THE GIRTH OF CUBIC GRAPHS

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ABSTRACT

We start with an account of the known bounds for $n(3,g)$, the number of vertices in the smallest trivalent graph of girth g, for $g \leq 12$, including the construction of the smallest known trivalent graph of girth 9. This particular graph has 58 vertices - the 32 known trivalent graphs with 60 vertices are also catalogued and in some cases constructed.

We prove the existence of vertextransitive trivalent graphs of arbitrarily high girth using Cayley graphs. The same result is proved for symmetric (that is vertextransitive and edgetransitive) graphs, and a family of 2-arctransitive graphs for which the girth is unbounded is exhibited. The excess of trivalent graphs of girth g is shown to be unbounded as a function of g.

A lower bound for the number of vertices in the smallest trivalent Cayley graph of girth g is then found for all $g \leq 9$, and in each case it is shown that this bound is attained. We also establish an upper bound for the girth of Cayley graphs of subgroups of Aff (p^f) the group of linear transformations of the form $x \to ax + b$ where a,b are members of the field with p^f elements and a is non-zero. This family contains the smallest known trivalent graphs of girth 13 and 14, which are exhibited.

Lastly a family of 4-arctransitive graphs for which the girth may be unbounded is constructed using "sextets". There is a graph in this family corresponding to each odd prime, and the family splits into several subfamilies depending on the congruency class of this prime modulo 16. The graphs corresponding to the primes congruent to 3,5,11,13

modulo 16 are actually 5-arctransitive. The girth of many of these graphs has been computed and graphs with girths up to and including 32 have been found.

CONTENTS

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Chapter 1

Introduction

1.1 Glossary

At the outset it is necessary to outline some of the basic concepts of graph theory and define some of the notation that will be used. In general we follow the notation used in R.J. Wilson's Introduction to Graph Theory [37] and N.L. Biggs Algebraic Graph Theory [5]

A graph G consists of a set V(G) of elements called vertices and a set E(G) of elements called edges together with a relation of incidence which associates with each edge two vertices called its ends. If none of the edges have coincident ends, and no two edges are incident with the same pair of vertices, then we say G is a simple graph, and indeed we shall be dealing exclusively with simple graphs, or more briefly graphs. The two ends of an edge are said to be adjacent. We define a path of length ℓ in G joining v_i to v_j to be a finite sequence of vertices of G

$$v_i = u_o, u_1, \dots , u_\ell = v_j$$

such that u_{t-1} and u_t are adjacent for $1 \leq t \leq \ell$, and u_{t-1} and u_{t+1} are distinct $1 \leq t \leq \ell-1$. A circuit or cycle is a path in which the endvertices coincide. An s-arc is the ordered set of vertices underlying a path of length s.

A subgraph of a graph G is simply a graph all of whose vertices belong to V(G) and all of whose edges belong to E(G). A graph G is connected if for each pair of vertices v_i, v_j in V(G), there is a v_i v_j path in G_j: a maximal connected subgraph of G is a component of G. The degree or valency of a vertex v is the number of edges incident with v, and if every vertex in G is of degree 3 G is said to be trivalent or cubic. The distance between two vertices x,y in graph G is the length of the shortest path between them and will be denoted $d_G(x,y)$ (or $d(x,y)$ if there is no ambiguity).

An automorphism Ø of a graph G is a one-to-one mapping of the vertex set v(G) onto itself with the property that Ø(v) and Ø(w) are adjacent if and only if v and w are. These automorphisms form a group under composition called the automorphism group. We say that a graph G is vertex-transitive if the automorphism group acts transitively on the vertices and edge-transitive if the automorphism group acts transitively on the edges. Further if for all vertices u,v,x,y of G such that u is adjacent to v and x is adjacent to y there is an automorphism Ø such that Ø(u) = x and Ø(w) = y, G is called symmetric. A graph G is s-arc-transitive ($s \geq 1$) if its automorphism group is transitive on the set of s-arcs in G, but not transitive on the (s + 1) arcs in G; thus every symmetric graph is at least 1-arctransitive. Lastly and most importantly the girth of a graph G (which is the subject of this thesis) is the length of the shortest cycle in G.

Motivation

It is not easy to find trivalent graphs with large girth. When this work was begun there were no published examples of trivalent graphs with girth more than 12, although the existence of trivalent graphs with arbitrarily high girth had been proved. Tutte [4] and Bollobás [8] have published proofs that are in some sense constructive.

