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Testable Restrictions of General Equilibrium Theory in Exchange Economies with Externalities.

Andrés Carvajal*
Yale University and Royal Holloway, University of London
andres.carvajal@yale.edu

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Abstract

The theory of general equilibrium was criticized for its apparent lack of testable implications, as seemingly implied by the results of Sonnenschein, Mantel and Debreu in the Seventies. This view was challenged by the results of Brown and Matzkin (1996), which showed the existence of testable restrictions on the equilibrium manifold of exchange economies. This paper studies a problem similar to the one posed by Brown and Matzkin, for the case of general equilibrium in the presence of externalities. The natural definition of equilibrium in such case is the Nash-Walras equilibrium concept. I first consider the case of strategic externalities, where I assume that each player chooses a consumption bundle, subject to some budget, and a strategy from a continuous domain, and where the utility of each individual depends on his consumption and on the strategies chosen by all the players. I also consider the case of consumption externalities, in which each individual's utility depends on his consumption of all commodities and on the consumption of some particular commodity by all individuals. The results obtained here are rather negative in that they point towards the unfalsifiability of the equilibrium hypothesis. Under the assumption that one can observe individual choices for the externality, I find that there exist some extremely mild testable restrictions. This, however, is not a pure extension of the Brown-Matzkin result, since some individual decisions are assumed to be observed. If there is no information on individual choices, I find that the equilibrium concept imposes no testable

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restrictions. This occurs unless one imposes further assumptions, such as weak separability.

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JEL classification: D12, D50, D62.

1 Introduction:

After its elegant and precise development by Arrow, Debreu and McKenzie in the mid Twentieth Century, the model of general equilibrium has become the foundation of most theoretical developments in economics. Issues of existence, determinacy and optimality were favorably solved and, with this, the model became the underpinning for theoretical analysis in practically all fields in economics and the basis of many an economic policy recommendation. All this, despite the troublesome result often referred to as the "Sonnenschein-Mantel-Debreu theorem," which many came to believe implied the unfalsifiability of the Arrow-Debreu theory and cast doubts about the scientific character of general equilibrium, at least according to a commonly accepted epistemologial position that maintains that theories are scinetific only if they are falsifiable. Such position, known as "falsificationism," was first and foremost defended by Karl Popper and was first brought to economics by Samuelson (1947).[1] According to it, it is precisely the existence of testable implications, which the researcher should *ex ante* expect the theory to be at odds with, what distinguishes scientific from esoteric knowledge. Thus, despite the simplicity and analytical appeal of the general equilibrium theory, the belief that it imposed no testable implications was problematic because, if correct, it would imply that the theory could only be believed out of faith in its assumptions and principles, but lacked any statements about empirical observations that could be conceivably refuted upon contrast with real data.

The position that from the Sonnenschein-Mantel-Debreu theorem it followed that general equilibrium theory imposed no, or hardly any, testable restrictions permeated to the literature. Arrow (1991) wrote that from this theorem one should conclude that "in the aggregate, the hypothesis of rational behavior has in general no implications," while according to Hansen and Heckman (1996), "...as a paradigm for organizing and synthesizing economic data, [general equilibrium theory] poses some arduous challenges. A widely accepted empirical counterpart of general equilibrium remains to be developed." Most explicitly, a basic textbook in microeconomic theory, Mas-Collel et al (1995), categorized the results of the Sonnenschein-Mantel-Debreu theorem as saying that "anything

[1] Both Antonelli (1886) and Slutsky (1915) had studied the problem of what conditions were implied by the hypothesis of individual rationality of the consumers, from a differential perspective. But it was Samuelson who emphasized the emptiness that economic theories exhibited when lacking refutable implications. He also tackled the problem from a different perspective, namely from algebraic, rather than differential, implications. Significant dontributions from this perspective were also Houthakker (1950), Richter (1966) and Afriat (1967).

goes," in the sense that "...anything satisfying..." the very mild restrictions of the Sonnenschein-Mantel-Debreu theorem "...can actually occur."

In contrast, it has recently been shown that this view of the problem of falsifiability of general equilibrium theory is overly pessimistic. Brown and Matzkin (1996) and Chiappori et al (2002) have shown that some information at the individual level may generate nontrivial testable restrictions, even if it reveals nothing about actual individual choices, and only describes individual constraints.

From a theoretical perspective, these results provide firm ground to establish the Arrow-Debreu model as scientific knowledge. From an economic policy perspective, it is also refreshing to know that the general equilibrium hypothesis is refutable from real data, as this implies that there exists tests to be performed before implementing policy recommendations based on the model itself.

In this paper, I argue that this optimistic perspective does not hold when one assumes a very simple change to the principles of general equilibrium, namely the presence of externalities. Specifically, I show that the natural concept of competitive equilibrium in this context, the Nash-Walras equilibrium concept, constitutes an unfalsifiable hypothesis, unless the researcher possesses specific information regarding actual individual choices, and that even in this case, the restrictions that the theory imposes are extremely mild.

The problem considered here appears important given the well-known results on welfare properties of Nash-Walras equilibrium, but in particular those recently obtained by Geanakoplos and Polemarchakis (2001), according to which the equilibrium allocations are typically constrained Pareto-suboptimal, a result that calls for policy interventions when there are external effects, subject to the policy authority having available the right information.

This paper is organized as follows. In the next section, I briefly review the relevant literature and distinguish my results form others that already exist. Then, in the following two sections, I state and solve the falsifiability problem considering two types of externalities. Strategic externalities are those in which the external effects come from individual actions other than consumption of commodities. These actions need not be physically comparable between individuals, their feasibility is not mediated by endowments or prices and decisions on their regard bear no effect on the budgets of individuals. In contrast, consumption externalities arise from the consumption of some commodities by individuals. These consumptions are aggregable, in equilibrium their markets must clear, given aggregate endowments, and individual decisions about them are subject to budgetary considerations, given prices. Although one can embed consumption externalities as strategic externalities, as I illustrate in subsection 4.2, this comes at a cost in terms of generality and simplicity of the results and I have therefore chosen to treat the two cases as entirely independent. For both types of externalities, after stating the problem and the conditions under which data are to be considered consistent with Nash-Walras equilibrium, I show that if individual choices of the externality are observed, then some restrictions, somewhat similar but much weaker than the ones obtained by Brown and Matzkin (1996), are imposed by the theory. Then, it is shown that without such individual level information the theory is unfalsifiable. This last issue is less relevant in the case of strategic externalities, where no observation of individual actions means no information about the externality at all. In the case of consumption externalities, observation of all prices implies that the researcher knows at least some "summary statistic" about individual choices. My result is that the theory imposes no restrictions on these summary statistics, even under full information about the constraints of all consumers, a result that contrasts with the ones of Brown and Matzkin (1996) and Chiappori et al (2002). The latter is true unless one adds further conditions to the hypothesis. In particular, I show that requiring weak separability on the private commodities restores the refutability of the equilibrium hypothesis.

All my results are based on the assumption that just a finite set of individual constraints and prices is observed, as in Brown and Matzkin (1996), and not on the knowledge of the whole equilibrium manifold, as in Chiappori et al (2002). I find this approach to be more convenient from an empirical perspective.

2 Review of the literature:

The first study of the problem of falsifiability of general equilibrium theory without observation of individual choices was Sonnenschein (1973), where the following problem was posed: suppose that one observes a function mapping prices into quantities of commodities; what conditions must this function satisfy if it is to be the aggregate excess demand function of an exchange economy under standard assumptions? Well-known necessary conditions are continuity, homogeneity of degree zero and Walras' law. The surprising result was that these very mild conditions exhaust all the restrictions of the theory, as shown by Mantel (1974) and Debreu (1974): for any function that satisfies these three conditions, there exists an economy, with at least as many consumers as commodities, such that, away from zero prices, the function is its aggregate excess demand function. This result is commonly referred to as the Sonnenschein-Mantel-Debreu theorem.[2]

The conclusion was formed that if the condition that there are at least as many consumers as there are commodities is acceptable, then the restrictions of utility maximization disappear when one does not observe individual choices. This interpretation was challenged by Brown and Matzkin (1996), who showed that general equilibrium theory is falsifiable, even without observing individual choices, provided that there exists information about individual budgets. The novelty of their approach resided in that they did not analyze the aggregate

[2] Mas-Collel (1977) showed that there are no restrictions on the set of equilibrium prices of an economy, Diewert (1977) showed that there are some restrictions on the derivatives of the aggregate excess demand and Geanakoplos and Polemarchakis (1980) showed that these are all the restrictions. A similar result for market demand functions was shown by Diewert (1977) and Mantel (1977). Andreu proved that a conclusion similar to the Sonnenschein-Mantel-Debreu applies to finite subsets of prices. Recently, Chiappori and Ekeland (1999) showed that the Sonnenschein-Mantel-Debreu extends to the whole market demand function, under smoothness assumptions. For a recount of the earlier part of this literature, see Shafer and Sonnenschein (1982).

excess demand function, which from an empirical point of view is inconvenient, as under the general equilibrium hypothesis it can only be observed precisely when it vanishes, but focused on the equilibrium manifold, where variations of individual endowments are accounted for. By varying individual endowments, Brown and Matzkin showed a conflict that may arise between the two principles that constitute the basis of general equilibrium: individual rationality and market clearing. Specifically, they showed an important tension between aggregate feasibility and individual-wise satisfaction of the axioms of revealed preference, the first of which is necessary condition for market clearing, and the second of which is equivalent to individual rationality. This tension implied that not every data set of individual endowments and prices can be rationalized as coming from observations of Walrasian equilibria in an exchange economy under standard assumptions.

A similar approach, where individual endowments are taken into account, was followed by Chiappori et al (2002), with the difference that they consider the whole of the equilibrium manifold, rather that just some finite subset of it. They find that "whenever data are available at the individual level, then utility maximization generates very stringent restrictions upon observed behavior, even if the observed variables are aggregate (e.g. aggregate excess demand or equilibrium prices)." Furthermore, under the extra assumption that individuals have preferences such that income effects do not vanish, they show that all the restrictions of individual rationality are preserved upon aggregation, since it is possible to recover individual preferences from the equilibrium manifold (at least locally), uniquely up to ordinal equivalence. They also show that some individual level information is necessary for falsification, since any smooth manifold can be locally rationalized as resulting from utility maximizing agents, whenever their number is at least as large as the number of commodities and redistribution of endowments is allowed.

In this paper, I take the same approach as in Brown and Matzkin (1996), which requires the observation of only a finite subset of the equilibrium manifold. However, I assume that there exist externalities, so that the equilibrium concept that applies is not simply the one of Walrasian equilibrium, but rather the one of Nash-Walras equilibrium, in which individuals are assumed individually rational, taking as given not only prices, but also the choices of everybody else. A related paper is Snyder (1999), where the problem of falsifiability of the hypothesis of Pareto-efficient provision of a public good is studied.[3] By an application of the methodology of Brown and Matzkin to the analysis of public goods via Lindahl prices, Snyder shows that the hypothesis is indeed falsifiable, whenever information on market prices (not on Lindahl prices), production levels and individual incomes is available. The differences between the contexts of Snyder and mine are straightforward. First of all, she considers a commodity which is a public good, whereas in my analysis of strategic externalities, the actions that

[3] Less related extensions of the results of Brown and Matzkin are Kubler (2001), to intertemporal problems under uncertainty, and Carvajal (2004b), where preferences of individuals are allowed to change randomly, but there is no uncertainty at the moment of making decisions, nor are there any intertemporal links.

generate the external effect need not even be physically comparable between individuals, and in the case of consumption externalities, although they come from a comparable commodity, my assumptions are that this good is rival.[4] More importantly, Snyder focuses on Pareto Efficient solutions, whereas here it is assumed that individuals act noncooperatively. Regarding the results, the differences are also important: Snyder finds that the Pareto-efficiency hypothesis is falsifiable without data on individual choices, whereas here I conclude that this is not the case for the Nash-Walras hypothesis, and that even if individual-level information about individual choices of the externality is available, the restrictions that the hypothesis imposes are *a priori* far from harsh.

3 Strategic externalities:

Strategic externalities do not occur through markets. In this case, actions chosen by individuals are abstract choices that need not even represent one common physical object, when compared across agents. For example, one player's action may be the volume at which he listens to his music, while his neighbor's action may be the time at which he decides to do the laundry on Sundays. Both decisions may affect both players' well-being, and the decisions are therefore subject to strategic interaction, but neither decision is a market choice: they are not affected by prices nor do they change the disposable income of individuals. Moreover, these actions are not subject to aggregation and no market is to clear for the decisions to constitute an equilibrium. In this section, I consider the problem of falsifiability of the hypothesis of Nash-Walras equilibrium under this kind of externalities.

3.1 The model:

Consider an economy with a finite set of consumers, which I denote by $\mathcal{I} = \{1, ..., I\}$, with $2 \leqslant I < \infty$. Consumers in this economy make two decisions. As in the standard general equilibrium model, they must choose a consumption bundle. I assume that there is a finite number, $L \in \mathbb{N}$, of commodities, so that the consumption set for each individual is $\mathbb{R}^L_+$. In this case, I assume that individuals must also choose an action. For simplicity and concreteness, I assume that each individual chooses his action from a nonempty interval of the real line. Hence, analogous to the consumption set, I assume that each individual $i \in \mathcal{I}$ has as space of conceivable actions the interval $A^i = \left[0, \overline{\overline{a}}_i\right]$, where $\overline{\overline{a}}_i \in \mathbb{R}_+$.[5] I will use $x \in \mathbb{R}^L_+$ to denote bundles of commodities, while for each $i \in \mathcal{I}$, $a_i \in A^i$ will denote an action.

[4] To some extent, this distinction is shallow for the case of consumption externalities, since the consumption of the commodities that cause the externality by each individual can be seen as public goods.

[5] Of course, individual i's actions impose an externality on others only if $\overline{\overline{a}}_i > 0$. The case $\overline{\overline{a}}_i = 0$, although less interesting, still fits in this context. Clearly, if, for all $i \in \mathcal{I}$, $\overline{\overline{a}}_i = 0$, then the case studied here reduces to the Brown-Matzkin problem.

Strategic externalities exist because each individual $i \in \mathcal{I}$ derives utility not only from his own consumption, $x \in \mathbb{R}_+^L$, and action, $a_i \in A^i$, but also from actions by all other players, which I will denote by

$$a_{-i} \in A^{-i} = \prod_{j \in \mathcal{I} \setminus \{i\}} A^j$$

Formally, I assume that each individual $i \in \mathcal{I}$ has preferences represented by

$$U^i : \mathbb{R}_+^L \times A^i \times A^{-i} \longrightarrow \mathbb{R}$$

As in the Walrasian framework, individuals make their decisions taking into account only their own constraints. When consumer $i \in \mathcal{I}$ is endowed with $w_i \in \mathbb{R}_+^L$ and prices are $p \in \mathbb{R}_{++}^L$, he chooses his demand for commodities without violating his budget constraint

$$p \cdot x \leq p \cdot w_i$$

Similarly, I will assume that there may be further constraints to the action that each $i \in \mathcal{I}$ may choose. These constraints are assumed to take the very simple form of upper bounds to choices. Specifically, I will denote by $\overline{a}_i \in \left[0, \overline{\overline{a}}_i\right]$ such upper bounds, and assume that individuals choose their actions from $[0, \overline{a}_i]$, which is analogous to the budget set.[6]

An economy is completely described by the set of players, $\mathcal{I}$, their preferences $\left(U^i : \mathbb{R}_+^L \times A^i \times A^{-i} \longrightarrow \mathbb{R}\right)_{i \in \mathcal{I}}$, their endowments, which I assume to be strictly positive, $(w_i)_{i \in \mathcal{I}} \in \left(\mathbb{R}_{++}^L\right)^I$, and their constraints to actions $(\overline{a}_i)_{i \in \mathcal{I}} \in A = \prod_{i \in \mathcal{I}} A^i$. Formally, an economy is a vector

$$\left\{\mathcal{I}, \left(U^i, w_i, \overline{a}_i\right)_{i \in \mathcal{I}}\right\}$$

The hypothesis whose falsifiability I want to study is that individuals act noncooperatively, in the sense that they behave as in the Nash solution concept regarding their strategic interaction, and according to the Walrasian principles regarding their market behavior. That is to say, I assume that each individual $i \in \mathcal{I}$, given his endowments $w_i \in \mathbb{R}_{++}^L$ and his constraints to actions $\overline{a}_i$, takes also as given the prices $p \in \mathbb{R}_{++}^L$ and the actions of all his opponents $a_{-i} \in A^{-i}$, and maximizes his well-being, by maximizing $U^i(\cdot, \cdot, a_{-i}) : \mathbb{R}_+^L \times A^i \longrightarrow \mathbb{R}$, subject only to his constraints

$$\begin{aligned} p \cdot x &\leq p \cdot w_i \\ a_i &\in [0, \overline{a}_i] \end{aligned}$$

When prices are such that under this kind of behavior all markets clear, the economy attains an equilibrium. Formally,

[6] I do not explore where these constraints come from and take them as given, as I do with the endowments. I do not rule out the possibility that they are redundant ($\overline{a}_i = \overline{\overline{a}}_i$), though.

Definition 1 *Given an economy*

$$\mathcal{E}=\left\{\mathcal{I},\left(U^i,w_i,\overline{a}_i\right)_{i\in\mathcal{I}}\right\}$$

a Nash-Walras equilibrium is a vector

$$\left(p^*,\left(x_i^*,a_i^*\right)_{i\in\mathcal{I}}\right)\in\mathbb{R}_{++}^L\times\prod_{i\in\mathcal{I}}\left(\mathbb{R}_+^L\times A^i\right)$$

such that

$$(\forall i\in\mathcal{I})\quad:\quad\left(x_i^*,a_i^*\right)\in Arg\max_{(x,a_i)}U^i\left(x,a_i,a_{-i}^*\right)$$
$$s.t.\ \begin{cases}p^*\cdot x\leqslant p^*\cdot\omega_i\\ a_i\in[0,\overline{a}_i]\\ x\geqslant 0\end{cases}$$

and

$$\sum_{i\in\mathcal{I}}x_i^*=\sum_{i\in\mathcal{I}}\omega_i$$

Given an economy $\mathcal{E}$, I will denote by $NW(\mathcal{E})$ the set of all its Nash-Walras equilibria. This set is the canonical definition of noncooperative equilibrium in economies with strategic interaction. It is a well-defined equilibrium concept, in the sense that it has been proven to be nonempty under standard assumptions.[7]

Given an economy $\mathcal{E}$, if $\left(p^*,\left(x_i^*,a_i^*\right)_{i\in\mathcal{I}}\right)\in NW(\mathcal{E})$, then p^* is said to be a Nash-Walras equilibrium price vector of $\mathcal{E}$, $\left(x_i^*\right)_{i\in\mathcal{I}}$ is a Nash-Walras equilibrium allocation of $\mathcal{E}$ and $\left(a_i^*\right)_{i\in\mathcal{I}}$ is a Nash-Walras equilibrium profile of strategies of $\mathcal{E}$.

My goal in this section is to study the falsifiability of the hypothesis of Nash-Walras equilibrium, based on data that does not include all the information on individual choices. If one has full information on individual choices, then the theory of revealed preference can be applied in a manner similar to its application to games on continuous sets, as in Carvajal (2004a). Since strategic externalities need not generate aggregate data, I will first assume that prices of commodities and individual choices of actions (but not individual demands) are observed and, hence, I will study the projection of the Nash-Walras set into the space of prices of commodities and individual actions. For an economy $\mathcal{E}$, this projection is formally defined as:

$$NWPS(\mathcal{E})=$$
$$\left\{\left(p,(a_i)_{i\in\mathcal{I}}\right)\in\mathbb{R}_{++}^L\times\prod_{i\in\mathcal{I}}A^i\;\middle|\;\left(\exists\,(x_i)_{i\in\mathcal{I}}\in\left(\mathbb{R}_+^L\right)^I\right):\left(p,(x_i,a_i)_{i\in\mathcal{I}}\right)\in NW(\mathcal{E})\right\}$$

Consistently, I define a data set as containing only information on prices, individual endowments, individual upper bounds to individual actions and individual chosen actions, for a finite number of observations. Formally,

[7] See Ghosal and Polemarchakis (1997), where it is also shown that Nash-Walras equilibria are typically determinate and Pareto suboptimal.

Definition 2 *A data set is a finite sequence*

$$\left(\left(p_t, \left(\omega_{i,t}, a^*_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

such that for each $t \in \{1, ..., T\}$

$$p_t \in \mathbb{R}^L_{++}$$

and for each $t \in \{1, ..., T\}$ *and each* $i \in \mathcal{I}$

$$\begin{aligned} \omega_{i,t} &\in \mathbb{R}^L_{++} \\ a^*_{i,t} &\in [0, \overline{a}_{i,t}] \subseteq A^i \end{aligned}$$

This means that at each observation $t \in \mathcal{T} = \{1, ..., T\}$, one has available, for each individual $i \in \mathcal{I}$, full information on his feasible set, as this set determined by prices, p_t, and endowments, $w_{i,t}$, regarding commodities, and by the constraint to his actions, $\overline{a}_{i,t}$. One also observes the action chosen by each individual $a^*_{i,t}$, but has no information about his chosen consumption, for which all one observes is an aggregate summary statistic, p_t.[8] It is worthwhile to point out that, despite the notation, I am not dealing with a dynamic problem here. I am, indeed, assuming that there is no intertemporal link between observations and that individual preferences are invariant weak orders over $\mathbb{R}^L_+ \times A^i \times A^{-i}$.[9]

Under finite data sets, it may not always be reasonable to assume that one has observed equilibria exhaustively. In that sense, it is convenient to take a weak approach to the falsifiability problem, in which one requires that the observed data be consistent with equilibrium, but does not assume or imply that there cannot be other equilibria. When doing so, however, it is customary to impose conditions that restrict the choice behavior of individuals in manners that are interesting from the point of view of the theory. In this case, I impose conditions that ensure that individuals spend all their wealth and that, given their feasible sets and the actions of others, their choices are uniquely determined. The exact sense in which data sets are going to be considered consistent with the hypothesis of Nash-Walras equilibrium is given by the following definition:

Definition 3 *A data set*

$$\left(\left(p_t, \left(\omega_{i,t}, a^*_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

is said to be Nash-Walras-rationalizable (NW-rationalizable) if for each $i \in \mathcal{I}$ *there exists a function* $U^i : \mathbb{R}^L_+ \times A^i \times A^{-i} \longrightarrow \mathbb{R}$, *continuous and satisfying that*

$$\begin{aligned} \left(\forall a_{-i} \in A^{-i}\right) &: \quad U^i\left(\cdot, \cdot, a_{-i}\right) \quad \textit{is Lipschitzian with constant } M_{a_{-i}} \\ \left(\forall a_{-i} \in A^{-i}\right) &: \quad U^i\left(\cdot, \cdot, a_{-i}\right) \quad \textit{is strongly concave} \\ \left(\forall \left(a_i, a_{-i}\right) \in A^i \times A^{-i}\right) &: \quad U^i\left(\cdot, a_i, a_{-i}\right) \quad \textit{is strictly monotone} \end{aligned}$$

[8] The assumption of finite data sets is identical to the one in Brown and Matzkin (1996) and Snyder (1999), but contrasts to the approach in Chiappori et al (2002). Because of it, if testable restrictions arise in this setting, they must be seen as nonparametrical.

[9] As mentioned in 2, the intertemporal problem (without externalities) was studied by Kubler (2001) with rather negative results.

such that for each $t \in \mathcal{T}$

$$\left(p_t, \left(a^*_{i,t}\right)_{i\in\mathcal{I}}\right) \in NWPS\left(\left\{\mathcal{I}, \left(U^i, w_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right\}\right)$$

In this case, it is said that $\left(U^i\right)_{i\in\mathcal{I}}$ NW-rationalizes the data set.

Continuity is always plausible for functions representing preferences, as general representability results have been found under this condition (Debreu (1954)). It is imposed here because it implies that individual choices are always defined (by Weierstrass' theorem) and depend continuously on parameters, specifically on actions by other consumers (by the theorem of the maximum). The Lipschitz condition is imposed for technical reasons. It is weaker than differentiability, but implies differentiability almost everywhere. Strong concavity and strict monotonicity also have technical use, but are actually imposed in order to rule out trivial rationalizations in which individuals are indifferent between all outcomes in their domains and, therefore, every triple composed by a vector of prices, an allocation where markets clear and a profile of actions is a Nash-Walras equilibrium. Both conditions have desirable implications: monotonicity implies Walras' law, whereas strong concavity implies that given prices, own endowments and the actions of other players, each consumer's choices are uniquely determined and are therefore subject to the same axioms of revealed used by Brown and Matzkin (1996).

3.2 Equilibrium inequalities:

The definition of rationalizability only states explicitly the requirement that there exist utility functions consistent with the observed data. Implicitly, however, it also requires that there exist demands for commodities consistent with the data and those utility functions. Revealed preference theory has provided conditions under which one does not need to work with those two mathematical objects, preferences and choices, in the sense that existence of the former is equivalent to existence of the latter. As a first step towards the derivation of testable restrictions, the following characterization of Nash-Walras equilibrium deals only with individual demands at each observation, imposing on them the conditions which make them equivalent to the approach via utility functions.

Before the characterization can be given, the following notation needs to be introduced. Given a data set

$$\left(\left(p_t, \left(\omega_{i,t}, a^*_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

define, for each $i \in \mathcal{I}$, the correspondence

$$\mathcal{T}^i : A^{-i} \rightrightarrows \mathcal{T}$$

by

$$\mathcal{T}^i\left(a_{-i}\right) = \left\{ t \in \mathcal{T} \middle| \, a^*_{-i,t} = a_{-i} \right\}$$

Theorem 1 *A data set*

$$\left(\left(p_t, \left(\omega_{i,t}, a^*_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

is NW-rationalizable if, and only if, for each $i \in \mathcal{I}$ and each $t \in \mathcal{T}$ there exist $x^*_{i,t} \in \mathbb{R}^L_+$, $V^i_t \in \mathbb{R}$, $v^i_t \in \mathbb{R}^L_{++}$, $\rho^i_t \in \mathbb{R}$, $\lambda^*_{i,t} \in \mathbb{R}_{++}$, $\varsigma^*_{i,t} \in \mathbb{R}^L_+$, $\mu^*_{i,t} \in \mathbb{R}_+$ and $\eta^*_{i,t} \in \mathbb{R}_+$ such that:

1. $(\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) : v^i_t = \lambda^*_{i,t} p_t - \varsigma^*_{i,t}$
2. $(\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) : \rho^i_t = \eta^*_{i,t} - \mu^*_{i,t}$
3. $(\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) : \varsigma^*_{i,t} \cdot x^*_{i,t} = 0,\ \mu^*_{i,t} a^*_{i,t} = 0$ *and* $\eta^*_{i,t}\left(\overline{a}_{i,t} - a^*_{i,t}\right) = 0$
4. $(\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) : p_t \cdot x^*_{i,t} = p_t \cdot \omega_{i,t}$
5. $(\forall i \in \mathcal{I})\left(\forall a_{-i} \in A^{-i}\right)\left(\forall t, t' \in \mathcal{T}^i\left(a_{-i}\right)\right) :$

   $$V^i_{t'} \leqslant V^i_t + v^i_t \cdot \left(x^*_{i,t'} - x^*_{i,t}\right) + \rho^i_t\left(a^*_{i,t'} - a^*_{i,t}\right)$$

   with strict inequality whenever

   $$\left(x^*_{i,t'}, a^*_{i,t'}\right) \neq \left(x^*_{i,t}, a^*_{i,t}\right)$$

6. $(\forall t \in \mathcal{T}) : \sum_{i\in\mathcal{I}} x^*_{i,t} = \sum_{i\in\mathcal{I}} \omega_{i,t}$

Before the proof of the theorem, it may be useful to obtain intuition about it, which is simple. Given a rationalizable data set, there have to exist, by definition, individual demands $x^*_{i,t}$, for each player $i \in \mathcal{I}$ and at each observation $t \in \mathcal{T}$, which are individually rational according to some preferences and clear markets. The latter is condition (6), whereas condition (4) follows from strict monotonicity (Walras' law). Suppose for a moment that all the utility functions that NW-rationalize the data set are differentiable with respect to consumption and own actions. Then, conditional on $a^*_{-i,t}$, condition (5) follows from concavity of the utility functions (using its characterization via tangents),[10] whereas

[10] This type of condition is usually known as "Afriat inequalities." Incidentally, notice that it implies that if for $i \in \mathcal{I}$ and $t, t' \in \mathcal{T}$, one has that $v^i_t = \lambda^*_{i,t} p_t$, $v^i_{t'} = \lambda^*_{i,t'} p_{t'}$, $a^*_{i,t} = a^*_{i,t'}$, $a^*_{-i,t} = a^*_{-i,t'}$, $p_t = p_{t'}$ and $p_t \cdot x^*_{i,t} = p_t \cdot x^*_{i,t'}$, then $x^*_{i,t} = x^*_{i,t'}$. To see this, suppose not: $x^*_{i,t} \neq x^*_{i,t'}$. Then, by this condition,

$$\begin{aligned} V^i_{t'} &< V^i_t + \lambda^*_{i,t} p_t \cdot \left(x^*_{i,t'} - x^*_{i,t}\right) \\ V^i_t &< V^i_{t'} + \lambda^*_{i,t'} p_{t'} \cdot \left(x^*_{i,t'} - x^*_{i,t}\right) \end{aligned}$$

and, therefore,

$$\begin{aligned} V^i_{t'} &< V^i_t \\ V^i_t &< V^i_{t'} \end{aligned}$$

an obvious contradiction.

conditions (1), (2) and (3) would follow from Kuhn-Tucker's theorem (as first-order necessary conditions of each individual's maximization problem). Now, I am not assuming that the cross-sections of the utility functions for consumption and own action are differentiable, but only that they are Lipschitzian. This is a technical problem, and its solution is the content of lemma 4, in appendix 6, which uses an analogous of the Kuhn-Tucker theorem via subdifferential calculus (lemma 3).

