}^{-\varepsilon_{i_4}}, x_{i_2}, x_{i_5}{}^{-\varepsilon_{i_5}}, \ldots, x_{i_n}{}^{-\varepsilon_{i_n}}], [x_{i_1}, x_{i_3}{}^{\varepsilon_{i_3}}]]$$

and proceed as with s^{-1} just above to show t^{-1} is a product of commutators satisfying (i), (ii), (iii) and (iv). We shall refer to this process of dealing with e as 'amalgamating' x_{i_2}.

Let $a = [[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]$

$i_3 \neq j$, $i_1 > i_3$, and suppose i_3 is not a repeated entry. By the same argument as that for the case of i_3 repeated we can take $i_1 \leq i_4$. Suppose $i_4 < i$. Then

$$a = [[x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_1}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]$$

$$[[x_{i_1}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]$$

= bc say.

Apply Lemma 4.6 and (A) to b to express it as a product of commutators of the correct form.

$$c = [[x_{i_1}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}, x_{i_4}^{-\varepsilon_{i_4}}]]$$

$$= [[x_{i_1}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_{i_4}^{-\varepsilon_{i_4}}, x_{i_2}, x_i]]$$

$$[[x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_1}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_4}^{-\varepsilon_{i_4}}, x_{i_2}]]$$

= de say.

$$d = [[x_{i_1}, x_{i_3}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}, x_i^{-1}], [x_{i_4}^{-\varepsilon_{i_4}}, x_{i_2}]]$$

$$= [[x_{i_1}, x_{i_2}, x_{i_3}{}^{\varepsilon_{i_3}}, x_{i_5}{}^{\varepsilon_{i_5}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}, x_i{}^{-1}], [x_{i_4}{}^{-\varepsilon_{i_4}}, x_{i_2}]]$$

$$[[x_{i_3}{}^{\varepsilon_{i_3}}, x_{i_2}, x_{i_1}, x_{i_5}{}^{\varepsilon_{i_5}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}, x_i{}^{-1}], [x_{i_4}{}^{-\varepsilon_{i_4}}, x_{i_2}]]$$

Apply Lemma 4.6 and (A) to these to express them as products of commutators satisfying (i), (ii), (iii) and (iv).

e can be dealt with by amalgamating x_{i_2}.

Hence we have in

$$a = [[x_{i_1}, x_{i_2}, x_{i_3}{}^{\varepsilon_{i_3}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]], \ (i_2 \neq i_3)$$

either $i_1 \leq i_3$ or else $i_3 < i_1 \leq i \leq i_4$. Suppose in a $i_3 < i_1 \leq i \leq i_4$ and $\varepsilon_{i_3} = -1$. Then

$$a = [[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}{}^{\varepsilon_{i_4}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]^{-1}$$

$$[[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_3}{}^{-1}, x_{i_4}{}^{\varepsilon_{i_4}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]^{-1}$$

$= b^{-1}c^{-1}$ say.

b^{-1} is okay.

$$c^{-1} = [[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}{}^{\varepsilon_{i_4}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_{i_2}, x_{i_3}]]$$

$$= [[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}{}^{\varepsilon_{i_4}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_{i_3}, x_{i_2}, x_i]]$$

$$[[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_3}, x_{i_2}]]$$

and proceed as for the case above when i_3 is a repeated entry.

We now show (iv) (β). Suppose

$$a = [[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]$$

with $i_1 = i_3 < i = i_4$ and $\varepsilon_{i_3} = -1$. Then

$$a = [[x_{i_1}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}, x_{i_1}]]$$

$$= [[x_{i_1}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_{i_1}, x_{i_2}, x_i]]$$

$$[[x_{i_1}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_1}, x_{i_2}]]$$

= bc say.

b can be expressed as a product of commutators of the required type as before.

$$c = [[x_{i_4}^{\varepsilon_{i_4}}, x_{i_2}, x_{i_1}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_1}, x_{i_2}]]$$

$$[[x_{i_4}^{\varepsilon_{i_4}}, x_{i_1}, x_{i_2}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_1}, x_{i_2}]]$$

$= d\ e^{-1}$ say.

e^{-1} can be dealt with by amalgamating x_{i_2}.

$$d = [[x_{i_4}^{\varepsilon_{i_4}}, x_{i_2}, x_{i_1}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}, x_{i_1}]]$$

$$[[x_{i_4}^{\varepsilon_{i_4}}, x_{i_2}, x_{i_1}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_{i_1}, x_{i_2}, x_i]]^{-1}$$

$= f\, g^{-1}$ say.

g is a product of commutators of the required type by a similar argument as before.

$$f = [[x_{i_4}^{\varepsilon_{i_4}}, x_{i_2}, x_{i_1}, x_{i_5}^{\varepsilon_{i_5}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]^{-1}$$

$$[[x_i, x_{i_2}, x_{i_1}, x_{i_5}^{-\varepsilon_{i_5}}, \ldots, x_{i_2}^{-\varepsilon_{i_n}}], [x_{i_4}^{\varepsilon_{i_4}}, x_{i_2}]]$$

and these are products of commutators of the required type (note $i = i_4$). This completes the proof.

<u>Lemma 4.8</u>: If a and b ε F, c ε F' then

$(a-1)(b-1)(c-1) \equiv (a-1)([c, b^{-1}] - 1)$ mod $\mathfrak{f}\mathfrak{a}\mathfrak{f}$

<u>Proof</u>: $(a - 1)([c, b^{-1}] - 1)$

$= (a - 1)\{c^{-1}b(cb^{-1} - b^{-1}c)\}$

$= (a-1)\{c^{-1}b((c-1)(b^{-1}-1) - (b^{-1}-1)(c-1))\}$

$\equiv -(a-1)c^{-1}b(b^{-1} - 1)(c-1)$ mod $\mathfrak{f}\mathfrak{a}\mathfrak{f}$

$= -(a-1)c^{-1}(1-b)(c-1)$

$= (a-1)c^{-1}(b-1)(c-1)$

$\equiv (a-1)(b-1)(c-1)$ mod $\mathfrak{f}\mathfrak{a}\mathfrak{f}$

<u>Lemma 4.9</u>: $a_1, a_2, \ldots, a_n \in F$ then

$[a_1, a_2, \ldots, a_n]-1 \equiv \{(a_1-1)(a_2-1) - (a_2-1)(a_1-1)\}$
$\{(a_3 - 1) \ldots (a_n - 1)\}$ mod $\mathfrak{f}\mathfrak{a}$.

<u>Proof</u>: $[a_1, a_2] - 1 = a_1^{-1} a_2^{-1}(a_1a_2 - a_2a_1)$

$\equiv a_1a_2 - a_2a_1$ mod $\mathfrak{f}\mathfrak{a}$
$= (a_2 - 1)(a_2 - 1) - (a_2 - 1)(a_1 - 1)$

Hence it is true for $n = 2$.

We now proceed by induction on n.

$[a_1,a_2, \ldots, a_n] -1 = [a_1,a_2, \ldots, a_{n-1}]^{-1} a_n^{-1}$

$\{([a_1,a_2,\ldots,a_{n-1}] -1)(a_n-1) - (a_n-1)([a_1,a_2,\ldots,a_{n-1}]-1)\}$

$\equiv ([a_1,a_2,\ldots,a_{n-1}]-1)(a_n-1)-(a_n-1)([a_1,a_2,\ldots,a_{n-1}]-1)$

$\equiv \{(a_1-1)(a_2-1)-(a_2-1)(a_2-1)\}(a_3-1) \ldots (a_n-1)$

by the inductive hypothesis. /

Let $a = [[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]]$

be a generator of $F''/[F'', F]$ as in Proposition 4.7. Call $[x_{i_1}, x_{i_2}]$ and $[x_i, x_j]$ the <u>heads</u> of a and call $[x_{i_1}, x_{i_2}]$ the <u>leading head</u> of a. For the following proposition a 'generator', with inverted commas, will mean a generator as in Proposition 4.7 in order to distinguish it from the terms <u>free generators</u> of $\mathfrak{f}$, $\mathfrak{a}$, or $\mathfrak{f}\mathfrak{a}$.

Proposition 4.10: $(1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}) \cap F = [F'', F]$.

Proof. First of all we show $[F'', F] \leq (1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}) \cap F$. Now $F'' \leq 1 + \mathfrak{a}^2$. Let $a \in F''$, $b \in F$ then

$[a, b] = 1 + a^{-1}b^{-1}\{(a-1)(b-1) - (b-1)(a-1)\} \in 1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}$,

since $a-1 \in \mathfrak{a}^2$ and $b - 1 \in \mathfrak{f}$.

Suppose $d \in (1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}) \cap F$. Then $d \in F''$ by Theorem 1.6.

