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FINITE COLLINEATION GROUPS IN PROJECTIVE SPACES OF ONE, TWO AND THREE DIMENSIONS.

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ABSTRACT.

The account given in this thesis of the finite collineation groups in projective spaces of one, two and three dimensions is divided into two main sections. Section I indicates some of the methods available for determining the orders of the primitive groups in these spaces, with particular reference to the work done in one and two dimensions by H.F. Blichfeldt, and in three dimensions by G. Bagnera. Section II is an investigation of some of the primitive groups in three dimensions which are generated by biaxial homographies. This latter section has four main paragraphs; in paragraph II the groups generated by biaxial homographies which leave fixed a quadric are determined, and we are concerned with those groups which are isomorphic with symmetric groups in paragraph III; the methods used in these two paragraphs are my own. The group of order 11520 which leaves fixed the Klein 60_{15} configuration is the subject of paragraph IV, the operations of this group and some of its subgroups are found by methods based on those used by J. Todd to determine a simple group of order 25920 in four dimensions. Similar methods are used in paragraph V to find the operations of a simple group of order 25920 leaving fixed a configuration of forty-five points and planes; the configuration has been described by G. Bagnera, but I can find no other account of the group in three dimensions.

<u>CONTENTS.</u>

INTRODUCTION

The problem of the investigation of finite collineation groups is not new, for as long ago as 1876 F. Klein produced a paper (Klein 14*) in which he determined the linear groups in two variables.** The same results were later obtained by P. Gordan (8), C. Jordan (13) and H. Valentiner (19). In these same papers, Jordan and Valentiner also enumerated the finite groups in three variables and in 1879, Jordan published an outline of a method which could be used to determine the orders of the finite primitive groups in any number of variables (Jordan 12). He illustrated his process by applying it to linear groups in four variables, but his results were only partially correct in that some groups were omitted.

Various methods have been used to determine particular classes of groups which occur in three dimensions. E. Goursat (9) obtained all the groups which leave fixed the quadric

$$xt - yz = 0.$$

The same set of groups was the subject of a paper by G. Bagnera (1) + in which he discussed all the groups that

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* A full list of references will be found at the end of Section II.

** There is a difference between a linear group and a collineation group: if T and T' are two transformations such that the matrix of one is a scalar multiple of the matrix of the other, then if they belong to a collineation group they are counted as the same operation, whereas if they belong to a linear group, then they are distinct.

arise in S_3 such that the variables are in the field of real numbers. Four years later he again produced a paper (Bagnera 2) in which he determines all the primitive groups in S_3 which contain homologies, where a homology in S_n is an operation having an isolated united point and also ~~leaves~~ leaving fixed every point of a prime.

The first complete account of the groups in four variables was published by H.F. Blichfeldt in 1905 (4). His method consisted of considering the effect that any particular invariant subgroup would have on the order of a finite group, and by this means he arrived at the conclusion that there are in all 30 primitive groups in three dimensions. (See also Blichfeldt (5) and Miller, Blichfeldt and Dickson (15)).

The work done on groups in spaces of dimension greater than three presents a picture which is by no means complete. Several papers have appeared discussing various isolated groups or the configurations which are invariant under the operations of a particular group; for instance in S_4, the quartic primal with 45 nodes which is left fixed by a simple group of ~~25290~~ 25920 transformations was the subject of a paper by H.F. Baker (3), and the operations of this group were enumerated by J. Todd in 1947 (18). In 1914 H.H. Mitchell shewed that no primitive group in a space of dimension higher than three could contain homologies of period greater than 2, or two homologies of period 2 whose product had period greater than 3 (Mitchell 17). He determines the orders of all the primitive groups in S_n $(n > 3)$ which contain homologies and the discussion of these groups has been continued more recently by C.M. Hamill (10). Mitchell also gave some results for the groups arising on a modular line and derived the finite projective plane groups by methods which can be applied to the modular plane (Mitchell 16).

The object of this paper is twofold; firstly to present an outline of some of the methods available for determining the nature of the primitive groups in projective spaces of one, two and three dimensions, and secondly to consider a number of the groups in S_3 which are generated by biaxial homographies, where a biaxial homography in S_{2n+1} is an operation which leaves fixed each point of two skew linear subspaces of dimension n. The second section also considers the generation of the simple group of order 25920 by means of the homologies of period 3 that it contains.

SECTION I.

§I. Preliminary Definitions and Notation.

The groups with which we shall be concerned are finite groups of collineations in a projective space S_n, in which a point is determined by the ratios between a set of (n + 1) coordinates $x_0, x_1, \ldots\ldots, x_n$, the x_i belonging to the field of complex numbers. A collineation $\underline{C}$ is a transformation of these variables and may be written briefly as

$$\underline{x}' = C\underline{x}$$

where $\underline{x}, \underline{x}'$ are column vectors and C is the matrix of the collineation; occasionally it is useful to write this in a slightly different way, for example in two dimensions the expression

$$\underline{C}: (c_{11}\ x + c_{12}\ y + c_{13}\ z, c_{21}\ x + c_{22}\ y + c_{23}\ z, c_{31}\ x + c_{32}\ y + c_{33}\ z)$$

is equivalent to saying that $\underline{C}$ is the collineation given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

Since it is the n ratios $\frac{x_0}{x_n}, \frac{x_1}{x_n}, \ldots\ldots, \frac{x_{n-1}}{x_n}$ which are important the collineation given by $\underline{x}' = C\underline{x}$ is the same as that given by $\underline{x}' = \lambda C\underline{x}$ where λ is any non-zero scalar quantity. Now if $\underline{C}$ belongs to some group $\mathcal{G}$ which is of finite order, $\underline{C}$ must be of finite period, and there is some positive, non-zero integer m such that $\underline{C}^m$ is the identical operation.

The determinant of $\underline{C}^m$ is $|C|^m$, hence $|C|^m = 1$ and $|C|$ is thus some root of unity. Hence it is always possible to choose the matrix of $\underline{C}$ to have unit determinant simply by introducing a suitable scalar multiplier.

Further restrictions may be found which limit the collineations which can occur in a finite group. For consider the case of plane collineations; there are just five distinct types that arise.

1. The collineation $\underline{C}$ has three distinct, non-collinear united points and by choosing these points as the triangle of reference, $\underline{C}$ takes the form

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

2. $\underline{C}$ has one isolated united point and a line of united points which does not contain the isolated point; the canonical form of $\underline{C}$ is

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

3. $\underline{C}$ has just two distinct united points; this is a specialisation of case 1. in which two of the three united points coincide and by a suitable choice of the triangle of reference $\underline{C}$ takes the form

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

4. In this case, which is a further specialisation of 1., in which all three united points are coincident, C has only one united point and may be expressed in canonical form as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

5. This is a specialisation of 2. in which the isolated united point lies on the line of united points; the canonical form of C is

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

Of these five classes only the collineations in 1. or 2. can be of finite period, so that groups of finite order in S_2 contain collineations having three distinct non-collinear self-corresponding points and collineations having a line of united points and an isolated united point. (The latter transformation is a plane homology.)

In S_1 the operations of a finite group must have two distinct self-corresponding points.

There are four distinct types of collineations of finite period occurring in S_3 ; either

1. a collineation, $\underline{C}$, has the vertices of a tetrahedron as united points;

or 2. $\underline{C}$ has a line of united points and two isolated united points whose join is a line skew to the invariant line;

or 3. $\underline{C}$ leaves fixed each point of a plane and a point not lying in that plane;

or 4. $\underline{C}$ leaves fixed each point of two skew lines.

The collineations in 3. and 4. are known as homologies and biaxial homographies respectively.

At this point, it is necessary to introduce the idea of a positive definite Hermitian form, J. This is an expression in the 2(n + 1) variables $x_i, \overline{x}_i$, i = 0,1,....,n ($\overline{x}_i$ is the complex conjugate of x_i) where

$$J = \sum_{s=0}^{n} \sum_{t=0}^{n} q_{st} x_s \overline{\overline{x}}_t$$

and J is zero only for $x_0 = x_1 = \ldots\ldots = x_n = 0$, and is otherwise strictly positive. By introducing new variables y_i, i = 0,1,.....,n, where

$$y_i = \rho_{i0} x_0 + \rho_{i1} x_1 + \ldots\ldots + \rho_{ii} x_i$$

for a suitable choice of the ρ_{ij}, J may be reduced to the form $\sum_{i=0}^{n} y_i \overline{y}_i$. (See Miller, Blichfeldt and Dickson (15) p.208.)

If we now consider a group $\mathcal{G}^{\times}$ which is obtained from a group $\mathcal{G}$ by considering the group of the operations $\underline{C}^{\times}$ arising from the elements $\underline{C}$ of $\mathcal{G}$, where

$$\underline{C}^{\times}: \begin{pmatrix} x_0' \\ \vdots \\ \bar{x}_0' \\ \vdots \end{pmatrix} = \left(\begin{array}{c|c} C & 0 \\ \hline 0 & \bar{C} \end{array}\right) \begin{pmatrix} x_0 \\ \vdots \\ \bar{x}_0 \\ \vdots \end{pmatrix}$$

and $\bar{C}$ is the complex conjugate of the matrix C, we find that there is a Hermitian form J associated with every $\mathcal{G}$ so that J is invariant under every operation of $\mathcal{G}^{\times}$; for suppose $\mathcal{G}^{\times}$ contains the elements $\underline{C}_1^{\times}, \underline{C}_2^{\times}, \ldots\ldots, \underline{C}_g^{\times}$, and let I be the form

$$I = \sum_{i=0}^{n} x_i \bar{x}_i .$$

Then consider the form

$$\begin{aligned} J &= \underline{C}_1^{\times}(I) + \underline{C}_2^{\times}(I) + \ldots\ldots + \underline{C}_g^{\times}(I). \\ &= \underline{C}_1^{\times}\Big(\sum_{i=0}^{n} x_i\bar{x}_i\Big) + \underline{C}_2^{\times}\Big(\sum_{i=0}^{n} x_i\bar{x}_i\Big) + \ldots\ldots + \underline{C}_g^{\times}\Big(\sum_{i=0}^{n} x_i\bar{x}_i\Big). \\ &= \sum_{j=1}^{g}\sum_{i=0}^{n} \underline{C}_j^{\times}(x_i\bar{x}_i). \end{aligned}$$

This is strictly positive except for the case in which all the transformed variables are zero, and so J is a Hermitian form. In addition the transform of J by an element $\underline{C}_i^{\times}$ of $\mathcal{G}^{\times}$ is given by

$$\underline{C}_i^{\times}(J) = \underline{C}_i^{\times}.\underline{C}_1^{\times}(I) + \underline{C}_i^{\times}.\underline{C}_2^{\times}(I) + \ldots\ldots + \underline{C}_i^{\times}.\underline{C}_g^{\times}(I).$$

But $C_i^{\times}.\underline{C}_j^{\times}$ is just another element of the group $\mathcal{G}^{\times}$ and so $\underline{C}_i^{\times}(J) = J$ for all $i = 1, 2, \ldots\ldots, g$. Hence J is an invariant Hermitian form for the group $\mathcal{G}$. (Miller, Blichfeldt and Dickson (15), p.209).

The last main definitions which must be given in this paragraph are those of an intransitive group, an imprimitive group and a primitive group (See Blichfeldt 5).

If the $(n+1)$ variable of a group G of collineations in S_n may be separated into two or more sets (either directly or after a suitable change of variable) such that under each operation of G the variables of any one set are transformed into linear combinations of the variables of the <u>same</u> set, then G is said to be intransitive. If such a division is not possible then G is transitive. In other words every operation of an intransitive group leaves fixed each one of a set of linear subspaces. For example, if a group in S_3 is such that each of its operations leaves a given point, 0, fixed and at the same time transforms each point of a plane π (not containing 0) into another point of π, then the group is intransitive.

Our next definition is that of an imprimitive group. If G is a transitive group such that (after a suitable choice of variable) its variables may be divided into two or more sets so that under each operation of G the variables of any particular set are transformed into linear combinations of the same or a different set, then G is said to be imprimitive. This may also be expressed alternatively: G is imprimitive if each of its operations permutes a set of linear subspaces among themselves. Otherwise G is primitive.

It will perhaps be helpful to list the restrictions that these definitions place on the primitive groups. In S_1, an intransitive group leaves fixed each point of a pair of points, while an imprimitive group leaves fixed a pair of points, each operation in the group either having these points as united points or interchanging them. In S_2 , an intransitive group either leaves fixed a point and a line not passing through that point, or else each operation of the group has the vertices of a given triangle as united points; an imprimitive group has an invariant triangle. For S_3 , it is sufficient to note that a primitive group does not leave fixed either a tetrahedron, or a pair of skew lines, or a plane ~~or~~ and a point not lying in that plane.

§II. Determination of the Finite Groups in S_1.

The method given here for enumerating the finite groups of collineations in S_1 is due to H.F. Blichfeldt (see Blichfeld (5) or Miller, Blichfeldt and Dickson (15)) and is based on ar original method by F. Klein ((14) p.183). It consists of setting up an isomorphism between a collineation group $\mathcal{G}$ in S_1 and a group $\mathcal{G}^*$ of rotations in ordinary Euclidean space; the problem is then reduced to that of determining all the finite rotational groups.

Suppose $\underline{C}$ is a collineation belonging to $\mathcal{G}$ where $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc = 1$, when the invariant Hermitian form for $\mathcal{G}$ is

$$J = x\bar{x} + y\bar{y}.$$

Then $(ax + by)(\bar{a}\bar{x} + \bar{b}\bar{y}) + (cx + dy)(\bar{c}\bar{x} + \bar{d}\bar{y}) = \rho(x\bar{x} + y\bar{y})$.

(ρ is some non-zero scalar quantity.)

Hence $\quad a\bar{a} + c\bar{c} = \rho = b\bar{b} + d\bar{d}$

and $\quad a\bar{b} + c\bar{d} = 0 = b\bar{a} + d\bar{c}$.

These equations may be solved to give

$$c = -\bar{b} \quad \text{and} \quad d = \bar{a}.$$

If now p and q are the positive square roots of $a\bar{a}$ and $b\bar{b}$ respectively, $p^2 + q^2 = 1$ and there is a v such that

$p = \cos v, \quad q = \sin v$.

In addition $\left|\dfrac{a}{p}\right| = 1$ and $\left|\dfrac{b}{q}\right| = 1$ so that h,k may be chosen to satisfy

$$\frac{a}{p} = \cos h - i \sin h$$

$$\frac{b}{q} = \cos k - i \sin k.$$

Finally, by putting $\frac{h+k}{2} = u$ and $\frac{h-k}{2} = w$, we may express C as the product $C_1 . C_2 . C_3$, where

$$C_1 = \begin{pmatrix} e^{-iu} & 0 \\ 0 & e^{iu} \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \cos v & \sin v \\ -\sin v & \cos v \end{pmatrix}$$

$$C_3 = \begin{pmatrix} e^{-iw} & 0 \\ 0 & e^{iw} \end{pmatrix}.$$

Defining the four transformations $\underline{C}^x, \underline{C}_1^x, \underline{C}_2^x, \underline{C}_3^x$ where

$$\underline{C}^x: \quad \begin{pmatrix} x' \\ y' \\ \bar{x}' \\ \bar{y}' \end{pmatrix} = \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ -\bar{b} & \bar{a} & 0 & 0 \\ \hline 0 & 0 & \bar{a} & \bar{b} \\ 0 & 0 & -b & a \end{array}\right) \begin{pmatrix} x \\ y \\ \bar{x} \\ \bar{y} \end{pmatrix}$$

and $\underline{C}_1^x, \underline{C}_2^x$ and $\underline{C}_3^x$ are similarly defined, it is seen that $\underline{C}^x = \underline{C}_1^x \underline{C}_2^x \underline{C}_3^x$.

Then if $X = x\bar{x} - y\bar{y}$

$$Y = x\bar{y} + \bar{x}y$$

$$Z = i(x\bar{y} - \bar{x}y) \qquad (i^2 = -1)$$

$\underline{C}_1^x, \underline{C}_2^x$ and $\underline{C}_3^x$ may be regarded as linear transformations in the real variables X,Y,Z and as such are given by

$$\underline{C}_1^x: = (X, Y \cos 2u - Z \sin 2u, Y \sin 2u + Z \cos 2u)$$
$$\underline{C}_2^x: = (X \cos 2v + Y \sin 2v, -X \sin 2v + Y \cos 2v, Z)$$
$$\underline{C}_3^x: = (X, Y \cos 2w - Z \sin 2w, Y \sin 2w + Z \cos 2w).$$

X,Y and Z may be taken as coordinates in a real Euclidean space and $\underline{C}_1^x, \underline{C}_2^x, \underline{C}_3^x$ are rotations about the X,Z and X axes respectively. Each operation $\underline{C}$ of $\mathcal{C}$ is thus related to a unique rotation $\underline{C}^x$ belonging to a group of rotations $\mathcal{C}^x$; as the correspondence between $\underline{C}$ and $\underline{C}^x$ is (1-1) and also is such that if $\underline{C}_1$ and $\underline{C}_2$ give rise to $\underline{C}_1^x$ and $\underline{C}_2^x$ then $\underline{C}_1 . \underline{C}_2$ gives rise to $\underline{C}_1^x . \underline{C}_2^x$, the groups $\mathcal{C}$ and $\mathcal{C}^x$ are isomorphic. Hence $\mathcal{C}$ is of finite order if and only if $\mathcal{C}^x$ is finite.

Clearly each operation $\underline{C}^x$ of $\mathcal{C}^x$ leaves fixed the origin of coordinates, O. It is therefore convenient to consider the effect of $\mathcal{C}^x$ on a sphere Σ, centre O, rather than its effect on the whole of space. If A and A$'$ are diametrically opposite points on Σ, a rotation through an angle θ about A is equivalent to a rotation $-\theta$ about A$'$.

Any operation $\underline{C}^x$ of $\mathcal{C}^x$ permutes the axes of rotation of the elements of $\mathcal{C}^x$ among themselves; $\underline{C}^x$ transforms an operation $\underline{T}_1^x$ into $\underline{C}^x \underline{T}_1^x \underline{C}^{x-1}$ and hence $\underline{C}^x$ sends the axis of $\underline{T}_1^x$ into the axis of $\underline{C}^x \underline{T}_1^x \underline{C}^{x-1}$. Now suppose the points in which the axis of $\underline{T}_1^x$ meets Σ are T_1 and T_1' and let the points $T_1, T_2, \ldots, T_t$ be all the transforms of T_1 by the operations of $\mathcal{C}^x$. Then since $\underline{T}_1^x$ and $\underline{C}^x \underline{T}_1^x \underline{C}^{x-1}$ have the same period, if $\underline{T}_1^x$ is of period m the lines $OT_2, OT_3, \ldots, OT_t$ will all be axes of rotations of period m belonging to $\mathcal{C}_1^x$; the distribution of the set $T_1, \ldots, T_t$ about any member of the set will be the same as the distribution about any other member.

Consider the arcs of the great circles joining T_1 to $T_2, T_3, \ldots, T_t$ (where t is greater than one). Since $\mathcal{C}_1^x$ is of finite order there will be an arc, T_1T_ℓ say, which is less than or equal to all the others. Suppose T_1T_ℓ is of arc length L, then the number of arcs of this length radiating from T_1 will be a multiple of m, since rotations about T_1 transform each point T_ℓ into (m - 1) further points at distance L from T_1. However there cannot be more than five such points; for if there are six or more the angle θ between two consecutive arcs T_1T_i, T_1T_j of length L will be less than or equal to $\pi/3$, thus if the arc length between T_i and T_j is L', we have that

$$\begin{aligned} \cos L' &= \cos^2 L + \sin^2 L \cos\theta \\ &\geq \cos^2 L + \tfrac{1}{2}\sin^2 L \\ &= \tfrac{1}{2}(1 + \cos^2 L) \\ &> \cos L \end{aligned}$$

and as $0 < L' < \pi/2$ this gives that $L' < L$. But the lengths of the arcs radiating from T_1 are the same as those radiating from T_i and so L' cannot be less than L. Hence there are at most five points T_ℓ so that T_1T_ℓ is of length L.

For m = 3,4 or 5 there are just m points T_ℓ, each arc T_1T_ℓ making an angle $2\pi/m$ with the adjacent arcs; as this is true for all the points $T_1, T_2, \ldots\ldots, T_t$, the sphere Σ will be divided by arcs of length L into a number of equal, regular polygons, and the T_i $(i = 1,2,\ldots,t)$ are therefore the vertices of a regular polyhedron inscribed in Σ, and $\mathcal{G}_1^x$ is just the group of rotations which leave that polyhedron fixed. (It should be noted that not all the T_i need be vertices of the same polyhedron; if T_s is not a vertex of the polyhedron $T_1, T_2, \ldots T_g$, then as rotation about OT_s must leave this polyhedron fixed, OT_s must be an axis of symmetry for $T_1, T_2, \ldots, T_g$. In addition, $\underline{T}_s^x$ is of period m as it is a transform of $\underline{T}_1^x$ by an operation of $\mathcal{G}_1^x$ and so T_s must be a central point of one of the faces of $T_1, T_2, \ldots, T_g$.)

If there is no axis of rotation for which m is greater than two, then either there is just one operation of period 2 and $\mathcal{G}_1^x$ is isomorphic with the cyclic group of order 2, or else there are at least two axes of rotation of period 2, say A_1OA_1' and A_2OA_2'. If the acute angle between these two axes is α, a rotation through π about one axis followed by a rotation through π about the second axis is equivalent to a rotation through 2α about an axis perpendicular to the plane A_1OA_2. Therefore as 2α must be a multiple of π and $OA_1 \neq OA_2$, α must be equal to $\pi/2$. Hence if there are no axes of rotations of period greater than 2 then either there is just one axis, or else there are three mutually perpendicular axes; in the latter case $\mathcal{G}_1^x$ is isomorphic with the dihedral group, $\mathcal{C}_2^2$, which is known as the 'Four group'.

The last case to consider is that in which $t = 1$ and $m \neq 2$. Here again either there is just one axis of rotation and $\mathcal{G}^{x}$ is isomorphic with a cyclic group of order m, or else any further axes lie in the plane perpendicular to the axis of period m; clearly if such axes exist they must be of period 2. By considering the transform of each operation of period 2 by that of period m it can be seen that there will be just m axes of period 2. Hence in these last cases $\mathcal{G}^{x}$ is isomorphic with either a cyclic group $\mathcal{C}_m$ or a dihedral group $\mathcal{C}_m^2$.

If $\mathcal{G}^{x}$ is isomorphic with a cyclic group of order m, $\mathcal{C}_m$, the corresponding collineation group $\mathcal{G}$ has two invariant points, which are the united points of the operation of period m which generates the group.

In the case in which $\mathcal{G}^{x}$ is isomorphic with a dihedral group $\mathcal{C}_m^2$, $\mathcal{G}$ is generated by two operations $\underline{A}$ and $\underline{B}$ where

$$\underline{A}^m = \underline{B}^2 = (\underline{A}\underline{B})^2 = e.$$

If the united points of $\underline{A}$ are M and N and

$$\underline{B}(M) = M', \quad \underline{B}(N) = N',$$

we have that

$$\begin{aligned} \underline{A}(M') = \underline{B}\,\underline{A}^{-1}\underline{B}(M') &= \underline{B}\,\underline{A}^{-1}(M) \\ &= \underline{B}(M) \\ &= M' \end{aligned}$$

and similarly $$\underline{A}(N') = N'$$

Thus $\underline{B}$ either interchanges the united points of $\underline{A}$ or else $\underline{A}$ and $\underline{B}$ have the same united points. In the latter instance there would be some operation $\underline{C}$ in $\mathcal{G}$ such that $\underline{A}$ and $\underline{B}$ are both powers of $\underline{C}$ and $\mathcal{G}$ is simply a cyclic group, so the operations of a dihedral group all leave fixed a pair of points and $\mathcal{G}$ is imprimitive.

The primitive groups in S_1 are those isomorphic with the rotational groups of the regular polyhedra. There are three distinct groups of this type.

1. A tetrahedral group $\mathcal{R}_{12}$ (the notation is taken from Bagnera (1)) which has 12 elements and is isomorphic with the alternating group on 4 letters, $\mathcal{A}_4$; besides the identity it contains 3 operations of period 2 forming a $\mathcal{C}_2^2$ which is invariant in $\mathcal{R}_{12}$ and 8 operations of period 3.

2. The octahedral group $\mathcal{R}_{24}$ is isomorphic with the symmetric group of degree 4, $\mathcal{S}_4$. Its 24 operations are those of $\mathcal{R}_{12}$ together with 6 more operations of period 2 and 6 of period 4.

3. The icosahedral group $\mathcal{R}_{60}$ is isomorphic with the alternating group of degree 5, $\mathcal{A}_5$; it contains the identity, 15 operations of period 2, 20 of period 3 and 24 of period 5.

§III. Finite Groups in Two Dimensions.

The determination of the finite linear groups in three variables was carried out before the end of the last century by both Jordan and Valentiner (Jordan (13) and Valentiner (19) but the method which will be reproduced in part here is one due to Blichfeldt. (A full account may be found in either Blichfeldt (5) or Miller, Blichfeldt and Dickson (15).)

A plane intransitive group either

(a) leaves fixed each vertex of a triangle

or (b) leaves fixed a point and a line not passing through that point.

In the first case, each operation of the group may be written in the form $(\alpha x, \beta y, \gamma z)$ by taking the invariant points as the vertices of the triangle of reference; in the second case, by taking the isolated point and the invariant line as a vertex and opposite side of the triangle of reference, each operation of the group can be expressed in the form $(x, ay + bz, cy + dz)$.

The imprimitive groups in S_2 all leave fixed a triangle, and there are again two cases to consider,

either (c) the operations of the group permute the vertices of the triangle cyclically,

or (d) the operations of the group subject the vertices to all the six possible permutations.

