CLASSIFICATION THEORY OF SIMPLE LOCALLY FINITE GROUPS

BY

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A thesis submitted in partial fulfillment

of the requirements for the degree of

DOCTOR OF PHILOSOPHY

at

BEDFORD COLLEGE, UNIVERSITY OF LONDON

1983

ABSTRACT

This thesis constitutes a contribution to applied stability theory. We consider the classification problem of the stable simple locally finite groups.

First the classification of the finite simple groups is used to reduce the problem to an identification problem for the simple locally finite groups of Lie type and an interpretation problem in model theoretic algebra.

In chapter three, the identification problem is solved. It is shown that the union of a chain of groups of the same Lie type over finite fields is a group of Lie type over a locally finite field. This result, together with the classification of the finite simple groups, implies that an infinite simple periodic linear group is a group of Lie type over a locally finite field.

The next two chapters solve the interpretation problem, and complete the proof that a stable simple locally finite group is a Chevalley group over an algebraically closed field. We also show that the class of Chevalley groups of a fixed Lie type is finitely axiomatisable.

Chapter six contains a partial classification of the nonsoluble locally finite groups of finite Morley rank.

In the final chapter, we show that a simple constructible group over an algebraically closed field is a Chevalley group. The proof is model theoretic, and makes no use of algebraic geometry or Lie algebras. This result can be regarded as a nonstandard corollary of the classification of the finite simple groups.

Acknowledgements

I have had the good fortune to have been given valuable assistance by many logicians and algebraists. I would especially like to thank G. Cherlin, U. Felgner, M. Roberts, A. Schofield and B.A.F. Wehrfritz.

It is a pleasure to thank my supervisor, Wilfrid Hodges, who gave more help and guidance than a research student could reasonably expect. My first results were obtained after he had spent many hours translating Zilber's Kemerovo paper for me. This, together with his enthusiastic encouragement, gave me the confidence necessary to do research.

Thanks are due to Kendal Anderson for the beautiful typing of this thesis, as well as apologies for the many multi-indexed formulae.

I would like to thank the Science Research Council for their financial assistance in the form of a Research Studentship.

This thesis is dedicated to my mother and father in gratitude for their love and support.

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Chapter One: Introduction

This thesis will present some results on the borders of stability theory and algebra. The stability theory necessary for an understanding of this work will be surveyed in section 1.2. This should make the thesis comparatively easy reading for any algebraist who is familiar with the theory of the simple groups of Lie type. Unfortunately a logician who wishes to understand the proofs will have to do some preparatory work. To make this thesis self-contained, it would be necessary to summarise the first thirteen chapters of Carter's "Simple groups of Lie type". As a compromise, section 1.4 provides a guided tour of the familiar group PSL(3,K). In this concrete setting, we shall explain the meaning of such important concepts as root subgroup, Weyl subgroup, etc. This section is included for two reasons.

i) It should make [6] and [7] more accessible to the reader.

ii) If he keeps this concrete example in mind, the reader will find the algebraic proofs fairly transparent.

We begin with a non-technical account of the classification programme. It is hoped that after reading this, the algebraist will be keen to learn the logical methods that occasionally allow this ambitious programme to be carried through.

1.1 The classification programme

Some of the most satisfying theorems in classical algebra describe all structures which possess certain properties. In the language of model theory, the algebraist begins with a fixed theory T and then describes all the models of T in terms of invariants.

Examples

1) T = algebraically closed fields

Structure theory

An algebraically closed field of fixed characteristic is determined up to isomorphism by the cardinality of its transcendence basis over the prime subfield.

2) T = divisible abelian groups

Structure theory

A divisible abelian group has the form

$$[\bigoplus_{i\in I_0} \mathbb{Q}] \oplus \ldots \oplus [\bigoplus_{i\in I_p} \mathbb{Z}(p^{\infty})] \oplus \ldots$$

and is determined up to isomorphism by the cardinalities of its index sets.

In the last 10 years, some logicians have been interested in this type of phenomenon. However, as usual, they have attacked the problem backwards. The classification programme is concerned with the question:

IF A THEORY T HAS A STRUCTURE THEORY, WHAT IS T?

In other words, have the algebraists already found all those theories which possess a structure theory? Perhaps surprisingly, it is sometimes possible to answer this question. But first it must be posed in a less vague form.

What is a theory?

Throughout this thesis, we will work with complete first order theories. (A theory T is complete if for every sentence ϕ in the language, either $\phi \in T$ or $\neg\phi \in T$.) A typical example is the theory of algebraically closed fields in some fixed characteristic.

What is a structure theory?

Here things are not so clear. However, it seems difficult to imagine a satisfactory structure theory existing for a theory T which has 2^{κ} nonisomorphic models in every uncountable cardinality κ. We shall take the existence of too many models as an indication that T does not possess a structure theory.

Thus one form of our original question is: if a complete first order theory T has less than 2^{κ} nonisomorphic models in some uncountable cardinality κ, what is T?

Less ambitiously, we could try to classify those theories which have very few models.

Definition 1

Let κ be an infinite cardinal. A complete first-order theory T is κ-categorical if it has a unique model (up to isomorphism) of cardinality κ.

The following theorem says that there are essentially two kinds of categoricity.

Theorem 2 (Morley [24])

Let T be a complete first-order theory in a countable language. If T is categorical in some uncountable cardinality, then T is categorical in every uncountable cardinality.

The first results in the classification programme were published by MacIntyre in 1971. (If M is a structure, ThM is the set of all sentences, in the appropriate first order language, which are satisfied in M. Clearly ThM is a complete theory.)

Theorem 3 (MacIntyre [20])

If G is an abelian group, then ThG is ω_1-categorical iff G is of one of the following forms:

i) $K \oplus H$, where H is finite and K is a direct sum of copies of a fixed finite cyclic group of prime-power order;

ii) $D \oplus H$, where H is finite and D is a divisible group with the property that for each prime p there are only finitely many elements of D of order p.

Theorem 4 (MacIntyre [21])

If F is a field, then ThF is ω_1-categorical iff F is algebraically closed.

The latter result was improved in [11].

Theorem 5 (Cherlin, Shelah)

If F is an infinite field, then ThF has less than 2^{κ} nonisomorphic models in some uncountable cardinality κ iff F is algebraically closed.

Complete classification of the ω_1-categorical theories have also been obtained for:

i) Noetherian commutative rings (Cherlin, Reineke [10], Zilber [37])

ii) semisimple rings (Felgner [14])

iii) skew fields (Shelah [26], Zilber [37], Cherlin [8]).

The study of the ω_1-categorical theories of nonabelian groups was initiated by Zilber [37]. He showed that if G is a simple algebraic matrix group over an algebraically closed field, then ThG is ω_1-categorical. He and Cherlin [9] also realised that, conversely, if ThG is ω_1-categorical then G seems to resemble an algebraic matrix group over an algebraically closed field. (More details will be given in a later section.) This led them to make the following conjecture.

Conjecture 6

Let G be a simple group. Then ThG is ω_1-categorical iff G is an algebraic matrix group over an algebraically closed field.

Unfortunately this appears to be very difficult to prove with the techniques currently available. In this thesis, following a suggestion of Cherlin, we shall restrict our attention to simple locally finite groups.

Definition 7

A group G is locally finite if for each finite subset $X \subseteq G$, the subgroup generated by X is finite.

In the next four chapters, we shall prove:

Theorem 8

Let G be an infinite simple locally finite group. Then the following are equivalent:

i) ThG has less than 2^{κ} nonisomorphic models in some uncountable cardinality κ.

ii) ThG has a unique model in every uncountable cardinality κ.

iii) G is a Chevalley group over $\bar{\mathbb{F}}_p$ for some prime $p > 0$.

($\bar{\mathbb{F}}_p$ denotes the algebraic closure of the field with p elements.)

1.2 A survey of stability theory

In this section, we shall discuss the model-theoretic techniques which are used in the classification programme. The reader is assumed to be familiar with the notions of a first order language L, a model for L, and first order satisfaction. A very clear account of these topics can be found in Barwise [3].

Notation and conventions

We use α, β, γ, ξ, η for ordinals; k, ℓ, m, n for natural numbers; κ, λ for cardinals. δ is reserved for limit ordinals. An ordinal is the set of preceding ordinals, i.e.

$$\alpha = \{\beta \mid \beta < \alpha\} = \{\beta \mid \beta \in \alpha\}.$$

ω_α denotes the αth infinite cardinal. ω is the cardinality of the natural numbers.

L always denotes a first order language with equality. Variables are denoted by x, y, z; finite sequences of variables by $\bar{x}$, $\bar{y}$, $\bar{z}$. We use $\phi(x_1,\ldots,x_n)$ to denote a formula ϕ whose free variables form a subset of $\{x_1,\ldots,x_n\}$. The language of groups is $\{x,1\}$ and the language of fields is $\{x,+,1,0\}$.

If M is a structure, we write $|M|$ for its cardinality. a, b, c, g, h will denote elements of M; $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{g}$, $\bar{h}$ finite sequences of elements. We often write $\bar{a} \in M$.

If $\phi(\bar{x})$ is a formula and $\bar{a} \in M$, we write $M \models \phi(\bar{a})$ to mean that $\phi(\bar{a})$ is true in M. ThM is the set of all sentences true in M. If $\phi(\bar{x},\bar{y})$ is a formula and $\bar{a} \in M$,

$$\phi(M,\bar{a}) = \{\bar{b} \in M \mid M \models \phi(\bar{b},\bar{a})\}.$$

We quite happily confuse formulas $\phi(\bar{x},\bar{a})$ with the subsets which they define $\phi(M,\bar{a})$.

When we say "it is easily shown that...", we mean that the result follows from simple applications of the following basic theorems.

<u>The compactness theorem</u>

A set of sentences Σ in a first order language L has a model iff every finite subset of Σ has a model.

<u>The Lowenheim-Skolem-Tarski theorem</u>

Let M be a model of cardinality κ and let $|L| \leq \lambda \leq \kappa$. Given any set $X \subseteq M$ of cardinality $\leq \lambda$, M has an elementary submodel of cardinality λ which contains X.

<u>Definition</u>

Let $M \subseteq N$ be two structures for the language L. M is an <u>elementary submodel</u> of N, written $M \prec N$, if for all formulas $\phi(x_1,\ldots,x_n)$ and all elements $a_1,\ldots,a_n \in M$,

$$M \models \phi(a_1,\ldots,a_n) \text{ iff } N \models \phi(a_1,\ldots,a_n).$$

Having established our notation, we can get down to business.

<u>Types and stability</u>

Suppose that $M \prec N$ and $b \in N$. The <u>type of b over M</u> is a complete description of the relation of b to M. Formally,

$$tp(b,M) = \{\phi(x,\bar{a}) \mid N \models \phi(b,\bar{a})\}$$

where $\bar{a} \in M$ and ϕ is a formula in the language L. Define

$$FM = \{\phi(x,\bar{a}) \mid \phi \in L, \bar{a} \in M\}.$$

Then, more generally, any maximal consistent subset of FM is called a <u>type over M</u>. It is easily shown that

<u>Theorem 1</u>

Let p be a type over M. There exists an elementary extension $N \succ M$ such that:

i) $|M| = |N|$

ii) There is an element $b \in N$ which realises p, i.e. for all $\phi(x,\bar{a}) \in p$, $N \models \phi(b,\bar{a})$.

Thus a type is a description of a way of adjoining an element in some elementary extension of M. Define

$$SM = \{p \mid p \text{ is a type over } M\}.$$

Example

Let K be an algebraically closed field. Then any formula $\phi(x,\bar{k})$ is equivalent to a boolean combination of polynomials in $\bar{k}$. Thus a type p is determined by the subsets

$$X = \{g(x,\bar{k}) = 0 \mid g(x,\bar{k}) \in K[x]\} \cap p$$

$$Y = \{g(x,\bar{k}) \neq 0 \mid g(x,\bar{k}) \in K[x]\} \cap p.$$

Suppose that $X \neq \emptyset$. Choose "$g(x,\bar{k}) = 0$" $\in p$ of minimal degree. Since K is algebraically closed, deg $g = 1$ and "$x-k = 0$" $\in p$ for some $k \in K$. Thus p is realised in K. On the other hand, if $X = \emptyset$ then p describes an element which is transcendental over K.

We are interested in theories T with less than the maximum number of models. It seems reasonable to expect that if there are many distinct ways of adjoining elements to models of T, then T has many nonisomorphic models. (Of course, life is not quite this simple. Consider countable dense linear orders!) This provides motivation for

Definition 2

Let λ be an infinite cardinal. The theory T is λ-stable if for all models $M \models T$,

$$|M| = \lambda \text{ implies } |SM| = \lambda.$$

The following theorem is due to Shelah [27].

The stability spectrum theorem

Let T be a complete countable first order theory. Then one of the following clauses holds:

i) T is λ-stable for all $\lambda \geq \omega$.

ii) T is λ-stable for all $\lambda \geq 2^{\omega}$.

iii) T is λ-stable for all λ of the form $\lambda^{\omega} = \lambda$.

iv) T is not λ-stable for any cardinal λ.

Definition 3

a) If (i) holds, T is ω-stable.

b) If (ii) holds, T is superstable.

c) If (iii) holds, T is stable.

d) If (iv) holds, T is unstable.

Notice that ω-stable $\Rightarrow$ superstable $\Rightarrow$ stable. In general, these are strict implications. The classification programme is based on the following very difficult theorem of Shelah [27].

Theorem 4

Let T be a complete countable first order theory. If T is non-superstable, then T has 2^{κ} nonisomorphic models in every uncountable cardinality κ.

For example, to prove theorem 5 of the previous section, Cherlin and Shelah classified the superstable theories of infinite fields. There is a very simply stated criterion for unstability.

Theorem 5 (Shelah [27])

A complete theory T is unstable iff there is a formula $\phi(\bar{x},\bar{y})$, a model $M \models T$, and an infinite subset $X = \{\bar{a}_n | n \in \omega\}$ such that

$$M \models \phi(\bar{a}_n,\bar{a}_m) \quad \text{iff} \quad n < m.$$

Unfortunately it is not always an easy task to determine whether a theory is unstable. It is not known if the theory of the free group on two generators is stable. Rather than use theorem 5 directly, we shall use the following consequence.

Theorem 6 (Baldwin, Saxl [2])

Let G be a group. If ThG is stable, then G satisfies the minimal condition on centralisers.

Definition 7

G is an M_c-group if it satisfies the minimal condition on centralisers, i.e. there is no infinite descending chain

$$C(X_0) \supsetneq C(X_1) \supsetneq \cdots \supsetneq C(X_n) \supsetneq \cdots$$

of centralisers, where each $X_n \subseteq G$.

Following the usual convention, we shall say that a structure M is λ-stable if its first order theory ThM is λ-stable. Similarly, we shall speak of ω_1-categorical structures.

The method of interpretations

After we have proved that a stable simple locally finite group is a group of Lie type, we shall show that the underlying field is algebraically closed. We do this via the method of interpretations.

Definition 8

Let M and N be structures and σ a mapping from a subset of M^k onto N (for some natural number k). We say that σ is an interpretation of N in M if:

i) The domain D_σ of σ is a subset of M^k definable with parameters, i.e. there exists $\bar{a} \in M$ and $\phi(\bar{x},\bar{y}) \in L$ such that

$$\bar{b} \in D_\sigma \text{ iff } M \models \phi(\bar{b},\bar{a})$$

ii) The preimages of the equality relation and all predicates, functions and constants in the language of N are definable in M with parameters.

Two examples

1) Let G be a group with centre Z(G). Let $\sigma: G \to G/Z(G)$ be the canonical homomorphism. Then σ is an interpretation of G/Z(G) in G.

i) D_σ is defined by the formula "x = x".

ii) For $g,h \in G$, $\sigma(g) = \sigma(h)$ iff

$$G \models (\exists x)(hx = 1 \wedge (\forall y)(xgy = yxg)).$$

iii) For $a,g,h \in G$, $\sigma(a) = \sigma(g).\sigma(h)$ iff

$$G \models (\exists x)(ax = 1 \wedge (\forall y)(xghy = yxgh)$$

iv) For $g \in G$, $\sigma(g) = 1$ iff

$$G \models (\forall x)(xg = gx).$$

2) The two structures may have different languages. We can interpret the group $\langle K,+,0\rangle$ in the field $\langle K,\times,+,1,0\rangle$.

Less trivial examples will be given in later chapters. The following theorem is due to Zilber [37].

Theorem 9

Let M be a λ-stable structure. If the structure N is interpretable in M, then N is λ-stable.

The Morley rank of a definable subset

Vaught has shown that if M is ω_1-categorical and $\phi(x,\bar{a})$ is a definable subset of M, then either $\phi(M,\bar{a})$ is finite or $|\phi(M,\bar{a})| = |M|$. So the cardinality of a definable subset of an ω_1-categorical structure is an extremely crude measure of its size. A more useful measure is provided by the Morley rank. First we must introduce the notion of λ-saturation.

Definition 10

Let $A \subseteq M$ be any subset. p is a type over A in M if:

i) p is a set of formulas of the form $\phi(x,\bar{a})$ where $\bar{a} \in A$.

ii) p is consistent with M, i.e. for every finite $q \subseteq p$,

$$M \models (\exists x) \bigwedge_{\phi(x,\bar{a})\in q} \phi(x,\bar{a}).$$

iii) For all formulas $\phi(x,\bar{a})$ where $\bar{a} \in A$, either $\phi(x,\bar{a}) \in p$ or $\neg\phi(x,\bar{a}) \in p$.

Thus a type over A in M is a maximal consistent set of formulas with parameters in A.

Definition 11

A model M is λ-saturated if every type in M over some $A \subseteq M$, $|A| < \lambda$, is realised in M.

The Morley rank of a definable subset is a measure of how finely we may partition it using other definable subsets. It is convenient to work with ω_1-saturated models.

Definition 12

Let M be an ω_1-saturated model and $\phi(x,\bar{a}) \in FM$. We define the Morley rank $R(\phi(x,\bar{a}))$ by defining inductively when $R(\phi(x,\bar{a})) \geq \alpha$, α an ordinal.

1) $R(\phi(x,\bar{a})) \geq 0$ if $\phi(M,\bar{a}) \neq \emptyset$. (If $\phi(M,\bar{a}) = \emptyset$ we stipulate that $R(\phi(x,\bar{a})) = -1$.)

2) $R(\phi(x,\bar{a})) \geq \delta$ for δ a limit ordinal if $R(\phi(x,\bar{a})) \geq \alpha$ for all $\alpha < \delta$.

3) $R(\phi(x,\bar{a})) \geq \alpha+1$ if for every $n \in \omega$ there exist $\phi_i(x,\bar{a}_i) \in FM$, $1 \leq i \leq n$, such that:

a) $R(\phi_i(x,\bar{a}_i)) \geq \alpha$ for $1 \leq i \leq n$.

b) $\phi(M,\bar{a}) = \bigsqcup_{1\leq i\leq n} \phi_i(M,\bar{a}_i)$, where $\bigsqcup$ denotes the disjoint union.

4) If $R(\phi(x,\bar{a})) \geq \alpha$ but not $R(\phi(x,\bar{a})) \geq \alpha+1$, we say that $R(\phi(x,\bar{a})) = \alpha$. If $R(\phi(x,\bar{a})) \geq \alpha$ for all α, we define $R(\phi(x,\bar{a})) = \infty$.

Definition 13

If $R(\phi(x,\bar{a})) = \alpha \neq \infty$, we define the Morley degree $\deg(\phi(x,\bar{a}))$ to be the greatest natural number n such that there exist formulas $\phi_i(x,\bar{a}_i) \in FM$, $1 \leq i \leq n$, with:

a) $R(\phi_i(x,\bar{a}_i)) = \alpha$ for $1 \leq i \leq n$.

b) $\phi(M,\bar{a}) = \bigsqcup_{1\leq i\leq n} \phi_i(M,\bar{a}_i)$.