Proof. Necessity: Let $\left(U^i\right)_{i\in\mathcal{I}}$ NW-rationalize the data set.

Fix $t \in \mathcal{T}$. Since

$$\left(p_t, \left(a^*_{i,t}\right)_{i\in\mathcal{I}}\right) \in NWPS\left(\left\{\mathcal{I}, \left(U^i_{\overline{a}_t}, \omega_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right\}\right)$$

it follows by definition that there exists $\left(x^*_{i,t}\right)_{i\in\mathcal{I}} \in \left(\mathbb{R}^L_+\right)^I$ such that

$$\left(p_t, \left(x^*_{i,t}, a^*_{i,t}\right)_{i\in\mathcal{I}}\right) \in NW\left(\left\{\mathcal{I}, \left(U^i_{\overline{a}_t}, \omega_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right\}\right)$$

Fix one such $\left(x^*_{i,t}\right)_{i\in\mathcal{I}}$. By definition,

$$\sum_{i\in\mathcal{I}} x^*_{i,t} = \sum_{i\in\mathcal{I}} \omega_{i,t}$$

which implies condition (6).

Now, fix also $i \in \mathcal{I}$. Again by definition,

$$\left(x^*_{i,t}, a^*_{i,t}\right) \in Arg\max_{(x,a_i)} U^i\left(x, a_i, a^*_{-i,t}\right)$$
$$\text{s.t.}\left\{\begin{array}{c} p^* \cdot x \leqslant p^* \cdot \omega_{i,t} \\ a_i \in [0, \overline{a}_{i,t}] \\ x \geqslant 0 \end{array}\right.$$

and, then, the other five conditions follow from lemma 4, using

$$V^i(\cdot,\cdot) = U^i\left(\cdot,\cdot,a^*_{-i,t}\right)$$

Sufficiency: Fix $i \in \mathcal{I}$. Define Γ^i as follows:

- $\gamma^i_1 = \{1\}$
- for $t \in \{2, ..., T\}$

$$\gamma^i_t = \left\{\begin{array}{cc} \varnothing & \text{if} \quad \left(\exists t' \in \{1, ..., t-1\}\right) : a^*_{-i,t'} = a^*_{-i,t} \\ & \{t\} \quad \text{otherwise} \end{array}\right.$$

- $\Gamma^i = \bigcup_{t\in\mathcal{T}} \gamma^i_t$

Clearly,

$$\left(\forall t, t' \in \Gamma^i : t \neq t'\right) : a^*_{-i,t} \neq a^*_{-i,t'}$$
$$\left(\forall t \in \mathcal{T}\right)\left(\exists t' \in \Gamma^i\right) : a^*_{-i,t'} = a^*_{-i,t}$$

the first of which implies that

$$\overline{r}_i = \min_{t,t' \in \Gamma^i : t \neq t'} \left\| a^*_{-i,t'} - a^*_{-i,t} \right\| > 0$$

Fix $r_i \in (0, \overline{r}_i)$, and define

$$C^i = \overline{\bigcup_{t \in \Gamma^i} B_{r_i/4}\left(a^*_{-i,t}\right)} \cap A^{-i}$$

which is a compact set.

Define the function $\Omega^i : A^{-i} \longrightarrow A^{-i}$ by

$$\Omega^i\left(a_{-i}\right) = lex - \min\left\{ Arg \min_{a^*_{-i,t} \in \bigcup_{t' \in \Gamma^i}\left\{a^*_{-i,t'}\right\}} \left\| a_{-i} - a^*_{-i,t} \right\| \right\}$$

where $lex-\min$ represents the component-wise lexicographic minimum on $\mathbb{R}^{I-1}$.

Define also, as in Matzkin and Richter (1991), the function $h : \mathbb{R}^L \times \mathbb{R} \longrightarrow \mathbb{R}$, by

$$h\left(x, a\right) = \sqrt{\left\|\left(x, a\right)\right\|^2 + 1} - 1$$

This function is differentiable and strongly convex, and satisfies the following properties

$$\begin{aligned} h\left(x,a\right) &= 0 \Longleftrightarrow \left(x,a\right) = 0 \\ h\left(x,a\right) &> 0 \Longleftrightarrow \left(x,a\right) \neq 0 \\ \left(\forall l \in \{1,...,L\}\right) &: \frac{\partial h}{\partial x_l}\left(\cdot,\cdot\right) \in [0,1) \\ \frac{\partial h}{\partial a}\left(\cdot,\cdot\right) &\in [0,1) \end{aligned}$$

The last two properties imply that h is Lipschitzian with constant $L+1$. To see this, let $\left(x,a\right), \left(x',a'\right) \in \mathbb{R}^L \times \mathbb{R}$. By the mean value theorem, for some $\left(\widehat{x}, \widehat{a}\right) \in \mathbb{R}^L \times \mathbb{R}$, it is true that

$$h\left(x,a\right) - h\left(x',a'\right) = \sum_{l \in \{1,\ldots,L\}} \frac{\partial h}{\partial x_l}\left(\widehat{x},\widehat{a}\right)\left(x_l - x'_l\right) + \frac{\partial h}{\partial a}\left(\widehat{x},\widehat{a}\right)\left(a - a'\right)$$

from where

$$
\begin{aligned}
\left|h(x,a)-h(x',a')\right| &= \left|\sum_{l\in\{1,\ldots,L\}}\frac{\partial h}{\partial x_l}(\widehat{x},\widehat{a})(x_l-x_l')+\frac{\partial h}{\partial a}(\widehat{x},\widehat{a})(a-a')\right| \\
&\leqslant \sum_{l\in\{1,\ldots,L\}}\left|\frac{\partial h}{\partial x_l}(\widehat{x},\widehat{a})(x_l-x_l')\right|+\left|\frac{\partial h}{\partial a}(\widehat{x},\widehat{a})(a-a')\right| \\
&= \sum_{l\in\{1,\ldots,L\}}\left|\frac{\partial h}{\partial x_l}(\widehat{x},\widehat{a})\right|\left|(x_l-x_l')\right|+\left|\frac{\partial h}{\partial a}(\widehat{x},\widehat{a})\right|\left|(a-a')\right| \\
&= \sum_{l\in\{1,\ldots,L\}}\frac{\partial h}{\partial x_l}(\widehat{x},\widehat{a})\left|(x_l-x_l')\right|+\frac{\partial h}{\partial a}(\widehat{x},\widehat{a})\left|(a-a')\right| \\
&\leqslant \sum_{l\in\{1,\ldots,L\}}\left|(x_l-x_l')\right|+\left|(a-a')\right| \\
&\leqslant (L+1)\max\left\{\max_{l\in\mathcal{L}}\left\{\left|(x_l-x_l')\right|\right\},\left|(a-a')\right|\right\} \\
&\leqslant (L+1)\left\|(x,a)-(x',a')\right\|
\end{aligned}
$$

Since $T<\infty$, by condition (5), there exists some $\varepsilon_i\in\mathbb{R}_{++}$ such that

$$
\left(\forall a_{-i}\in A^{-i}\right)\left(\forall t,t'\in\mathcal{T}^i(a_{-i}):\left(x^*_{i,t'},a^*_{i,t'}\right)\neq\left(x^*_{i,t},a^*_{i,t}\right)\right):
$$

$$
V^i_{t'}<V^i_t+v^i_t\cdot\left(x^*_{i,t'}-x^*_{i,t}\right)+\rho^i_t\left(a^*_{i,t'}-a^*_{i,t}\right)-\varepsilon_i h\left(\left(x^*_{i,t'},a^*_{i,t'}\right)-\left(x^*_{i,t},a^*_{i,t}\right)\right)
$$

whereas,

$$
\left(\forall a_{-i}\in A^{-i}\right)\left(\forall t,t'\in\mathcal{T}^i(a_{-i}):\left(x^*_{i,t'},a^*_{i,t'}\right)=\left(x^*_{i,t},a^*_{i,t}\right)\right):
$$

$$
V^i_{t'}=V^i_t
$$

Now, for each $t\in\mathcal{T}$, define the function $\phi^i_t:\mathbb{R}^L\times\mathbb{R}\longrightarrow\mathbb{R}$ by

$$
\phi^i_t(x,a_i)=V^i_t+v^i_t\cdot\left(x-x^*_{i,t}\right)+\rho^i_t\left(a_i-a^*_{i,t}\right)-\varepsilon_i h\left((x,a_i)-\left(x^*_{i,t},a^*_{i,t}\right)\right)
$$

which is strongly concave. Since it is the sum of two Lipschitzian functions (every affine function is Lipschitzian), it follows that ϕ^i_t is Lipschitzian with some constant M^i_t.

Moreover, notice that

$$
\begin{aligned}
\left(\forall a_i\in A^i\right)(\forall l\in\{1,...,L\}):\frac{\partial\phi^i_t}{\partial x_l}(\cdot,a_i) &= v^i_{t,l}-\varepsilon_i\frac{\partial h}{\partial x_l}\left((x,a_i)-\left(x^*_{i,t},a^*_{i,t}\right)\right) \\
&> v^i_{t,l}-\varepsilon_i
\end{aligned}
$$

so that, since $L<\infty$ and $T<\infty$, one can choose ε_i small enough so that

$$
\left(\forall a_i\in A^i\right):\phi^i_t(\cdot,a_i)\quad\text{is strictly monotone}
$$

and, therefore, ϕ_t^i is bounded below on $\mathbb{R}_+^L \times A^i$: let $Q_1 = [0,1]^L$ and let

$$\underline{\phi}_{i,t} = \min_{(x,a)\in Q_1\times A^i} \phi_t^i(x,a)$$

which is well defined, since ϕ_t^i is continuous and $Q_1 \times A^i$ is compact; then $\forall (x,a) \in \mathbb{R}_+^L \times A^i$, it is true that $\phi_t^i(x,a) \geqslant \phi_t^i(0,a) \geqslant \underline{\phi}_{i,t}$, by monotonicity and since $x \geqslant 0$.

Define the function $V^i : \mathbb{R}_+^L \times A^i \times C^i \longrightarrow \mathbb{R}$ by

$$V^i(x, a_i, a_{-i}) = \min_{t\in\mathcal{T}^i(\Omega^i(a_{-i}))} \left\{\phi_t^i(x, a_i)\right\}$$

This function is bounded below and unbounded above (because $T < \infty$). To see that it is continuous, fix $(x, a_i, a_{-i}) \in \mathbb{R}_+^L \times A^i \times C^i$ and let $\left(x^n, a_i^n, a_{-i}^n\right)_{n=1}^{\infty}$ be a sequence on $\mathbb{R}_+^L \times A^i \times C^i$ such that

$$\left(x^n, a_i^n, a_{-i}^n\right) \longrightarrow (x, a_i, a_{-i})$$

Then

$$(\exists N \in \mathbb{N})\,(\forall n \geqslant N) : \left\|\left(x^n, a_i^n, a_{-i}^n\right) - (x, a_i, a_{-i})\right\| < \frac{r_i}{4}$$

Fix one such N. Then,

$$(\forall n \geqslant N) : \left\|a_{-i}^n - a_{-i}\right\| < \frac{r_i}{4}$$

Since, by assumption $a_{-i} \in C^i$, one has, by construction, that

$$\left(\exists t \in \Gamma^i\right) : \left\|a_{-i} - a_{-i,t}^*\right\| \leqslant \frac{r_i}{4}$$

Fix one such $t \in \Gamma^i$. By triangle inequality,

$$(\forall n \geqslant N) : \left\|a_{-i}^n - a_{-i,t}^*\right\| < \frac{r_i}{2}$$

Now, suppose that for some $t' \in \mathcal{T}$, such that $a_{-i,t'}^* \neq a_{-i,t}^*$, it is true that

$$(\exists n' \geqslant N) : \left\|a_{-i}^{n'} - a_{-i,t'}^*\right\| \leqslant \frac{r_i}{2}$$

Then, by triangle inequality,

$$\left\|a_{-i,t'}^* - a_{-i,t}^*\right\| < r_i < \overline{r}_i$$

which is a contradiction, since there exists $\widehat{t} \in \Gamma^i$ such that $a_{-i,\widehat{t}}^* = a_{-i,t'}^*$ and, by definition,

$$\left\|a_{-i,\widehat{t}}^* - a_{-i,t}^*\right\| \geqslant \overline{r}_i$$

Then, it follows that

$$(\forall n \geqslant N) : \Omega^i\left(a_{-i}^n\right) = a_{-i,t}^*$$

and, therefore, $(\forall n \geqslant N)$:

$$\begin{aligned} V^i\left(x^n, a_i^n, a_{-i}^n\right) &= \min_{t \in \mathcal{T}^i\left(a_{-i,t}^*\right)} \left\{\phi_t^i\left(x^n, a_i^n\right)\right\} \\ &\longrightarrow \min_{t \in \mathcal{T}^i\left(a_{-i,t}^*\right)} \left\{\phi_t^i\left(x, a_i\right)\right\} \\ &= V^i\left(x, a_i, a_{-i}\right) \end{aligned}$$

It is also clear that for each $a_{-i} \in C^i$, $V^i\left(\cdot, \cdot, a_{-i}\right)$ is strongly concave, since $\#\mathcal{T}^i\left(\Omega^i\left(a_{-i}\right)\right) \leqslant T < \infty$ and each ϕ_t^i is strongly concave.

Since each ϕ_t^i is Lipschitzian with some constant M_t^i, and $T < \infty$, define

$$M^i = \max_{t \in \mathcal{T}} \left\{M_t^i\right\}$$

Fix $a_{-i} \in C^i$, $\left(x, a_i\right)$, $\left(x', a_i'\right) \in \mathbb{R}_+^L \times A^i$. By definition, there exist $t, t' \in \mathcal{T}^i\left(\Omega^i\left(a_{-i}\right)\right)$ such that

$$\begin{aligned} V^i\left(x, a_i, a_{-i}\right) &= \phi_t^i\left(x, a_i\right) \leqslant \phi_{t'}^i\left(x, a_i\right) \\ V^i\left(x', a_i', a_{-i}\right) &= \phi_{t'}^i\left(x', a_i'\right) \leqslant \phi_t^i\left(x', a_i'\right) \end{aligned}$$

and

$$\begin{aligned} \left|\phi_t^i\left(x, a_i\right) - \phi_t^i\left(x', a_i'\right)\right| &\leqslant M_t^i \left\|\left(x, a_i\right) - \left(x', a_i'\right)\right\| \leqslant M^i \left\|\left(x, a_i\right) - \left(x', a_i'\right)\right\| \\ \left|\phi_{t'}^i\left(x, a_i\right) - \phi_{t'}^i\left(x', a_i'\right)\right| &\leqslant M_{t'}^i \left\|\left(x, a_i\right) - \left(x', a_i'\right)\right\| \leqslant M^i \left\|\left(x, a_i\right) - \left(x', a_i'\right)\right\| \end{aligned}$$

If $\phi_{t'}^i\left(x', a_i'\right) \leqslant \phi_t^i\left(x, a_i\right)$, then

$$\phi_{t'}^i\left(x', a_i'\right) \leqslant \phi_t^i\left(x, a_i\right) \leqslant \phi_{t'}^i\left(x, a_i\right)$$

implies that

$$\left|\phi_t^i\left(x, a_i\right) - \phi_{t'}^i\left(x', a_i'\right)\right| \leqslant M^i \left\|\left(x, a_i\right) - \left(x', a_i'\right)\right\|$$

whereas if $\phi_t^i\left(x, a_i\right) < \phi_{t'}^i\left(x', a_i'\right)$, then

$$\phi_t^i\left(x, a_i\right) < \phi_{t'}^i\left(x', a_i'\right) \leqslant \phi_t^i\left(x', a_i'\right)$$

implies that

$$\left|\phi_t^i\left(x, a_i\right) - \phi_{t'}^i\left(x', a_i'\right)\right| \leqslant M^i \left\|\left(x, a_i\right) - \left(x', a_i'\right)\right\|$$

In any case,

$$\left|V^i\left(x, a_i, a_{-i}\right) - V^i\left(x', a_i', a_{-i}\right)\right| \leqslant M^i \left\|\left(x, a_i\right) - \left(x', a_i'\right)\right\|$$

and $V^i\left(\cdot, \cdot, a_{-i}\right)$ is Lipschitzian with constant M^i.

Finally, it is clear that for each $a_i \in A^i$ and each $a_{-i} \in C^i$, $V^i\left(\cdot, a_i, a_{-i}\right)$ is strictly monotone, since so are all the $\phi_t^i\left(\cdot, a_i\right)$ functions.

Define

$$\underline{v}^i = \inf_{\mathbb{R}_+^L \times A^i \times C^i} V^i(x, a_i, a_{-i}) \in \mathbb{R}$$

and the truncated logistic functions $\ell^i : [\underline{v}^i, \infty) \longrightarrow [1, 2)$ by

$$\ell^i(v) = \frac{2}{1 + \exp(\underline{v}^i - v)}$$

ℓ^i is strongly concave, strictly increasing, differentiable and Lipschitzian with constant 1. Then, the function $W^i : \mathbb{R}_+^L \times A^i \times C^i \longrightarrow [1, 2)$, defined by

$$W^i(x, a_i, a_{-i}) = (\ell^i \circ V^i)(x, a_i, a_{-i})$$

which is bounded and continuous, has also the following properties:

First, notice that for each $a_{-i} \in C^i$, $W^i(\cdot, \cdot, a_{-i})$ is strongly concave. To see this, fix $a_{-i} \in C^i$, $(x, a_i), (x', a_i') \in \mathbb{R}_+^L \times A^i$, $(x, a_i) \neq (x', a_i')$ and $\alpha \in (0, 1)$. Then,

$$V^i(\alpha(x, a_i) + (1 - \alpha)(x', a_i')) > \alpha V^i(x, a_i) + (1 - \alpha) V^i(x', a_i')$$

implies

$$\begin{aligned} \ell^i\left(V^i(\alpha(x, a_i) + (1 - \alpha)(x', a_i'))\right) &> \ell^i\left(\alpha V^i(x, a_i) + (1 - \alpha) V^i(x', a_i')\right) \\ &\geqslant \alpha \ell^i\left(V^i(x, a_i)\right) + (1 - \alpha) \ell^i\left(V^i(x', a_i')\right) \end{aligned}$$

from where

$$W^i(\alpha(x, a_i) + (1 - \alpha)(x', a_i')) > \alpha W^i(x, a_i) + (1 - \alpha) W^i(x', a_i')$$

Secondly, for each $a_{-i} \in C^i$, $W^i(\cdot, \cdot, a_{-i})$ is Lipschitzian with constant M^i. This follows since fixing $a_{-i} \in C^i$ and $(x, a_i), (x', a_i') \in \mathbb{R}_+^L \times A^i$

$$\begin{aligned} \left|\ell^i\left(V^i(x, a_i)\right) - \ell^i\left(V^i(x', a_i')\right)\right| &\leqslant \left|V^i(x, a_i) - V^i(x', a_i')\right| \\ &\leqslant M^i \|(x, a_i) - (x', a_i')\| \end{aligned}$$

Finally, it is clear that for each $a_{-i} \in C^i$ and each $a_i \in A^i$, $W^i(\cdot, a_i, a_{-i})$ is strictly monotone.

Define now the function $U^i : \mathbb{R}_+^L \times A^i \times A^{-i} \longrightarrow \mathbb{R}$ by

$$U^i(x, a_i, a_{-i}) = \begin{cases} W^i(x, a_i, a_{-i}) & \text{if } a_{-i} \in C^i \\ \frac{\inf_{\widehat{a}_{-i} \in C^i} W^i(x, a_i, \widehat{a}_{-i}) \|\widehat{a}_{-i} - a_{-i}\|}{dis(a_{-i}, C^i)} & \text{otherwise} \end{cases}$$

It follows from corollary 1 in Carvajal (2004c) that U^i is continuous and satisfies that for each $a_{-i} \in A^{-i}$, $U^i(\cdot, \cdot, a_{-i})$ is strongly concave and Lipschitzian with some constant $M^i_{a_{-i}}$ and that for each $a_{-i} \in C^i$, $U^i(\cdot, \cdot, a_{-i}) = W^i(\cdot, \cdot, a_{-i})$. Moreover, for each $a_{-i} \in A^{-i}$ and each $a_i \in A^i$, $U^i(\cdot, \cdot, a_{-i})$ is strictly monotone. To see this, let $a_{-i} \in A^{-i}$, $a_i \in A^i$ and $x, x' \in \mathbb{R}_+^L$ such that

$x > x'$. If $a_{-i} \in C^i$, the result follows by strict monotonicity of $W^i(\cdot, a_i, a_{-i})$. Else, since W^i is continuous and C^i is compact, there exists $\widehat{a}_{-i} \in C^i$ such that

$$U^i(x, a_i, a_{-i}) = \frac{W^i(x, a_i, \widehat{a}_{-i}) \|\widehat{a}_{-i} - a_{-i}\|}{dis(a_{-i}, C^i)}$$

Fix one such $\widehat{a}_{-i} \in C^i$. Since $W^i(\cdot, a_i, \widehat{a}_{-i})$ is monotone, $\|\widehat{a}_{-i} - a_{-i}\| > 0$ and $dis\left(a_{-i}, C^i\right) > 0$,

$$\begin{aligned} U^i(x, a_i, a_{-i}) &= \frac{W^i(x, a_i, \widehat{a}_{-i}) \|\widehat{a}_{-i} - a_{-i}\|}{dis(a_{-i}, C^i)} \\ &> \frac{W^i(x', a_i, \widehat{a}_{-i}) \|\widehat{a}_{-i} - a_{-i}\|}{dis(a_{-i}, C^i)} \\ &\geqslant \inf_{\widehat{\widehat{a}}_{-i} \in C^i} \frac{W^i\left(x', a_i, \widehat{\widehat{a}}_{-i}\right) \left\|\widehat{\widehat{a}}_{-i} - a_{-i}\right\|}{dis(a_{-i}, C^i)} \\ &= U^i(x', a_i, a_{-i}) \end{aligned}$$

I now show that

$$(\forall t \in \mathcal{T}) : \left(x^*_{i,t}, a^*_{i,t}\right) \in Arg \max_{(x,a_i)} U^i\left(x, a_i, a^*_{-i,t}\right) \quad \text{s.t.} \begin{cases} p_t \cdot x \leq p_t \cdot \omega_{i,t} \\ a_i \in [0, \overline{a}_{i,t}] \\ x \geqslant 0 \end{cases}$$

Fix $t \in \mathcal{T}$. Obviously, $t \in \mathcal{T}^i\left(a^*_{i,t}\right)$, and by construction

$$\left(\exists t' \in \Gamma^i\right) : a^*_{-i,t'} = a^*_{-i,t}$$

from where $\Omega\left(a^*_{-i,t}\right) = a^*_{-i,t}$ and $a^*_{-i,t} \in C^i$. Then,

$$U^i\left(\cdot, \cdot, a^*_{-i,t}\right) = W^i\left(\cdot, \cdot, a^*_{-i,t}\right)$$

and, since ℓ^i is strictly monotone, it is true that

$$\left(x^*_{i,t}, a^*_{i,t}\right) \in Arg \max_{(x,a_i)} U^i\left(x, a_i, a^*_{-i,t}\right) \quad \text{s.t.} \begin{cases} p_t \cdot x \leq p_t \cdot \omega_{i,t} \\ a_i \in [0, \overline{a}_{i,t}] \\ x \geqslant 0 \end{cases}$$

if, and only if,

$$\left(x^*_{i,t}, a^*_{i,t}\right) \in Arg \max_{(x,a_i)} V^i\left(x, a_i, a^*_{-i,t}\right) \quad \text{s.t.} \begin{cases} p_t \cdot x \leq p_t \cdot \omega_{i,t} \\ a_i \in [0, \overline{a}_{i,t}] \\ x \geqslant 0 \end{cases}$$

By the definition of data set and condition (4), it follows that $\left(x_{i,t}^{*}, a_{i,t}^{*}\right)$ is feasible for the problem. Moreover, notice that

$$V^{i}\left(x_{i,t}^{*}, a_{i,t}^{*}, a_{-i,t}^{*}\right) \geqslant V_{t}^{i}$$

since, otherwise, $\left(\exists t' \in \mathcal{T}^{i}\left(a_{i,t}^{*}\right)\right):$

$$\begin{aligned}
\phi_{t'}^{i}\left(x_{i,t}^{*}, a_{i,t}^{*}\right) &= V_{t'}^{i} + v_{t'}^{i} \cdot \left(x_{i,t}^{*} - x_{i,t'}^{*}\right) + \rho_{t'}^{i}\left(a_{i,t}^{*} - a_{i,t'}^{*}\right) \\
&\quad -\varepsilon_{i} h\left(\left(x_{i,t}^{*}, a_{i,t}^{*}\right) - \left(x_{i,t'}^{*}, a_{i,t'}^{*}\right)\right) \\
&< V_{t}^{i}
\end{aligned}$$

contradicting the definition of ε_{i} or the properties of V_{t}^{i}.