Hence by choosing the invariant triangle as triangle of reference, it is seen that groups in class (c) are generated by an intransitive group of class (a) together with an operation (ay, bz, cx), while groups in class (d) are generated by an operation $(a'x, b'z, c'y)$ together with a group of class (c).

The question now arises, is it possible for an imprimitive group to have more than one invariant triangle? By considering the transforms of the triangle

$$t \equiv (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z)(a_3x + b_3y + c_3z) = 0$$

by the operations of a group $\mathcal{C}$ of class (c) and a group $\mathcal{D}$ of class (d) it is found that it is identical with the triangle

$$t_1 \equiv xyz = 0$$

except in two cases. These two exceptional groups are

1. a group $\mathcal{C}_0$ generated by the operations $(x, \varpi y, \varpi z)$ where $\varpi^3 = 1$ and (y, z, x)

and 2. a group $\mathcal{D}_0$ generated by $\mathcal{C}_0$ and the operation (x, z, y).

Both $\mathcal{C}_0$ and $\mathcal{D}_0$ have four invariant triangles,

$$t_1 \equiv xyz = 0$$

and $$t_2, t_3, t_4 \equiv (x + y + \theta z)(x + \theta y + \varpi^2\theta z)(x + \varpi^2 y + \varpi\theta z) \quad 0$$

where $\theta = 1, \varpi, \varpi^2$, for t_2, t_3, t_4 respectively.

It may now be proved that any plane group, $\mathcal{G}$, which has an intransitive, invariant subgroup, $\mathcal{G}'$, must itself be intransitive or imprimitive. For suppose $\mathcal{G}'$ is of type (b) with an invariant point, X say. Let $\underline{S}$ be any operation of $\mathcal{G}'$ and $\underline{T}, \underline{T}^{-1}$ be operations of $\mathcal{G}$ not belonging to $\mathcal{G}'$. Then $\underline{S}_1 = \underline{T}^{-1}\underline{S}\,\underline{T}$ also belongs to $\mathcal{G}'$, since $\mathcal{G}'$ is invariant in $\mathcal{G}$, and this is true for all $\underline{T}$, so that the operations $\underline{S}_1$ have just one invariant point, X.

If $\underline{T}(X) = X'$ then

$$\begin{aligned} \underline{S}(X') = \underline{T}\,\underline{S}_1\underline{T}^{-1}(X') &= \underline{T}\,\underline{S}_1(X) \\ &= \underline{T}(X) \\ &= X'. \end{aligned}$$

It follows that X' is an invariant point for all $\underline{S}$, and so X' is coincident with X ; but this means that X is an invariant point for all $\underline{T}$ in $\mathcal{G}$, so that $\mathcal{G}$ cannot possibly be primitive.

If $\mathcal{G}'$ is a group of type (a) then any operation $\underline{S}$ of $\mathcal{G}'$ has three invariant points, X,Y and Z say. As before, if $\underline{T}$ is any element of $\mathcal{G}$ not belonging to $\mathcal{G}'$, and $\underline{T}(X) = X'$, $\underline{T}(Y) = Y'$, $\underline{T}(Z) = Z'$ then $\underline{S}(X') = X'$, $\underline{S}(Y') = Y'$, $\underline{S}(Z') = Z'$ and the triangle $X'Y'Z'$ is coincident with the triangle XYZ and is invariant for all $\underline{T}$ in $\mathcal{G}$, and again $\mathcal{G}$ is not primitive.

With this restriction on the invariant subgroups of a primitive plane group, all such groups must be contained in one of the following three classes:

1. Primitive simple groups (that is, primitive groups with no invariant subgroups.)

2. Primitive groups with imprimitive invariant subgroups.

3. Primitive groups with primitive invariant subgroups.

Blichfeldt's method for determining the groups in class 1. depends on a number of theorems shewing the effect of the presence of an invariant subgroup on a group $\mathcal{G}$ and so restricting the possible orders of the simple groups. He deduces that there are just three primitive simple groups, Γ_{60} and Γ_{360} of orders 60 and 360 respectively, which are isomorphic with the alternating groups on 5 and 6 letters, and Γ_{168} which is isomorphic with the subgroup of order 168 of the symmetric group on 7 letters, generated by the three cycles (1234567), (142)(356), (12)(35). (A fuller description of these groups will be found at the end of this paragraph.)

Before attempting to determine the primitive groups having imprimitive invariant subgroups, it should be noted that if the subgroup has only one invariant triangle then the main group, $\mathcal{G}$, cannot be primitive; this follows by reasoning similar to that used in invariant intransitive subgroups. It may be assumed therefore that the invariant subgroup is either $\mathcal{C}_0$ or $\mathcal{D}_0$ and leaves fixed each of the triangles t_1, t_2, t_3 and t_4. Each operation of $\mathcal{G}$ permutes these four triangles among themselves; for if $\underline{T}$ is an element of $\mathcal{G}$ not in the invariant subgroup, then $\underline{T}.\underline{S}.\underline{T}^{-1}$ has only the given triangles as invariant triangles, if $\underline{S}$ belongs to the subgroup; but if $\underline{T}(t_i) = t_i'$ for i = 1,2,3,4, then

$$\begin{aligned} \underline{T}\,\underline{S}\,\underline{T}^{-1}(t_i') &= \underline{T}\,\underline{S}(t_i) \\ &= \underline{T}(t_i) \\ &= t_i'. \end{aligned}$$

and t_i' must be one of the four given triangles. We have therefore associated a permutation of the four letters t_1, t_2, t_3 and t_4 with every operation of $\mathcal{G}$ and $\mathcal{G}$ is homomorphic with some substitution group, K, on four letters, in which the invariant subgroup in $\mathcal{G}$ corresponds to the identity in K. No one letter can be left fixed by every substitution in K (or else the corresponding triangle is invariant for $\mathcal{G}$) and also, no operation of $\mathcal{G}$ can interchange two of the triangles and leave the other two fixed (as may be directly verified.) There are thus only three possible forms for K. These are

K_1: e, $(t_1t_2)(t_3t_4)$.

K_2: e, $(t_1t_2)(t_3t_4)$, $(t_1t_3)(t_2t_4)$, $(t_1t_4)(t_2t_3)$.

K_3: the alternating group, $\mathcal{A}_4$, generated by $(t_1t_2)(t_3t_4)$ and $(t_2t_3t_4)$.

Now the group $\mathcal{D}_0$ contains all the transformations which leave fixed each of the four triangles. Further, if a given transformation $\underline{T}$ permutes the triangles in a given way then any operation $\underline{T}'$ permuting them in the same way may be expressed as $\underline{T}' = \underline{X}\,\underline{T}$, where $\underline{X}$ belongs to $\mathcal{D}_0$, for $\underline{T}'\underline{T}^{-1}$ must leave each triangle fixed and is thus an operation belonging to $\mathcal{D}_0$.

The three operations $\underline{U}$, $\underline{V}$, $\underline{U}\,\underline{V}\,\underline{U}^{-1}$ where

$$\underline{U}: \quad (x, y, \varpi z)$$

$$\underline{V}: \quad (x + y + z,\ x + \varpi y + \varpi^2 z,\ x + \varpi^2 y + \varpi z)$$

$$\underline{U}\,\underline{V}\,\underline{U}^{-1}: \quad (x + y + \varpi^2 z,\ x + \varpi y + \varpi z,\ \varpi x + y + z)$$

correspond to $(t_2 t_3 t_4)$, $(t_1 t_2)(t_3 t_4)$, $(t_1 t_4)(t_2 t_3)$ respectively. Since all the required groups $\mathcal{G}$ contain at least one element corresponding to $(t_1 t_2)(t_3 t_4)$, every such group must contain an element $\underline{X}.\underline{V}$, where $\underline{X}$ belongs to $\mathcal{D}_0$. Hence, if $\mathcal{G}$ has $\mathcal{D}_0$ as a subgroup, then $\mathcal{G}$ always contains the element $\underline{V}$. On the other hand, if $\mathcal{G}$ has $\mathcal{C}_0$ as a subgroup, but does not contain $\mathcal{D}_0$, then we can only say that $\mathcal{G}$ contains either the element $\underline{V}$ or the element $\underline{X}.\underline{V}$, where $\underline{X}$ does not belong to $\mathcal{G}$; however in the latter case, $\underline{X}$ must be an operation of the type $\underline{X}_1\underline{W}$, where $\underline{X}_1$ belongs to $\mathcal{C}_0$ and $\underline{W}$: $(x, z\ y)$; $\mathcal{G}$ then contains $\underline{X}_1$ and hence the operation $\underline{W}.\underline{V}$; but $(\underline{W}.\underline{V})^2 = \underline{W}$ and $\mathcal{G}$ thus contains $\underline{W}, \underline{W}^{-1}$ and hence $\underline{V}$. So that in each case, $\mathcal{G}$ contains $\underline{V}$ and also $\underline{X}$; in other words, the invariant subgroup in $\mathcal{G}$ is always $\mathcal{D}_0$.

In an exactly similar way it can be shewn that if $\mathcal{G}$ contains an element corresponding to either $(t_1 t_4)(t_2 t_3)$ or $(t_2 t_3 t_4)$, then it contains the element $\underline{U}\,\underline{V}\,\underline{U}^{-1}$ or $\underline{U}$.

From this information it can be seen that the three groups corresponding to K_1, K_2 and K_3 are

Γ_{36}: of order 36, generated by $\mathcal{T}_0$ and V.

Γ_{72}: of order 72, generated by $\mathcal{T}_0$, V and UVU.

Γ_{216}: of order 216, generated by $\mathcal{T}_0$, V and U.

The last stage of Blichfeldt's determination shews that class 3. contains no plane groups that have not already occurred; we have thus just six primitive groups in the plane, Γ_{60}, Γ_{360}, Γ_{168}, Γ_{36}, Γ_{72} and Γ_{216}.

The simple group Γ_{360} contains 45 plane homologies of period 2 which generate the group. (This description, and those of the other plane groups, is taken from Hamill (10), pp. 21-49.) The fixed point of a homology will be referred to as its centre, and the invariant line as its polar line. In Γ_{360}, the following relations exist between the centres and polar lines. Each polar line passes through just 4 centres, so that a given centre, P, is joined to 12 other centres by polar lines; the joins of P to the remaining 32 centres are of two kinds, there are 4 ϕ-lines passing through P, where a ϕ-line contains 5 centres in all, and 8 K-lines, each of which contains 2 centres other than P. Thus we have $\frac{45 \times 4}{5} = 36$ ϕ-lines and $\frac{45 \times 8}{3} = 120$ K-lines in the configuration left fixed by Γ_{360}. A suitable choice of the trangle of reference gives the coordinates of the 45 centres as

$(1,0,0)$, $(0,1,\pm\varpi)$, $(1,\pm\alpha,\ \pm\alpha^2)$ where $\alpha^2+\alpha-1=0$

$(1,\ \pm\varpi^2\alpha^2,\ \pm\varpi\alpha)$, $(1,\ \pm(1-\varpi\alpha), \pm\varpi^2)$ together with the points obtained by cyclic permutation of the coordinates. Taken in this form, the group Γ_{360} has an invariant Hermitian form $\underline{J}$, where

$$J = x\bar{x} + y\bar{y} + z\bar{z},$$

and the polar line of a centre (ξ, η, ζ) is the line

$$\bar{\xi} x + \bar{\eta} y + \bar{\zeta} z = 0.$$

The operations of Γ_{360}, besides the identical collineation and the 45 homologies, are 90 collineations of period 4, 80 of period 3 and 144 of period 5.

Γ_{60} is the subgroup of Γ_{360} which is generated by the 15 homologies whose centres are given by permutations of the coordinates in (1,0,0) and $(1, \pm\alpha, \pm\alpha^2)$. The configuration left fixed by Γ_{60} contains 10 κ-lines and 6 ϕ-lines; each centre lies on 2 polar lines, 2 κ-lines and 2 ϕ-lines. Its operations are the identity, 15 homologies of period 2, 20 collineations of period 3 and 24 of period 5.

Γ_{168}, the remaining simple group, contains 21 involutary homologies which generate the group. The 21 centres may be given by cyclic permutations of the coordinates in (1,0,0), $(0,1,\pm 1)$, $(\lambda, \pm 1, \pm 1)$ where $\lambda^2 - \lambda + 2 = 0$. Again in this form Γ_{168} has a Hermitian invariant J where

$$J = x\bar{x} + y\bar{y} + z\bar{z},$$

and the polar line of a centre (ξ, η, ζ) is the line $\bar{\xi} x + \bar{\eta} y + \bar{\zeta} z = 0$. The group configuration has just two sorts of lines, polar lines and 28 K-lines; each polar line passes through 4 centres and so each centre lies on 4 polar lines, also each centre lies on 4 K-lines. Besides the identity and the 21 homologies, Γ_{168} contains 42 operations of period 4, 56 of period 3 and 48 of period 7.

The configuration left fixed by Γ_{216} is the Hessian figure of the 9 inflexions of a plane cubic curve; ~~#~~ Γ_{216} contains 9 involutary homologies whose centres are the inflexional points, but these do not generate the group. It also contains 12 homologies of period 3 and their squares which do generate the group. By taking three of their centres as the triangle of reference the remainder have coordinates

$$\begin{array}{lll} (\varpi\ 1,\ 1), & (\varpi^2, 1,\ 1), & (1,\ 1,\ 1), \\ (1, \varpi, 1), & (1,\ \varpi^2,\ 1), & (1, \varpi^2, \varpi), \\ (1,\ 1, \varpi), & (1,\ 1, \varpi^2), & (1, \varpi, \varpi^2), \end{array}$$

and once again, the polar line of the centre (ξ, η, ζ) is the line $\bar{\xi} x + \bar{\eta} y + \bar{\zeta} z = 0$ and the Hermitian invariant is $(x\bar{x} + y\bar{y} + z\bar{z})$. Each centre of a homology of period 3 lies on 3 of the polar lines of the involutary homologies, and each of these 9 polar lines contains 4 of the above 12 centres. The other lines in the figure are e-lines, each of which contains 2 centres of homologies of period 3; there are 12 e-lines and they are the polar lines of the 12 homologies of period 3. The remaining operations of Γ_{216} are the identity, 8 collineations of period 3, 54 of period 4, 72 of period 6 and a further set of 48 collineations of period 3.

Γ_{72} and Γ_{36} are both subgroups in Γ_{216} ; Γ_{72} contains the identity, 9 homologies of period 2, the first set of 8 collineations of period 3 together with the 54 collineations of period 4. Γ_{36} contains the identity, 9 involutary homologies and 8 collineations of period 3, together with 18 of the operations of period 4.

§IV. Primitive Groups in S_3.

The method used by H.F. Blichfeldt to obtain all the finite primitive groups in three dimensions is an extension of that used for the plane groups. (Blichfeldt (4) or Blichfeldt (5).) He lists 30 distinct primitive groups.

1. There are 12 primitive groups having invariant intransitive subgroups; of these there are 3 of order 288, 3 of order 576 and one each of order 144, 720, 1440, 3600, 1152 and 7200.

2. The primitive groups having an invariant imprimitive subgroup are 9 in number; 2 of these are of order 960, 2 are of order 1920 and the other 5 have orders 80, 160, 320, 5760 and 11520.

3. There are just 6 simple primitive groups, 2 of order 60 and one each of order 360, 2520, 168 and 25920.

4. The last three groups which have not appeared in any of the previous classes all contain primitive groups as invariant subgroups; they are of orders 120, 120 and 720.

Apart from Blichfeldt's account of the groups in S_3 special methods have been used by various people to determine certain classes of these 30 groups (references will be found in the introduction.) Bagnera's method for obtaining all the groups generated by homologies (or containing homologies) is purely geometrical, and as some of his results will be needed in the next section it will be useful to reproduce part of his work here (see Bagnera (2).)

The first step is to shew that if a primitive group, $\mathcal{G}$, in S_3 contains a homology of period ν, then ν is either 2 or 3. Suppose $\mathcal{G}$ contains a homology $\underline{O}_1$, centre O_1, polar plane Π_1, and period ν, and that $O_1, O_2, \ldots, O_j$ are the transforms of O_1 by means of the operations of $\mathcal{G}$, then if $\mathcal{G}$ is primitive, j must be greater than 4. (Otherwise $\mathcal{G}$ leaves a tetrahedron fixed.) Further if $\underline{A}(O_1) = O_i$ where $\underline{A}$ belongs to $\mathcal{G}$ and under $\underline{A}$ the points of Π_1 go into the points of a plane Π_i, then O_i and Π_i are the centre and polar plane of a homology $\underline{O}_i$ of period ν where $\underline{O}_i = \underline{A}\,\underline{O}_1\underline{A}^{-1}$;

$$\begin{aligned} \text{for } \underline{A}\,\underline{O}_1\underline{A}^{-1}(O_i) &= \underline{A}\,\underline{O}_1(O_1) \\ &= \underline{A}(O_1) \\ &= O_i \end{aligned}$$

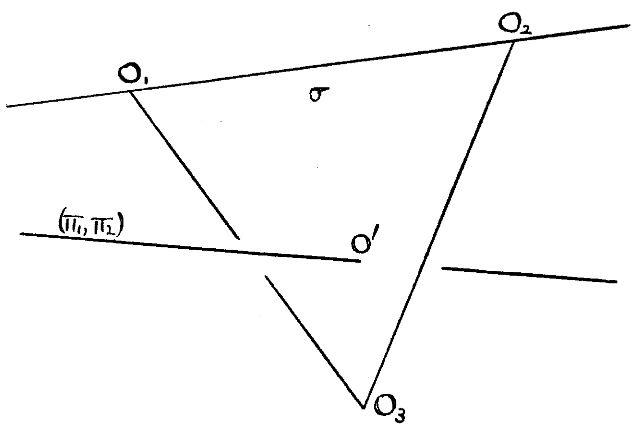
and if Q is a point of Π_1 and $\underline{A}(Q) = P$ then P lies in Π_i, so that

$$\begin{aligned} \underline{A}\,\underline{O}_1\underline{A}^{-1}(P) &= \underline{A}\,\underline{O}_1(Q) \\ &= \underline{A}(Q) \\ &= P \qquad \text{for all points of } \Pi_i . \end{aligned}$$

Thus $\underline{O}_i$ is a homology, centre O_i and polar plane Π_i, and clearly $\underline{O}_i$ has the same period as $\underline{O}$. Hence all the points O_i, $i = 1,2,\ldots,j$, are centres of homologies of period ν.

If $\nu \geqslant 5$ then the two homologies $\underline{O}_1$, $\underline{O}_2$, which both leave fixed the line O_1O_2, induce on that line one of the groups obtained in § II. The only finite groups on a line which are generated by two operations of period greater than 5 are cyclic groups, and O_2 must therefore lie in Π_1 and vice versa. Similarly it can be shewn that each O_i,-i = 1,2,....,j, lies in the polar plane of each of the other homologies and thus there can be at most 4 distinct points O_i, and $\mathcal{G}$ is not primitive.

Now let us consider $\nu = 5$. It may be assumed that not all the transforms of O_1 lie in Π_1, or else $\mathcal{G}$ is not primitive as before. Suppose O_2 is a centre not lying in Π_1, then $\underline{O}_1$ and $\underline{O}_2$ generate an R_{60} on the line O_1O_2. $\underline{O}_1$ and $\underline{O}_2$ both leave invariant each point of the line (Π_1,Π_2), which is the line of intersection of the two planes Π_1 and Π_2; therefore the space group generated by $\underline{O}_1$ and $\underline{O}_2$ leaves fixed every plane passing through O_1O_2, since every such plane contains a point of (Π_1, Π_2). If all the remaining transforms of O_1 lie on either O_1O_2 or (Π_1, Π_2) $\mathcal{G}$ cannot be primitive for it has a pair of invariant lines; let us suppose, if possible, that O_3 does not lie on either of these lines and consider the plane σ, containing O_1, O_2 and O_3 and meeting (Π_1, Π_2) in a point O'. $\underline{O}_1$, $\underline{O}_2$ and $\underline{O}_3$ all leave σ fixed and



must therefore generate a plane group in it. This plane group will contain an intransitive subgroup, having

O' as invariant point and O_1O_2 as an invariant line, which is generated by $\underline{O}_1$ and $\underline{O}_2$. But it may be seen from §III that there is no plane primitive or imprimitive group having such a subgroup and the only intransitive group is one leaving fixed just one point and a line. The only point and line in σ left fixed by $\underline{O}_1$ and $\underline{O}_2$ are O' and O_1O_2 so these must also be left fixed by $\underline{O}_3$. Thus either O_3 coincides with O' or else lies on O_1O_2. Both these alternatives are contrary to our initial assumption and it is therefore impossible to find a transform of O_1 not lying on either O_1O_2 or (Π_1, Π_2) if $\nu = 5$; hence in this case also, $\mathcal{G}$ is not primitive.

If $\nu = 4$, then by the same reasoning as that used for $\nu = 5$, $\mathcal{G}$ can be shewn to be imprimitive or intransitive.

On the other hand there are primitive groups of finite order containing homologies of period 2 or 3; for suppose $\underline{O}_1$ is of period 3. We may discard the case in which for $i = 1, 2, \ldots j$ O_i lies in Π_1 and also the case in which $\underline{O}_1$ and $\underline{O}_i$ generate an $\mathcal{R}_{60}$ on O_1O_i for all i, for both these assumptions lead to a non-primitive group $\mathcal{G}$. Suppose therefore that $\underline{O}_1$ and $\underline{O}_2$ generate an $\mathcal{R}_{12}$ on O_1O_2. Now a space group which generates a $\mathcal{C}_2^2$ on one line and leaves fixed each point of a second line which is skew to the first is of order $4 \times 2 = 8$. (This may be illustrated by considering the two operations $\underline{S}:(x, -y, z, t)$ and $\underline{T}:(y, x, z, t)$ which leave fixed each point of $x = 0 = y$ and generate an $\mathcal{C}_2^2$ on $z = 0 = t$. The operation

$\underline{S}\underline{T}$:(y,-x,z,t) is of period 2 on $z = 0 = t$ but period 4 in space and the group in S_3 generated by $\underline{S}$ and $\underline{T}$ contains a biaxial homography (-x,-y,z,t) and is of order 8) Thus any space group generating a group of order n which contains a $\mathcal{C}_2^2$ on one line, at the same time leaving each point of a second line fixed, will be of order 2n. $\underline{O}_1$ and $\underline{O}_2$ leave fixed every point of (Π_1, Π_2) and induce an R_{12} which contains a $\mathcal{C}_2^2$ on O_1O_2 and thus the group generated in S_3 by $\underline{O}_1$ and $\underline{O}_2$ will be of order 24. It contains the identity, 4 homologies of period 3 and their squares, 8 axial homographies of period 6 (an axial homography has 2 isolated united points and a line of united points), 6 axial homographies of period 4 and one involutary biaxial homography having O_1O_2 and (Π_1, Π_2) as axes.

If $\mathcal{G}$ is primitive then there must be at least one transform of O_1 not laying on either O_1O_2 or (Π_1, Π_2); if O_3 is such a point then Π_3 does not contain either O_1O_2 or (Π_1, Π_2). $\underline{O}_1$, $\underline{O}_2$ and $\underline{O}_3$ all leave fixed the plane σ which contains O_1, O_2 and O_3 and meets (Π_1, Π_2) in a point O' say. Now the only non-primitive plane groups containing the group generated in σ by $\underline{O}_1$ and $\underline{O}_2$ as a subgroup are those leaving fixed O_1, O_2 and O', but the group generated by $\underline{O}_1$, $\underline{O}_2$ and $\underline{O}_3$ in σ does not have these three points as invariant points and so must be primitive. This group contains plane homologies of period 3 and therefore can only be Γ_{216} .

Then the space group generated by $\underline{O}_1$, $\underline{O}_2$ and $\underline{O}_3$ contains a homology of period 3, centre $O_4 = (\Pi_1, \Pi_2, \Pi_3)$ and polar plane σ and is of order 648. (Choose coordinates so that σ is the plane $t = 0$, O_4 the point $(0,0,0,1)$ and three of the homologies of period 3 whose centres lie in σ are given by

$\underline{O}_1$: $(\varpi x, y, z, t)$

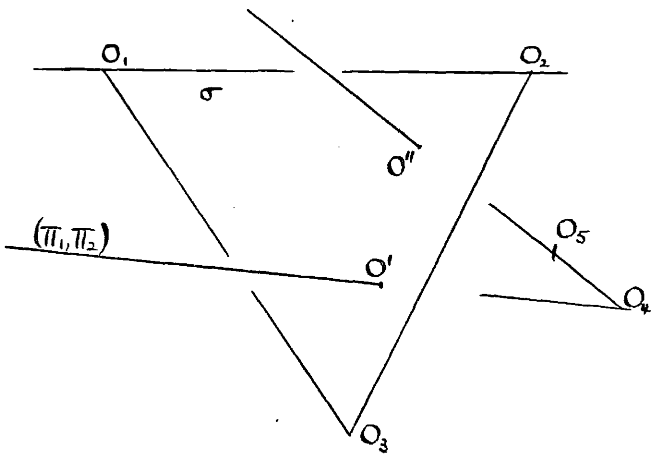
$\underline{O}_2$: $(\varpi^2 x + y + z, x + \varpi^2 y + z, x + y + \varpi^2 z, -\varpi(\varpi^2 + 1)t)$

$\underline{A}$: $(x, \varpi y, z, t)$

(The coordinate system is taken from Hamill (10) p.39)

Then $\underline{O}_1\underline{A}\,\underline{O}_2$: $(\varpi x + \varpi^2 y + \varpi^2 z, \varpi x + y + \varpi z, x + y + \varpi^2 z, -\varpi(2\varpi^2 + 1)t)$ is such that $(\underline{O}_1\underline{A}\,\underline{O}_2)^3$ is the operation $(\varpi x, \varpi y, \varpi z, t)$ so that $\underline{O}_1\underline{A}\,\underline{O}_2$ is of period 3 in σ and 9 in S_3, and its cube is a homology, centre O_4 and polar plane σ, and period 3. It follows that the order of the group generated in S_3 by $\underline{O}_1$, $\underline{O}_2$ and $\underline{A}$ is $3 \times 216 = 648$.)