Suppose $d \notin [F'', F]$.

$$d \equiv \Pi[[x_{i_1}, x_{i_2}, x_{i_3}{}^{\varepsilon_{i_3}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_j]]^{\alpha_i}$$

mod$[F'', F]$, $\alpha_i \in \mathbb{Z} - \{0\}$, (by Proposition 4.7) where the commutators of the product are as in the Proposition. Call this product (A).

Since $[F'', F] \leq 1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}$, and $d \in 1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}$,

$$\Longrightarrow \Pi\, [[x_{i_1}, x_{i_2}, x_{i_3}{}^{\varepsilon_{i_3}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_j]]^{\alpha_i}$$

$\in 1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}$

$$\Longrightarrow \Sigma\, \alpha_i([[x_{i_1}, x_{i_2}, x_{i_3}{}^{\varepsilon_{i_3}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}], [x_i, x_j]] - 1)$$

$\in \mathfrak{f}\mathfrak{a}\mathfrak{f}$

Let $a = [x_{i_1}, x_{i_2}, x_{i_3}{}^{\varepsilon_{i_3}}, \ldots, x_{i_n}{}^{\varepsilon_{i_n}}]$ and $b = [x_i, x_j]$.

All congruences, unless otherwise stated, will be mod $\mathfrak{f}\mathfrak{a}\mathfrak{f}$.

$$[a, b] - 1 = a^{-1}b^{-1}\{(a-1)(b-1) - (b-1)(a-1)\}$$

$$\equiv (a-1)(b-1) - (b-1)(a-1)$$

$$\equiv \{(x_{i_1}-1)(x_{i_2}-1) - (x_{i_2}-1)(x_{i_1}-1)\}\{(x_{i_3}^{\varepsilon_{i_3}} - 1) \ldots (x_{i_n}^{\varepsilon_{i_n}} - 1)\}([x_i, x_j] -1) - \{(x_i-1)(x_j-1) - (x_j-1)(x_i-1)\}([x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}] - 1)$$

by Lemma 4.9.

$$\equiv (x_{i_1}-1)([x_i,x_j,x_{i_3}^{-\varepsilon_{i_3}},\ldots,x_{i_n}^{-\varepsilon_{i_n}}, x_{i_2}^{-1}] - 1)$$

$$- (x_{i_2}-1)([x_i, x_j, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}, x_{i_1}^{-1}] - 1)$$

$$- (x_i-1)([x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}, x_j^{-1}] - 1)$$

$$+ (x_j-1)([x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}, x_i^{-1}] - 1)$$

$= \beta_i$ say (by Lemmas 4.8 and 4.9).

We can now express the commutators in the expression for β_i as a product of free generators of F' modulo F" using Lemmas 4.1 and 4.5, and hence we can express β_i as the sum of free generators of $\mathfrak{fa}$ modulo $\mathfrak{faf}$. (We note that $\mathfrak{fa}/\mathfrak{faf}$ is free abelian on (X-1)(W-1) by previous remarks.) We indicate how this is done for

$$s = (x_{i_1}-1)([x_i, x_j, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}, x_{i_2}^{-1}] - 1).$$

The others are similar.

Case (a): $i_2 < j$. Then

$$[x_i, x_j, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}, x_{i_2}^{-1}]$$

$$\equiv [x_i, x_{i_2}^{-1}, x_j, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}] \cdot$$

$$[x_j, x_{i_2}^{-1}, x_i, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}]^{-1}$$

modulo F", by Lemma 4.1.

We can now apply Lemma 4.5 to express the commutators on the right hand side as products of free generators of F' (mod F") and hence we can express s as a sum of free generators of $\mathfrak{f}\mathfrak{a}$ modulo $\mathfrak{f}\mathfrak{a}\mathfrak{f}$. There are a few things to note about this expression for s as a sum of free generators of $\mathfrak{f}\mathfrak{a}/\mathfrak{f}\mathfrak{a}\mathfrak{f}$. (i) No index is lost, i.e., the free generators of $\mathfrak{f}\mathfrak{a}/\mathfrak{f}\mathfrak{a}\mathfrak{f}$ produced involve $i_1, i_2, \ldots, i_n, i, j$. (ii) The length of the commutator part of the free generator of $\mathfrak{f}\mathfrak{a}$ does not exceed n+1. (iii) Distinct free generators of $\mathfrak{f}\mathfrak{a}$ are produced. (iv) The only time it is possible for the sign of x_i or x_j to be -1 in the commutator part of the free generator of $\mathfrak{fa}$ is when an amalgamation (i.e., a reduction of the length) of x_i or x_j is necessary in s to order to express it as the sum of free generators of $\mathfrak{fa}$.

(v) We shall be interested in the free generators of greatest length produced from s and we note that the commutator part of these have length at least n - 2. (In this particular case at least n - 1 but for general purposes at least n - 2, e.g., for t = $(x_{i_2} - 1)$.

$([x_i, x_j, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}, x_{i_1}^{-1}] - 1)$; amalgamations of x_i, x_j and x_{i_1} may be necessary. Note also for t that the commutator part of the free generators of $\mathfrak{fa}$ produced from t have x_{i_1} with a minus sign except when an amalgamation of x_{i_1} is necessary.) (vi) The non-commutator part of the free generators of $\mathfrak{fa}$ come from entries of the heads of the 'generators' of (A). (vii) The entries of the head of the commutator part of the free generators of $\mathfrak{fa}$ produced also come from the heads of the 'generators' of (A), and the first entry comes from a different head than x_{i_1} does.

Case (b): $i_2 = j$. We need only apply Lemma 4.5. Notes (i) to (vii) hold in this case also.

Since $\mathfrak{fa}/\mathfrak{faf}$ is free abelian on (X-1)(W-1), for every free generator α of $\mathfrak{fa}$ produced from the 'generators' of the product (A) we must have its inverse produced from (A) as well. When we shall say 'look for an inverse for α' we mean try to find a 'generator' from (A) which will produce an inverse for α. What we are going to do is choose a 'generator' of greatest length from (A) and we look at the free generators it produces. We shall be particularly interested in the free generators of greatest length that it produces. We shall show that there is at least one free generator which does not have an inverse for it produced, thereby getting a contradiction and hence showing that our original assumption that $d \not\equiv 1 \bmod [F'', F]$ is incorrect.

We have four cases to consider: (α), (β), (γ) and (δ) below.
We must consider case (γ) first.

(α). Suppose we can choose a 'generator'

$$p = [[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]] \text{ of}$$

greatest length with $i_2 < j$. If no amalgamation of x_i or x_j is necessary in $t = (x_{i_1}-1)([x_i, x_j, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}, x_{i_2}^{-1}] - 1)$ then look for an inverse for

$$q = (x_{i_1}-1)([x_i, x_{i_2}^{-1}, x_j, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}] - 1).$$

By (vi) and (vii) x_{i_1} and x_i must be entries of distinct heads of the 'generator' that produces an inverse for q; x_{i_2} must be the second entry of the leading head by the ordering but it cannot occur in the same head as x_i in order to produce an inverse for q. Hence the heads must be $[x_{i_1}, x_{i_2}]$ and $[x_i, x_j]$ with $[x_{i_1}, x_{i_2}]$ the leading head and (see the ordering of the indices) thus we see that there is no inverse for q. If an amalgamation of x_j but not x_i is necessary in t then look for an inverse for

$$q = (x_{i_1} - 1)([x_i, x_{i_2}^{-1}, x_j^{-1}, x_{i_4}^{-\varepsilon_{i_4}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}] - 1)$$

and noting (v) above we see as before there is no inverse for q. If an amalgamation of x_i but not x_j is necessary in t (suppose $i = i_k$) then look for an inverse for $q = (x_{i_1} -1)$

$$([x_i^{-1}, x_{i_2}^{-1}, x_j, \underbrace{x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}}_{\text{missing } i_k}] - 1)$$ and again we

find (see (v)) there is no inverse for q. If amalgamations of both x_i and x_j are necessary in t then look for an inverse for

$$q = (x_{i_1} -1)([x_i^{-1}, x_{i_2}^{-1}, x_j^{-1}, \underbrace{x_{i_4}^{-\varepsilon_{i_4}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}}_{\text{missing } i_k}] - 1)$$

(where $i = i_k$) and as before find there is no inverse for q. Hence we have no 'generators' of greatest length of form p with $i_2 < j$.

(β). Suppose we can choose 'generators' of greatest length of the form $p = [[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]]$

with $i_2 = j$, $i_2 = i_3$ and $n \geq 3$.