Finally suppose that $\phi(x,\bar{a}) \in FM$, where M is not ω_1-saturated. Then we can find an ω_1-saturated $N > M$, and calculate $R(\phi(x,\bar{a}))$, $\deg(\phi(x,\bar{a}))$ inside N. It is easily proved that the rank and degree are independent of the choice of N. The following properties are almost immediate consequences of the definition.

Proposition 14

Let $\phi(x,\bar{a})$, $\psi(x,\bar{b}) \in FM$

i) $R(\phi(x,\bar{a}) \vee \psi(x,\bar{b})) = \max\{R(\phi(x,\bar{a})), R(\psi(x,\bar{b}))\}$

ii) If $R(\phi(x,\bar{a})) = R(\psi(x,\bar{b}))$ and $\phi(M,\bar{a}) \cap \psi(M,\bar{b}) = \emptyset$, then

$$\deg(\phi(x,\bar{a}) \vee \psi(x,\bar{b})) = \deg(\phi(x,\bar{a})) + \deg(\psi(x,\bar{b})).$$

iii) Every formula of rank α is equivalent in M to a finite disjunction of formulas of rank α and degree 1.

iv) If there is a formula of rank α, α an ordinal, then there are formulas of every rank $\beta < \alpha$.

It follows that the valid formula "x = x" has the largest rank. If M is a model, we call $R(x = x)$ the <u>rank of the model M</u>. If T is a complete countable theory, then any two ω_1-saturated models of T have the same rank. We call this the <u>rank of T</u>. T is called totally <u>transcendental</u> iff its rank is an ordinal, rather than ∞. In [24], Morley proved:

<u>Theorem 15</u>

Let T be a complete countable theory.

i) T is ω-stable iff T is totally transcendental.

ii) If T is totally transcendental, then its rank is a countable ordinal.

iii) If T is ω_1-categorical, then T is ω-stable.

Later Baldwin [1] and Zilber [36] proved:

<u>Theorem 16</u>

Let T be a complete countable theory. If T is ω_1-categorical, then the rank of T is finite.

The converse is not true. As our first application of rank and degree, we shall prove:

<u>Theorem 17</u> (MacIntyre [20])

If G is an ω-stable group, then G satisfies the minimal condition on definable subgroups.

Let $\phi(x,\bar{a})$, $\psi(x,\bar{b}) \in FG$. A bijection π: $\phi(G,\bar{a}) \to \psi(G,\bar{b})$ is said to be definable if there exists a formula $\sigma(x,y,\bar{c})$, $\bar{c} \in G$, such that for all $g \in \phi(G,\bar{a})$ and $h \in \psi(G,\bar{b})$,

$$\pi(g) = h \quad \text{iff} \quad G \models \sigma(g,h,\bar{c}).$$

In this situation, any partition of $\phi(G,\bar{a})$ by definable subsets is mapped onto a corresponding partition of $\psi(G,\bar{b})$. An easy induction gives $R(\phi(x,\bar{a})) \leqslant R(\psi(x,\bar{b}))$. But π^{-1} is also a definable bijection, and hence $R(\phi(x,\bar{a})) = R(\psi(x,\bar{b}))$. Similarly $\deg(\phi(x,\bar{a})) = \deg(\psi(x,\bar{b}))$.

In particular, let H be a definable subgroup of G. Then for all $g \in G$, the right translation map $\pi_g: H \to Hg$ is a definable bijection. Thus

$$R(Hg) = R(H)$$
$$\deg(Hg) = \deg(H).$$

Lemma 18

Let $H_0 \subsetneq H_1$ be definable subgroups of the ω-stable group G. Then either (i) or (ii) holds.

i) $R(H_0) < R(H_1)$.

ii) $R(H_0) = R(H_1)$ and $\deg(H_0) < \deg(H_1)$.

Further, $R(H_0) = R(H_1)$ iff $[H_1:H_0]$ is finite.

Proof

Let $H_1 = \bigsqcup_{\alpha<\beta} H_0 g_\alpha$ be a coset decomposition. Then for all $\alpha < \beta$, $R(H_0 g_\alpha) = R(H_0)$. Hence if β is infinite, $R(H_1) > R(H_0)$.

Suppose that β is finite, say $\beta = n$. Then $H_1 = H_0 g_0 \sqcup \ldots \sqcup H_0 g_{n-1}$. By proposition 14(i),(ii), $R(H_1) = R(H_0)$ and $\deg(H_1) = n \deg(H_0)$.

□

Proof of theorem 17

This is an immediate consequence of theorem 15, lemma 18 and the well-ordering of the ordinal numbers.

□

If G has finite Morley rank, then many important subgroups, such as the commutator subgroup [G,G], are definable. This will be proved in the next section, where we discuss some of the similarities between groups of finite Morley rank and algebraic matrix groups over algebraically closed fields.

1.3 Algebraic matrix group and groups of finite Morley rank

In this section, we will discuss some of Zilber's results on groups of finite Morley rank. First we remind the reader of some basic properties of affine groups. Throughout K will be an algebraically closed field.

Definition 1

A subset S of K^n is closed if it is the set of common zeros of some polynomials $\{f_i\}$ in $K[x_1,\ldots,x_n]$.

Definition 2

An affine group over K is a closed set S with a group law on it in which mult: $S\times S \to S$ and inv: $S \to S$ are polynomial maps.

For example, K^3 may be regarded as an affine group with the noncommutative group law

$$\langle x,y,z\rangle\langle x',y',z'\rangle = \langle x+x',y+y',z+z'+xy'\rangle.$$

Suppose that the affine group S is defined by the polynomials $\{f_i\}$. Let I be the radical of the ideal generated by $\{f_i\}$. The K-algebra of polynomial functions

$$K[S] = K[x_1,\ldots,x_n]/I$$

is an essential tool in the analysis of S. For instance, by considering a certain natural action of S on K[S], it can be shown that S is isomorphic to an algebraic matrix group over K.

Definition 3

An algebraic matrix group over K is a subgroup of SL(n,K) whose matrix entries are the set of common zeros of some polynomials $\{f_i\}$ in $K[x_1,\ldots,x_{n^2}]$.

The set of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \qquad x,y,z \in K$$

is an algebraic matrix group. Note that

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' & z+z'+xy' \\ 0 & 1 & z+z' \\ 0 & 0 & 1 \end{pmatrix}.$$

This is precisely the group law which we defined using polynomials in the previous example.

It is not immediately clear that various important subgroups of the affine group G, such as the commutator subgroup [G,G], are closed. The proof requires the following notion.

Definition 4

Let G be an affine group over K. The closed subset $X \subseteq G$ is irreducible if:

i) $X \neq \emptyset$

ii) whenever X is expressed in the form $X = F_1 \cup F_2$ where F_1, F_2 are closed subsets of G, then either $X = F_1$ or $X = F_2$.

If H is a closed subgroup of G, then H is irreducible iff H is connected.

Definition 5 (Nonstandard)

The affine group H is connected if it has no proper closed subgroup of finite index.

Theorem 6 (The irreducibility theorem)

Let G be an affine group over K. Let S_i, $i \in I$, be a family of irreducible subsets of G, such that $1 \in S_i$ for all $i \in I$. Then:

a) The subgroup $\langle S_i \mid i \in I \rangle$ is closed and connected.

b) For some finite sequence $i_1,\dots,i_n \in I$, $\langle S_i \mid i \in I\rangle = S_{i_1}^{\varepsilon_1} \dots S_{i_n}^{\varepsilon_n}$ where $\varepsilon_j \in \{1,-1\}$ for $1 \leqslant j \leqslant n$.

Many of the basic results in the theory of affine groups can be proved using only the irreducibility theorem and the minimal condition on closed subgroups. Zilber managed to extend these results to the more general setting of groups of finite Morley rank. The following table is the key to this subject.

affine groups	groups of finite Morley rank
closed subset	definable subset
connected subgroup	A subgroup $H \subseteq G$ is connected if it has no proper definable subgroup of finite index.
irreducible subset	indecomposable subset

This correspondence is not completely faithful. While groups of finite Morley rank satisfy the minimal condition on definable subgroups, they do not satisfy the minimal condition on definable subsets. However, it is very suggestive, and has guided most of the research in this area. Zilber [37] solved the problem of finding the correct analogue of an irreducible subset.

Defintion 7

Let H be a definable subgroup of G. The definable subset $\phi(G)$ is H-decomposable if there exists $g_1,\ldots,g_n \in G$, $n > 1$, such that

$$\phi(G) \subseteq g_1H \sqcup \ldots \sqcup g_nH$$

and n is the least such integer for which such a decomposition exists.

$\phi(G)$ is indecomposable if $\phi(G)$ is not H-decomposable for any definable subgroup H.

Thus a definable subgroup is indecomposable iff it is connected.

Theorem 8 (Zilber [37])

Let G be a group of finite Morley rank. Let S_i, $i \in I$, be a family of indecomposable definable subsets of G, such that $1 \in S_i$ for all $i \in I$. Then:

a) The subgroup $\langle S_i \mid i \in I\rangle$ is definable and connected.

b) For some finite sequence $i_1,\ldots,i_n \in I$,

$$\langle S_i \mid i \in I\rangle = S_{i_1}^{\varepsilon_1} \ldots S_{i_n}^{\varepsilon_n}$$

where $\varepsilon_j \in \{1,-1\}$ for $1 \leq j \leq n$.

An English translation of Zilber's proof may be found in [32]. We shall use theorem 8 and the minimal condition on definable subgroups to prove those theorems which are needed in later chapters.

Theorem 9 (Baur/Cherlin/MacIntyre [4])

Let G be an ω-stable group. There exists a smallest definable subgroup $G^0 \triangleleft G$ of finite index. G^0 is called the connected component of G.

Proof

Let $\{H_i \mid i \in I\}$ be the set of definable subgroups of finite index. For each $g \in G$ and $i \in I$, H_i^g is a definable subgroup. Thus $G^0 = \bigcap_{i\in I} H_i$ is a normal subgroup. Since G satisfies the minimal condition on definable subgroups, there exist $j_1,\ldots,j_n \in I$ such that $G^0 = \bigcap_{k=1}^{n} H_{j_k}$. Hence G^0 is definable and $[G:G^0]$ is finite.

□

Lemma 10 (Zilber [38])

Let $\phi(G)$ be a definable subset of the ω-stable group G. Suppose that $H \subseteq N(\phi(G))$. If $\phi(G)$ is P-indecomposable for all definable subgroups P such that $H \subseteq N(P)$, then $\phi(G)$ is indecomposable.

Proof

Suppose that $\phi(G)$ is Q-decomposable for some definable subgroup Q. Let

$$\phi(G) \subset g_1Q \cup \ldots \cup g_nQ$$

where $n > 1$ is minimal. For any $h \in H$,

$$\phi(G) \subset g_1^hQ^h \cup \ldots \cup g_n^hQ^h.$$

Let $P = \bigcap_{h\in H} Q^h$. Then $H \subseteq N(P)$. There exist $h_1,\ldots,h_m \in H$ such that

$P = \bigcap_{i=1}^{m} Q^{h_i}$. Thus P is definable, and there are k cosets of P which partition $\phi(G)$ where $n \leqslant k \leqslant n^m$. ✱

□

Theorem 11 (Zilber [38])

Let G be an infinite nonabelian group of finite Morley rank. If G has no proper normal definable subgroup, then G is simple.

Proof

Let $g \neq 1$ be any element. We shall show that $W = g^G \cup \{1\} \cup (g^{-1})^G$ is indecomposable. By lemma 10, it is enough to consider normal definable subgroups . Clearly g^G is infinite and so W is not 1-decomposable. It is clear that W is not G-decomposable! Thus $\langle g^G \rangle = \langle W \rangle$ is a definable normal subgroup of G. Hence $\langle g^G \rangle = G$ and G is simple.

□

Theorem 12 (Zilber [38])

Let G be a group of finite Morley rank and A be a definable connected subgroup. If B is any subgroup, then [B,A] is definable and connected.

Proof

Let $b \in B$. We shall show that $b[b,A] = b^A$ is indecomposable. By lemma 10, it suffices to show that b^A is Q-indecomposable for all definable subgroups Q such that $A \subseteq N(Q)$. Suppose that

$$b^A \subset h_1Q \cup \dots \cup h_mQ$$

where m is minimal. For $a \in A$,

$$(h_1Q)^a = h_1^aQ = h_iQ$$

for some $i \leqslant m$. This defines an action of A on $\{h_1Q,\dots,h_mQ\}$. Since m is minimal, the action is transitive. The stabiliser of h_1Q under this action is the definable subgroup $N_A(h_1Q)$. Hence $[A:N_A(h_1Q)] = m$ and so m = 1. Since b[b,A] is indecomposable, the same is true of

$[b,A]$. Thus $[B,A] = \langle [b,A] \mid b \in B \rangle$ is definable and connected.

□

Finally we mention a remarkable result of Zilber. A proof is given in [32].

Theorem 13 (Zilber [38])

Let G be a connected soluble nonnilpotent group of finite Morley rank. Then an algebraically closed field can be interpreted in G.

If it could be shown that a simple ω_1-categorical group has a connected soluble nonnilpotent subgroup, then conjecture 6 of section 1.1 would seem to be almost certainly true. Unfortunately the existence of such a subgroup is still undecided. In [32], the author used the characterisation [19] of PSL(2,F), F a locally finite field of odd characteristic, to prove:

Theorem 14

Let G be a connected nonsoluble locally finite group of finite Morley rank. Then G has a connected soluble nonnilpotent subgroup.

However, the role of the algebraically closed field remained elusive. In particular, it was not even possible to show that a simple ω_1-categorical locally finite group is linear. So this approach was abandoned. The method then adopted makes essential use of the classification of the finite simple groups, and forms the subject of this thesis.

In this section, we have described how affine groups have guided research on groups of finite Morley rank. In the final chapter, we shall attempt to repay a little of this debt. There is a natural generalisation of the notion of an affine group.

Definition 15

The group G is constructible over the field K if:

i) There exists a formula $\phi(\bar{x},\bar{k})$, $\bar{k} \in K$, such that $G = \{\bar{g} \mid K \models \phi(\bar{g},\bar{k})\}$.

ii) The group operation is given by a definable function $\psi(\bar{x},\bar{y},\bar{z},\bar{k})$, i.e.

$$K \models (\forall \bar{x}\, \bar{y})[\phi(\bar{y},\bar{k}) \wedge \phi(\bar{x},\bar{k}) \rightarrow (\exists! \bar{z})\psi(\bar{x},\bar{y},\bar{z},\bar{k}) \wedge \phi(\bar{z},\bar{k})]$$

A constructible group is not necessarily isomorphic to an affine group. However, we shall prove:

Theorem 16

Let K be an algebraically closed field and G be an infinite simple constructible group over K. Then G is isomorphic to a Chevalley group over an algebraically closed field F.

Unfortunately the proof is too crude to enable us to identify the underlying field F. Recently van den Dries has discovered an algebraic proof of this theorem. His proof, which makes use of a number of deep results in algebraic geometry, shows that $F \cong K$.

1.4 A guided tour of PSL(3,K)

In this section, we shall describe the important structural features of the typical Chevalley group PSL(3,K).

Notation

Throughout this thesis, we will use the notation of Carter [7]. For example, if $X \subseteq G$, then

C(X) is the centraliser of X,

N(X) is the normaliser of X,

$X^g = \{g^{-1} x g \mid x \in X\}$.

We shall write $\gamma_i(G)$, $Z_i(G)$ for the ith member of the lower, upper central series of G respectively.

Definition 1

a) SL(3,K) is the group of 3×3 matrices with determinant 1 over the field K.

b) PSL(3,K) = SL(3,K)/Centre.

Our first intention is to explain the following insight of Chevalley:

PSL(3,K) consists of three copies of SL(2,K), together with a finite symmetry group which permutes these copies.

Throughout, K will be a locally finite field of characteristic $p > 0$.

1. Sylow p-subgroups and Borel subgroups

Let $P < PSL(3,K)$ denote the subgroup of strictly upper triangular matrices. Then P is a Sylow p-subgroup. Let

$$N(P) = \left\{ \begin{pmatrix} \alpha & a & b \\ 0 & \beta & c \\ 0 & 0 & \gamma \end{pmatrix} \middle| \; a,b,c \in K, \; \alpha,\beta,\gamma \in K^* \right\}$$

Then $B = N(P)$. B is called a Borel subgroup of PSL(3,K). It is easily checked that $B = P \dot{\times} H$, where

$$H = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \middle| \; \alpha,\beta,\gamma \in K^* \right\}$$

The Borel subgroup B is, in many senses, the most important subgroup of PSL(3,K).

2. Transvections and root subgroups

P is generated by the following three subgroups of transvections:

$$X_a = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \middle| \; k \in K \right\}$$

$$X_b = \left\{ \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; k \in K \right\}$$

$$X_{a+b} = \left\{ \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; k \in K \right\}$$

We shall write

$$x_a(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

and use a similar notation for $x_b(k)$ and $x_{a+b}(k)$. It is immediate that for all $k,\ell \in K$,

$$x_a(k)x_a(\ell) = x_a(k+\ell).$$

Thus $X_a \simeq \langle K,+,0\rangle$, the additive group of the field K. It is suggestive to imagine these three subgroups lying in the following configuration:

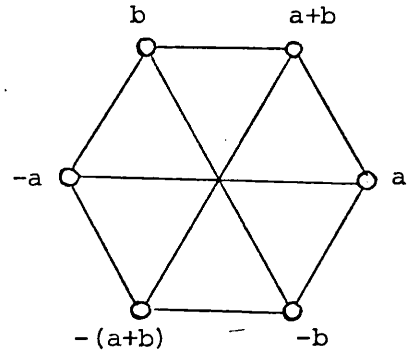


Fig.1

We associate groups of transvections with the points -a,-b,-(a+b) as follows:

$$X_{-a} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix} \middle| k \in K \right\}$$

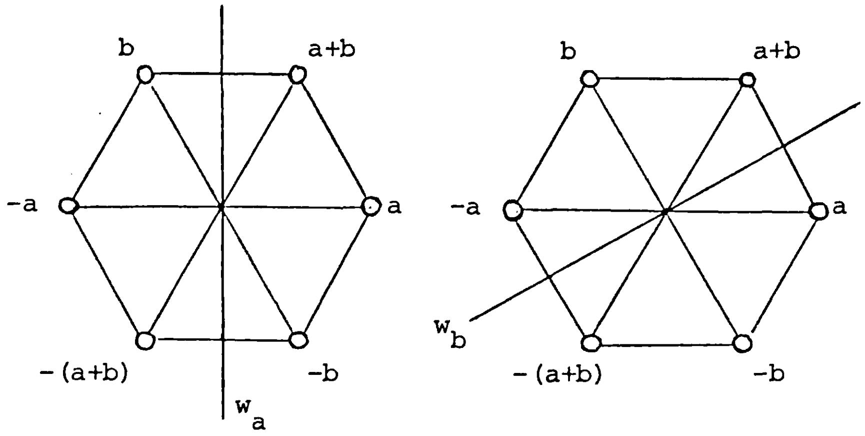
$$X_{-b} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| k \in K \right\}$$

$$X_{-(a+b)} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix} \middle| k \in K \right\}$$

(The reader will soon see why this assignment, rather than the more obvious one, is correct.) The subgroups X_r, $r \in \{\pm a, \pm b, \pm(a+b)\}$ are known as the <u>root subgroups</u> of PSL(3,K). $\Phi = \{\pm a, \pm b, \pm(a+b)\}$ is the set of <u>roots</u>. PSL(3,K) is generated by the root subgroups X_r, $r \in \Phi$.

3. <u>The Weyl subgroup</u>

It is a remarkable fact that the entire structure of PSL(3,K) is coded within fig.1. Consider the two reflections w_a and w_b, as shown below.