Now for the group $\mathcal{G}$ to be primitive it must contain at least one homology which does not have O_4 as a fixed point. In addition $\mathcal{G}$ cannot contain more than the given 648 operations which leave O_4 fixed, for suppose $\underline{B}$ does not belong to the group generated by $\underline{O}_1$, $\underline{O}_2$ and $\underline{O}_3$, but $\underline{B}$ leaves O_4 fixed. Then $\underline{B}$ will also leave σ fixed (for $\mathcal{G}$ is a finite group) and as the Γ_{216} generated in σ is not contained in a group of higher order, $\underline{B}$ must be a homology centre O_4 and polar plane σ. If $\underline{B}$ has period greater than 3 then $\mathcal{G}$ is immediately not primitive and if $\underline{B}$ is of period 2, the operation $\underline{O_4 B}$, which belongs to $\mathcal{G}$, is of period 6, and again $\mathcal{G}$ is either intransitive or imprimitive. We may therefore suppose that $\underline{O}_5$ is a homology with polar plane Π_5 and $\underline{O}_5(O_4) \neq O_4$. Then $\underline{O}_4$ and $\underline{O}_5$ generate an R_{12} on O_4O_5 and a group in S_3 which leaves fixed every point of (σ, Π_5); this group therefore contains an involutary biaxial homography with axes (σ, Π_5) and O_4O_5. This biaxial homography clearly leaves O_4 and the plane σ fixed and so must be one of the 648 operations generated by $\underline{O}_1, \underline{O}_2$ and $\underline{O}_3$; hence the plane homology of period 2 whose centre and polar line are



$O'' = (O_4O_5, \sigma)$ and (σ, Π_5) respectively must be one of the 9 involutary homologies contained in Γ_{216}. There are therefore only 9 possible positions for O'' and 9 lines passing through O_4 like O_4O_5. Each of these 9 lines will contain 3 centres of homologies of period 3 other than O_4 and none of these 27 points will lie in σ. Thus we have a total of 40 homologies of period 3 which generate $\mathcal{G}_{25920}$, a simple group of order 25920, which apart from its subgroups, is the only primitive group containing homologies of period 3 in S_3 ; it does not contain any homologies of period 2.

Bagnera proceeds in a similar way to obtain the primitive groups containing homologies of period 2, which are the groups $\mathcal{G}_{11520}$ and $\mathcal{G}_{7200}$, of orders 11520 and 7200 respectively, together with certain of their subgroups.

SECTION II.

§I. It is proposed to investigate in this section certain of the thirty primitive groups that occur in S_3 , where the main emphasis will be on those groups which can be generated by biaxial homographies. In §II. we shall determine the groups generated by biaxial homographies whose axes all lie on a given quadric, and in §III. we shall investigate those groups which are simply isomorphic with symmetric groups of degree n. The remaining paragraphs will be concerned with particular groups and their subgroups, such as the group of order 11520, $\mathcal{C}_{11520}$, which leaves fixed the Klein 60_{15} configuration, and the simple group of order 25920, $\mathcal{C}_{25920}$, which leaves fixed a configuration of 40 points and planes.

Before proceeding to the main work of this section, it is necessary to note some general results about the finite groups generated by biaxial homographies.

If <u>A</u> is a biaxial homography of period ν and axes a, a' and under <u>A</u>, the transform of a line x is a line x', then if x is skew to both a and a so is x', and the four lines a, a', x, x' are all generators of one system on a quadric.

If <u>B</u> is a second biaxial homography, with axes b and b', then if a meets b, a' meets b' and if a is skew to b then a' is skew to b'. For suppose a meets b in a point X but a' is skew to b', then if x is the transversal from X to a' and b', the operations <u>A</u> and <u>B</u> generate a group on x

containing two operations having just one common united point; this line group cannot therefore be finite and so the space group generated by $\underline{A}$ and $\underline{B}$ is not finite either. By a similar argument it can be shewn that if a and b are the same line, then a' and b' are also coincident.

From this it can be seen that if $\underline{A}$ and $\underline{B}$ belong to a finite group, either a and a' meet both of b and b', or a meets b and is skew to b' while a' meets b' and is skew to b, or a, a', b, b' are all skew.

If $\underline{A}$ and $\underline{B}$ are both of period 2 and $\underline{A}.\underline{B}$ is also of period 2, then either a, a', b, b' are all skew and lie on a quadric or else a and a' meet both of b and b'. For if $\underline{A}(b) = b_1$ and $\underline{A}(b') = b_1'$ and a, a', b, b' are all skew, then a, a', b, b_1 are generators of one system on a quadric, and a, a', b', b_1' are generators of one system on another quadric; let B be a point of b such that $\underline{A}(B) = B_1$, then

$$\begin{aligned} B_1 = \underline{A}(B) &= \underline{B}.\underline{A}\underline{B}(B) \quad \text{since } (\underline{A}.\underline{B})^2 = e. \\ &= \underline{B}.\underline{A}(B) \\ &= \underline{B}(B_1) \end{aligned}$$

so that B_1 is a point of either b or b'. However, $\underline{A}$ does not leave any of the points of b fixed, as b is skew to a and a', so that $b_1 = b'$ and the lines a, a', b, b' lie on a quadric. On the other hand, if a meets b then we know that a' meets b'; let the points (a,b), (a',b') be Y, Y'. Then as before, if $\underline{A}(b) = b_1$, b_1 is coincident with either b or b';

as $\underline{A}$ leaves Y fixed and Y lies on b, b_1 and b must be coincident. But this implies that b also meets a', and so b' meets a, and hence if $\underline{A}.\underline{B}$ is of period 2 and a, a', b, b' are not skew, then a, a' meet both of b, b'.

Finally there is a theorem restricting the period ν of a biaxial homography contained in a primitive group. (Bagnera (2)) No primitive group in S_3 can contain a biaxial homography of period ν, if ν is greater than five. For suppose $\underline{A}$ is a biaxial homography of period $\nu > 5$, with axes a, a', contained in a group $\mathcal{G}$. If $\mathcal{G}$ is to be primitive there must be at least one operation, $\underline{S}$ say, that does not leave the pair of lines a, a' fixed. If $\underline{S}(a) = b$ and $\underline{S}(a') = b'$ then $\underline{B} = \underline{S}.\underline{A}.\underline{S}^{-1}$ is a biaxial homography of period ν and axes b, b'; for if B is a point of b and $\underline{S}(A) = B$, A is a point of a, and we see that

$$\begin{aligned} \underline{B}(B) &= \underline{S}.\underline{A}.\underline{S}^{-1}(B) \\ &= \underline{S}.\underline{A}(A) \\ &= \underline{S}(A) \\ &= B \end{aligned}$$

and similarly, $\underline{B}$ leaves each point of b' fixed, and $\underline{B}$ is thus a biaxial homography of period ν on b, b' as axes. If a, a', b, b' are all skew, there are at least two transversals to these four lines, and the group cut on either of the transversals by $\underline{A}$ and $\underline{B}$ contains two operations of period $\nu > 5$ with distinct united points; this group is therefore not finite and the space group generated by $\underline{A}$ and $\underline{B}$ cannot be finite. Since $\underline{S}$ does not leave a, a' fixed we may assume that a and b are not the same line; also as a, a', b, b' cannot all be skew, let us suppose that a meets b in a point Y, then a' and b' will meet in a point, Y' say. If a is skew to b', $\underline{A}$ and $\underline{B}$ cut a group on the line of intersection of the two planes (a,b), (a',b') which contains two operations of period $\nu > 5$ with distinct united points, and so a must meet b' in a point Y_1 say, and similarly a' meets b in a point Y_1'. If $\underline{T}$ is another operation in $\mathcal{G}$ such that $\underline{T}(a) = c$ and $\underline{T}(a') = c'$, the operation $\underline{C} = \underline{T}.\underline{A}.\underline{T}^{-1}$ will be a biaxial homography with axes c, c' and period ν, and by the same argument as that used above, c, c' will meet both of a, a'. But

$$\underline{T}.\underline{A}.\underline{T}^{-1} = \underline{T}.\underline{S}^{-1}.\underline{S}.\underline{A}.\underline{S}^{-1}.\underline{S}.\underline{T}^{-1}$$
$$= (\underline{T}.\underline{S}^{-1}).\underline{B}.(\underline{T}.\underline{S}^{-1})^{-1}$$

and so c, c' also meet b, b'; that is, a, a', b, b', c, c' are the six edges of a tetrahedron, and b, b', c, c' are the only possible transforms of a, a' by means of the operations of $\mathcal{G}$. $\mathcal{G}$ thus leaves fixed a tetrahedron and cannot be primitive, if it contains a biaxial homography of period $\nu > 5$.

§II. Groups Generated by Biaxial Homographies whose Axes all Lie on a Quadric.

Since we know that if $\underline{A}$ is a biaxial homography with axes a, a' and x is a line skew to a, a', then $a, a', x, \underline{A}(x)$ all lie on a quadric, it is natural to investigate the primitive groups which can be generated by biaxial homographies whose axes all lie on the same quadric.

If $\underline{A}$ is a biaxial homography whose axes a, a' lie on a quadric Q then $\underline{A}$ leaves fixed each line meeting a and a'; that is to say that if a, a' belong to the λ system of generators on Q, $\underline{A}$ leaves fixed each line of the μ ~~systems~~ system of generators. Thus any group which is generated by a set of biaxial homographies whose axes are all λ-generators on Q must be intransitive.

If $\underline{B}$ is a second biaxial homography whose axes b, b' lie on Q and are skew to a, a', then the product $\underline{A}.\underline{B}$ is also a biaxial homography, for $\underline{A}.\underline{B}$ has two fixed points on every μ-generator. $\underline{A}, \underline{B}$ and $\underline{A}.\underline{B}$ cannot therefore be of period greater than five. Thus the only possibility for the group generated on the μ-generators by the homographies whose axes are λ-generators is that it is one of the eleven groups R_{12}, R_{24}, R_{60}, $\mathcal{C}_n$, $\mathcal{C}_n^2$ where n = 2,3,4 or 5. In fact we see that if the group, $\mathcal{G}_\lambda$, generated by the λ-homographies (that is, the homographies whose axes are λ-generators on Q) is contained in a group $\mathcal{G}$, where $\mathcal{G}$ is generated by biaxial homographies leaving Q fixed and $\mathcal{G}$ is primitive, then $\mathcal{G}_\lambda$ cannot cut a $\mathcal{C}_n$ or a $\mathcal{C}_n^2$ on the μ-generators. For suppose $\mathcal{G}_\lambda$ cuts a $\mathcal{C}_n$ on each μ-generator, then there will be two points on each μ-generator which are invariant for the group $\mathcal{G}_\lambda$; hence $\mathcal{G}_\lambda$ leaves fixed each of two ~~μ~~ λ-generators, and these two lines are also left fixed by $\mathcal{G}_\mu$, the group generated by the μ-homographies. But $\mathcal{G}$ is generated by the λ- and μ-~~generators~~ homographies, and so $\mathcal{G}$ leaves these two λ-generators fixed and cannot be primitive. Similarly, it may be proved that $\mathcal{G}$ is imprimitive if $\mathcal{G}_\lambda$ cuts a $\mathcal{C}_n^2$ on every μ-generator.

If $\underline{C}$ is a biaxial homography whose axes c, c' belong to the μ-system on Q, c, c' both meet a and a'. By taking the four points of intersection of a, a', c, c' as the vertices of the tetrahedron of reference, it may be seen that

$$\underline{A}.\underline{C} = \underline{C}.\underline{A}$$

regardless of the periods of $\underline{A}$ and $\underline{C}$. Thus $\mathcal{G}$ is simply the direct product of $\mathcal{G}_\lambda$ and $\mathcal{G}_\mu$ and we have the following possibilities for $\mathcal{G}$:

$\mathcal{G}_{3600}$: where $\mathcal{G}_\lambda$ cuts an $\mathcal{R}_{60}$ on the μ-generators and $\mathcal{G}_\mu$ cuts an $\mathcal{R}_{60}$ on the λ-generators.

$\mathcal{G}_{1440}$: $\mathcal{G}_\lambda$ cuts an $\mathcal{R}_{60}$ on the μ-generators while $\mathcal{G}_\mu$ cuts an $\mathcal{R}_{24}$ on the λ-generators.

$\mathcal{G}_{720}$: $\mathcal{G}_\lambda$ cuts an $\mathcal{R}_{60}$ on the μ-generators while $\mathcal{G}_\mu$ cuts an $\mathcal{R}_{12}$ on the λ-generators.

$\mathcal{C}_{576}$: $\mathcal{C}_{1\lambda}$ cuts an R_{24} on the μ-generators while $\mathcal{C}_{1\mu}$ cuts an R_{24} on the λ-generators.

$\mathcal{C}_{288}$: $\mathcal{C}_{1\lambda}$ cuts an R_{24} on the μ-generators while $\mathcal{C}_{1\mu}$ cuts an R_{12} on the λ -generators.

$\mathcal{C}_{144}$: $\mathcal{C}_{1\lambda}$ cuts an R_{12} on the μ-generators while $\mathcal{C}_{1\mu}$ cuts an R_{12} on the λ-generators.

$\mathcal{C}_{3600}$ contains $\mathcal{C}_{720}$ and $\mathcal{C}_{144}$ as subgroups but not $\mathcal{C}_{1440}$; the two groups $\mathcal{C}_{288}$ and $\mathcal{C}_{144}$ occur as subgroups in both $\mathcal{C}_{1440}$ and $\mathcal{C}_{576}$.

Suppose the vertices of the tetrahedron of reference are taken as the four points of intersection of a, a', c, c', then $\underline{A}$ and $\underline{C}$ will assume the forms $\underline{A}:(\alpha x, \alpha y, z, t)$

$\underline{C}:(x, \gamma y, \gamma z, t)$

where α and γ are some roots of unity and $\alpha, \gamma \neq 1$.

The product of $\underline{A}$ with $\underline{C}$ is given by

$$\underline{A}.\underline{C}:(\alpha x, \alpha\gamma y, \gamma z, t),$$

and it follows that

1. $\underline{A}.\underline{C}$ cannot possibly be a homology.

2. $\underline{A}.\underline{C}$ is a biaxial homography if and only if $\alpha = \gamma$ and $\alpha\gamma = 1$.

3. $\underline{A}.\underline{C}$ is an axial homography either if $\alpha = \gamma$ and $\alpha\gamma \neq 1$ or if $\alpha \neq \gamma$ and $\alpha\gamma = 1$.

Otherwise $\underline{A}.\underline{C}$ is a general collineation with just four distinct united points. From 2. we see that if $\underline{A}.\underline{C}$ is a biaxial homography then both $\underline{A}$ and $\underline{C}$ are of period 2.

We are now in a position to write down all the operations of the groups $\mathcal{C}_{13600}$, $\mathcal{C}_{1440}$, $\mathcal{C}_{1720}$, $\mathcal{C}_{1576}$, $\mathcal{C}_{1288}$, $\mathcal{C}_{1144}$ and tables will be found on the next page. Since $\mathcal{C}_{1\lambda}$ cuts either an $\mathcal{R}_{60}$ or an $\mathcal{R}_{24}$ or an $\mathcal{R}_{12}$ on every μ-generator, it is isomorphic with $\mathcal{A}_5$, $\mathcal{S}_4$ or $\mathcal{A}_4$, and similarly so is $\mathcal{C}_{1\mu}$; it is therefore convenient to denote any operation of $\mathcal{C}_{1\lambda}$ by λ(ij....) where (ij...) is the cycle to which the operation corresponds under the isomorphism, while the operations of $\mathcal{C}_{1\mu}$ are denoted by μ(ij....).

In the tables, p is the period of the operation and under the column headed $\mathcal{C}_{1n}$ appears the number of that type of operation occurring in $\mathcal{C}_{1n}$.

Footnote. These six groups were determined by different methods by Goursat (9) as some of the groups leaving fixed the quadric xt - yz = 0; the same set of groups were again investigated by Bagnera (1) using yet another method.

The Operations of $\mathcal{G}_{1576}$, $\mathcal{G}_{1288}$ and $\mathcal{G}_{1144}$.

p.	Description of the Operation.		$\mathcal{G}_{1576}$	$\mathcal{G}_{1288}$	$\mathcal{G}_{1144}$
1.	Identity		1	1	1
2.	Biaxial homography.	$\lambda(ij)$ or $\mu(ij)$	12	6	-
3.	Biaxial homography.	$\lambda(ijk)$ or $\mu(ijk)$	16	16	16
2.	Biaxial homography.	$\lambda(ij)(kl)$ or $\mu(ij)(kl)$	6	6	6
4.	Biaxial homography.	$\lambda(ijkl)$ or $\mu(ijkl)$	12	6	-
2.	Biaxial homography.	$\lambda(ij) \times \mu(ij)$	36	-	-
6.		$\lambda(ij) \times \mu(ijk)$	96	48	-
2.	Biaxial homography.	$\lambda(ij) \times \mu(ij)(kl)$.	36	18	-
4.		$\lambda(ij) \times \mu(ijkl)$	72	-	-
3.	Axial homography.	$\lambda(ijk) \times \mu(ijk)$	64	64	64
6.		$\lambda(ijk) \times \mu(ij)(kl)$	48	48	48
12.		$\lambda(ijk) \times \mu(ijkl)$	96	48	-
2.	Biaxial homography.	$\lambda(ij)(kl) \times \mu(ij)(kl)$	9	9	9
4.		$\lambda(ij)(kl) \times \mu(ijkl)$	36	18	-
4.	Axial homography.	$\lambda(ijkl) \times \mu(ijkl)$	36	-	-

The Operations of $\mathcal{C}_{3600}$ and $\mathcal{C}_{720}$.

p.	Description of the Operation.		$\mathcal{C}_{3600}$	$\mathcal{C}_{720}$
1.	Identity		1	1
2.	Biaxial homography.	$\lambda(ij)(kl)$ or $\mu(ij)(kl)$	30	18
3.	Biaxial homography.	$\lambda(ijk)$ or $\mu(ijk)$	40	28
5.	Biaxial homography.	$\lambda(ijklm)$ or $\mu(ijklm)$	48	24
2.	Biaxial homography.	$\lambda(ij)(kl) \times \mu(ij)(kl)$	225	45
6.		$\lambda(ij)(kl) \times \mu(ijk)$	600	180
10.		$\lambda(ij)(kl) \times \mu(ijklm)$	720	72
3.	Axial homography.	$\lambda(ijk) \times \mu(ijk)$	400	160
15.		$\lambda(ijk) \times \mu(ijklm)$	960	192
5.	Axial homography.	$\lambda(ijklm) \times \mu(ijklm)$	576	-

The Operations of G_{1440}.

p.	Description of the Operation.		G_{1440}
1.	Identity.		1
2.	Biaxial homography.	$\lambda(ij)$	6
2.	Biaxial homography.	$\lambda(ij)(kl)$	3
3.	Biaxial homography.	$\lambda(ijk)$	8
4.	Biaxial homography.	$\lambda(ijkl)$	6
2.	Biaxial homography.	$\mu(ij)(kl)$	15
3.	Biaxial homography.	$\mu(ijk)$	20
5.	Biaxial homography.	$\mu(ijklm)$	24
2.	Biaxial homography.	$\lambda(ij) \times \mu(ij)(kl)$	90
6.		$\lambda(ij) \times \mu(ijk)$	120
10.		$\lambda(ij) \times \mu(ijklm)$	144
2.	Biaxial homography.	$\lambda(ij)(kl) \times \mu(ij)(kl)$	45
6.		$\lambda(ij)(kl) \times \mu(ijk)$	60
10.		$\lambda(ij)(kl) \times \mu(ijklm)$	72
6.		$\lambda(ijk) \times \mu(ij)(kl)$	120
3.	Axial homography.	$\lambda(ijk) \times \mu(ijk)$	160
15.		$\lambda(ijk) \times \mu(ijklm)$	192
4.		$\lambda(ijkl) \times \mu(ij)(kl)$	90
12.		$\lambda(ijkl) \times \mu(ijk)$	120
20.	Axial homography.	$\lambda(ijkl) \times \mu(ijklm)$	144

§III. The Collineation Groups Isomorphic with Symmetric Groups.

The symmetric group $\mathfrak{S}_n$ contains a set of $\frac{1}{2}n(n - 1)$ cycles of period 2, such as (ij) where i and j are any two of the numbers $1,2,\ldots,n$, $i \neq j$ and $(ij) = (ji)$. The subset $(1n),(2n),\ldots,((n-1)n)$ is sufficient to generate $\mathfrak{S}_n$ and the following relations exist:

1. $(ij)(ik) = (ijk)$
2. $(ij)(ik)(ij) = (jk)$
3. $(ij)(kl) = (kl)(ij)$

where i,j,k,l are any four different numbers taken from $1,2,\ldots,n$.

Suppose now that there is an isomorphism H between a collineation group $\mathfrak{G}$ in S_3 and the symmetric group $\mathfrak{S}_n$, in which a set of $(n - 1)$ biaxial homographies $\underline{A}_1,\underline{A}_2,\ldots,\underline{A}_{n-1}$ of period 2 correspond to $(1n),(2n),\ldots,((n-1)n)$; then the $\underline{A}_i$ will generate $\mathfrak{G}$ and will be such that $\underline{A}_i\underline{A}_j$ is of period 3 for $i \neq j$ and $i,j = 1,2,\ldots,n - 1$, and $\underline{A}_i\underline{A}_j\underline{A}_k\underline{A}_j$ is of period 2 for i,j,k all different and $i,j,k = 1,2,\ldots,n - 1$.

There are just three possibilities for the relations between the axes a_i, a'_i of $\underline{A}_i$ and a_j, a'_j of $\underline{A}_j$.

1. a_i and a'_i meet both of a_j, a'_j.
2. a_i meets a_j and is skew to a'_j, while a'_i meets a'_j and is skew to a_j.
3. a_i, a'_i, a_j, a'_j are all skew.

Case 1. may be immediately discarded, for if a_i, a_i' meet both of a_j, a_j' then $\underline{A}_i.\underline{A}_j$ is of period 2, not period 3.

In both 2. and 3. $\underline{A}_i.\underline{A}_j.\underline{A}_i$ is another involutary biaxial homography, for suppose

$$\underline{A}_i(a_j) = x \text{ and } \underline{A}_i(a_j') = x',$$

then if X_j is a point of a_j and $\underline{A}_i(X_j) = X$, X is a point of x; then we have that

$$\begin{aligned} \underline{A}_i.\underline{A}_j.\underline{A}_i(X) &= \underline{A}_i\underline{A}_j(X_j) \\ &= \underline{A}_i(X_j) \\ &= X \end{aligned}$$

and similarly, $\underline{A}_i.\underline{A}_j.\underline{A}_i$ leaves each point of x' fixed. Thus $\underline{A}_i.\underline{A}_j.\underline{A}_i$ is an involutary biaxial homography with axes x, x'. $\underline{A}_i.\underline{A}_j.\underline{A}_i$ will be denoted by $\underline{A}_{ij}$ and its axes by a_{ij}, a_{ij}'.

If a_i, a_i', a_j and a_j' are all skew and

$$\underline{A}_i(a_j) = b, \qquad \underline{A}_i(a_j') = b'$$

then a_i, a_i', a_j, b are generators of the same system on one quadric, while a_i, a_i', a_j' and b' are generators of the same system on another quadric. Suppose B is a point of b such that

$$\underline{A}_j(B) = C,$$

then since $(\underline{A}_i.\underline{A}_j)^3 = e$

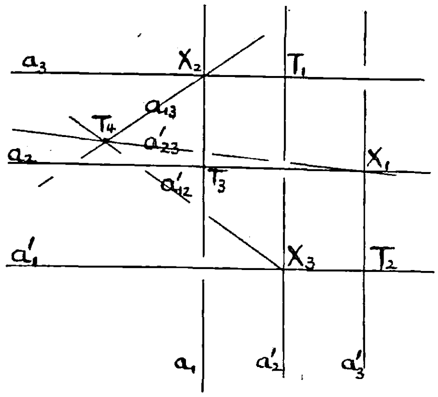
$$\begin{aligned} C = \underline{A}_j(B) &= \underline{A}_i.\underline{A}_j.\underline{A}_i.\underline{A}_j.\underline{A}_i(B) \\ &= \underline{A}_i.\underline{A}_j.\underline{A}_i.\underline{A}_j(A) \text{ (where A is a point of } a_j) \\ &= \underline{A}_i.\underline{A}_j.\underline{A}_i(A) \\ &= \underline{A}_i.\underline{A}_j.(B) \\ &= \underline{A}_i(C) \end{aligned}$$

and C is a point of either a_i or a_i' ; that is to say that either a_j, a_j', a_i, b or a_j, a_j', a_i', b are all generators of one system on a quadric; either supposition leads to the conclusion that $a_i, a_i', a_j, a_j', b, b'$ all lie on a quadric. Hence if the axes of two biaxial homographies of period 2 are skew, and their product is of period 3, the axes all lie on a quadric.

Let us consider 2. and 3. separately.

1. Let a_1 meet a_2 and be skew to a_2' , while a_1' meets a_2' and is skew to a_2. If the biaxial homographies $\underline{A}_3, \underline{A}_4, \ldots, \underline{A}_{n-1}$ are such that one axis of each meets both a_1 and a_2 then the group $\mathcal{G}$ cannot be primitive. We may therefore assume that the $\underline{A}_i$ (i = 3,4,...,n - 1) have been chosen so that if a_i meets a_1 it also meets a_2' but is skew to a_1' and a_2; it follows that if a_i meets a_j and a_k, then a_j meets a_k' and is skew to a_k, for i,j,k all different and i,j,k = 1,2,....,n - 1. In addition, since $\underline{A}_i.\underline{A}_{12}$ is of period 2 for i = 3,4,...,n - 1, either $a_i, a_i', a_{12}, a_{12}'$ are all generators of one system on a quadric or else a_i, a_i' meet both of a_{12}, a_{12}' ; however, as a_{12}, a_1, a_2 are coplanar, if a_i meets a_1 and a_{12} it also meets a_2, thus the latter alternative cannot hold and $a_i, a_i', a_{12}, a_{12}'$ are all skew and lie on a quadric.

