Longest paths in strong spanning oriented subgraphs of strong semicomplete multipartite digraphs

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Abstract

A digraph obtained by replacing each edge of a complete multipartite graph by an arc or a pair of mutually opposite arcs with the same end vertices is called a semicomplete multipartite digraph. L. Volkmann (1998) raised the following question: Let D be a strong semicomplete multipartite digraph with a longest path of length l. Does there exist a strong spanning oriented subgraph of D with a longest path of length l? We provide examples which show that the answer to this question is negative. We also demonstrate that every strong semicomplete multipartite digraph D, which is not bipartite with a partite set of cardinality one, has a strong spanning oriented subgraph of D with a longest path of length at least $l-2$. This bound is sharp.

Keywords: Semicomplete multipartite digraph; Path; Spanning subgraph

1 Introduction, terminology and results

We assume that the reader is familiar with the standard terminology on graphs and digraphs and refer the reader to [3]. We will define necessary (less standard) terms below.

By a *cycle* and a *path* in a digraph we mean a directed simple cycle and path, respectively. A *biorientation* of an undirected graph G is a digraph obtained from G by replacing each edge $\{x, y\}$ of G with either the arc xy or the arc yx or both xy and yx. A biorientation D of G is *complete* if $xy \in D$ implies $yx \in D$ for every pair x, y of distinct vertices of G. The complete biorientation of G is denoted by $\vec{G}$. The complete biorientations of stars are of importance in this paper: $\mathcal{ST} = \{\vec{K}_{1,n-1} : \ n \geq 2\}$.

A digraph D is *strong* if, for every pair x, y of distinct vertices of D, D has both a path from x to y and a path from y to x. A digraph D is *connected* if its underlying graph is connected.

An *oriented graph* is a digraph with no cycle of length two. If a digraph D has an arc xy, then we often use the notation $x \rightarrow y$ and say that x *dominates* y and y *is dominated by* x. A *semicomplete multipartite digraph* is a biorientation of a complete multipartite graph. Semicomplete multipartite digraphs are well studied, see, e.g., [1, 4, 6, 8, 10, 11].

L. Volkmann [9] observed that a strong semicomplete multipartite digraph D has a strong spanning oriented subgraph if and only if $D \notin \mathcal{ST}$. He raised the following question: Let D be a strong semicomplete multipartite digraph, $D \notin \mathcal{ST}$, with a longest path of length l. Does there exist a strong spanning oriented subgraph of D with a longest path of length l? The following related result from [5] might suggest that the answer to this question is positive. Let G be a bridgeless graph, let D be a complete biorientation of G, and let l be the length of a longest path in D. Then D contains a strong spanning oriented subgraph with a path of length l. Another assertion which might lead to the same suggestion is due to Volkmann [9]. Let D be a strong semicomplete multipartite digraph, $D \notin \mathcal{ST}$, with a longest cycle C. Then D contains a strong spanning oriented subgraph which also has the cycle C.

However, the above suggestion turns out to be false. We provide examples which show that the answer to Volkmann's question is negative (see Proposition 1.3). We also prove (in the next section) the following result.

Theorem 1.1 *Let D be a strong semicomplete multipartite digraph, $D \notin \mathcal{ST}$, and let $P = p_0 p_1 \dots p_l$ be a longest path in D. Define $\delta_1(P, D)$ and $\delta_2(P, D)$ as follows:*

If $p_0 p_1 p_0$ is a 2-cycle in D and p_0 is dominated by only one vertex (p_1), then set $\delta_1(P, D) = 1$, otherwise set $\delta_1(P, D) = 0$. Analogously, if $p_l p_{l-1} p_l$ is a 2-cycle and p_l dominates only one vertex (p_{l-1}), then set $\delta_2(P, D) = 1$, otherwise set $\delta_2(P, D) = 0$.