Also, by definition $\left(\exists t' \in \mathcal{T}^{i}\left(a_{i,t}^{*}\right)\right):$

$$\begin{aligned}
V^{i}\left(x_{i,t}^{*}, a_{i,t}^{*}, a_{-i,t}^{*}\right) &= \phi_{t'}^{i}\left(x_{i,t}^{*}, a_{i,t}^{*}\right) \\
&\leqslant \phi_{t}^{i}\left(x_{i,t}^{*}, a_{i,t}^{*}\right) \\
&= V_{t}^{i}
\end{aligned}$$

This establishes that

$$V^{i}\left(x_{i,t}^{*}, a_{i,t}^{*}, a_{-i,t}^{*}\right) = V_{t}^{i}$$

Now, suppose that $(x, a_{i}) \in \mathbb{R}^{L} \times \mathbb{R} \setminus \left\{\left(x_{i,t}^{*}, a_{i,t}^{*}\right)\right\}$ satisfies that

$$\begin{aligned}
p_{t} \cdot x &\leq p_{t} \cdot \omega_{i,t} \\
a_{i} &\in [0, \overline{a}_{i,t}] \\
x &\geqslant 0
\end{aligned}$$

Then, since $t \in \mathcal{T}^{i}\left(a_{i,t}^{*}\right)$, it follows that

$$\begin{aligned}
V^{i}\left(x, a_{i}, a_{-i,t}^{*}\right) &= \min_{t' \in \mathcal{T}^{i}\left(a_{i,t}^{*}\right)} \left\{\phi_{t'}^{i}\left(x, a_{i}\right)\right\} \\
&\leqslant \phi_{t}^{i}\left(x, a_{i}\right) \\
&= V_{t}^{i} + v_{t}^{i} \cdot \left(x - x_{i,t}^{*}\right) + \rho_{t}^{i}\left(a_{i} - a_{i,t}^{*}\right) - \varepsilon_{i} h\left(\left(x, a_{i}\right) - \left(x_{i,t}^{*}, a_{i,t}^{*}\right)\right)
\end{aligned}$$

By condition (1),

$$\begin{aligned}
V^{i}\left(x, a_{i}, a_{-i,t}^{*}\right) &\leqslant V_{t}^{i} + \left(\lambda_{i,t}^{*} p_{t} - \varsigma_{i,t}^{*}\right) \cdot \left(x - x_{i,t}^{*}\right) \\
&\quad + \rho_{t}^{i}\left(a_{i} - a_{i,t}^{*}\right) - \varepsilon_{i} h\left(\left(x, a_{i}\right) - \left(x_{i,t}^{*}, a_{i,t}^{*}\right)\right) \\
&= V_{t}^{i} + \lambda_{i,t}^{*} p_{t} \cdot \left(x - x_{i,t}^{*}\right) - \varsigma_{i,t}^{*} \cdot \left(x - x_{i,t}^{*}\right) \\
&\quad + \rho_{t}^{i}\left(a_{i} - a_{i,t}^{*}\right) - \varepsilon_{i} h\left(\left(x, a_{i}\right) - \left(x_{i,t}^{*}, a_{i,t}^{*}\right)\right)
\end{aligned}$$

Now, since $\lambda_{i,t}^{*} \in \mathbb{R}_{++}$, by condition (4) and assumption, $p_{t} \cdot x \leq p_{t} \cdot \omega_{i,t}$, one has that

$$\lambda_{i,t}^{*} p_{t} \cdot \left(x - x_{i,t}^{*}\right) = \lambda_{i,t}^{*}\left(p_{t} \cdot x - p_{t} \cdot \omega_{i,t}\right) \leqslant 0$$

Since $\varsigma_{i,t}^* \in \mathbb{R}_+^L$, by condition (3) and assumption $x \geqslant 0$, it is true that

$$-\varsigma_{i,t}^* \cdot \left(x - x_{i,t}^*\right) = -\varsigma_{i,t}^* \cdot x \leqslant 0$$

Suppose that $\rho_t^i > 0$. Then, since $\mu_{i,t}^* \geqslant 0$, by condition (2), it must be that $\eta_{i,t}^* > 0$, and then, by condition (3), $a_{i,t}^* = \overline{a}_{i,t}$. Since $a_i \in [0, \overline{a}_{i,t}]$, it follows that

$$\rho_t^i \left(a_i - a_{i,t}^*\right) \leqslant 0$$

If, on the other hand, $\rho_t^i < 0$. Then, since $\eta_{i,t}^* \geqslant 0$, by condition (2), it must be that $\mu_{i,t}^* > 0$, and then, by condition (3), $a_{i,t}^* = 0$. Since $a_i \in [0, \overline{a}_{i,t}]$, it follows that

$$\rho_t^i \left(a_i - a_{i,t}^*\right) \leqslant 0$$

If $\rho_t^i = 0$, trivially

$$\rho_t^i \left(a_i - a_{i,t}^*\right) \leqslant 0$$

Finally, since $\varepsilon_i \in \mathbb{R}_{++}$ and by assumption $(x, a_i) \neq \left(x_{i,t}^*, a_{i,t}^*\right)$

$$-\varepsilon_i h \left((x, a_i) - \left(x_{i,t}^*, a_{i,t}^*\right)\right) < 0$$

from where

$$\begin{aligned} V^i \left(x, a_i, a_{-i,t}^*\right) &< V_t^i \\ &= V^i \left(x_{i,t}^*, a_{i,t}^*, a_{-i,t}^*\right) \end{aligned}$$

as needed.

Since the former is true for each $i \in \mathcal{I}$, and since, by condition (6),

$$\sum_{i \in \mathcal{I}} x_{i,t}^* = \sum_{i \in \mathcal{I}} \omega_{i,t}$$

it follows that

$$\left(p_t, \left(x_{i,t}^*, a_{i,t}^*\right)_{i \in \mathcal{I}}\right) \in NW \left(\left(\mathcal{I}, \left(U^i, w_i, \overline{a}_i\right)_{i \in \mathcal{I}}\right)\right)$$

and therefore that

$$\left(p_t, \left(a_{i,t}^*\right)_{i \in \mathcal{I}}\right) \in NWPS \left(\left(\mathcal{I}, \left(U^i, w_i, \overline{a}_i\right)_{i \in \mathcal{I}}\right)\right)$$

■

The previous characterization can now be used to derive testable restrictions, in a way similar to what Brown and Matzkin (1996) do with their equilibrium inequalities, via quantifier elimination. This is done in subsections 3.3 and 3.4. It must be noticed, though, that these restrictions will be extremely mild, as they will be equivalent to the conditions of the previous theorem, the crucial one of which,[11] condition (5), need only hold for pairs of observations for which the

[11] Because it is the only one that involves more than one observations.

opponents of the player to whom the condition is being applied for evaluation are all playing the same strategies. In this sense, whatever testable restrictions are derived will be of "zero measure," in a manner similar to the ones found in Carvajal (2004a): assuming that no individual has degenerate domains ($\overline{\overline{a}}_i > 0$), if one generates a data set randomly, using nonatomic probability measures, then the probability of finding a nonrationalizable data set is zero, despite the fact that the data set in no way has been derived fron the equilibrium concept. This weakness implies that unless players are known to have degenerate domains of strategies, any researcher applying tests based on this theory, and with this kind of information, should expect, before observing the data set, that the hypothesis of Nash-Walras equilibrium will not be rejected. Given this, the fact that the conditions of theorem 1 are not only necessary but sufficient becomes of great relevance, because it implies that there are no stronger restrictions.[12]

3.3 Testable restrictions:

The characterization of theorem 1 uses existential quantifiers, which perhaps the researcher can deal with in specific cases of data sets via computational procedures, but which also appear uncomfortable from a theoretical perspective. In particular, one would like to know whether there exist restrictions on the data set only and what form these restrictions have. I now use the theory of quantifier elimination to show that there exists an equivalent characterization of rationalizable data sets, in terms of the observed variables only and that the abstract form of this characterization is relatively simple. Specifically,

Theorem 2 *Given a vector*

$$d = \left(\left(w_{i,t}, a^*_{i,t}, \overline{a}_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T} \in \left(\prod_{i \in \mathcal{I}} \mathbb{R}^L_{++} \times A^i \times A^{-i}\right)^T$$

there exists a semialgebraic set $\Delta(d) \subseteq \left(\mathbb{R}^L_{++}\right)^T$ *such that* $(p_t)_{t=1}^T \in \Delta(d)$ *if, and only if,*

$$\left(p_t, \left(w_{i,t}, a^*_{i,t}, \overline{a}_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T}$$

is NW-rationalizable.[13]

Proof. Define the functions $sgn : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ by

$$sgn(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

[12] In terms of the falsificationist position in epistemology, what the theorem says is that all the testable restrictions of the theory still fail to generate a test that should *ex ante* be considered harsh.

[13] Semialgebraic sets are defined in appendix 7.

and $\overrightarrow{sgn} : \mathbb{R}^L \longrightarrow \{-1,0,1\}^L$ by

$$\overrightarrow{sgn}(x) = (sgn(x_l))_{l=1}^L$$

Define

$$\Xi = \left(\left(\mathbb{R}_+^L \times \mathbb{R} \times \mathbb{R}_+^L \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}\right)^I\right)^T$$

and denote its generic element by

$$\xi = \left(\left(x_{i,t}^*, V_t^i, v_t^i, \rho_t^i, \lambda_{i,t}^*, \varsigma_{i,t}^*, \mu_{i,t}^*, \eta_{i,t}^*\right)_{i\in\mathcal{I}}\right)_{t=1}^T$$

Let $\widehat{\Delta} \subseteq \left(\mathbb{R}_+^L\right)^T \times \Xi$ be the set of all vectors

$$\left((p_t)_{t=1}^T, \xi\right)$$

that satisfy the following six conditions

1. $(\forall t \in \mathcal{T})(\forall i \in \mathcal{I})$:

$$\overrightarrow{sgn}\left(v_t^i - \lambda_{i,t}^* p_t - \varsigma_{i,t}^*\right) = (0)_{l=1}^L$$

2. $(\forall t \in \mathcal{T})(\forall i \in \mathcal{I})$:

$$sgn\left(\rho_t^i - \eta_{i,t}^* - \mu_{i,t}^*\right) = 0$$

3. $(\forall t \in \mathcal{T})(\forall i \in \mathcal{I})$:[14]

$$sgn\left(\sum_{l=1}^L \varsigma_{l,i,t}^* x_{l,i,t}^*\right) = 0$$
$$sgn\left(\mu_{i,t}^* a_{i,t}^*\right) = 0$$
$$sgn\left(\eta_{i,t}^*\left(\overline{a}_{i,t} - a_{i,t}^*\right)\right) = 0$$

4. $(\forall t \in \mathcal{T})(\forall i \in \mathcal{I})$:[15]

$$sgn\left(\sum_{l=i}^L p_{l,t} x_{l,i,t}^* - \sum_{l=i}^L p_{l,t} w_{l,i,t}\right) = 0$$

5. $(\forall i \in \mathcal{I})\left(\forall t, t' \in \mathcal{T} : a_{-i,t}^* = a_{-i,t'}^*\right)$:

$$sgn\left(V_{t'}^i - V_t^i - \sum_{l=1}^L v_{l,t}^i x_{l,i,t'}^* + \sum_{l=1}^L v_{l,t}^i x_{l,i,t}^* - \rho_t^i a_{i,t'}^* + \rho_t^i a_{i,t}^*\right) = s_{t,t'}^i$$

where

$$s_{t,t'}^i = \begin{cases} 0 \text{ if } \left(x_{i,t}^*, a_{i,t}^*\right) = \left(x_{i,t'}^*, a_{i,t'}^*\right) \\ -1 \text{ otherwise} \end{cases}$$

[14] When three subindices are used, they come in the order commodity-individual-observation: l, i, t.

[15] For prices, when two subindices are used, they come in the order commodity-observation: l, t.

6. $(\forall t \in \mathcal{T})$:

$$\overrightarrow{sgn}\left(\sum_{i\in\mathcal{I}} x^*_{i,t} - \sum_{i\in\mathcal{I}} w_{i,t}\right) = (0)^L_{l=1}$$

By definition, $\widehat{\Delta}$ is a semialgebraic set. Let $\Delta(d)$ be the projection of $\widehat{\Delta}$ into the space of prices:

$$\Delta(d) = \left\{ (p_t)^T_{t=1} \in \left(\mathbb{R}^L_+\right)^T \middle| (\exists \xi \in \Xi) : \left((p_t)^T_{t=1}, \xi\right) \in \widehat{\Delta} \right\}$$

By corollary 3 in appendix 7, $\Delta(d)$ is semialgebraic. Now, since conditions (1) to(6) above are equivalent to the conditions of theorem 1, it follows that $(p_t)^T_{t=1} \in \Delta(d)$ if, and only if,

$$\left(p_t, \left(w_{i,t}, a^*_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)^T_{t=1}$$

is NW-rationalizable. ■

This theorem implies that for given data on endowments, individual actions and individual constraints to actions,

$$\left(\left(w_{i,t}, a^*_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)^T_{t=1}$$

there do exist conditions on observed prices only, $(p_t)^T_{t=1}$, which are equivalent to the Nash-Walras rationalizability of the data set. These conditions have a relatively simple functional form, since they can be expressed as polynomial inequalities.

3.4 A non-NW-rationalizable data set:

The result of the previous subsection implies that one can characterize NW-rationalizability by conditions on prices only, given endowments, actions and constraints to actions. It does not, however, provide any information about the set of rationalizable prices, beyond the formal fact that it is semialgebraic. Nothing in that result implies that such set is nonempty, or prevents the possibility that it is the whole space of prices, $\left(\mathbb{R}^L_{++}\right)^T$. That the set is never empty follows from the existence results of Ghosal and Polemarchakis (1997). I now show that it need not always be equal to $\left(\mathbb{R}^L_{++}\right)^T$, by the following example:

Example 1 *Suppose that $I = L = T = 2$ and $\overline{\overline{a}}_1 = \overline{\overline{a}}_2 = 4$. The information of the data set is:*

$$\begin{array}{rr} \omega_{1,1} = (2,4) & \omega_{1,2} = (4,2) \\ \omega_{2,1} = (1,1) & \omega_{2,2} = (1,1) \\ \overline{a}_{1,1} = 3 & \overline{a}_{1,2} = 4 \\ \overline{a}_{2,1} = 2 & \overline{a}_{2,2} = 3 \\ p_1 = (1,10) & p_2 = (10,1) \\ a^*_{1,1} = 2 & a^*_{1,2} = 2 \\ a^*_{2,1} = 2 & a^*_{2,2} = 2 \end{array}$$

Notice that if the data is in $NWPS(\mathcal{E})$ for some economy $\mathcal{E}$, since $a^*_{2,1} = a^*_{2,2}$, at both observations, consumer 1 is maximizing the same utility function $U^1\left(\cdot,\cdot,a^*_{2,1}\right) = U^1\left(\cdot,\cdot,a^*_{2,2}\right)$. Since $p_1 \cdot \omega_{1,1} = p_2 \cdot \omega_{1,2} = 42$, $\sum_{i=1}^{2} \omega_{i,1} = (3,5)$, and $\sum_{i=1}^{2} \omega_{i,2} = (5,3)$, feasible values of $x^*_{1,1}$ and $x^*_{1,2}$ can only be, respectively, in

$$\begin{aligned} X_1 &= \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \middle| \, x \geqslant 0, p_1 \cdot x = 42, x_1 \leq 3 \right\} \\ &= \left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| \, x_1 \in [0,3], x_2 = 4.2 - 0.1 x_1 \right\} \end{aligned}$$

and

$$\begin{aligned} X_2 &= \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \middle| \, x \geqslant 0, p_1 \cdot x = 42, x_2 \leq 3 \right\} \\ &= \left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| \, x_1 \in [3.9, 4.2], x_2 = 42 - 10 x_1 \right\} \end{aligned}$$

Obviously, $X_1 \cap X_2 = \varnothing$, which implies that any candidates to $x^*_{1,1}$ and $x^*_{1,2}$ satisfy $x^*_{1,1} \neq x^*_{1,2}$. Suppose that $x^*_{1,1}$ and $x^*_{1,2}$ rationalize the behavior of consumer 1. Since $x^*_{1,1} \in X_1$,[16] then

$$\begin{aligned} p_2 \cdot x^*_{1,1} &= 10 x^*_{1,1,1} + x^*_{2,1,1} \\ &= 10 x^*_{1,1,1} + 4.2 - 0.1 x^*_{1,1,1} \\ &= 9.9 x^*_{1,1,1} + 4.2 \\ &\leqslant 9.9\,(3) + 4.2 \\ &< 42 \\ &= p_2 \cdot \omega_{1,2} \end{aligned}$$

whereas since $x^*_{1,2} \in X_2$, then

$$\begin{aligned} p_1 \cdot x^*_{1,2} &= x^*_{1,1,2} + 10 x^*_{2,1,2} \\ &= x^*_{1,1,2} + 10\left(42 - 10 x^*_{1,1,2}\right) \\ &= 420 - 99 x^*_{1,1,2} \\ &\leqslant 420 - 99\,(3.9) \\ &< 42 \\ &= p_1 \cdot \omega_{1,1} \end{aligned}$$

Also, since $a^*_{1,2} \in [0, \overline{a}_{1,1}]$ and $a^*_{1,1} \in [0, \overline{a}_{1,2}]$, it must be that $U^1\left(x^*_{1,1}, a^*_{1,1}, a^*_{2,1}\right) = U^1\left(x^*_{1,2}, a^*_{1,2}, a^*_{2,2}\right)$. But then, by strongly concavity of $U^1\left(\cdot,\cdot,a^*_{2,1}\right)$ for any $\lambda \in (0,1)$ one would have that, letting

$$(x_\lambda, a_\lambda) = \lambda\left(x^*_{1,1}, a^*_{1,1}\right) + (1-\lambda)\left(x^*_{1,2}, a^*_{1,2}\right)$$

it is true that $(x_\lambda, a_\lambda) \in \left\{ x \in \mathcal{R}^2_+ \middle| \, p_1 \cdot x \leqslant p_1 \cdot \omega_{1,1} \right\} \times [0, \overline{a}_{1,1}]$ and still

$$U^1\left(x_\lambda, a_\lambda, a^*_{2,1}\right) > U^1\left(x^*_{1,1}, a^*_{1,2}, a^*_{2,1}\right)$$

[16] Recall that for consumption of bundle x, two subindices i,t are taken in the order consumer-observation, whereas three subindices l,i,t are taken in the order commodity-consumer-observation.

contradicting the fact that $x^*_{1,1}$ rationalizes the behavior of consumer 1 at observation 1. Notice that this occurs in spite of the fact that nothing in the observed data reveals immediately that consumer 1 must be violating the axioms of revealed preferences, given the equality $a^*_{1,1} = a^*_{1,2}$. The fact of the matter is that, given the aggregate endowments and the prices, one can prove that his choices, although unknown, cannot be consistent with maximization of a single utility function. The example still holds if one assume that $a^*_{1,2} = 3$, but in this case the violation of the axioms of revealed preferences are more obvious from the data set.

The example indicates that there exist vectors

$$d = \left(\left(w_{i,t}, a^*_{i,t}, \overline{a}_{i,t} \right)_{i \in \mathcal{I}} \right)_{t=1}^{T} \in \left(\prod_{i \in \mathcal{I}} \mathbb{R}^L_{++} \times A^i \times A^{-i} \right)^T$$

for which the set of rationalizable prices, $\Delta(d)$ in the notation of subsection 3.3, is a proper subset of $\left(\mathbb{R}^L_{++}\right)^T$. For this example, the theory imposes nontautological testable restrictions. My comments after the proof of theorem 1 can then be restated as follows: for almost every vector d, $\Delta(d) = \mathbb{R}^L_{++}$.

3.5 Data sets with no observation of strategies:

My results so far indicate that, even with full observation of individual choices of actions, the restrictions imposed by the theory are very weak. The appeal of the results of Brown and Matzkin (1996) is their restrictions did not require observation of any individual choices, but only of individual budgets and summary statistics of choices in the form of prices. Suppose now that one does not observe

$$\left(\left(a^*_{i,t} \right)_{i \in \mathcal{I}} \right)_{t=1}^{T}$$

It seems reasonable to conjecture that all testable restrictions now disappear. In this case, the relevant set to study is the projection of the Nash-Walras set into the space of prices, defined as

$$NWP(\mathcal{E}) =$$

$$\left\{ p \in \mathbb{R}^L_{++} \times \prod_{i \in \mathcal{I}} A^i \,\middle|\, \left(\exists \left(x_i, a_i \right)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} \left(\mathbb{R}^L_{+} \times A^i \right) \right) : \left(p, \left(x_i, a_i \right)_{i \in \mathcal{I}} \right) \in NW(\mathcal{E}) \right\}$$

The conjecture in confirmed by the following theorem:

Theorem 3 *Let a finite sequence*

$$\left(p_t, \left(w_{i,t}, \overline{a}_{i,t} \right)_{i \in \mathcal{I}} \right)_{t=1}^{T}$$

in $\mathbb{R}^L_{++} \times \prod_{i \in \mathcal{I}} \left(\mathbb{R}^L_{+} \times A^i \right)$ *be given. If for each* $t \in \mathcal{T}$*, there exist* $i, i' \in \mathcal{I}$*,* $i \neq i'$*, such that* $\overline{a}_{i,t} \neq 0$ *and* $\overline{a}_{i',t} \neq 0$*, then for each* $i \in \mathcal{I}$ *there exists a* *function* $U^i : \mathbb{R}^L_+ \times A^i \times A^{-i} \longrightarrow \mathbb{R}$, *continuous and satisfying that*

$$\begin{array}{rcl} \left(\forall a_{-i} \in A^{-i}\right) & : & U^i\left(\cdot,\cdot,a_{-i}\right) \quad \textit{is Lipschitzian with constant } M_{a_{-i}} \\ \left(\forall a_{-i} \in A^{-i}\right) & : & U^i\left(\cdot,\cdot,a_{-i}\right) \quad \textit{is strongly concave} \\ \left(\forall \left(a_i, a_{-i}\right) \in A^i \times A^{-i}\right) & : & U^i\left(\cdot,a_i,a_{-i}\right) \quad \textit{is strictly monotone} \end{array}$$

such that for each $t \in \mathcal{T}$

$$p_t \in NWP\left(\left\{\mathcal{I}, \left(U^i, w_i, \overline{a}_i\right)_{i\in\mathcal{I}}\right\}\right)$$

Proof. Fix $i \in \mathcal{I}$.

If $\forall t \in \mathcal{T}$, $\overline{a}_{i,t} = 0$, then define $\forall t \in \mathcal{T}$, $a^*_{i,t} = 0$. Else, define $\alpha_i = \min_{t\in\mathcal{T}}\left\{\overline{a}_{i,t} \middle| \overline{a}_{i,t} > 0\right\}$, which exists since $T < \infty$, and define $\forall t \in \mathcal{T}$,

$$a^*_{i,t} = \begin{cases} \frac{\alpha_i}{t} & \text{if } \overline{a}_{i,t} > 0 \\ 0 & \text{otherwise} \end{cases}$$

Consider the sequence

$$\left(p_t, \left(w_{i,t}, a^*_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)_{t=1}^T$$

By assumption,

$$\begin{array}{rcl} \left(\forall t \in \mathcal{T}\right) & : & p_t \in \mathbb{R}^L_{++} \\ \left(\forall t \in \mathcal{T}\right)\left(\forall i \in \mathcal{I}\right) & : & w_{i,t} \in \mathbb{R}^L_{++} \\ \left(\forall t \in \mathcal{T}\right)\left(\forall i \in \mathcal{I}\right) & : & \left[0, \overline{a}_{i,t}\right] \subseteq A^i \end{array}$$

and by construction,

$$\left(\forall t \in \mathcal{T}\right)\left(\forall i \in \mathcal{I}\right) : a^*_{i,t} \in \left[0, \overline{a}_{i,t}\right]$$

because if $\forall t \in \mathcal{T}$, $\overline{a}_{i,t} = 0$, then $a^*_{i,t} = 0$, whereas if $\exists t' \in \mathcal{T}$ such that $\overline{a}_{i,t'} > 0$, then $a^*_{i,t} = 0$ if $\overline{a}_{i,t} = 0$, and if $\overline{a}_{i,t} > 0$, then

$$0 < a^*_{i,t} = \frac{\alpha_i}{t} \leqslant \alpha_i \leqslant \overline{a}_{i,t}$$

All this implies that

$$\left(p_t, \left(w_{i,t}, a^*_{i,t}, \overline{a}_{i,t}\right)_{i\in\mathcal{I}}\right)_{t=1}^T$$

is a data set.

Define also $\forall i \in \mathcal{I}$, $\forall t \in \mathcal{T}$, $x^*_{i,t} = w_{i,t}$, $V^i_t = 1$, $v^i_t = p_t$, $\rho^i_t = 0$, $\lambda^*_{i,t} = 1$, $\varsigma^*_{i,t} = (0)^L_{l=1}$, $\mu^*_{i,t} = 0$ and $\eta^*_{i,t} = 0$.

Conditions (1) to (4) and (6) of theorem 1 are immediate.

Now fix $i \in \mathcal{I}$ and $a_{-i} \in A^{-i}$. If $\mathcal{T}^i\left(a_{-i}\right) = \varnothing$, condition (5) is satisfied. Now, suppose that $\mathcal{T}^i\left(a_{-i}\right) \neq \varnothing$ and let $t \in \mathcal{T}^i\left(a_{-i}\right)$. Then $a^*_{-i,t} = a_{-i}$. For every $t' \in \mathcal{T} \setminus \{t\}$, by assumption of the theorem $\exists i' \in \mathcal{I} \setminus \{i\}$ such that $\overline{a}_{i',t'} \neq 0$ and hence

$$a^*_{i',t'} = \frac{\alpha_{i'}}{t'} > 0$$

If $\overline{a}_{i',t} = 0$, then $a^*_{i',t} = 0$, whereas if $\overline{a}_{i',t} > 0$, then $a^*_{i',t} = \frac{\alpha_{i'}}{t}$ and, in any case, $a^*_{i',t} \neq a^*_{i',t'}$ and therefore $t' \notin \mathcal{T}^i(a_{-i})$, from where condition (5) of theorem 1 is also satisfied.

It follows then that

$$\left(p_t, \left(w_{i,t}, a^*_{i,t}, \overline{a}_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T}$$

is NW-rationalizable, which implies the result. ■

The theorem implies that every sequence that contains no information about individual choices of the externality (and in this case not even some aggregate information) is rationalizable as coming from Nash-Walras equilibria of an economy under the assumptions imposed here, unless the domains for strategies happen to be very degenerate. This result is easy to obtain in this case, since no aggregate information is restricting the possible choices of

$$\left(\left(a^*_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T}$$

and the assumption on the constraints still leaves enough room to construct these choices with enough variations so that no player ever needs to be assumed to maximize the same utility function at two different observations.

4 Consumption externalities:

Consumption externalities arise when consumption of some commodity by one individual affects the well-being of others. These commodities are traded in markets, in the same way as any other goods. As they represent clearly defined physical objects, decisions about this kind of externalities can be compared across individuals. Moreover, these decisions are market choices: they are made with consideration to prices and endowments and different demands by an agent do change the amount of the commodity available to others (although no player takes this consideration into account when making his decisions). In this sense, these commodities are not public goods.[17] Also, this demands are subject to aggregation and their markets have to clear for an equilibrium to be attained.

In this section, I consider whether the hypothesis of Nash-Walras equilibrium is falsifiable under consumption externalities. In contrast to strategic externalities, even if only prices and not individual demands are observed, one still has an aggregate summary of all the decisions of individuals. This is why in this section, unlike in the previous one, I have carefully distinguished the cases of partial observability (some information on individual choices) and no observability

[17] Although they could enter the utility functions as aggregates.

(none). In the first subsection, I introduce the specific problem under consideration and then in subsection 4.2 I show that, with some limitations, one can deal with consumption externalities as if they were strategic externalities. Since these limitations weaken the results obtained by that approach, subsections 4.3 and 4.4 solve the problem as entirely independent of the results of section 3. Finally, subsection 4.5 solves the problem under the extra assumption of weak separability of the utility functions.