Both start with a graph G on 2^g vertices with girth g in which every vertex has degree 2 or 3 and show that if there are any vertices of degree 2 in G a graph with more edges also of girth g and every vertex of degree 2 or 3 may be constructed on the same number of vertices. Pisanski and Shawe Taylor [30] have also produced a construction that develops a trivalent graph of girth g+1 from a cycle permutation graph of girth g, while the number of vertices in the new graph is roughly the square of the number of vertices in the original. The central problem examined in this thesis is the enumeration of n(3,g), the number of vertices in the smallest trivalent graph of girth g. It is known that this value must exceed a number close to $2^{\frac{1}{2}g}$ [34], and as we have seen it is bounded by 2^g, so significance will be attached to the value

$$c(g) = \frac{\log_2(n(3,g))}{g}$$

which in turn must lie between $\frac{1}{2}$ and 1. Although it remains a mystery what happens to c(g) as g tends to infinity, in Chapter 5 we will exhibit some trivalent graphs with girth up to 32, and so obtain some upper bounds for c(g), $g < 32$.

Contents

In Chapter 2 there is an account of the known bounds for n(3,g) for $g \leq 12$, and the smallest known trivalent graph of girth 9 is derived. This particular graph has 58 vertices - the thirty two known graphs of girth 9 with 60 vertices are also catalogued and in some cases constructed.

Chapters 3 and 4 are largely concerned with Cayley graphs. A Cayley graph can be obtained from a group G with a set of generators Ω not containing the identity satisfying the additional property;

$$x \in \Omega \Rightarrow x^{-1} \in \Omega .$$

The Cayley graph $\Gamma = \Gamma(G,\Omega)$ is the simple graph whose vertexset and edgeset are

$$V(\Gamma) = G; \; E(\Gamma) = \{(g,b) \mid g^{-1}b \in \Omega\}.$$

If Ω consists of three involutions, or an involution and an element of order greater than 2 and its inverse, the resulting Cayley graph will be trivalent. A trivalent Cayley graph will be said to be Type I if its generating set consists of three involutions and Type II otherwise.

Chapter 3 contains a proof that there exist trivalent graphs that are Cayley of arbitrarily large girth and a similar result for symmetric graphs (Cayley graphs are all vertextransitive [5]). It also contains a result concerning the number of vertices in a vertextransitive graph with valency k and girth g.

Chapter 4 contains the construction of the smallest trivalent Cayley graphs of girth g where $g \leq 9$, and some examples of groups and generating sets giving trivalent Cayley graphs of girth up to 17. One particular area of investigation will be the Cayley graphs of the groups denoted $Z(p, \frac{p-1}{2}, k)$ by Coxeter, Frucht and Powers [11] where p is an odd prime and k a primitive root modulo p. There is an upper bound on the girth of such graphs which is established and attained.

Because the girth of an s-arctransitive graph must be at least 2s-2 [34], it would seem that highly arctransitive graphs would be a fertile area to look for graphs of large girth. However there is a wellknown theorem of Tutte which states that there are no trivalent s-arctransitive graphs with $s > 5$ [35]. In Chapter 5 we show how to construct a family of graphs that are at least 4-arctransitive for which it is conjectured that the girth is unbounded. There is a one-to-one correspondence between members of this family and the odd primes. The family subdivides into several subfamilies depending on the congruency class of the prime modulo 16. The subfamily of graphs corresponding to primes congruent to 1 or 15 modulo 16 is the same set of graphs as that defined by Wong in terms of primitive subgroups of the projective special linear group PSL(2,p) [38] . As shall be shown the number of vertices in the graph corresponding to prime p is of the order of p^3 if p is congruent to 1 or 7 modulo 8 and of the order of p^6 otherwise.

Some of the most interesting results are those portrayed in the numerical tables to be found at the back of the thesis. Firstly there is a table showing the smallest known trivalent graphs of girth $g \leq 17$ and some of their properties. Secondly there is a table giving the girths and degree of arctransitivity of the Cayley graphs of Z(p,(p-1)/2,k) where p is a prime less than or equal to 23; finally there are various tables associated with the sextet construction of Chapter 5.

Chapter 2

The (3,g) cages $2 \leq g \leq 12$

A (3,g)-cage is defined as a trivalent graph with girth g such that there are no other graphs with less vertices with these properties. This chapter will be devoted to the search for (3,g)-cages in the cases $2 \leq g \leq 12$ in particular the case $g = 9$.