$w_a(a) = -a$ $\qquad\qquad$ $w_b(b) = -b$

The Weyl subgroup of PSL(3,K) is defined to be $W = \langle w_a, w_b \rangle$. W is a finite symmetry group which acts transitively on Φ. It is the symmetry group which was mentioned at the beginning of this section. At this point, the reader may be worried that W is not actually a subgroup of PSL(3,K). This will soon be clarified!

4. The subgroup $\langle X_a, X_{-a} \rangle$

Define $H_a = \langle X_a, X_{-a} \rangle \cap H$. Then

$$H_a = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \middle| \; \alpha \in k^* \right\}$$

We write

$$h_a(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix}.$$

A glance at the matrices concerned will convince the reader that there is an isomorphism π: $SL(2,K) \to \langle X_a, X_{-a} \rangle$ such that

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \longrightarrow x_a(k)$$

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \longrightarrow x_{-a}(k)$$

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \to h_a(\alpha)$$

Similar remarks hold for the subgroups $\langle X_b, X_{-b}\rangle$ and $\langle X_{a+b}, X_{-(a+b)}\rangle$. So we may regard PSL(3,K) as consisting of three copies of SL(2,K) lying in the configuration shown in Fig.1.

Let $w \in W$. Since W is generated by reflections, if $w(a) = r$ then $w(-a) = -r$. Hence the action of w on fig.1 induces a permutation of the three copies of SL(2,K), given by

$$\langle X_r, X_{-r}\rangle \longrightarrow \langle X_{w(r)}, X_{-w(r)}\rangle$$

for $r \in \Phi^+ = \{a,b,a+b\}$.

5. (B,N)-pairs

Finally we show "where" W lies inside PSL(3,K). Define $N = N(H)$. It turns out that $N/H \cong W$. For example, consider

$$n_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in N.$$

Then $n_a X_a n_a^{-1} = X_{-a}$, $n_a X_b n_a^{-1} = X_{a+b}$. So n_a acts on the roots subgroups in the same way that w_a acts on the roots. Similarly,

$$n_b = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in N$$

corresponds to the reflection w_b.

The subgroups $B = N(P)$ and $N = N(H)$ are called a (B,N)-pair for PSL(3,K). They completely determine its structure.

The other groups of Lie type have a similar structure. For a full account, the reader is referred to [6] or [7].

Chapter Two: The Reduction Lemma

In this chapter, we reduce the classification of the stable simple locally finite groups to two concrete problems, one in algebra and one in model theory. By theorem 1.2.6, if G is stable then $G \in M_c$. Our immediate target is the following result.

Theorem 1

An infinite simple locally finite M_c-group G is a countable linear group.

(A linear group is a group of invertible matrices over a field.) We require two results from algebra. The first is an immediate consequence of 4.8 of [19] and the classification of the finite simple groups.

Fact A

If for the fixed prime p, every p-subgroup of the countable simple locally finite group G is soluble, then G is linear.

Fact B [5]

A locally nilpotent M_c-group is soluble.

We remind the reader that a group is locally nilpotent if every finitely generated subgroup is nilpotent. We can now easily obtain:

Lemma 2

A countable simple locally finite M_c-group G is a linear group.

Proof

Fix any prime p. Let P be a p-subgroup of G. Then any finitely generated subgroup of P is a finite p-group, and hence nilpotent. By Fact B, P is soluble. By Fact A, G is linear.

□

To remove the hypothesis of countability, we make use of the model theoretic transfer method.

Proof of theorem 1

Let $H \prec G$ be a countable elementary submodel. Then H is a locally finite M_c-group. We claim that H is also simple. Remember that H is simple iff for every nonidentity element $h \in H$, the conjugacy class h^H generates H. Let $a \in H$ be any element. Since G is simple, there exist $g_1,\ldots,g_n \in G$ such that

$$G \models a = (h^{\varepsilon_1})^{g_1} \ldots (h^{\varepsilon_n})^{g_n}$$

where $\varepsilon_i \in \{1,-1\}$ for $1 \leq i \leq n$. Thus

$$G \models (\exists x_1 \ldots x_n)(a = (h^{\varepsilon_1})^{x_1} \ldots (h^{\varepsilon_n})^{x_n}).$$

As $H \prec G$, there must also exist $h_1,\ldots,h_n \in H$ such that

$$H \models a = (h^{\varepsilon_1})^{h_1} \ldots (h^{\varepsilon_n})^{h_n}.$$

Thus h^H generates H, and H is simple. By lemma 2, H is linear of degree n, say. In [23], Mez showed that the class of linear groups of degree n is first order axiomatizable. So G is also linear. Finally, Winter [35] has proved that a simple locally finite linear group is countable.

□

This enables us to use:

Theorem (Kegel [18])

Let G be a simple locally finite linear group. Then $G = \bigcup_{i \in \omega} G_i$, where each G_i is a finite simple group.

(Recently, Hickin [16] has shown that the hypothesis of linearity cannot be omitted from this theorem.) Since the finite simple groups are known, it seems natural to try to identify G by an examination of the approximating chain $\{G_i \mid i \in \omega\}$. We require another application of the classification to see which groups may appear in $\{G_i \mid i \in \omega\}$. Every finite simple group lies in one of the following families.

a) The sporadic groups

There are only 26 of these, and so we may safely forget about them.

b) The cyclic groups of prime order

If $p \neq q$ are distinct primes, then $\mathbb{Z}(p)$ does not embed into $\mathbb{Z}(q)$. We may also forget about these groups.

c) The alternating groups

It is quite easy to show that an infinite chain of alternating groups cannot yield a linear group. A proof will be given after we have dealt with the remaining families.

d) The Chevalley groups

These occur as a finite set of families:

A_n $(n \geq 1)$; $A_n(k) = PSL(n+1,k)$.

B_n $(n \geq 2)$

C_n $(n \geq 2)$

D_n $(n \geq 4)$

E_n $(6 \leq n \leq 8)$

F_4

G_2

Of course, the underlying field k must be finite.

e) The twisted Chevalley groups

Once again, these occur as a finite set of families.

Classes (d) and (e) are known collectively as the groups of Lie type. We say that $A_n(k)$ is a group of Lie type A_n, etc. By a slightly educated pigeon-hole principle, we can deduce:

Theorem 3 (Kegel)

If G is a simple locally finite linear group, then $G = \bigcup_{i \in \omega} G_i$, where each G_i is isomorphic to a group of the same Lie type L over a finite field.

Sketch proof

For a rigorous proof, see [19] pages 120-123. Suppose, for example, that each G_i is isomorphic to a group of Lie type $A_n = PSL(n+1, \_)$ over a finite field. We must show that groups of Lie type A_n cannot occur for arbitrarily large values of n.

Let $p > 0$ be the characteristic of the underlying field. Then we may assume that the underlying field of each G_i has characteristic p. Since G is linear, its Sylow p-subgroups are nilpotent. The Sylow p-subgroups of $PSL(n+1, \mathbb{F}_{p^m})$ are conjugates of the subgroup of strictly upper triangular matrices. Hence as n increases, the nilpotency class of its Sylow p-subgroups increases. Thus there is a bound on the values of n for which groups of Lie type A_n occur in $\{G_i | i \in \omega\}$.

□

We can now explain why G cannot be a limit of alternating groups. Every finite simple group embeds in some alternating group. In particular, for each $n \in \omega$, $A_n(\mathbb{F}_p)$ embeds in an alternating group. So a limit of alternating groups will not have nilpotent Sylow p-subgroups.

Theorem 3 suggests that we consider:

<u>The identification problem</u>

If $G = \bigcup_{i \in \omega} G_i$, where each G_i is isomorphic to a group of Lie type L over a finite field, is G isomorphic to a group of Lie type L over a locally finite field?

(A locally finite field is a subfield of $\bar{\mathbb{F}}_p$, the algebraic closure of the field with p elements.) In chapter three, we shall show that this is indeed the case. Notice that we will then have proved:

<u>Theorem 4</u>

An infinite simple locally finite M_c-group is a group of Lie type over a locally finite field.

<u>Corollary 5</u>

An infinite simple periodic linear group is a group of Lie type over a locally finite field.

<u>Proof</u>

It is well known that a periodic linear group is locally finite and M_c.

□

Corollary 5 was proved independently by Shute [28]. Thus a stable simple locally finite group G is a group of Lie type. To complete the proof of Cherlin's conjecture, we must show that:

a) the underlying field K is algebraically closed;

b) G is a Chevalley group (i.e. it cannot be a twisted Chevalley group).

It is enough to prove (a). For then $k = \bar{\mathbb{F}}_p$ for some prime $p > 0$, and there are no twisted Chevalley groups with underlying field $\bar{\mathbb{F}}_p$. We shall prove (a) in chapters four and five by showing that if G is a locally finite group of Lie type, then the underlying field K may be interpreted in G. Hence by theorem 1.2.9, if G is stable then K is a stable locally finite field. We can then apply:

Theorem 6 (Duret [12])

A stable infinite locally finite field is algebraically closed.

This will give us:

Theorem 7

An infinite stable simple locally finite group is a Chevalley group over an algebraically closed field.

An aside: ω-categorical simple groups

The reader may be wondering why we are only interested in κ-categorical simple groups for uncountable κ. In [15], Felgner used the classification of the finite simple groups to prove that there are no ω-categorical simple groups. (By definition an ω-categorical theory has a model of cardinality ω.) We shall deduce this result from theorem 4.

The study of ω-categorical theories is based upon the following result.

Theorem (Engeler [13], Ryll-Nardzewski [25], Svenonius [31]).

Let T be a complete countable theory. Then the following are equivalent:

a) T is ω-categorical.

b) For each $n \in \omega$, there are only finitely many (parameter-free) formulas $\phi(x_1,\ldots,x_n)$ up to logical equivalence with respect to T.

To illustrate the use of the above theorem, we shall show that an ω-categorical group G has bounded exponent. Consider the set of formulas

$$v_0^n = v_1 , \qquad n \in \omega.$$

Since G is ω-categorical, for some distinct n,k, we have

$$\mathrm{Th}\,G \vdash (v_0^n = v_1) \leftrightarrow (v_0^k = v_1).$$

Thus $\mathrm{Th}\,G \vdash v_0^n = v_0^k$, and G has bounded exponent. Similarly, it can be shown that an ω-categorical group G is locally finite.

The following result is an immediate consequence of 4.8 of [19] and the classification of the finite simple groups.

Fact C

If for the fixed prime p, every p-subgroup of the countable simple locally finite group G is of bounded exponent, then G is linear.

Theorem 8 (Felgner)

There are no ω-categorical simple groups.

Proof

Suppose that G is a countable ω-categorical group. Then G is locally finite of bounded exponent. By fact C, G is linear and so $G \in M_c$. By theorem 4, G is a group of Lie type over an infinite locally finite field. But then the subgroup of diagonal matrices does not have bounded exponent.

□

Chapter Three: The Identification Theorem

In this chapter, we shall prove:

Theorem

Let $G = \bigcup_{i\in\omega} G_i$, where each G_i is isomorphic to a group of Lie type L over a finite field. Then G is isomorphic to a group of Lie type L over a locally finite field.

In 1967, Kegel [18] dealt with the cases of PSL(n,_), the Suzuki groups Sz(_), and the projective unitary groups PSU(3,_). In Thomas [33], using a different approach, the theorem was proved for the nontwisted Chevalley groups. More recently [34], we have discovered a much simpler proof, which also works for the twisted groups. Before becoming involved in the details, we shall outline the main ideas of the proof.

For simplicity, assume that $G_i \simeq PSL(3,p^{n_i})$, the 3-dimensional projective linear group over the Galois field $GF(p^{n_i})$. It is enough to show that there is a commuting system of maps:

$$\begin{array}{ccc}
\cup\!| & & \uparrow c_3 \\
G_3 & \xrightarrow[\phi_3]{\approx} & PSL(3,p^{n_3}) \\
\cup\!| & & \uparrow c_2 \\
G_2 & \xrightarrow[\phi_2]{\approx} & PSL(3,p^{n_2}) \\
\cup\!| & & \uparrow c_3 \\
G_1 & \xrightarrow[\phi_1]{\approx} & PSL(3,p^{n_3})
\end{array}$$

where c_i is the canonical embedding, i.e. it sends each matrix in $PSL(3,p^{n_i})$ to the corresponding matrix in $PSL(3,p^{n_{i+1}})$. For then,

$$G \simeq \lim_{n\in\omega} PSL(3,p^{n_i}) = PSL(3,K)$$

where $K = \bigcup_{i\in\omega} GF(p^{n_i})$.

This will be done in two stages. First we shall show that isomorphisms τ_i and embeddings π_i can be chosen so that each of the following diagrams commutes:

$$\begin{array}{ccc} G_{i+1} & \xrightarrow[\tau_{i+1}]{\approx} & PSL(3,p^{n_{i+1}}) \\ \cup\!| & & \uparrow \pi_i \\ G_i & \xrightarrow[\tau_i]{\approx} & PSL(3,p^{n_i}) \end{array}$$

and such that the embedding π_i is "almost canonical".

Definition

For each $i \in \omega$ and $r \in \Phi$,

$${}_iX_r = \{x_r(t) \mid t \in GF(p^{n_i})\}.$$

Definition

π_i: $PSL(3,p^{n_i}) \to PSL(3,p^{n_{i+1}})$ is almost canonical if for each $r \in \Phi$, $\pi_i[{}_iX_r] \subseteq {}_{i+1}X_r$.

For example, for each $t \in GF(p^{n_i})$,

$$\pi_i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$$

for some $s \in GF(p^{n_{i+1}})$. At this stage, there is no reason to suppose that $t = s$. Finally, we shall "unwrap" the almost canonical embeddings $\{\pi_i \mid i \in \omega\}$ to obtain canonical embeddings c_i; i.e. we shall find isomorphisms ψ_i such that

$$\begin{array}{ccccc} G_{i+1} & \xrightarrow[\tau_{i+1}]{\approx} & PSL(3,p^{n_{i+1}}) & \xrightarrow[\psi_{i+1}]{\approx} & PSL(3,p^{n_{i+1}}) \\ \cup\!| & & \uparrow \pi_i & & \uparrow c_i \\ G_i & \xrightarrow[\tau_i]{\approx} & PSL(3,p^{n_i}) & \xrightarrow[\psi_i]{\approx} & PSL(3,p^{n_i}) \end{array}$$

commutes. Putting $\phi_i = \psi_i \tau_i$, the theorem follows.

3.1 Chevalley groups

Throughout this section, let $G = \bigcup_{i \in \omega} G_i$ where each G_i is isomorphic to a (nontwisted) Chevalley group of type L over a finite field of characteristic $p > 0$. Let Φ be the associated root system and Π be the set of fundamental roots. We suppose that $G_i \simeq L(q_i)$, the Chevalley group of Lie type L over the Galois field $GF(q_i)$. Let $\{{}_iX_r \mid r \in \Phi\}$ be the set of root subgroups of $L(q_i)$.

Definition 1

$\pi: L(q_i) \to L(q_j)$ is a natural embedding if for each $r \in \Phi$, $\pi[{}_iX_r] \subseteq {}_jX_r$. If $i = j$, we shall say that π is a natural automorphism.

(The term "almost canonical embedding" is too awkward for repeated use.) We begin the construction of the isomorphisms $\{\tau_i \mid i \in \omega\}$. In view of the importance of (B,N) pairs, it is natural to attempt to select $B_i, N_i \leq G_i$ such that $B_i \leq B_{i+1}$ and $N_i \leq N_{i+1}$. First we show that this is possible.

By 1.L.6. of [19], G is linear. Hence every Sylow p-subgroup of G is nilpotent of class c, say. By 1.D.3. of [19], we may choose a Sylow p-subgroup P such that $P_i = P \cap G_i$ is a Sylow p-subgroup of G_i. Since G is linear, it satisfies the minimal condition on centralisers, and we may make use of the results of Bryant [5]. By the proof of lemma 2.1 of [5], there exists a finite subgroup $X \leq P$ such that $C(\gamma_i(X)) = C(\gamma_i(P))$ for $i = 1,2,\ldots,c$. By lemma 3.2 of [5], for any p-subgroup $H \supseteq X$ we have $H \leq P$. Without loss of generality, we may assume that $X = P_1$.

Lemma 2

$N(P) = \bigcup_{i \in \omega} N_{G_i}(P_i)$.

Proof

Clearly $N(P) \cap G_i \subseteq N_{G_i}(P_i)$ and so $N(P) \subseteq \bigcup_{i \in \omega} N_{G_i}(P_i)$. Conversely, suppose that $g \in N_{G_i}(P_i)$. Then $P_1 \leq P^g$ and by the above remark $P^g \leq P$. Thus $g \in N(P)$.

□

From now on, we write $B_i = N_{G_i}(P_i)$. We have shown that $B = N(P) = \bigcup_{i\in\omega} B_i$. From the structure theory of the finite Chevalley groups,

$$B_i = P_i \dot{\times} H_i$$

where H_i is an abelian p'-group,

$$H_i \leqslant H_{i+1}^{g_{i+1}}$$

for some $g_{i+1} \in B_{i+1}$, and

$$B_{i+1} = P_{i+1} \dot{\times} H_{i+1}^{g_{i+1}}.$$

So we may assume that the H_i have been chosen so that $H_i \leqslant H_{i+1}$ for all $i \in \omega$. Define $H = \bigcup_{i\in\omega} H_i$. Then $B = P \dot{\times} H$ and H is a Sylow p'-subgroup of B. Since H is abelian, there is a finite subgroup $Y \leqslant H$ such that $C_B(\gamma_i(Y)) = C_B(\gamma_i(H))$ for $i \in \omega$. Again by lemma 2.1 of [5], if $L \supseteq Y$ is any p'-subgroup of B, then $L \leqslant H$. We may assume that $Y = H_1$. Define $N = N(H)$ and $N_i = N \cap G_i$. It is immediate that $N_i \subseteq N_{G_i}(H_i)$.

<u>Lemma 3</u>

$$N(H) = \bigcup_{i\in\omega} N_{G_i}(H_i).$$

<u>Proof</u>

Suppose that $g \in N_{G_i}(H_i)$. Then $H_1 \leqslant H_i \leqslant B_i \cap B_i^g$. We define H_j' for $j \geqslant i$ inductively.

a) $H_i' = H_i$

b) Suppose that H_j' has been defined so that $H_j' = H_j^{g_j} \leqslant B_j \cap B_j^g$ and $H_i' \leqslant H_k' \leqslant H_j'$ for all $i \leqslant k \leqslant j$. There exists a maximal torus H_{j+1}^a of G_{j+1} such that $H_{j+1}^a \leqslant B_{j+1} \cap B_{j+1}^g$. H_j' is included in some maximal p'-subgroup $K \leqslant B_{j+1} \cap B_{j+1}^g$. Since $B_{j+1} \cap B_{j+1}^g$ is soluble, K and H_{j+1}^a are conjugate. Put $H_{j+1}' = K = H_{j+1}^{g_{j+1}}$.

Define $H' = \bigcup_{j\in\omega} H_j'$. H' is a Sylow p'-subgroup of B containing H_1. Thus $H = H' \leqslant B \cap B^g$. Since H is the unique Sylow p'-subgroup of B which contains H_1, it must also be the unique Sylow p'-subgroup of B^g

containing H_1. Hence $H^g = H$ and $g \in N(H)$.