4.1 The model:

The economy considered here is again populated by a finite set of consumers $\mathcal{I} = \{1, ..., I\}$, with $2 \leqslant I < \infty$. The decisions that each consumer has to make now relate only to consumption bundles. I assume that there exist $L+1$ commodities, where $L \in \mathbb{N}$, and define the consumption set of each individual to be $\mathbb{R}_+^L \times \mathbb{R}_+$. A consumption bundle is now denoted by (x, y), where $x \in \mathbb{R}_+^L$ and $y \in \mathbb{R}_+$.

Consumption externalities exist because the well-being of each individual is affected not only by his own consumption, but also by the consumption of commodity y by all the other consumers. For each player $i \in \mathcal{I}$, I will denote by $(x_i, y_i) \in \mathbb{R}_+^L \times \mathbb{R}_+$ his own consumption and by $y_{-i} \in \mathbb{R}_+^{I-1}$ the consumption of commodity y by the rest of the consumers. Formally, then, I assume that each individual has preferences that can be represented by

$$U^i : \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}$$

which means that I am assuming that there is one common commodity that generates the external effects. From now on, this commodity will be referred to as the externality.

When making decisions, individuals only take into account their own constraints, which in this case reduce to the standard budget constraint. That is to say that if agent $i \in \mathcal{I}$ is endowed with $w_i \in \mathbb{R}_+^L$ of bundle x and with $\kappa_i \in \mathbb{R}_+$ of the externality y, and prices are $p \in \mathbb{R}_{++}^L$ and $q \in \mathbb{R}_{++}$ respectively, then he chooses his demand (x_i, y_i) subject only to his budget constraint:

$$p \cdot x_i + q y_i \leqslant p \cdot w_i + q \kappa_i$$

An economy is completely described by the set of agents, $\mathcal{I}$, their preferences, $\left(U^i : \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}\right)_{i \in \mathcal{I}}$, and their endowments of bundle x, $(w_i)_{i \in \mathcal{I}} \in \left(\mathbb{R}_{++}^L\right)^I$ and of the externality y, $(\kappa_i)_{i \in \mathcal{I}} \in (\mathbb{R}_{++})^I$, where I am assuming strictly positive endowments. Formally, an economy is a vector

$$\left(\mathcal{I}, \left(U^i, w_i, \kappa_i\right)_{i \in \mathcal{I}}\right)$$

The hypothesis whose falsifiability I want to study is again whether agents behave noncooperatively, as in Nash-Walras equilibrium. In this case, the definition of equilibrium states that agents choose their demands so as to maximize their own well-being, given their endowments, and taking as given all prices and the demands of the externality by all other consumers. Equilibrium occurs when prices are such that under this behavior all markets clear. Formally:

Definition 4 *Given an economy*

$$\mathcal{E} = \left\{ \mathcal{I}, \left(U^i, w_i, \kappa_i\right)_{i \in \mathcal{I}} \right\}$$

a Nash-Walras equilibrium is a vector

$$\left(p^*, q^*, (x_i^*, y_i^*)_{i \in \mathcal{I}}\right) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++} \times \left(\mathbb{R}_+^L \times \mathbb{R}_+\right)^I$$

such that

$$\begin{aligned} (\forall i \in \mathcal{I}) : (x_i^*, y_i^*) \quad & \in \quad \underset{(x,y)}{Argmax} U^i \left(x, y, y_{-i}^*\right) \\ & \quad s.t. \begin{cases} p^* \cdot x + q^* y \leqslant p^* \cdot \omega_i + q^* \kappa_i \\ x \geqslant 0 \\ y \geqslant 0 \end{cases} \end{aligned}$$

and

$$\sum_{i \in \mathcal{I}} (x_i^*, y_i^*) = \sum_{i \in \mathcal{I}} (\omega_i, \kappa_i)$$

The set of Nash-Walras equilibria of economy $\mathcal{E}$ is denoted $NW(\mathcal{E})$.

4.2 Embedding consumption as strategic externalities:

Before proceeding any further, it is worthwhile to explore whether, and at what cost, one can reduce the problem of consumption externalities to just a particular case of strategic externalities, maintaining the framework under which the results of section 3 can be applied directly.

The following theorem shows that indeed this can be done, under further smoothness assumptions and restricting attention to compact subsets of the domain.

Theorem 4 *Suppose that $U : \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}$ is continuously differentiable and that for each $y_{-i} \in \mathbb{R}_+^{I-1}$, $U(\cdot, \cdot, y_{-i})$ is strongly concave, strictly monotone and Lipschitzian with constant $M_{y_{-i}}$. Let $D \subseteq \mathbb{R}_+^L$ be nonempty, convex and compact, and let $\overline{\overline{a}} \in \mathbb{R}_+$. Denote $A^i = \left[0, \overline{\overline{a}}\right]$ and $\widehat{X} = D \times A^i$. There exists a function $\alpha : \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}_{++}$ such that the function $V : \widehat{X} \times A^{-i} \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}$, defined by*

$$V\left((x_i, y_i), a_i, y_{-i}\right) = U\left(x_i, y_i, y_{-i}\right) - \alpha\left(y_{-i}\right)\left(y_i - a_i\right)^2$$

is continuous, satisfies that

$$\begin{aligned} \left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) \quad &: \quad V^i(\cdot, \cdot, y_{-i}) \quad \text{is Lipschitzian with } \widehat{M}_{y_{-i}} \geqslant M_{y_{-i}} \\ \left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) \quad &: \quad V^i(\cdot, \cdot, y_{-i}) \quad \text{is strongly concave} \\ \left(\forall (a_i, y_{-i}) \in A^i \times \mathbb{R}_+^{I-1}\right) \quad &: \quad V^i(\cdot, a_i, y_{-i}) \quad \text{is strictly monotone} \end{aligned}$$

and is such that for all $y^*_{-i} \in \mathbb{R}^{I-1}_+$*, it is true that*

$$\begin{aligned} (x_i^*, y_i^*) &\in Arg \max_{(x_i, y_i)} U\left(x_i, y_i, y^*_{-i}\right) \\ & s.t: \begin{cases} x_i \in \mathbb{R}^L_+ \\ y_i \in \mathbb{R}_+ \\ p \cdot x_i + q y_i \leqslant W \end{cases} \end{aligned}$$

if, and only if,

$$\begin{aligned} ((x_i^*, y_i^*), y_i^*) &\in Arg \max_{(x_i, y_i, a_i)} V\left((x_i, y_i), a_i, y^*_{-i}\right) \\ & s.t: \begin{cases} (x_i, y_i) \in \widehat{X} \\ a_i \in A^i \\ (p, q) \cdot (x_i, y_i) \leqslant W \end{cases} \end{aligned}$$

for all $W \in \mathbb{R}_{++}$*,* $p \in \mathbb{R}^L_{++}$ *and* $q \in \mathbb{R}_{++}$ *satisfying that*

$$\left\{ (x, y) \in \mathbb{R}^L_+ \times \mathbb{R}_+ \middle| \, p \cdot x + q y \leqslant W \right\} \subseteq \widehat{X}$$

Proof. If $\overline{\overline{a}} = 0$, then the result is trivial, as any continuous function

$$\alpha : \mathbb{R}^{I-1}_+ \longrightarrow \mathbb{R}_{++}$$

will have the desired properties. Hence, I assume that $\overline{\overline{a}} > 0$.

Define the function $\alpha : \mathbb{R}^{I-1}_+ \longrightarrow \mathbb{R}_{++}$ by

$$\alpha\left(y_{-i}\right) = \frac{1}{4\overline{\overline{a}}} \min_{(x_i, y_i) \in \widehat{X}} \frac{\partial U}{\partial y_i}\left(x_i, y_i, y_{-i}\right)$$

Since D is compact, so is $\widehat{X}$ and then, by Weierstrass' theorem, α is well defined because U is continuously differentiable (and hence continuously partially differentiable), and $\forall y_{-i} \in \mathbb{R}^{I-1}_+$, $\alpha\left(y_{-i}\right) > 0$ by monotonicity of U. Moreover, by the theorem of the maximum, it follows that α is continuous and hence that V is continuous.

Fix $(a_i, y_{-i}) \in A^i \times \mathbb{R}^{I-1}_+$. Since U is differentiable, so is $V\left((\cdot, \cdot), a_i, y_{-i}\right)$ and for each $(\widehat{x}_i, \widehat{y}_i) \in \widehat{X}$,

$$D_{(x,y)} V\left((\widehat{x}_i, \widehat{y}_i), a_i, y_{-i}\right) = \begin{bmatrix} D_x U\left(\widehat{x}_i, \widehat{y}_i, y_{-i}\right) \\ D_y U\left(\widehat{x}_i, \widehat{y}_i, y_{-i}\right) \end{bmatrix} - 2\alpha\left(y_{-i}\right) \begin{bmatrix} \mathbf{0}_{L \times 1} \\ y_i - a_i \end{bmatrix}$$

Since, by construction, for each $(\widehat{x}_i, \widehat{y}_i) \in \widehat{X}$,

$$\begin{aligned} D_y U\left(\widehat{x}_i, \widehat{y}_i, y_{-i}\right) - 2\alpha\left(y_{-i}\right)\left(y_i - a_i\right) &\geqslant D_y U\left(\widehat{x}_i, \widehat{y}_i, y_{-i}\right) - 2\alpha\left(y_{-i}\right)\overline{\overline{a}} \\ &> D_y U\left(\widehat{x}_i, \widehat{y}_i, y_{-i}\right) - 4\alpha\left(y_{-i}\right)\overline{\overline{a}} \\ &= D_y U\left(\widehat{x}_i, \widehat{y}_i, y_{-i}\right) \\ &\quad - \min_{(x_i, y_i) \in \widehat{X}} \frac{\partial U}{\partial y_i}\left(x_i, y_i, y_{-i}\right) \\ &\geqslant 0 \end{aligned}$$

it follows that $D_{(x,y)}V\left(\left(\widehat{x}_i,\widehat{y}_i\right),a_i,y_{-i}\right)\in\mathbb{R}^{L+1}_{++}$, and hence that

$$\left(\forall\left(a_i,y_{-i}\right)\in A^i\times\mathbb{R}^{I-1}_{+}\right):V^i\left(\cdot,a_i,y_{-i}\right)\quad\text{is strictly monotone}$$

Now, fix $y_{-i}\in\mathbb{R}^{I-1}_{+}$. Let $\left(x_i,y_i\right),\left(x'_i,y'_i\right)\in\widehat{X}$ and $a_i,a'_i\in A^i$ be such that $\left(\left(x_i,y_i\right),a_i\right)\neq\left(\left(x'_i,y'_i\right),a'_i\right)$ and let $\theta\in(0,1)$. Then, denoting

$$\begin{array}{rcl}\left(\widehat{x}_i,\widehat{y}_i\right)&=&\theta\left(x_i,y_i\right)+(1-\theta)\left(x'_i,y'_i\right)\\ \widehat{a}_i&=&\theta a_i+(1-\theta)\,a'_i\end{array}$$

it follows that

$$V\left(\left(\widehat{x}_i,\widehat{y}_i\right),\widehat{a}_i,y_{-i}\right)=U\left(\widehat{x}_i,\widehat{y}_i,y_{-i}\right)-\alpha\left(y_{-i}\right)\left(\widehat{y}_i-\widehat{a}_i\right)^2$$

Suppose first that $y_i=y'_i$. Then,

$$U\left(\widehat{x}_i,\widehat{y}_i,y_{-i}\right)\leqslant\theta U\left(x_i,y_i,y_{-i}\right)+(1-\theta)\,U\left(x'_i,y'_i,y_{-i}\right)$$

and

$$\begin{array}{rcl}-\left(\widehat{y}_i-\widehat{a}_i\right)^2&=&-\left(y_i-\theta a_i-(1-\theta)\,a'_i\right)^2\\ &\leqslant&-\theta\left(y_i-a_i\right)^2-(1-\theta)\left(y_i-a'_i\right)^2\\ &=&-\theta\left(y_i-a_i\right)^2-(1-\theta)\left(y'_i-a'_i\right)^2\end{array}$$

with at least one of the previous inequalities being strict. Then, since $\alpha\left(y_{-i}\right)>0$, it follows that

$$V\left(\left(\widehat{x}_i,\widehat{y}_i\right),\widehat{a}_i,y_{-i}\right)<\theta V\left(\left(x_i,y_i\right),a_i,y_{-i}\right)+(1-\theta)\,V\left(\left(x'_i,y'_i\right),a'_i,y_{-i}\right)$$

If, on the other hand, $y_i\neq y'_i$, then

$$U\left(\widehat{x}_i,\widehat{y}_i,y_{-i}\right)<\theta U\left(x_i,y_i,y_{-i}\right)+(1-\theta)\,U\left(x'_i,y'_i,y_{-i}\right)$$

whereas

$$-\left(\widehat{y}_i-\widehat{a}_i\right)^2\leqslant-\theta\left(y_i-a_i\right)^2-(1-\theta)\left(y'_i-a'_i\right)^2$$

and, again, since $\alpha\left(y_{-i}\right)>0$,

$$V\left(\left(\widehat{x}_i,\widehat{y}_i\right),\widehat{a}_i,y_{-i}\right)<\theta V\left(\left(x_i,y_i\right),a_i,y_{-i}\right)+(1-\theta)\,V\left(\left(x'_i,y'_i\right),a'_i,y_{-i}\right)$$

implying that

$$\left(\forall y_{-i}\in\mathbb{R}^{I-1}_{+}\right):V^i\left(\cdot,\cdot,y_{-i}\right)\quad\text{is strongly concave}$$

Now, for any $\alpha>0$, consider the mapping $h_\alpha:\left[0,\overline{\overline{a}}\right]^2\longrightarrow\mathbb{R}$ defined by

$$h_\alpha\left(y,a\right)=-\alpha\left(y-a\right)^2$$

Fix $(y,a),(y',a') \in \left[0,\overline{\overline{a}}\right]^2$. By the mean value theorem, $\exists\left(\widehat{y},\widehat{a}\right) \in \left[0,\overline{\overline{a}}\right]^2$ such that

$$\begin{aligned} h_\alpha(y',a') &= h_\alpha(y,a) + Dh_\alpha\left(\widehat{y},\widehat{a}\right) \cdot \begin{bmatrix} y'-y \\ a'-a \end{bmatrix} \\ &= h_\alpha(y,a) - 2\alpha \begin{bmatrix} \widehat{y}-\widehat{a} \\ \widehat{a}-\widehat{y} \end{bmatrix} \cdot \begin{bmatrix} y'-y \\ a'-a \end{bmatrix} \end{aligned}$$

from where

$$h_\alpha(y',a') - h_\alpha(y,a) = -2\alpha\left(\widehat{y}-\widehat{a}\right)\left((y'-y)-(a'-a)\right)$$

and hence

$$\begin{aligned} \left|h_\alpha(y',a') - h_\alpha(y,a)\right| &= 2\alpha\left|\widehat{y}-\widehat{a}\right|\left|(y'-y)-(a'-a)\right| \\ &\leqslant 2\alpha\overline{\overline{a}}\left|(y'-y)-(a'-a)\right| \\ &\leqslant 2\alpha\overline{\overline{a}}\left(|y'-y|+|a'-a|\right) \\ &\leqslant \sqrt{8}\alpha\overline{\overline{a}}\sqrt{|y'-y|^2+|a'-a|^2} \\ &= \sqrt{8}\alpha\overline{\overline{a}}\left\|(y,a)-(y',a')\right\| \end{aligned}$$

where the first inequality follows since $\widehat{y},\widehat{a} \in \left[0,\overline{\overline{a}}\right]$, the second one from triangle inequality and the last one since $\forall k \in \mathbb{R}_+$,

$$\left(\max_{(u,v)\in\mathbb{R}^2} u+v \quad s.t. \quad \sqrt{u^2+v^2}=k\right) = \sqrt{2}k$$

This implies that h_α is Lipschitzian with constant $\sqrt{8}\alpha\overline{\overline{a}}$.

Again, fix $y_{-i} \in \mathbb{R}_+^{I-1}$ and let $(x_i,y_i),(x_i',y_i') \in \widehat{X}$ and $a_i,a_i' \in A^i$. Then,

$$\begin{aligned} & \left|V\left((x_i,y_i),a_i,y_{-i}\right) - V\left((x_i',y_i'),a_i',y_{-i}\right)\right| \\ = & \left|U(x_i,y_i,y_{-i}) - h_{\alpha(y_{-i})}(y_i,a_i) - U(x_i',y_i',y_{-i}) + h_{\alpha(y_{-i})}(y_i',a_i')\right| \\ \leqslant & \left|U(x_i,y_i,y_{-i}) - U(x_i',y_i',y_{-i})\right| + \left|h_{\alpha(y_{-i})}(y_i,a_i) - h_{\alpha(y_{-i})}(y_i',a_i')\right| \\ \leqslant & M_{y_{-i}}\left\|(x_i,y_i)-(x_i',y_i')\right\| + \sqrt{8}\alpha(y_{-i})\overline{\overline{a}}\left\|(y_i,a_i)-(y_i',a_i')\right\| \\ \leqslant & \left(M_{y_{-i}} + \sqrt{8}\alpha(y_{-i})\overline{\overline{a}}\right)\left\|(x_i,y_i,a_i)-(x_i',y_i',a_i')\right\| \end{aligned}$$

which means that $\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) : V^i(\cdot,\cdot,y_{-i})$ is Lipschitzian with constant

$$\widehat{M}_{y_{-i}} = M_{y_{-i}} + \sqrt{8}\alpha(y_{-i})\overline{\overline{a}} \geqslant M_{y_{-i}}$$

Finally, fix $y_{-i}^* \in \mathbb{R}_+^{I-1}$, $W \in \mathbb{R}_{++}$, $p \in \mathbb{R}_{++}^L$ and $q \in \mathbb{R}_{++}$ such that

$$\left\{(x,y) \in \mathbb{R}_+^L \times \mathbb{R}_+ \,\middle|\, p\cdot x + qy \leqslant W\right\} \subseteq \widehat{X}$$

Suppose first that

$$\begin{aligned}
(x_i^*, y_i^*) &\in Arg \max_{(x_i,y_i)} U\left(x_i, y_i, y_{-i}^*\right) \\
& s.t: \begin{cases} x_i \in \mathbb{R}_+^L \\ y_i \in \mathbb{R}_+ \\ p \cdot x_i + q y_i \leqslant W \end{cases}
\end{aligned}$$

By construction, $(x_i^*, y_i^*) \in \widehat{X}$ and $\forall (x_i^*, y_i^*) \in \widehat{X}$ such that $p \cdot x_i + q y_i \leqslant W$ and $\forall a_i \in A^i$,

$$\begin{aligned}
V((x_i, y_i), a_i, y_{-i}) &= U(x_i, y_i, y_{-i}) - \alpha(y_{-i})(y_i - a_i)^2 \\
&\leqslant U(x_i, y_i, y_{-i}) \\
&\leqslant U(x_i^*, y_i^*, y_{-i}) \\
&= V((x_i^*, y_i^*), y_i^*, y_{-i})
\end{aligned}$$

from where

$$\begin{aligned}
((x_i^*, y_i^*), y_i^*) &\in Arg \max_{(x_i,y_i,a_i)} V\left((x_i, y_i), a_i, y_{-i}^*\right) \\
& s.t: \begin{cases} (x_i, y_i) \in \widehat{X} \\ a_i \in A^i \\ (p,q) \cdot (x_i, y_i) \leqslant W \end{cases}
\end{aligned}$$

If, on the other hand,

$$\begin{aligned}
((x_i^*, y_i^*), y_i^*) &\in Arg \max_{(x_i,y_i,a_i)} V\left((x_i, y_i), a_i, y_{-i}^*\right) \\
s.t &: \begin{cases} (x_i, y_i) \in \widehat{X} \\ a_i \in A^i \\ (p,q) \cdot (x_i, y_i) \leqslant W \end{cases}
\end{aligned}$$

Then, $x_i^* \in \mathbb{R}_+^L$, $y_i^* \in \mathbb{R}_+$ and $\forall (x_i, y_i) \in \mathbb{R}_+^L \times \mathbb{R}_+$ such that $p \cdot x_i + q y_i \leqslant W$, we have that $(x_i, y_i) \in \widehat{X}$ and

$$\begin{aligned}
U(x_i, y_i, y_{-i}) &= V((x_i, y_i), y_i, y_{-i}) \\
&\leqslant V((x_i^*, y_i^*), y_i^*, y_{-i}) \\
&= U(x_i^*, y_i^*, y_{-i})
\end{aligned}$$

from where

$$\begin{aligned}
(x_i^*, y_i^*) &\in Arg \max_{(x_i,y_i)} U\left(x_i, y_i, y_{-i}^*\right) \\
& s.t: \begin{cases} x_i \in \mathbb{R}_+^L \\ y_i \in \mathbb{R}_+ \\ p \cdot x_i + q y_i \leqslant W \end{cases}
\end{aligned}$$

■

This result is useful in that it implies that the existence results in Ghosal and Polemarchakis (1997), which are intended for strategic externalities, also hold in the case of consumption externalities. The theorem also shows that by embedding consumption externalities in the framework of section 3, one would need to restrict attention to the class of continuously differentiable utility functions with Lipschitzian cross sections, and that the rationalizations would hold for compact subdomains.

As it turns out, these restrictions are unnecessary and only weaken the results. Although from a practical perspective this may be an irrelevance, I have chosen not to pay the cost, and will develop the results of this section independently, and not as corollaries of the ones in section 3.

4.3 Partial observability:

I first consider the case in which there exists information about individual demands for the externality. In this case, one is interested in equilibrium values of prices and individual demands for y, for which it is convenient to define the projection of the set of Nash-Walras equilibria of an economy $\mathcal{E}$ into the space of prices and demands for the externality. Let $NWPE(\mathcal{E})$ denote this set, whose formal definition is:

$$\begin{aligned} NWPE(\mathcal{E}) &= \Big\{ \left(p, q, (y_i)_{i\in\mathcal{I}}\right) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++} \times (\mathbb{R}_+)^I \Big| \\ &\quad \left(\exists (x_i)_{i\in\mathcal{I}} \in \left(\mathbb{R}^L_+\right)^I\right) : \left(p, q, (x_i, y_i)_{i\in\mathcal{I}}\right) \in NW(\mathcal{E}) \Big\} \end{aligned}$$

Again, I consider finite data sets of prices of all commodities, individual endowments of all commodities and individual demands of the externality, as follows.

Definition 5 *A data set with partial observability is a finite sequence*

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^T$$

such that for each $t \in \mathcal{T} = \{1, ..., T\}$

$$\begin{aligned} p_t &\in \mathbb{R}^L_{++} \\ q_t &\in \mathbb{R}_{++} \\ \sum_{i\in\mathcal{I}} y^*_{i,t} &= \sum_{i\in\mathcal{I}} \kappa_{i,t} \end{aligned}$$

and for each $t \in \mathcal{T}$ *and each* $i \in \mathcal{I}$

$$\begin{aligned} \omega_{i,t} &\in \mathbb{R}^L_{++} \\ \kappa_{i,t} &\in \mathbb{R}_{++} \\ y^*_{i,t} &\in \mathbb{R}_+ \end{aligned}$$

$$p_t \cdot \omega_{i,t} + q_t\left(\kappa_{i,t} - y^*_{i,t}\right) \geqslant 0$$

That is, for each observation $t \in \mathcal{T}$, one observes strictly positive prices for the externality, q_t, and the other commodities, p_t, and for each individual one observes strictly positive endowments of the externality, $\kappa_{i,t}$, and all other commodities, $w_{i,t}$, and a demand for the externality, $y_{i,t}^*$. These observed demands are assumed feasible form both the individual point of view (budget constraints) and from the market clearing perspective (given the endowments).[18]

As before, I want to test whether a data set is weakly consistent with Nash-Walras equilibrium, in conditions under which this does not occur trivially. These conditions are imposed in the following definition

Definition 6 *A data set with partial observability*

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y_{i,t}^*\right)_{i \in \mathcal{I}}\right)\right)_{t=1}^{T}$$

is said to be Nash-Walras-rationalizable if for each $i \in \mathcal{I}$ there exists $U^i : \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}$, continuous, satisfying that

$$\begin{array}{lcll} \left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) & : & U^i\left(\cdot,\cdot,y_{-i}\right) & \textit{is strongly concave} \\ \left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) & : & U^i\left(\cdot,\cdot,y_{-i}\right) & \textit{is strictly monotone} \end{array}$$

such that

$$\left(\forall t \in \mathcal{T}\right) : \left(p_t, q_t, \left(y_{i,t}^*\right)_{i \in \mathcal{I}}\right) \in NWPE\left(\left\{\mathcal{I}, \left(U^i, w_{i,t}, \kappa_{i,t}\right)_{i \in \mathcal{I}}\right\}\right)$$

In this case, it is said that $\left(U^i\right)_{i \in \mathcal{I}}$ NW-rationalizes the data.

The conditions here play the same roles as in section 3, but I no longer need to impose the Lipschitz condition, because I will derive the results here independently of the analysis of strategic externalities, using standard results in revealed preference theory, instead of the approach through subdifferential calculus.

4.3.1 Equilibrium Inequalities:

The following characterization of rationalizable data sets is analogous to theorem 1 in section 3: it dispenses with the utility functions in the definition of rationalizability, substituting them by individual choices satisfying conditions that make them "equivalent" to the utility functions.

For the theorem, the following notation needs to be introduced. Given a data set with partial observability

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y_{i,t}^*\right)_{i \in \mathcal{I}}\right)\right)_{t=1}^{T}$$

define, for each $i \in \mathcal{I}$, the correspondence

$$\mathcal{T}^i : \mathbb{R}_+^{I-1} \rightrightarrows \mathcal{T}$$

[18] Recall that I am assuming that the problem lacks any intertemporal character.

by

$$\mathcal{T}^i\left(y_{-i}\right)=\left\{t\in\mathcal{T}\,|\,y^*_{-i,t}=y_{-i}\right\}$$

Theorem 5 *A data set with partial observability*

$$\left(\left(p_t,q_t,\left(\omega_{i,t},\kappa_{i,t},y^*_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

is NW-rationalizable if, and only if, for each $i\in\mathcal{I}$ and each $t\in\mathcal{T}$ there exist $x^*_{i,t}\in\mathbb{R}^L_+$, $V^i_t\in\mathbb{R}$, $v^i_t\in\mathbb{R}^L_{++}$, $\rho^i_t\in\mathbb{R}_{++}$ and $\lambda^*_{i,t}\in\mathbb{R}_{++}$ such that:

1. $(\forall i\in\mathcal{I})\,(\forall t\in\mathcal{T}):v^i_t=\lambda^*_{i,t}p_t$
2. $(\forall i\in\mathcal{I})\,(\forall t\in\mathcal{T}):\rho^i_t=\lambda^*_{i,t}q_t$
3. $(\forall i\in\mathcal{I})\,(\forall t\in\mathcal{T}):p_t\cdot x^*_{i,t}=p_t\cdot\omega_{i,t}+q_t\left(\kappa_{i,t}-y^*_{i,t}\right)$
4. $(\forall i\in\mathcal{I})\left(\forall y_{-i}\in\mathbb{R}^{I-1}_+\right)\left(\forall t,t'\in\mathcal{T}^i\left(y_{-i}\right)\right):$

$$V^i_{t'}\leqslant V^i_t+v^i_t\cdot\left(x^*_{i,t'}-x^*_{i,t}\right)+\rho^i_t\left(y^*_{i,t'}-y^*_{i,t}\right)$$

   with strict inequality whenever

$$\left(x^*_{i,t'},y^*_{i,t'}\right)\neq\left(x^*_{i,t},y^*_{i,t}\right)$$

5. $(\forall t\in\mathcal{T}):\sum_{i\in\mathcal{I}}x^*_{i,t}=\sum_{i\in\mathcal{I}}\omega_{i,t}$

Proof. Necessity: Let $\left(U^i\right)_{i\in\mathcal{I}}$ NW-rationalize the data set.