The group generated by $\underline{A}_1, \underline{A}_2$ and $\underline{A}_3$ is isomorphic with $\mathfrak{S}_4$ but is not primitive. For suppose the points



$X_1, X_2, X_3, T_1, T_2, T_3$ are the points of intersection of a_1, a'_2, a'_3 with a'_1, a_2 and a_3, (see diagram) then, as X_1 lies on a_2 and a'_3, if $\underline{A}_1(X_1) = X_4$, X_4 must lie on the transforms of a_2 and a'_3 under $\underline{A}_1$, that is to say that a_{12} and a'_{13} meet in a point, X_4.

Similarly a'_{12} and a_{13} meet in a point, T_4 say. Now

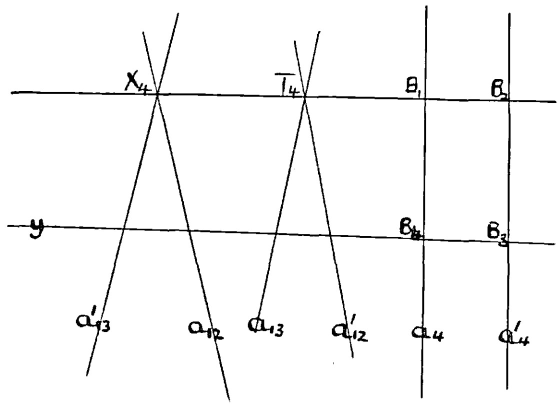
$$\begin{aligned} \underline{A}_{12}.\underline{A}_{13}.\underline{A}_{12} &= \underline{A}_1.\underline{A}_2.\underline{A}_1.\underline{A}_1.\underline{A}_3.\underline{A}_1.\underline{A}_1.\underline{A}_2.\underline{A}_1 \\ &= \underline{A}_1.\underline{A}_2.\underline{A}_3.\underline{A}_2.\underline{A}_1 \\ &= \underline{A}_2.\underline{A}_3.\underline{A}_2 \\ &= \underline{A}_{23} \end{aligned}$$

so that a_{23}, a'_{23} pass through X_4 and T_4 respectively, and the two tetrahedra $X_1 X_2 X_3 X_4$ and $T_1 T_2 T_3 T_4$ ~~and~~ are each left fixed by $\underline{A}_1, \underline{A}_2$ and $\underline{A}_3$ and the group isomorphic with $\mathfrak{S}_4$ cannot be primitive.

If $\mathcal{G}$ is to be primitive there must be at least one biaxial homography which does not leave either $X_1 X_2 X_3 X_4$ or $T_1 T_2 T_3 T_4$ fixed, and it may be assumed that none of the $\underline{A}_i$ $(i = 4, 5, \ldots, n-1)$ leave these two tetrahedra fixed.

Now a_i, a'_i $(i = 4, 5, \ldots, n-1)$ are skew to a_{12}, a'_{12} and these four lines all lie on a quadric, similarly $a_i, a'_i, a_{13}, a'_{13}$ all lie on a quadric and there are thus two lines meeting each of $a_i, a'_i, a_{12}, a'_{12}, a_{13}, a'_{13}$; but the only lines meeting

$a_{12}, a'_{12}, a_{13}, a'_{13}$ are X_4T_4 and y, where y is the line of intersection of the two planes (a_{12}, a_{13}) and (a'_{12}, a'_{13}), therefore a_i, a'_i meet X_4T_4 and y (for all $i = 4, 5, \ldots, n-1$). If a_4, a'_4 meet X_4T_4 and y in B_1, B_2, B_3, B_4 (see diagram) then as



a_5 meets X_4T_4, y and a'_4 and is skew to a_4, a_5 must pass through either B_2 or B_3 and similarly a'_5 passes through either B_1 or B_4. If a_5 meets a'_4 in B_2 then a'_5 meets a_4 in B_1, otherwise $\underline{A}_4$ and $\underline{A}_5$ would generate a group on X_4T_4 containing two operations with just one common united point which would therefore not be finite; hence either a_5 and a'_5 pass through B_2 and B_1 respectively, or else they pass through B_3 and B_4 respectively. It can now be seen that it is impossible to choose a line a_6 meeting X_4T_4, y, a'_4 and a'_5 and skew to a_4 and a_5.

Thus n cannot be greater than six, and there are at most five biaxial homographies of period 2, generating a group $\mathcal{G}_{720}$ which is isomorphic with $\mathfrak{S}_6$.

Now suppose a_5 and a'_5 meet a'_4 and a_4 in B_1 and B_2 respectively; then $\underline{A}_{12}, \underline{A}_{13}, \underline{A}_4, \underline{A}_5$ generate a group on y containing two operations of period 6, $\underline{A}_{12}\cdot\underline{A}_{13}\cdot\underline{A}_4$; $\underline{A}_{12}\cdot\underline{A}_{13}\cdot\underline{A}_5$, which have distinct united points. But there is no finite group in S_1 of this type and a_4, a'_5 must therefore meet in B_4.

Since $\underline{A}_4$ does not leave either of the tetrahedra $X_1X_2X_3X_4$, $T_1T_2T_3T_4$ fixed, the groups generated by $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ and $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4, \underline{A}_5$ are both primitive.

We may now enumerate the operations of the primitive group $\mathcal{G}_{720}$, isomorphic with $\mathfrak{S}_6$. We have already

I. The identical collineation.

II. 15 biaxial homographies of period 2, corresponding to the cycles conjugate with (12).

A product such as $\underline{A}_1.\underline{A}_2$ is of period 3 and corresponds to (126) under H; it is an axial homography and we have

III. 40 axial homographies of period 3.

The axes of $\underline{A}_1$ and $\underline{A}_{23}$ lie on a quadric and are all skew; $\underline{A}_1.\underline{A}_{23}$ is of period 2 and corresponds to (16)(23). Thus

IV. 45 biaxial homographies of period 2.

The operation $\underline{A}_1.\underline{A}_2.\underline{A}_3$ corresponds to (1236) and is of period 4. We have

V. 90 operations of period 4 corresponding to the cycles conjugate with (1234).

Another operation which is the product of three biaxial homographies of type II is $\underline{A}_{12}.\underline{A}_3.\underline{A}_4$. Under H this corresponds to (12)(346) and is of period 6. We have

VI 120 operations of period 6 corresponding to the cycles conjugate with (12)(345).

Now the product $\underline{A}_1.\underline{A}_{23}$ is a biaxial homography of period 2 and type IV, so that $\underline{A}_1.\underline{A}_{23}.\underline{A}_{45}$ is also a biaxial homography for it is of period 2. Hence

VII. 15 biaxial homographies of period 2 corresponding to the cycles conjugate with (12)(34)(56).

An operation such as $\underline{A}_1.\underline{A}_2.\underline{A}_3\underline{A}_4$ is of period 5, for $\underline{A}_1.\underline{A}_2.\underline{A}_3.\underline{A}_4$ corresponds to (12346) under H, and we have

VIII. 144 operations of period 5.

A further set of operations of period 4 are those similar to $\underline{A}_{12}.\underline{A}_3.\underline{A}_4\underline{A}_5$, which corresponds to (12)(3456) under H. We have

IX. 90 operations of period 4 corresponding to the cycles conjugate with (12)(3456).

The operation $\underline{A}_{12}.\underline{A}_{13}.\underline{A}_4\underline{A}_5$ corresponds to (123)(456) and is of period 3; we have

X. 40 operations of period 3, whose images under H are the set of cycles conjugate with (123)(456).

The last type of operation in $\mathcal{G}_{1720}$ is the set similar to $\underline{A}_1.\underline{A}_2.\underline{A}_3\underline{A}_4\underline{A}_5$; we have

XI. 120 operations of period 6, corresponding to the cycles conjugate with (123456).

The operations in class X are biaxial homographies; for consider $\underline{A}_4\underline{A}_5\underline{A}_{12}.\underline{A}_{13}$, this may be regarded as the product of two axial homographies of period 3, $\underline{A}_4.\underline{A}_5$ and $\underline{A}_{12}.\underline{A}_{13}$ where the axis and fixed points of $\underline{A}_{12}.\underline{A}_{13}$ are X_4T_4 and B'_3, B'_4, two points of y, while those of $\underline{A}_4.\underline{A}_5$ are y and X'_4, T'_4, two points of X_4T_4. We may choose the tetrahedron $X'_4T'_4B'_3B'_4$ as the tetrahedra of reference, and then $\underline{A}_4.\underline{A}_5$ and $\underline{A}_{12}.\underline{A}_{13}$ take the form

$$\underline{A}_4\underline{A}_5: \quad (x, \varpi y, \varpi^2 z, t)$$

$$\underline{A}_{12}.\underline{A}_{13}: \quad (\varpi x, y, z, \varpi^2 t)$$

$\underline{A}_4.\underline{A}_5.\underline{A}_{12}.\underline{A}_{13}$ is then the operation $(\varpi x, \varpi y, \varpi^2 z, \varpi^2 t)$ and is a biaxial homography of period 3.

The group $\mathcal{G}_{120}$ generated by $\underline{A}_1, \underline{A}_2, \underline{A}_3, \underline{A}_4$ is a primitive subgroup of $\mathcal{G}_{720}$ which is isomorphic with $\mathcal{S}_5$. It contains a subgroup $\mathcal{G}_{60}$, isomorphic with $\mathcal{A}_5$, which is also primitive and which is generated by the 15 biaxial homographies of period 2 in $\mathcal{G}_{120}$ which correspond to the set of cycles conjugate with (12)(34) in $\mathcal{S}_5$; it is easy to show that $\mathcal{G}_{60}$ is primitive, for it contains the operations $\underline{A}_1.\underline{A}_2$, $\underline{A}_1.\underline{A}_3$ and $\underline{A}_1.\underline{A}_4$, of which the first two collineations belong to the group generated by $\underline{A}_1, \underline{A}_2$ and $\underline{A}_3$ and leaving fixed just the two tetrahedra $X_1X_2X_3X_4$ and $T_1T_2T_3T_4$; but $\underline{A}_4$ was chosen so that it does not leave either of these tetrahedra fixed, and so $\underline{A}_1.\underline{A}_4$ does not leave them fixed and the group $\mathcal{G}_{60}$ is therefore primitive.

The group generated by the set of all the biaxial homographies of period 2 and type IV in $\mathcal{G}_{720}$ is similarly primitive; it will be known as $\mathcal{G}_{360}$ and is isomorphic with $\mathcal{A}_6$.

Now it is known that the cycles conjugate with (12)(34)(56) generate $\mathfrak{S}_6$ and so the set of biaxial homographies in type VII generate $\mathcal{G}_{720}$; in fact the five operations $\underline{B}_i$, i = 1,...,5, where

$$\underline{B}_1 = \underline{A}_1.\underline{A}_{23}.\underline{A}_{45}$$
$$\underline{B}_2 = \underline{A}_2.\underline{A}_{14}.\underline{A}_{35}$$
$$\underline{B}_3 = \underline{A}_3.\underline{A}_{15}.\underline{A}_{24}$$
$$\underline{B}_4 = \underline{A}_4.\underline{A}_{13}.\underline{A}_{25}$$
$$\underline{B}_5 = \underline{A}_5.\underline{A}_{12}.\underline{A}_{34}$$

are sufficient to generate $\mathcal{G}_{720}$. The product of any two of these, say $\underline{B}_1.\underline{B}_2$, is an operation of type X and period 3, for

$$\begin{aligned}\underline{B}_1.\underline{B}_2 &= \underline{A}_1.\underline{A}_{23}.\underline{A}_{45}.\underline{A}_2.\underline{A}_{14}.\underline{A}_{35}\\ &= \underline{A}_1.\underline{A}_2.\underline{A}_3.\underline{A}_{45}.\underline{A}_{14}.\underline{A}_{35}\\ &= \underline{A}_{12}.\underline{A}_{13}.\underline{A}_{45}.\underline{A}_4.\underline{A}_{35}.\underline{A}_1\\ &= \underline{A}_{45}.\underline{A}_4.\underline{A}_{12}.\underline{A}_{35}.\underline{A}_{15}.\underline{A}_1\\ &= \underline{A}_4.\underline{A}_5.\underline{A}_{12}.\underline{A}_{35}.\underline{A}_5.\underline{A}_{15}\\ &= \underline{A}_4.\underline{A}_3.\underline{A}_{12}.\underline{A}_{15},\end{aligned}$$

so that the ~~five~~ operation~~s~~ $\underline{B}_i$, i = 1,...,5 is a biaxial homography of period 3. But we already know that if the product of two biaxial homographies, $\underline{A}$ and $\underline{B}$, of period 2 is of period 3, then $\underline{A}.\underline{B}$ is an axial homography if each of the axes of $\underline{A}$ meets one of the axes of $\underline{B}$, and that $\underline{A}.\underline{B}$ is a biaxial homography only if the axes of $\underline{A}$ and $\underline{B}$ are all skew and lying on a quadric. Hence the biaxial homographies

$\underline{B}_i$, i = 1,...,5 are such that the axes of any two are skew and lie on a quadric. In addition

$$
\begin{aligned}
\underline{B}_3 \cdot \underline{B}_1 \cdot \underline{B}_2 \cdot \underline{B}_1 &= \underline{A}_3 \cdot \underline{A}_{15} \cdot \underline{A}_{24} \cdot \underline{A}_1 \cdot \underline{A}_{23} \cdot \underline{A}_{45} \cdot \underline{A}_2 \cdot \underline{A}_{14} \cdot \underline{A}_{35} \cdot \underline{A}_1 \cdot \underline{A}_{23} \cdot \underline{A}_{45} \\
&= \underline{A}_3 \cdot \underline{A}_{15} \cdot \underline{A}_{24} \cdot \underline{A}_4 \underline{A}_3 \cdot \underline{A}_{12} \cdot \underline{A}_{15} \cdot \underline{A}_1 \cdot \underline{A}_{23} \cdot \underline{A}_{45} \\
&= \underline{A}_{24} \cdot \underline{A}_{43} \cdot \underline{A}_{25} \cdot \underline{A}_1 \cdot \underline{A}_{23} \cdot \underline{A}_{45} \\
&= \underline{A}_{43} \cdot \underline{A}_{23} \cdot \underline{A}_{25} \cdot \underline{A}_1 \cdot \underline{A}_{23} \cdot \underline{A}_{45}. \\
&= \underline{A}_{43} \cdot \underline{A}_{35} \cdot \underline{A}_1 \cdot \underline{A}_{45}. \\
&= \underline{A}_1 \cdot \underline{A}_{35}.
\end{aligned}
$$

which is a biaxial homography of period 2, belonging to type IV; so either the axes of $\underline{B}_3$ and $\underline{B}_1 . \underline{B}_2 . \underline{B}_1$ are skew and lie on a quadric or else they have four points of intersection. The latter alternative cannot hold, for by symmetry this gives that the axes of $\underline{B}_4 \underline{B}_5 \underline{B}_4$ meet the axes of both $\underline{B}_3$ and $\underline{B}_1 . \underline{B}_2 . \underline{B}_1$, and so $\underline{B}_3 . \underline{B}_1 . \underline{B}_2 . \underline{B}_1 = \underline{B}_4 . \underline{B}_5 . \underline{B}_4 = \underline{A}_1 . \underline{A}_{24} . \underline{A}_{35}$, which we know is not true.

Hence $\mathcal{C}_{1720}$ is also generated by a set of five biaxial homographies of period 2, which are such that the axes of any two are skew and lie on a quadric, while the axes of the two homographies $\underline{B}_i$, $\underline{B}_j . \underline{B}_k . \underline{B}_j$ (for i,j,k all different, and i,j,k = 1,...,5) are also skew and lie on a quadric. Clearly it is impossible to choose six such operations, $\underline{B}_i$, generating a group $\mathcal{C}_1'$ isomorphic with $\mathcal{S}_7$, for such a group would contain more than five biaxial homographies of the set $\underline{A}_1, \underline{A}_2, \ldots, \underline{A}_{n-1}$, which we have seen cannot occur.

The group $\mathcal{G}'_{60}$ containing the elements of $\mathcal{G}_{1720}$ which correspond to $\mathcal{A}'_5$, where $\mathcal{A}'_5$ is generated by the cycles (16523) and (125)(364), is also primitive; for $\mathcal{G}'_{60}$ contains the operations $\underline{B}_1.\underline{B}_2$, $\underline{B}_1.\underline{B}_3$, $\underline{B}_1.\underline{B}_4$, corresponding to the cycles (125)(364), (134)(265) and (142)(356) respectively; now $\underline{B}_1, \underline{B}_2, \underline{B}_3$ are such that the axes b_1, b_1', b_2, b_2' are all skew and lie on a quadric while b_1, b_1', b_3, b_3' also lie on a quadric, and so the group generated by $\underline{B}_1, \underline{B}_2$ and $\underline{B}_3$ leaves fixed two lines, x and x$'$ say, which meet each of the six axes, and cuts an $\mathcal{R}_{24}$ on each of these lines; but $\underline{B}_4$ cannot leave either of these lines fixed, for if it did, then $\underline{B}_1, \underline{B}_2, \underline{B}_3$ and $\underline{B}_4$ would generate a group on either x or x which, if it were finite, would be isomorphic with $\mathcal{S}_5$, as such a line group does not exist, $\underline{B}_4$ does not leave either x or x$'$ fixed and the space group $\mathcal{G}'_{60}$, containing $\underline{B}_1.\underline{B}_4$ must be primitive.

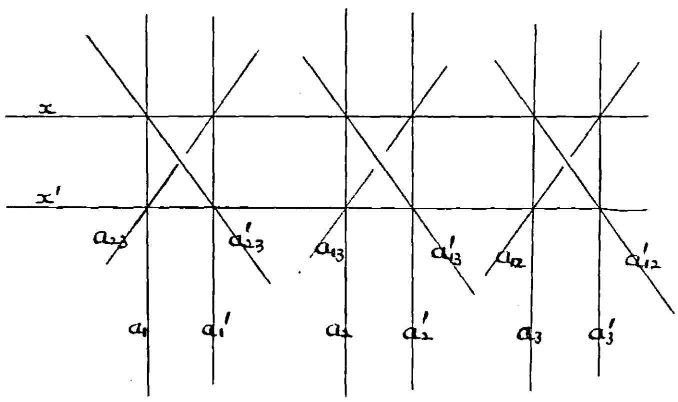
Similarly the space group $\mathcal{G}'_{120}$ generated by $\underline{B}_1, \underline{B}_2, \underline{B}_3$ and $\underline{B}_4$ is primitive and is isomorphic with $\mathcal{S}_5$.

A table of operations for $\mathcal{G}_{1720}$ and its primitive subgroups will be found at the end of this paragraph.

3. We have already investigated the case in which a set of biaxial homographies $\underline{A}_1, \underline{A}_2, \ldots, \underline{A}_{n-1}$, whose axes are all skew, are such that the axes of $\underline{A}_i$ and $\underline{A}_{jk}$ are also skew, and we have seen that the $\underline{A}_i$ generate a primitive group isomorphic with $\mathcal{S}_n$ only if n = 5,6. It remains to investigate the

case in which the axes of $\underline{A}_i$ meet those of $\underline{A}_{jk}$.

Now suppose a_1, a_1' meet a_{23}, a_{23}' ; a_2, a_2' meet a_{13}, a_{13}'; and a_3, a_3' meet a_{12}, a_{12}'. Then as $a_1, a_1', a_2, a_2', a_{12}, a_{12}'$ are all skew and lie on a quadric and $a_1, a_1', a_3, a_3', a_{13}, a_{13}'$ are also skew and lie on a quadric, we have two lines, x, x' say, which meet all of $a_1, a_1', a_2, a_2', a_3, a_3', a_{12}, a_{12}', a_{13}, a_{13}'$. (We are assuming that not all the axes a_i, a_i', $i = 1, \ldots, n-1$ lie on the same quadric.) Since $a_{12}, a_{12}', a_{13}, a_{13}'$ lie on a quadric with a_{23}, a_{23}', x and x' will also meet a_{23}, a_{23}'. Then the only possible configuration for the six pairs of axes and x, x' is that x, x' are the remaining two sides of the three tetrahedra formed by $a_1, a_1', a_{23}, a_{23}'$; $a_2, a_2', a_{13}, a_{13}'$ and $a_3, a_3', a_{12}, a_{12}'$. Thus x and x' are the axes for operations $\underline{A}_1 \cdot \underline{A}_{23}$; $\underline{A}_2 \cdot \underline{A}_{13}$; $\underline{A}_3 \cdot \underline{A}_{12}$; these biaxial homographies are of period 2, and so



$\underline{A}_1 \cdot \underline{A}_{23} = \underline{A}_2 \cdot \underline{A}_{13} = \underline{A}_3 \cdot \underline{A}_{12}$.

This gives that

$$\underline{A}_2 \cdot \underline{A}_1 \cdot \underline{A}_3 \cdot \underline{A}_2 \cdot \underline{A}_3 \cdot \underline{A}_3 \cdot \underline{A}_1 \cdot \underline{A}_3 = e$$

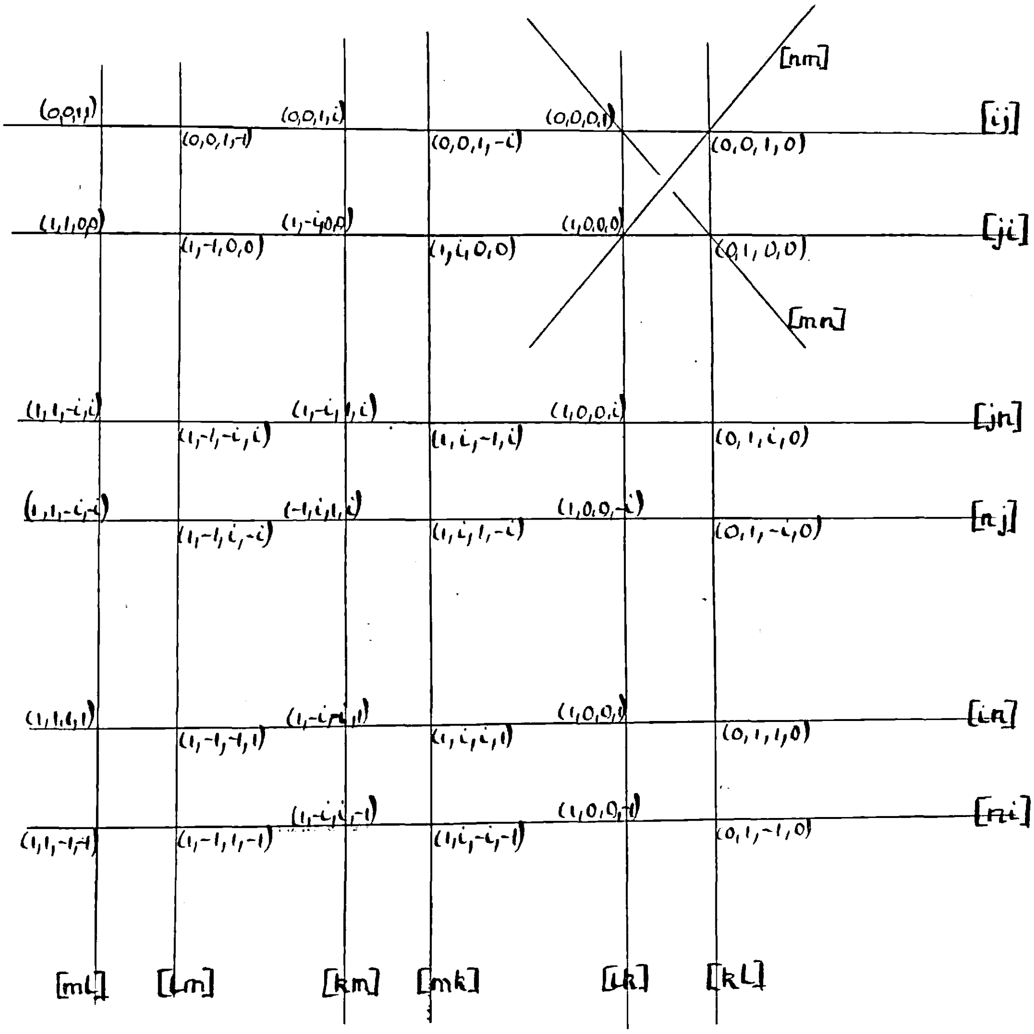
$$\text{or} \qquad \underline{A}_2 \cdot \underline{A}_1 \cdot \underline{A}_3 \cdot \underline{A}_2 \cdot \underline{A}_1 \cdot \underline{A}_3 = e.$$

But this is not so, for $\underline{A}_2 \cdot \underline{A}_1 \cdot \underline{A}_3$ is an operation corresponding to a cycle conjugate with (1234) and is of period 4. Hence the assumption that a_i, a_i' meet a_{jk}, a_{jk}' is not valid.

Thus there are just three primitive collineation groups which are generated by biaxial homographies of period 2 and are isomorphic with a symmetric group S_n.

The Operations of G_{720} and its Primitive Subgroups.

Type	p.	Image under H	G_{720}	G_{360}	G_{120}	G'_{120}	G_{60}	G'_{60}
I	1	(1)	1	1	1	1	1	1
II	2	(ij)	15	-	10	-	-	-
III	3	(ijk)	40	40	20	-	20	-
IV	2	(ij)(kl)	45	45	15	15	15	15
V	4	(ijkl)	90	-	30	30	-	-
VI	6	(ij)(klm)	120	-	20	-	-	-
VII	2	(ij)(kl)(mn)	15	-	-	10	-	-
VIII	5	(ijklm)	144	144	24	24	24	24
IX	4	(ij)(klmn)	90	90	-	-	-	-
X	3	(ijk)(lmn)	40	40	-	20	-	20
XI	6	(ijklmn)	120	-	-	20	-	-



Klein 60_{15} Configuration.