□

Define the following subgroups of $L(q_i)$ by:

$$\bar{P}_i = \langle {}_iX_r \mid r \in \Phi^+ \rangle$$

$$\bar{B}_i = N(\bar{P}_i)$$

$$\bar{H}_i = \langle h_r(t) \mid r \in \Phi,\ t \in GF(q_i) \rangle$$

$$\bar{N}_i = N(\bar{H}_i).$$

Let W be the Weyl subgroup associated with each Chevalley group of type L. Then $\bar{N}_i/\bar{H}_i \simeq W$. Suppose that $\bar{n}_r \in \bar{N}_i$, $r \in \Pi$, is mapped to w_r under the natural homomorphism and $\bar{n}_0 \in \bar{N}_i$ is mapped to w_0, where w_0 is the element of W of maximum length. Then for each $r \in \Pi$,

$${}_iX_r = \bar{P}_i \cap \bar{n}_r\, \bar{n}_0\, \bar{P}_i\, \bar{n}_0^{-1}\, \bar{n}_r^{-1}.$$

For each $i \in \omega$, let $\tau_i : G_i \to L(q_i)$ be an isomorphism such that

$$P_i \longrightarrow \bar{P}_i$$

$$H_i \longrightarrow \bar{H}_i$$

Let $\{n_0,\ldots,n_m\} \subseteq N_1$ be a complete set of coset representatives for N_1/H_1. Then $\{n_0,\ldots,n_m\}$ is also a complete set of coset representatives for N_i/H_i. τ_i induces a homomorphism $N_i \xrightarrow{\sigma_i} W$. There are only finitely many bijections $\{n_0,\ldots,n_m\} \to W$, so we may assume that for each $n \in \{n_0,\ldots,n_m\}$, $\sigma_i(n) = \sigma_j(n)$ for all $i,j \in \omega$.

Let $\sigma_1(n_r) = w_r$, $r \in \Pi$

$$\sigma_1(n_0) = w_0.$$

Define ${}_iY_r = P_i \cap n_r\, n_0\, P_i\, n_0^{-1}\, n_r^{-1}$, $r \in \Pi$. By proposition 2.1.8 of Carter [7], every root in Φ is the image of some root in Π under some element in W. Let $s \in \Phi$ and $w(r) = s$, $r \in \Pi$, $w \in W$. We define

$${}_iY_s = n\; {}_iX_r\; n^{-1}$$

where $\sigma_1(n) = w$. Then for each $i \in \omega$, $r \in \Phi$, we have

$${}_iY_r \xrightarrow{\ \tau_i\ } {}_iX_r .$$

So if π_i completes the following diagram

$$\begin{array}{ccc} G_{i+1} & \xrightarrow[\tau_{i+1}]{!\ \approx} & L(q_{i+1}) \\ \cup\!| & & \uparrow \pi_i \\ G_i & \xrightarrow[\tau_i]{!\ \approx} & L(q_i) \end{array}$$

then for each $r \in \Phi$,

$$\pi_i[{}_iX_r] \subseteq {}_{i+1}X_r .$$

The following lemma sums up our work so far. Intuitively, it says that eventually the "bad" embeddings die out.

<u>Lemma 4</u>

Without loss of generality, we may assume that there exist isomorphisms $\tau_i : G_i \to L(q_i)$ and natural embeddings $\pi_i : L(q_i) \to L(q_{i+1})$ such that each of the following diagrams commute:

$$\begin{array}{ccc} G_{i+1} & \xrightarrow[\tau_{i+1}]{} & L(q_{i+1}) \\ \cup\!| & & \uparrow \pi_i \\ G_i & \xrightarrow[\tau_i]{} & L(q_i) \end{array}$$

□

We shall use automorphisms to "unwrap" the embeddings $\{\pi_i \mid i \in \omega\}$. First we remind the reader of the classification of the automorphisms of PSL(3,K), for a finite field K. There are four basic types of automorphism.

a) <u>Inner automorphisms</u>

For each $g \in PSL(3,K)$, the map

$$x \longrightarrow g^{-1}xg \quad , \quad x \in PSL(3,K)$$

is an automorphism.

b) Diagonal automorphisms

For each $k_a, k_b \in K^*$, there is an automorphism d such that

$$x_a(c) \rightarrow x_a(ck_a)$$

$$x_b(c) \rightarrow x_b(ck_b)$$

for all $c \in K$. The automorphism d can be induced by conjugating by a diagonal matrix $h \in PSL(3,\bar{K})$ for some extension $\bar{K} \supseteq K$. We shall give an example to clarify this last sentence.

Consider the diagonal automorphism d of PSL(3,7) such that

$$x_a(1) \rightarrow x_a(2)$$

$$x_b(1) \rightarrow x_b(3).$$

We shall show that d is not an inner automorphism. A typical element of H is $h = h_a(\alpha)h_b(\beta)$ where $\alpha,\beta \in GF(7)^*$. Note that

$$hx_a(1)h^{-1} = x_a(\alpha^2\beta^{-1})$$

$$hx_b(1)h^{-1} = x_b(\beta^2\alpha^{-1}).$$

Suppose that $2 = \alpha^2\beta^{-1}$ and $3 = \beta^2\alpha^{-1}$. Then $5 = \alpha^3$. But 1,6 are the only cubes in GF(7)*. ※ Thus d fails to be inner because 5 has no cubic root in GF(7). (If K is algebraically closed, then all diagonal automorphisms are inner.)

Since $d[X_r] = X_r$ for all $r \in \Phi$, diagonal automorphisms are useful whenever we need to "normalise" $x_a(1)$ and $x_b(1)$.

c) Field automorphisms

Let ξ be an automorphism of the field K. Then the map

$$x_r(t) \rightarrow x_r(\xi(t)) \qquad r \in \Phi,\ t \in K$$

can be extended to an automorphism of PSL(3,K).

d) Graph automorphisms

These correspond to symmetries of the root system. For example, there is a graph automorphism θ induced by the map $a \rightarrow b$ and $b \rightarrow a$. Under this automorphism, $\theta[X_a] = X_b$ and $\theta[X_b] = X_a$. The twisted

Chevalley group ${}^2A_2(K)$ is defined in terms of θ. (See Carter [7], Chapter 13.)

The following result is due to Steinberg [29].

Theorem

Let G be a finite Chevalley group and $\theta \in \mathrm{Aut}\,G$. Then there exist inner, diagonal, graph and field automorphisms $i, d, g, \mathfrak{f}$ such that $\theta = idg\mathfrak{f}$.

Actually we shall not use the full theorem, but rather one of Steinberg's lemmas.

Lemma 5 (Steinberg)

Let G be a finite Chevalley group. Suppose that the natural automorphism σ satisfies

$$\sigma(x_r(1)) = x_r(1)$$

for each $r \in \Pi$. Then σ is a field automorphism.

Proof

This is merely a restatement of 5.7 of [29].

□

We are now ready to prove the key result of this section.

Lemma 6 (The unwrapping lemma)

Suppose that:

i) π: $L(q_1) \to L(q_2)$ is a natural embedding;

ii) ϕ: $L(q_1) \to L(q_1)$ is a natural automorphism.

Then the following diagram can be completed:

$$\begin{array}{ccc} L(q_2) & \xrightarrow{\ \psi\ } & L(q_2) \\ \pi \uparrow & & \uparrow c \\ L(q_1) & \xrightarrow{\ \phi\ } & L(q_1) \end{array}$$

where ψ is a natural automorphism and c is the canonical embedding.

Proof

For clarity, we shall write

$$_1X_r = \{x_r(t) \mid t \in GF(q_1)\}$$

$$_2X_r = \{\bar{x}_r(t) \mid t \in GF(q_2)\}$$

for each $r \in \Phi$. Note that $\pi\phi^{-1}$ is a natural embedding. Suppose that for each $r \in \Pi$,

$$\pi\phi^{-1}(x_r(1)) = \bar{x}_r(c_r).$$

Then there exists a homomorphism h of the additive group generated by the roots into $GF(q_2)^*$ such that $h(r) = c_r^{-1}$ for each $r \in \Pi$. Application of the corresponding diagonal automorphism d now yields

$$d\pi\phi^{-1}(x_r(1)) = \bar{x}_r(1)$$

for each $r \in \Pi$. It is easily seen (e.g. 5.2 [29]) that for each $r \in \Pi$, we also have

$$d\pi\phi^{-1}(x_{-r}(1)) = \bar{x}_{-r}(1)$$

and

$$d\pi\phi^{-1}(n_r) = \bar{n}_r$$

where $n_r = x_r(1)x_{-r}(-1)x_r(1)$.

Claim

$d\pi\phi^{-1}[L(q_1)]$ is the canonical subgroup of $L(q_2)$.

Proof of claim

For each $t \in GF(q_1)^*$ and $r \in \Pi$,

$$h_r(t)^{q_1} = h_r(t).$$

Let $d\pi\phi^{-1}(h_r(t)) = \bar{h}_r(s)$. Then

$$\bar{h}_r(s)^{q_1} = \bar{h}_r(s)$$

and so $s \in GF(q_1)^*$. Now let $r \in \Pi$ and $t \in GF(q_1)$. There exists $t_1, t_2 \in GF(q_1)^*$ such that $t = t_1^2 + t_2^2$. Thus

$$x_r(t) = \prod_{i=1}^{2} h_r(t_i)x_r(1)h_r(t_i)^{-1}.$$

Hence $d\pi\phi^{-1}(x_r(t))$ is in the canonical subgroup. Since $L(q_1)$ is generated by $\{x_r(t), n_r \mid r \in \Pi,\ t \in GF(q_1)\}$, the claim is proved.

Thus $d\pi\phi^{-1}$ induces a natural automorphism σ of $L(q_1)$ such that $\sigma(x_r(1)) = x_r(1)$ for each $r \in \Pi$. By lemma 5, σ is a field automorphism. Suppose that σ is induced by the field map $F\colon GF(q_1) \overset{\simeq}{\to} GF(q_1)$. This extends to a field map $\bar{F}\colon GF(q_2) \overset{\simeq}{\to} GF(q_2)$. Let $\mathfrak{F}$ be the corresponding field automorphism of $L(q_2)$. Then $\psi = \mathfrak{F}^{-1}d$ is the required natural automorphism.

□

Theorem 7

Let $G = \bigcup_{i\in\omega} G_i$, where G_i is isomorphic to a Chevalley group of Lie type L over a finite field. Then G is isomorphic to a Chevalley group of Lie type L over a locally finite field.

Proof

By lemmas 4 and 6, we may complete the following diagram, where $c_i\colon L(q_i) \to L(q_{i+1})$ is the canonical embedding. This implies that $G \simeq L(K)$, where $K = \bigcup_{i\in\omega} GF(q_i)$.

$$
\begin{array}{ccccc}
\cup\!\!| & & \uparrow & & \uparrow \\
G_3 & \xrightarrow[\tau_3]{} & L(q_3) & \xrightarrow[\psi_3]{} & L(q_3) \\
\cup\!\!| & & \uparrow{\scriptstyle \pi_2} & & \uparrow{\scriptstyle c_2} \\
G_2 & \xrightarrow[\tau_2]{} & L(q_2) & \xrightarrow[\psi_2]{} & L(q_2) \\
\cup\!\!| & & \uparrow{\scriptstyle \pi_1} & & \uparrow{\scriptstyle c_1} \\
G_1 & \xrightarrow[\tau_1]{} & L(q_1) & \xrightarrow[\psi_1=\text{identity}]{} & L(q_1)
\end{array}
$$

□

3.2 Natural embeddings

In the next five sections, we shall consider the twisted Chevalley groups. Throughout these sections, let $G = \bigcup_{i\in\omega} G_i$, where each G_i is isomorphic to a twisted Chevalley group of type T over a suitable finite field of characteristic $p > 0$. We shall suppose that:

$G_i \simeq T(q_i^2)$ $T = {}^2A_n,\ {}^2D_n,\ {}^2E_6.$

$G_i \simeq T(q_i^3)$ $T = {}^3D_4.$

$G_i \simeq T(2^{2m_i+1})$ $T = {}^2F_4.$

In order to limit the length of this chapter, we have not included the proofs for the groups of type ${}^2A_2$, ${}^2B_2$ or ${}^2G_2$. These cases have already been dealt with in [19] and [30]. The reader should have no diffulty in providing proofs for these cases using the methods of this chapter.

First we shall give a brief account of the (B,N)-pair structure of $T(p^n)$. We follow the notation of chapters 13 and 14 of [7]. Suppose that $T(p^n)$ is the twisted form of the Chevalley group $L(p^n)$, defined in terms of the graph symmetry ρ of the Dynkin diagram of L and the nontrivial field automorphism $\mathfrak{F}$. Let $\sigma = g\mathfrak{F}$, where g is the graph automorphism of $L(p^n)$ induced by ρ. Let $U = \langle X_r | r \in \Phi^+ \rangle$ and $H = \{h_r(t) | r \in \Phi,\ t \in GF(p^n)^*\}$.

i) Let U^1 be the subgroup of $U \subseteq L(p^n)$ fixed under σ. Then U^1 is a Sylow p-subgroup of $T(p^n)$.

ii) Let $H^1 = H \cap T(p^n)$.

iii) Then $B^1 = N(U^1) = U^1 \dot{\times} H^1$.

iv) Let $N^1 = N(H^1)$.

(B^1,N^1) form a (B,N)-pair for $T(p^n)$ and, as with the Chevalley groups, the Weyl subgroup W is given by

$$W^1 = N^1/H^1 = N^1/B^1 \cap N^1.$$

The analogue of the root subgroups for the twisted group $T(p^n)$ is the collection of subgroups of the form X_S^1, where $S = w(\Phi_J^+)$ for some $w \in W^1$ and some ρ-orbit J of Π. The Weyl subgroup W^1 allows us to "extract" the subgroup X_S^1 from U^1, as follows:

i) Let $S = \Phi_J^+$, where J is a ρ-orbit of Π, and let $w_0^J \in W^1$ be the element defined by proposition 13.1.2 of [7]. Then there exists $n_S \in N^1$ such that n_S is mapped to w_0^J under the natural homomorphism $N^1 \to W^1$.

ii) Let $w_0 \in W^1$ be the unique element of maximal length, and suppose that $n_0 \in N^1$ is mapped to w_0 under the natural homomorphism.

iii) Then $X^1_S = U^1 \cap n_S n_0 U^1 n_0^{-1} n_S^{-1}$.

iv) Finally let $S' = w(\Phi^+_J)$ for some $w \in W^1$ and some ρ-orbit J of Π. Suppose that $n \in N^1$ is mapped to w under the natural homomorphism. Then $X^1_{S'} = n X^1_S n^{-1}$.

An examination of the proof of lemma 4 shows that we only use the (B,N)-pair structure and the associated Bruhat decomposition of the Chevalley groups. (The Bruhat decomposition is used to prove that the intersection of two Borel subgroups always contains a maximal torus. Exactly the same proof works in the twisted case.) So we are now able to define the notion of a natural embedding for the twisted groups, and state the analogue of lemma 4.

Definition 8

$\pi: T(p^{n_i}) \to T(p^{n_j})$ is a natural embedding if for each $S = w(\Phi^+_J)$,

$$\pi[{}_iX^1_S] \subseteq {}_jX^1_S .$$

Lemma 9

Without loss of generality, we may assume that there exist isomorphisms $\tau_i: G_i \to T(p^{n_i})$ and natural embeddings $\pi_i: T(p^{n_i}) \to T(p^{n_{i+1}})$ such that each of the following diagrams commute:

$$\begin{array}{ccc} G_{i+1} & \xrightarrow{\tau_{i+1}} & T(p^{n_{i+1}}) \\ \cup\!\!\mid & & \uparrow \pi_i \\ G_i & \xrightarrow{\tau_i} & T(p^{n_i}) \end{array}$$

□

3.3 The case of $T = {}^2A_{2n-1}$, ${}^2D_n$ or ${}^2E_6$.

Once again, we shall construct natural automorphisms so that the following diagram commutes:

$$\begin{array}{ccc} T(q_{i+1}^2) & \xrightarrow[\psi_{i+1}]{} & T(q_{i+1}^2) \\ \uparrow \pi_i & & \uparrow c_i \\ T(q_i^2) & \xrightarrow[\psi_i]{} & T(q_i^2) \end{array}$$

where c_i is the canonical embedding. Our first requirement is that $T(q_{i+1}^2)$ should contain a canonical copy of $T(q_i^2)$.

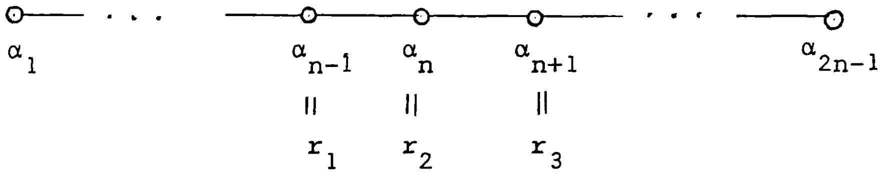
Lemma 10

Suppose that $T = {}^2A_{2n-1}$, ${}^2D_n$ or ${}^2E_6$. Then for all sufficiently large q, $T(q^2)$ does not embed naturally into $T(q^{4m})$.

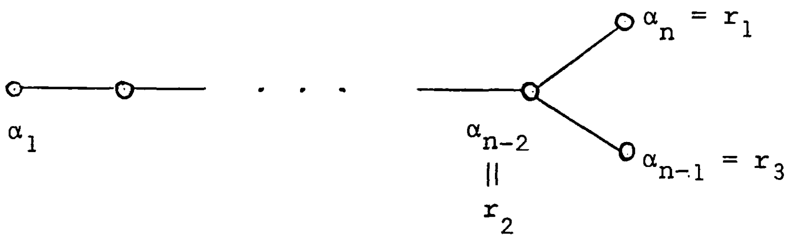
Proof

Select fundamental roots r_i $(1 \leqslant i \leqslant 3)$ as follows:

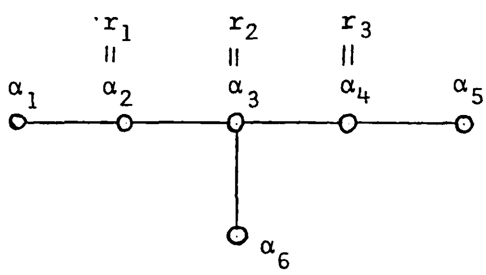
${}^2A_{2n-1}$



${}^2D_n$



${}^2E_6$



Let $h = h_{r_1}(\lambda)h_{r_3}(\lambda^q) \in T(q^2)$, where λ is a generator of $GF(q^2)^*$. Fix a nonidentity $x_{r_2}(s) \in T(q^2)$. Then

$$h^{-1}x_{r_2}(s)h = x_{r_2}(s\lambda^{q+1}).$$

Since $\lambda^{q+1} \in GF(q)$, we obtain

$$h^{-(q-1)} x_{r_2}(s) h^{q-1} = x_{r_2}(s).$$

Suppose there is a natural embedding $\pi\colon T(q^2) \to T(q^{4m})$, and that $\pi(h) = \bar{h}$ and $\pi(x_{r_2}(s)) = x_{r_2}(u) \in T(q^{4m})$. Then since

$$h^{q^2-1} = 1$$

we must have $\bar{h} = h_{r_1}(t) h_{r_3}(t)$ for some generator t of $GF(q^2)^*$. Thus

$$\bar{h}^{-1} x_{r_2}(u) \bar{h} = x_{r_2}(ut^2)$$

and so $t^{2(q-1)} = 1$. But this means that q^2-1 divides $2(q-1)$, which is false for $q \geq 2$.

$\square$

So we may assume that for each $i \in \omega$, $[GF(q_{i+1}^2) : GF(q_i^2)]$ is odd. Let $\mathfrak{F}_i \in \operatorname{Aut} GF(q_i^2)$ be defined by $\lambda \longrightarrow \lambda^{q_i}$. Then for each $i \in \omega$, $\mathfrak{F}_{i+1} \restriction GF(q_i^2) = \mathfrak{F}_i$. This has two important consequences.

A) $T(q_{i+1}^2)$ contains a canonical copy of $T(q_i^2)$.

B) Let $K = \bigcup_{i \in \omega} GF(q_i^2)$ and $K_0 = \bigcup_{i \in \omega} GF(q_i)$. Then K has an automorphism $\mathfrak{F}$ of order 2 such that $\mathfrak{F} \restriction GF(q_i^2) = \mathfrak{F}_i$ with fixed subfield K_0.

Thus if we take the direct limit of the system

$$T(q_1^2) \xrightarrow{c_1} T(q_2^2) \to \dots \to T(q_i^2) \xrightarrow{c_i} T(q_{i+1}^2) \longrightarrow \dots$$

where c_i is the canonical embedding, we obtain the twisted Chevalley group $T(K)$.