Lower bound for n(3,g)

There is a lower bound on the number of vertices in a trivalent graph of girth g either obtained by counting the number of vertices at distance strictly less than $(g+1)/_2$ from a given vertex or by counting those vertices at distances less than $g/_2$ from either endvertex of a given edge [34]. If graph G is trivalent and has girth g and n vertices then

$$n \geq 3(2^{(g-1)/2}) - 2 \quad \text{if } g \text{ is odd, and}$$

$$n \geq 2^{g/2+1} - 2 \quad \text{if } g \text{ is even.}$$

This minimum is rarely attained. The <u>excess e(3,g)</u> is defined as the difference between n(3,g), the number of vertices in a (3,g)cage, and the minimum $n_o(3,g)$ where

$$n_o = 3.(2^{(g-1)/2}) - 2 \quad \text{if } g \text{ is odd, and}$$

$$n_o = 2^{g/2+1} - 2 \quad \text{if } g \text{ is even .}$$

The Known (3,g)cages

By considering the multiplicities of the eigenvalues of the collapsed adjacency matrices it has now been shown by various authors that the excess $e(3,g)$ can be zero only if g is equal to 3,4,5,6,8 or 12 (see [5]).

All these values of g correspond to unique cages with excess zero. The (3,g)cages for $g = 3,4,5,6,8$ respectively are the complete graph K_4, the complete bipartite graph $K_{3,3}$, the Petersen graph on 10 vertices, the Heawood graph which has 14 vertices, and the Tutte graph on 30 vertices. Their uniqueness is proved by Tutte [34], as is the uniqueness of the McGee graph which has 24 vertices and girth 7 and consequently has excess 2. The (3,12)cage on 126 vertices is described by Biggs [5] and Benson [4] and was proved unique by Rees [33] and others. O'Keefe and Wong [29] have proved that a (3,10)cage must have 70 vertices and excess 8 and that there are at least 3 of these cages. One of them was found by Balaban, and the other two were discovered by O'Keefe and Wong and independently by Harries and will be referred to here as X and Y.

Girth 9 and "Tree-Removal"

From the three graphs with 70 vertices, graphs with 60 vertices and girth 9 may be constructed as follows.

Let v be a vertex in a trivalent graph G of girth 10 and let $v_1, v_2, \ldots, v_{12}$ be the 12 vertices at distance 3 from v such that v_i is at distance 2 from v_{i+1} if i is odd.

See Fig. 2.1.

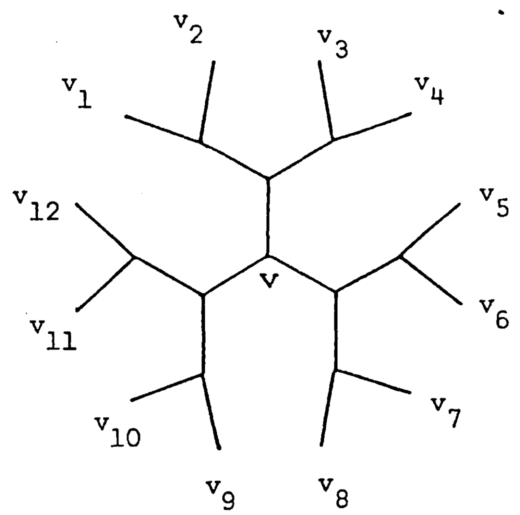


Figure 2.1

Define a new graph H whose vertex-set and edge set are

$$V(H) = V(G) \setminus \{x \in V(G) \mid d_G(v,x) \leqq 2\}$$

and $$E(H) = [E(G) \setminus (V(H) \times V(H))] \cup A$$

where A is the set of edges $\{(v_1,v_2), \ldots ,(v_{11},v_{12})\}$.

The new graph H is trivalent; we now prove every cycle in H is of length at least 9.

Let C be a cycle in the graph H.

If no edges in C are in A, then C is a cycle of G and consequently of length at least 10.

If there is just one edge (v_i,v_{i+1}) say which is in both C and A, then there is a circuit C' in G corresponding to C with the edge (v_i,v_{i+1}) replaced by two edges since v_i and v_{i+1} are at distance 2 in G. But G has girth 10 so C' has at least 10 edges and C must contain at least nine edges.

If C contains two or more edges in A it must also contain 2 paths $v_a v_b$ and $v_c v_d$ in H where v_a, v_b, v_c, and v_d are all in $\{v_1, v_2, \ldots, v_{12}\}$.

There are paths from v_a and v_b to v of length 3 in G, so there is a $v_a v_b$ path of length at most 6 in G but not in H. If there was a $v_a v_b$ path of length less than 4 in H, G would contain a cycle with less than 10 edges, so $d_H(v_a, v_b)$ must be at least 4. Similarly $d_H(v_c, v_d)$ must also be at least 4. Hence C contains at least 10 edges, and H has girth at least 9.

This result may be generalized to obtain an upper bound for $n(g,3)$ in terms of $n(g+k,3)$ for all $g \geq 6$ as follows.

Proposition

$n(g,3) \leq n(g+1,3) - n_o(\lfloor (g+2)/2 \rfloor ,3)$.