§IV. A Primitive Group Leaving Fixed the Klein 60_{15} Configuration.

The Klein 60_{15} configuration contains a set of fifteen pairs of lines, known as directrices. (A full account of this configuration may be found in Hudson (11) chap. IV). These thirty lines may be denoted by the symbols $[ij]$ where i and j are any two of the numbers 1,....,6 and $[ij] \neq [ji]$; $[ij]$ and $[ji]$ form one of the fifteen pairs and the following relations exist between the lines.

1. $[ij]$ is skew to any line whose symbol contains either i or j, including $[ji]$, and meets each of the remaining twelve lines.

2. $[ij]$, $[ji]$, $[jk]$, $[kj]$, $[ik]$, $[ki]$ are all generators of one system on a quadric, and $[lm]$, $[ml]$, $[mn]$, $[nm]$, $[ln]$, $[nl]$ are generators of the opposite system on the same quadric.

3. If three pairs of lines are generators of the same system on a quadric then they form a set such that any pair harmonically separates each of the other pairs.

4. The four points of intersection of $[ij]$, $[ji]$ with $[kl]$, $[lk]$ form a tetrahedron of which the remaining two edges are $[mn]$ and $[nm]$.

The sixty points determined by the intersections of the directrices will be known as vertices; they form fifteen tetrahedra whose edges are the directrices. Twenty desmic sets may be chosen from the tetrahedra; two of these sets will be said to be associated if the twenty-four edges of one set are also the edges of the second set; for example, if the tetrahedron with edges $[il]$, $[li]$, $[jm]$, $[mj]$, $[kn]$, $[nk]$ is denoted by $\{il, jm, kn\}$, then the three tetrahedra $\{il, jm, kn\}$, $\{in, jl, km\}$, $\{im, jn, kl\}$ form a desmic set, and the associated set is $\{il, km, jn\}$, $\{in, kl, jm\}$, $\{im, kn, jl\}$

A suitable coordinate system is set out on page 59a.

Let us now consider the groups generated by biaxial homographies of period ν, having a pair of directrices as axes.

1. $\underline{\nu = 2}$. The biaxial homography having the pair of directrices $[ij]$, $[ji]$ as axes can be denoted by

$$ij = ji ,$$

$$\text{where} \quad ij^2 = e.$$

Then, since $[ij]$, $[ji]$ harmonically separate $[jk]$ and $[kj]$, $[ik]$ and $[ki]$, we see that

$$ij.jk = ki$$

$$\text{and} \quad ij.kl = mn.$$

Hence the group, $\mathcal{G}_1$, generated by these homographies is simply the group of order 16 consisting of the identity and the given set of involutary biaxial homographies; as each pair of directrices is left fixed by every operation of the group, $\mathcal{G}_1$ cannot be primitive.

2. $\underline{\nu = 3,5}$. Let $\underline{A}$ be the biaxial homography $(x, y, \varpi z, \varpi t)$ $\varpi^3 = 1$, with the pair of lines $x = 0 = y$, $z = 0 = t$ as axes. Then the transforms of the points $(0,1,i,0)$, $(0,1,-i,0)$, $(0,1,1,0)$, $(0,1,-1,0)$ by $\underline{A}$ and $\underline{A}^2$ are $(0,1,\varpi i,0)$, $(0,1,-\varpi i,0)$, $(0,1,\varpi,0)$, $(0,1,-\varpi,0)$, $(0,1,\varpi^2 i,0)$, $(0,1,-\varpi^2 i,0)$, $(0,1,\varpi^2,0)$ and $(0,1,-\varpi^2,0)$ respectively, and so the group generated by those of the set of fifteen biaxial homographies of period 3 which leave the line $x = 0 = t$ fixed, cut a group on that line containing at least ten operations of period 3. Similarly if $\underline{B}$ is the biaxial homography of period 3 having as axes the lines whose general points are $(1,\mu,\mu,1)$ and $(1,\mu,-\mu,1)$, then the group generated by $\underline{A}$ and $\underline{B}$ cuts a group on $x = 0 = t$ containing at least twenty operations of period 3, ten of which have distinct united points. Hence if $\underline{C}$ is the biaxial homography of period 3 having as axes the lines whose general points are $(1,\mu,i\mu,i)$ and $(1,\mu,-i\mu,-i)$, then the group generated on $x = 0 = t$ by $\underline{A}$, $\underline{B}$ and $\underline{C}$ contains at least twenty-two distinct operations of period 3 and cannot therefore be finite. Thus the group generated by the set of fifteen biaxial homographies of period 3 having the pairs of directrices as axes cannot be finite.

In exactly the same way it can be seen that if $\nu = 5$ the group generated by the set of biaxial homographies is not finite.

3. $\underline{\nu = 4}$. The two biaxial homographies of period 4 whose axes are the pair of directrices [ij], [ji] will be denoted by ij and ji, where $ij^3 = ji$ and $ij^2 = ji^2$. If the set of thirty biaxial homographies are chosen in the way indicated on page 88, it can be readily seen that the relations existing between the biaxial homographies are

1a. ij.jk = jk.ki = ki.ij.

1b. ij.ki = ki.kj = kj.ij.

2. ij.kl = kl.ij.

3. $ij^2.jk^2 = jk^2.ij^2 = ik^2$.

4. $ij^2.lk^2 = lk^2.ij^2 = mn^2$.

The rules for determining $\underline{C}$, given that $\underline{A}.\underline{B} = \underline{B}.\underline{C} = \underline{C}.\underline{A}$ (where $\underline{A}.\underline{B}$ is a product such as ij.jk or ij.ki) are that

(a) if the repeated letter in the product $\underline{A}.\underline{B}$ occupies the second and third positions, as in ij.jk, then in both $\underline{B}.\underline{C}$ and $\underline{C}.\underline{A}$ the second and third letters are the same.

(b) if the repeated letter in $\underline{A}.\underline{B}$ occurs in the first and fourth positions as in ij.ki, then in $\underline{B}.\underline{C}$ the first and third letters are the same, and in $\underline{C}.\underline{A}$ the second and fourth letters are the same.

In both (a) and (b) the last position is filled with the remaining letter of the trio i,j,k.

These thirty biaxial homographies of period 4 are contained in the group G_{115520}, the group of maximum order leaving the Klein configuration invariant (Hamill (10) page 58a). Hence the group $\mathcal{G}'$ which they generate is either coincident with $\mathcal{G}_{11520}$ or else is a primitive subgroup of $\mathcal{G}_{11520}$.

The operation ij leaves fixed each point of [ij] and [ji], interchanges the pair of lines [ik], [ki] with the pair [jk], [kj] and leaves fixed every other directrix. Thus the effect of ij on any directrix (other than [ij] and [ji]) is to interchange i and j in the symbol for that directrix, and there is thus a homomorphism, $\mathcal{H}$, between $\mathcal{G}'$ and $\mathfrak{S}_6$, the symmetric group of degree 6, in which ij and ji both map onto the cycle (ij). The order of $\mathcal{G}'$ is therefore some integral multiple of 720.

Before proceeding to enumerate the operations of $\mathcal{G}'$ it must be remarked that each directrix contains six vertices, and that the join of a vertex on [ij] say to any vertex on the associated directrix [ji] is either a directrix or an e-line containing just two vertices, and there are $\frac{60 \times 4}{2} = 120$ e-lines in the configuration. If the join of two vertices is neither a directrix nor an e-line then it is a K-line and contains one further vertex. The K-lines are the perspective lines of the sets of desmic tetrahedra and there are thus $20 \times 16 = 320$ K-lines.

We have already 46 operations of $\mathcal{G}'$:

I The identical operation.

II The 30 biaxial homographies of period 4, whose axes are the pairs of associated directrices.

III The 15 biaxial homographies of period 2 denoted by ij^2, which are the squares of the operations of type II and also have the pairs of directrices as axes.

In the homomorphism, $\mathcal{H}$, each operation ij^2 of type III leaves each pair of directrices fixed and maps onto the identity in $\mathcal{S}_6$; thus the order of $\mathcal{G}'$ must be at least as great as 16 x 720 = 11520, and so $\mathcal{G}'$ is coincident with $\mathcal{G}_{11520}$.

The product of any two biaxial homographies of period 4 whose axes belong to one system of generators on a quadric Q is also a biaxial homography, whose axes belong to the same system. A typical operation is ij.jk, whose axes lie on Q_{ijk} and which is of period 3, for

$$\begin{aligned}(ij.jk)^3 &= ij.jk.ij.jk.ij.jk.\\ &= ij^2.ik.ij.ik.jk.\\ &= ji.kj.ik.jk.\\ &= ji.kj.jk.ij.\\ &= e\end{aligned}$$

Now the axes of lm, mn and nl also lie on Q_{ijk}, and the group cut on these lines by ij,jk,ki must be an $\mathcal{R}_{24}$, since it contains six operations of period 4 which do not all have the same united points; the group generated by ij,jk,ki,lm,mn and nl is therefore a $\mathcal{C}_{576}$ and thus contains 16 biaxial homographies of period 3 whose axes all lie on Q_{ijk} and which are all the same type as ij.jk. Clearly none of these 16 operations can have their axes lying on any of the other nine quadrics like Q_{ijk}, and so we have

IV 160 biaxial homographies of period 3.

There is one further operation which is a product of two biaxial homographies of period 4; a typical example is ij.kl. Since ij and kl commute, ij.kl is of period 4, and we may easily see that such an operation has not arisen before; for from page 88, we may take the forms of ij and kl as ij:(x,y,iz,it) and kl:(x,iy,iz,t), then ij.kl:(x,iy,-z,it) is an axial homography having [mn] as its axis and the two vertices of {ij,kl,mn} not lying on [mn] as its isolated united points.

$$(ij.kl)^3 = ij^3.kl^3 = ji.lk$$

and so the cube of ij.kl is the same type of operation as ij.kl and we have two axial homographies of period 4 associated with [mn] and {ij,kl,mn}. By symmetry there will be two such operations associated with each directrix and any one of the three tetrahedra of which that directrix is an edge. Hence

V 180 axial homographies of period 4.

There are four distinct types of operations which are products of three biaxial homographies of type II; typical operations of these types are

$$ij^2.jk.$$
$$ij^2.kl.$$
$$ij.kl.lm.$$
$$ij.ik.il.$$
$$ij.kl.mn.$$

Any other product of three biaxial homographies of period 4 may be reduced to one of these five types or to one involving fewer factors.

A product such as $ij^2.jk.$ is a biaxial homography belonging to the $\mathcal{R}_{14}$ cut on $[lm]$, $[ml]$ and $[mn]$ by ij, jk, ik. It is of period 2 for

$$\begin{aligned}(ij^2.jk)^2 &= ij^2.jk.ij^2.jk.\\ &= kj.jk.\\ &= e\end{aligned}$$

Under $\mathcal{H}$ this operation maps onto (jk) and so, since the only other operations of period 2 which we have already had map onto the identity, $ij^2.jk$ must be a new operation; in fact it is one of the remaining six operations of period 2 in the $\mathcal{R}_{14}$ generated by ij, jk, ki. There are thus twelve distinct operations of this type arising from any set of biaxial homographies of period 4 whose axes all lie on one of the ten quadrics Q_{ijk} . Clearly as the axes of $ij^2.jk$ lie on Q_{ijk} , they cannot lie on any of the other quadrics and there are

VI 120 biaxial homographies of period 2.

The operation $ij^2.kl$ is of period 4 since

$$(ij^2.kl)^2 = ij^2.kl.ij^2.kl$$
$$= kl^2$$

and kl^2 is of type III and period 2. Under $\mathcal{H}$, $ij^2.kl$ maps onto (kl) and if it has arisen before, it can only be as an operation of type II which maps onto the same cycle; in other words, if $ij^2.kl$ is not a new operation then

either $ij^2.kl = kl$

or $ij^2.kl = lk$.

Either of these suppositions leads to a contradiction and we may therefore assume that $ij^2.kl$ is a new operation. $ij^2.kl$ can only arise (as an operation of this type) as a product of biaxial homographies whose axes pass through the vertices of $\{ij, kl, mn\}$ and it may be easily seen that it arises in just two ways, in fact

$$ij^2.kl = ij^2.kl^2.lk.$$
$$= mn^2.lk.$$

and we have

VII 90 $\left(= \dfrac{15 \times 12}{2}\right)$ operations of period 4.

The next product to consider is $ij.kl.lm$. Now

$$(ij.kl.lm)^3 = ij.kl.lm.ij.kl.lm.ij.kl.lm.$$
$$= ji.(kl.lm)^3.$$
$$= ji.$$

and $(ij.kl.lm)^4 = ji.ij.kl.lm.$

$= kl.lm.$

Thus $ij.kl.lm$ is an operation whose cube is of period 4 and whose fourth power is of period 3, and $ij.kl.lm$ must therefore be of period 12, and so is a new operation. Now we have already seen that ij, jn, ni, kl, lm and mk generate a $\mathcal{G}_{576}$, which we know contains 96 operations of the type $ij.kl.lm$; also the image of $ij.kl.lm$ under $\mathcal{H}$ is $(ij)(kml)$ and so it is clear that $ij.kl.lm$ cannot appear in any of the other nine groups of order 576 which are subgroups of $\mathcal{G}_{11520}$. Hence

VIII 960 operations of period 12, whose cubes are of type II and fourth powers of type IV.

Since the square of an operation like $ij.kl.lm$ is of period 6, it cannot have arisen before.

Now $(ij.kl.lm)^2 = ij^2.kl.lm.kl.lm.$

$= ij^2.mk.kl^2.lm.$

$= mn^2.km.lm.$

and so this new operation of period 6 may be expressed as a product of four biaxial homographies of type II. Its image under $\mathcal{H}$ is (klm) and the only operations of this type having the same image are $mn^2.km.lm$; $mn^2.km.ml$; $mn^2.mk.lm$; $mn^2.mk.ml$, and 32 similar operations derived in the same way from $mi^2.km.lm$; $mj^2.km.lm$; $kn^2.kl.km$; $ki^2.kl.km$; $kj^2.kl.km$; $ln^2.lm.lk$; $li^2.lm.lk$; $lj^2.lm.lk$. From this we see that such an operation arises in just three ways; in fact

$$mn^2.km.lm = kn^2.lk.mk = ln^2.ml.kl,$$

and we have therefore

IX 480 $\left(=\dfrac{15 \times 16 \times 6}{3}\right)$ operations of period 6.

The operation $ij.ik.il$ is a new operation of period 8, for

$$\begin{aligned}(ij.ik.il)^2 &= ij.ik.il.ij.ik.il.\\ &= ij^2.kj.lj.ik.il.\\ &= ij^2.lj.kl.kl.ik.\\ &= mn^2.jl.ik.\\ &= jl^2.ik^2.jl.ik.\\ &= lj.ki.\end{aligned}$$

$lj.ki$ is of type V and period 4 and therefore $ij.ik.il$ is of period 8. Under $\mathcal{H}$, $ij.ik.il$ maps onto the cycle $(ijkl)$ and can only arise as an operation mapping onto the same cycle. Hence $ij.ik.il$ can only arise as one of the operations

$ij.ik.il$; $ij.ik.li$; $ij.ki.il$; $ij.ki.li$;

$ji.ik.il$; $ji.ik.li$; $ji.ki.il$; $ji.ki.li$.

or as one of the twenty-four operations arising in the same way from $jk.jl.ji$; $kl.ki.kj$; $li.lj.lk$. Suppose $\underline{A}$ is an operation of type IV then if

$ij.\underline{A} = ij.ik.il$, $\underline{A} = ik.il$ and if

$ji.\underline{A} = ij.ik.il$, $\underline{A} = ij^2.ik.il$, so that we see that the first eight operations listed above are all distinct; hence there are at most three other operations of this type equal to $ij.ik.i\ell$, one from each of the sets of eight operations deriving from $jk.j\ell.ji$; $k\ell.ki.kj$; $\ell i.\ell j.\ell k$. In fact we find that

$$
\begin{aligned}
1.\quad ij.ik.i\ell &= jk.ij.i\ell. \\
&= jk.j\ell.ij. \\
2.\quad ij.ik.i\ell &= ij.k\ell.ik. \\
&= k\ell.ij.ik. \\
&= k\ell.ik.jk. \\
3.\quad ij.ik.i\ell &= ij.i\ell.k\ell. \\
&= i\ell.j\ell.k\ell.
\end{aligned}
$$

and so each of this type of operation arises in four ways and we have

$$X\ 720 \ \left(= \frac{30 \times 16 \times 6}{4}\right) \text{ operations of period 8.}$$

The last distinct product of three biaxial homographies of period 4 is $ij.k\ell.mn$.

$$
\begin{aligned}
(ij.k\ell.mn)^2 &= ij^2.k\ell^2.mn^2 \\
&= e
\end{aligned}
$$

$ij.k\ell.mn$ is an operation of period 2 whose image under $\mathcal{H}$ is $(ij)(k\ell)(mn)$; hence it is a new operation as the only other operations of period 2 that we have either map onto the identity, or else onto the cycles of the set conjugate to (ij). Taking ij, $k\ell$, mn as the operations (x,y,iz,it), (x,iy,iz,t), (x,iy,z,it) respectively, we find that $ij.k\ell.mn$ is the homology $(x,-y,-z,-t)$, centre $X(1, 0, 0, 0)$

and polar plane $x = 0$. The directrices $[ji]$, $[lk]$ and $[nm]$ all pass through X while $[ij]$, $[kl]$ and $[mn]$ all lie in $x = 0$. By symmetry there is such a homology having any given vertex of the configuration as centre, and we have

XI 60 homologies of period 2.

These sixty homologies are the set of operations used in Hamill (10) chap. IV to generate $\mathcal{C}_{11520}$, and we have a further proof that the group generated by the biaxial homographies of type II is in fact $\mathcal{C}_{11520}$.

We have already considered one product of four biaxial homographies, which is the square of an operation of type VIII; any other product of four biaxial homographies may either be reduced to one involving fewer factors or to one of the following five types:

$ij.ik.lm.ln.$

$ij.ik.il.im.$

$ij^2.ik.jl.$

$ij^2.ik.lm.$

$ij.kl.km.kn.$

A product such as $ij.ik.lm.ln$ is of period 3, for

$$(ij.ik.lm.ln)^3 = (ij.ik)^3.(lm.ln)^3$$
$$= e.$$

Under $\mathcal{H}$, $ij.ik.lm.ln$ maps onto $(ijk)(lmn)$ and so has not arisen before. Also if $\underline{A}$ and $\underline{B}$ are two operations of type IV such that

$$\underline{A}.\underline{B} = ij.ik.lm.ln,$$

then $\underline{A}.\underline{B}$ maps onto $(ijk)(lmn)$ under $\mathcal{H}$, and so the axes of $\underline{A}$ and $\underline{B}$ must lie on Q_{ijk} but belong to opposite systems of generators on Q_{ijk}. There are 64 operations of this type in the $\mathcal{G}_{576}$ generated by ij, ik, lm, ln and none of these operations are contained in any of the other nine groups $\mathcal{G}_{576}$ in $\mathcal{G}_{11520}$. Hence

XII 640 operations of period 3.

Any product of four biaxial homographies whose symbols all contain a given letter, such as ij.ik.il.im is of period 5, for

$$\begin{aligned}(ij.ik.il.im)^2 &= ij.ik.il.im.ij.ik.il.im.\\ &= ij^2.kj.lj.mj.ik.il.im.\\ &= ij^2.lj.kl.mj.kl.ik.im.\\ &= ij^2.lj.kl^2.mj.ik.im.\\ &= mn^2.jl.mj.ik.im.\\ &= jl^2.ik^2.jl.mj.ik.im.\\ &= lj.mj.ki.im.\\ &= mj.lm.im.mk.\end{aligned}$$

and mj.lm.im.mk is the same type of operation as ij.ik.il.im.

$$\begin{aligned}\text{Also } (ij.ik.il.im)^3 &= ij.ik.il.im.mj.lm.im.mk.\\ &= ij.ik.il.im^2.ij.li.mk.\\ &= im^2.ji.ki.li.ij.li.mk.\\ &= im^2.jk.jl.li.mk.\\ &= ij^2.jm^2.jk.li.ij.mk.\end{aligned}$$

$= ij^2 . jm^2 . jm.jk.li.ij.$

$= ij^2 .mj.jk.li.ij.$

$= jm.kj.il.ji$

$= jm.kj.ji.jl.$

and $(ij.ik.il.im)^3$ is the same type of operation as ij.ik.il.im.

Finally $(ij.ik.il.im)^5 = mj.lm.im.mk.jm.kj.ji.jl.$

$= lj.ij.jk.kj.ji.jl.$

$= e.$

and we see that ij.ik.il.im is a new operation of period 5 whose powers are all of the same type. The image of ij.ik.il.im under $\mathcal{H}$ is (ijklm) and clearly an operation of this type whose symbol involves the letter n cannot possibly be the same as ij.ik.il.im; also if ij.$\underline{A}$ is an operation of this type, $\underline{A}$ must be of class X and

ij.$\underline{A}$ = ij.ik.il.im only if $\underline{A}$ = ik.il.im.

If however ji.$\underline{A}$ = ij.ik.il.im, then $\underline{A} = ij^2$.ik.il.im and $\underline{A}$ can be shewn to be of period 4, and so $\underline{A}$ cannot belong to type X. Hence the operations

ij.ik.il.im ;	ji.ik.il.im;
ij.ik.il.mi ;	ji.ik.il.mi;
ij.ik.li.im. ;	ji.ik.li.im;
ij.ik.li.mi ;	ji.ik.li.mi;
ij.ki.il.im ;	ji.ki.il.im;
ij.ki.il.mi ;	ji.ki.il.mi;

$$ij.ki.li.im; \qquad ji.ki.li.im;$$
$$ij.ki.li.mi; \qquad ji.ki.li.mi.$$

are all distinct and all map onto $(ijklm)$ under $\mathcal{H}$. This implies that all the images of all the cycles conjugate to $(ijklm)$ belong to this type and we have

XIII 2304 (= 144 x 16) operations of period 5.

Since we would normally expect five times as many operations of type XIII, each such operation must arise in five ways. In fact we find that the operations $ij.ik.il.im$; $jk.jl.jm.ij$; $kl.km.ik.jk$; $lm.il.jl.kl$ and $im.jm.km.lm$ are all the same operation.

The next product of four biaxial homographies of type II that has to be considered is one such as $ij^2.ik.jl$. This is of period 2, for

$$\begin{aligned}(ij^2.ik.jl)^2 &= ij^2.ik.jl.ij^2.ik.jl.\\ &= ki.lj.ik.jl.\\ &= e.\end{aligned}$$

Under $\mathcal{H}$, $ij^2.ik.jl$ maps onto $(ik)(jl)$ and therefore is a new operation of period 2. We may easily see that the four operations $ij^2.ik.jl$; $ij^2.ik.lj$; $ij^2.ki.jl$, and $ij^2.ki.lj$ are all distinct; the only other operations of this type which map onto $(ik)(jl)$ are the twelve operations deriving from $il^2.ik.jl$; $kj^2.ik.jl$; $kl^2.ik.jl$ in the same way that the four operations given above derive from $ij^2.ik.jl$.

It follows that $ij^2.ik.jl$ can arise in at most three other ways; in fact

$$\begin{aligned}
1.\ ij^2.ik.jl &= ij^2.jl.ik. \\
&= il^2.lj.ik. \\
&= il^2.ik.lj. \\
2.\ ij^2.ik.jl &= kj^2.ki.jl. \\
3.\ ij^2.ik.jl &= kj^2.ki.jl. \\
&= kj^2.jl.ki. \\
&= kl^2.lj.ki. \\
&= kl^2.ki.lj.
\end{aligned}$$

so that each operation of this type arises in four distinct ways; hence, remembering that $ik.jl = jl.ik$,

XIV 180 $\left(= \dfrac{15 \times 16 \times 6}{4 \times 2}\right)$ operations of period 2.