Since $\forall t\in\mathcal{T}$,

$$\left(p_t,q_t,\left(y^*_{i,t}\right)_{i\in\mathcal{I}}\right)\in NWPE\left(\left(U^i,\omega_{i,t},\kappa_{i,t}\right)_{i\in\mathcal{I}}\right)$$

by definition, there exists $\left(x^*_{i,t}\right)_{i\in\mathcal{I}}\in\left(\mathbb{R}^L_+\right)^I$ such that

$$\left(p_t,q_t,\left(x^*_{i,t},y^*_{i,t}\right)_{i\in\mathcal{I}}\right)\in NW\left(\left(\mathcal{I},\left(U^i,w_{i,t},\kappa_{i,t}\right)_{i\in\mathcal{I}}\right)\right)$$

Fix one such $\left(x^*_{i,t}\right)_{i\in\mathcal{I}}\in\left(\mathbb{R}^L_+\right)^I$. By definition,

$$\sum_{i\in\mathcal{I}}x^*_{i,t}=\sum_{i\in\mathcal{I}}\omega_{i,t}$$

which is condition (5). Also, since for each $i\in\mathcal{I}$, $U^i\left(\cdot,\cdot,y^*_{-i,t}\right)$ is monotone, we must have that

$$(\forall i\in\mathcal{I}):p_t\cdot x^*_{i,t}+q_ty^*_{i,t}=p_t\cdot\omega_{i,t}+q_t\kappa_{i,t}$$

from where condition (3) is obvious.

Now, fix $i\in\mathcal{I}$. Define $\Gamma^i\subseteq\mathbb{N}$ as follows:

- $\gamma_1^i = \{1\}$
- for $t \in \{2, ..., T\}$

$$\gamma_t^i = \left\{ \begin{array}{cc} \varnothing \quad \text{if} & (\exists t' \in \{1, ..., t-1\}) : y^*_{-i,t'} = y^*_{-i,t} \\ \{t\} & \text{otherwise} \end{array} \right.$$

- $\Gamma^i = \bigcup_{t \in \mathcal{T}} \gamma_t^i$

Clearly,

$$\begin{array}{rcl} (\forall t \in \mathcal{T}) \left(\exists \widehat{t} \in \Gamma^i\right) & : & y^*_{-i,\widehat{t}} = y^*_{-i,t} \\ \left(\forall \widehat{t}, \widetilde{t} \in \Gamma^i : \widehat{t} \neq \widetilde{t}\right) & : & y^*_{-i,\widehat{t}} \neq y^*_{-i,\widetilde{t}} \end{array}$$

so that $\left\{\mathcal{T}^i \left(y^*_{-i,\widehat{t}}\right)\right\}_{\widehat{t} \in \Gamma^i}$ is a partition of $\mathcal{T}$: the first condition implies that

$$\mathcal{T} \subseteq \bigcup_{\widehat{t} \in \Gamma^i} \mathcal{T}^i \left(y^*_{-i,\widehat{t}}\right)$$

and since

$$\bigcup_{\widehat{t} \in \Gamma^i} \mathcal{T}^i \left(y^*_{-i,\widehat{t}}\right) \subseteq \mathcal{T}$$

it is clear that

$$\mathcal{T} = \bigcup_{\widehat{t} \in \Gamma^i} \mathcal{T}^i \left(y^*_{-i,\widehat{t}}\right)$$

Whereas from the second condition $\forall \widehat{t}, \widetilde{t} \in \Gamma^i$,

$$\widehat{t} \neq \widetilde{t} \Longrightarrow \mathcal{T}^i \left(y^*_{-i,\widehat{t}}\right) \cap \mathcal{T}^i \left(y^*_{-i,\widetilde{t}}\right) = \varnothing$$

Now, for each $\widehat{t} \in \Gamma^i$, define

$$W^i_{\widehat{t}} : \mathbb{R}^L_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}; W^i_{\widehat{t}} (x_i, y_i) = U^i \left(x_i, y_i, y^*_{-i,\widehat{t}}\right)$$

Since $W^i_{\widehat{t}}$ is continuous, strongly concave and strictly monotone, and

$$\begin{array}{rcl} \left(\forall t \in \mathcal{T}^i \left(y^*_{-i,\widehat{t}}\right)\right) & : & \left(x^*_{i,t}, y^*_{i,t}\right) \in Arg \max\limits_{(x,y) \in \mathbb{R}^L_+ \times \mathbb{R}_+} W^i_{\widehat{t}} (x_i, y_i) \\ \text{s.t.} \quad p_t \cdot x + q_t y & \leqslant & p_t \cdot \omega_{i,t} + q_t \kappa_{i,t} \end{array}$$

it then follows from theorem 2 (b$\Longrightarrow$c) in Matzkin and Richter (1991)[19] that $\forall t \in \mathcal{T}^i \left(y^*_{-i,\widehat{t}}\right)$, there exist $V^i_{t,\widehat{t}} \in \mathbb{R}$, $v^i_{t,\widehat{t}} \in \mathbb{R}^L_{++}$, $\rho^i_{t,\widehat{t}} \in \mathbb{R}_{++}$, $\lambda^*_{i,t,\widehat{t}} \in \mathbb{R}_{++}$ such that:

[19] In order to match their notation, one may define

$$\begin{array}{rcl} \mathcal{B}^i_{\widehat{t}} & = & \bigcup_{t \in \mathcal{T}^i \left(y^*_{-i,\widehat{t}}\right)} \{((p_t, q_t), p_t \cdot \omega_{i,t} + q_t \kappa_{i,t})\} \\ \left(\forall t \in \mathcal{T}^i \left(y^*_{-i,\widehat{t}}\right)\right) & : & h^i_{\widehat{t}} (((p_t, q_t), p_t \cdot \omega_{i,t} + q_t \kappa_{i,t})) = \left(x^*_{i,t}, y^*_{i,t}\right) \end{array}$$

(i) $\left(\forall t \in \mathcal{T}^i\left(y^*_{-i,\widehat{t}}\right)\right) : v^i_{t,\widehat{t}} = \lambda^*_{i,t,\widehat{t}} p_t$

(ii) $\left(\forall t \in \mathcal{T}^i\left(y^*_{-i,\widehat{t}}\right)\right) : \rho^i_{t,\widehat{t}} = \lambda^*_{i,t,\widehat{t}} q_t$

(iii) $\left(\forall t, t' \in \mathcal{T}^i\left(y^*_{-i,\widehat{t}}\right)\right) :$

$$V^i_{t',\widehat{t}} \leqslant V^i_{t,\widehat{t}} + v^i_{t,\widehat{t}} \cdot \left(x^*_{i,t'} - x^*_{i,t}\right) + \rho^i_{t,\widehat{t}}\left(y^*_{i,t'} - y^*_{i,t}\right)$$

with strict inequality whenever

$$\left(x^*_{i,t'}, y^*_{i,t'}\right) \neq \left(x^*_{i,t}, y^*_{i,t}\right)$$

However, since $\left\{\mathcal{T}^i\left(y^*_{-i,\widehat{t}}\right)\right\}_{\widehat{t}\in\Gamma^i}$ is a partition of $\mathcal{T}$, one can define, unambiguously, for each $t \in \mathcal{T}$,

$$\begin{aligned} V^i_t &= V^i_{t,\widehat{t}} \\ v^i_t &= v^i_{t,\widehat{t}} \\ \rho^i_t &= \rho^i_{t,\widehat{t}} \\ \lambda^*_{i,t} &= \lambda^*_{i,t,\widehat{t}} \end{aligned}$$

where $\widehat{t} \in \Gamma^i$ is such that $t \in \mathcal{T}^i\left(y^*_{-i,\widehat{t}}\right)$.

Conditions (1) and (2) of the theorem follow then immediately from (i) and (ii). Condition (4) is obvious for

$$y_{-i} \in \mathbb{R}^{I-1}_+ \setminus \left\{y^*_{-i,\widehat{t}}\right\}_{\widehat{t}\in\Gamma^i}$$

since, in such case $\mathcal{T}^i\left(y_{-i}\right) = \varnothing$, and follows from (iii) otherwise.

Sufficiency: Fix $i \in \mathcal{I}$, and define Γ^i as in the proof of necessity. As before,

$$\left(\forall \widehat{t}, \widetilde{t} \in \Gamma^i : \widehat{t} \neq \widetilde{t}\right) : y^*_{-i,\widehat{t}} \neq y^*_{-i,\widetilde{t}}$$
$$\left(\forall t \in \mathcal{T}\right)\left(\exists \widehat{t} \in \Gamma^i\right) : y^*_{-i,\widehat{t}} = y^*_{-i,t}$$

Fix $\widehat{t} \in \Gamma^i$. By condition (4), $\forall t, t' \in \mathcal{T}^i\left(y^*_{-i,\widehat{t}}\right) :$

$$V^i_{t'} \leqslant V^i_t + v^i_t \cdot \left(x^*_{i,t'} - x^*_{i,t}\right) + \rho^i_t\left(y^*_{i,t'} - y^*_{i,t}\right)$$

with strict inequality whenever

$$\left(x^*_{i,t'}, y^*_{i,t'}\right) \neq \left(x^*_{i,t}, y^*_{i,t}\right)$$

which implies, given conditions (1), (2) and (3), by theorem 2 (c$\Longrightarrow$b) in Matzkin and Richter (1991), that there exists $W^i_{\widehat{t}} : \mathbb{R}^L_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$, continuous, strongly concave and strictly monotone, such that

$$\begin{aligned} \left(\forall t \in \mathcal{T}^i\left(y^*_{-i,\widehat{t}}\right)\right) : \left(x^*_{i,t}, y^*_{i,t}\right) \quad &\in \quad Arg \max_{(x,y)\in\mathbb{R}^L_+\times\mathbb{R}_+} W^i_{\widehat{t}}(x,y) \\ &\text{s.t.} \quad p_t \cdot x + q_t y \leqslant p_t \cdot \omega_{i,t} + q_t \kappa_{i,t} \end{aligned}$$

By strong monotonicity,

$$\left(\forall\left(x,y\right)\in\mathbb{R}_{+}^{L}\times\mathbb{R}_{+}\right):W_{\widehat{t}}^{i}\left(x,y\right)\geqslant W_{\widehat{t}}^{i}\left(0,0\right)$$

Let

$$\underline{w}^{i}=\min\left\{W_{\widehat{t}}^{i}\left(0,0\right)\right\}_{\widehat{t}\in\Gamma^{i}}\in\mathbb{R}$$

Define the truncated logistic function $\ell^{i}:\left[\underline{w}^{i},\infty\right)\longrightarrow[1,2)$ by

$$\ell^{i}\left(w\right)=\frac{2}{1+\exp\left(\underline{w}^{i}-w\right)}$$

which is continuous, strictly increasing and strongly concave

Define

$$C^{i}=\left\{y_{-i,\widehat{t}}^{*}\right\}_{\widehat{t}\in\Gamma^{i}}\subseteq\mathbb{R}_{+}^{I-1}$$

and $W^{i}:\mathbb{R}_{+}^{L}\times\mathbb{R}_{+}\times C^{i}\longrightarrow[1,2)$ by

$$W^{i}\left(x_{i},y_{i},y_{-i}\right)=\ell^{i}\left(W_{\widehat{t}}^{i}\left(x_{i}.y_{i}\right)\right)$$

where $\widehat{t}\in\Gamma^{i}$ is such that $y_{-i,\widehat{t}}^{*}=y_{-i}$. By construction, W^{i} is bounded. Also, $\forall y_{-i}\in C^{i}$, it is clear that $W^{i}\left(\cdot,\cdot,y_{-i}\right)$ is strictly monotone, since so are all $W_{\widehat{t}}^{i}$, for $\widehat{t}\in\Gamma^{i}$, and ℓ^{i}. Moreover, $\forall y_{-i}\in C^{i}$, $W^{i}\left(\cdot,\cdot,y_{-i}\right)$ is strongly concave: let $\left(x,y\right),\left(x',y'\right)\in\mathbb{R}_{+}^{L}\times\mathbb{R}_{+}$, $\left(x,y\right)\neq\left(x',y'\right)$ and $\theta\in(0,1)$; then, for $\widehat{t}\in\Gamma^{i}$ such that $y_{-i,\widehat{t}}^{*}=y_{-i}$

$$W_{\widehat{t}}^{i}\left(\theta x+\left(1-\theta\right)x',\theta y+\left(1-\theta\right)y'\right)>\theta W_{\widehat{t}}^{i}\left(x,y\right)+\left(1-\theta\right)W_{\widehat{t}}^{i}\left(x',y'\right)$$

by strong concavity of $W_{\widehat{t}}^{i}$. This implies that

$$\begin{aligned}\ell^{i}\left(W_{\widehat{t}}^{i}\left(\theta x+\left(1-\theta\right)x',\theta y+\left(1-\theta\right)y'\right)\right) &> \ell^{i}\left(\theta W_{\widehat{t}}^{i}\left(x,y\right)+\left(1-\theta\right)W_{\widehat{t}}^{i}\left(x',y'\right)\right)\\ &\geqslant \theta\ell^{i}\left(W_{\widehat{t}}^{i}\left(x,y\right)\right)+\left(1-\theta\right)\ell^{i}\left(W_{\widehat{t}}^{i}\left(x',y'\right)\right)\end{aligned}$$

by strict monotonicity and concavity of ℓ^{i}.

Furthermore, W^{i} is continuous, as C^{i} contains no limit points. To see this, let $\left(x_{i},y_{i},y_{-i}\right)\in\mathbb{R}_{+}^{L}\times\mathbb{R}_{+}\times C^{i}$ and consider any sequence $\left(\left(x_{i}^{n},y_{i}^{n},y_{-i}^{n}\right)\right)_{n=1}^{\infty}$ such that $\forall n\in\mathbb{N}$, $\left(x_{i}^{n},y_{i}^{n},y_{-i}^{n}\right)\in\mathbb{R}_{+}^{L}\times\mathbb{R}_{+}\times C^{i}$ and $\left(x_{i}^{n},y_{i}^{n},y_{-i}^{n}\right)\longrightarrow\left(x_{i},y_{i},y_{-i}\right)$. Then, since $\#C^{i}=\#\Gamma^{i}\leqslant T<\infty$, $\exists N\in\mathbb{N}$ such that

$$\left(\forall n\geqslant N\right):y_{-i}^{n}=y_{-i}$$

and hence

$$\left(\forall n\geqslant N\right):W^{i}\left(x_{i}^{n},y_{i}^{n},y_{-i}^{n}\right)=\ell^{i}\left(W_{\widehat{t}}^{i}\left(x_{i}^{n},y_{i}^{n}\right)\right)$$

where $\widehat{t}\in\Gamma^{i}$ is such that $y_{-i,\widehat{t}}^{*}=y_{-i}$. But, by continuity of ℓ^{i} and $W_{\widehat{t}}^{i}$ it follows that

$$\begin{aligned}\ell^{i}\left(W_{\widehat{t}}^{i}\left(x_{i}^{n},y_{i}^{n}\right)\right) &\longrightarrow \ell^{i}\left(W_{\widehat{t}}^{i}\left(x_{i},y_{i}\right)\right)\\ &= W^{i}\left(x_{i},y_{i},y_{-i}\right)\end{aligned}$$

Now, since C^i is compact, it follows from corollary 1 in Carvajal (2004c) that there exists $U^i : \mathbb{R}^L_+ \times \mathbb{R}_+ \times \mathbb{R}^{I-1}_+ \longrightarrow \mathbb{R}$ such that:

(i) U^i is continuous.

(ii) $\left(\forall y_{-i} \in \mathbb{R}^{I-1}_+\right) : U^i\left(\cdot,\cdot,y_{-i}\right)$ is strongly concave.

(iii) $\left(\forall y_{-i} \in \mathbb{R}^{I-1}_+\right) : U^i\left(\cdot,\cdot,y_{-i}\right)$ is strictly monotone.

(iv) $\left(\forall \left(x_i,y_i,y_{-i}\right) \in \mathbb{R}^L_+ \times \mathbb{R}_+ \times C^i\right) : U^i\left(x_i,y_i,y_{-i}\right) = W^i\left(x_i,y_i,y_{-i}\right)$.

By construction, and since ℓ^i is strictly monotone, it is true that $\forall t \in \mathcal{T}$,

$$\begin{array}{rcl}\left(x^*_{i,t}, y^*_{i,t}\right) & \in & Arg \displaystyle\max_{(x,y)\in\mathbb{R}^L_+\times\mathbb{R}_+} \ell^i\left(W^i_{\widehat{t}}\left(x,y\right)\right) \\ & & \text{s.t. } \ p_t \cdot x + q_t y \leqslant p_t \cdot \omega_{i,t} + q_t \kappa_{i,t}\end{array}$$

where $\widehat{t} \in \Gamma^i$ is such that $y^*_{i,\widehat{t}} = y^*_{-i,t}$, which exists and is unique because $\left\{\mathcal{T}^i\left(y^*_{-i,\widehat{t}}\right)\right\}_{\widehat{t}\in\Gamma^i}$ is a partition of $\mathcal{T}$. From here, it follows that $\forall t \in \mathcal{T}$,

$$\begin{array}{rcl}\left(x^*_{i,t}, y^*_{i,t}\right) & \in & Arg \displaystyle\max_{(x,y)\in\mathbb{R}^L_+\times\mathbb{R}_+} W^i\left(x,y,y^*_{-i,t}\right) \\ & & \text{s.t. } \ p_t \cdot x + q_t y \leqslant p_t \cdot \omega_{i,t} + q_t \kappa_{i,t}\end{array}$$

and, hence, by (iv), $\forall t \in \mathcal{T}$,

$$\begin{array}{rcl}\left(x^*_{i,t}, y^*_{i,t}\right) & \in & Arg \displaystyle\max_{(x,y)\in\mathbb{R}^L_+\times\mathbb{R}_+} U^i\left(x,y,y^*_{-i,t}\right) \\ & & \text{s.t. } \ p_t \cdot x + q_t y \leqslant p_t \cdot \omega_{i,t} + q_t \kappa_{i,t}\end{array}$$

By condition (5) of the theorem, the latter implies that for each $t \in \mathcal{T}$,

$$\left(p_t, q_t, \left(x^*_{i,t}, y^*_{i,t}\right)\right) \in NW\left(\left(U^i, \omega_{i,t}, \kappa_{i,t}\right)\right)$$

and hence that

$$\left(p_t, q_t, \left(y^*_{i,t}\right)\right) \in NWE\left(\left(U^i, \omega_{i,t}, \kappa_{i,t}\right)\right)$$

It then follows from (i), (ii) and (iii) that $\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^T$ is NW-rationalizable. ■

Again, the conditions of theorem 5 are intuitive. Given a rationalizable data set, if one also observed individual demands for commodities other than the externality, $x^*_{i,t}$, for each $i \in \mathcal{I}$ and each $t \in \mathcal{T}$, then condition 5 would come by definition of equilibrium (market clearing), and condition 3 from strict monotonicity of the utility functions (Walras' law), whereas conditions 1, 2 and 4, which are Afriat inequalities, are equivalent to the Strong Axiom of Revealed Preference (see Matzkin and Richter (1991)), conditional on $y^*_{-i,t}$.

Similarly to what was done in section 3, I will use this characterization to derive testable restrictions. Before doing so, though, it must again be noticed that whatever restrictions are found to be imposed by the theory, they will be very mild in the same sense as in the case of strategic externalities: they are restrictive only for pairs of observations for which all the opponents of a consumer maintain their demand for the externality unchanged. In this sense, the test implicit in these restrictions lacks any power, in the sense that, before observing the data, one should expect it to past such test.

4.3.2 Testable restrictions:

As in subsection 3.3, the theory of quantifier elimination can be applied here to show that there exist restrictions, on observed data only, that are equivalent to Nash-Walras rationalizability with partial observability. The result is:

Theorem 6 *Given a vector*

$$d = \left(\left(w_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T} \in \left(\left(\mathbb{R}^L_{++} \times \mathbb{R}_{++} \times \mathbb{R}_+\right)^I\right)^T$$

there exists a semialgebraic set $\Delta(d) \subseteq \left(\mathbb{R}^L_{++} \times \mathbb{R}_{++}\right)^T$ *such that* $(p_t, q_t)_{t=1}^T \in \Delta(d)$ *if, and only if,*

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i \in \mathcal{I}}\right)\right)_{t=1}^{T}$$

is NW-rationalizable.

Proof. The argument is similar to the proof of theorem 2, and is therefore omitted. ■

4.3.3 A non-NW-rationalizable data set with partial observability:

I now show that the restrictions derived above need not always be vacuous, in the sense that there exist vectors

$$d = \left(\left(w_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T} \in \left(\left(\mathbb{R}^L_{++} \times \mathbb{R}_{++} \times \mathbb{R}_+\right)^I\right)^T$$

for which the set $\Delta(d)$ defined in theorem 6 is a proper subset of $\left(\mathbb{R}^L_{++} \times \mathbb{R}_{++}\right)^T$.[20]

Example 2 *Suppose that* $I = L = T = 2$. *The information of the data set with partial observation is:*

$$\begin{array}{cc}
\omega_{1,1} = (1,4) & \omega_{1,2} = (4,1) \\
\omega_{2,1} = (2,1) & \omega_{2,2} = (1,2) \\
\kappa_{1,1} = 1 & \kappa_{1,2} = 0.5 \\
\kappa_{2,1} = 1 & \kappa_{2,2} = 1.5 \\
p_1 = (1,10) & p_2 = (10,1) \\
q_1 = 1 & q_2 = 2 \\
y^*_{1,1} = 0 & y^*_{1,2} = 0 \\
y^*_{2,1} = 2 & y^*_{2,2} = 2
\end{array}$$

[20] The results of Ghosal and Polemarchakis (1997) and theorem 4 above imply that this set is nonempty.

Suppose that the data set lies in $NWPE(\mathcal{E})$, for some economy $\mathcal{E}$ with all the properties of definition 6. Since $y^*_{2,1} = y^*_{2,2}$, consumer 1 maximizes the same utility function $U^1\left(\cdot,\cdot,y^*_{2,1}\right) = U^1\left(\cdot,\cdot,y^*_{2,2}\right)$ at both observations. Since

$$\begin{aligned} p_1 \cdot \omega_{1,1} + q_1\left(\kappa_{1,1} - y^*_{1,1}\right) &= 42 \\ p_2 \cdot \omega_{1,2} + q_2\left(\kappa_{1,2} - y^*_{1,2}\right) &= 42 \end{aligned}$$

$\sum_{i=1}^{2} \omega_{i,1} = (3,5)$ and $\sum_{i=1}^{2} \omega_{i,2} = (5,3)$, feasible values of $x^*_{1,1}$ and $x^*_{1,2}$ can only be, respectively, in

$$\begin{aligned} X_1 &= \left\{ x = (x_1,x_2) \in \mathbb{R}^2 \middle| x \geqslant 0, p_1 \cdot x = 42, x_1 \leq 3 \right\} \\ &= \left\{ (x_1,x_2) \in \mathbb{R}^2 \middle| x_1 \in [0,3], x_2 = 4.2 - 0.1x_1 \right\} \end{aligned}$$

and

$$\begin{aligned} X_2 &= \left\{ x = (x_1,x_2) \in \mathbb{R}^2 \middle| x \geqslant 0, p_1 \cdot x = 42, x_2 \leq 3 \right\} \\ &= \left\{ (x_1,x_2) \in \mathbb{R}^2 \middle| x_1 \in [3.9,4.2], x_2 = 42 - 10x_1 \right\} \end{aligned}$$

As before, $X_1 \cap X_2 = \varnothing$ implies that any candidates to $x^*_{1,1}$ and $x^*_{1,2}$ satisfy $x^*_{1,1} \neq x^*_{1,2}$. Suppose that $x^*_{1,1}$ and $x^*_{1,2}$ rationalize the behavior of consumer 1. Since $x^*_{1,1} \in X_1$,[21] then

$$\begin{aligned} p_2 \cdot x^*_{1,1} + q_2 y^*_{1,1} &= 10x^*_{1,1,1} + x^*_{2,1,1} + 2y^*_{1,1} \\ &= 10x^*_{1,1,1} + 4.2 - 0.1x^*_{1,1,1} \\ &= 9.9x^*_{1,1,1} + 4.2 \\ &\leqslant 9.9(3) + 4.2 \\ &< 42 \\ &= p_2 \cdot \omega_{1,2} + q_2\kappa_{1,2} \end{aligned}$$

whereas since $x^*_{1,2} \in X_2$, then

$$\begin{aligned} p_1 \cdot x^*_{1,2} + q_1 y^*_{1,2} &= x^*_{1,1,2} + 10x^*_{2,1,2} + y^*_{1,2} \\ &= x^*_{1,1,2} + 10\left(42 - 10x^*_{1,1,2}\right) + 0.5 \\ &= 420 - 99x^*_{1,1,2} \\ &\leqslant 420 - 99(3.9) \\ &< 42 \\ &= p_1 \cdot \omega_{1,1} + q_1\kappa_{1,1} \end{aligned}$$

This implies that $U^1\left(x^*_{1,1}, y^*_{1,1}, y^*_{2,1}\right) = U^1\left(x^*_{1,2}, y^*_{1,2}, y^*_{2,2}\right)$. But then, by strong concavity of $U^1\left(\cdot,\cdot,y^*_{2,1}\right)$ for any $\lambda \in (0,1)$ one would have that, letting

$$(x_\lambda, y_\lambda) = \lambda\left(x^*_{1,1}, y^*_{1,1}\right) + (1-\lambda)\left(x^*_{1,2}, y^*_{1,2}\right)$$

[21] Once again, for consumption of bundle x, two subindices i,t are taken in the order consumer, observation, whereas three subindices l,i,t are taken in the order commodity, consumer, observation.

it is true that $(x_\lambda, y_\lambda) \in \left\{(x,y) \in \mathbb{R}^2_+ \times \mathbb{R}_+ \middle| p_1 \cdot x + q_1 y \leqslant p_1 \cdot \omega_{1,1} + q_1 \kappa_{1,1}\right\}$ and still $U^1\left(x_\lambda, y_\lambda, y^*_{2,1}\right) > U^1\left(x^*_{1,1}, y^*_{1,1}, y^*_{2,1}\right)$, contradicting the fact that $x^*_{1,1}$ rationalizes the behavior of consumer 1 at observation 1. Again, this occurs in spite of the fact that nothing on the observed choices of individual 1 is immediately inconsistent with the axioms of revealed preferences, since $y^*_{1,1} = y^*_{1,2}$. The example is less interesting, but still holds with minor changes if one lets $\kappa_{1,2} = 1$ and $y^*_{1,2} = 0.5$.

4.4 No observability:

In the last section I showed that, although *ex ante* very mild, Nash-Walras equilibrium hypothesis imposes testable restrictions whenever prices, individual endowments and individual consumptions of the externality are observed. This approach assumes that some individual choices are being observed and does not, therefore, follow the spirit of Brown and Matzkin (1996). When no individual choice is observed, one must only deal with prices and endowments.