If an amalgamation of x_i is not necessary in

$$t = (x_{i_1} - 1)([x_i, x_{i_2}, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}, x_{i_2}^{-1}] - 1)$$

and if $\varepsilon_{i_3} = +1$ then look for an inverse for

$$q = (x_{i_1} - 1)([x_i, x_{i_2}^{-1}, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}] - 1).$$

x_{i_1} and x_i must be entries of distinct heads by (vi) and (vii) and since the index i_2 occurs more than once in q (note $i_2 = i_3$) then the heads must be $[x_{i_1}, x_{i_2}]$ and $[x_i, x_{i_2}]$. If $i_1 < i$ then $[x_{i_1}, x_{i_2}]$ is the leading head and noting that $[[x_{i_1}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]$ does not give an inverse for q we find there is no inverse for q produced from any other generator. If $i_1 = i$ then noting that $[[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_{i_1}, x_{i_2}]]$ is not a generator when $\varepsilon_{i_3} = -1$ (see condition (iii) of proposition 4.7) and that $[[x_{i_1}, x_{i_2}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_{i_1}, x_{i_2}]]$ does not give an inverse for q, we find there is no inverse for q. If an amalgamation of x_i is not necessary in t and if $\varepsilon_{i_3} = -1$ look for an inverse for $(x_{i_1} - 1)$ $([x_i, x_{i_2}, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}] - 1)$ and again get a contradiction. If an amalgamation of x_i is necessary in t and if $\varepsilon_{i_3} = +1$ look for an inverse for $(x_{i_1} - 1)$ $([x_i^{-1}, x_{i_2}^{-1}, \underbrace{x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}}_{\text{missing } i_k}] - 1)$ (where $i = i_k$);

if an amalgamation of x_i is necessary and if $\varepsilon_{i_3} = -1$, look for an inverse for $(x_{i_1} - 1)([x_i^{-1}, x_{i_1}, \underbrace{x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}}_{\text{missing } i_k}] - 1)$ (where $i = i_k$). In both cases we get a contradiction. Hence we may suppose there are no 'generators' of greatest length in the product (A) of the forms (α) or (β).

(γ). Suppose we can choose a 'generator' of greatest length of the form $p = [[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]]$, with $i_2 = j$ and $i_2 \neq i_3$ and $i_1 < i$. If $n = 2$ look for an inverse for $(x_{i_2} - 1)([x_i, x_{i_2}, x_{i_1}^{-1}] - 1)$ and get a contradiction. We take three sub-cases.

(1) Suppose further i_3 is a repeated entry of p. This implies by condition (iv) of proposition that $i_1 \leq i_3$. If an amalgamation of neither x_{i_1} nor x_i is necessary in $t = (x_{i_2} - 1)([x_i, x_{i_2}, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}, x_{i_1}^{-1}] - 1$, look

for an inverse for $(x_{i_2}-1)([x_i, x_{i_2}, x_{i_1}^{-1}, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}]-1$; if an amalgamation of x_{i_1} but not x_i is necessary in t and if $i_1 = i_4$ (or $n = 3$) look for an inverse for $(x_{i_2} - 1)$ $([x_i, x_{i_2}, x_{i_1}, x_{i_4}^{-\varepsilon_{i_4}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}] - 1)$; if an amalgamation of x_i but not x_{i_1} is necessary in t look for an inverse for $(x_{i_2}-1)([x_i^{-1}, x_{i_2}, x_{i_1}^{-1}, \underbrace{x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}}_{\text{missing } i_k}]-1)$

(where $i_k = i$); if amalgamations of both x_{i_1} and x_i are necessary in t and if $i_1 = i_4$ (or $n = 3$) look for an inverse for $(x_{i_2}-1)([x_i^{1}, x_{i_2}, \underbrace{x_{i_1}, x_{i_4}^{-\varepsilon_{i_4}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}}_{\text{missing } i_k}]-1)$

(where $i = i_k$). In all these cases we get a contradiction. If further in p, $i_1 = i_3$ $\varepsilon_{i_3} = -1$, $i_3 \neq i_4$ and $i \neq i_4$ (Case $i_1 = i_3, \varepsilon_{i_3} = -1$, $i_3 \neq i_4$ and $i = i_4$ does not arise by condition (iv) (β) of Proposition 4.7) look for an inverse for $(x_{i_2}-1)([x_{i_1}^{-1}, x_{i_2}, x_i^{-1}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}] - 1)$ if $i < i_4$ so that we can assume we can choose a 'generator' of the same form as p of greatest length for which $i_4 < i$ or else $q = [[x_{i_1}, x_{i_2}, x_i, x_{i_4}^{-\varepsilon_{i_4}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}], [x_{i_1}, x_{i_2}]]$ is a 'generator' of greatest length; now look for an inverse for

$(x_{i_2}-1)([x_{i_1}^{-1}, x_{i_2}, x_i, x_{i_4}^{-\varepsilon_{i_4}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}] - 1)$ and we see that the only possibility is a 'generator' of the same form as p but with $i > i_4$. So we can assume we can choose a 'generator' of greatest length of the same form as p with $i > i_4$. Now look for an inverse for $(x_{i_2} - 1)$ $([x_i, x_{i_2}, x_{i_1}, x_{i_4}^{-\varepsilon_{i_4}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}] - 1)$ if an amalgamation of x_i is not necessary in t or look for an inverse for $(x_{i_2}-1)([x_i^{-1}, x_{i_2}, x_{i_1}, \underbrace{x_{i_4}^{-\varepsilon_{i_4}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}}_{\text{missing } i_k}]-1)$ (where $i = i_k$) if an amalgamation of x_i is necessary in t. We now get our contradiction on noting carefully condition (iv) of Proposition 4.7.

(2) Suppose further in p, $i_1 < i_3$. Look for an inverse for $(x_{i_2} - 1)([x_i, x_{i_2}, x_{i_1}^{-1}, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}]-1$ if an amalgamation of x_i is not necessary in

$t = (x_{i_2}-1)([x_i, x_{i_2}, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}, x_{i_1}^{-1}] - 1)$ and look for an inverse for $(x_{i_2} - 1)$

$([x_i^{-1}, x_{i_2}, x_{i_1}^{-1}, \underbrace{x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}}_{\text{missing } i_k}] - 1$ (where $i = i_k$)

if an amalgamation of x_i is necessary in t. The only possibility, in both these cases, is the generator

$$q = [[x_{i_3}, x_{i_2}, x_{i_1}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_{i_2}]]$$

with $i_1 < i_3 \leq i \leq i_4$. If an amalgamation of neither x_{i_3} nor i is necessary in $u = (x_{i_2}-1)$

$([x_{i_3}, x_{i_2}, x_{i_1}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}, x_i^{-1}] - 1)$ we must now look for an inverse for $(x_{i_2} - 1)$

$([x_{i_3}, x_{i_2}, x_{i_1}, x_i^{-1}, x_{i_4}^{\varepsilon_{i_4}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}] - 1)$ and look for the appropriate inverses when amalgamations are necessary and we see that we have a contradiction.

(3) Suppose further in p, $i_3 < i_1 \leq i \leq i_4$. This can be dealt with in a similar manner as (2).

(δ). We can now suppose that the only generators of greatest length are of the form $p = [[x_{i_1}, x_{i_2}, x_{i_3}^{\varepsilon_{i_3}}, \ldots, x_{i_n}^{\varepsilon_{i_n}}], [x_i, x_j]]$ with $i_2 = j \neq i_3$ and $i_1 = i$. ($\varepsilon_{i_3} = +1$). Look for an inverse for $(x_{i_2}-1)([x_{i_1}^{-1}, x_{i_2}, x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}] - 1)$. The only possibility is

$$q = [[x_{i_1}, x_{i_2}, x_{i_1}^{-1}, \underbrace{x_{i_3}^{-\varepsilon_{i_3}}, \ldots, x_{i_n}^{-\varepsilon_{i_n}}}_{\text{missing } i_k}], [x_{i_k}, x_{i_2}]]$$

with $i_k \neq i_1$ (we must of course have had $\varepsilon_{i_k} = +1$). But q is of the same length as p and is of a different form to p. Hence we have no such q in our product (A).

This completes the proof.

Corollary. $F''/[F'', F] \simeq \dfrac{\mathfrak{a}^{[2]} + \mathfrak{f}\mathfrak{a}\mathfrak{f}}{\mathfrak{f}\mathfrak{a}\mathfrak{f}}$ and hence is free abelian (being a subgroup of $\mathfrak{f}\mathfrak{a}/\mathfrak{f}\mathfrak{a}\mathfrak{f}$) where $\mathfrak{a}^{[2]} = \mathrm{Ker}\, \mathbb{Z}F \to \mathbb{Z}(F/F'')$. Further the free generators of $F''/[F'', F]$ are as in Proposition 4.7.