An operation of the type $ij^2.ik.lm$ is of period 4 for

$$\begin{aligned}
(ij^2.ik.lm)^2 &= ij^2.ik.lm.ij^2.ik.lm. \\
&= ki.lm.ik.lm. \\
&= lm^2.
\end{aligned}$$

The image of $ij^2.ik.lm$ under $\mathcal{H}$ is $(ik)(lm)$, and so if $ij^2.ik.lm$ is not a new operation, it must have arisen as an operation of type V; but the only operations of type V having this image are $ik.lm$; $ki.lm$; $ik.ml$ and $ki.ml$ and the assumption that $ij^2.ik.lm$ is equal to any one of these leads to a direct contradiction, and so $ij^2.ik.lm$ is a new operation. Again it is one of the 576 operations of the group generated by those biaxial homographies of type II whose axes lie on Q_{ijk}. The four operations deriving from $ij^2.ik.\ell m$ are $ij^2.ik.\ell m$; $ij^2.ik.m\ell$; $ij^2.ki.\ell m$; $ij^2.ki.m\ell$ and these are all distinct and all map onto $(ik)(\ell m)$ under $\mathcal{H}$; there are 28 further operations of this type having the same image under $\mathcal{H}$, deriving from $in^2.ki.m\ell$; $kj^2.ki.m\ell$; $kn^2.ki.m\ell$; $mj^2.m\ell.ki$; $mn^2.m\ell.ki$; $\ell j^2.m\ell.ki$; $\ell n^2.m\ell.ki$. Now suppose $\underline{B}$ is one of the eight operations $mn^2.m\ell$; $mn^2.\ell m$; $mj^2.m\ell$; $mj^2.\ell m$; $\ell n^2.m\ell$; $\ell n^2.\ell m$; $\ell j^2.\ell m$; $\ell j^2.m\ell$, then the assumption that either $\underline{B}.ki$ or $\underline{B}.ik$ is the same operation as $ij^2.ik.m\ell$ leads to a contradiction; it follows that $ij^2.ik.m\ell$ can arise in at most three more ways. In fact we find that

$$\begin{aligned}
1.\ ij^2.ik.m\ell &= in^2.jn^2.ik.m\ell. \\
&= in^2.ik^2.m\ell^2.ik.m\ell. \\
&= in^2.ki.\ell m. \\
2.\ ij^2.ik.m\ell &= jk^2.ki.m\ell. \\
3.\ ij^2.ik.m\ell &= kn^2.m\ell^2.ik.m\ell. \\
&= kn^2.ik.\ell m.
\end{aligned}$$

Hence each operation of this type arises in exactly four ways and we have

$$\text{XV} \quad 360 \left(= \frac{15 \times 16 \times 6}{4}\right) \text{ operations of period 4.}$$

The last distinct product of four biaxial homographies is one of the same type as $ij.k\ell.km.kn$.

$$
\begin{aligned}
(ij.k\ell.km.kn)^2 &= ij.k\ell.km.kn.ij.k\ell.km.kn. \\
&= ij^2.(k\ell.km.kn)^2. \\
&= ij^2.k\ell^2.m\ell.n\ell.km.kn. \\
&= mn^2.n\ell.mn.mn.km. \\
&= \ell n.km.
\end{aligned}
$$

Thus $(ij.k\ell.km.kn)^2$ is of type V and period 4, so $ij.k\ell.km.kn$ is of period 8; it has not arisen before as its image under $\mathcal{H}$ is $(ij)(k\ell mn)$ and the images of the other operations of period 8 (type X) all belong to the conjugate set of $(ijk\ell)$. If now $\underline{A}$ is an operation of type X so that $ij.\underline{A}$ and $ji.\underline{A}$ are of the same type as $ij.k\ell.km.kn$, then $ij.\underline{A}$ and $ij.k\ell.km.kn$ are the same only if $\underline{A} = k\ell.km.kn$, and

$$
\begin{aligned}
ji.\underline{A} &= ij.k\ell.km.kn \quad \text{only if} \\
\underline{A} &= ij^2.k\ell^2.km.kn. \\
&= k\ell^2.mn^2.k\ell.km.kn. \\
&= \ell k.nm.km. \\
&= \ell k.km.nk.
\end{aligned}
$$

Clearly all the operations of this type whose images under $\mathcal{H}$ are $(ij)(k\ell mn)$ are of the form $ij.\underline{A}$ or $ji.\underline{A}$ and since $\underline{A}$ can arise in exactly four ways, (see X page 71.) we have that an operation such as $ij.k\ell.km.kn$ arises in eight distinct ways. Thus

XVI. 720 $\left(= \dfrac{30 \times 24 \times 8 \times 2}{8} \right)$ operations of period 8.

Four distinct products involving five biaxial homographies of type II may be formed which have not arisen before; they are typified by

$$ij^2.ik.lm.ln.$$
$$ij.ik.il.im.in.$$
$$ij^2.ik.il.im.$$
$$ij.kl^2.km.ln.$$

All other products of five biaxial homographies may be expressed as one of the above or can be reduced to a product with less factors.

An operation such as $ij^2.ik.lm.ln$ is of period 6, for

$$\begin{aligned}(ij^2.ik.lm.ln)^2 &= ij^2.ik.lm.ln.ij^2.ik.lm.ln.\\ &= ki.lm.ln.ik.lm.ln.\\ &= (lm.ln)^2\\ &= nl.ml.\end{aligned}$$

$$\text{and } \begin{aligned}(ij^2.ik.lm.ln)^3 &= ij^2.ik.lm.ln.nl.ml\\ &= ij^2.ik.\end{aligned}$$

and so the square and the cube of $ij^2.ik.lm.ln$ are of periods 3 and 2 respectively, and $ij^2.ik.lm.ln$ must be of period 6; under $\mathcal{H}$ it maps onto $(ik)(lmn)$ and so has not arisen before. In fact, this operation belongs to the last type of collineation generated by biaxial homographies of type II whose axes all lie on a quadric Q_{ijk} and which generate a $\mathcal{C}_{576}$ in S_3; clearly such an operation cannot be generated by biaxial homographies whose axes do not lie on Q_{ijk} and as there are 96 such distinct operations in each of the ten $\mathcal{C}_{576}$ contained in $\mathcal{C}_{11520}$, we have

XVII 960 operations of period 6, whose squares are of type IV and cubes of type VI.

An operation such as ij.ik.il.im.in is also of period 6, for

$$
\begin{aligned}
(ij.ik.il.im.in)^2 &= ij.ik.il.im.in.ij.ik.il.im.in.\\
&= ij^2.kj.lj.mj.mj.ik.il.im.in.\\
&= ij^2.lj.kl.nj.mn.kl.ik.mn.im.\\
&= ij^2.lj.kl^2.nj.mn^2.ik.im.\\
&= jl.jn.ik.im.
\end{aligned}
$$

and

$$
\begin{aligned}
(ij.ik.il.im.in)^3 &= jl.jn.ik.im.ij.ik.il.im.in.\\
&= jl.jn.ik.im^2.jm.km.lm.in.\\
&= im^2.jl.jn.ki.jl.jm.kl.in.\\
&= im^2.jl^2.nl.ki.jm.kl.in.\\
&= kn^2.kl.nk.il.in.jm.\\
&= lk.kn.il.in.jm.\\
&= kn.nl.ln.il.jm.\\
&= kn.il.jm.
\end{aligned}
$$

so that the square and cube of ij.ik.il.im.in are of types XII and XI respectively and ij.ik.il.im.in is itself of period 6; it is a new operation as the squares of the only other operations of period 6 which have already arisen are of type IV. Now there are 32 products of five biaxial homographies whose symbols all involve i and which map onto (ijklmn) under $\mathcal{H}$; we know already that the 16 operations of type XIII which map onto (iklmn) and the symbols of whose factors all involve the letter i are all distinct (see page 74), hence if ij.$\underline{A}$ is of the same type as ij.ik.il.im.in. and ij.$\underline{A}$ = ij.ik.il.im.in, then $\underline{A}$ = ik.il.im.in, and we have 16 distinct operations of this type all mapping onto (ijklmn); thus this class contains all the operations whose images under $\mathcal{H}$ are conjugate to (ijklmn) and we have

XVIII 1920 (= 16 x 120) new operations of period 6.

An operation such as $ij^2.ik.il.im$ is of period 4 since

$$\begin{aligned}(ij^2.ik.il.im)^2 &= ij^2.ik.il.im.ij^2.ik.il.im.\\ &= ki.li.mi.ik.il.im.\\ &= kl.km.il.im.\\ &= km.lm.lm.il.\\ &= lm^2.mk.il.\end{aligned}$$

$lm^2.mk.il$ is of type XIV and period 2 so that $ij^2.ik.il.im$ is of period 4; its image under $\mathcal{H}$ is (iklm) and so it is a new operation. We have already, in class X, eight distinct operations which all have (iklm) as their image under $\mathcal{H}$, so there can be at most eight distinct operations of the type $ij^2.ik.il.im$ having the same image; but we know that there are eight operations of type X (see page 70) which map onto (iklm) and are distinct and the symbols of the factors of these operations all involve the letter i. Hence we have exactly eight distinct operations of the same type as $ij^2.ik.i\ell.im$ mapping onto $(ik\ell m)$ under $\mathcal{H}$. Thus

XIX 720 (= 8 x 90) operations of period 4 whose images under $\mathcal{H}$ are conjugate to $(ijk\ell)$.

The last typical product of five biaxial homographies is $ij.k\ell^2.km.\ell n$.

$$\begin{aligned}(ij.k\ell^2.km.\ell n)^2 &= ij.k\ell^2.km.\ell n.ij.k\ell^2.km.\ell n.\\ &= ij^2.mk.n\ell.km.\ell n.\\ &= ij^2.\end{aligned}$$

so that the square of $ij.k\ell^2.km.\ell n$ is of type III and period 2 and $ij.k\ell^2.km.\ell n$ is itself of period 4; it is a new operation for its image under $\mathcal{H}$ is $(ij)(n\ell)(mk)$ and none of the other operations of period 4 have this type of image. In class XI we have four distinct operations with the same image $(ij)(n\ell)(mk)$ under $\mathcal{H}$, and there can be at most twelve operations of the type $ij.k\ell^2.km.\ell n$ having the same image under $\mathcal{H}$. Since an operation such as $k\ell^2.km.\ell n$ can arise in four distinct ways (as products the symbols of whose factors do not involve either i or j, see page 76) we see that of the sixteen operations of the type under consideration whose first factor is ij and whose image under $\mathcal{H}$ is $(ij)(n\ell)(mk)$, only four are distinct. Now suppose that $km.\underline{A}$ is an operation such that

$$km.\underline{A} = ij.k\ell^2.km.\ell n.$$

$$\begin{aligned}\text{then } \underline{A} &= mk.ij.k\ell^2.km.\ell n.\\ &= ij.m\ell^2.\ell n.\end{aligned}$$

and so A is not of type XIV and we see that the four distinct products of this type whose first factor is km, and which map onto (ij)(km)(ℓn) are all distinct from ij.kℓ^2.km.ℓn. Similarly there are four products of this type whose first factor is ℓn and which are all distinct from ij.kℓ^2.km.ℓn. We have therefore exactly twelve distinct products of this type mapping onto (ij)(km)(ℓn) and thus

XX 180 (= 15 x 12) new operations of period 4, mapping onto the set conjugate to (ij)(kℓ)(mn).

There is just one product of six biaxial homographies which cannot be reduced to a product involving fewer factors. A typical example is ij.jk^2.kℓ.km.kn and it is another operation of period 4, for

$$\begin{aligned}(ij.jk^2.k\ell.km.kn)^2 &= ij.jk^2.k\ell.km.kn.ij.jk^2.k\ell.km.kn.\\ &= \ell k.mk.nk.k\ell.km.kn.\\ &= \ell m.\ell n.km.kn.\\ &= \ell n.mn.mn.km.\\ &= mn^2.n\ell.km.\end{aligned}$$

mn^2.nℓ.km is of type XIV and period 2 and so ij.jk^2.kℓ.km.kn is of period 4. It is a new operation for its image under $\mathcal{H}$ is (ij)(kℓmn). In class XVI we had eight distinct operations mapping onto (ij)(kℓmn) and so in this new class we can have at most eight further operations with the same image. But we have already seen (page 70) that the eight operations

$k\ell.km.kn$; $k\ell.km.nk$; $k\ell.mk.kn$; $k\ell.mk.nk$; $\ell k.km.kn$; $\ell k.km.nk$; $\ell k.mk.kn$; $\ell k.mk.nk$, are all distinct and so the eight operations $ij.jk^2.\underline{A}$ (where $\underline{A}$ is one of the eight operations listed above) are all distinct, and under $\mathcal{H}$ they all map onto $(ij)(k\ell mn)$. Hence there are exactly eight distinct operations of this type mapping onto $(ij)(k\ell mn)$ and we have

XXI 720 (= 8 x 90) new operations of period 4.

All the products of seven biaxial homographies of type II may be reduced to a product with fewer factors and we see that the group $\mathcal{G}_{11520}$, generated by thirty biaxial homographies of period 4 whose axes are the pairs of associated directrices of the Klein 60_{15} configuration, contains 11520 operations which fall into twenty-one distinct classes.

$\mathcal{G}_{11520}$ contains a number of primitive subgroups which have not already been considered in this section, though these are not all generated by the biaxial homographies they contain.

The group of operations in $\mathcal{G}_{11520}$ which consists of all the images under $\mathcal{H}$ of $(ijk\ell m)$, $(ikmj\ell)$, $(i\ell jmk)$, $(im\ell kj)$ and (i) is of order 80 and is known as $\mathcal{G}_{80}$. It contains the involutary biaxial homographies of type III together with a set of 64 of the collineations of type XIII. $\mathcal{G}_{80}$ does not leave any pair of lines fixed, or a point and a plane, or a tetrahedron and so it is primitive; it is one group leaving fixed the set of five mutually skew pairs of directrices [an], [na] for a = i, j, k, ℓ, m, where any operation of $\mathcal{C}_{180}$ which leaves fixed one of these five pairs of directrices also leaves fixed each of the other four pairs. $\mathcal{C}_{180}$ is not generated by biaxial homographies.

$\mathcal{C}_{180}$ is contained as an invariant subgroup in a group $\mathcal{C}_{1160}$, of order 160, which contains all the operations of $\mathcal{C}_{11520}$ whose images under $\mathcal{H}$ are the cycles of the $\mathcal{C}_5^2$ generated by (ijkℓm) and (iℓ)(kj).

To the group in $\mathcal{S}_6$ of order 20 generated by the cycles (ijkℓm), (iℓ)(kj), (ikjm) corresponds $\mathcal{C}_{1320}$, of order 320. $\mathcal{C}_{1320}$ contains operations of types III, V, X, XIII, XIV, XV and XIX.

As both $\mathcal{C}_{1160}$ and $\mathcal{C}_{1320}$ contain $\mathcal{C}_{180}$ as a subgroup the former two groups must also be primitive.

The simple group $\mathcal{A}_5$ of order 60 which is contained in $\mathcal{S}_6$ and is generated by (ijkℓm) and (ikj) is homomorphic with a group $\mathcal{C}_{1960}$ contained in $\mathcal{C}_{11520}$. $\mathcal{S}_6$ contains a second simple subgroup of order 60, $\mathcal{A}_5'$, which is generated by (ijkℓm) and (ijm)(knℓ) and is isomorphic with $\mathcal{A}_5$. The group in $\mathcal{C}_{11520}$ corresponding to $\mathcal{A}_5'$ is again of order 960 and will be known as $\mathcal{C}_{1960}'$.

The two groups $\mathcal{S}_5$ and $\mathcal{S}_5'$ which are generated by (ij); (ik); (iℓ); (im) and $\mathcal{A}_5'$; (ij)(kℓ)(mn) respectively are isomorphic, and the corresponding groups in $\mathcal{C}_{11520}$ are

$\mathcal{G}_{1920}$ and $\mathcal{G}'_{1920}$, both of order 1920.

The group $\mathcal{G}_{5760}$, which is homomorphic with $\mathcal{A}_6$, the alternating group of degree 6 contained in $\mathcal{S}_6$, is the last subgroup of $\mathcal{G}_{11520}$ whose operations will be found and tabulated.

$\mathcal{G}_{80}$ appears as a subgroup in each of $\mathcal{G}_{960}$, $\mathcal{G}'_{960}$, $\mathcal{G}_{1920}$, $\mathcal{G}'_{1920}$, $\mathcal{G}_{5760}$ and these last five groups are therefore primitive; $\mathcal{G}_{960}$, $\mathcal{G}'_{960}$, $\mathcal{G}_{1920}$, $\mathcal{G}^{+}_{1920}$, $\mathcal{G}_{5760}$ are generated by biaxial homographies, $\mathcal{G}_{80}$, $\mathcal{G}_{320}$ and $\mathcal{G}'_{1920}$ are not.

The operations of $\mathcal{G}_{11520}$ and its eight primitive subgroups which are discussed above are tabulated on page 87. The column headed 'n' gives the least number of homographies of type II required to generate an operation of the required type.

In conclusion, it will be noticed that $\mathcal{G}_{11520}$ and the eight subgroups are those groups which appear in Blichfeldt's list (see page 28) as having an invariant, imprimitive subgroup; the imprimitive group is the group of order 16 generated by the involutary biaxial homographies in class III, and which is invariant in all the nine primitive groups.

The Operations of $\mathcal{G}_{11520}$ and its Subgroups.

Type.	p	n	Description.	Image under $\mathcal{H}$.	Type of powers.	$\mathcal{G}_{11520}$	$\mathcal{G}_{5760}$	$\mathcal{G}_{1920}$	$\mathcal{G}'_{1920}$	$\mathcal{G}'_{960}$	$\mathcal{G}_{960}$	$\mathcal{G}_{320}$	$\mathcal{G}_{160}$	$\mathcal{G}_{80}$
I	1	1	Identity.	e.		1	1	1	1	1	1	1	1	1
II	4	2	ij.	(ij).	2nd. III.	30	-	20	-	-	-	-	-	-
III	2	2	ij^2.	e.		15	15	15	15	15	15	15	15	15
IV	3	2	ij.ik.	(ijk).		160	160	80	-	-	80	-	-	-
V	4	2	ij.kl.	(ij)(kl).	2nd. III.	180	180	60	60	60	60	20	20	-
VI	2	3	ij^2.ik.	(ik).		120	-	80	-	-	-	-	-	-
VII	4	3	ij^2.kl.	(kl).	2nd. III.	90	-	60	-	-	-	-	-	-
VIII	12	3	ij.kl.km.	(ij)(klm).	2nd. IX, 3rd. II, 4th. IV, 6th. III.	960	-	160	-	-	-	-	-	-
IX	6	4	ij^2.ik.il.	(ikl).	2nd. IV, 3rd. III.	480	480	240	-	-	240	-	-	-
X	8	3	ij.ik.il.	(ijkl).	2nd. V.	720	-	240	240	-	-	80	-	-
XI	2	3	ij.kl.mn.	(ij)(kl)(mn).		60	-	-	40	-	-	-	-	-
XII	3	4	ij.ik.lm.ln.	(ijk)(lmn).		640	640	-	320	320	-	-	-	-
XIII	5	4	ij.ik.il.im.	(ijklm).		2304	2304	384	384	384	384	64	64	64
XIV	2	4	ij^2.ik.jl.	(ik)(jl).		180	180	60	60	60	60	20	20	-
XV	4	4	ij^2.ik.ml.	(ik)(ml).	2nd. III.	360	360	120	120	120	120	40	40	-
XVI	8	4	ij.kl.km.kn.	(ij)(klmn).	2nd. V.	720	720	-	-	-	-	-	-	-
XVII	6	5	ij^2.jk.lm.ln.	(jk)(lmn).	2nd. IV, 3rd. VI.	960	-	160	-	-	-	-	-	-
XVIII	6	5	ij.ik.il.im.in.	(ijklmn).	2nd. XII, 3rd. XI.	1920	-	-	320	-	-	-	-	-
XIX	4	5	ij^2.ik.il.im.	(iklm).	2nd. XIV.	720	-	240	240	-	-	80	-	-
XX	4	5	$ij.kl^2$.km.ln.	(ij)(km)(ln).	2nd. III.	180	-	-	120	-	-	-	-	-
XXI	4	6	$ij.jk^2$.kl.km.kn.	(ij)(klmn).	2nd. XIV.	720	720	-	-	-	-	-	-	-

Notation for Case in which $\nu = 4$.

ij: (x, y, iz, it).

ik: $(x - z, y - t, x + z, y + t)$.

il: $(x + iz, y - it, ix + z, -iy + t)$.

im: $(x - t, y + z, -y + z, x + t)$.

in: $(x + it, y + iz, iy + z, ix + t)$.

jk: $(x + iz, y + it, ix + z, iy + t)$.

jl: $(x + z, y - t, -x + z, y + t)$.

jm: $(x + it, y - iz, -iy + z, ix + t)$.

jn: $(x + t, y + z, -y + z, -x + t)$.

kl: (x, iy, iz, t).

km: $(x + y, -x + y, z - t, z + t)$.

nk: $(x + iy, ix + y, z - it, -iz + t)$.

lm: $(x - iy, -ix + y, z - it, -iz + t)$.

nl: $(x - y, x + y, z - t, z + t)$.

mn: (x, iy, z, it).

§V. The Simple Group $\mathcal{G}_{25920}$.

In Section I we shewed that there is a group of 25920 operations in S_3 which is simple and is generated by 40 homologies of period 3 (see page 31). This group is isomorphic with the group in S_4 which leaves fixed a quartic primal having 45 nodes and which has been the subject of papers by both H.F. Baker (3) and J. Todd (18). In this paragraph $\mathcal{G}_{25920}$ will be generated in two ways: 4

(a) by the 40 homologies it contains,

and (b) by a set of 45 involutary biaxial homographies.

A coordinate system may be set up in which one of the 40 homologies of period 3, say $\underline{O}$, has its centre, O, at the point (0,0,0,1) and the plane t = 0 as its polar plane; there are then 12 centres of homologies lying in the polar plane of $\underline{O}$ and these may be taken as

A_0 (1,0,0,0), A_1(0, 1,0,0), A_2(0,0, 1,0),
B_0 (ϖ,1,1,0), B_1(1,ϖ,1,0), B_2(1,1,ϖ,0), ($\varpi^3 = 1$)
C_0 (ϖ^2,1,1,0), C_1(1,ϖ^2,1,0), C_2(1,1,ϖ^2,0),
D_0 (1,1,1,0), D_1(1,ϖ^2,ϖ,0), D_2(1,ϖ,ϖ^2,0),

where if $\underline{\alpha_i}$ is the homology with centre α_i then the polar plane of $\underline{\alpha_i}$ passes through α_j, α_k and 0, for α = A, B, C or D, i, j, k all different and i, j, k = 0, 1 or 2. (This part of the notation is adapted from that given in Hamill (10), page 39, for the centres of the homologies of period 3 generating Γ_{216} .)

The centres of the 9 involutary homologies in the Γ_{216} generated in $t = 0$ by $\underline{A}_0$, $\underline{B}_0$ and $\underline{C}_1$ have coordinates

$$(0, 1, -1, 0), \quad (0, 1, -\varpi, 0), \quad (0, 1, -\varpi^2, 0),$$
$$(-1, 0, 1, 0), \quad (-\varpi, 0, 1, 0), \quad (-\varpi^2, 0, 1, 0),$$
$$(1, -1, 0, 0), \quad (1, -\varpi, 0, 0), \quad (1, -\varpi^2, 0, 0).$$

Then the remaining 27 homologies have their centres at the points

$P_1\,(-1, 0, 1, 1)$, $P_2\,(-1, 0, 1, \varpi)$, $P_3\,(-1, 0, 1, \varpi^2)$,
$Q_1\,(-1, 0, \varpi^2, 1)$, $Q_2\,(-1, 0, \varpi^2, \varpi)$, $Q_3\,(-1, 0, \varpi^2, \varpi^2)$,
$R_1\,(-1, 0, \varpi, 1)$, $R_2\,(-1, 0, \varpi, \varpi)$, $R_3\,(-1, 0, \varpi, \varpi^2)$,
$U_1\,(0, 1, -1, 1)$, $U_2\,(0, 1, -1, \varpi)$, $U_3\,(0, 1, -1, \varpi^2)$,
$V_1\,(0, 1, -\varpi, 1)$, $V_2\,(0, 1, -\varpi, \varpi)$, $V_3\,(0, 1, -\varpi, \varpi^2)$,
$W_1\,(0, 1, -\varpi^2, 1)$, $W_2\,(0, 1, -\varpi^2, \varpi)$, $W_3\,(0, 1, -\varpi^2, \varpi^2)$,
$X_1\,(1, -1, 0, 1)$, $X_2\,(1, -1, 0, \varpi)$, $X_3\,(1, -1, 0, \varpi^2)$,
$Y_1\,(1, -\varpi, 0, 1)$, $Y_2\,(1, -\varpi, 0, \varpi)$, $Y_3\,(1, -\varpi, 0, \varpi^2)$,
$Z_1\,(1, -\varpi^2, 0, 1)$, $Z_2\,(1, -\varpi^2, 0, \varpi)$, $Z_3\,(1, -\varpi^2, 0, \varpi^2)$,

and the polar plane of the homology centre (ξ, η, ζ, τ) is given by

$$\bar{\xi} x + \bar{\eta} y + \bar{\zeta} z + \bar{\tau} t = 0.$$

The four points O, β_1, β_2, β_3 will be seen to be collinear, for β one of the letters P, Q, R, U, V, W, X, Y, Z.

The configuration formed by the centres of the 40 homologies contains two sorts of lines, e-lines and a-lines. Any centre, say O, is joined to the 12 centres in its polar plane by lines containing no further centres, that is to say, 12 e-lines pass through every centre; hence there are 240 e-lines in the configuration. The join of O to any centre not in the plane $t = 0$ is a line containing exactly four centres and will be known as an a-line; the figure thus has 90 a-lines. The line of intersection of the polar planes of two homologies whose centres lie on an e-line is itself an e-line; two such e-lines will be known as polar e-lines. The four centres determined by a pair of polar e-lines form a tetrahedron whose edges are three pairs of polar e-lines; such a tetrahedron will be called a polar tetrahedron. The polar planes of the homologies whose centres lie on an a-line all pass through a second a-line, which will be said to be polar to the first.

There are two types of planes in the figure, j-planes and t-planes. A j-plane is simply one of the polar planes of the generating homologies, and there are thus 40 of them; each j-plane contains 12 centres, 9 a-lines and 12 e-lines. The second type of plane is one containing just 5 centres, 4 e-lines and an a-line, and is a plane joining a given a-line to one of the centres on the polar a-line; there are 360 t-planes.

For convenience the details of the configuration will be summarised in the following table. The numbers above the shaded portions refer to the number of centres, e-lines and a-lines lying in a given subconfiguration, while those below the shaded area refer to the numbers of any given subconfiguration passing through a given centre, e-line or a-line. (For example, there are 4 centres lying on an a-line, and 45 t-planes passing through a given centre.)

Incidence Table.