Definition 7 *A data set with no observability is a finite sequence*

$$\left(\left(p_t, q_t, (\omega_{i,t}, \kappa_{i,t})_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

such that for each $t \in \mathcal{T} = \{1, ..., T\}$

$$\begin{aligned} p_t &\in \mathbb{R}^L_{++} \\ q_t &\in \mathbb{R}_{++} \end{aligned}$$

and for each $t \in \mathcal{T}$ *and each* $i \in \mathcal{I}$

$$\begin{aligned} \omega_{i,t} &\in \mathbb{R}^L_{++} \\ \kappa_{i,t} &\in \mathbb{R}_{++} \end{aligned}$$

The definition of consistency with Nash-Walras equilibrium of a data set must now require that there exist utility functions (with the same conditions that were required under partial observability) such that for each observation $t \in \mathcal{T}$, the observed prices lie in the projection into the space of prices of the Nash-Walras set of the economy configured by those preferences and the observed corresponding endowments.

Definition 6 provides with a shortcut for the definition of Nash-Walras rationalizability in this case,

Definition 8 *A data set with no observability*

$$\left(\left(p_t, q_t, (\omega_{i,t}, \kappa_{i,t})_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

is said to be Nash-Walras-rationalizable if for each $i \in \mathcal{I}$ *and each* $t \in \mathcal{T}$ *there exists* $y^*_{i,t} \in \mathbb{R}_+$ *such that the sequence*

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

is a data set with partial observability and is NW-rationalizable.

In this case, in contrast to strategic externalities, even if one does not observe individual decisions on the externality, there is a summary statistic about this decision, in the form of prices $(q_t)_{t=1}^T$. The interesting result is that, the observation of those prices notwithstanding, the theory again fails to impose any testable restrictions.

Theorem 7 *Every data set with no observability*

$$\left(\left(p_t, q_t, (\omega_{i,t}, \kappa_{i,t})_{i\in\mathcal{I}}\right)\right)_{t=1}^T$$

is NW-rationalizable.

Proof. Introduce the following algorithm

Algorithm 1 [22] *Input:* $\left(\left(p_t, q_t, (\omega_{i,t}, \kappa_{i,t})_{i\in\mathcal{I}}\right)\right)_{t=1}^T$

1. *Let* $t = 1$.
2. *If*
$$(\forall t' \in \{1, ..., t-1\})\,(\forall i \in \mathcal{I}) : y^*_{i,t'} \neq \kappa_{i,t}$$
define
$$\begin{aligned}(\forall i \in \mathcal{I}) \quad &: \quad x^*_{i,t} = \omega_{i,t}\\ (\forall i \in \mathcal{I}) \quad &: \quad y^*_{i,t} = \kappa_{i,t}\end{aligned}$$
and go to step 6.
3. *Let* $J = \left\{ i \in \mathcal{I} |\, (\exists t' \in \{1, ..., t-1\}) : y^*_{i,t'} \neq \kappa_{i,t} \right\}$. *If* $J = \varnothing$, *let* $\varepsilon = 1$ *and go to step 5.*
4. *Define*
$$\varepsilon = \min_{i\in\mathcal{J}} \left\{ \min_{t'\in\{1,\ldots,t-1\}: y^*_{i,t'}\neq\kappa_{i,t}} \left| y^*_{i,t'} - \kappa_{i,t} \right| \right\}$$
5. *Define*
$$\gamma = \min_{i\in\mathcal{I}\setminus\{1\}} \left\{ \frac{(I-1)\, p_{1,t}\omega_{1,i,t}}{q_t} \right\}$$
$$\delta = \min\left\{\frac{\varepsilon}{2}, \kappa_{1,t}, \gamma\right\}$$
$$y^*_{1,t} = \kappa_{1,t} - \delta$$
$$(\forall i \in \mathcal{I}\setminus \{1\}) : y^*_{i,t} = \kappa_{i,t} + \frac{\delta}{I-1}$$

[22] Recall that for consumption of bundle x, three subindices l, i, t are taken in the order commodity-individual-observation. For prices, two subindices l, t are taken in the order commodity-observation.

$$x^*_{l,1,t} = \begin{cases} \omega_{l,1,t} + \frac{q_t \delta}{p_{1,t}} & \textit{if} \quad l = 1 \\ \omega_{l,1,t} & \textit{otherwise} \end{cases}$$

$$(\forall i \in \mathcal{I} \backslash \{1\}) : x^*_{l,i,t} = \begin{cases} \omega_{l,i,t} - \frac{q_t \delta}{p_{1,t}(I-1)} & \textit{if} \quad l = 1 \\ \omega_{l,i,t} & \textit{otherwise} \end{cases}$$

6. If $t = T$ stop. Else, $t = t + 1$ and go to step 2.

Output: $\left(\left(x^*_{i,t}, y^*_{i,t} \right)_{i \in \mathcal{I}} \right)_{t=1}^{T}$

This algorithm runs in finite time, since $T < \infty$. I now show that its output has the following properties:

(i) $(\forall t \in \mathcal{T}) (\forall i \in \mathcal{I}) : p_t \cdot x^*_{i,t} + q_t y^*_{i,t} = p_t \cdot \omega_{i,t} + q_t \kappa_{i,t}$

(ii) $(\forall t \in \mathcal{T}) (\forall i \in \mathcal{I}) : x^*_{i,t} \in \mathbb{R}^L_+$ and $y^*_{i,t} \in \mathbb{R}_+$

(iii) $(\forall t \in \mathcal{T})$:

$$\sum_{i \in \mathcal{I}} \left(x^*_{i,t}, y^*_{i,t} \right) = \sum_{i \in \mathcal{I}} \left(\omega_{i,t}, \kappa_{i,t} \right)$$

(iv) $(\forall i \in \mathcal{I}) (\forall t, t' \in \mathcal{T} : t \neq t') : y^*_{i,t} \neq y^*_{i,t'}$.

Let $e_1 = (1, 0, ..., 0) \in \mathbb{R}^L_+$. Notice that, since $T < \infty$, if at some pass through the algorithm, δ is defined, it satisfies $\delta > 0$.

Property (i) is straightforward if $\left(x^*_{i,t}, y^*_{i,t} \right)_{i \in \mathcal{I}} = \left(\omega_{i,t}, \kappa_{i,t} \right)_{i \in \mathcal{I}}$, in which case it has been defined by step 2. Else, $\left(x^*_{i,t}, y^*_{i,t} \right)_{i \in \mathcal{I}}$ must have been defined by step 5, and

$$\begin{aligned} p_t \cdot x^*_{1,t} + q_t y^*_{1,t} &= p_t \cdot \left(\omega_{1,t} + \frac{q_t \delta}{p_{1,t}} e_1 \right) + q_t \left(\kappa_{1,t} - \delta \right) \\ &= p_t \cdot \omega_{1,t} + q_t \kappa_{1,t} \end{aligned}$$

whereas for each $i \in \mathcal{I} \backslash \{1\}$,

$$\begin{aligned} p_t \cdot x^*_{1,t} + q_t y^*_{1,t} &= p_t \cdot \left(\omega_{1,t} - \frac{q_t \delta}{p_{1,t} (I-1)} e_1 \right) + q_t \left(\kappa_{1,t} + \frac{\delta}{I-1} \right) \\ &= p_t \cdot \omega_{1,t} + q_t \kappa_{1,t} \end{aligned}$$

Property (ii) follows by definition if $\left(x^*_{i,t}, y^*_{i,t} \right)_{i \in \mathcal{I}} = \left(\omega_{i,t}, \kappa_{i,t} \right)_{i \in \mathcal{I}}$. Else, since $\delta > 0$,

$$\begin{aligned} x^*_{1,t} &= \omega_{1,t} + \frac{q_t \delta}{p_{1,t}} e_1 \\ &> \omega_{1,t} \\ &\gg 0 \end{aligned}$$

whereas for each $i \in \mathcal{I} \setminus \{1\}$,

$$\begin{aligned} x^*_{1,i,t} &= \omega_{1,i,t} - \frac{q_t \delta}{p_{1,t}(I-1)} \\ &\geqslant \omega_{1,i,t} - \frac{q_t \gamma}{p_{1,t}(I-1)} \\ &\geqslant \omega_{1,i,t} - \frac{q_t}{p_{1,t}(I-1)} \left(\frac{(I-1) p_{1,t} \omega_{1,i,t}}{q_t} \right) \\ &= 0 \end{aligned}$$

and for each $l \in \{2, ..., L\}$, $x^*_{l,i,t} = \omega_{l,i,t} > 0$.

Also,

$$\begin{aligned} y^*_{1,t} &= \kappa_{1,t} - \delta \\ &\geqslant \kappa_{1,t} - \kappa_{1,t} \\ &= 0 \end{aligned}$$

and for each $i \in \mathcal{I} \setminus \{1\}$,

$$\begin{aligned} y^*_{i,t} &= \kappa_{i,t} + \frac{\delta}{I-1} \\ &> \kappa_{i,t} \\ &> 0 \end{aligned}$$

Property (iii) is straightforward if $\left(x^*_{i,t}, y^*_{i,t}\right)_{i \in \mathcal{I}} = \left(\omega_{i,t}, \kappa_{i,t}\right)_{i \in \mathcal{I}}$. Else,

$$\begin{aligned} \sum_{i \in \mathcal{I}} \left(x^*_{i,t}, y^*_{i,t}\right) &= \left(x^*_{1,t}, y^*_{1,t}\right) + \sum_{i \in \mathcal{I} \setminus \{1\}} \left(x^*_{i,t}, y^*_{i,t}\right) \\ &= \left(\omega_{1,t} + \frac{q_t \delta}{p_{1,t}} e_1, \kappa_{1,t} - \delta\right) \\ &\quad + \sum_{i \in \mathcal{I} \setminus \{1\}} \left(\omega_{i,t} - \frac{q_t \delta}{p_{1,t}(I-1)} e_1, \kappa_{i,t} + \frac{\delta}{I-1}\right) \\ &= \sum_{i \in \mathcal{I}} \left(\omega_{1,t}, \kappa_{1,t}\right) \end{aligned}$$

Finally, in order to show property (iv), since $T < \infty$, it suffices to show that if at the t^{th} pass through the algorithm

$$\left(\forall t', t'' \in \{1, ..., t-1\} : t' \neq t''\right) \left(\forall i \in \mathcal{I}\right) : y^*_{i,t'} \neq y^*_{i,t''}$$

then

$$\left(\forall t', t'' \in \{1, ..., t\} : t' \neq t''\right) \left(\forall i \in \mathcal{I}\right) : y^*_{i,t'} \neq y^*_{i,t''}$$

To establish this, it suffices to show that

$$\left(\forall t' \in \{1, ..., t-1\}\right) \left(\forall i \in \mathcal{I}\right) : y^*_{i,t'} \neq y^*_{i,t}$$

This is tautological if $t=1$, and follows from step 2 if $\left(x_{i,t}^*, y_{i,t}^*\right)_{i\in\mathcal{I}} = (\omega_{i,t}, \kappa_{i,t})_{i\in\mathcal{I}}$. Hence, I now consider $t \geqslant 2$ and assume that $\left(x_{i,t}^*, y_{i,t}^*\right)_{i\in\mathcal{I}}$ is given by steps 3, 4 and 5. I consider three different cases:

Case 1: $t=2$ and $\mathcal{J}=\varnothing$. Then, for each $i\in\mathcal{I}$, $\kappa_{i,2}=y_{i,1}^*$. Since $\delta>0$, it is straightforward that for each $i\in\mathcal{I}$, $y_{i,2}^* \neq \kappa_{i,2} = y_{i,1}^*$.

Case 2: $t=2$ and $\mathcal{J}\neq\varnothing$. Then, if $1\notin\mathcal{J}$, one has that $y_{1,1}^*=\kappa_{1,2}$, and, since $\delta>0$, it follows that

$$\begin{aligned} y_{1,2}^* &= \kappa_{1,2}-\delta \\ &= y_{1,1}^*-\delta \\ &\neq y_{1,1}^* \end{aligned}$$

Else, if $1\in\mathcal{J}$, one has that if $y_{1,1}^*=y_{1,2}^*$, then, since $y_{1,2}^*=\kappa_{1,2}-\delta$, it would follow that

$$\begin{aligned} \left|y_{1,1}^*-\kappa_{1,2}\right| &= \delta \\ &\leqslant \frac{\varepsilon}{2} \\ &< \varepsilon \\ &\leqslant \left|y_{1,1}^*-\kappa_{1,2}\right| \end{aligned}$$

an obvious contradiction.

On the other hand, for each $i\in\mathcal{I}\backslash(\mathcal{J}\cup\{1\})$, it is true that $y_{i,1}^*=\kappa_{i,2}$. Since $\delta>0$, it follows that

$$\begin{aligned} y_{i,2}^* &= \kappa_{i,2}+\frac{\delta}{I-1} \\ &= y_{i,1}^*+\frac{\delta}{I-1} \\ &\neq y_{i,1}^* \end{aligned}$$

whereas for each $i\in\mathcal{J}\backslash\{1\}$, if one had that $y_{i,1}^*=y_{i,2}^*$, then, since $y_{i,2}^*=\kappa_{i,2}+\frac{\delta}{I-1}$ and $\delta>0$, it would be true that

$$\begin{aligned} \left|y_{i,1}^*-\kappa_{i,2}\right| &= \frac{\delta}{I-1} \\ &\leqslant \delta \\ &\leqslant \frac{\varepsilon}{2} \\ &< \varepsilon \\ &\leqslant \left|y_{i,1}^*-\kappa_{i,2}\right| \end{aligned}$$

again, a contradiction.

Case 3: $t\geqslant 3$. In this case, by the induction assumption and definition, $\mathcal{J}=\mathcal{I}$, from where, if

$$(\exists t'\in\{1,...,t-1\}) : y_{1,t'}^*=y_{1,t}^*$$

then $y^*_{1,t} = \kappa_{1,t} - \delta$ and $\delta > 0$ would imply that $y^*_{1,t'} \neq \kappa_{1,t}$ and

$$\begin{aligned} \left|y^*_{1,t'} - \kappa_{1,t}\right| &= \delta \\ &\leqslant \frac{\varepsilon}{2} \\ &< \varepsilon \\ &\leqslant \min_{t'' \in \{1,...,t-1\}: y^*_{1,t''} \neq \kappa_{1,t}} \left\{\left|y^*_{1,t''} - \kappa_{1,t}\right|\right\} \end{aligned}$$

an obvious contradiction. Finally, if for some $i \in \mathcal{I} \setminus \{1\}$, one had that

$$(\exists t' \in \{1, ..., t-1\}) : y^*_{i,t'} = y^*_{i,t}$$

then, $y^*_{i,t} = \kappa_{i,t} + \frac{\delta}{I-1}$ and $\delta > 0$ would imply that $y^*_{i,t'} \neq \kappa_{i,t}$ and

$$\begin{aligned} \left|y^*_{i,t'} - \kappa_{i,t}\right| &= \frac{\delta}{I-1} \\ &\leqslant \delta \\ &\leqslant \frac{\varepsilon}{2} \\ &< \varepsilon \\ &\leqslant \min_{t'' \in \{1,...,t-1\}: y^*_{i,t''} \neq \kappa_{i,t}} \left\{\left|y^*_{i,t''} - \kappa_{i,t}\right|\right\} \end{aligned}$$

again, a contradiction.

Now, taking $\left((y^*_{i,t})_{i \in \mathcal{I}}\right)_{t=1}^{T}$ from the output of the algorithm, since, by property (iii),

$$(\forall t \in \mathcal{T}) : \sum_{i \in \mathcal{I}} y^*_{i,t} = \sum_{i \in \mathcal{I}} \kappa_{i,t}$$

and by properties (i) and (ii)

$$\begin{aligned} (\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) &: y^*_{i,t} \in \mathbb{R}_+ \\ (\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) &: p_t \cdot \omega_{i,t} + q_t \left(\kappa_{i,t} - y^*_{i,t}\right) \geqslant 0 \end{aligned}$$

it follows by definition that

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i \in \mathcal{I}}\right)\right)_{t=1}^{T}$$

is a data set with partial observability.

Moreover, taking $\left(\left(x^*_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^{T}$ from the output, it follows from property (ii) that

$$(\forall t \in \mathcal{T})\,(\forall i \in \mathcal{I}) : x^*_{i,t} \in \mathbb{R}^L_+$$

whereas, defining

$$\begin{aligned} (\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) &: V^i_t = 1 \\ (\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) &: v^i_t = p_t \\ (\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) &: \rho^i_t = q_t \\ (\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) &: \lambda^*_{i,t} = 1 \end{aligned}$$

it is obvious that

$$(\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) : V_t^i \in \mathbb{R},\ v_t^i \in \mathbb{R}_{++}^L,\ \rho_t^i \in \mathbb{R} \text{ and } \lambda_{i,t}^* \in \mathbb{R}_{++}$$

and that

(a) $(\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) : v_t^i = \lambda_{i,t}^* p_t$

(b) $(\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) : \rho_t^i = \lambda_{i,t}^* q_t$

From property (i) of the output,

(c) $(\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) : p_t \cdot x_{i,t}^* = p_t \cdot \omega_{i,t} + q_t \left(\kappa_{i,t} - y_{i,t}^*\right)$

And from property (iv) of the output, since, by construction,

$$(\forall i \in \mathcal{I})\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) : \#\mathcal{T}^i\left(y_{-i}\right) \leqslant 1$$

it is clear that

(d) $(\forall i \in \mathcal{I})\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right)\left(\forall t, t' \in \mathcal{T}^i\left(y_{-i}\right)\right) :$

$$V_{t'}^i \leqslant V_t^i + v_t^i \cdot \left(x_{i,t'}^* - x_{i,t}^*\right) + \rho_t^i\left(y_{i,t'}^* - y_{i,t}^*\right)$$

with strict inequality whenever

$$\left(x_{i,t'}^*, y_{i,t'}^*\right) \neq \left(x_{i,t}^*, y_{i,t}^*\right)$$

Finally, from property (iii) of the output,

(e) $(\forall t \in \mathcal{T}) : \sum_{i \in \mathcal{I}} x_{i,t}^* = \sum_{i \in \mathcal{I}} \omega_{i,t}$

Then, by theorem 5, (a) to (e) imply that

$$\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y_{i,t}^*\right)_{i \in \mathcal{I}}\right)_{t=1}^T$$

is NW-rationalizable, and, therefore, that

$$\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}\right)_{i \in \mathcal{I}}\right)_{t=1}^T$$

is NW-rationalizable. ■

Hence, even under observation of prices of the externality, strictly positive endowments give enough room to construct individual demands for the externality such that no individual has to be assumed to maximize the same utility function at two different observations, which destroys all possibilities of testable restrictions, as these are derived from revealed preference theory.

4.5 Weak separability:

The results of the previous two subsections, in particular the ones of 4.4, imply that, in general, externalities are able to destroy all the restrictions existing on the equilibrium manifold of exchange economies, specifically those found by Brown and Matzkin (1996). In many cases, this result may appear extreme in the sense that the researcher may be of the opinion that the externalities, although a reality, are not of great relevance, either because the aggregate endowment of the externality is relatively "small" or because the effects that the externality has, both in the individual who is consuming it and on others are somewhat "insignificant".

I now show that this view is correct in the sense that if one assumes that the consumption of the externality, both by the individual and by the rest of consumers, has no effect on the ordinal relations between the rest of the commodities, there do exist testable restrictions, with and without observation of the profiles of consumption of the externality. In other words, I now show that if one has reasons to impose, besides the hypothesis of Nash Equilibrium, the one that all consumers have preferences that are separable in the consumption of the private commodities, then there do exist testable restrictions. Specifically:

Definition 9 *A function* $U : \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}, (x, y_i, y_{-i}) \longmapsto U(x, y_i, y_{-i})$ *is said to be weakly separable (in x) if there exist* $u : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}$*, which is continuous and monotonically increasing in its first argument, and* $V : \mathbb{R}_+^L \longrightarrow \mathbb{R}$*, which is continuous, such that*

$$\left(\forall (x, y_i, y_{-i}) \in \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1}\right) : U(x, y_i, y_{-i}) = u(V(x), y_i, y_{-i}).$$

As before, for simplicity it turns out to be convenient to consider first the case in which the profiles of consumption of the externality are assumed to be observed.

4.5.1 Partial observability:

In analogy to definition 6, under the separability assumption the hypothesis of Nash-Walras behavior is:

Definition 10 *A data set with partial observability*

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y_{i,t}^*\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^T$$

is said to be Nash-Walras-rationalizable with weak separability if for each $i \in \mathcal{I}$ *there exists* $U^i : \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}$*, continuous and weakly separable, satisfying that*

$$\begin{array}{lcll}\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) & : & U^i(\cdot, \cdot, y_{-i}) & \textit{is concave} \\ \left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) & : & U^i(\cdot, \cdot, y_{-i}) & \textit{is strictly monotone}\end{array}$$

such that

$$(\forall t \in \mathcal{T}) : \left(p_t, q_t, \left(y_{i,t}^*\right)_{i\in\mathcal{I}}\right) \in NWPE\left(\left(\mathcal{I}, \left(U^i, w_{i,t}, \kappa_{i,t}\right)_{i\in\mathcal{I}}\right)\right)$$

In this case, for technical reasons, the characterization via equilibrium inequalities, analogous to theorem 5, is given by the following two results, which are partial converse of one another. Both theorems are based on Varian (1983).

Theorem 8 *Suppose that a data set with partial observability*

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

is Nash-Walras-rationalized with weak separability by $\left(U^i\right)_{i\in\mathcal{I}}$ *and for each* $i \in \mathcal{I}$ *there exist* $u^i : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}$*, which is continuous and monotonically increasing in its first argument, and* $V^i : \mathbb{R}_+^L \longrightarrow \mathbb{R}$*, which is continuous, such that:*

(i) $\left(\forall \left(x, y_i, y_{-i}\right) \in \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1}\right) : U^i\left(x, y_i, y_{-i}\right) = u^i\left(V^i\left(x\right), y_i, y_{-i}\right);$

(ii) $\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) : U^i\left(\cdot, \cdot, y_{-i}\right)$ *is differentiable;*

(iii) $\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) : u^i\left(\cdot, \cdot, y_{-i}\right)$ *is differentiable and concave;*

(iv) V^i *is differentiable and concave;*

(v) $\left(\forall \left(x, y_i\right) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}\right)\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) :$

$$\left\{\left(x', y_i'\right) \in \mathbb{R}_+^L \times \mathbb{R}_+ \middle| U^i\left(x', y_i', y_{-i}\right) = U^i\left(x, y_i, y_{-i}\right)\right\} \subseteq \mathbb{R}_{++}^L \times \mathbb{R}_{++}$$

then for each $i \in \mathcal{I}$ *and each* $t \in \mathcal{T}$ *there exist* $x^*_{i,t} \in \mathbb{R}_{++}^L$*,* $V_t^i \in \mathbb{R}$*,* $U_t^i \in \mathbb{R}$*,* $v_t^i \in \mathbb{R}_{++}^L$*,* $\rho_t^i \in \mathbb{R}_{++}$*,* $\lambda^*_{i,t} \in \mathbb{R}_{++}$ *and* $\mu^*_{i,t} \in \mathbb{R}_{++}$ *such that:*

1. $\left(\forall i \in \mathcal{I}\right)\left(\forall t \in \mathcal{T}\right) : v_t^i = \lambda^*_{i,t} p_t$

2. $\left(\forall i \in \mathcal{I}\right)\left(\forall t \in \mathcal{T}\right) : \rho_t^i = \mu^*_{i,t} q_t$

3. $\left(\forall i \in \mathcal{I}\right)\left(\forall t \in \mathcal{T}\right) : p_t \cdot x^*_{i,t} = p_t \cdot \omega_{i,t} + q_t\left(\kappa_{i,t} - y^*_{i,t}\right)$

4. $\left(\forall i \in \mathcal{I}\right)\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right)\left(\forall t, t' \in \mathcal{T}^i\left(y_{-i}\right)\right) :$

$$U_{t'}^i \leqslant U_t^i + \rho_t^i\left(y^*_{i,t'} - y^*_{i,t}\right) + \frac{\mu^*_{i,t}}{\lambda^*_{i,t}}\left(V_{t'}^i - V_t^i\right)$$

5. $\left(\forall i \in \mathcal{I}\right)\left(\forall t, t' \in \mathcal{T}\right) :$

$$V_{t'}^i \leqslant V_t^i + v_t^i \cdot \left(x^*_{i,t'} - x^*_{i,t}\right)$$

6. $\left(\forall t \in \mathcal{T}\right) : \sum_{i\in\mathcal{I}} x^*_{i,t} = \sum_{i\in\mathcal{I}} \omega_{i,t}$

Proof. Since $\forall t \in \mathcal{T}$,

$$\left(p_t, q_t, \left(y^*_{i,t}\right)_{i\in\mathcal{I}}\right) \in NWPE\left(\left(\mathcal{I}, \left(U^i, w_{i,t}, \kappa_{i,t}\right)_{i\in\mathcal{I}}\right)\right)$$

then, by definition, there exists $\left(x^*_{i,t}\right)_{i\in\mathcal{I}} \in \left(\mathbb{R}_+^L\right)^I$ such that

$$\left(p_t, q_t, \left(x^*_{i,t}, y^*_{i,t}\right)_{i\in\mathcal{I}}\right) \in NW\left(\left(\mathcal{I}, \left(U^i, w_{i,t}, \kappa_{i,t}\right)_{i\in\mathcal{I}}\right)\right)$$

Fix one such $\left(x^*_{i,t}\right)_{i\in\mathcal{I}} \in \left(\mathbb{R}^L_+\right)^I$. By definition,

$$\sum_{i\in\mathcal{I}} x^*_{i,t} = \sum_{i\in\mathcal{I}} \omega_{i,t}$$

which is condition (6). Also, since for each $i \in \mathcal{I}$, $U^i\left(\cdot,\cdot,y^*_{-i,t}\right)$ is monotone, we must have that

$$(\forall i \in \mathcal{I}) : p_t \cdot x^*_{i,t} + q_t y^*_{i,t} = p_t \cdot \omega_{i,t} + q_t \kappa_{i,t}$$

from where condition (3) is obvious.

Fix $i \in \mathcal{I}$. Since $\forall t \in \mathcal{T}$,

$$\begin{aligned}\left(x^*_{i,t}, y^*_{i,t}\right) \quad &\in \quad Arg \max_{(x,y_i)\in\mathbb{R}^L_+\times\mathbb{R}_+} U^i\left(x, y_i, y^*_{-i,t}\right) \\ &\qquad \text{s.t.} \quad p_t \cdot x + q_t y_i \leqslant p_t \cdot \omega_{i,t} + q_t \kappa_{i,t}\end{aligned}$$

we have by (v), since $p_t \cdot \omega_{i,t} + q_t\kappa_{i,t} > 0$, that[23]

$$\left(x^*_{i,t}, y^*_{i,t}\right) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$$

and also, from (i), that $\forall t \in \mathcal{T}$,

$$\begin{aligned}x^*_{i,t} \quad &\in \quad Arg \max_{x\in\mathbb{R}^L_+} V^i(x) \\ &\qquad \text{s.t.} \quad p_t \cdot x \leqslant p_t \cdot \omega_{i,t} + q_t\left(\kappa_{i,t} - y^*_{i,t}\right)\end{aligned}$$

Fix $t \in \mathcal{T}$. From (ii) and (iv) and the Kuhn-Tucker theorem[24] it follows that $\exists v^i_t \in \mathbb{R}^L_{++}$, $\rho^i_t \in \mathbb{R}_{++}$, $\lambda^*_{i,t} \in \mathbb{R}_{++}$ and $\mu^*_{i,t} \in \mathbb{R}_{++}$ such that:

- $v^i_t = \lambda^*_{i,t} p_t$
- $\rho^i_t = \mu^*_{i,t} q_t$
- $\left(\forall (x, y_i) \in \mathbb{R}^L_+ \times \mathbb{R}_+\right)$:

  $$U^i\left(x, y_i, y^*_{-i,t}\right) \leqslant U^i\left(x^*_{i,t}, y^*_{i,t}, y^*_{-i,t}\right) + \rho^i_t\left(y_i - y^*_{i,t}\right) + \mu^*_{i,t} p_t \cdot \left(x - x^*_{i,t}\right)$$

---

[23] One can obtain the same implication with a weaker assumption:

$$\left(\forall \{(x_n, y_{i,n})\}^\infty_{n=1} \overset{\text{seq}}{\subseteq} \mathbb{R}^L_{++} \times \mathbb{R}_{++}\right)\left(\forall (\overline{x}, \overline{y}_i) \in \left(\mathbb{R}^L_+ \times \mathbb{R}_+\right) \setminus \left(\mathbb{R}^L_{++} \times \mathbb{R}_{++}\right)\right)\left(\forall y_{-i} \in \mathbb{R}^{I-1}_+\right)$$

$$\lim_{n\longrightarrow\infty} (x_n, y_{i,n}) = (\overline{x}, \overline{y}_i) \Longrightarrow \lim_{n\longrightarrow\infty} \frac{D_{x,y_i} U^i(x_n, y_{i,n}, y_{-i})}{\|D_{x,y_i} U^i(x_n, y_{i,n}, y_{-i})\|} = \left((0)^L_{l=1}, 0\right)$$

where $D_{x,y_i}U^i$ represents the gradient of U^i with respect to x and y_i and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{L+1}$.