Proof: The isomorphism is given by $a \mapsto a-1$ and we note that what proposition 4.10 does exactly is prove the generators of proposition 4.7 are linearly independent modulo $\mathfrak{f}\mathfrak{a}\mathfrak{f}$.

Note $F''/[F'', F]$ is the Schur Multiplier of F/F'' ($H_2(F/F'', \mathbb{Z})$) (see, e.g., Gruenberg [6] Chapter 3, Proposition 7), and so we have $F''/[F'', F] \simeq \dfrac{\mathfrak{a}^{[2]} + \mathfrak{f}\mathfrak{a}\mathfrak{f}}{\mathfrak{f}\mathfrak{a}\mathfrak{f}} \simeq \dfrac{\mathfrak{a}^{[2]}}{\mathfrak{f}\mathfrak{a}^{[2]} + \mathfrak{a}^{[2]}\mathfrak{f}}$.

Theorem 4.11: Let $P_{1,1}$ be the power series ring in X over subject to $x_{i_1}(x_{i_2}x_{i_3} - x_{i_3}x_{i_2})x_{i_4} = 0$ then subgroup G of $U(P_{1,1})$ generated by 1+X is isomorphic to $F/[F'', F]$.

Proof: Immediate from Lemma 2.19 Corollary and Proposition 4.10.

Theorem 4.12: $F/[F'', F]$ is residually torsion-free nilpotent.

Proof: Use Theorem 4.8 with minor alterations to Theorem 3.5.

Corollary: $F''/[F'', F]$ is residually a finite p-group for all primes p.

This theorem has been proved by Ridley [13], for the case where F has rank 2.

Section 2: F will again denote the free group of X. Let S denote the set of free generators of $F''/[F'', F]$ derived in Proposition 4.7.

Lemma 4.13: $[F'', F]/[F'', F, F]$ is generated by

$$\{[s, x_i]/s \in S, x_i \in X\}.$$

Proof: Follows easily from $[ab, c] = [a, c][a, c, b][b, c]$.

Proposition 4.14:

$$(1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}^2) \cap F = [F'', F, F].$$

Proof: $[F'', F] \leq 1 + \mathfrak{f}\mathfrak{a}^2 + \mathfrak{a}^2\mathfrak{f}$.

Let $a \in [F'', F]$, $b \in F$ then

$$[a, b] = 1 + a^{-1}b^{-1}\{(a-1)(b-1) - (b-1)(a-1)\}$$

$$\in 1 + (\mathfrak{f}\mathfrak{a}^2 + \mathfrak{a}^2\mathfrak{f})\mathfrak{f} + \mathfrak{f}(\mathfrak{f}\mathfrak{a}^2 + \mathfrak{a}^2\mathfrak{f})$$

$$\leq 1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}^2 .$$

Hence $[F'', F, F] \leq (1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}^2) \cap F$. (Note also $[F'', F, F] \leq (1 + \mathfrak{f}^2\mathfrak{a}\mathfrak{f}) \cap F$). Suppose $a \in (1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}^2) \cap F$. Then $a \in [F'', F]$ by Proposition 4.10. Suppose $a \notin [F'',F,F]$. Then

$a \equiv \prod_{i,j} [s_{j(i)}, x_i]^{\alpha_{i,j}}$ modulo $[F'', F, F]$ with $s_{j(i)} \in S$, $x_i \in X$, $\alpha_{i,j} \in \mathbb{Z}$ and if $x_i = x_k$, $s_{j(i)} \neq s_{j(k)}$ (by Lemma 4.13).

Since $a \in 1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}^2$ and $[F'', F, F] \leq 1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}^2$

$\Rightarrow \prod_{i,j} [s_{j(i)}, x_i]^{\alpha_{i,j}} \in 1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}^2$.

$[s_{j(i)}, x_i] = 1 + s_{j(i)}^{-1} x_i^{-1}\{(s_{j(i)}-1)(x_i-1) - (x_i - 1)(s_{j(i)} - 1)\}$.

$$\equiv 1 + (s_{j(i)} - 1)(x_i - 1) - (x_i - 1)(s_{j(i)} - 1),$$

modulo $\mathfrak{f}\mathfrak{a}\mathfrak{f}^2$ since $s_{j(i)} - 1 \in \mathfrak{f}\mathfrak{a}^2 + \mathfrak{a}^2\mathfrak{f}$

$$\equiv 1 + (s_{j(i)} - 1)(x_i - 1) \text{ modulo } \mathfrak{f}\mathfrak{a}\mathfrak{f}^2.$$

Hence $\sum_{i,j} \alpha_{i,j}(s_{j(i)} - 1)(x_i - 1) \in \mathfrak{f}\mathfrak{a}\mathfrak{f}^2$

$\Rightarrow D_k \sum_{i,j} \alpha_{i,j} (s_{j(i)} - 1)(x_i - 1) \in \mathfrak{f}\mathfrak{a}\mathfrak{f}$

for all k by Theorem 1.7.

$\Rightarrow \sum_j \alpha_{k,j}(s_{j(k)} - 1) \in \mathfrak{f}\mathfrak{a}\mathfrak{f}$ for all k $\Rightarrow \alpha_{k,j} = 0$ for all k

by Proposition 4.10. Hence $a \in [F'', F, F]$.

<u>Corollary 1</u>: $[F'', F]/[F'', F, F] \simeq \dfrac{\mathfrak{u} + \mathfrak{f}\mathfrak{a}\mathfrak{f}^2}{\mathfrak{f}\mathfrak{a}\mathfrak{f}^2}$ where $\mathfrak{u} = \mathrm{Ker}\, \mathbb{Z}F \to \mathbb{Z}(F/[F'', F])$ and hence is free abelian with free generators given by Lemma 4.13. (Note $[F''/F]/[F'', F, F]$ is also the Schur Multiplier of $F/[F'', F]$.)

Proof: See proof of Corollary to Proposition 4.10.

Corollary 2: Let $P_{1,2}$ be the power series ring in X over $\mathbb{Z}$ subject to the relations $x_{i_1}(x_{i_2} x_{i_3} - x_{i_3} x_{i_2})x_{i_4} x_{i_5} = 0$, then subgroup of $U(P_{1,2})$ generated by 1 + X is isomorphic to F/[F", F, F].

Proof: From Lemma 2.19.

Theorem 4.15: F/[F", F, F] is residually torsion-free nilpotent.

Proof: See proof of Theorem 3.5.

Corollary: F/[F", F, F] is residually a finite p-group for all primes p.

Section 3:

Lemma 4.16: If a ε F", b and c ε F then $[a, b, c] \equiv [a, c, b]$ modulo $(F')_3$.

Proof:
$$\begin{aligned}[a, bc] &= [a, c][a, b][a, b, c] \\ &= [a, cb[b, c]] \\ &\equiv [a, cb] \\ &= [a, b][a, c][a, c, b]\end{aligned}$$

Hence result.

Lemma 4.17: [F", F, F] modulo $[F", F, F, F](F')_3$ is generated by $\{[s, x_i, x_j]/s \in S, x_i \text{ and } x_j \in X, \text{ and } i \leq j\}$.

Proof: Apply Lemma 4.16 for the condition $i \leq j$.

<u>Proposition 4.18</u>: $(1 + \mathfrak{f}^2\mathfrak{a}\mathfrak{f}^2) \cap F = [F'', F, F, F](F')_3$.

<u>Proof</u>: $(F')_3 \leq (1 + \mathfrak{a}^3) \cap F \leq (1 + \mathfrak{f}^2\mathfrak{a}\mathfrak{f}^2) \cap F$.

$[F'', F, F] \leq \{(1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}^2) \cap F\} \cap \{(1 + \mathfrak{f}^2\mathfrak{a}\mathfrak{f}) \cap F\}$ (see Proposition 4.14).