Then, there is a strong spanning oriented subgraph of D which has a path of length l', where $l - \delta_1(P, D) - \delta_2(P, D) \leq l' \leq l$.

This theorem immediately implies the following:

Corollary 1.2 *Let D be a strong semicomplete multipartite digraph, $D \notin \mathcal{ST}$, and let l be the length of a longest path in D. Then D contains a strong spanning oriented subgraph with a path of length at least $l-2$.*

It follows from the next result that the bound in the corollary is sharp. Proposition 1.3 is proved in the next section.

Proposition 1.3 *For every integer $p \geq 3$, there exists an infinite family $\mathcal{F}_p$ of strong semicomplete p-partite digraphs such that every digraph D in $\mathcal{F}_p$ contains hamiltonian paths, yet, a longest path of any strong spanning oriented subgraph of D has $n-2$ vertices, where n is the order of D.*

Let $\mathcal{P}$ be a property of a strong semicomplete multipartite digraph D, $D \notin \mathcal{ST}$. Then $\mathcal{P}$ is said to be *2-cycle-independent* if there exists a strong spanning oriented subgraph of D with property $\mathcal{P}$. Volkmann [9] suggested to find interesting 2-cycle-independent properties of strong semicomplete multipartite digraphs. We saw earlier that the property to contain a longest cycle of length l is 2-cycle-independent, while the property to have a longest path of length l is not.

2 Proofs

In our proofs we will use some additional terminology and notation. For disjoint sets X and Y of vertices in a digraph D, we say that X *strongly dominates* Y, and use the notation $X \Rightarrow Y$, if there is no arc from Y to X. This means that for every pair $x \in X, y \in Y$ of adjacent vertices x dominates y, but y does not dominate x (there can be non-adjacent pairs $x \in X, y \in Y$). A *k-path-cycle factor*, F, ($k \geq 0$) of a digraph D is a vertex-disjoint collection of k paths and some number of cycles such that every vertex of D is in F. Note that if $k \geq 1$, then a k-path-cycle factor does not have to contain any cycles. For a vertex x of a semicomplete multipartite digraph D, $PS(x)$ is the partite set of D, which contains the vertex x.

To provide a short proof of Proposition 1.3, we will use the following lemma established in [7].

Lemma 2.1 *A digraph D has no k-path-cycle factor ($k \geq 0$) if and only if its vertex set $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$, the set Y is independent (i.e., contains no adjacent vertices) and $|Y| > |Z| + k$.*

Proof of Proposition 1.3:

Let $s \geq 7$ be an odd integer. We first construct $\mathcal{F}_3$. We define its member D_s as follows. The vertex set of D_s is $\{x_1, x_2, ..., x_s, y, z\}$; its partite sets are

$$V_1 = \{x_2, x_4, ..., x_{s-1}, y, z\}, V_2 = \{x_{4i+1} : \ 1 \leq 4i+1 \leq s\}, V_3 = \{x_{4i+3} : \ 1 \leq 4i+3 \leq s\}.$$

There are only two 2-cycles in D_s: the cycles x_1yx_1 and x_szx_s. For every $1 \leq i < j \leq s$, if $PS(x_i) \neq PS(x_j)$, then $x_i \rightarrow x_j$ unless $j = i+2$, in which case $x_j \rightarrow x_i$. For every $x_i \notin V_1$, $y \rightarrow x_i \rightarrow z$. Also, $x_1 \rightarrow y$ and $z \rightarrow x_s$.

Clearly, D_s has a hamiltonian path, $yx_1x_2....x_sz$. The only strong spanning oriented subgraph of D is the digraph $D' = D - \{yx_1, x_sz\}$. Set $Y = V_1$, $R_1 = \{x_1\}$, $R_2 = \{x_s\}$ and $Z = \{x_3, x_5, x_7, ..., x_{s-2}\}$. It is easy to verify that, in D', the above four sets satisfy the conditions of Lemma 2.1 for $k = 2$. Thus, D' has no 2-path-cycle factor. This implies that D' has no path with $s+1$ vertices (such a path and the remaining vertex would form a 2-path-cycle factor of D'). Hence, $x_1x_2....x_s$ is a longest path in D'.