The following lemma is implicit in [29]. The proof is virtually the same as 5.7 of [29].

<u>Lemma 11</u>

Let G be a finite twisted Chevalley group of type ${}^2A_{2n-1}$, ${}^2D_n$, ${}^2E_6$ or ${}^3D_4$. Suppose that the natural automorphism σ satisfies

$$\sigma(x_S(1)) = x_S(1)$$

for each $S = \Phi_J^+$, where J is a ρ-orbit of Π. Then σ is a field automorphism.

Finally we state the unwrapping lemma for this case. The proof is almost identical to that of lemma 6. The identification theorem for $T = {}^2A_{2n-1}$, ${}^2D_n$ or ${}^2E_6$ now follows easily.

Lemma 12

Let $T = {}^2A_{2n-1}$, ${}^2D_n$ or ${}^2E_6$. Suppose that:

i) $[GF(q_2^2) : GF(q_1^2)]$ is odd.

ii) $\pi: T(q_1^2) \to T(q_2^2)$ is a natural embedding.

iii) $\phi: T(q_1^2) \to T(q_1^2)$ is a natural automorphism.

Then the following diagram can be completed:

$$\begin{array}{ccc} T(q_2^2) & \xrightarrow{\psi} & T(q_2^2) \\ \pi \uparrow & & \uparrow c \\ T(q_1^2) & \xrightarrow{\phi} & T(q_1^2) \end{array}$$

where ψ is a natural automorphism and c is the canonical embedding.

3.4 The case of $T = {}^3D_4$

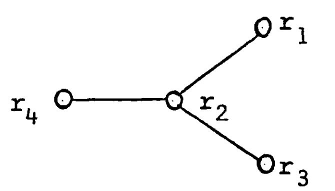
It is enough to show that $T(q_{i+1}^3)$ contains a canonical copy of $T(q_i^3)$. The unwrapping lemma and the identification theorem then follow from lemma 11.

Lemma 13

Suppose that $T = {}^3D_4$. Then for all sufficiently large q, $T(q^3)$ does not embed naturally into $T(q^{9m})$.

Proof

${}^3D_4$



We argue as in lemma 10 using

$$h = h_{r_1}(\lambda) h_{r_3}(\lambda^q) h_{r_4}(\lambda^{q^2})$$

where λ is a generator of $GF(q^3)^*$.

□

So we may assume that for each $i \in \omega$, $[GF(q^3_{i+1}): GF(q^3_i)] \equiv 1,2$ (mod 3). Since $2.2 \equiv 1$ (mod 3), we may further assume that $[GF(q^3_{i+1}): GF(q^3_i)] \equiv 1$ (mod 3). If we define $\mathfrak{F}_i \in \mathrm{Aut}\, GF(q^3_i)$ by $\lambda \rightarrow \lambda^{q_i}$, then $\mathfrak{F}_{i+1} \restriction GF(q^3_i) = \mathfrak{F}_i$. As a consequence:

A) $T(q^3_{i+1})$ contains a canonical copy of $T(q^3_i)$.

B) Let $K = \bigcup_{i\in\omega} GF(q^3_i)$ and $K_0 = \bigcup_{i\in\omega} GF(q_i)$.
Then K has an automorphism $\mathfrak{F}$ of order 3 such that $\mathfrak{F} \restriction GF(q^3_i) = \mathfrak{F}_i$ with fixed subfield K_0.

3.5 <u>The case of $T = {}^2A_{2n}$, where $n \geqslant 2$.</u>

We have to do a little more work in this section. The difficulty arises from the existence of a two parameter root subgroup.

<u>Lemma 14</u>

Suppose that $T = {}^2A_{2n}$. Then for all sufficiently large q, $T(q^2)$ does not embed into $T(q^{4m})$.

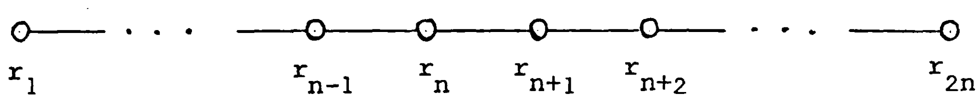
<u>Proof</u>

Argue as in lemma 10, and derive a contradiction from equation (*) on page 48.

□

As in section 3, this ensures that $T(q^2_{i+1})$ contains a canonical copy of $T(q^2_i)$, and that $K = \bigcup_{i\in\omega} GF(q^2_i)$ has the necessary automorphism of order 2.

A_{2n} has Dynkin diagram:



Thus each ρ-orbit of Π has type $A_1 \times A_1$ or A_2. Let $T = \{r_{n-1}, r_{n+2}\}$ and $S = \{r_n, r_{n+1}, r_n + r_{n+1}\}$. We shall write

$$x_T(t) = x_{r_{n-1}}(t)\, x_{r_{n+2}}(\bar{t})$$

$$x_S(t,u) = x_{r_n}(t)\, x_{r_{n+1}}(\bar{t})\, x_{r_n+r_{n+1}}(u)$$

where $u+\bar{u} = -N_{r_n,r_{n+1}}\, t\,\bar{t}$. (We follow the convention of [7] and write $\bar{t}$ for the image of t under the field automorphism.) Note that $Z(X_S^1) = \{x_S(0,u) \mid u+\bar{u} = 0\}$ and $x_S(t,u_1)^{-1}x_S(t,u_2) \in Z(X_S^1)$. Hence if σ is a natural automorphism of $T(q^2)$, the map α defined by

$$\sigma(x_S(t,u)) = x_S(t^{\alpha},u')$$

is single-valued.

The following lemma is implicit in section 7 of [29].

<u>Lemma 15</u>

Let G be a finite twisted Chevalley group of type ${}^2A_{2n}$. Suppose that σ is a natural automorphism such that:

i) $\sigma(x_R(1)) = x_R(1)$ for each $R = \Phi_J^+$ where J is a ρ-orbit of Π of type $A_1 \times A_1$.

ii) $\sigma(x_S(1,u)) = x_S(1,v)$ for some u satisfying $u+\bar{u} = -N_{r_n,r_{n+1}}$.

Then σ is a field automorphism.

We shall now prove the unwrapping lemma. A difficulty arises when, after normalisation by a diagonal automorphism, we want to show that $T(q_1^2)$ is mapped onto the canonical subgroup of $T(q_2^2)$. The trick which we use is borrowed from section 7 of [29].

<u>Lemma 16</u>

Let $T = {}^2A_{2n}$, where $n \geq 2$. Suppose that:

i) $[GF(q_2^2)\colon GF(q_1^2)]$ is odd.

ii) $\pi\colon T(q_1^2) \to T(q_2^2)$ is a natural embedding.

iii) $\phi\colon T(q_1^2) \to T(q_1^2)$ is a natural automorphism.

Then the following diagram can be completed:

$$\begin{array}{ccc} T(q_2^2) & \xrightarrow{\ \psi\ } & T(q_2^2) \\ \pi\uparrow & & \uparrow c \\ T(q_1^2) & \xrightarrow{\ \phi\ } & T(q_1^2) \end{array}$$

where ψ is a natural automorphism and c is the canonical embedding.

Proof

As in the proof of lemma 6, we shall write $x_T(t)$, $\bar{x}_T(t)$ for the elements of $T(q_1^2)$, $T(q_2^2)$ respectively. Fix an element u such that $u+\bar{u} = -N_{r_n,r_{n+1}}$. Suppose that

$$\pi\phi^{-1}(x_S(1,u)) = \bar{x}_S(c_S, v)$$

$$\pi\phi^{-1}(x_R(1)) = \bar{x}_R(c_R)$$

for each $R = \Phi_J^+$ where J is a ρ-orbit of Π of type $A_1 \times A_1$. Then there is a diagonal automorphism d of $T(q_2^2)$ such that

$$d\pi\phi^{-1}(x_S(1,u)) = \bar{x}_S(1,\ell)$$

$$d\pi\phi^{-1}(x_R(1)) = \bar{x}_R(1).$$

For each R, the argument of lemma 6 shows that $d\pi\phi^{-1}[\langle {}_1X_R^1, {}_1X_{-R}^1\rangle]$ is contained in the canonical subgroup of $T(q_2^2)$.

Claim

$d\pi\phi^{-1}[\langle {}_1X_S^1, {}_1X_{-S}^1\rangle]$ is contained in the canonical subgroup of $T(q_2^2)$.

Proof of claim

We assume that structure constants have been chosen so that $N_{r_n,r_{n+1}} = 1$. Thus

$$x_S(t_1,u_1)x_S(t_2,u_2) = x_S(t_1+t_2, u_1+u_2-\bar{t}_1t_2).$$

Choose $k \in GF(q_1^2)-GF(q_1)$ and put $j = \bar{k}-k$. Let $m = uk\bar{k}$. Then an easy calculation shows that

$$[x_S(1,u),\ h_T(k)^{-1}x_S(1,u)h_T(k)] = x_S(0,j). \qquad (*)$$

Applying $d\pi\phi^{-1}$, we obtain

$$d\pi\phi^{-1}(x_S(0,j)) = [\bar{x}_S(1,\ell), \bar{h}_T(\lambda)^{-1}\ \bar{x}_S(1,\ell)\bar{h}_T(\lambda)]$$

for some $\lambda \in GF(q_1^2)-GF(q_1)$. Thus

$$d\pi\phi^{-1}(x_S(0,j)) = \bar{x}_S(0,j_1)$$

where $j_1 = \bar{\lambda}-\lambda$.

Let $H_S = \{h_S(t) \mid t \in GF(q_1^2)\}$, and n_S be the element defined in section 2. Then $x_S(0,j)x_{-S}(0,-j^{-1})x_S(0,j) \in n_S H_S$. Applying $d\pi\phi^{-1}$, we obtain

$$\bar{x}_S(0,j_1)\bar{x}_{-S}(t,v)\bar{x}_S(0,j_1) \in \bar{n}_S\bar{H}_S.$$

Using the unitary identification, it follows that

$$\bar{x}_{-S}(t,v) = \bar{x}_{-S}(0,-j_1^{-1}).$$

Define $n_S(j) = x_S(0,j)x_{-S}(0,-j_1^{-1})x_S(0,j)$ and $\bar{n}_S(j_1) = \bar{x}_S(0,j_1)\bar{x}_{-S}(0,-j_1^{-1})\bar{x}_S(0,j_1)$. Choose any element of the form $x_S(1,v)$. Using the unitary identification again, we obtain

$$x_S(1,v)n_S(j)x_S(k,m) \in {}_1X^1_{-S}H_S$$

iff $k = \bar{j}\bar{v}^{-1}$ and $m = j\bar{j}\bar{v}^{-1}$. Suppose that these conditions are met, and that

$$d\pi\phi^{-1}(x_S(1,v)) = \bar{x}_S(1,v_1)$$

$$d\pi\phi^{-1}(x_S(k,m)) = \bar{x}_S(k_1,m_1).$$

Then $k_1 = \bar{j}_1\bar{v}_1^{-1}$. So to show that $x_S(1,v)$ is mapped into the canonical subgroup of $T(q_2^2)$, it is enough to show that $k_1 \in GF(q_1^2)$. But $x_S(k,m) = h_T(k)^{-1}x_S(1,mk^{-1}\bar{k}^{-1})h_T(k)$ and so $\bar{x}_S(k_1,m_1) = \bar{h}_T(k_1)^{-1}\bar{x}_S(1,m_1k_1^{-1}\bar{k}_1^{-1})\bar{h}_T(k_1)$. Thus $k_1 \in GF(q_1^2)$.

We have shown:

i) every element of the form $x_S(1,v)$ is mapped into the canonical subgroup;

ii) $n_S(j)$ and $x_S(0,j)$ are mapped into the canonical subgroup.

We can now easily prove the claim. Suppose that $k \neq 0$. Then

$$x_S(k,m) = h_T(k)^{-1}x_S(1,mk^{-1}\bar{k}^{-1})h_T(k)$$

is mapped into the canonical subgroup. On the other hand there are precisely q_1-1 elements of the form $x_S(0,m)$, where $m \neq 0$. The possible values of m are $\{vj \mid v \in GF(q_1)^*\}$ and each such $v = k\bar{k}$ for some

$k \in GF(q_1^2)^*$. Hence

$$x_S(0,vj) = h_T(k)^{-1} x_S(0,j) h_T(k)$$

is mapped into the canonical subgroup. Since $_1X^1_{-S} = n_S(j)\,{}_1X^1_S\, n_S(j)^{-1}$, the claim is proved.

By proposition 13.6.5 of [7], $d\pi\phi^{-1}$ maps $T(q_1^2)$ into the canonical subgroup of $T(q_2^2)$. By lemma 15, $d\pi\phi^{-1} \restriction T(q_1^2)$ is a field automorphism. Extend this to a field automorphism $\mathfrak{F}$ of $T(q_2^2)$. Then $\psi = \mathfrak{F}^{-1}d$ is the required natural automorphism.

□

3.6 <u>The case of $T = {}^2F_4$</u>

Again the existence of a two parameter root subgroup forces us to do a little work. The twisted Chevalley group $T(2^{2m_i+1})$ is defined in terms of a field automorphism

$$\theta_i : GF(2^{2m_i+1}) \to GF(2^{2m_i+1})$$
$$\lambda \to \lambda^{2^{m_i}}$$

Note that $[GF(2^{2m_{i+1}+1}) : GF(2^{2m_i+1})]$ is odd. Thus for every $\lambda \in GF(2^{2m_i+1})$,

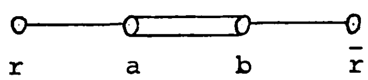
$$\lambda^{2^{m_{i+1}}} = \lambda^{2^{m_i}}.$$

It follows that:

A) $T(2^{2m_{i+1}+1})$ contains a canonical copy of $T(2^{2m_i+1})$.

B) Let $K = \bigcup_{i\in\omega} GF(2^{2m_i+1})$. Then K has an automorphism θ such that $2\theta^2 = 1$ and $\theta \restriction GF(2^{2m_i+1}) = \theta_i$.

F_4 has Dynkin diagram:



Let $S = \{r, \bar{r}\}$ be of type $A_1 \times A_1$, where r is a short root and $\bar{r}$ is a long root. Let $T = \{a, b, a+b, 2a+b\}$ be of type B_2, where a is a short root and b is a long root. We shall write t^θ instead of $\theta(t)$. Define $K_i = GF(2^{2m_i+1})$. We shall prove the unwrapping lemma for $T = {}^2F_4$ via a series of claims.

Lemma 17

Let $T = {}^2F_4$. Suppose that:

i) π: $T(K_1) \to T(K_2)$ is a natural embedding.

ii) ϕ: $T(K_1) \to T(K_1)$ is a natural automorphism.

Then the following diagram can be completed:

$$\begin{array}{ccc} T(K_2) & \xrightarrow{\ \psi\ } & T(K_2) \\ \pi \uparrow & & \uparrow c \\ T(K_1) & \xrightarrow{\ \phi\ } & T(K_1) \end{array}$$

where ψ is a natural automorphism and c is the canonical embedding.

Proof

From the above remarks, we may write θ for θ_1 and θ_2. For each $u,t \in K_1$ define

$$x_S(t) = x_r(t^\theta)x_{\bar{r}}(t)$$

$$h_S(t) = h_r(t^\theta)h_{\bar{r}}(t)$$

$$\alpha(t) = x_a(t^\theta)x_b(t)x_{a+b}(t^{\theta+1})$$

$$\beta(u) = x_{a+b}(u)x_{2a+b}(u^{2\theta})$$

$$x_T(t,u) = \alpha(t)\beta(u)$$

$$h_T(t) = h_a(t^\theta)h_b(t).$$

We shall write $\bar{x}_S(t)$ etc. for the corresponding elements of $T(K_2)$. Suppose that

$$\pi\phi^{-1}(x_S(1)) = \bar{x}_S(c)$$

$$\pi\phi^{-1}(x_T(1,0)) = \bar{x}_T(\ell,m)$$

There exists a diagonal automorphism d of $T(K_2)$ such that

$$d\pi\phi^{-1}(x_S(1)) = \bar{x}_S(1)$$

$$d\pi\phi^{-1}(x_T(1,0)) = \bar{x}_T(1,u)$$

for some $u \in K_2$.

Claim 1

$d\pi\phi^{-1}[T(K_1)]$ is the canonical subgroup of $T(K_2)$.

Proof of claim 1

Once again, it is clear that $<{}_1X^1_S, {}_1X^1_{-S}>$ is mapped to the canonical subgroup of $T(K_2)$. The difficulty is to show that this is also true of $<{}_1X^1_T, {}_1X^1_{-T}>$.

Choose an element $k \in K_1$ such that $k^\theta \neq k$. For any $\ell, m \in K_1$,

$$[x_T(1,\ell),\ x_T(k,m)] = x_T(0,k-k^\theta).$$

Thus

$$[x_T(1,0),\ h_S(k^{-1})x_T(1,0)h_S(k)] = x_T(0,k-k^\theta).$$

Hence $d\pi\phi^{-1}(x_T(0,k-k^\theta)) = \bar{x}_T(0,\lambda-\lambda^\theta)$ for some $\lambda \in K_1$. Put $j = k-k^\theta$ and $j^\eta = \lambda-\lambda^\theta$. We have shown that $d\pi\phi^{-1}(\beta(j)) = \bar{\beta}(j^\eta)$. Let $j\ell \neq 0$ be any element of K_1^*. Then

$$\beta(j\ell) = h_T(\ell)\beta(j)h_T(\ell)^{-1}.$$

Thus each element of the form $\beta(u)$, $u \in K_1$, is mapped into the canonical subgroup. For later use, define the function η by the equation

$$d\pi\phi^{-1}(\beta(u)) = \bar{\beta}(u^\eta).$$

We shall now make use of the matrix realisation of ${}^2B_2(K_1)$ given on page 246 of [7]. Let n_T be the element of the Weyl subgroup such that

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \longrightarrow n_T .$$

The following equation holds in $<{}_1X^1_T, {}_1X^1_{-T}>$.

$$\alpha(1)n_T\beta(1)n_T = n_T\alpha(1).$$

Under $d\pi\phi^{-1}$, we have

$$\alpha(1) \longrightarrow \bar{x}_T(1,u)$$

$$\beta(1) \longrightarrow \bar{\beta}_T(j) \text{ for some } j \in K_1$$

$$n_T \longrightarrow \bar{n}_T\bar{h}_T(t) \text{ for some } t \in K_2.$$

Thus we obtain

$$x_T(1,u)\bar{n}_T\bar{h}_T(t)\bar{\beta}(j)\bar{h}_T(t)^{-1}\bar{n}_T = \bar{n}_T\bar{h}_T(t)\bar{x}_T(1,u).$$

Using the matrix realisation, a calculation shows that $u = 0$ and $t = j = 1$. So under $d\pi\phi^{-1}$, we have

$$\alpha(1) \longrightarrow \bar{\alpha}(1)$$

$$\beta(1) \longrightarrow \bar{\beta}(1)$$

$$n_T \longrightarrow \bar{n}_T .$$

Let $t \in K_1^*$ be any nonzero element. Then

$$\alpha(t) = h_S(t^{-1})\alpha(1)h_S(t^{-1})^{-1}.$$

Thus every element of the form $\alpha(t)$, $t \in K_1$, is mapped into the canonical subgroup. For later use, define the function μ by the equation

$$d\pi\phi^{-1}(\alpha(t)) = \bar{\alpha}(t^{\mu}).$$

We have shown that ${}_1X_T$ and ${}_1X^1_{-T} = n_T\,{}_1X^1_T n_T$ are mapped into the canonical subgroup, and so claim 1 is proved.