	40 centres	240 e-lines	90 a-lines	40 j-planes	360 t-planes
40 centres.		2	4	12	5
240 e-lines.	12			12	4
90 a-lines.	9			9	1
40 j-planes.	12	2	4		
360 t-planes.	45	6	4		

(a) Each of the 40 homologies in $\mathcal{G}_{25920}$ leaves fixed the figure formed by the centres, transforming any given centre into itself or one of the other centres; we may therefore represent a homology as a product of cycles of the 40 centres. By selecting the collineation $(\omega x, y, z, t)$ as the form for $\underline{A}_o$ we find that the following 18 homologies may be represented by the products of cycles given below:

$$\underline{A}_0 \equiv \{(B_0 C_0 D_0)(B_1 C_2 D_1)(B_2 C_1 D_2)(P_1 Q_3 R_2)(P_2 Q_1 R_3)(P_3 Q_2 R_1)(X_1 Z_3 Y_2)(X_2 Z_1 Y_3)(X_3 Z_2 Y_1)\}.$$

$$\underline{A}_1 \equiv \{(B_0 C_2 D_2)(B_1 C_1 D_0)(B_2 C_0 D_1)(U_1 W_3 V_2)(U_2 W_1 V_3)(U_3 W_2 V_1)(X_1 Y_1 Z_1)(X_2 Y_2 Z_2)(X_3 Y_3 Z_3)\}.$$

$$\underline{A}_2 \equiv \{(B_0 C_1 D_1)(B_1 C_0 D_2)(B_2 C_2 D_0)(P_1 R_1 Q_1)(P_2 R_2 Q_2)(P_3 R_3 Q_3)(U_1 V_1 W_1)(U_2 V_2 W_2)(U_3 V_3 W_3)\}.$$

$$\underline{B}_0 \equiv \{(A_0 D_0 C_0)(A_1 D_2 C_2)(A_2 D_1 C_1)(P_1 Y_2 W_2)(P_2 Y_3 W_3)(P_3 Y_1 W_1)(R_1 V_2 X_3)(R_2 V_3 X_1)(R_3 V_1 X_2)\}.$$

$$\underline{B}_1 \equiv \{(A_0 D_1 C_2)(A_1 D_0 C_1)(A_2 D_2 C_0)(Q_1 U_3 Z_3)(Q_2 U_1 Z_1)(Q_3 U_2 Z_2)(R_1 X_2 V_3)(R_2 X_3 V_1)(R_3 X_1 V_2)\}.$$

$$\underline{B}_2 \equiv \{(A_0 D_2 C_1)(A_1 D_1 C_0)(A_2 D_0 C_2)(P_1 W_1 Y_3)(P_2 W_2 Y_1)(P_3 W_3 Y_2)(Q_1 Z_2 U_1)(Q_2 Z_3 U_2)(Q_3 Z_1 U_3)\}.$$

$$\underline{C}_0 \equiv \{(A_0 B_0 D_0)(A_1 B_2 D_1)(A_2 B_1 D_2)(P_1 V_3 Z_3)(P_2 V_1 Z_1)(P_3 V_2 Z_2)(Q_1 X_2 W_3)(Q_2 X_3 W_1)(Q_3 X_1 W_2)\}.$$

$$\underline{C}_1 \equiv \{(A_0 B_2 D_2)(A_1 B_1 D_0)(A_2 B_0 D_1)(R_1 Y_2 U_2)(R_2 Y_3 U_3)(R_3 Y_1 U_1)(Q_1 W_2 X_3)(Q_2 W_3 X_1)(Q_3 W_1 X_2)\}.$$

$$\underline{C}_2 \equiv \{(A_0 B_1 D_1)(A_1 B_0 D_2)(A_2 B_2 D_0)(P_1 Z_2 V_1)(P_2 Z_3 V_2)(P_3 Z_1 V_3)(R_1 U_1 Y_3)(R_2 U_2 Y_1)(R_3 U_3 Y_2)\}.$$

$$\underline{D}_0 \equiv \{(A_0 C_0 B_0)(A_1 C_1 B_1)(A_2 C_2 B_2)(Q_1 V_1 Y_3)(Q_2 V_2 Y_1)(Q_3 V_3 Y_2)(R_1 Z_2 W_1)(R_2 Z_3 W_2)(R_3 Z_1 W_3)\}.$$

$$\underline{D}_1 \equiv \{(A_0 C_2 B_1)(A_1 C_0 B_2)(A_2 C_1 B_0)(Q_1 Y_2 V_2)(Q_2 Y_3 V_3)(Q_3 Y_1 V_1)(P_1 U_2 X_3)(P_2 U_3 X_1)(P_3 U_1 X_2)\}.$$

$$\underline{D}_2 \equiv \{(A_0 C_1 B_2)(A_1 C_2 B_0)(A_2 C_0 B_1)(P_1 X_2 U_3)(P_2 X_3 U_1)(P_3 X_1 U_2)(R_1 W_3 Z_3)(R_2 W_1 Z_1)(R_3 W_2 Z_2)\}.$$

$$\underline{O} \equiv \{(P_1 P_2 P_3)(Q_1 Q_2 Q_3)(R_1 R_2 R_3)(U_1 U_2 U_3)(V_1 V_2 V_3)(W_1 W_2 W_3)(X_1 X_2 X_3)(Y_1 Y_2 Y_3)(Z_1 Z_2 Z_3)\}.$$

$$\underline{P}_1 \equiv \{(OP_1 P_2)(A_0 R_2 Q_3)(A_2 Q_1 R_1)(B_0 W_2 Y_2)(B_2 Y_3 W_1)(C_0 Z_3 V_3)(C_2 V_1 Z_2)(D_1 X_3 U_2)(D_2 U_3 X_2)\}.$$

$$\underline{P}_2 \equiv \{(OP_1 P_3)(A_0 R_3 Q_1)(A_2 Q_2 R_2)(B_0 W_3 Y_3)(B_2 Y_1 W_2)(C_0 Z_1 V_1)(C_2 V_2 Z_3)(D_1 X_1 U_3)(D_2 U_1 X_3)\}.$$

$$\underline{P}_3 \equiv \{(OP_2 P_1)(A_0 R_1 Q_2)(A_2 Q_3 R_3)(B_0 W_1 Y_1)(B_2 Y_2 W_3)(C_0 Z_2 V_2)(C_2 V_3 Z_1)(D_1 X_2 U_1)(D_2 U_2 X_1)\}.$$

$$\underline{U}_3 \equiv \left\{ \begin{array}{l} (OU_2U_1)(A_1V_1W_2)(A_2W_3V_3)(B_1Q_1Z_3)(B_2Z_1Q_3)(C_1Y_3R_2)(C_2R_3Y_2) \\ \qquad (D_1P_2X_1)(D_2X_2P_1) \end{array} \right\}.$$

$$\underline{V}_3 \equiv \left\{ \begin{array}{l} (OV_2V_1)(A_1W_1U_2)(A_2U_3W_3)(B_0R_2X_1)(B_1X_2R_1)(C_0P_1Z_3)(C_2Z_1P_3) \\ \qquad (D_0Q_3Y_2)(D_1Y_3Q_2) \end{array} \right\}.$$

Under this representation we find that if K, L, M, N are any four centres on an a-line such that $\underline{K}(L) = M$, $\underline{K}(M) = N$, $\underline{K}(N) = L$, then

$$\underline{K}.\underline{L} = \underline{L}.\underline{M} = \underline{M}.\underline{K}.$$

$$\underline{K}.\underline{M} = \underline{M}.\underline{N} = \underline{N}.\underline{K}.$$

$$\underline{K}.\underline{N} = \underline{N}.\underline{L} = \underline{L}.\underline{K}.$$

$$\underline{M}.\underline{L} = \underline{L}.\underline{N} = \underline{N}.\underline{M}.$$

Also if $\underline{K}$, $\underline{L}$, $\underline{M}$, $\underline{N}$ are any four homologies whose centres form the vertices of a polar tetrahedron, then $\underline{K}.\underline{L}.\underline{M}.\underline{N}$ is the identical collineation.

We are now in a position to be able to enumerate the operations of $\mathcal{G}_{25920}$. Since the squares of the 40 generating homologies are also homologies of period 3 we have already 81 operations of the group.

I. The identical collineation.

II. 80 homologies of period 3.

There are two distinct types of product arising from an e-line, and both these are of period 3. For consider the typical e-line OA_0; $\underline{O}$ and $\underline{A}_0$ have been chosen to have the forms

$$\underline{O} : (x, y, z, \varpi t)$$

$$\underline{A}_0 : (\varpi x, y, z, t).$$

Then $\underline{O}.\underline{A}_0 : (\varpi x, y, z, \varpi t)$ and $\underline{O}^2.\underline{A}_0 : (\varpi x, y, z, \varpi^2 t)$ are respectively a biaxial homography and an axial homography of period 3. Clearly all the operations generated by $\underline{O}$ and $\underline{A}_0$ must be of one of these two types. $\underline{O}.\underline{A}_0$ has the pair of polar e-lines OA_0 and A_1A_2 as its axes, and there will be one of these biaxial homographies of period 3 and its square associated in this way with every pair of polar e-lines. Thus

III 240 biaxial homographies of period 3 whose axes are a pair of polar e-lines.

$\underline{O}^2.\underline{A}_0$ has its two isolated united points at the points O and A_0 and the line A_1A_2 as its axis; since $(\underline{O}^2.\underline{A}_0)^2$ is the same type as $\underline{O}^2.\underline{A}_0$, we see that there are just two axial homographies of period 3 in $\mathcal{C}_{25920}$ having a given e-line as axis and isolated united points coinciding with the centres on the polar e-line. Hence there are two operations of this type associated with every e-line in the configuration, and we have

IV 480 axial homographies of period 3.

The homologies whose centres lie on a given a-line generate an $\mathcal{R}_{12}$ on that line and leave each point of the polar a-line fixed; they therefore generate a group in S_3 containing 24 operations. We already have 9 of these operations, the identity and the 4 homologies and their squares, whose centres lie on the given a-line, and the remaining collineations fall into three distinct classes. For consider the a-line $A_0B_0C_0D_0$, then $\underline{A}_0 . \underline{B}_0$; $\underline{A}_0^2 . \underline{B}_0$; $\underline{D}_0 . \underline{A}_0 . \underline{B}_0$ are operations of different periods arising from products of $\underline{A}_0$, $\underline{B}_0$, $\underline{C}_0$, $\underline{D}_0$, and we shall see that all other products of these four homologies fall into one of the classes determined by the given products.

$\underline{A}_0 . \underline{B}_0$ is of period 6 for

$$\underline{A}_0 . \underline{B}_0 \equiv \left\{ \begin{array}{l} (A_0B_0C_0)(A_1B_2C_1A_2B_1C_2)(D_1D_2)(P_1X_1)(P_2X_2)(P_3X_3)(Q_1R_3V_1Z_1Y_3W_3) \\ (Q_2R_1V_2Z_2Y_1W_1)(Q_3R_2V_3Z_3Y_2W_2). \end{array} \right\}$$

Since $\underline{A}_0 . \underline{B}_0$ leaves fixed just the centres D_0, U_1, U_2, U_3 and O and OU_1 is the a-line polar to A_0B_0, we see that no product of this type from an a-line other than A_0B_0 can give rise to this particular operation. As $(\underline{A}_0 . \underline{B}_0)^5$ is the same type of operation as $\underline{A}_0 . \underline{B}_0$ there will be, by symmetry, 8 operations of this type arising from each a-line. In addition, since this operation has a line of united points and is of period 6, it can only be an axial homography. Hence

V 720 ($= 8 \times 90$) axial homographies of period 6.

The squares of operations of type V have already occurred, for

$$\begin{aligned} (\underline{A}_0 . \underline{B}_0)^2 &= \underline{A}_0 . \underline{B}_0 . \underline{A}_0 . \underline{B}_0 \\ &= \underline{A}_0^2 . \underline{D}_0 . \underline{B}_0 \\ &= \underline{A}_0^3 . \underline{D}_0 \\ &= \underline{D}_0 . \end{aligned}$$

The cubes of operations of type V are of period 2 and must therefore be new operations.

$$(\underline{A}_0 . \underline{B}_0)^3 = \underline{D}_0 . \underline{A}_0 . \underline{B}_0$$

so that $\underline{D}_0 . \underline{A}_0 . \underline{B}_0$ is of period 2 and leaves fixed each point of A_0B_0 and OU_1. By symmetry there will be an operation of this type associated with each pair of polar a-lines; hence

VI 45 biaxial homographies of period 2, each having a pair of polar a-lines as axes.

The operation $\underline{A}_0^2 . \underline{B}_0$ is of period 4 and so has not occurred before, for

$$\underline{A}_0^2 . \underline{B}_0 \equiv \left\{ \begin{array}{l} (A_0 C_0)(B_0 D_0)(A_1 C_1 A_2 C_2)(B_1 D_1 B_2 D_2)(P_1 Z_3 X_1 Q_3)(P_2 Z_1 X_2 Q_1) \\ \quad (P_3 Z_2 X_3 Q_2)(R_1 V_2 Y_1 W_1)(R_2 V_3 Y_2 W_2)(R_3 V_1 Y_3 W_3). \end{array} \right\}$$

It leaves fixed each point of OU_1 and two points (not centres) on A_0B_0, interchanging A_0 with C_0, B_0 with D_0, and so cannot arise as a similar product of homologies whose centres lie on any a-line other than A_0B_0. $(\underline{A}_0^2 . \underline{B}_0)^3$ is the same type of operation and has the same effect on A_0B_0, so there will be just four further operations of this class arising from products of $\underline{A}_0$, $\underline{B}_0$, $\underline{C}_0$, $\underline{D}_0$, two of these collineations will interchange A_0 with B_0, C_0 with D_0, and two will interchange A_0 with D_0 and B_0 with C_0. Hence each a-line gives rise to 6 distinct axial homographies of this type; hence

VII 540 (= 6 x 90) axial homographies of period 4.

We have now obtained 24 operations of the group generated by $\underline{A}_0$, $\underline{B}_0$, $\underline{C}_0$ and $\underline{D}_0$ and so any further products of these four homologies (or products of any set of homologies whose centres lie on an a-line) must be contained in one of the classes I-VII.

The products we have to consider next are those of homologies whose centres all lie in a j-plane. The group generated by the homologies whose centres lie in any j-plane, say $t = 0$, is of order 648, and we have already the identity, 26 homologies, 24 biaxial homographies of type III, 48 axial homographies of type IV, 90 operations of type V, 9 involutary biaxial homologies of type VI and 54 axial homographies of type VII belonging to this group, so there are only 396 further operations to be found.

The operation $\underline{A}_1^2 . \underline{A}_0^2 . \underline{B}_0 = \underline{A}_2 . \underline{O} . \underline{B}_0$ is of period 6, for

$$\underline{A}_1^2 . \underline{A}_0^2 . \underline{B}_0 \equiv \left\{ \begin{array}{l} (A_0 B_2 C_2 A_1 B_1 C_0)(A_2 B_0 C_1)(D_0 D_2)(P_1 Y_3 U_1 V_2 X_1 Q_3) \\ (P_2 Y_1 U_2 V_3 X_2 Q_1)(P_3 Y_2 U_3 V_1 X_3 Q_2)(R_1 W_3 R_3 W_2 R_2 W_1) \\ (Z_1 Z_2 Z_3). \end{array} \right\}$$

It is not of type V for it only leaves fixed two centres, D_1 and O, and so $\underline{A}_1^2 . \underline{A}_0^2 . \underline{B}_0$ has not occurred before. Since $(\underline{A}_1^2 . \underline{A}_0^2 . \underline{B}_0)^5$ is the same type of operation as $\underline{A}_1^2 . \underline{A}_0^2 . \underline{B}_0$, leaving fixed the same pair of polar a-lines A_2B_0 and OZ_1 and also having D_1 and O as fixed points, we see that there are just two operations in $\mathcal{G}_{25920}$ of this class having a given pair of polar centres as fixed points and leaving fixed a given pair of polar a-lines through those two centres. Hence each pair of polar a-lines will be associated to 32 such operations, and we have

VIII 1440 (= 32 × 45) operations of period 6.

The squares and cubes of an operation of this type have already occurred.

In addition we may see that $\underline{A}_1^2 . \underline{A}_0^2 . \underline{B}_0$ leaves fixed just two j-planes, the planes $t = 0$ and $OZ_1 R_1 W_1$; therefore there are 72 operations of type VIII leaving fixed any given j-plane.

An operation such as $\underline{A}_2 . \underline{A}_1 . \underline{A}_0^2 . \underline{B}_0 = \underline{C}^2 . \underline{A}_0 . \underline{B}_0$ is also of period 6, for

$$\underline{A}_2 . \underline{A}_1 . \underline{A}_0^2 . \underline{B}_0 \equiv \left\{ \begin{array}{l} (A_0 B_0 C_0)(A_1 B_2 C_1 A_2 B_1 C_2)(D_1 D_2)(P_1 X_3 P_2 X_1 P_3 X_2) \\ (Q_1 R_2 V_2 Z_1 Y_2 W_1)(Q_2 R_3 V_3 Z_2 Y_3 W_2)(Q_3 R_1 V_1 Z_3 Y_1 W_3) \\ \qquad (U_1 U_3 U_2), \end{array} \right\}$$

and this also only leaves two centres fixed, D_0 and O, but besides leaving fixed each of the pair of polar a-lines $A_0 B_0$, OU_1 it interchanges three a-lines with their polars, $Q_1 V_2$ with $R_2 Z_1$, $Q_2 V_3$ with $R_3 Z_2$ and $Q_3 V_1$ with $R_1 Z_3$. Operations of type VIII have only one fixed pair of polar a-lines, and so $\underline{A}_2 . \underline{A}_1 . \underline{A}_0^2 . \underline{B}_0$ has not occurred before. Again there will be 32 of these operations associated with each pair of polar a-lines and thus

IX 1440 operations of period 6.

The square and cube of $\underline{A}_2 . \underline{A}_1 . \underline{A}_0^2 . \underline{B}_0$ have already occurred and the operation again leaves fixed just two j-planes. Hence there are 72 operations of this type leaving fixed any given j-plane.

An operation such as $\underline{A}_1 . \underline{A}_0 . \underline{B}_0$ is of period 12, for

$$\underline{A}_1 . \underline{A}_0 . \underline{B}_0 \equiv \left\{ \begin{array}{l} (A_0 C_2 A_1 C_0)(A_2 C_1)(B_0 D_1)(B_1 D_2 B_2 D_0) \\ (P_1 Y_1 V_3 X_3 P_3 Y_3 V_2 X_2 P_2 Y_2 V_1 X_1)(Q_1 R_3 U_3 W_2 Q_3 R_2 U_2 W_1 Q_2 R_1 U_1 W_3) \\ (Z_1 Z_3 Z_2), \end{array} \right\}$$

and thus cannot have occurred before. It can only arise as a product of homologies whose centres are in the plane $t = 0$, since the only centre that it leaves fixed is 0. Now the fifth, seventh and eleventh powers of this operation all leave 0 fixed and the line $A_2 C_1 B_0 D_1$, interchanging A_2 with C_1 and B_0 with D_1. $\mathcal{C}_{25920}$ will therefore contain eight further operations each leaving 0 fixed, four of which interchange A_2 with B_0, C_1 with D_1, the remainder interchanging A_2 with D_1, B_0 with C_1. Thus each j-plane gives rise to 108 operations of this type which cannot arise from any other j-plane. Hence in $\mathcal{C}_{25920}$ we have

X 4320 (= 108 × 40) operations of period 12.

The remaining powers of operations of type X have already occurred.

The operation $\underline{A}_1^2 . \underline{A}_0 . \underline{B}_0$ is another new operation leaving 0 fixed, for

$$\underline{A}_1^2 . \underline{A}_0 . \underline{B}_0 \equiv \left\{ \begin{array}{l} (A_0 D_2 C_0)(A_1 D_1 C_2)(A_2 D_0 C_1)(B_0 B_2 B_1)(P_1 Z_1 X_3 P_3 Z_3 X_2 P_2 Z_2 X_1) \\ (Q_1 R_3 W_2 Q_3 R_2 W_1 Q_2 R_1 W_3)(U_1 V_2 Y_2 U_3 V_1 Y_1 U_2 V_3 Y_3), \end{array} \right\}$$

so that $\underline{A}_1^2 . \underline{A}_0 . \underline{B}_0$ is of period 9 and has not arisen before. The second, fourth, fifth, seventh and eighth powers of $\underline{A}_1^2 . \underline{A}_0 . \underline{B}_0$ are the same type as $\underline{A}_1^2 . \underline{A}_0 . \underline{B}_0$ itself, but only the fourth and seventh powers have the same effect in the polar plane of 0 as $\underline{A}_1^2 . \underline{A}_0 . \underline{B}_0$. Now $\underline{A}_1^2 . \underline{A}_0 . \underline{B}_0$ leaves the triangle $^{\Delta}B_0B_1B_2$ fixed and permutes the triangles $^{\Delta}A_0A_1A_2$, $^{\Delta}C_0C_1C_2$, $^{\Delta}D_0D_1D_2$ in such a way that if it transforms A_i into D_j, D_j into C_k and C_k into A_i, then $^{\Delta}A_iD_jC_k$ is a proper triangle; in addition, the effect of $\underline{A}_1^2 . \underline{A}_0 . \underline{B}_0$ in the plane polar to O is determined by the two cycles $(B_0B_2B_1)$ and $(A_0D_2C_0)$. The operations of this type whose effects in the plane t = 0 are determined by the cycle $(B_0B_2B_1)$ together with one of the cycles $(A_0D_2C_2)$, $(A_0D_0C_1)$, $(A_0D_0C_2)$, $(A_0D_1C_0)$, $(A_0D_1C_1)$ are all distinct from $\underline{A}_1^2 . \underline{A}_0 . \underline{B}_0$, and we see that there are at least six operations of this type whose effects in the plane t = 0 are all distinct and which leave one of the triangles $\alpha_0\alpha_1\alpha_2$ (α = A, B, C, D) fixed. Thus there are at least 36 of these operations in the group generated by homologies whose centres lie in t = 0 which leave a given one of the four triangles fixed. Hence there are at least 4 × 36 = 144 such operations arising from a given j-plane. But there can be no more than this, for we already have 504 of the operations of the group of order 648 leaving fixed a j-plane. In addition, since $\underline{A}_1^2 . \underline{A}_0 . \underline{B}_0$ only leaves one j-plane fixed, it cannot arise from any other j-plane, hence

XI 5760 (= 144 × 40) operations of period 9.

There are just four more different types of operation in $\mathcal{C}_{25920}$ and typical examples of these are

$$\underline{P}_3 . \underline{O} . \underline{A}_0 . \underline{B}_0.$$
$$\underline{P}_3 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0.$$
$$\underline{P}_3 . \underline{O} . \underline{R}_2 . \underline{V}_3.$$
$$\underline{P}_3^2 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0.$$

$$\underline{P}_3.\underline{O}.\underline{A}_0.\underline{B}_1 \equiv \left\{\begin{matrix}(A_0V_1B_3O\,P_2)(A_1W_1W_2B_2C_1)(A_2Y_2U_1B_0Z_2)(C_0Q_3U_3Y_3X_3)\\(C_2R_1V_2P_3R_3)(D_0U_2W_3Y_1D_2)(D_1X_2V_1P_1Q_1)(Q_2X_1Z_1R_2Z_3)\end{matrix}\right\}.$$

$\underline{P}_3.\underline{O}.\underline{A}_0.\underline{B}_1$ is thus a new operation of period 5. It leaves fixed eight non-planar pentagons of which $A_2Y_2U_1B_0Z_2$ and $D_0U_2W_3Y_1D_2$ are such that the edges A_2Y_2, Y_2U_1, U_1B_0, B_0Z_2, Z_2A_2 and D_0U_2, U_2W_3, W_3Y_1, Y_1D_2, D_2D_0 are all e-lines, while all the diagonals are a-lines. Now $\underline{P}_3.\underline{O}.\underline{A}_0.\underline{B}_1$ is completely determined by the cycle $(A_2Y_2U_1B_0Z_2)$ and so the number of such operations in $\mathcal{G}_{25920}$ depends on the number of pentagons such as $A_2Y_2U_1B_0Z_2$ which can be formed from the 40 centres. The point A_2 may be chosen in 40 ways and there are 12 centres which are polar to A_2, so that Y_2 may be chosen in 12 ways. The third vertex must be joined to A_2 by an a-line and to Y_2 by an e-line, and there are 9 centres satisfying these conditions. If U_1 is chosen, it is found that there are 6 centres which are joined to A_2 and Y_2 by a-lines and U_1 by e-lines. The last vertex of the pentagon must be polar to A_2 and the fourth vertex and be joined to Y_2 and U_1 by a-lines, and there are just 2 centres satisfying all these conditions. We see that there are thus $\dfrac{40 \times 12 \times 9 \times 6 \times 2}{5}$ such pentagons. However $\underline{P}_3.\underline{O}.\underline{A}_0.\underline{B}_1$ is equally well determined by the cycle $(D_0U_2W_3Y_1D_2)$ and so we have

XII 5184 $\left(= \dfrac{40 \times 12 \times 9 \times 6 \times 2}{10}\right)$ operations of period 5.

All the powers of $\underline{P}_3.\underline{O}.\underline{A}_0.\underline{B}_1$ are of type XII.

The operation $\underline{P}_3.\underline{O}^2.\underline{A}_0.\underline{C}_0$ is a new operation of period 4, for

$$\underline{P}_3 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0 \equiv \left\{ \begin{array}{l} (A_0 Z_2 Q_1 Z_3)(A_1 C_1 U_2 D_1)(A_2 V_3 B_0 W_1)(B_1 Y_2 X_3 B_2) \\ (C_0 D_0 R_1 P_1)(C_2 X_2 W_2 Q_2)(D_2 Q_3 V_2 Y_3)(O\ P_2 Z_1 R_3) \\ \qquad\qquad\qquad\qquad\quad (P_3 V_1 W_3 R_2)(U_1 U_3 X_1 Y_1) \end{array} \right\}$$

so $\underline{P}_3 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0$ leaves no centres fixed and cannot have arisen before. It leaves fixed just two non-planar quadrilaterals which are such that their eight vertices are the eight centres lying on a pair of polar a-lines; these two quadrilaterals are $B_1 Y_2 X_3 B_2$ and $P_3 V_1 W_3 R_2$. Since $\underline{P}_3 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0$ is completely determined by the cycle $(B_1 Y_2 X_3 B_2)$, the number of such operations in $\mathcal{C}_{25920}$ depends on the number of such quadrilaterals which can be formed from the 40 centres. There are in fact $\dfrac{4 \times 4 \times 3 \times 3 \times 2}{4}$ such quadrilaterals which have a given pair of polar a-lines as diagonals. As $\underline{P}_3 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0$ is equally well defined by the cycle $(P_3 V_1 W_3 R_2)$ we have

XIII 3240 $\left(= \dfrac{45 \times 4 \times 4 \times 3 \times 3 \times 2}{4 \times 2} \right)$ operations of period 4.