Now let $p \geq 4$ be an integer. For every sufficiently large odd s, one can easily transform D_s into a semicomplete p-partite digraph $D(s,p)$, a member of $\mathcal{F}_p$, by introducing a transitive tournament on some $p-2$ vertices of V_3. The digraph $D(s,p)$ has a hamiltonian path as D_s does. At the same time, the only strong spanning oriented graph of $D(s,p)$ has a longest path on s vertices, which can be seen as above using Lemma 2.1. □

To prove Theorem 1.1, we need the next lemma:

Lemma 2.2 *Let D be a strong semicomplete multipartite digraph, $D \notin \mathcal{ST}$. If xyx is a 2-cycle in D, then either $D - xy$ or $D - yx$ (or both) are strong.*

This lemma follows from the next reformulation of a theorem by Boesch and Tindell [2] whose short proof is given by Volkmann [9]: Let D be a strong digraph and let uvu be a 2-cycle in D. Then at least one of the digraphs $D - uv$ and $D - vu$ is strong if and only if $D - \{uv, vu\}$ is connected. Indeed, since $D \notin \mathcal{ST}$, the deletion of any 2-cycle from D leaves D connected (the underlying graph of D is bridgeless as this graph is complete multipartite but not a star).

Proof of Theorem 1.1:

We prove this by induction on the number of 2-cycles in D. Let Q be the set of all 2-cycles in D. Clearly the theorem is true if $Q = \emptyset$, so assume that $|Q| = m > 0$, and that the theorem is true for all digraphs with $m-1$ 2-cycles. Let $P = p_0p_1...p_l$ be a longest path in D.

If there is any 2-cycle in D, not of the form $p_ip_{i+1}p_i$ ($i \in \{0, 1, \ldots, l-1\}$), then by Lemma 2.2 we may delete one arc in the 2-cycle, and still have a strong semicomplete multipartite digraph. Now we are done by our induction hypothesis. If there is a 2-cycle in D, of the form $p_ip_{i+1}p_i$ ($i \in \{0, 1, \ldots, l-1\}$) such that $D - p_{i+1}p_i$ is strong, then we may delete the arc $p_{i+1}p_i$, and thereby obtain the desired result by our induction hypothesis.

Therefore we may assume that Q consists of 2-cycles of the form $p_ip_{i+1}p_i$ ($i \in \{0, 1, \ldots, l-1\}$), and that $D - p_{i+1}p_i$ is not strong and $D - p_ip_{i+1}$ is strong (see Lemma 2.2). For $p_ip_{i+1}p_i \in Q$, it is hence not difficult to see that there can be no path from p_{i+1} to p_i in $D - p_{i+1}p_i$. This implies that

$$\{p_0, p_1, \ldots, p_i\} \Rightarrow \{p_{i+1}, p_{i+2}, \ldots, p_l\} \text{ in } D - p_{i+1}p_i. \tag{1}$$

We now consider the following two cases:

Case 1: There is an i, such that $p_ip_{i+1}p_i \in Q$ and $1 \leq i \leq l-2$.

Let $D' = D - p_{i+1}p_i$ and let $D'_1, \ldots, D'_t$ be the strong components of D', $t \geq 2$, such that if $q < j$, then no arc goes from D'_j to D'_q. Since D is strong, $p_{i+1} \in D'_t$ and $p_i \in D'_1$. As $p_{i-1} \to p_i$, we have $p_{i-1} \in D'_1$. Similarly, we see that $p_0, p_1, \ldots, p_i \in D'_1$ and $p_{i+1}, p_{i+2}, \ldots, p_l \in D'_t$. Let $p_iR_1p_a$ be a shortest path from p_i to a vertex p_a belonging to $\{p_0, p_1, \ldots, p_{i-1}\}$ in D'_1, and let $p_bR_2p_{i+1}$ be a shortest path from a vertex p_b, belonging to $\{p_{i+2}, p_{i+3}, \ldots, p_l\}$, to p_{i+1} in D'_t.