[24] The following step one could give, without assuming differentiability and just imposing Lipschitz continuity, via theorem 4 in appendix 6. However, a further step below will require the use a chain rule and I am unaware of one that applies for subdifferential calculus in the direction needed here.

- $\left(\forall x \in \mathbb{R}_{+}^{L}\right)$:
$$V^{i}\left(x\right) \leqslant V^{i}\left(x_{i,t}^{*}\right) + v_{t}^{i} \cdot \left(x - x_{i,t}^{*}\right)$$

Conditions (1) and (2) are hence satisfied.

Now, by (iii) and the chain rule,

$$\begin{aligned}
\mu_{i,t}^{*} p_{t} &= \frac{\partial u^{i}}{\partial V}\left(V\left(x_{i,t}^{*}\right), y_{i,t}^{*}, y_{-i,t}^{*}\right) v_{t}^{i} \\
&= \frac{\partial u^{i}}{\partial V}\left(V\left(x_{i,t}^{*}\right), y_{i,t}^{*}, y_{-i,t}^{*}\right) \lambda_{i,t}^{*} p_{t}
\end{aligned}$$

from where
$$\frac{\partial u^{i}}{\partial V}\left(V\left(x_{i,t}^{*}\right), y_{i,t}^{*}, y_{-i,t}^{*}\right) = \frac{\mu_{i,t}^{*}}{\lambda_{i,t}^{*}}$$

and hence, by (iii) again (concavity of $u^{i}\left(\cdot, \cdot, y_{-i,t}^{*}\right)$),

$$\begin{aligned}
\left(\forall \left(V, y_{i}\right) \in \mathbb{R}^{L} \times \mathbb{R}_{+}\right) &: \\
u^{i}\left(V, y_{i}, y_{-i,t}^{*}\right) &\leqslant u^{i}\left(V\left(x_{i,t}^{*}\right), y_{i,t}^{*}, y_{-i,t}^{*}\right) + \rho_{t}^{i}\left(y_{i} - y_{i,t}^{*}\right) \\
&\quad + \frac{\mu_{i,t}^{*}}{\lambda_{i,t}^{*}}\left(V - V\left(x_{i,t}^{*}\right)\right)
\end{aligned}$$

Define $\forall t' \in \mathcal{T}$,

$$\begin{aligned}
U_{t'}^{i} &= u^{i}\left(V\left(x_{i,t'}^{*}\right), y_{i,t'}^{*}, y_{-i,t'}^{*}\right) \\
V_{t'}^{i} &= V\left(x_{i,t'}^{*}\right)
\end{aligned}$$

and let $t' \in \mathcal{T}$ be such that $y_{-i,t'}^{*} = y_{-i,t}^{*}$. By construction,

$$\begin{aligned}
U_{t'}^{i} &= u^{i}\left(V\left(x_{i,t'}^{*}\right), y_{i,t'}^{*}, y_{-i,t'}^{*}\right) \\
&= u^{i}\left(V\left(x_{i,t'}^{*}\right), y_{i,t'}^{*}, y_{-i,t}^{*}\right) \\
&\leqslant u^{i}\left(V\left(x_{i,t}^{*}\right), y_{i,t}^{*}, y_{-i,t}^{*}\right) + \rho_{t}^{i}\left(y_{i,t'} - y_{i,t}^{*}\right) + \frac{\mu_{i,t}^{*}}{\lambda_{i,t}^{*}}\left(V\left(x_{i,t'}^{*}\right) - V\left(x_{i,t}^{*}\right)\right) \\
&= U_{t}^{i} + \rho_{t}^{i}\left(y_{i,t'}^{*} - y_{i,t}^{*}\right) + \frac{\mu_{i,t}^{*}}{\lambda_{i,t}^{*}}\left(V_{t'}^{i} - V_{t}^{i}\right)
\end{aligned}$$

which is condition (4).

Also, letting $t' \in \mathcal{T}$, it follows that

$$\begin{aligned}
V_{t'}^{i} &= V\left(x_{i,t'}^{*}\right) \\
&\leqslant V^{i}\left(x_{i,t}^{*}\right) + v_{t}^{i} \cdot \left(x_{i,t'}^{*} - x_{i,t}^{*}\right) \\
&\leqslant V_{t}^{i} + v_{t}^{i} \cdot \left(x_{i,t'}^{*} - x_{i,t}^{*}\right)
\end{aligned}$$

which is condition (5). ■

The partial converse is:

Theorem 9 *Given a data set with partial observability*

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

if for each $i \in \mathcal{I}$ and each $t \in \mathcal{T}$ there exist $x^*_{i,t} \in \mathbb{R}^L_+$, $V^i_t \in \mathbb{R}$, $U^i_t \in \mathbb{R}$, $v^i_t \in \mathbb{R}^L_{++}$, $\rho^i_t \in \mathbb{R}_{++}$, $\lambda^*_{i,t} \in \mathbb{R}_{++}$ and $\mu^*_{i,t} \in \mathbb{R}_{++}$ such that:

1. $(\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) : v^i_t = \lambda^*_{i,t} p_t$
2. $(\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) : \rho^i_t = \mu^*_{i,t} q_t$
3. $(\forall i \in \mathcal{I})\,(\forall t \in \mathcal{T}) : p_t \cdot x^*_{i,t} = p_t \cdot \omega_{i,t} + q_t\left(\kappa_{i,t} - y^*_{i,t}\right)$
4. $(\forall i \in \mathcal{I})\left(\forall y_{-i} \in \mathbb{R}^{I-1}_+\right)\left(\forall t, t' \in \mathcal{T}^i\left(y_{-i}\right)\right) :$

$$U^i_{t'} \leqslant U^i_t + \rho^i_t\left(y^*_{i,t'} - y^*_{i,t}\right) + \frac{\mu^*_{i,t}}{\lambda^*_{i,t}}\left(V^i_{t'} - V^i_t\right)$$

5. $(\forall i \in \mathcal{I})\left(\forall t, t' \in \mathcal{T}\right) :$

$$V^i_{t'} \leqslant V^i_t + v^i_t \cdot \left(x^*_{i,t'} - x^*_{i,t}\right)$$

6. $(\forall t \in \mathcal{T}) : \sum_{i\in\mathcal{I}} x^*_{i,t} = \sum_{i\in\mathcal{I}} \omega_{i,t}$

then

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

is Nash-Walras-rationalizable with weak separability.

Proof. Fix $i \in \mathcal{I}$ and define $C^i = \left\{y^*_{-i,t}\right\}_{t\in\mathcal{T}}$. Clearly, $C^i \subseteq \mathbb{R}^{I-1}_+$ is compact. Define the function $w^i : \mathbb{R}_+ \times \mathbb{R}_+ \times C^i \longrightarrow \mathbb{R}$ by:

$$\left(\forall\left(V, y_i, y^*_{-i,t'}\right) \in \mathbb{R}_+ \times \mathbb{R}_+ \times C^i\right) :$$

$$w^i\left(V, y_i, y^*_{-i,t'}\right) = \min_{t\in\mathcal{T}: y^*_{-i,t} = y^*_{-i,t'}} \left\{U^i_t + \rho^i_t\left(y_i - y^*_{i,t}\right) + \frac{\mu^*_{i,t}}{\lambda^*_{i,t}}\left(V - V^i_t\right)\right\}$$

This function satisfies that $\forall y^*_{-i,t'} \in C^i$, $w^i\left(\cdot, \cdot, y^*_{-i,t'}\right)$ is continuous, strictly monotone and concave. Moreover, w^i is continuous, as C^i contains no limit points (see the proof of theorem 5). Also, by monotonicity, $\forall\left(V, y_i, y^*_{-i,t'}\right) \in \mathbb{R}_+ \times \mathbb{R}_+ \times C^i$,

$$w^i\left(V, y_i, y^*_{-i,t'}\right) \geqslant \underline{w}^i = \min_{y^*_{-i,t'} \in C^i} \left\{w^i\left(0, 0, y^*_{-i,t'}\right)\right\} \in \mathbb{R}$$

Define the truncated logistic function $\ell^i : \left[\underline{w}^i, \infty\right) \longrightarrow [1, 2)$ by

$$\left(\forall w \in \left[\underline{w}^i, \infty\right)\right) : \ell^i(w) = \frac{2}{1 + \exp\left(\underline{w}^i - w\right)}$$

which is continuous, strictly increasing and strongly concave. Let $W^i = \ell^i \circ w^i : \mathbb{R}_+ \times \mathbb{R}_+ \times C^i \longrightarrow [1, 2)$. By construction, W^i is bounded, continuous and $\forall \overset{*}{y}_{-i,t'} \in C^i$, $W^i\left(\cdot, \cdot, \overset{*}{y}_{-i,t'}\right)$ is strictly monotone and concave. Since C^i is compact, it follows from corollary 1 in Carvajal (2004c) that there exists $u^i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}$ such that:

- u^i is continuous;
- $\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) : u^i(\cdot, \cdot, y_{-i})$ is concave;
- $\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) : u^i(\cdot, \cdot, y_{-i})$ is strictly monotone;
- $\left(\forall \left(V, y_i, \overset{*}{y}_{-i,t'}\right) \in \mathbb{R}_+ \times \mathbb{R}_+ \times C^i\right) : u^i\left(V, y_i, \overset{*}{y}_{-i,t'}\right) = W^i\left(V, y_i, \overset{*}{y}_{-i,t'}\right)$.

Without loss of generality, assume that $\forall i \in \mathcal{I}$ and $\forall t \in \mathcal{T}$,

$$V_t^i \geqslant v_t^i x_{i,t}^*$$

Define also $V^i : \mathbb{R}_+^L \longrightarrow \mathbb{R}_+$ by

$$\left(\forall x \in \mathbb{R}_+^L\right) : V^i(x) = \min_{t \in \mathcal{T}} \left\{V_t^i + v_t^i \cdot \left(x - x_{i,t}^*\right)\right\}$$

Clearly, V^i is continuous, concave and strictly monotone. Now, define the function $U^i : \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}$ by:

$$\left(\forall (x, y_i, y_{-i}) \in \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1}\right) : U^i(x, y_i, y_{-i}) = u^i\left(V^i(x), y_i, y_{-i}\right)$$

which is continuous, weakly separable and satisfies that

$$\begin{array}{lcll} \left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) & : & U^i(\cdot, \cdot, y_{-i}) & \text{is concave} \\ \left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) & : & U^i(\cdot, \cdot, y_{-i}) & \text{is strictly monotone} \end{array}$$

Now, let $t \in \mathcal{T}$. By condition (5),

$$V^i\left(x_{i,t}^*\right) = V_t^i$$

and then, by condition (4),

$$w^i\left(V^i\left(x_{i,t}^*\right), y_{i,t}^*, \overset{*}{y}_{-i,t}\right) = U_t^i$$

from where

$$\begin{array}{rcl} U^i\left(V^i\left(x_{i,t}^*\right), y_{i,t}^*, \overset{*}{y}_{-i,t}\right) & = & W^i\left(V^i\left(x_{i,t}^*\right), y_{i,t}^*, \overset{*}{y}_{-i,t}\right) \\ & = & \ell^i\left(U_t^i\right) \end{array}$$

Now, suppose that $(x, y_i) \in \mathbb{R}_+^L \times \mathbb{R}_+$ is such that

$$\begin{aligned} p_t \cdot x + q_t y_i &\leqslant p_t \cdot \omega_{i,t} + q_t \kappa_{i,t} \\ &= p_t \cdot x_{i,t}^* + q_t y_{i,t}^* \end{aligned}$$

where the equality follows from condition (3). Then, using conditions (1) and (2), one has that

$$\begin{aligned} U^i\left(x, y_i, \overset{*}{y}_{-i,t}\right) &= u^i\left(V^i(x), y_i, \overset{*}{y}_{-i,t}\right) \\ &= W^i\left(V^i(x), y_i, \overset{*}{y}_{-i,t}\right) \\ &= \ell^i\left(w^i\left(V^i(x), y_i, \overset{*}{y}_{-i,t}\right)\right) \\ &= \ell^i\left(w^i\left(\min_{t' \in \mathcal{T}}\left\{V_{t'}^i + v_{t'}^i \cdot \left(x - x_{i,t'}^*\right)\right\}, y_i, \overset{*}{y}_{-i,t}\right)\right) \\ &= \ell^i\left(\min_{t'' \in \mathcal{T}: y_{-i,t''}^* = y_{-i,t}^*}\left\{U_{t''}^i + \rho_{t''}^i\left(y_i - y_{i,t''}^*\right)\right.\right. \\ &\quad \left.\left. + \frac{\mu_{i,t''}^*}{\lambda_{i,t''}^*}\left(\min_{t' \in \mathcal{T}}\left\{V_{t'}^i + v_{t'}^i \cdot \left(x - x_{i,t'}^*\right)\right\} - V_{t''}^i\right)\right\}\right) \\ &\leqslant \ell^i\left(\min_{t'' \in \mathcal{T}: y_{-i,t''}^* = y_{-i,t}^*}\left\{U_{t''}^i + \rho_{t''}^i\left(y_i - y_{i,t''}^*\right)\right.\right. \\ &\quad \left.\left. + \frac{\mu_{i,t''}^*}{\lambda_{i,t''}^*}\left(V_t^i + v_t^i \cdot \left(x - x_{i,t}^*\right) - V_{t''}^i\right)\right\}\right) \\ &\leqslant \ell^i\left(U_t^i + \rho_t^i\left(y_i - y_{i,t}^*\right) + \frac{\mu_{i,t}^*}{\lambda_{i,t}^*} v_t^i \cdot \left(x - x_{i,t}^*\right)\right) \\ &= \ell^i\left(U_t^i + \mu_{i,t}^* q_t\left(y_i - y_{i,t}^*\right) + \frac{\mu_{i,t}^*}{\lambda_{i,t}^*} \lambda_{i,t}^* p_t \cdot \left(x - x_{i,t}^*\right)\right) \\ &= \ell^i\left(U_t^i + \mu_{i,t}^*\left(\left(q_t y_i + p_t \cdot x\right) - \left(q_t y_{i,t}^* + p_t x_{i,t}^*\right)\right)\right) \\ &\leqslant \ell^i\left(U_t^i\right) \\ &= U^i\left(V^i\left(x_{i,t}^*\right), y_{i,t}^*, \overset{*}{y}_{-i,t}\right) \end{aligned}$$

where the inequality in the sixth line follows because ℓ^i is strictly monotone, $t \in \mathcal{T}$ and $\left(\forall t'' \in \mathcal{T} : y_{-i,t''}^* = y_{-i,t}^*\right) : \mu_{i,t''}^*, \lambda_{i,t''}^* \in \mathbb{R}_{++}$; the inequality in the seventh line follows because ℓ^i is strictly monotone, $t \in \mathcal{T}$ and $y_{-i,t}^* = y_{-i,t}^*$, and the last inequality follows since, by construction,

$$p_t \cdot x + q_t y_i \leqslant p_t \cdot x_{i,t}^* + q_t y_{i,t}^*$$

Since, by definition, $\forall t \in \mathcal{T}$, $\sum_{i \in \mathcal{I}} y_{i,t}^* = \sum_{i \in \mathcal{I}} \kappa_{i,t}$ and, by condition 6, $\sum_{i \in \mathcal{I}} x_{i,t}^* = \sum_{i \in \mathcal{I}} \omega_{i,t}$, it follows that $\left(U^i\right)_{i \in \mathcal{I}}$ NW-rationalizes the data set with weak separability. ■

Notice that the conditions that arise in both theorems are identical. Two features of these conditions deserve to be highlighted. The first one is that these is no longer a set of polynomial inequalities, given that condition (4) involves a ratio between variables. The implication of this fact is that one can no longer argue that the set of values of the variables that satisfy the conditions, or its projections, are semialgebraic. Put in other words, one cannot use Tarski-Seidenberg quantifier elimination (appendix 7) to get rid of the quantified variables.

The other feature that is important is that although condition (4) is still a conditional inequality (i.e. it has to be satisfied for pairs of observations where the rest of players keep their actions constant), this is not the case for condition (5). The characterization involves a set of Afriat inequalities that must hold for every pair of observations: these are not "zero-measure" restrictions, and hence allow for a test with positive power, in the sense that randomly generated data sets may be at odds with the null hypothesis with nonzero probability..

4.5.2 No observability:

In this case, the hypothesis that I want to study is given by:

Definition 11 *A data set with no observability*

$$\left(\left(p_t, q_t, (\omega_{i,t}, \kappa_{i,t})_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

is said to be NW-rationalizable with weak separability if for each $i \in \mathcal{I}$ and each $t \in \mathcal{T}$ there exists $y^*_{i,t} \in \mathbb{R}_+$ such that the sequence

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

is a data set with partial observability and is NW-rationalizable with weak separability.

I will argue below that a result analogous to theorem 7 does not hold. Alternatively, the following characterization can be derived:

Corollary 1 *Suppose that a data set with no observability*

$$\left(\left(p_t, q_t, (\omega_{i,t}, \kappa_{i,t})_{i\in\mathcal{I}}\right)\right)_{t=1}^{T}$$

is Nash-Walras-rationalized with weak separability by $\left(U^i\right)_{i\in\mathcal{I}}$ and for each $i \in \mathcal{I}$ there exist $u^i : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1} \longrightarrow \mathbb{R}$, which is continuous and monotonically increasing in its first argument, and $V^i : \mathbb{R}_+^L \longrightarrow \mathbb{R}$, which is continuous, such that:

(i) $\left(\forall\left(x, y_i, y_{-i}\right) \in \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+^{I-1}\right) : U^i\left(x, y_i, y_{-i}\right) = u^i\left(V^i\left(x\right), y_i, y_{-i}\right)$;

(ii) $\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) : U^i\left(\cdot, \cdot, y_{-i}\right)$ *is differentiable;*

(iii) $\left(\forall y_{-i} \in \mathbb{R}_+^{I-1}\right) : u^i\left(\cdot, \cdot, y_{-i}\right)$ *is differentiable and concave;*

(iv) V^i *is differentiable and concave;*
(v) $\left(\forall (x, y_i) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}\right)\left(\forall y_{-i} \in \mathbb{R}^{I-1}_{+}\right)$:

$$\left\{ (x', y_i') \in \mathbb{R}^L_{+} \times \mathbb{R}_{+} \middle| U^i (x', y_i', y_{-i}) = U^i (x, y_i, y_{-i}) \right\} \subseteq \mathbb{R}^L_{++} \times \mathbb{R}_{++}$$

then for each $i \in \mathcal{I}$ *and each* $t \in \mathcal{T}$ *there exist* $x^*_{i,t} \in \mathbb{R}^L_{++}$, $y^*_{i,t} \in \mathbb{R}_{++}$, $V^i_t \in \mathbb{R}$, $U^i_t \in \mathbb{R}$, $v^i_t \in \mathbb{R}^L_{++}$, $\rho^i_t \in \mathbb{R}_{++}$, $\lambda^*_{i,t} \in \mathbb{R}_{++}$ *and* $\mu^*_{i,t} \in \mathbb{R}_{++}$ *such that:*

1. $(\forall i \in \mathcal{I})(\forall t \in \mathcal{T}) : v^i_t = \lambda^*_{i,t} p_t$
2. $(\forall i \in \mathcal{I})(\forall t \in \mathcal{T}) : \rho^i_t = \mu^*_{i,t} q_t$
3. $(\forall i \in \mathcal{I})(\forall t \in \mathcal{T}) : p_t \cdot x^*_{i,t} + q_t y^*_{i,t} = p_t \cdot \omega_{i,t} + q_t \kappa_{i,t}$
4. $(\forall i \in \mathcal{I})\left(\forall y_{-i} \in \mathbb{R}^{I-1}_{+}\right)\left(\forall t, t' \in \mathcal{T}^i (y_{-i})\right)$:
$$U^i_{t'} \leqslant U^i_t + \rho^i_t \left(y^*_{i,t'} - y^*_{i,t}\right) + \frac{\mu^*_{i,t}}{\lambda^*_{i,t}} \left(V^i_{t'} - V^i_t\right)$$
5. $(\forall i \in \mathcal{I})\left(\forall t, t' \in \mathcal{T}\right)$:
$$V^i_{t'} \leqslant V^i_t + v^i_t \cdot \left(x^*_{i,t'} - x^*_{i,t}\right)$$
6. $(\forall t \in \mathcal{T}) : \sum_{i \in \mathcal{I}} x^*_{i,t} = \sum_{i \in \mathcal{I}} \omega_{i,t}$
7. $(\forall t \in \mathcal{T}) : \sum_{i \in \mathcal{I}} y^*_{i,t} = \sum_{i \in \mathcal{I}} \kappa_{i,t}$

Proof. This follows straightforwardly from theorem 8. ■

Its partial converse is:

Corollary 2 *Given a data set with partial observability*

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y^*_{i,t}\right)_{i \in \mathcal{I}}\right)\right)_{t=1}^{T}$$

if for each $i \in \mathcal{I}$ *and each* $t \in \mathcal{T}$ *there exist* $x^*_{i,t} \in \mathbb{R}^L_{+}$, $y^*_{i,t} \in \mathbb{R}_{+}$, $V^i_t \in \mathbb{R}$, $U^i_t \in \mathbb{R}$, $v^i_t \in \mathbb{R}^L_{++}$, $\rho^i_t \in \mathbb{R}_{++}$, $\lambda^*_{i,t} \in \mathbb{R}_{++}$ *and* $\mu^*_{i,t} \in \mathbb{R}_{++}$ *such that:*

1. $(\forall i \in \mathcal{I})(\forall t \in \mathcal{T}) : v^i_t = \lambda^*_{i,t} p_t$
2. $(\forall i \in \mathcal{I})(\forall t \in \mathcal{T}) : \rho^i_t = \mu^*_{i,t} q_t$
3. $(\forall i \in \mathcal{I})(\forall t \in \mathcal{T}) : p_t \cdot x^*_{i,t} + q_t y^*_{i,t} = p_t \cdot \omega_{i,t} + q_t \kappa_{i,t}$
4. $(\forall i \in \mathcal{I})\left(\forall y_{-i} \in \mathbb{R}^{I-1}_{+}\right)\left(\forall t, t' \in \mathcal{T}^i (y_{-i})\right)$:
$$U^i_{t'} \leqslant U^i_t + \rho^i_t \left(y^*_{i,t'} - y^*_{i,t}\right) + \frac{\mu^*_{i,t}}{\lambda^*_{i,t}} \left(V^i_{t'} - V^i_t\right)$$

5. $(\forall i \in \mathcal{I}) \left(\forall t, t' \in \mathcal{T}\right)$:

$$V_{t'}^i \leqslant V_t^i + v_t^i \cdot \left(x_{i,t'}^* - x_{i,t}^*\right)$$

6. $(\forall t \in \mathcal{T}) : \sum_{i \in \mathcal{I}} x_{i,t}^* = \sum_{i \in \mathcal{I}} \omega_{i,t}$
7. $(\forall t \in \mathcal{T}) : \sum_{i \in \mathcal{I}} y_{i,t}^* = \sum_{i \in \mathcal{I}} \kappa_{i,t}$

   then

$$\left(\left(p_t, q_t, \left(\omega_{i,t}, \kappa_{i,t}, y_{i,t}^*\right)_{i \in \mathcal{I}}\right)\right)_{t=1}^T$$

   is Nash-Walras-rationalizable with weak separability.

Proof. This follows straightforwardly from theorem 9. ■

4.5.3 Nonrationalizable data sets:

Although the results obtained under separability are not as strong as without such assumption, in the sense that the necessary conditions are weaker than the sufficient conditions and that no argument about quantifier elimination has been attempted, I now show, via an example, that the intuition that under this assumption, if aggregate endowments of the externality are small enough, one can refute the hypothesis of Nash-Walras behavior is correct. That is, I now show that under weak separability there are data sets with no observability that cannot be NW-rationalized with weak separability. This, of course, implies that there exist nonrationalizable data sets with partial observability.