$(\underline{P}_3 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0)^3$ is also of type XIII and the square of an operation of type XIII is of period 2, but such an operation has not previously occurred as we find that

$$(\underline{P}_3 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0)^2 \equiv \left\{ \begin{array}{l} (A_0 Q_1)(Z_2 Z_0)(A_1 U_2)(C_1 D_1)(A_2 B_0)(V_3 W_1)(B_1 X_3) \\ (Y_2 B_2)(C_0 R_1)(D_0 P_1)(C_2 W_2)(X_2 Q_2)(D_2 V_2)(Q_3 Y_3) \\ (O\ Z_1)(P_2 R_3)(P_3 W_3)(V_1 R_2)(U_1 X_1)(U_3 Y_1) \end{array} \right\}$$

and so $(\underline{P}_3 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0)^2$ does not leave any centres fixed. We may write such an operation as a product involving only four homologies, for

$$
\begin{aligned}
(\underline{P}_3.\underline{O}^2.\underline{A}_0.\underline{C}_0)^2 &= \underline{P}_3.\underline{O}^2.\underline{A}_0.\underline{C}_0.\underline{P}_3.\underline{O}^2.\underline{A}_0.\underline{C}_0\\
&= \underline{P}_3^2.\underline{P}_1^2.\underline{Q}_2.\underline{V}_2.\underline{O}^2.\underline{A}_0.\underline{C}_0\\
&= \underline{P}_3^2.\underline{P}_1^2.\underline{O}^2.\underline{Q}_3.\underline{V}_3.\underline{A}_0.\underline{C}_0\\
&= \underline{P}_3.\underline{P}_1.\underline{P}_2.\underline{Q}_3.\underline{V}_3.\underline{A}_0.\underline{C}_0\\
&= \underline{P}_3.\underline{P}_1.\underline{P}_2.\underline{Q}_3.\underline{A}_0.\underline{V}_3.\underline{C}_0\\
&= \underline{P}_3.\underline{P}_1.\underline{P}_2.\underline{P}_1.\underline{Q}_3.\underline{P}_1.\underline{V}_3\\
&= \underline{P}_3.\underline{O}.\underline{P}_1^2.\underline{P}_1.\underline{R}_2.\underline{V}_3\\
&= \underline{P}_3.\underline{O}.\underline{R}_2.\underline{V}_3.
\end{aligned}
$$

$\underline{P}_3.\underline{O}.\underline{R}_2.\underline{V}_3$ leaves fixed each of the six a-lines A_0P_2, A_1U_2, OZ_1, A_2B_0, B_2Y_2, B_1R_2 and interchanges C_0W_2 with R_1C_2 and D_0V_2 with D_2P_1. In addition it leaves fixed just four polar tetrahedra, $C_0R_1Y_1U_3$, $W_2C_2Q_2X_2$, $Q_3Y_3V_2D_2$, $X_1U_1D_0P_1$ where these are such that their sixteen vertices are the centres lying on two pairs of polar a-lines, each pair of a-lines being left fixed by the operation. There will be just one operation of this type leaving fixed each such set of polar tetrahedra. There are 45 pairs of polar a-lines and just two centres are sufficient to determine the particular polar tetrahedron to which they belong; suppose we consider the pair of polar a-lines C_0W_2, R_1C_2. Then there are four polar tetrahedra having C_0 and a centre on R_1C_2 as vertices and so the tetrahedra $C_0R_1Y_1U_3$ may be chosen in 45 x 4 ways. There are then three polar tetrahedra having W_2 as one vertex and a second vertex, distinct from R_1, lying on R_1C_2, and the two tetrahedra $C_0R_1Y_1U_3$, $W_2C_2Q_2X_2$ may be chosen in 45 x 4 x 3 ways. But we have now uniquely determined the other two tetrahedra in the set, as the four unknown vertices must lie on $Y_1Q_2V_2D_0$ and $U_3X_2D_2P_1$ and be distinct from Y_1, Q_2, U_3, X_2. Clearly this same set of tetrahedra may equally well be determined in the same way from the pair of polar a-lines Q_2Y_1, X_2U_3, and so we have

XIV 270 $\left(= \dfrac{45 \times 4 \times 3}{2}\right)$ operations of period 2.

$\underline{P}_3^2 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0$ is of period 6 for

$$\underline{P}_3^2 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0 \equiv \left\{\begin{array}{l}(A_0V_2Y_3U_2X_2W_2)(A_1C_1X_1B_0Y_1D_1)(B_1W_3R_2P_3V_1B_2) \\ (A_2Z_1)(X_3Y_2)(C_0D_0Q_2V_3W_1Q_3)(C_2U_1U_3D_2R_3P_2) \\ \qquad\qquad\qquad\qquad\qquad\qquad (OP_1Z_2Q_1Z_3R_1)\end{array}\right\}$$

and since it leaves no centre fixed, it is a new operation. It leaves fixed just one polar tetrahedron, $A_2Z_1X_3Y_2$, and the pairs of polar a-lines A_2U_2, A_0Z_1 ; B_1X_3, B_2Y_2 and permutes the tetrahedra $U_2A_0V_1W_3$, $X_2V_2B_2R_2$, $W_2Y_3B_1P_3$ among themselves, where these three polar tetrahedra together with $A_2Z_1X_3Y_2$ again form a set whose vertices lie on two pairs of polar a-lines. The operation is determined by the fixed tetrahedra and the method of permutation of the remainder among themselves so that the two pairs of polar a-lines are left fixed. We can choose one pair of polar a-lines and the fixed tetrahedron in 45 x 4 x 4 ways. We may then choose one vertex of the remaining three tetrahedra and its transform under $\underline{P}_3^2 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0$ in 3×3 ways. This will uniquely determine the operation, and also the remaining pair of polar a-lines left fixed by $\underline{P}_3^2 . \underline{O}^2 . \underline{A}_0 . \underline{C}_0$, and since the operation could be derived from the second pair of a-lines in the same way, we have

XV 2160 $\left(= \dfrac{45 \times 4 \times 4 \times 3 \times 3}{2} \right)$ operations of period 6.

We have now determined the 25920 operations in $\mathcal{G}_{25920}$; a table giving these operations will be found on page 116.

(b) In Todd (18) there is an account of the generation of the simple group in S_4 which is isomorphic with $\mathcal{G}_{25920}$ by means of a set of 45 homologies of period 2. These 45 homologies correspond to the involutary biaxial homographies of type VI in $\mathcal{G}_{25920}$ and the group may therefore be generated by this set of operations.

The biaxial homographies may be represented by the sets $(ij.kl.mn)$, (ij) where i, j, k, l, m, n are all different and chosen from the numbers 1,...,6 and the following relations exist between them:

1. $(ij) = (ji)$ and $(ij.kl.mn) = (kl.mn.ij) = (mn.ij.kl)$.

2. If three biaxial homographies $\underline{A}$, $\underline{B}$, $\underline{C}$ have axes a, a', b, b', c, c' then

(a) if a, b and c are coplanar and concurrent, then so are a', b' and c',

(b) if a, b and c are coplanar but not concurrent, then a', b', c' are concurrent but not coplanar.

3. (a) $(ij)(jk) = (jk)(ki) = (ki)(ij)$.

(b) $(ij)(ik.j\ell.mn) = (ik.j\ell.mn)(jk.i\ell.mn) = \left\{ \begin{matrix} (jk.i\ell.mn) \\ (ij) \end{matrix} \right\}$

(c) $(i\ell.jm.kn)(im.jn.k\ell) = (im.jn.k\ell)(in.j\ell.km)$

$= (in.j\ell.km)(i\ell.jm.kn)$.

4. All other products of two biaxial homographies not covered by 3. are of period 2.

Three biaxial homographies satisfying 2.(a) will be said to be concurrent and coplanar, while if they satisfy 2.(b) they will be said to be either concurrent and non-coplanar or coplanar and non-concurrent.

If we consider the set of biaxial homographies whose axes are the pairs of lines $A_0 B_0 C_0 D_0$ and $U_1 U_2 U_3 O$; $A_0 B_1 C_2 D_1$ and $W_1 W_2 W_3 O$; $B_0 P_1 Y_2 W_2$ and $B_1 U_1 Q_2 Z_1$; $C_0 W_1 X_3 Q_2$ and $C_2 U_3 Y_3 R_3$, $D_0 W_3 R_3 Z_1$ and $D_1 P_1 U_2 X_3$ we find that they generate a $\mathcal{G}_{720}$ (see page 47.) and may be denoted by the symbols (ij), (ik), $(i\ell)$, (im), (in). The forty-five biaxial homographies yield thirty-six sets of this type which generate distinct subgroups of $\mathcal{G}_{25920}$ all isomorphic with $\mathcal{S}_6$.

The set of biaxial homographies

(ij), (ik), (jk),

$(i\ell.jm.kn)$, $(im.jn.k\ell)$, $(in.j\ell.km)$,

$(im.kn.j\ell)$, $(i\ell.km.jn)$, $(in.k\ell.jm)$.

is such that one axis of each lies in a j-plane and the remaining nine axes all pass through the centre polar to the given plane. In the array given above, the twelve rows, columns and diagonals give the twelve sets of concurrent, coplanar biaxial homographies.

The four faces of a polar tetrahedron contain in all the thirty-six axes of eighteen biaxial homographies. One such set consists of the homographies whose symbols are (ij), (ik), (jk), (lm), (ln), (mn), (il.jm.kn), (im.jn.kl), (in.jl.km), (il.jn.km), (in.jm.kl), (im.jl.kn), (il.kn.jm), (im.kl.jn), (in.km.jl), (il.km.jn), (in.kl.jm), (im.kn.jl). From these eighteen homographies it is possible to choose three sets of six, such as (ij), (ik), (jk), (lm), (ln), (mn), so that their twelve axes lie three in each face and three through each vertex of the given polar tetrahedron; the other two such sets are (il.jm.kn), (im.jn.kl), (in.jl.km), (il.kn.jm), (im.kl.jn), (in.km.jl) and (il.jn.km), (in.jm.kl), (im.jl.kn), (il.km.jn), (in.kl.jm), (im.kn.jl).

It is possible to choose sets of five biaxial homographies whose ten axes are mutually skew; one such set is that consisting of biaxial homographies whose axes are $A_0B_0C_0D_0$ and $U_1U_2U_3O$; $B_1R_1X_2V_3$ and $B_2P_2W_2Y_1$; $A_1X_3Y_3Z_3$ and $A_2P_3Q_3R_3$; $C_1Q_2W_3X_1$ and $C_2P_1V_1Z_2$, $D_1Q_1Y_2V_2$ and $D_2R_2W_1Z_1$, and these homographies may be denoted by (ij), (kl), (mn), (ij.kl.mn), (ij.mn.kl). There are twenty-seven such sets and they will be known as Jordan sets (after Todd (18), page 328 (i).). Any product of five biaxial homographies which all belong to a Jordan set leaves fixed each of the forty centres of the configuration, for

$$
\begin{aligned}
&(ij).(kl).(mn).(ij.kl.mn).(ij.mn.kl)\ \{A_0\}\\
&= (ij).(kl).(mn).(ij.kl.mn)\ \{U_3\}\\
&= (ij).(kl).(mn)\ \{B_0\}\\
&= (ij).(kl)\ \{U_1\}\\
&= (ij)\ \{A_0\}\\
&= \{A_0\}
\end{aligned}
$$

and similarly for all the remaining centres. Hence any product such as (ij)(kl)(mn).(ij.kl.mn).(ij.mn.kl) is the identical operation.

We are now in a position to be able to enumerate the operations of $\mathcal{C}_{125920}$ as products of biaxial homographies of period 2, and an indication of the process will be given here.

Any group isomorphic with $\mathcal{S}_6$ contains ten different sets of elements, each set being a conjugate set within the given group, and corresponding to the conjugate sets of the cycles (12), (123), (1234), (12345), (123456), (12)(34), (12)(345), (12)(3456), (123)(456), (12)(34)(56). Let us now consider the operation (ij).(kl).(mn) which is an element of the last type in the $\mathcal{G}_{720}$ generated by (ij), (ik), (il), (im), (in). Since (ij), (kl) and (mn) belong to a Jordan set, we have that

$$(ij).(kl).(mn) = (ij.kl.mn).(ij.mn.kl)$$

so that this operation is also the product of two biaxial homographies whose axes are skew belonging to one of the remaining groups of order 720 contained in $\mathcal{G}_{25920}$. Similarly the operation (ij)(kl)(km)(kn) arises in $\mathcal{G}_{25920}$ as a product of fewer biaxial homographies, for

$$\begin{aligned}
&(ij).(kl).(km).(kn)\\
&\quad = (mn).(ij.kl.mn).(ij.mn.kl).(km).(kn)\\
&\quad = (mn).(ij.kl.mn).(ij.mn.kl).(mn)(km)\\
&\quad = (ij.kl.mn).(ij.mn.kl)(km).
\end{aligned}$$

Hence the thirty-six groups in $\mathcal{G}_{25920}$ isomorphic with $\mathcal{S}_6$ yield only eight distinct types of operations in $\mathcal{G}_{25920}$. It is found that the operations of types VI, IV, XIII, XII, XV, XIV, VIII and III correspond to the conjugate sets of (12), (123), (1234), (12345), (123456), (12)(34), (12)(345), (123)(456) respectively.

The nine biaxial homographies which are such that one axis of each lies in a given j-plane π, and the other axes all pass through the centre polar to π yield two new types of operation. A product such as (ij).(ik).(im.jn.kl) is of period 6 and is not of type XV or type VIII for

$\{(ij).(ik).(im.jn.kl)\}^3$

$= (ij).(ik).(im.jn.kl).(ij)(ik).(im.jn.kl).(ij).(ik).(im.jn.kl)$

$= (jk).(jm.in.kl).(km.jn.il).(jk).(im.jn.kl)$

$= (km.in.jl).(jm.kn.il).(im.jn.kl)$

$= (jm.kn.il).(im.jn.kl).(im.jn.kl)$

$= (jm.kn.il)$

so that $(ij).(ik).(im.jn.kl)$ is of period 6.

Also $\{(ij).(ik).(im.jn.kl)\}^2$

$= (ij).(ik).(im.jn.kl).(ij).(ik).(im.jn.kl)$

$= (jk).(jm.in.kl).(ik).(im.jn.kl)$

$= (jk).(ik).(jm.kn.il).(im.jn.kl)$

and $(jk).(ik).(jm.kn.il)(im.jn.kl)$ leaves fixed each point of π, for consider the effect of the operation on the point $\{(ij),(ik)\}$ which is the point of intersection of the axes of (ij) and (ik) lying in π:

$(jk).(ik).(jm.kn.il).(im.jn.kl)\ \{(ij),(ik)\}.$

$= (jk).(ik).(jm.kn.il)\ \{(jm.in.kl),(km.jn.il)\}$

$= (jk).(ik)\ \{(ik),(jk)\}$

$= (jk)\ \{(ik),(ij)\}$

$= \{(ij),(ik)\}$

and similarly for every other point of π. Hence $(ij).(ik).(im.jn.kl)$ and $(jk).(ik).(jm.kn.il)(im.jn.kl)$ are both new operations belonging to types V and II respectively.

The set of thirteen biaxial homographies which are all the members of the three Jordan sets containing one particular homography yield one further operation. Such a set of homographies is

$$
\begin{array}{lllll}
(ij), & (k\ell), & (mn), & (ij.k\ell.mn), & (ij.mn.k\ell), \\
 & (km), & (\ell n), & (ij.km.\ell n), & (ij.\ell n.km), \\
 & (kn), & (\ell m), & (ij.kn.\ell m), & (ij.\ell m.kn).
\end{array}
$$

Then $(km).(\ell n).(k\ell).(ij.k\ell.mn)$ is an operation of period 4 whose square is of type VI, for

$$
\begin{aligned}
&\{(km).(\ell n).(k\ell).(ij.k\ell.mn)\}^2 \\
&= (km).(\ell n).(k\ell).(ij.k\ell.mn).(km)(\ell n).(k\ell).(ij.k\ell.mn) \\
&= (\ell n).(m\ell).(ij.\ell m.kn).(\ell n).(k\ell).(ij.k\ell.mn) \\
&= (mn).(ij.mn.k\ell).(k\ell).(ij.k\ell.mn) \\
&= (ij).
\end{aligned}
$$

Thus $(km).(\ell n).(k\ell).(ij.k\ell.mn)$ is an operation of period 4 not belonging to type XIII, and it is found that it belongs to type VII.

The eighteen biaxial homographies whose axes all lie in the faces of a polar tetrahedron yield two new types of operation. $(ij).(ik).(in.k\ell.mn).(\ell m)$ is one of these operations and is of period 9 for

$$
\begin{aligned}
&\{(ij).(ik).(in.k\ell.mn).(\ell m)\}^3 \\
&= \{(ij).(ik).(in.k\ell.jm).(\ell m).(ij).(ik).(in.k\ell.jm).(\ell m).(ij).(ik).(in.k\ell.jm).(\ell m)\} \\
&= (jk).(jn.k\ell.im).(\ell m).(kn.i\ell.jm).(\ell m).(jk).(in.k\ell.jm).(\ell m) \\
&= (kn.j\ell.im).(\ell m).(jn.i\ell.km).(in.k\ell.jm) \\
&= (\ell m).(\ell n).(k\ell.jm.in).(jn.i\ell.km)
\end{aligned}
$$

which is of type II and period 3; hence operations such as $(ij).(ik).(in.k\ell.jm).(\ell m)$ are of period 9 and type XI.

An operation of the second new type arising from a polar tetrahedron is of period 6. A typical example is

$(ij).(ik).(i\ell.jm.kn).(in.j\ell.km).(\ell m).$

$$\{(ij).(ik).(i\ell.jm.kn).(in.j\ell.km).(\ell m)\}^2$$
$$= \{(ij).(ik).(i\ell.jm.kn).(in.j\ell.km).(\ell m).(ij).(ik).(i\ell.jm.kn).(in.j\ell.km).(\ell m)\}$$
$$= (jk).(j\ell.im.kn).(jn.i\ell.km).(ik).(im.j\ell.kn).(in.jm.k\ell)$$
$$= (jk).(j\ell.im.kn).(jn.k\ell.im).(jn.i\ell.km).(i\ell.jn.km).(im.j\ell.kn)$$
$$= (j\ell.im.kn).(k\ell.im.jn).(jn.k\ell.im).(jn.i\ell.km).(i\ell.jn.km).(im.j\ell.kn)$$
$$= (j\ell.im.kn).(jn.i\ell.km).(i\ell.jn.km).(im.j\ell.kn)$$

which is of type II and period 3, so that either $(ij).(ik).(i\ell.jm.kn).(in.j\ell.km).(\ell m)$ is of type III or else it is a new operation of period 6.

$$\{(ij).(ik).(i\ell.jm.kn).(in.j\ell.km).(\ell m)\}^3$$
$$= \{(ij).(ik).(i\ell.jm.kn).(in.j\ell.km).(\ell m).(j\ell.im.kn).(jn.i\ell.km).(i\ell.jn.km).(im.j\ell.kn)\}$$
$$= \{(\ell m).(ij).(ik).(im.j\ell.kn).(in.jm.k\ell).(j\ell.im.kn).(jn.i\ell.km).(i\ell.jn.km).(im.j\ell.kn)\}$$
$$= (\ell m).(ij).(ik).(i\ell.jn.km).(j\ell.im.kn).(jn.i\ell.km).(in.jm.k\ell)$$
$$= (\ell m).(ij)(k\ell.jn.im).(j\ell.km.in).(jn.k\ell.im).(in.jm.k\ell).(kn.jm.i\ell)$$
$$= (\ell m).(k\ell.jn.im).(k\ell.in.jm).(in.jm.k\ell).(\ell m).(mn)(kn.jm.i\ell)$$
$$= (km.jn.i\ell).(mn).(kn.jm.i\ell)$$
$$= (mn).(kn.jm.i\ell).(kn.jm.i\ell)$$
$$= (mn).$$

$(ij).(ik).(im.j\ell.kn).(in.jm.k\ell).(\ell m)$ is thus of period 6 and operations of this sort belong to type IX.

The last remaining type of operation of $\mathcal{C}_{25920}$ is again a product of five biaxial homographies. A typical example is $(i\ell).(ij.mn.k\ell).(in.km.j\ell).(in).(in.m\ell.jk)$ where $(i\ell).(ij.mn.k\ell).(in.km.j\ell)$ is a product leaving fixed the pair of axes of $(m\ell)$ while (in) and $(in.m\ell.jk)$ belong to one of the Jordan sets containing $(m\ell)$.

The square of such an operation belongs to type V for

$$
\begin{aligned}
&(i\ell).(ij.mn.k\ell).(in.km.j\ell).(in).(in.\ell m.jk) \\
&= (\ell n)(i\ell).(jn.im.k\ell).(in.km.j\ell).(in.\ell m.jk) \\
&= (\ell n)(i\ell).(k\ell).(jn.im.k\ell).(in.km.j\ell)
\end{aligned}
$$

so that

$$
\begin{aligned}
&\{(i\ell).(ij.mn.k\ell).(in.km.j\ell).(in).(in.\ell m.jk)\}^2 \\
&= \{(\ell n).(i\ell).(k\ell).(jn.im.k\ell).(in.km.j\ell).(\ell n)(i\ell).(k\ell).(jn.im.k\ell).(in.km.j\ell)\} \\
&= (in).(kn).(j\ell.im.kn).(i\ell.km.jn).(ik).(i\ell).(jn.im.k\ell).(in.km.j\ell) \\
&= (kn).(j\ell.km.in).(k\ell.im.jn).(i\ell).(jn.im.k\ell).(in.km.j\ell) \\
&= (kn).(i\ell).(ij.km.\ell n).(ik.\ell m.jn)(jn.im.k\ell).(in.km.j\ell) \\
&= (kn).(i\ell).(km).(ij.km.\ell n).(ik.\ell m.jn).(in.km.j\ell) \\
&= (kn).(i\ell).(km).(ij.km.\ell n).(in.km.j\ell).(i\ell.mn.jk) \\
&= (kn).(i\ell).(km).(jn).(ij.km.\ell n).(i\ell.mn.jk) \\
&= (mn).(km).(jn).(i\ell).(i\ell.mn.kj).(ij.km.\ell n) \\
&= (jm).(mn).(jk).(i\ell).(i\ell.mn.jk).(ij.km.\ell n) \\
&= (jm).(i\ell.jk.mn).(ij.km.\ell n).
\end{aligned}
$$

$(jm).(il.jk.mn).(ij.km.ln)$ is of type V and period 6 and so $(il).(ij.mn.kl).(in.km.jl).(in).(in.lm.jk)$ is of period 12 and is an operation of type X.

In the table of operations of G_{25920} given on the next page, the last column gives the form of a typical operation of a particular type expressed as a product of biaxial homographies of type VI. It should be noted that an operation of type XV can be expressed as a product of four biaxial homographies, for a typical operation of this type is $(ij).(ik).(il).(im).(in)$ and we find that

$$\begin{aligned}(ij).(ik).(il).(im).(in) &= (ij).(kl).(ik).(mn).(im)\\ &= (ij).(kl).(mn).(ik).(im)\\ &= (ij.kl.mn).(ij.mn.kl).(ik).(im).\end{aligned}$$

Operations of G_{25920}.

Type.	p.	n.	Type of powers.	N.	
I	1	-		1.	$(ij)^2$.
II	3	1		80.	(ij).(ik).(in.kl.jm).(il.km.jn).
III	3	2		240.	(ij).(ik).(lm).(ln).
IV	3	3		480.	(ij).(ik).
V	6	2	2^{nd}.II, 3^{rd}.VI.	720.	(ij).(ik).(im.jn.kl).
VI	2	3		45.	(ij).
VII	4	3	2^{nd}.VI.	540.	(km).(ln).(kl).(ij.kl.mn).
VIII	6	3	2^{nd}.IV, 3^{rd}.VI.	1440.	(ij).(kl).(km).
IX	6	4	2^{nd}.III, 3^{rd}.VI.	1440.	{(ij).(ik).(il.jm.kn).(in.jl.km).(lm).}
X	12	3	2^{nd}.V, 3^{rd}.VII, 4^{th}.II, 6^{th}.VI.	4320.	{(in).(ln).(kl).(jn.im.kl).(in.km.jl)}.
XI	9	4	3^{rd}.II.	5760.	(ij).(ik).(in.kl.jm).(lm).
XII	5	4		5184.	(ij).(ik).(il).(im).
XIII	4	5	2^{nd}.XIV.	3240.	(ij).(ik).(il).
XIV	2	4		270.	(ij).(kl).
XV	6	6	2^{nd}.III, 3^{rd}..XIV.	2160	(ij).(ik).(il).(im)(in).

§VI. The Remaining Primitive Groups in S_3.

The twelve groups in S_3 which have invariant intransitive subgroups have been obtained in at least two different ways (Bagnera (1) and Goursat (9)) and are found to be the set of groups leaving fixed a quadric. We have already discussed the groups generated by biaxial homographies which leave fixed a quadric and found that these are six in number; it follows that the remaining six groups with invariant, intransitive subgroups cannot be generated by biaxial homographies and we will not therefore investigate them further.

There remain just two primitive groups in S_3 which we have not discussed and will briefly mention here; these are of orders 168 and 2520 and are both isomorphic with simple subgroups of $\mathfrak{S}_7$. $\mathfrak{C}_{1168}$ is isomorphic with Γ_{168} and has been extensively studied by W.L. Edge (7); it contains a set of twenty-one involutary biaxial homographies which generate it. $\mathfrak{C}_{12520}$ is isomorphic with the alternating group of degree seven; sets of generators for $\mathfrak{C}_{1168}$ and $\mathfrak{C}_{12520}$ are given in Blichfeldt (5) page 142.

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