Let $a \in [F'', F, F]$, $b \in F$ then

$$[a, b] = 1 + a^{-1}b^{-1}\{(a-1)(b-1) - (b-1)(a-1)\}$$

$$\leq 1 + \mathfrak{f}^2\mathfrak{a}\mathfrak{f}^2 \text{ since } a - 1 \in \mathfrak{f}^2\mathfrak{a}\mathfrak{f} \cap \mathfrak{f}\mathfrak{a}\mathfrak{f}^2 .$$

Hence $[F'', F, F, F] \leq (1 + \mathfrak{f}^2\mathfrak{a}\mathfrak{f}^2) \cap F$. Suppose $a \in (1 + \mathfrak{f}^2\mathfrak{a}\mathfrak{f}^2) \cap F$. Then $a \in [F'', F, F]$ by Proposition 4.14. Suppose $a \notin [F'', F, F, F]\,(F')_3$ then by Lemma 4.17

$$a \equiv \prod_{i,j,k} [s_{k(i,j)}, x_i, x_j]^{\alpha_{k,i,j}} \text{ modulo } [F'', F, F, F](F')_3$$

with $s_{k(i,j)} \in S$ x_i and $x_j \in X$, $i \leq j$, $\alpha_{k,i,j} \in \mathbb{Z} - \{0\}$ and if $x_i = x_t$ and $x_j = x_u$ then $s_{k(i,j)} \neq s_{k(t,u)}$.

$$[s_{k(i,j)}, x_i, x_j] = 1 + [s_{k(i,j)}, x_i]^{-1} x_j^{-1}$$

$$\{([s_{k(i,j)}, x_i] - 1)(x_j-1) - (x_j-1)([s_{k(i,j)}, x_j] - 1)\}$$

$$\equiv 1 + ([s_{k(i,j)}, x_i] - 1)(x_j - 1) - (x_j - 1)\cdot$$

$([s_{k(i,j)}, x_j] - 1)$, modulo $\mathfrak{f}^2\mathfrak{a}\mathfrak{f}^2$ since $[F'', F] \leq 1 + \mathfrak{f}\mathfrak{a}\mathfrak{f}^2$.

Now $[s_{k(i,j)}, x_i] = 1 + s_{k(i,j)}^{-1} x_i^{-1}\{(s_{k(i,j)} - 1)(x_i - 1) - (x_i - 1)(s_{k(i,j)} - 1)\}$. Hence

$[s_{k(i,j)}, x_i, x_j] \equiv 1 + \{(s_{k(i,j)}-1)(x_i-1) - (x_i-1)(s_{k(i,j)}-1)\}$

$(x_j-1) - (x_j-1)\{(s_{k(i,j)}-1)(x_i-1) - (x_i-1)(s_{k(i,j)} - 1)\}$,

modulo $\mathfrak{f}^2\mathfrak{a}\mathfrak{f}^2$ since $F'' \leq 1 + \mathfrak{a}^2$.

$\equiv 1 - (x_i - 1)(s_{k(i,j)} - 1)(x_j - 1)$

$- (x_j-1)(s_{k(i,j)} - 1)(x_i - 1)$ modulo $\mathfrak{f}^2\mathfrak{a}\mathfrak{f}^2$

$= 1 + \gamma_{i,j,k}$ say.

$\Rightarrow \sum_{i,j,k} \alpha_{i,j,k}\,\gamma_{i,j,k} \in \mathfrak{f}^2\mathfrak{a}\mathfrak{f}^2$

$\Rightarrow \Sigma\, (D_t\, \alpha_{i,j,k}\, \gamma_{i,j,k})d_u \in \mathfrak{f}\mathfrak{a}\mathfrak{f}$

for all t and u by Theorem 1.7. If $u < t \Rightarrow \sum_k - \alpha_{u,t,k}$ $(s_{k(u,t)} - 1) \in \mathfrak{f}\mathfrak{a}\mathfrak{f}$ and if $u = t \Rightarrow \sum_k - 2\alpha_{t,t,k}(s_{k(t,t)} - 1) \in$ $\mathfrak{f}\mathfrak{a}\mathfrak{f}$. In any case this implies that $\alpha_{u,t,k} = 0$ for all u,t,k by Proposition 4.10.

$\Rightarrow a \in [F'', F, F, F](F')_3$.

Corollary 1: $[F'', F, F]/[F'', F, F, F](F')_3 \simeq \dfrac{\mathfrak{L} + \mathfrak{f}^2\mathfrak{a}\mathfrak{f}^2}{\mathfrak{f}^2\mathfrak{a}\mathfrak{f}^2}$

and hence is free abelian with free generators given by Lemma 4.15, *where* $\mathfrak{L} = \mathrm{Ker}\ \mathbb{Z}F \to \mathbb{Z}(F/[F'', F, F])$

Proof: See proof of Corollary to Proposition 4.10.

Corollary 2: Let $P_{2,2}$ be the power series ring in X over $\mathbb{Z}$ subject to the relations

$x_{i_1}x_{i_2}(x_{i_3}x_{i_4} - x_{i_4}x_{i_3})x_{i_5}x_{i_6}$ then subgroup of U(P) generated by 1 + X is isomorphic to $F/[F'', F, F, F](F')_3$.

Theorem 4.19: $F/[F'', F, F, F](F')_3$ is residually torsion-free nilpotent.

Proof: See proof of Theorem 3.5.

Corollary: $F/[F'', F, F, F](F')_3$ is residually a finite p-group for all primes p.

CHAPTER 5

Section 1: In this section we prove analogues of Magnus' Theorem (Theorem 1.5) for the groups F/F'', $F/(F')_3(F_3)'$ and $F/(F')_3(F_3)'$ and $F/(F')_3(F_4)'$ and compute the structure of the lower central factors of these groups. It seems probable that the methods devised here can be used to obtain the structure of the lower central factors of $F/[F'',F]$ and $F/[F'',F,F]$ but a Theorem like Theorems 5.3, 5.8 and 5.13 below is not true for $F/[F'',F]$ since Ridley [13] has shown that the lower central factors of $F/[F'',F]$ contain torsion elements.

Let $Q = P_{1,0}$, i.e. the power series ring in X over Z subject to $x_{i_1}(x_{i_2}x_{i_3} - x_{i_3}x_{i_2}) = 0$. Identify F/F'' with its (isomorphic) image G in Q. This representation of F/F'' is very similar to that obtained by Baumslag [1]. Compare Theorem 5.3 below with Theorem 2 in [1]. In fact $x_i \mapsto x_{2,i} + x_{1,i}$ gives a homomorphism from Q to Baumslag's power series. Every element s in the multiplicative semigroup of Q generated by X can be written uniquely in the form

$$s = x_{i_1}x_{i_2}\dots x_{i_n}, \quad i_2 \leq i_3 \leq \dots \leq i_n \qquad (1)$$

Let K_i be the ideal of elements in Q of order $\geq i$.

Lemma 5.1: In Q, $[1 + x_{i_1}, 1 + x_{i_2}, \ldots, 1 + x_{i_n}]$

$= 1 + (x_{i_1}x_{i_2} - x_{i_2}x_{i_1})x_{i_3}\ldots x_{i_n}$ for $n \geq 2$.

Proof: $[1 + x_{i_1}, 1 + x_{i_2}] = 1 + (1 + x_{i_1})^{-1}(1 + x_{i_2})^{-1}$

$\{(x_{i_1}x_{i_2} - x_{i_2}x_{i_1})\}$.

$= 1 + \{1-(1+x_{i_1})^{-1}x_{i_1}\}\{1-(1+x_{i_2})^{-1}x_{i_2}\}\{(x_{i_1}x_{i_2} - x_{i_2}x_{i_1})\}$.

$= 1 + x_{i_1}x_{i_2} - x_{i_2}x_{i_1}$

Hence we have result for n=2. Let $r \geq 3$. Then

$[1 + x_{i_1}, 1 + x_{i_2}, \ldots, 1 + x_{i_r}] = 1 + [1 + x_{i_1}, 1 + x_{i_2}, \ldots$

$\ldots, 1 + x_{i_{r-1}}]^{-1}(1 + x_{i_r})^{-1}\{([1 + x_{i_1}, 1 + x_{i_2}, \ldots, 1 + x_{i_{r-1}}]-1$

$x_{i_r} - x_{i_r}([1 + x_{i_1}, 1 + x_{i_2}, \ldots, 1 + x_{i_{r-1}}]-1)\}$

$= 1 + \{1 + (x_{i_2}x_{i_1} - x_{i_1}x_{i_2})x_{i_3}\ldots x_{i_{r-1}}\}\{1-(1+x_{i_r})^{-1}x_{i_r}\}$

$\{(x_{i_1}x_{i_2} - x_{i_2}x_{i_1})x_{i_3}\ldots x_{i_r} - x_{i_r}(x_{i_1}x_{i_2} - x_{i_2}x_{i_1})x_{i_3}\ldots$

$\ldots x_{i_{r-1}}\}$ by an inductive argument.

$= 1 + (x_{i_1}x_{i_2} - x_{i_2}x_{i_1})x_{i_3}\ldots x_{i_r}$.

Theorem 5.2: The basic commutators of G weight n freely generate modulo K_{n+1} a free abelian group.

Proof: A basic commutator of G weight n is of the form $a = [1 + x_{i_1}, 1 + x_{i_2}, \ldots, 1 + x_{i_n}]$, $i_1 > i_2$, $i_2 \leq i_3 \leq \ldots \leq i_n$. By Lemma 5.1, $a = 1 + (x_{i_1}x_{i_2} - x_{i_2}x_{i_1})x_{i_3} \ldots x_{i_n}$ and result follows from (1).