Define the function ξ by the equation

$$d\pi\phi^{-1}(x_S(u)) = \bar{x}_S(u^{\xi}).$$

Then $d\pi\phi^{-1}$ induces a natural automorphism σ of $T(K_1)$ such that:

i) $\sigma(x_S(u)) = x_S(u^{\xi})$, with $1^{\xi} = 1$.

ii) $\sigma(\alpha(t)) = \alpha(t^{\mu})$, with $1^{\mu} = 1$.

iii) $\sigma(\beta(u)) = \beta(u^{\eta})$, with $1^{\eta} = 1$.

iv) $\sigma(n_S) = n_S$ and $\sigma(n_T) = n_T$.

Claim 2

σ is a field automorphism of $T(K_1)$.

Proof of claim 2

Using the isomorphism $SL(2,K_1) \to \langle {}_1X_S, {}_1X_{-S} \rangle$ and lemma 5, it follows that ξ: $K_1 \to K_1$ is an automorphism. We also have that $\sigma(h_S(t)) = h_S(t^{\xi})$. Let $t \in K_1^*$ be any nonzero element. Then

$$\sigma(\alpha(t)) = h_S(t^{\xi})^{-1}\alpha(1)h_S(t^{\xi}) = \alpha(t^{\xi})$$

Thus $\mu = \xi$. The equation

$$x_S(t) = h_T(t)^{-1}x_S(1)h_T(t)$$

implies that $\sigma(h_T(t)) = h_T(t^{\xi})$. Hence

$$\sigma(\beta(t)) = h_T(t^{\xi})\beta(1)h_T(t^{\xi})^{-1}$$

and so $\eta = \xi$. From (i), (ii), (iii) and (iv), σ agrees with the field automorphism induced by ξ on ${}_1X_S^1$, ${}_1X_T^1$, n_S and n_T. Since these elements generate $T(K_1)$, σ is a field automorphism and claim 2 is proved.

It is now easy to complete the proof of lemma 17.

□

Chapter Four: Elementary Properties of Chevalley Groups

To complete the proof of Cherlin's conjecture, we must show that if $G = L(K)$ is the group of Lie type L over the locally finite field K, then K may be interpreted in G. In this chapter, we will deal with the (nontwisted) Chevalley groups. As well as describing the interpretations, we shall show that certain classes of groups are finitely axiomatizable. These results, which will be useful in the final chapter, continue the work begun by Malcev in [22].

Definition 1

Let M, N be structures for the languages L_1, L_2 respectively. M is syntactically equivalent to N iff there exist two effective algorithms $\Gamma_1: L_1 \to L_2$ and $\Gamma_2: L_2 \to L_1$ such that:

a) for each sentence $\phi \in L_1$, $M \models \phi$ iff $N \models \Gamma_1(\phi)$.

b) for each sentence $\psi \in L_2$, $N \models \psi$ iff $M \models \Gamma_2(\psi)$.

In [22], Malcev proved

Theorem 2

For $n > 2$, $SL(n,K)$, $PSL(n,K)$ and K are syntactically equivalent for any field K of characteristic 0.

Malcev was unable to prove this result for $n = 2$. We shall first show that the theorem is true for $n = 2$. Then we shall prove the corresponding result for the other Chevalley groups. (The required interpretations and finite axiomatizations will form the major part of the proof.) Throughout, K is a field of arbitrary characteristic.

The following lemmas are almost trivial.

Lemma 3 (Malcev)

There is an algorithm Γ_1 such that for each group sentence ϕ,

$$SL(2,K) \models \phi \text{ iff } K \models \Gamma_1(\phi).$$

□

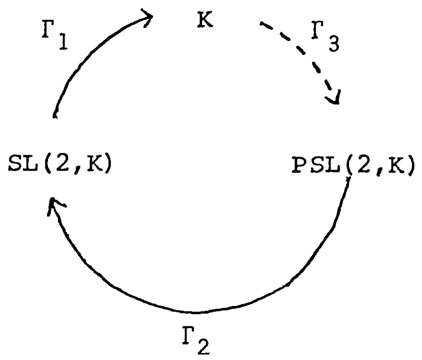
<u>Lemma 4</u> (Malcev)

There is an algorithm Γ_2 such that for each group sentence ϕ,

$$PSL(2,K) \models \phi \text{ iff } SL(2,K) \models \Gamma_2(\phi).$$

□

The problem is to find an algorithm Γ_3 which completes the following diagram.

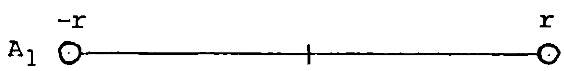


<u>Lemma 5</u>

Let $|K| > 3$. Then K is interpretable in $PSL(2,K)$.

<u>Proof</u>

Remember that $PSL(2,K)$ has the following root system.



We shall make use of the parameters

$$a = x_r(1)$$

$$b = h_r(\lambda), \text{ where } \lambda^2 \neq 1$$

$$c = n_r.$$

It is easily checked that

$$C(a) = \{x_r(k) \mid k \in K\}$$

$$C(b) = \{h_r(k) \mid k \in K^*\}.$$

<u>Step One</u>

We can interpret $\langle K, +, *, 1, 0\rangle$ in $PSL(2,K)$, where $t*u = t^2u$.

<u>Proof of step one</u>

We define the interpretation as follows:

i) The underlying set is $C(a)$.

ii) Field addition $\oplus$ is given by the group operation on $C(a)$.

iii) a represents $1 \in K$ and the identity element $1 \in PSL(2,K)$ represents $0 \in K$.

iv) Finally we require a formula $\phi(x,y,z,a,b,c)$ such that for each $x_r(t)$, $x_r(u)$, $g \in PSL(2,K)$

$$g = x_r(t^2u) \text{ iff}$$
$$PSL(2,K) \models \phi(x_r(t), x_r(u), g, a, b, c).$$

So suppose that $x_r(t)$, $x_r(u) \in C(a)$ are nonidentity elements. An easy calculation shows that $x_{-r}(-t^{-1})$ is the unique element $g \in c\, C(a)c^{-1}$ such that

$$x_r(t)\, g\, x_r(t) \in C(b)c.$$

Thus $h_r(t) = x_r(t)\, g\, x_r(t)c^{-1}$ and $x_r(t^2u) = h_r(t)x_r(u)h_r(t)^{-1}$ are uniformly definable from $x_r(t)$, $x_r(u)$. If $x_r(t)$ or $x_r(u)$ is the identity element, we put $x_r(t) \circledast x_r(u) = 1$. Thus such a $\phi(x,y,z,a,b,c)$ exists.

It is immediate that

$$\langle C(a), \oplus, \circledast, a, 1\rangle \simeq \langle K, +, *, 1, 0\rangle .$$

<u>Step Two</u>

$\langle K, +, \times, 1, 0\rangle$ is interpretable in $\langle K, +, *, 1, 0\rangle$.

<u>Proof of step two</u>

<u>Case 1</u> char $K \neq 2$

$x = tu$ iff

$$(\exists y)[x+x = y \wedge y = t*u + u - (t-1)*u]$$

<u>Case 2</u> char $K = 2$

$$x = tu \text{ iff } x*1 = t*(u*1).$$

□

Unfortunately we are not allowed to use parameters in $\Gamma_3(\phi)$. However, a simple trick allows us to eliminate them.

Theorem 6

Suppose that $|K| > 5$. There is an algorithm Γ_3 such that for each field sentence ϕ,

$$K \models \phi \text{ iff } PSL(2,K) \models \Gamma_3(\phi).$$

Proof

Rather than writing $\Gamma_3(\phi)$ explicitly, we shall show that such an algorithm exists. Let a,b,c be the parameters used in the previous lemma. This lemma provides an algorithm for constructing a formula $\psi(x,y,z)$ such that

$$K \models \phi \text{ iff } PSL(2,K) \models \psi(a,b,c).$$

$\Gamma_3(\phi)$ will be a formula of the form

$$(\exists xyz)[P(x,y,z) \wedge \psi(x,y,z)].$$

First we state some of the first order properties of $\langle a,b,c \rangle$.

i) $b^2 \neq 1$

ii) $(\forall x \in C(a))(\exists! y \in c\,C(a)c^{-1})(xyx \in C(b)c)$.

iii) The interpretation $\langle C(a), \oplus, \otimes, a, 1 \rangle$ is a field.

iv) We can use the interpretation of K on C(a) to assign field elements to elements of $c\,C(a)c^{-1}$ in a definable manner. Suppose that $x_r(t) \in C(a)$, i.e. $x_r(t)$ represents $t \in K$. Then $x_{-r}(-t) = c\,x_r(t)\,c^{-1}$. Let

$$n_s(t) = x_s(t)x_{-s}(-t^{-1})x_s(t)$$

$$h_s(t) = n_s(t)c^{-1}$$

for $t \in K^*$ and $s \in \{r,-r\}$. Then we can use the field interpetation to say that the following identities hold:

$$x_s(t_1)x_s(t_2) = x_s(t_1+t_2)$$

$$n_s(t)x_s(u)n_s(t)^{-1} = x_s(-t^{-2}u), \quad t \neq 0$$

$$h_s(t_1)h_s(t_2) = h_s(t_1t_2),\ t_1t_2 \neq 0$$

for $s \in \{r,-r\}$. (These formulas are rather complicated. As an example, we will explain how to say

$$n_{-r}(t)x_{-r}(u)n_{-r}(t) = x_{-r}(-t^{-2}u)$$

with a first order formula. For each $x,y \in c\,C(a)c^{-1}$ with $x \neq 1$, there is a unique $z \in C(a)$ such that $z \otimes c^{-1}xc = a$. If $g = z \otimes z \otimes c^{-1}yc$, then $xzxyxzx = (cgc^{-1})^{-1}$.)

v) Every element of PSL(2,K) is the product of at most four elements from $C(a)$ and $c\,C(a)c^{-1}$.

Let $P(x,y,z)$ state the first order properties (i) to (v). Put

$$\Gamma_3(\phi) = (\exists xyz)[P(x,y,z) \wedge \psi(x,y,z)].$$

It only remains to show that if $PSL(2,K) \models \Gamma_3(\phi)$, then $K \models \phi$. So suppose that $\alpha,\beta,\gamma \in PSL(2,K)$ and $PSL(2,K) \models P(\alpha,\beta,\gamma) \wedge \psi(\alpha,\beta,\gamma)$. Define

$$\bar{X}_r = C(\alpha),\quad \bar{X}_{-r} = \gamma C(\alpha)\gamma^{-1}.$$

By (iii),

$$\langle \bar{X}_r, \oplus, \otimes, \alpha, 1\rangle \overset{\pi}{\cong} \langle F, +, \times, 1, 0\rangle$$

for some field F. Clearly $F \models \phi$. We shall show that $F \cong K$. For $t \in F$, define

$$\bar{x}_r(t) = \pi^{-1}(t)$$

$$\bar{x}_{-r}(-t) = \gamma\,\bar{x}_r(t)\gamma^{-1}.$$

Then the relations in (iv) hold for $\bar{x}_s(t)$, $s \in \{r,-r\}$. By (v) and Carter [7] page 198,

$$PSL(2,K) = \langle \bar{X}_r, \bar{X}_{-r}\rangle \cong PSL(2,F).$$

Since $|K| > 5$, $K \cong F$.

□

If $|K| \leq 5$, it is trivially true that PSL(2,K) and K are syntactically equivalent. So the theorem for n = 2 is proved.

Definition 7

Let C be a class of groups. C is finitely axiomatizable iff there is a sentence ϕ in the language of groups such that for any group G,

$$G \models \phi \quad \text{iff} \quad G \in C.$$

Corollary 8

The class $\{PSL(2,K) \mid K \text{ is a field}\}$ is finitely axiomatizable.

Proof

There is a sentence ψ such that

$$G \models \psi \quad \text{iff} \quad G \simeq PSL(2,2) \text{ or } PSL(2,3).$$

Let ϕ be the sentence

$$\psi \vee [(\exists xyz)P(x,y,z) \wedge (\forall x\neq 1)(\exists y)([x,y]\neq 1)]$$

where $P(x,y,z)$ is the formula used in the proof of theorem 6.

□

Now we consider the Chevalley groups of Lie type $L \neq A_1$. We shall make use of the following theorem of Steinberg.

Theorem (Carter [7] page 190)

Let L be a simple Lie algebra with $L \neq A_1$ and let K be a field. For each root r of L and each element t of K introduce a symbol $\bar{x}_r(t)$. Let $\bar{G}$ be the abstract group generated by the elements $\bar{x}_r(t)$ subject to the relations

(a) $\bar{x}_r(t_1)\bar{x}_r(t_2) = \bar{x}_r(t_1+t_2)$

(b) $[\bar{x}_s(u),\bar{x}_r(t)] = \prod_{i,j>0} \bar{x}_{ir+js}(C_{ijrs}(-t)^i u^j)$

(c) $\bar{h}_r(t_1)\bar{h}_r(t_2) = \bar{h}_r(t_1t_2)$, $t_1t_2 \neq 0$ where $\bar{h}_r(t) = \bar{n}_r(t)\bar{n}_r(-1)$ and $\bar{n}_r(t) = \bar{x}_r(t)\bar{x}_{-r}(-t^{-1})\bar{x}_r(t)$. Let $\bar{Z}$ be the centre of $\bar{G}$. Then $\bar{G}/\bar{Z}$ is isomorphic to the Chevalley group $G = L(K)$.

As in the proof of theorem 6, we shall show that for each Lie type L, there is a sentence ϕ_L which describes the generators and relations of L(K). For ϕ_L to be independent of K, it is important that a bounded number of parameters are needed. When $L = A_1$, explicit calculations show that <a,b,c> suffice for any field. The following lemma allows us to avoid tedious calculations in the remaining cases.

Lemma 9

There is a constant $N_n \in \omega$ such that for any linear group G of degree n and any subset $A \subseteq G$, there is a subset $A_0 \subseteq A$ of cardinality $\leqslant N_n$ with $C(A_0) = C(A)$.

Proof

Suppose not. Then for any $m \in \omega$, there exists a linear group G of degree n and a subset $A \subseteq G$ such that if $A_0 \subseteq A$ and $C(A_0) = C(A)$, then $|A_0| > m$. For such a group, choose a set $B \subseteq A$ of minimal cardinality such that $C(B) = C(A)$. Then if $B = \{b_i | 1 \leqslant i \leqslant N\}$, $N > m$, we have

$$C(\{b_i | i \leqslant j\}) \supsetneq C(\{b_i | i \leqslant j+1\})$$

for all $1 \leqslant j \leqslant N-1$.

Extend the language of groups by adding new constants $\{c_m | m \in \omega\}$, and consider the theory T consisting of the following sentences:

$$C(\{c_i | i \leqslant j\}) \supsetneq C(\{c_i | i \leqslant j+1\}), \quad j \in \omega$$

$$\Phi$$

where Φ is the set of sentences which axiomatizes the class of linear groups of degree n. By compactness, there exists a model G of T. But this means that G is a linear group which fails to satisfy the minimal condition on centralisers. ⋇

□

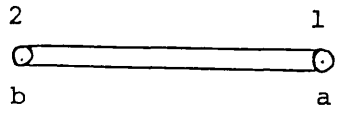
Fix a Lie type L. First we show that for any field K, the root subgroups of L(K) are definable. We remind the reader that we are following the notation of Carter [7].

Lemma 10

If char K > 3, then X_r is definable for each root $r \in \Phi$.

Proof

Let $H = \langle h_r(t) \mid r \in \Phi,\ t \in K\rangle$. Since $L(K)$ satisfies the minimal condition on centralisers, $H = C(H)$ is definable. If Φ has two distinct root lengths, select $a,b \in \Pi$ which lie in one of the following configurations:



or

.

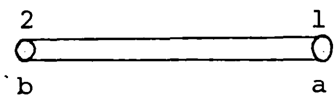
Otherwise, select $a,b \in \Pi$ such that:

In each case, $\langle X_b, X_{-b}\rangle \simeq SL(2,K)$ and

$$h_b(t)x_a(1)h_b(t)^{-1} = x_a(t^{-1}).$$

Thus $X_a = \{hx_a(1)h^{-1} \mid h \in H\}$ is definable. If r is a short root, there exists $g \in L(K)$ such that $X_a^g = X_r$. If all the roots have the same length, the proof is finished. If not, it is enough to show that X_r is definable for some long root r.

Case 1



The integral combinations of a,b which are in Φ form a root system of type B_2. Thus Chevalley's commutator formula gives

$$[x_{a+b}(1), x_a(t)] = x_{2a+b}(-C_{11a,a+b}t)$$

where $C_{11a,a+b} = N_{a,a+b}$. The a-chain of roots through $a+b$ is

$$-a+(a+b),\ a+b,\ a+(a+b).$$

Hence $C_{11a,a+b} \in \{\pm 2\}$, and

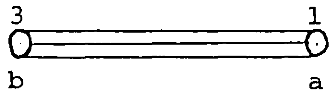
$$[x_{a+b}(1),\ x_a(-t\ C_{11a,a+b}^{-1})] = x_{2a+b}(t).$$

Thus the long root subgroup

$$X_{2a+b} = \{[x_{a+b}(1),x] | x \in X_a\}$$

is definable.

Case 2



The integral combinations of a,b which are in Φ form a root system of type G_2. By examining the (a+b)-chain through 2a+b, we see that $C_{11a+b,2a+b} \in \{\pm 3\}$. Chevalley's commutator formula gives

$$[x_{2a+b}(1),x_{a+b}(-t\ C^{-1}_{11a+b,2a+b})] = x_{3a+2b}(t).$$

Since a+b is a short root, the long root subgroup

$$X_{3a+2b} = \{[x_{2a+b}(1),x] | x \in X_{a+b}\}$$

is definable.

□

If K is quadratically closed, then the above proof can be simplified. In this case, for each $r \in \Phi$,

$$X_r = \{h\ x_r(1)h^{-1} | h \in H\}.$$

Note that this also works for char K = 2,3. The application in the final chapter only requires our result for Chevalley groups over quadratically closed fields. However, for the sake of completeness, we will prove the strongest possible result. This forces us to do some work when K is a field of characteristic 2 or 3 which is not quadratically closed. The technique which we use is borrowed from Bryant [5].

Definition 11

Let P be a subgroup of G. Define subgroups $C^n_G(P)$, $n \geq 0$, as follows.

i) $C^0_G(P) = 1$.

ii) For $n \geq 1$, let $C^n_G(P)$ be the set of all elements x of G which normalise $C^0_G(P),\ldots,C^{n-1}_G(P)$ and satisfy $[x,y] \in C^{n-1}_G(P)$ for all $y \in P$.

It is easy to prove, by induction on n, that $C_G^n(P)$ is a subgroup of G and that P normalises $C_G^n(P)$. Also, $C_G^n(P) \cap P = Z_n(P)$, the nth-centre of P. We now generalise this notion by considering $C_G^n(X)$ for arbitrary subsets $X \subseteq G$.

Lemma 12

For all $n \in \omega$, $C_G^n(X) = C_G^n(\langle X\rangle)$.

Proof

When n = 0, the result holds trivially. Suppose that for each $i \leq n$, we have $C_G^i(X) = C_G^i(\langle X\rangle)$. It is enough to show that $[x,X] \subseteq C_G^n(X)$ iff $[x,\langle X\rangle] \subseteq C_G^n(X)$. Suppose that $[x,X] \subseteq C_G^n(X)$. Let $g,h \in X$. Then $[x,gh] = [x,h][x,g]^h$ and $[x,h],[x,g] \in C_G^n(X) = C_G^n(\langle X\rangle)$. Since $\langle X\rangle$ normalises $C_G^n(\langle X\rangle)$, $[x,g]^h \in C_G^n(\langle X\rangle)$. Hence $[x,gh] \in C_G^n(X)$, and by induction we obtain $[x,\langle X\rangle] \subseteq C_G^n(X)$. The other direction is trivial.

□

Lemma 13

Let G = L(K), where char K = p > 0. For all $n \in \omega$, $C_G^n(U) \subseteq N(U)$.