We now consider the following subcases.

Subcase 1.1: Let $PS(p_{i\;1}) = PS(p_{i+2})$. Then, by (1),

$$P' = p_0p_1 \ldots p_{i-1}p_{i+1}p_ip_{i+2}p_{i+3} \ldots p_l$$

is a path in $D - p_ip_{i+1}$, of length l. Therefore we may use our induction hypothesis for the digraph $D - p_ip_{i+1}$.

Subcase 1.2: Let $PS(p_{i\;1}) \neq PS(p_{i+2})$ and $a = 0$. Observe that by (1)

$$P' = p_{i+1}p_iR_1p_0p_1 \ldots p_{i-1}p_{i+2}p_{i+3} \ldots p_l$$

is a path of length l in $D - p_ip_{i+1}$. (This means that, in particular, $R_1 = \emptyset$ in this subcase.) Furthermore, observe that $\delta_1(P', D - p_ip_{i+1}) = 0$ and $\delta_2(P', D - p_ip_{i+1}) = \delta_2(P, D)$. We may therefore use our induction hypothesis for the digraph $D - p_ip_{i+1}$, which completes this subcase.

Subcase 1.3: Let $PS(p_{i\;1}) \neq PS(p_{i+2})$ and $b = l$. This can be handled analogously to the previous case.

Subcase 1.4: Let $PS(p_{i\;1}) \neq PS(p_{i+2})$, $a > 0$ and $b < l$. If $PS(p_{a-1}) \neq PS(p_{i+1})$ then observe that $P' = p_0p_1 \ldots p_{a-1}p_{i+1}p_iR_1p_ap_{a+1} \ldots p_{i-1}p_{i+2}p_{i+3} \ldots p_l$ is a path in $D - p_ip_{i+1}$, of length l. Therefore we may use our induction hypothesis on the digraph $D - p_ip_{i+1}$. If $PS(p_{b+1}) \neq PS(p_i)$ then we proceed analogously to the case when $PS(p_{a-1}) \neq PS(p_{i+1})$. We may therefore assume that $PS(p_{a-1}) = PS(p_{i+1})$ and $PS(p_{b+1}) = PS(p_i)$. We now observe that

$$P' = p_0p_1 \dots p_{a-1}p_{i+2}p_{i+3} \dots p_bR_2p_{i+1}p_iR_1p_ap_{a+1} \dots p_{i-1}p_{b+1}p_{b+2} \dots p_l,$$

is a path in $D - p_ip_{i+1}$ of length l. Therefore we may use our induction hypothesis for the digraph $D - p_ip_{i+1}$.

Case 2: There is no i, such that $p_ip_{i+1}p_i \in Q$ and $1 \leq i \leq l-2$.

If $p_0p_1p_0 \in Q$ then we note that $p_0 \Rightarrow \{p_2, p_3, \dots, p_l\}$ in D as $D - p_{i+1}p_i$ is not strong. Furthermore $p_0 \Rightarrow V(D) - V(P)$, since otherwise we get a contradiction to P being a longest path in D. Therefore $\delta_1(P, D) = 1$. We may now consider the path $P' = p_1p_2 \dots p_l$ in $D - p_0p_1$, which has $\delta_1(P', D - p_0p_1) = 0$ and $\delta_2(P', D - p_0p_1) = \delta_2(P, D)$. We may use our induction hypothesis for $D - p_0p_1$, which gives us the desired result. (Even if P' is not a longest path in $D - p_0p_1$, we are still done). If $p_lp_{l-1}p_l \in Q$ then we may proceed analogously to the case $p_0p_1p_0 \in Q$. □

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