Corollary: (a theorem of Magnus, see Neumann (Chap. 3) [12]). The basic commutators in the free metabelian group are linearly independent.

Theorem 5.3: $(1 + K_i) \cap G = G_i$

Proof: Clearly $G_i \leq (1 + K_i) \cap G$ and $(1 + K_1) \cap G = G_1 = G$. We proceed by induction on i. Suppose $a \in (1 + K_{i+1}) \cap G$ and $a \notin G_{i+1}$. By induction $a \in G_i \Rightarrow a = bc$, $c \in G_{i+1}$ $b(\neq 1) \in G_i$, and is a product of basic commutators weight i. $a-1 \in K_{i+1} \Rightarrow b-1 \in K_{i+1}$, since $G_{i+1} \subseteq 1 + K_{i+1}$. This contradicts Theorem 5.2.

Corollary: $(1 + \mathfrak{g}^n) \cap G = G_n$ where $\mathfrak{g}$ is the augmentation ideal of G.

Proof: Clearly $G_n \leq (1 + \mathfrak{g}^n) \cap G$. The map $\phi' : F/F'' \to U(Q)$ given by $y_i \mapsto 1 + x_i$ can be extended (uniquely) to a map $\phi : \mathbb{Z}(F/F'') \to Q$. Then $\phi : \mathfrak{g}^n \to K_n$. If $a \in (1 + \mathfrak{g}^n) \cap F \Rightarrow (a-1)\phi = a\phi - 1 = a\phi' - 1 \in K_n$. $\Rightarrow a\phi' \in G_n$ by the theorem. /

Let $P = P_{3,0}$, the power series ring in X over $\mathbb{Z}$ subject to $x_{i_1}x_{i_2}x_{i_3}(x_{i_4}x_{i_5} - x_{i_5}x_{i_4}) = 0$. Every element s in the multiplicative semigroup of P generated by X can be written uniquely in the form

$$s = x_{i_1}x_{i_2}x_{i_3}x_{i_4}\cdots x_{i_n}, \quad i_4 \leq i_5 \leq \cdots \leq i_n \qquad (2)$$

By Lemma 3.2 Corollary $F/(F')_3(F_3)'$ is embedded in P and we identify $F/(F')_3(F_3)'$ with its image H in P. Let R_i be the ideal of elements in P of order $\geq$ i and let ψ be the natural homomorphism from P to Q. When $1 + x_i$ occurs as an entry of a commutator we shall write x_i instead of $1 + x_i$, it being clear that $1 + x_i$ is meant. (We shall continue to use this convention from now on). What do the basics of H look like?

Lemma 5.4: H_n modulo H_{n+1} is generated by the basics of the forms either

(i) $[x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$, $i_1 > i_2$, $i_2 \leq i_3 \leq \cdots \leq i_n$

or

(ii) $\Big[[x_{i_1}, x_{i_2}, \ldots, x_{i_{n-2}}], [x_i, x_j]\Big]$, $i_1 > i_2$

$i_2 \leq i_3 \leq \cdots \leq i_{n-2}$, $i > j$, $n \geq 4$ and when $n = 4$ $x_{i_1}, x_{i_2} > x_i, x_j$ in the ordering of the basic 2-commutators.

Proof: Is clear since $[H_3', H_3] = 1$ and $[H_2, H_2, H_2] = 1$, i.e. the basics of any other type vanish.

We shall call the basics in the statement of the Lemma basics of type (i) and type (ii) respectively.

Lemma 5.5: The nth homogeneous component (in P) of $\left[\left[x_{i_1},x_{i_2},\ldots,x_{i_{n-2}}\right],\left[x_i,x_j\right]\right]-1$ is

(a) $(x_{i_1}x_{i_2} - x_{i_2}x_{i_1})(x_ix_j - x_jx_i) - (x_ix_j - x_jx_i)$

$(x_{i_1}x_{i_2} - x_{i_2}x_{i_1})$, for $n=4$.

(b) $-(x_ix_j - x_jx_i)(x_{i_1}x_{i_2} - x_{i_2}x_{i_1})x_{i_3}\ldots x_{i_{n-2}}$, for $n > 4$.

Proof: (a) is clear. The nth homogeneous component of $\left[\left[x_{i_1},x_{i_2},\ldots,x_{i_n}\right],\left[x_i,x_j\right]\right]-1$

$= ((x_{i_1},x_{i_2},\ldots,x_{i_{n-2}}),(x_i,x_j))$

$= (x_{i_1},x_{i_2},\ldots,x_{i_{n-2}})(x_i,x_j)-(x_i,x_j)(x_{i_1},x_{i_2},\ldots,x_{i_{n-2}})$

$= -(x_i,x_j)(x_{i_1},x_{i_2},\ldots,x_{i_{n-2}})$, when $n > 4$.

$= -(x_ix_j-x_jx_i)\{(x_{i_1},x_{i_2},\ldots,x_{i_{n-3}})x_{i_{n-2}}$ $-$

$x_{i_{n-2}}(x_{i_1},x_{i_2},\ldots,x_{i_{n-3}})\}$.

$= -(x_ix_j - x_jx_i)(x_{i_1},x_{i_2},\ldots,x_{i_{n-3}})x_{i_{n-2}}$, since

$(x_{i_1},x_{i_2},\ldots,x_{i_{n-3}})$ involves $(x_{i_1}x_{i_2}-x_{i_2}x_{i_1})$ implicitly

$= -(x_i x_j - x_j x_i)(x_{i_1} x_{i_2} - x_{i_2} x_{i_1}) x_{i_3} \dots x_{i_{n-2}}$, by an inductive argument.

<u>Theorem 5.6</u>: The basic commutators of H of weight n, types (i) and (ii), free generate modulo R_{n+1} a free abelian group.

<u>Proof</u>: Need only prove linear independence. By taking the map ψ from P to Q we see that it suffices to prove that the basic commutators type (ii) are linearly independent (by Theorem 5.2).

By Lemma 5.5, the leading term of $a = \left[[x_{i_1}, x_{i_2}, \dots, x_{i_{n-2}}], [x_i, x_j] \right]$ is

$(x_{i_1} x_{i_2} - x_{i_2} x_{i_1})(x_i x_j - x_j x_i) - (x_i x_j - x_j x_i)(x_{i_1} x_{i_2} - x_{i_2} x_{i_1})$

for n = 4 and $-(x_i x_j - x_j x_i)(x_{i_1} x_{i_2} - x_{i_2} x_{i_1}) x_{i_3} \dots x_{i_{n-2}}$

for $n > 4$.

The proof for the case n = 4 follows from (2) and Theorem 1.9 (i.e. for case n = 4 we have the same situation as for the absolutely free case). So need only consider $n > 4$. If not linearly independent we must try to find a commutator not a which will give an inverse for $p = x_i x_j x_{i_1} x_{i_2} \dots x_{i_{n-2}}$. By (2) the 2-commutator part of this basic (which is to give an inverse for p) must be $[x_i, x_j]$. Also by (2) x_{i_1} must be an entry of the head of the other part of this basic and $x_{i_2}, x_{i_3}, \dots, x_{i_{n-2}}$ must be the other entries of this basic. By the ordering of the indices we get a contradiction.

As corollaries to this we get the following theorems.

Theorem 5.7: H_n modulo H_{n+1} is free abelian, freely generated by the basics type (i) and type (ii).

Theorem 5.8: $(1 + R_i) \cap H = H_i$

Proof: See proof of Theorem 5.3.

Corollary: $(1 + \mathcal{H}^n) \cap H = H_n$, where $\mathcal{H}$ is the augmentation ideal of H.

Proof: See proof of Corollary to Theorem 5.3. /

Let $S = P_{4,0}$ be the power series ring in X over $\mathbb{Z}$ subject to $x_{i_1}x_{i_2}x_{i_3}x_{i_4}(x_{i_5}x_{i_6} - x_{i_6}x_{i_5}) = 0$. Then $F/(F')_3(F_4)'$ is embedded in S by Lemma 3.3 Corollary, and we identify $F/(F')_3(F_4)'$ with its image L in S. Every element w in the multiplicative semigroup of S generated by X can be written uniquely in the form

$$w = x_{i_1}x_{i_2}x_{i_3}x_{i_4}x_{i_5}\dots x_{i_n}, \quad i_5 \le i_6 \le \dots \le i_n \qquad (3)$$

Let T_i be the ideal of elements in S of order $\geq$ i and let θ be the natural homomorphism from S to P.