Proof

For n = 0, the result is trivial. Suppose that the result holds for $n \geq 0$. If $x \in C_G^{n+1}(U)$, then $[x,U] \subseteq C_G^n(U) \subseteq N(U)$. Thus for each $u \in U$, $u^x \in N(U)$. But U is the set of p-elements of N(U). Hence $u^x \in U$ and $x \in N(U)$.

□

Lemma 14

Let char K = p > 0. Then X_r is definable for each $r \in \Phi$.

Proof

Let G = L(K) and suppose that U is nilpotent of class n. For each $i \leq n$, there is a finite subset $T_i \subseteq \gamma_i(U)$ such that $C(\gamma_i(U)) = C(T_i)$. There is a finite subset $X_i \subseteq U$ such that $T_i \subseteq \gamma_i(Y_i)$, where $Y_i = \langle X_i\rangle$. Let $X = \bigcup_{1\leq i\leq n} X_i$ and $Y = \langle X\rangle$. Then for each i, we have

$$C(\gamma_i(U)) \leq C(\gamma_i(Y)) \leq C(T_i) = C(\gamma_i(U)).$$

Thus $C(\gamma_i(U)) = C(\gamma_i(Y))$. By lemma 2.5 of Bryant [5], $C_G^n(U) = C_G^n(Y)$. By lemma 12, $C_G^n(U) = C_G^n(X)$. Hence $C_G^n(U)$ is definable. (Also note that regardless of the field, we can choose the set of parameters X so that

$$|X| \leq \sum_{i=1}^{n} i\,N$$

where N is the constant given by lemma 9.) We have $C_G^n(U) \cap U = Z_n(U) = U$ and $C_G^n(U) \subseteq N(U)$. Since U has bounded exponent and is the set of p-elements of N(U), it follows that U is definable. Let $r \in \Pi$. Then

$$X_r = n_r\, n_0\, U\, n_0^{-1}\, n_r^{-1} \cap U$$

is definable. The result now follows easily.

□

The argument in the next lemma is the same for all fields K such that $|K| > 3$.

<u>Lemma 15</u>

Let $|K| > 3$. Then K may be interpreted in L(K).

<u>Proof</u>

Let $a,b \in \Pi$ be the roots defined in lemma 10. Then X_a, X_b, X_{-b} are definable. Since every element of SL(2,K) is the product of at most four transvections, $\langle X_b, X_{-b}\rangle$ is definable. Let $\lambda \in K$ with $\lambda^2 \neq 1$. Then

$$H_b = C(h_b(\lambda)) \cap \langle X_b, X_{-b}\rangle$$

is definable. For each $t \in K^*$,

$$h_b(t)x_a(1)h_b(t)^{-1} = x_a(t^{-1}).$$

We interpret K inside L(K) as follows:

i) The underlying set is X_a.

ii) Addition $\oplus$ in X_a is the group operation.

iii) Suppose that $g_i = x_a(t_i) \in X_a$ $(i = 1,2)$ are nonidentity elements.

There exist unique $h_i = h_b(t_i^{-1}) \in H_b$ such that $g_i = h_i x_a(1) h_i^{-1}$. We define $g_1 \otimes g_2 = h_2 h_1 x_a(1) h_1^{-1} h_2^{-1}$. If either $g_1 = 1$ or $g_2 = 1$, we define $g_1 \otimes g_2 = 1$. Clearly $\otimes$ is defined by a first order formula.

It is immediate that

$$\langle X_a, \oplus, \otimes, x_a(1), 1\rangle \simeq \langle k, +, \times, 1, 0\rangle.$$

□

Lemma 16

The class $\{L(K)\,|\,\text{char } K > 3\}$ is finitely axiomatizable.

Proof

We shall use the interpretation of K on X_a to assign field elements to X_r, $r \in \Phi$, in a definable manner. First suppose that r is a short root. Then there exist fundamental roots $r_1, \ldots, r_m \in \Pi$ such that

$$w_{r_m} \ldots w_{r_1}(a) = r.$$

Let $\eta_{r,r'}$ $(r, r' \in \Phi)$ be the constants defined in 6.4.2 of [7]. Thus $\eta_{r,r'} = \pm 1$. Then

$$x_r(\eta t) = n_{r_m} \ldots n_{r_1} x_a(t) n_{r_1}^{-1} \ldots n_{r_m}^{-1}$$

where

$$\eta = \eta_{r_m, s_{m-1}} \ldots \eta_{r_1, s_0}$$

$$s_0 = a$$

$$s_i = w_{r_i}(s_{i-1}).$$

Thus if $x \in X_r$, we can discover which field element is associated with it, as follows. There is a unique $h \in H_b$ such that

$$x = n_{r_m} \ldots n_{r_1} h\, x_a(1) h^{-1} n_{r_1}^{-1} \ldots n_{r_m}^{-1}.$$

If $h = h_b(t^{-1})$, then $x = x_r(\eta t)$.

Next we use the interpretation of K on X_a to assign field elements to X_b in a definable manner. For each $x \in X_b$, there exists a unique

$y \in X_{-b}$ such that $xyxn_b^{-1} \in H_b$. Thus $x = x_b(t)$ iff there exists $y \in X_{-b}$ such that $h = xyxn_b^{-1} \in H_b$ and $h^{-1}x_a(1)h = x_a(t)$.

Finally we can again use the action of the Weyl subgroup to assign field elements to the other long root subgroups.

Now note that every element $g \in L(K)$ has an expression in the form

$$g = u_1 h n_w u_2$$

for some $u_1, u_2 \in U$, $h \in H$ and $w \in W$. Since $H = \prod_{r \in \Pi} H_r$, this is a first order property which says that $\{X_r | r \in \Phi\}$ generates $L(K)$.

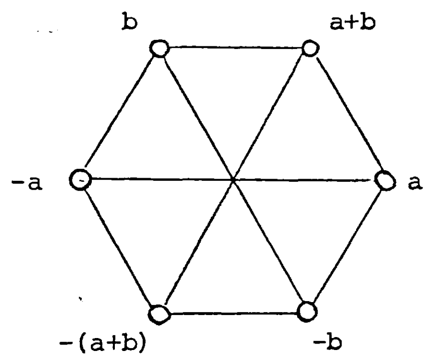
We collect together some of the first order properties of $L(K)$.

i) The definable subgroups X_r, $r \in \Phi$, generate $L(K)$.

ii) $Z(L(K)) = 1$.

iii) $\langle X_a, \oplus, \otimes, x_a(1), 1 \rangle$ is a field of characteristic greater than 3.

iv) Steinberg's relations (a), (b), (c) are satisfied, where "$x_r(t)$" is defined by means of one of the above assignments. (Once again, we give an example. Suppose that $L = A_2$. The associated root system is:



Putting $n_1 = n_b$ and $n_2 = n_a n_b$, we have

$$n_1 x_a(t) n_1^{-1} = x_{a+b}(t)$$

$$n_2 x_a(t) n_2^{-1} = x_b(t)$$

$$[x_a(t), x_b(s)] = x_{a+b}(-st).$$

This final equation has a first order expression as: for each $x \in X_a$ and $y \in X_b$, there exist unique $h_1, h_2 \in H_b$ such that

$$x = h_1 \, x_a(1) h_1^{-1}$$

$$y = n_2 h_2 x_a(1) h_2^{-1} n_2^{-1}$$

and we have

$$[x,y] = n_1(h_1 h_2 x_a(1) h_2^{-1} h_1^{-1})^{-1} n_1^{-1}).$$

Notice that our treatment of the groups L(K) in lemmas 10, 15 and 16 was uniform and used a bounded number of parameters. Thus they all satisfy the first order sentence ϕ_1 which says: "There exist parameters such that (i) to (iv) hold."

Conversely, suppose that $G \models \phi_1$. Then arguing as in the proof of theorem 6, we can use Steinberg's theorem to deduce that $G \simeq L(K)$, for some field K of characteristic greater than 3.

□

Lemma 17

The class $\{L(K) | \text{char } K = 2,3\}$ is finitely axiomatizable.

Proof

As above, there is a sentence ϕ_2 such that $GF \models \phi_2$ iff $G \simeq L(K)$ for some field K of characteristic 2 or 3 with $|K| > 3$. There is also a sentence ϕ_3 such that $G \models \phi_2$ iff $G \simeq L(2)$ or $L(3)$.

□

Summing up, we have proved

Theorem 18

For each nontwisted Lie type L, the class $\{L(K) | K \text{ is a field}\}$ is finitely axiomatizable.

□

Arguing as in the proof of theorem 6, we obtain

Theorem 19

For each field K and nontwisted Lie type L, L(K) and K are syntactically equivalent.

□

We also see that if $L(K) \equiv L(F)$, then $K \equiv F$. This provides another proof of a theorem of Zilber.

Theorem 20 (Zilber)

Let G = L(K), the Chevalley group of Lie type L over the algebraically closed field K. Then Th G is ω_1-categorical.

□

Chapter Five: The Classification of the Stable Simple Locally Finite Groups

In chapters 2 and 3, we proved that if G is a stable simple locally finite group then $G \simeq L(K)$, the group of Lie type L over some locally finite field K. In chapter 4, we showed that if G is a Chevalley group, then $K = \bar{\mathbb{F}}_p$ for some prime $p > 0$. It only remains to show that G cannot be a twisted Chevalley group. The twisted Chevalley group $L(K)$ is defined in terms of a nontrivial automorphism $\mathfrak{f} : K \to K$. Let K_0 be the fixed subfield of K under $\mathfrak{f}$. In some cases, it is easier to interpret K_0 in $L(K)$. This is sufficient, since we then have $\bar{\mathbb{F}}_p = K_0 \subsetneq K \subseteq \bar{\mathbb{F}}_p$ for some prime $p > 0$. ※

Lemma 1

Let $G = L(K)$ be a twisted Chevalley group over a locally finite field K. Let $S = \Phi_J^+$, where J is a ρ-orbit of Π. Then X_S^1 is definable.

Proof

The proofs in lemmas 13 and 14 of chapter 4 go through without change.

□

From now on, let G be a fixed stable simple locally finite group.

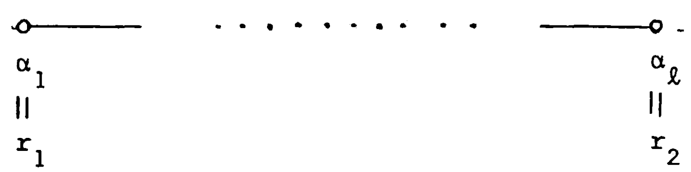
Lemma 2

G is not of type ${}^2A_\ell$ ($\ell \geq 3$), ${}^2D_\ell$ ($\ell \geq 4$), ${}^2E_6$, ${}^2F_4$ or ${}^3D_4$.

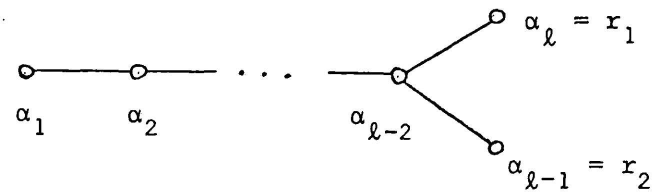
Proof

Suppose that the lemma is false. We select ρ-orbits J of Π as follows.

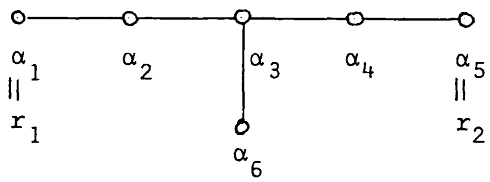
${}^2A_\ell$, $\ell \geq 3$



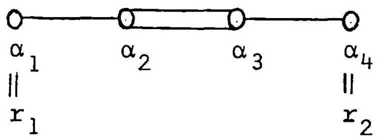
$^2D_\ell$, $\ell \geq 4$



$^2E_6$

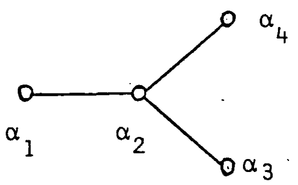


$^2F_4$



In each of the above cases, let $J = \{r_1, r_2\}$.

$^3D_4$



In this final case, let $J = \{\alpha_1, \alpha_3, \alpha_4\}$. Let $S = \Phi_J^+$. By lemma 1, X_S^1 and X_{-S}^1 are definable. Since $\langle X_S^1, X_{-S}^1 \rangle \simeq SL(2,K)$, $\langle X_S^1, X_{-S}^1 \rangle$ is definable. Thus we can interpret PSL(2,K) in G, and so by lemma 4.5 we can interpret K in G. By theorems 1.2.9 and 2.6, $K = \bar{\mathbb{F}}_p$ for some prime $p > 0$. #

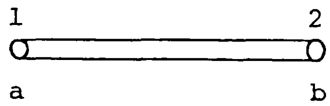
□

Lemma 3

G is not of type $^2B_2$.

Proof

Suppose that $G = {}^2B_2(K)$. Again we shall show that K may be interpreted in $^2B_2(K)$. B_2 has Dynkin diagram:



and $U^1 = X_S^1$, where $S = \{a,b,a+b,2a+b\}$. We remind the reader that $^2B_2(K)$ is defined in terms of an automorphism $\theta: K \to K$ such that $2\theta^2 = 1$. If we write

$$\alpha(t) = x_a(t^\theta)x_b(t)x_{a+b}(t^{\theta+1})$$

$$\beta(u) = x_{a+b}(u)x_{2a+b}(u^{2\theta})$$

then a typical element of U^1 has the form

$$x_S(t,u) = \alpha(t)\beta(u) \qquad (u,t \in K)$$

and

$$Z(U^1) = \{\beta(u) \mid u \in K\}.$$

We note that for $u_1, u_2 \in K$,

$$\beta(u_1)\beta(u_2) = \beta(u_1+u_2).$$

The subgroup of diagonal matrices is

$$H^1 = \{h_a(t^\theta)h_b(t) \mid t \in K^*\}.$$

If we write $h(t) = h_a(t^\theta)h_b(t)$, then we have

$$h(t)\beta(u)h(t)^{-1} = \beta(ut).$$

We can thus interpret K inside $^2B_2(K)$, as follows.

i) The underlying set is $Z(U^1)$.

ii) The field addition $\oplus$ corresponds to the group operation in $Z(U^1)$.

iii) Suppose that $\beta(t_1), \beta(t_2) \in Z(U^1)$ are nonidentity elements. Then there exist $g_i \in N(U^1) = U^1 \rtimes H^1$ such that

$$g_i \beta(1) g_i^{-1} = \beta(t_i) \qquad (i = 1,2).$$

(Clearly $g_i = uh(t_i)$ for some $u \in U^1$.) We define $\beta(t_1) \otimes \beta(t_1) = g_2 g_1 \beta(1) g_1^{-1} g_2^{-1}$. If $\beta(t_1) = 1$ or $\beta(t_2) = 1$, then we define $\beta(t_1) \otimes \beta(t_2) = 1$.

It is immediate that

$$\langle Z(U^1), \oplus, \otimes, \beta(1), 1\rangle \cong \langle K, +, \times, 1, 0\rangle.$$

□

Lemma 4

G is not of type $^2A_2$.

Proof

Suppose that $G = {}^2A_2(K)$. The group $^2A_2(K)$ is defined in terms of an automorphism $\mathfrak{F} : K \to K$ of order 2. We shall write $\mathfrak{F}(t) = \bar{t}$, and let K_0 be the subfield fixed by $\mathfrak{F}$. We shall show that K_0 may be interpreted in $^2A_2(K)$.

A_2 has Dynkin diagram:

and $U^1 = X_S^1$, where $S = \{a, b, a+b\}$. A typical element of U^1 has the form

$$x_a(t) x_b(\bar{t}) x_{a+b}(u)$$

where $u + \bar{u} = -N_{a,b} t\bar{t}$. It is easily checked that

$$Z(U^1) = \{x_{a+b}(u) \mid u + \bar{u} = 0\}.$$

Fix some element $x_{a+b}(u_0) \in Z(U^1)$. If $x_{a+b}(u) \in Z(U^1)$, there exists $t \in K_0^{\cdot}$ such that $u = u_0 t$. (Note that $\bar{u} = -u$ and $\bar{u}_0 = -u_0$. Thus

$\overline{uu_0^{-1}} = uu_0^{-1}$.) We now describe how to interpret K_0 in ${}^2A_2(K)$.

i) The underlying set is $Z(U^1)$.

ii) The field addition $\oplus$ corresponds to the group operation on $Z(U^1)$.

iii) Suppose that $g_i = x_{a+b}(s_i) \in Z(U^1)$ are nonidentity elements for $i = 1,2$. There exist $t_i \in K_0^*$ such that $s_i = u_0 t_i$. Hence there exist $h_j \in N(U^1) = U^1 \dot{\times} H^1$ $(1 \leq j \leq 4)$ such that

$$g_1 = h_1 x_{a+b}(u_0) h_1^{-1} h_2 x_{a+b}(u_0) h_2^{-1}$$

$$g_2 = h_3 x_{a+b}(u_0) h_3^{-1} h_4 x_{a+b}(u_0) h_4^{-1}.$$

(A typical set of examples is $h_a(z_j)h_b(z_j)$ for $z_j \in K_0$ such that $z_1^2+z_2^2 = t_1$ and $z_3^2+z_4^2 = t_2$.) We define

$$g_1 \otimes g_2 = \prod_{\langle i,j\rangle \in I} h_i h_j x_{a+b}(u_0) h_j^{-1} h_i^{-1}$$

where $I = \{1,2\} \times \{3,4\}$. If $g_1 = 1$ or $g_2 = 1$, we define $g_1 \otimes g_2 = 1$.

It is easily checked that the map $\langle Z(U^1), \oplus, \otimes, x_{a+b}(u_0), 1\rangle \to \langle K_0, +, \times, 1, 0\rangle$

$$x_{a+b}(u_0 t) \longrightarrow t$$

is an isomorphism. ⧣

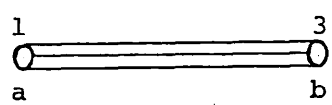
□

<u>Lemma 5</u>

G is not of type ${}^2G_2$.

<u>Proof</u>

Suppose that $G = {}^2G_2(K)$. We remind the reader that ${}^2G_2(K)$ is defined in terms of an automorphism $\theta: K \to K$ such that $3\theta^2 = 1$. G_2 has Dynkin diagram:



and $U^1 = X_S^1$, where $S = \{a,b,a+b,2a+b,3a+b,3a+2b\}$. Define

$$\alpha(t) = x_a(t^\theta)x_b(t)x_{a+b}(t^{\theta+1})x_{2a+b}(t^{2\theta+1})$$

$$\beta(u) = x_{a+b}(u^\theta)x_{3a+b}(u)$$

$$\gamma(v) = x_{2a+b}(v^\theta)x_{3a+2b}(v).$$

Then a typical element of U^1 has the form $\alpha(t)\beta(u)\gamma(v)$ for $t,u,v \in K$. An easy computation shows that

$$Z(U^1) = \{\gamma(v) \mid v \in K\}$$

and

$$\gamma(v_1)\gamma(v_2) = \gamma(v_1+v_2).$$

A typical element of H^1 has the form

$$h(u) = h_a(u^\theta)h_b(u), \qquad u \in K^*$$

and

$$h(u)\gamma(v)h(u)^{-1} = \gamma(uv).$$

We shall now explain how to interpret K inside ${}^2G_2(K)$.

i) The underlying set is $Z(U^1)$.

ii) The field addition $\oplus$ corresponds to the group operation in $Z(U^1)$.

iii) Let $g_1, g_2 \in Z(U^1)$ be nonidentity elements. There exist $h_1, h_2 \in N(U^1) = U^1 \dot{\times} H^1$ such that $g_i = h_i\gamma(1)h_i^{-1}$ $(i = 1,2)$. We define $g_1 \otimes g_2 = h_1h_2\gamma(1)h_2^{-1}h_1^{-1}$. If $g_1 = 1$ or $g_2 = 1$, we define $g_1 \otimes g_2 = 1$.