Example 3 *Suppose that $I = L = T = 2$. The information of the data set with partial observation is:*

$$\begin{array}{cc}
\omega_{1,1} = (1,4) & \omega_{1,2} = (4,1) \\
\omega_{2,1} = (2,1) & \omega_{2,2} = (1,2) \\
\kappa_{1,1} = 0.01 & \kappa_{1,2} = 0.005 \\
\kappa_{2,1} = 0.01 & \kappa_{2,2} = 0.005 \\
p_1 = (1,10) & p_2 = (10,1) \\
q_1 = 0.1 & q_2 = 0.2
\end{array}$$

Suppose that the data set is NW-rationalized with weak separability by $\left(U^1, U^2\right)$. Let, for each $i \in \{1,2\}$ and each $t \in \{1,2\}$,

$$\begin{aligned}
\left(x_{i,t}^*, y_{i,t}^*\right) \quad \in \quad & Arg \max_{(x,y_i) \in \mathbb{R}_+^L \times \mathbb{R}_+} U^i\left(x, y_i, y_{-i,t}^*\right) \\
& \text{s.t.} \quad p_t \cdot x + q_t y_i \leqslant p_t \cdot \omega_{i,t} + q_t \kappa_{i,t}
\end{aligned}$$

Let U^1 be representable as $U^1(x, y_1, y_2) = u^1\left(V^1(x), y_1, y_2\right)$, where u^1 is monotonically increasing in its first argument (and, therefore, V^1 is strictly monotone). Then, it must be that for each $t \in \{1,2\}$

$$\begin{aligned}
x_{1,t}^* \quad \in \quad & Arg \max_{x \in \mathbb{R}_+^L} V^1(x) \\
& \text{s.t.} \quad p_t \cdot x \leqslant T_{1,t}^x = p_t \cdot \omega_{1,t} + q_t\left(\kappa_{1,t} - y_{1,t}^*\right)
\end{aligned}$$

Since, by aggregate feasibility,

$$(\forall t \in \{1,2\}) : T^x_{1,t} \in [p_t \cdot \omega_{1,t} - q_t \kappa_{2,t}, p_t \cdot \omega_{1,t} + q_t \kappa_{1,t}]$$

it follows that

$$(\forall t \in \{1,2\}) : T^x_{1,t} \in [40.999, 41.001]$$

Also, since $\sum_{i=1}^{2} \omega_{i,1} = (3,5)$ and $\sum_{i=1}^{2} \omega_{i,2} = (5,3)$, feasible values of $x^*_{1,1}$ and $x^*_{1,2}$ can only be, respectively, in

$$\begin{aligned} X_1 &= \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \middle| \, x \geqslant 0, p_1 \cdot x = T^x_{1,1}, x_1 \leq 3 \right\} \\ &= \left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| \, x_1 \in [0,3], x_2 = \frac{T^x_{1,1}}{10} - 0.1 x_1 \right\} \end{aligned}$$

and

$$\begin{aligned} X_2 &= \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \middle| \, x \geqslant 0, p_1 \cdot x = T^x_{1,2}, x_2 \leq 3 \right\} \\ &= \left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| \, x_2 \in [0,3], x_1 = \frac{T^x_{1,2}}{10} - 0.1 x_2 \right\} \\ &\subset \left\{ (x_1, x_2) \in \mathbb{R}^2 \middle| \, x_1 \in [3.7999, 4.2], x_2 = T^x_{1,2} - 10 x_1 \right\} \end{aligned}$$

As before, $X_1 \cap X_2 = \varnothing$ implies that any candidates to $x^*_{1,1}$ and $x^*_{1,2}$ satisfy $x^*_{1,1} \neq x^*_{1,2}$. Since $x^*_{1,1} \in X_1$,[25] then

$$\begin{aligned} p_2 \cdot x^*_{1,1} &= 10 x^*_{1,1,1} + x^*_{2,1,1} \\ &= 10 x^*_{1,1,1} + \frac{T^x_{1,1}}{10} - 0.1 x^*_{1,1,1} \\ &\leqslant 9.9 x^*_{1,1,1} + 4.1001 \\ &\leqslant 9.9\,(3) + 4.1001 \\ &< 40.999 \\ &\leqslant T^x_{1,2} \\ &= p_2 \cdot x^*_{1,2} \end{aligned}$$

whereas since $x^*_{1,2} \in X_2$, then

$$\begin{aligned} p_1 \cdot x^*_{1,2} &= x^*_{1,1,2} + 10 x^*_{2,1,2} \\ &= x^*_{1,1,2} + 10 \left(T^x_{1,2} - 10 x^*_{1,1,2} \right) \\ &\leqslant 410.01 - 99 x^*_{1,1,2} \\ &\leqslant 410.01 - 99\,(3.7999) \\ &< 40.999 \\ &\leqslant T^x_{1,1} \\ &= p_1 \cdot x^*_{1,1} \end{aligned}$$

[25] Once again, for consumption of bundle x, two subindices i, t are taken in the order consumer, observation, whereas three subindices l, i, t are taken in the order commodity, consumer, observation.

which is a violation of SARP and, hence contradicts the fact that for each $t \in \{1,2\}$

$$\begin{aligned} x_{1,t}^{*} \quad \in \quad & Arg \max_{x \in \mathbb{R}_{+}^{L}} V^{1}(x) \\ & \text{s.t.} \quad p_t \cdot x \leqslant T_{1,t}^{x} \end{aligned}$$

(See theorem 2 in Matzkin and Richter, 1991.)

5 Concluding remarks:

This paper considered the question of whether or not the hypothesis of Nash-Walras equilibrium in economies with externalities can be refuted based on finite data sets that do not contain all the information on individual decisions. The answer to this question is for the most part negative. Whether the external effects come form abstract actions or from consumption of commodities, if the data set contains information on prices, all individual constraints and individual choices of the externality, there do exist some extremely mild testable restrictions. These restrictions, however, exhaust all the empirical implications of the theory, since they are not only necessary but also sufficient for the data to be consistent with the equilibrium concept. If a researcher is going to apply tests based on these restrictions, before observing the data, he or she should expect to be unable to refute the hypothesis. If the domains that are defined are non-degenerate and one randomly simulates a data set using nonatomic measures, and , the chances of finding a data set which is at variance with the hypothesis of Nash-Walras equilibrium are nil. This result is similar with the findings of Carvajal (2004a) for the case of games under continuous domains.

Moreover, if there is no information about individual choices, so that only the prices and the individual constraints are observed, then the hypothesis is unfalsifiable: for any feasible data set, there exists a profile of preferences such that the data set arises as Nash-Walras equilibria of the economy with such preferences and endowments. Furthermore, these preferences can be taken to be such that Walras' law is satisfied and individual actions, for given feasible sets and actions of the opponents, are uniquely defined. The result is particularly strong in the case of consumption externalities, in which the price of the externality appears as an observed summary statistic of the individual decisions.

These negative results stand in contrast with the ones obtained by Brown and Matzkin (1996) for standard exchange economies (and by Snyder (1999) for Pareto-efficient provision of public goods). The conceptual reason why the results obtain is that under strategic interaction the preferences of each individual with respect to what he can decide depend on the actions of his opponents. Hence, when opponents change their actions, the objective function that the individual maximizes changes and, unless further assumptions are being imposed, revealed preference theory completely loses its grip.

These are, therefore, general results for which the effects in each individual's utility function of actions of his opponents are allowed a great deal of arbitrariness. If assumptions that restrict these effects are plausible, the restrictions that the theory imposes may strengthen.[26] For this to be the case, however, it is apparent that the assumptions have to either specify how actions by each individual's opponents change the preorder given by him to his own actions (sub- or super-modularity being examples of this case), or imply that, at least over some sets of actions of the opponents that have positive measure, such a preorder does not change.

For example, in the case of consumption externalities, if one can assume that the externality is relatively insignificant in the sense that it does not affect the ordinality of consumption of all other commodities for all individuals, then the hypothesis of Nash-Walras behavior is refutable, even without observation of individual demands for the externality.

6 Appendix: Subdifferential Calculus

The goal of this appendix is to argue that a conclusion similar to the one arising from Kühn-Tucker's necessity theorem can be obtained if the objective function of an optimization problem is smooth enough, even if it is not differentiable.

Let $\Sigma \subseteq \mathbb{R}^M$, with $M \in \mathbb{N}$ and $\Phi \subseteq \mathbb{R}$. Given a function $\pi : \Sigma \longrightarrow \Phi; \sigma \longmapsto \pi(\sigma)$, define for $\sigma^* \in \Sigma$

$$
\begin{aligned}
L^{\pi}(\sigma^*) &= \{\sigma \in \Sigma |\, \pi(\sigma) \leqslant \pi(\sigma^*)\} \\
SL^{\pi}(\sigma^*) &= \{\sigma \in \Sigma |\, \pi(\sigma) < \pi(\sigma^*)\} \\
\partial\pi(\sigma^*) &= \left\{\nu \in \mathbb{R}^M \middle| (\forall \sigma \in \Sigma) : \nu \cdot (\sigma - \sigma^*) \leqslant \pi(\sigma) - \pi(\sigma^*)\right\} \\
\partial^{\leqslant}\pi(\sigma^*) &= \left\{\nu \in \mathbb{R}^M \middle| (\forall \sigma \in L^{\pi}(\sigma^*)) : \nu \cdot (\sigma - \sigma^*) \leqslant \pi(\sigma) - \pi(\sigma^*)\right\} \\
\partial^{<}\pi(\sigma^*) &= \left\{\nu \in \mathbb{R}^M \middle| (\forall \sigma \in SL^{\pi}(\sigma^*)) : \nu \cdot (\sigma - \sigma^*) \leqslant \pi(\sigma) - \pi(\sigma^*)\right\}
\end{aligned}
$$

$L^{\pi}(\sigma^*)$ is the lower contour set of π at (σ^*), $SL^{\pi}(\sigma^*)$ is its strict lower contour set, $\partial\pi(\sigma^*)$ its subdifferential, $\partial^{\leqslant}\pi(\sigma^*)$ its infradifferential (Gutierrez) and $\partial^{<}\pi(\sigma^*)$ its lower subdifferential (Plastria)[27]. The following lemmas are going to be used.

Lemma 1 *(Penot) If there is no local minimizer of π on $\pi^{-1}(\pi(\sigma^*))$, but σ^*, in particular if*

$$\pi(\sigma^*) > \inf_{\Sigma} \pi(\sigma)$$

and if any local minimizer of π is a global minimizer, then

$$\partial^{<}\pi(\sigma^*) = \partial^{\leqslant}\pi(\sigma^*)$$

[26] Of course, under observability of individual decisions for the externality, the restrictions derived here will continue to be imposed, as they will still be necessary conditions. Under further assumptions, however, the sufficiency claims made in the theorems here may no longer be true, in which case the exhaustion of the restrictions of the theory would require stronger conditions.

[27] See Penot (1998).

Proof. This is Proposition 9 in Penot (1998), pages 20-21. ■

Lemma 2 *(Gutierrez) Let π be a convex function. Suppose that π does not attain a minimum at σ^*. Then*

$$\partial^{\leqslant}\pi\left(\sigma^*\right) = \bigcup_{\tau\geqslant 1} \tau\partial\pi\left(\sigma^*\right)$$

Proof. This is Theorem 2 in Gutierrez (1984), page 529. ■

Lemma 3 *(Penot) Let $f : Z \longrightarrow \mathbb{R}$, and for each $k \in \mathcal{K}$ let $g_k : \widehat{Z} \longrightarrow \mathbb{R}$, where $\#\mathcal{K} < \infty$. Suppose that z^* is a solution to the problem*

$$\min f(z)$$

$$s.t. \quad (\forall k \in \mathcal{K}) : g_k(z) \leqslant 0$$

and that f and g_k, for each $k \in \mathcal{K}$, are continuous, quasiconvex, $g_k(z^*) = 0$ for each $k \in \mathcal{K}$, f (resp. g_k) is Lipschitzian on $SL^f(z^*)$ (resp on $g_k^{-1}(\mathbb{R}_-)$). Suppose that there exists $z \in Z$ with $g_k(z) < 0$ for each $k \in \mathcal{K}$. Then, there exists $\alpha \in \partial^{<} f(z^*)$ and for each $k \in \mathcal{K}$ there exist $\beta_k \in \partial^{<} g_k(z^*)$ and $\gamma_k \in \mathbb{R}_+$ such that

$$\alpha + \sum_{k\in\mathcal{K}} \gamma_k \beta_k = 0$$

Proof. This is Corollary 3 in Penot (1998), page 36. ■

Lemma 4 *Let $\overline{\overline{a}} \in \mathbb{R}_+$. Suppose that $V : \mathbb{R}^L_+ \times \left[0, \overline{\overline{a}}\right] \longrightarrow \mathbb{R}$ is continuous, concave (resp. strongly concave) and Lipschitzian, and for each $a \in \left[0, \overline{\overline{a}}\right]$, $V(\cdot, a)$ is strictly monotone. Given $p \in \mathbb{R}^L_{++}$, $m \in \mathbb{R}_{++}$ and $\overline{a} \in \left[0, \overline{\overline{a}}\right]$, if (x^*, a^*) is a solution to the problem*

$$\max V(x, a)$$

$$s.t. \begin{cases} x \geqslant 0 \\ a \in [0, \overline{a}] \\ p \cdot x \leq m \end{cases}$$

then $p \cdot x^* = m$, and there exist $v \in \mathbb{R}^L_+ \setminus \{0\}$, $\rho \in \mathbb{R}$, $\lambda^* \in \mathbb{R}_{++}$, $\varsigma^* \in \mathbb{R}^L_+$, $\mu^* \in \mathbb{R}_+$ and $\eta^* \in \mathbb{R}_+$ such that:

1. *$v = \lambda^* p - \varsigma^*$*
2. *$\rho = \eta^* - \mu^*$*
3. *$\varsigma^* \cdot x^* = 0$, $\mu^* a^* = 0$ and $\eta^*(\overline{a} - a^*) = 0$*
4. *$\left(\forall (x, a) \in \mathbb{R}^L_+ \times \left[0, \overline{\overline{a}}\right]\right)$:*

$$V(x, a) \leqslant V(x^*, a^*) + v \cdot (x - x^*) + \rho(a - a^*)$$

(resp. with strict inequality whenever $(x', a') \neq (x, a)$.)

Proof. That $p \cdot x^* = m$ follows by strict monotonicity.

Define $f : \mathbb{R}^L_+ \times \left[0, \overline{\overline{a}}\right] \longrightarrow \mathbb{R}$ by

$$f(x,a) = -V(x,a)$$

f is continuous, convex, Lipschitzian, and

$$\left(\forall a \in \left[0, \overline{\overline{a}}\right]\right)\left(\forall x \in \mathbb{R}^L_+\right) : x' > x \Longrightarrow f(x',a) < f(x,a)$$

(I will refer to this last property as strict monotonicity.)

Let $\mathcal{J} = \{1, ..., L+3\}$, and define for $j \in \mathcal{J}$, $g_j : \mathbb{R}^L \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g_j(x,a) = \begin{cases} -x_j & \text{if} \quad j \in \{1,...,L\} \\ -a & \text{if} \quad j = L+1 \\ a - \overline{a} & \text{if} \quad j = L+2 \\ p \cdot x - m & \text{if} \quad j = L+3 \end{cases}$$

Then, (x^*, a^*) is a solution to the problem

$$\min f(x,a)$$

$$\text{s.t.} \quad (\forall j \in \mathcal{J}) : g_j(x,a) \leqslant 0$$

Let $\mathcal{K} = \{ j \in \mathcal{J} | \, g_j(x^*, a^*) = 0\}$. By monotonicity, $L + 3 \in \mathcal{K}$. Notice that (x^*, a^*) is a solution to the problem

$$\min f(x,a)$$

$$\text{s.t.} \quad (\forall k \in \mathcal{K}) : g_k(x,a) \leqslant 0$$

for, if not, there would exist $(x', a') \in \mathbb{R}^L \times \mathbb{R}$ such that

$$f(x', a') < f(x^*, a^*)$$

$$(\forall k \in \mathcal{K}) : g_k(x', a') \leqslant 0$$

by construction,

$$(\forall j \in \mathcal{J} \backslash \mathcal{K}) : g_j(x^*, a^*) < 0$$

Now, for each $\theta \in (0,1)$, define $(x_\theta, a_\theta) = \theta(x', a') + (1-\theta)(x^*, a^*)$. By convexity,

$$(\forall \theta \in (0,1)) : f(x_\theta, a_\theta) \leqslant \theta f(x', a') + (1-\theta) f(x^*, a^*) < f(x^*, a^*)$$

However, by continuity of g_j, for θ small enough

$$(\forall j \in \mathcal{J} \backslash \mathcal{K}) : g_j(x_\theta, a_\theta) \leqslant 0$$

whereas by convexity of g_k

$$(\forall k \in \mathcal{K}) : g_k(x_\theta, a_\theta) \leqslant \theta g_k(x', a') + (1-\theta) g_k(x^*, a^*) \leqslant 0$$

which is a contradiction.

Then, by lemma 3, it follows that $\exists\alpha \in \partial^{<} f(x^*, a^*)$ and $\forall k \in \mathcal{K}$ $\exists\beta_k \in \partial^{<} g_k(x^*, a^*)$ and $\exists\gamma_k \in \mathbb{R}_+$ such that

$$\alpha + \sum_{k\in\mathcal{K}} \gamma_k \beta_k = 0$$

Fix $k \in \mathcal{K}$. It is obvious that

$$g_k(x^*, a^*) > \inf_{\mathbb{R}^L\times\mathbb{R}} g_k(x, a)$$

and that every local minimizer of g_k is also a global minimizer, so that, by lemma 1, $\partial^{<} g_k(x^*, a^*) = \partial^{\leqslant} g_k(x^*, a^*)$. Moreover, since g_k is convex and attains no minimum at (x^*, a^*), it follows from lemma 2 that

$$\partial^{\leqslant} g_k(x^*, a^*) = \bigcup_{\tau\geqslant 1} \tau\partial g_k(x^*, a^*)$$

and hence, that

$$(\exists\vartheta_k \in \partial g_k(x^*, a^*))\,(\exists\tau_k \geqslant 1) : \tau_k\vartheta_k = \beta_k$$

whereas, by its definition,

$$\partial g_k(x^*, a^*) = \begin{cases} \{-e_k\} & \text{if } k \in \{1, ..., L\} \\ \{-e_{L+1}\} & \text{if } k = L+1 \\ \{e_{L+1}\} & \text{if } k = L+2 \\ \{(p, 0)\} & \text{if } k = L+3 \end{cases}$$

where, for $\ell \in \{1, ..., L+1\}$, e_ℓ represents the ℓ^{th} canonical unit vector in $\mathbb{R}^{L+1}$.

On the other hand, it follows from monotonicity, and the fact that $m > 0$, that

$$f(x^*, a^*) > \inf_{\mathbb{R}^L_+\times[0,\overline{a}]} f(x, a)$$

whereas by convexity, every local minimizer of f is also a global minimizer. Then, by lemma 1, $\partial^{<} f(x^*, a^*) = \partial^{\leqslant} f(x^*, a^*)$. Also, since f is convex and attains no minimum at (x^*, a^*), it follows from lemma 2 that

$$\partial^{\leqslant} f(x^*, a^*) = \bigcup_{\kappa\geqslant 1} \kappa\partial f(x^*, a^*)$$

and hence, that

$$(\exists\zeta \in \partial f(x^*, a^*))\,(\exists\kappa \geqslant 1) : \kappa\zeta = \alpha$$

Now, suppose that for $\widehat{l} \in \{1, ..., L\}$, $\zeta_{\widehat{l}} \geqslant 0$. Then, define x by

$$x_l = \begin{cases} x_l^* & \text{for } l \in \{1, ..., L\} \setminus \left\{\widehat{l}\right\} \\ x_l^* + \Delta & \text{for } l = \widehat{l} \end{cases}$$

For $\Delta > 0$, by monotonicity, $f(x, a^*) < f(x^*, a^*)$, whereas

$$\zeta \cdot ((x, a^*) - (x^*, a^*)) \geqslant 0$$

contradicting the fact that $\zeta \in \partial f(x^*, a^*)$.

Define and partition:

$$\begin{array}{rcl} (\upsilon, \rho) & = & -\zeta \\ (\varsigma^*, 0) & = & \left\{ \begin{array}{c} -\frac{1}{\kappa} \sum_{k \in \mathcal{K} \cap \{1,...,L\}} \gamma_k \tau_k \vartheta_k \quad \text{if } \ \mathcal{K} \cap \{1,...,L\} \neq \varnothing \\ 0 \quad \text{otherwise} \end{array} \right. \\ (0_L, \mu^*) & = & \left\{ \begin{array}{c} -\frac{1}{\kappa} \gamma_{L+1} \tau_{L+1} \vartheta_{L+1} \quad \text{if } \ L+1 \in \mathcal{K} \\ 0 \quad \text{otherwise} \end{array} \right. \\ (0_L, \eta^*) & = & \left\{ \begin{array}{c} \frac{1}{\kappa} \gamma_{L+2} \tau_{L+2} \vartheta_{L+2} \quad \text{if } \ L+2 \in \mathcal{K} \\ 0 \quad \text{otherwise} \end{array} \right. \end{array}$$

where $\nu \in \mathbb{R}^L_{++}$, $\varsigma^* \in \mathbb{R}^L_+$, $0_L = (0)^L_{l=1} \in \mathbb{R}^L$ and $\mu^*, \eta^* \in \mathbb{R}_+$. Define also $\lambda^* = \gamma_{L+3}$.

Since

$$-\alpha = \sum_{k \in \mathcal{K}} \gamma_k \beta_k$$

it follows that

$$\begin{array}{rcl} \upsilon & = & \lambda^* p - \varsigma^* \\ \rho & = & \eta^* - \mu^* \end{array}$$

By monotonicity, there exists $\widehat{l} \in \{1, ..., L\}$ with $p_{\widehat{l}} > 0$ and $x^*_{\widehat{l}} > 0$. By construction, $\widehat{l} \notin \mathcal{K}$, so that $\varsigma^*_{\widehat{l}} = 0$, and, therefore, $\lambda^* \in \mathbb{R}_{++}$.

By construction,

$$(\forall l \in \{1, ..., L\}) : x^*_l > 0 \Longrightarrow l \notin \mathcal{K} \Longrightarrow \varsigma^*_l = 0$$

from where $\varsigma^* \cdot x^* = 0$. Similar analysis implies that $\mu^* a^* = 0$ and $\eta^* (\overline{a} - a^*) = 0$.

Finally, notice that, by definition,

$$\left(\forall (x, a) \in \mathbb{R}^L_+ \times \left[0, \overline{\overline{a}}\right]\right) : \zeta \cdot ((x, a) - (x^*, a^*)) \leqslant f(x, a) - f(x^*, a^*)$$

and, therefore,

$$\left(\forall (x, a) \in \mathbb{R}^L_+ \times \left[0, \overline{\overline{a}}\right]\right) : (\upsilon, \rho) \cdot ((x, a) - (x^*, a^*)) \geqslant V(x, a) - V(x^*, a^*)$$

or, equivalently,

$$\left(\forall (x, a) \in \mathbb{R}^L_+ \times \left[0, \overline{\overline{a}}\right]\right) : V(x, a) \leqslant V(x^*, a^*) + \upsilon \cdot (x - x^*) + \rho (a - a^*)$$

Moreover, notice that if V is strongly concave, f is strongly convex and, therefore, if there exists $(x, a) \in \mathbb{R}^L_+ \times \left[0, \overline{\overline{a}}\right] \setminus \{(x^*, a^*)\}$ such that

$$\zeta \cdot ((x, a) - (x^*, a^*)) = f(x, a) - f(x^*, a^*)$$

then, defining

$$(x', a') = \frac{1}{2}(x, a) + \frac{1}{2}(x^*, a^*) \in \mathbb{R}_+^L \times [0, \overline{\overline{a}}]$$

one gets that

$$\begin{aligned} \zeta \cdot ((x', a') - (x^*, a^*)) &= \frac{1}{2}\zeta \cdot ((x, a) - (x^*, a^*)) \\ &= \frac{1}{2}(f(x, a) - f(x^*, a^*)) \\ &= \frac{1}{2}f(x, a) + \frac{1}{2}f(x^*, a^*) - f(x^*, a^*) \\ &> f(x', a') - f(x^*, a^*) \end{aligned}$$

contradicting the fact that $\zeta \in \partial f(x^*, a^*)$. Hence, it follows that

$$\begin{aligned} \left(\forall (x, a) \in \left(\mathbb{R}_+^L \times [0, \overline{\overline{a}}]\right) \setminus \{(x^*, a^*)\}\right) &: \\ \zeta \cdot ((x, a) - (x^*, a^*)) &< f(x, a) - f(x^*, a^*) \end{aligned}$$

and therefore that

$$\begin{aligned} \left(\forall (x, a) \in \left(\mathbb{R}_+^L \times [0, \overline{\overline{a}}]\right) \setminus \{(x^*, a^*)\}\right) &: \\ V(x, a) &< V(x^*, a^*) + v \cdot (x - x^*) + \rho(a - a^*) \end{aligned}$$

■

7 Appendix: Tarski-Seidenberg quantifier elimination.

Some of the logical statements in the paper contain existential quantifiers on unobserved (and even unobservable) variables of their models. Although modern computational algorithms have proven useful to deal with this kind of situation, from a purely theoretical perspective it is convenient to argue that these quantifiers can be eliminated and to obtain as much information as possible regarding equivalent statements that are free of quantifiers. This section takes concepts from Mishra (1993).

Definition 12 *A function $\mu : \mathbb{R}^K \longrightarrow \mathbb{R}$, where $K \in \mathbb{N}$, is a (Real) Multivariate Monomial if there exists $\{\alpha_k\}_{k=1}^K \overset{\text{seq}}{\subseteq} \mathbb{N} \cup \{0\}$ such that for every $x \in \mathbb{R}^K$,*

$$\mu(x) = \prod_{k=1}^{K} x_i^{\alpha_k}$$

The degree of the monomial is

$$\deg(\mu) = \sum_{k=1}^{K} \alpha_k$$

Definition 13 *A function $\rho : \mathbb{R}^K \longrightarrow \mathbb{R}$, where $K \in \mathbb{N}$, is a (Real) Multivariate Polynomial if for some $M \in \mathbb{N}$, there exist Multivariate Monomials*

$$\left\{\mu_m : \mathbb{R}^K \longrightarrow \mathbb{R}\right\}_{m=1}^{M}$$

and $\{a_m\}_{m=1}^{M} \overset{seq}{\subseteq} \mathbb{R}\backslash \{0\}$ such that,

$$\rho = \sum_{m=1}^{M} a_m \mu_m$$

The degree of the polynomial is

$$\deg(\rho) = \max_{m \in \{1,\ldots,M\}} \{\deg(\mu_m)\}$$

Definition 14 *A set $A \subseteq \mathbb{R}^K$, where $K \in \mathbb{N}$, is a semialgebraic set if it can be determined by a set theoretic expression of the form*

$$A = \bigcup_{m=1}^{M} \bigcap_{n=1}^{N_m} \left\{ x \in \mathbb{R}^K \middle| \, sgn\left(\rho_{m,n}(x)\right) = s_{m,n} \right\}$$

where for each $m \in \{1, ..., M\}$, $M \in \mathbb{N}$ and each $n \in \{1, ..., N_m\}$, $N_m \in \mathbb{N}$, $\rho_{m,n} : \mathbb{R}^K \longrightarrow \mathbb{R}$ is a Multivariate Polynomial and $s_{n,m} \in \{-1, 0, 1\}$.

Definition 15 *A function $\eta : A \longrightarrow B$, where $A \subseteq \mathbb{R}^{K_A}$ and $B \subseteq \mathbb{R}^{K_B}$ are semialgebraic sets ($K_A, K_B \in \mathbb{N}$), is a semialgebraic map if its graph,*

$$Graph(\eta) = \left\{ (x, y) \in \mathbb{R}^{K_A} \times \mathbb{R}^{K_B} \middle| \, y = \eta(x) \right\}$$

is semialgebraic.

Theorem 10 *(The Tarski-Seidenberg Theorem:) Let $A \subseteq \mathbb{R}^K$, where $K \in \mathbb{N}$, be a semialgebraic set and let $\eta : \mathbb{R}^K \longrightarrow \mathbb{R}^{K'}$, where $K' \in \mathbb{N}$, be a semialgebraic map. Then,*

$$\eta[A] = \left\{ y \in \mathbb{R}^{K'} \middle| \, (\exists x \in A) : \eta(x) = y \right\}$$

is a semialgebraic set.

Proof. This is theorem 8.6.6 in Mishra (1993), pp. 345. ■

Corollary 3 *Let $A \subseteq \mathbb{R}^{K_1} \times \mathbb{R}^{K_2}$, where $K_1, K_2 \in \mathbb{N}$, be a semialgebraic set and let $\overrightarrow{A}^1$ be its projection into $\mathbb{R}^{K_1}$, defined as*

$$\overrightarrow{A}^1 = \left\{ x \in \mathbb{R}^{K_1} \middle| \left(\exists y \in \mathbb{R}^{K_2}\right) : (x, y) \in A \right\}$$

Then, $\overrightarrow{A}^1$ is semialgebraic.

Proof. Define the function $\eta_1 : \mathbb{R}^{K_1} \times \mathbb{R}^{K_2} \longrightarrow \mathbb{R}^{K_1}$ by

$$\eta_1(x, y) = x$$

Its graph, $G(\eta_1) = \left(\mathbb{R}^{K_1} \times \mathbb{R}^{K_2}\right) \times \mathbb{R}^{K_1}$ is clearly semialgebraic. Since A is semialgebraic, it follows from the Tarski-Seidenberg theorem that

$$\begin{aligned}\left\{x \in \mathbb{R}^{K_1} \middle| (\exists (x', y) \in A) : \eta_1(x', y) = x\right\} &= \left\{x \in \mathbb{R}^{K_1} \middle| (\exists (x', y) \in A) : x' = x\right\} \\ &= \left\{x \in \mathbb{R}^{K_1} \middle| \left(\exists y \in \mathbb{R}^{K_2}\right) : (x, y) \in A\right\} \\ &= \overrightarrow{A}^1\end{aligned}$$

is semialgebraic. ■

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