Lemma 5.9: L_n modulo L_{n+1} is generated by the basic commutators of the forms either

(i) $[x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$, $i_1 > i_2$, $i_2 \leq i_3 \leq \cdots \leq i_n$

or

(ii) $[[x_{i_1}, x_{i_2}, \ldots, x_{i_{n-2}}], [x_i, x_j]]$, $i_1 > i_2$,

$i_2 \leq i_3 \leq \cdots \leq i_{n-2}$, $i > j$, $n \geq 4$ and for $n=4$

$[x_{i_1}, x_{i_2}] > [x_i, x_j]$ in the ordering of the basic 2-commutators.

or

(iii) $[[x_{i_1}, x_{i_2}, \ldots, x_{i_{n-3}}], [x_i, x_j, x_k]]$,

$i_1 > i_2$, $i_2 \leq i_3 \leq \cdots \leq i_{n-3}$, $i > j$, $j \leq k$, $n \geq 6$ and for $n = 6$, $[x_{i_1}, x_{i_2}, x_{i_3}] > [x_i, x_j, x_k]$ in the ordering of the basic 3-commutators.

Proof: Is clear since $[L_4, L_4] = 1$ and $[L_2, L_2, L_2] = 1$.

We shall call the basics in the statement of the Lemma basics of type (i), type (ii) and type (iii) respectively.

Lemma 5.10: In S, the nth homogeneous component of $[[x_{i_1}, x_{i_2}, \ldots, x_{i_{n-3}}], [x_i, x_j, x_k]]-1$ is

(α) $(x_{i_1}, x_{i_2}, x_{i_3})(x_i x_j - x_j x_i)x_k - (x_i, x_j, x_k)(x_{i_1} x_{i_2} - x_{i_2} x_{i_1})x_{i_3}$, for $n = 6$.

(β) $-(x_i, x_j, x_k)(x_{i_1} x_{i_2} - x_{i_2} x_{i_1})x_{i_3} \cdots x_{i_{n-3}}$, for $n > 6$.

Proof: (α):- the 6th homogeneous component of

$[[x_{i_1},x_{i_2},x_{i_3}],[x_i,x_j,x_k]]-1$ is

$((x_{i_1},x_{i_2},x_{i_3}),(x_i,x_j,x_k))$

$= (x_{i_1},x_{i_2},x_{i_3})(x_i,x_j,x_k) - (x_i,x_j,x_k)(x_{i_1},x_{i_2},x_{i_3})$

$= (x_{i_1},x_{i_2},x_{i_3})\{(x_ix_j-x_jx_i)x_k-x_k(x_ix_j-x_jx_i)\}$

$- (x_i,x_j,x_k)\{(x_{i_1}x_{i_2}-x_{i_2}x_{i_1})x_{i_3}-x_{i_3}(x_{i_1}x_{i_2}-x_{i_2}x_{i_1})\}$

$= (x_{i_1},x_{i_2},x_{i_3})(x_ix_j-x_jx_i)x_k - (x_i,x_j,x_k)(x_{i_1}x_{i_2}-x_{i_2}x_{i_1})x_{i_3}$

(β):- nth homogeneous component of

$[[x_{i_1},x_{i_2},\ldots,x_{i_{n-3}}],[x_i,x_j,x_k]]-1$ is

$((x_{i_1},x_{i_2},\ldots,x_{i_{n-3}}),(x_i,x_j,x_k))$

$= (x_{i_1},x_{i_2},\ldots,x_{i_{n-3}})(x_i,x_j,x_k)-(x_i,x_j,x_k)(x_{i_1},x_{i_2},\ldots,x_{i_{n-3}}).$

$= -(x_i,x_j,x_k)(x_{i_1},x_{i_2},\ldots,x_{i_{n-3}})$, since $n > 6$ and

(x_i,x_j,x_k) contains $(x_ix_j-x_jx_i)$ implicitly.

$= -(x_i,x_j,x_k)\{(x_{i_1},x_{i_2},\ldots,x_{i_{n-4}})x_{i_{n-3}} - x_{i_{n-3}}(x_{i_1},x_{i_2},\ldots$

$\ldots,x_{i_{n-4}})\}.$

$= -(x_i,x_j,x_k)(x_{i_1},x_{i_2},\ldots,x_{i_{n-4}})x_{i_{n-3}}.$

$= -(x_i,x_j,x_k)(x_{i_1}x_{i_2}-x_{i_2}x_{i_1})\,x_{i_3}\ldots x_{i_{n-3}}$, by an inductive argument.

For basics type (iii) call $[x_{i_1}, x_{i_2}, \ldots, x_{i_{n-3}}]$ the <u>leading part</u> of the double commutator

$$[[x_{i_1}, x_{i_2}, \ldots, x_{i_{n-3}}], [x_i, x_j, x_k]]$$

<u>Theorem 5.11</u>: The basic commutators of L of weight n, types (i), (ii) and (iii), freely generate modulo T_{n+1} a free abelian group.

<u>Proof</u>: Need only prove linear independence. By taking the map θ from S to P we need only show that the basics type (iii) are linearly independent, by Theorem 5.6. By Lemma 5.10, the leading term **t** of $a = [[x_{i_1}, x_{i_2}, \ldots, x_{i_{n-3}}], [x_i, x_j, x_k]]$ is

(α) $(x_{i_1}, x_{i_2}, x_{i_3})(x_i x_j - x_j x_i)x_k - (x_i, x_j, x_k)(x_{i_1} x_{i_2} - x_{i_2} x_{i_1}) x_{i_3}$

for n=6,

(β) $-(x_i, x_j, x_k)(x_{i_1} x_{i_2} - x_{i_2} x_{i_1})x_{i_3} \ldots x_{i_{n-3}}$ for $n > 6$.

First of all we shall take case n=6.

$t = \{(x_{i_1} x_{i_2} - x_{i_2} x_{i_1})x_{i_3} - x_{i_3}(x_{i_1} x_{i_2} - x_{i_2} x_{i_1})\}$

$(x_i x_j - x_j x_i)x_k - \{(x_i x_j - x_j x_i) x_k - x_k(x_i x_j - x_j x_i)\}$

$(x_{i_1} x_{i_2} - x_{i_2} x_{i_1})x_{i_3}$.

If not linearly independent, then there exists a basic type (iii) (length 6) not a which gives an inverse for $p = x_{i_2}x_{i_1}x_{i_3}x_ix_jx_k$. By (3) x_{i_2}, x_{i_1} and x_{i_3} must be entries of the same part of this basic (which is to give an inverse for p). Thus one part must be either $[x_{i_1}, x_{i_2}, x_{i_3}]$ or $[x_{i_3}, x_{i_2}, x_{i_1}]$. For $i_3 \neq i_1$ $(x_{i_3}, x_{i_2}, x_{i_1})$ does not involve $x_{i_3}, x_{i_2}, x_{i_1}$ in the sequence i_2, i_1, i_3. $\Rightarrow$ one part of the basic must be $x_{i_1}, x_{i_2}, x_{i_3}$.

By (3) x_i must be an entry of the head of another part and x_{i_2} and x_{i_3} must also be entries of this part. $\Rightarrow$ The other part must be $[x_i, x_j, x_k]$. Hence there is no inverse for p. ($[x_{i_1}, x_{i_2}, x_{i_3}] > [x_i, x_j, x_k]$ in the ordering of the basic 3-commutators).

We now consider case n > 6.

$(x_i, x_j, x_k) = (x_ix_j - x_jx_i)x_k - x_k(x_ix_j - x_jx_i)$. If not linearly independent, then there exists a basic commutator type (iii) (length n), not a, which gives an inverse for $p = x_jx_ix_kx_{i_1}x_{i_2}x_{i_3}\ldots x_{i_{n-3}}$. By (3) x_j, x_i and x_k must be the entries of the last part of this basic (which is to give an inverse for p). Hence last part must be either $[x_i, x_j, x_k]$ or $[x_k, x_j, x_i]$. For $i \neq k$ (x_k, x_j, x_i) does not involve x_k, x_j, x_i in the sequence j,i,k.

$\Rightarrow$ last part must be $[x_i, x_j, x_k]$. By (3) x_{i_1} must be an entry of the head of the other part and $x_{i_2}, \ldots, x_{i_{n-2}}$ must be the other entries of this part.

$\Rightarrow$ other part must be $[x_{i_1}, x_{i_2}, \ldots, x_{i_{n-3}}]$. Hence we have no inverse for p which is a contradiction.

As corollaries to this we get the following:

Theorem 5.12: L_n modulo L_{n+1} is free abelian freely generated by the basic types (i), (ii) and (iii).

Theorem 5.13: $(1 + T_n) \cap L = L_n$

Proof: See proof of Theorem 5.3.

Corollary: $(1 + \mathcal{L}^i) \cap L = L_i$ where $\mathcal{L}$ is the augmentation ideal of L.