Once again, it is immediate that

$$\langle Z(U^1), \oplus, \otimes, \gamma(1), 1\rangle \simeq \langle K, +, \times, 1, 0\rangle.$$

□

We have now eliminated all of the twisted Chevalley groups, and the proof of Cherlin's conjecture is complete.

Chapter Six: Nonsoluble Locally Finite Groups of Finite Morley Rank

In [32], it was shown that an algebraically closed field F can always be interpreted in a connected nonsoluble locally finite group G of finite Morley rank. However, the proof was indirect and gave no indication of the role of F inside G. In this chapter, we clear up the mystery by proving:

Theorem 1

Let G be a connected nonsoluble locally finite group of finite Morley rank. Then there exists a definable soluble normal subgroup N such that

$$G/N \simeq S_1 \oplus \dots \oplus S_n$$

where each S_i is a Chevalley group over an algebraically closed field (possibly of different characteristics).

This result confirms the feeling that groups of finite Morley rank resemble algebraic matrix groups over algebraically closed fields. It is perhaps worth noting that $PSL(2, \bar{\mathbb{F}}_2) \oplus PSL(7, \bar{\mathbb{F}}_5)$ has finite Morley rank. So we are now dealing with a strictly larger class of groups.

We shall prove theorem 1 by imitating the development of the theory of semisimple algebraic matrix groups. At various points, it is necessary to change the arguments as we do not have as much information available. For example, in the proof of the corresponding result for algebraic matrix groups, use is made of:

Fact 2 (Humphreys [17] page 166)

Let G be semisimple.

(a) Aut G = (Inn G)D, where D consists of those automorphisms which leave stable a given maximal torus T and a Borel subgroup B containing it.

(b) Inn G has finite index in Aut G.

In this result, Aut G refers to the group consisting of all algebraic group automorphisms of G, not to its full automorphism group as an abstract group. This result is not available when we try to prove theorem 1. However, we already know that the simple locally finite groups of finite Morley rank are Chevalley groups over algebraically closed fields. Full use will be made of this information. (Once again, the logician travels in a different direction from the algebraist.)

Definition 3

An ω-stable group is semisimple if it has no nontrivial connected normal soluble subgroup.

As usual, we understand "connected" to mean "definable and connected."

Lemma 4

Let G be a group of finite Morley rank. Then G has a maximum connected normal soluble subgroup S(G).

Proof

Let A,B be any two connected normal soluble subgroups. By theorem 1.3.6, AB is a connected normal soluble subgroup. If $A \nleq B$ and $B \nleq A$, then the Morley rank $R(AB) > \max\{R(A),R(B)\}$. Since the rank cannot increase forever, there is actually a largest such subgroup S(G).

□

Note that S(G) is a characteristic subgroup of G. (Let $S(G) = \phi(G,\bar{a})$ and $\pi \in \text{Aut}\,G$. Then $S(G)^{\pi} = \phi(G,\bar{a}^{\pi})$ is also a maximum connected normal soluble subgroup, and hence $S(G)^{\pi} = S(G)$.) We call S(G) the radical of G. Suppose that $S(G) = 1$, i.e. G is semisimple. Any connected normal subgroup of G is also semisimple; its radical is a characteristic subgroup, and hence is normal in G.

Lemma 5

Let G be a connected semisimple group of finite Morley rank.

i) G has a finite centre.

ii) If $H = G/Z(G)$, then H is a connected centreless semisimple group of finite Morley rank.

Proof

(i) By definition of semisimplicity.

(ii) We shall show that H is centreless. The other statements are obvious. Since H is semisimple, Z(H) is finite. Let $Z(H) = P/Z(G)$. Then P is a finite normal subgroup of G. Hence $[G : C(P)]$ is finite, and so $G = C(P)$. Hence $P \subseteq Z(G)$ and $Z(H) = 1$.

□

So we have shown:

Lemma 6

Let G be a connected nonsoluble group of finite Morley rank. Then there exists a definable normal soluble subgroup N such that G/N is a connected centreless semisimple group of finite Morley rank.

□

It is at this point that we need to add the hypothesis that G is locally finite. The following proposition will complete the proof of theorem 1.

Proposition 4

Let G be a connected centreless locally finite semisimple group of finite Morley rank. Then

$$G \simeq S_1 \oplus \ldots \oplus S_n$$

where each S_i is a Chevalley group over an algebraically closed field (possibly of different characteristics).

Proof (cf. [17] pages 167-168)

We proceed by induction on R(G), the Morley rank of G. The inductive hypothesis is:

Let G be a connected centreless locally finite semisimple group of rank r, and let $\{G_i | i \in I\}$ be the minimal connected nontrivial normal subgroups of G. Then:

a) I is finite, say $I = \{1,\dots,n\}$, and $G = G_1 \oplus \dots G_n$.

b) For each $i \in I$, G_i is simple.

So suppose that the result holds for all $r < R(G)$. First note that we may assume that $R(G_i) < R(G)$ for all $i \in I$. Otherwise, G has no proper connected normal subgroups. But since G is connected and centreless, any definable normal subgroup must be infinite. Thus G has no definable proper normal subgroups, and so by theorem 1.3.11 G is simple.

Claim 1

If $i \neq j \in I$, $[G_i, G_j] = 1$.

Proof of claim 1

By 1.3.12, $[G_i, G_j]$ is a connected normal subgroup contained in both G_i and G_j. By minimality, $[G_i, G_j] = 1$.

□

Let $I_0 = \{i_1,\dots,i_n\} \subseteq I$ be any finite subset. Then $G_{i_1}\dots G_{i_n}$ is a connected normal subgroup of G, and hence is semisimple. $Z(G_{i_1}\dots G_{i_n})$ is a characteristic subgroup of $G_{i_1}\dots G_{i_n}$, and hence is normal in G. Since G is connected centreless and $Z(G_{i_1}\dots G_{i_n})$ is a finite normal subgroup, we must have $Z(G_{i_1}\dots\dots G_{i_n}) = 1$.

Suppose that $i \notin I_0$. Then by claim 1, $[G_{i_1}\dots G_{i_n}, G_i] = 1$. Thus $G_i \cap G_i \dots G_{i_n} = 1$. These remarks have two important consequences.

Claim 2

For any finite $I_0 = \{i_1,\dots,i_n\} \subseteq I$, $G_{i_1}\dots G_{i_n} = G_{i_1} \oplus \dots \oplus G_{i_n}$.

□

Claim 3

I is finite, say $I = \{1,\dots,n\}$.

Proof of claim 3

By claim 2, and the finiteness of $R(G)$.

□

Claim 4

For each $i \in I$, G_i is simple.

Proof of claim 4

By the preceding discussion, G_i is a connected centreless semi-simple group of finite Morley rank. Since $R(G_i) < R(G)$, $G_i \simeq H_1 \oplus \ldots \oplus H_m$, where each H_j $(1 \leq j \leq m)$ is a simple group. Further, $\{H_j | 1 \leq j \leq m\}$ is the set of minimal G_i-connected normal subgroups of G_i. Note that every automorphism $\pi \in \text{Aut}\, G_i$ permutes the H_j. To see this, note that $H_j = \phi(G_i, \bar{g})$ is a minimal connected normal subgroup, and hence so is $H_j^{\pi} = \phi(G_i, \bar{g}^{\pi})$. Hence $H_j^{\pi} = H_k$ for some $k \leq m$.

The action of G on G_i by inner automorphisms thus induces a homomorphism $\psi: G \to S_m$, the symmetric group on m elements. Since $[G : \ker\psi]$ is finite and $\ker\psi$ is definable, $G = \ker\psi$. Hence each $H_j \triangleleft G$ and $m = 1$.

□

Claim 5

$G = G_1 \oplus \ldots \oplus G_n$.

Proof of claim 5

Suppose otherwise. Put $H = G_1 \oplus \ldots \oplus G_n$. Then $[G : H]$ is infinite. The action of G on H by inner automorphisms induces a homomorphism $\psi: G \to \text{Aut}\, H$, with $\ker\psi = C_G(H)$. Let $K = (\ker\psi)^0$, the connected component.

Suppose that $K \neq 1$. Let $K_0 \subseteq K \triangleleft G$ be a minimal connected normal subgroup of G. Then $K_0 = G_i$ for some $i \leq n$. Hence

$$[G_i, G_i] = [K_0, G_i] \subseteq [K_0, H] = 1.$$

But this means that G_i is abelian. #

Thus $K = 1$. So $\ker\psi$ is a finite normal subgroup, and hence is trivial. So ψ is an embedding. The argument used in the proof of claim 4 can now be repeated to show that ψ is actually an embedding

$$\psi: G \to \text{Aut } G_1 \oplus \ldots \oplus \text{Aut } G_n.$$

Clearly $\psi[H] = \text{Inn } G_1 \oplus \ldots \oplus \text{Inn } G_n$, and so we have an embedding

$$\phi: G/H \to \text{Aut } G_1/\text{Inn } G_1 \oplus \ldots \oplus \text{Aut } G_n/\text{Inn } G_n.$$

We now consider $A_i = \text{Aut } G_i$. Since G_i is a Chevalley group over an algebraically closed field, the structure of A_i is well known. (e.g. see Steinberg [29].) A normal sequence for A_i is

$$\text{Inn } G_i \triangleleft B_i \triangleleft A_i$$

where B_i is the group generated by inner and field automorphisms. Further, we have:

a) A_i/B_i is a finite group;

b) $B_i/\text{Inn } G_i \simeq \text{Aut}(\bar{\mathbb{F}}_{p_i})$, where $\bar{\mathbb{F}}_{p_i}$ is the underlying field of G_i.

Thus $\text{Aut } G_1/\text{Inn } G_1 \oplus \ldots \oplus \text{Aut } G_n/\text{Inn } G_n$ has a torsion-free subgroup of finite index. By the second isomorphism theorem, $\phi[G/H]$ has a torsion-free subgroup of finite index, contradicting the fact that G is locally finite. Hence G = H.

□

We have now shown that G satisfies the inductive hypothesis, and the proposition is proved.

□

Chapter Seven: Simple Constructible Groups Over Algebraically Closed Fields

In this final chapter, we shall prove:

Theorem 1

Let K be an algebraically closed field, and G be an infinite simple constructible group over K. Then G is isomorphic to a Chevalley group over an algebraically closed field F.

First we show that the theorem is true for $\bar{\mathbb{F}}_p$, $p > 0$.

Lemma 2

Suppose that the group G is constructible over $\bar{\mathbb{F}}_p$. Then G is locally finite.

Proof

Let the notation be as in definition 1.3.15. Suppose that $\bar{a},\bar{b},\bar{c} \in G$ and $\bar{\mathbb{F}}_p \models \psi(\bar{a},\bar{b},\bar{c},\bar{k})$. Let F_1,F_2 be the finite fields generated by the co-ordinates of $\{\bar{a},\bar{b},\bar{k}\}$, $\{\bar{a},\bar{b},\bar{c},\bar{k}\}$ respectively. Suppose that $F_1 \subsetneq F_2$. Since F_2 is a Galois extension of F_1, there exists $\alpha \in \text{Aut } F_2$ with $\alpha \mid F_1 = \text{id}$ and $\alpha(\bar{c}) \neq \bar{c}$. Extend α to an automorphism π of $\bar{\mathbb{F}}_p$. Then

$$\bar{\mathbb{F}}_p \models \psi(\bar{a},\bar{b},\pi(\bar{c}),\bar{k}). \;\#$$

Hence if $\bar{c}_1,\ldots,\bar{c}_n \in G$ and $\bar{d} \in gp\{\bar{c}_1,\ldots,\bar{c}_n\}$, then all the co-ordinates of $\bar{d}$ lie in the finite field generated by the co-ordinates of $\{\bar{c}_1,\ldots,\bar{c}_n,\bar{k}\}$. Hence G is locally finite.

□

We intend to use the following transfer principle.

Theorem (Robinson)

Let ϕ be a first order statement in the language of fields. Then the following are equivalent:

i) ϕ is true in all algebraically closed fields of characteristic 0.

ii) for every n, there is an algebraically closed field of characteristic $p > n$ in which ϕ is true.

First we introduce some notation. For any formulas $\phi(\bar{x},\bar{w})$ and $\psi(\bar{x},\bar{y},\bar{z},\bar{w})$, let $G_{\phi,\psi}(\bar{w})$ be the obvious formula in the language of fields such that:

for any field K and $\bar{k} \in K$,

$K \models G_{\phi,\psi}(\bar{k})$ iff $\phi(\bar{x},\bar{k})$, $\psi(\bar{x},\bar{y},\bar{z},\bar{k})$

defines a constructible group over K.

We shall also use $G_{\phi,\psi}(\bar{k})$ to denote the defined group. Then if P is any first order property of groups, there is a sentence in the language of fields which says "$G_{\phi,\psi}(\bar{k})$ has the property P." We shall use this expression as an abbreviation for the corresponding sentence. For example, by 4.8 and 4.18, for each nontwisted Lie type L there is a sentence in the language of fields which says "$G_{\phi,\psi}(\bar{k}) \simeq L(F)$ for some field F." Let Ch denote the set of nontwisted Lie types. By lemma 2, the following statement is true in $\bar{\mathbb{F}}_p$ for each prime $p > 0$:

$(\forall\bar{w})$["$G_{\phi,\psi}(\bar{w})$ is infinite and simple"

$\rightarrow \bigvee_{L\in Ch}$ ["$G_{\phi,\psi}(\bar{w}) \simeq L(F)$ for some field F"]]

Unfortunately this is clearly not a first order statement. We shall mention the main difficulties.

a) How can we say "simple"?

In general, simplicity is not a first order concept. An easy compactness argument shows that A_∞, the group of finite even permutations on $\mathbb{N}$, has an elementary extension which is not simple.

b) Can we make the infinite disjunction " $\bigvee_{L\in Ch}$ " into a finite disjunction?

c) How do we say "infinite"?

To solve problem (a), we shall use:

Theorem (Zilber [38])

Let G be an infinite simple group of finite Morley rank. Then all models of ThG are simple.

Proof

This is an immediate consequence of theorem 1.3.11.

□

For each $n \in \omega$, let cong(n) be the statement:

$$(\forall x \neq 1)(\forall y)(\exists x_1 \dots x_n)[y = (x^{\pm 1})^{x_1} \dots (x^{\pm 1})^{x_n}].$$

If G satisfies cong(n), then every nontrivial conjugacy class generates G, and so G is simple. However, the infinite simple group A_∞ does not satisfy cong(n) for any $n \in \omega$.

Lemma 3

Let G be an infinite simple group which is constructible over an algebraically closed field. Then $G \models$ cong(n) for some $n \in \omega$.

Proof

It is well-known (e.g. Zilber [37]) that such a group has finite Morley rank. Suppose that $G \models \neg$cong(n) for all $n \in \omega$. Let T be the theory

$$\mathrm{Th}(G,g)_{g \in G}$$

$$\neg(\exists x_1 \dots x_n)[c = (d^{\pm 1})^{x_1} \dots (d^{\pm 1})^{x_n}]$$

$$d \neq 1$$

for each $n \in \omega$, where c and d are new constants. Then T is consistent and hence has a model $G^* > G$. But clearly G^* is not simple. #

□

Thus in each case that we consider, we shall be able to say "simple". Next we deal with problem (c).

Lemma 4

Let $\phi(\bar{x},\bar{w})$ be a formula in the language of fields. There exists a formula $\inf(\bar{w})$ such that:

for <u>any</u> algebraically closed field K and $\bar{k} \in K$, $K \models inf(\bar{k})$ iff $\{\bar{m} \in K | K \models \phi(\bar{m},\bar{k})\}$ is infinite.

Proof

We shall use the fact that ACF, the theory of algebraically closed fields, admits elimination of quantifiers. Following the usual convention, we shall identify formulas and the subsets which they define. Let K be any algebraically closed field and $\bar{k} \in K$. Then $\phi(\bar{x};\bar{k})$ is infinite iff one of the projections

$$P_i(x_i;\bar{k}) = (\exists x_1 \ldots x_{i-1} x_{i+1} \ldots x_n)\phi(\bar{x};\bar{k})$$

is infinite. By the elimination of quantifiers in ACF,

$$P_i(x_i;\bar{w}) \equiv \bigvee [\Lambda p(x_i;\bar{w}) = 0 \wedge \Lambda q(x_i;\bar{w}) \neq 0]$$

where each p,q is a polynomial. (We have suppressed a number of indices, but it should be obvious what is going on.) Suppose that $P_i(x_i;\bar{k})$ is infinite. Then one of the above disjuncts must be infinite, say

$$[\Lambda p(x_i;\bar{k}) = 0 \wedge \Lambda q(x_i;\bar{k}) \neq 0].$$

Note that $[\Lambda p(x_i;\bar{k}) = 0]$ is a closed subset of affine 1-space. The only closed subsets are the finite subsets and the whole space. Hence

$$[\Lambda p(x_i;\bar{k}) = 0] \equiv [x_i = x_i].$$

Also $[\Lambda q(x_i;\bar{k}) \neq 0]$ is an open subset of affine 1-space. The only open subsets are the empty set and the cofinite sets. Hence

$$[\Lambda q(x_i;\bar{k}) = 0] \not\equiv [x_i \neq x_i].$$

Thus $inf(\bar{w})$ can be chosen to say:- for some $i \leq n$ and some disjunct of P_i,

$$[\Lambda p(x_i;\bar{w}) = 0] \leftrightarrow [x_i = x_i]$$

and

$$[\Lambda q(x_i;\bar{w}) \neq 0] \not\leftrightarrow [x_i \neq x_i].$$

□

Finally we deal with problem (b).

Lemma 5

For each formula $\phi(\bar{x},\bar{w})$ and $n \in \omega$, there exists a finite subset $L_{\phi,n} \subseteq Ch$ such that for every prime $p > 0$,

$$\bar{\mathbb{F}}_p \models (\forall \bar{w})[["G_{\phi,\psi}(\bar{w}) \text{ satisfies cong}(n)" \wedge \inf(\bar{w})] \rightarrow \bigvee_{L \in L_{\phi,n}} ["G_{\phi,\psi}(\bar{w}) \simeq L(F) \text{ for some field } F"]]$$

Proof

Suppose that

$$\bar{\mathbb{F}}_p \models "G_{\phi,\psi}(\bar{k}) \text{ satisfies cong}(n)" \wedge \inf(\bar{k}).$$

Since $G_{\phi,\psi}(\bar{k}) \subseteq \bar{\mathbb{F}}_p^m$, where $\bar{x} = \langle x_1,\ldots,x_m\rangle$, it follows that the Morley rank $R(G_{\phi,\psi}(\bar{k})) \leqslant m$. By lemma 2 and the classification of the stable simple locally finite groups, $G_{\phi,\psi}(\bar{k})$ is a Chevalley group. Let $G_{\phi,\psi}(\bar{k}) \simeq L(F)$. Then the Morley rank $R(L(F)) \leqslant m$. Let $\Pi = \{r_1,\ldots,r_\ell\}$ be a set of fundamental roots for L. Then for each $i \geqslant \ell$, H_{r_i} is definable and hence $R(H) \geqslant \ell$. Thus $\ell \leqslant R(H) \leqslant R(L(F)) \leqslant m$.

□

Proof of theorem 1

To make matters as awkward as possible, we shall assume that char $K = 0$. (The reader will have no difficulty providing a proof for char $K = p > 0$.) Let G be defined by $\phi(\bar{x};\bar{k})$ and $\psi(\bar{x},\bar{y},\bar{z};\bar{k})$. By lemma 3, there exists $n \in \omega$ such that $G \models \text{cong}(n)$. Hence

$$K \models "G_{\phi,\psi}(\bar{k}) \text{ satisfies cong}(n)" \wedge \inf(\bar{k}).$$

Let $\Phi(\phi,n)$ be the sentence given by lemma 5. By Robinson's transfer principle, $K \models \Phi(\phi,n)$. Hence G is a Chevalley group. Since we can interpret the underlying field in G, it must be ω-stable and hence algebraically closed.

□

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