Proof: See proof of corollary to Theorem 5.3.

Section 2:

An old problem of Fox [4] is the determination of $\mathfrak{f}^n\mathfrak{r}$ i.e. to give an explicit form for $(1 + \mathfrak{f}^n\mathfrak{r}) \cap F$.

Theorem 1.6 shows $(1 + \mathfrak{f}\mathfrak{r}) \cap F = R'$ and in this section we make a small contribution by showing $(1 + \mathfrak{f}^2\mathfrak{r}) \cap F = [R \cap F', R \cap F'] R_3$.

Proposition 5.14: $(1 + \mathfrak{f}^2\mathfrak{r}) \cap F = [R \cap F', R \cap F'] R_3$

Proof: Now $R_3 \leq 1 + \mathfrak{r}^3 \leq 1 + \mathfrak{f}^2\mathfrak{r}$

Hence $R_3 \leq (1 + \mathfrak{f}^2\mathfrak{r}) \cap F$.

Let $a \in R \cap F'$ and $b \in R \cap F'$ then

$$[a,b] = 1 + a^{-1}b^{-1}\{(a-1)(b-1)-(b-1)(a-1)\}$$
$$\in 1 + \mathfrak{f}^2\mathfrak{r}.$$

Suppose $a \in (1 + \mathfrak{f}^2\mathfrak{r}) \cap F$. Then by Theorem 1.6 $a \in R'$. Let R be free on W (F is free on X). Then $a \equiv \Pi\, [w_i, w_j] \bmod R_3$. Call this product (1). We use induction on the number of distinct free generators w that occur in the product (1) to show that $a \equiv 1 \bmod [R \cap F', R \cap F']R_3$. If there is no free generator in the product we are through. Let w be a particular free generator of R occuring in the product. We can now collect in one commutator mod R_3 all the commutators involving w thus:-

$$a \equiv \left[w_{i_1}^{\alpha_{i_1}} w_{i_2}^{\alpha_{i_2}} \cdots w_{i_n}^{\alpha_{i_n}},\ w\right] \Pi\, [w_i, w_j]$$

where the w_k's in the product do not involve w, and $i_1 < i_2 < \ldots < i_n$ (This latter condition is not necessary for the argument). If b and $c \in R$ then $[b,c] \equiv 1 + (b-1)(c-1)-(c-1)(b-1) \bmod \mathfrak{f}^2\mathfrak{r}$. Hence since $a-1 \in \mathfrak{f}^2\mathfrak{r}$ and $R_3 \leq 1 + \mathfrak{f}^2\mathfrak{r}$ this implies that

$$(w_{i_1}^{\alpha_{i_1}} w_{i_2}^{\alpha_{i_2}} \cdots w_{i_n}^{\alpha_{i_n}} - 1)(w-1)-(w-1)(w_{i_1}^{\alpha_{i_1}} w_{i_2}^{\alpha_{i_2}} \cdots w_{i_n}^{\alpha_{i_n}} -1)$$
$$+ \sum\{(w_i-1)(w_j-1)-(w_j-1)(w_i-1)\} = f \in \mathfrak{f}^2\mathfrak{r}.$$ This

implies by Theorem 1.7 that $f\, d_k \in \mathfrak{f}\mathfrak{W}$ for all k.

$$\Rightarrow (w_{i_1}^{\alpha_{i_1}} w_{i_2}^{\alpha_{i_2}} \cdots w_{i_n}^{\alpha_{i_n}} - 1)d_k(w-1)$$

$$- (w-1)d_k(w_{i_1}^{\alpha_{i_1}} w_{i_2}^{\alpha_{i_2}} \cdots w_{i_n}^{\alpha_{i_n}} - 1) + \sum\{(w_i-1)d_k(w_j-1) -$$

$$(w_j-1)d_k(w_i-1)\}.$$

We note that $q \equiv q\varepsilon \bmod \mathfrak{f}$ for any $q \in ZF$ (where ε = the augmentation) and hence since $\mathfrak{W}/\mathfrak{f}\mathfrak{W}$ is free abelian on W-1 by Lemma 1.4,

$$(w_{i_1}^{\alpha_{i_1}} w_{i_2}^{\alpha_{i_2}} \cdots w_{i_n}^{\alpha_{i_n}} - 1)d_k \in \mathfrak{f}$$

for all k $\Rightarrow$ by theorem 1.7 that

$$w_{i_1}^{\alpha_{i_1}} w_{i_2}^{\alpha_{i_2}} \cdots w_{i_n}^{\alpha_{i_n}} - 1 \in \mathfrak{f}^2 \Rightarrow w_{i_1}^{\alpha_{i_1}} w_{i_2}^{\alpha_{i_2}} \cdots w_{i_n}^{\alpha_{i_n}} \in F'$$

by Magnus' Theorem 1.5. In a similar manner we can collect in one commutator all the commutators of the product (1) involving w_{i_j} for $1 \leq j \leq n$ and by a similar argument we get that

$w^{\alpha_{i_j}} t_{i_j} \in F'$ for some t_{i_j} which is a product of w_t which are involved in the product (1), $w_t \neq w$ (and $w_t \neq w_{i_j}$). Let d be the highest common factor of $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}$. Then there exist integers

$s_{i_1}, s_{i_2}, \ldots, s_{i_n}$ such that $\alpha_{i_1} s_{i_1} + \alpha_{i_2} s_{i_2} + \ldots$

$\ldots + \alpha_{i_n} s_{i_n} = d$. Since $w^{\alpha_{i_j}} t_{i_j} \in F'$

$\Rightarrow w^{\alpha_{i_j} s_{i_j}} t_{i_j}^{s_{i_j}} \in F'$. Also since $w_{i_1}^{\alpha_{i_1}} w_{i_2}^{\alpha_{i_2}} \ldots w_{i_n}^{\alpha_{i_n}} \in F'$

$\Rightarrow (w_{i_1}^{\alpha_{i_1}/d} w_{i_2}^{\alpha_{i_2}/d} \ldots w_{i_n}^{\alpha_{i_n}/d})^d \in F'$ but since F/F' is

torsion free $\Rightarrow w_{i_1}^{\alpha_{i_1}/d} w_{i_2}^{\alpha_{i_2}/d} \ldots w_{i_n}^{\alpha_{i_n}/d} \in F'$. All

congruences from here on are mod $[R \cap F', R \cap F'] R_3$.

$$[w_{i_1}^{\alpha_{i_1}} w_{i_2}^{\alpha_{i_2}} \ldots w_{i_n}^{\alpha_{i_n}}, w].$$

$$\equiv [w_{i_1}^{\alpha_{i_1}/d} w_{i_2}^{\alpha_{i_2}/d} \ldots w_{i_n}^{\alpha_{i_n}/d}, w^d]$$

$$= [w_{i_1}^{\alpha_{i_1}/d} w_{i_2}^{\alpha_{i_2}/d} \ldots w_{i_n}^{\alpha_{i_n}/d}, w^{\alpha_{i_1} s_{i_1} + \ldots + \alpha_{i_n} s_{i_n}}]$$

$$\equiv \prod_{j=1}^{n} [w_{i_1}^{\alpha_{i_1}/d} w_{i_2}^{\alpha_{i_2}/d} \ldots w_{i_n}^{\alpha_{i_n}/d}, w^{\alpha_{i_j} s_{i_j}}]$$

$$\equiv \Pi [w_{i_1}^{\alpha_{i_1}/d} w_{i_2}^{\alpha_{i_2}/d} \ldots w_{i_n}^{\alpha_{i_n}/d}, t_{i_j}^{-s_{i_j}}]$$

This implies that $a \equiv \Pi' [w_i, w_j]$ where now the product involves one less distinct free generator of R. Since

$[R\cap F', R\cap F']R_3 \le 1 + \mathfrak{f}^2\mathfrak{r} \Rightarrow \Pi'[w_i, w_j] \in 1 + \mathfrak{f}^2\mathfrak{r}$ and hence by inductive hypothesis $\Pi'[w_i, w_j] \equiv 1 \Rightarrow$ $a \equiv 1$.

<u>Corollary</u>: $R'/[R\cap F', R\cap F']R_3 \cong \dfrac{\mathfrak{r}^{[2]} + \mathfrak{f}^2\mathfrak{r}}{\mathfrak{f}^2\mathfrak{r}}$ (where $\mathfrak{r}^{[2]} = \operatorname{Ker} \mathbb{Z}F \to \mathbb{Z}(F/R')$), and hence is free abelian, being a subgroup of $\mathfrak{f}\mathfrak{r}/\mathfrak{f}^2\mathfrak{r}$. (See Lemma 1.4)

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