Proof

Let G be a trivalent graph of girth g and let

$$\Delta_r(x) = \{ v \varepsilon V(G) \mid d_G(v,x) = r \}.$$

Then $\bigcup_{r=0}^{s} \Delta_r(x)$ is a tree consisting of all vertices at distance less than s+1 from x if $g > 2s$ and we shall say it is rooted at x, and has radius s. If $\lfloor g/2 \rfloor$ is odd it is possible to create a graph H of girth at most g-1 by replacing the tree rooted at a given vertex x with radius $s = \lfloor (g-4)/2 \rfloor$ $\bigcup_{r=0}^{s} \Delta_r(x)$ with edges joining those vertices in Δ_{s+1} that were at distance 2 from each other.

If $\lfloor g/2 \rfloor$ is even, if a given edge (y,x) in E(G) is contracted to single vertex x of valency 4, and then the tree $\bigcup_{i=0}^{s} \Delta_i(x)$

where $s = \lfloor (g-6)/2 \rfloor$ and those vertices in $\Delta_{s+1}(x)$ that were at distance 2 in G joined, the new graph will again be at least g-1 in girth.

Balaban used this method, starting from the (3,12) cage and removing fourteen vertices to find the smallest known trivalent graph of girth 11 which has 112 vertices [1]. The (3,12)cage is edgetransitive [4], and the tree to be removed is rooted on an edge so only one such graph can be produced in this way.

Trivalent Graphs of Girth 9 with 60 Vertices.

More trivalent graphs of girth 9 with 60 vertices can be created from the (3,10)cages by tree removal as the tree to be removed is rooted at a vertex and the three (3,10)cages Balaban, X and Y have 3,4 and 8 vertex orbits respectively under the action of their automorphism groups. Just two of the resulting graphs are isomorphic, so 14 trivalent 60 vertex girth 9 graphs have been obtained (Harries, unpublished). Previously five such graphs were known, two of which are Cayley graphs and will be described in Chapter 4. The other three are named after Foster, Evans and Balaban/Biggs respectively. The Evans graph is the only known trivalent graph of girth 9 on 60 vertices that is vertextransitive but is not a Cayley graph. Only one of these graphs, the Cayley graph named after Foster and Frucht [18], has the property that its diameter, which is the maximum distance between two vertices in a given graph, is 5. Graphs with the property that their diameter is less than or equal to $\frac{1}{2}(g+1)$ where g is their girth, are known as generalized Moore graphs. This is the largest known trivalent generalized Moore graph.

Table 4 contains various details about the thirtytwo known 60 vertex trivalent graphs of girth 9 including the number of 9-cycles they contain, the automorphism group and the value of the smallest eigenvalues of their adjacency matrices.

A Trivalent Graph with 58 Vertices and Girth 9

Only one trivalent graph with 58 vertices and girth 9 is known, that being described in a paper by Biggs and Hoare [6]. This was discovered while examining edgereplacement schemes and can be derived from a 60 vertex trivalent graph (itself derived from X) as follows. In this graph XC there exists a subgraph A,B,C,D,E,F,G,H shown in Figure 2.4 with the property that through the 2-arcs ABC and DEF there are no nine-cycles.

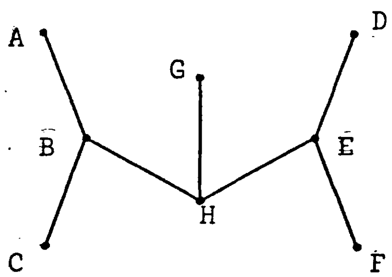


Fig. 2.4

It is possible to remove the vertices B,E,H and add the edges (A,C) and (D,F) to obtain a graph Γ on 57 vertices with girth 9, in which every vertex is of degree three except one vertex (G) which has valency 2. Elsewhere in the graph there is an edge (X,Y) such that $d_\Gamma(X,A) = d_\Gamma(Y,A) = 7$. By adding a vertex Z to the vertex set of Γ, and replacing the edge (X,Y) by the three edges (X,Z), (Y,Z) and (G,Z) a trivalent graph of girth 9 on 58 vertices is obtained.

The Value n(3,9).

In the known graph on 58 vertices described above there are 2 2-arcs which are not contained in any 9-cycle but unfortunately no means of removing either of them and reconstructing to obtain a trivalent graph on 56 vertices with girth 9 has yet been discovered. Hence the upper bound for n(3,9) remains 58. Using a computer McKay has shown that n(3,9) is at least 54 [28], but at present it cannot be said which of the three possible values 54,56,58 corresponds to the true number n(3,9).

Chapter 3

Some Families with Increasing Girth

In this chapter we shall investigate families of cubic graphs with the property that the girth is increasing. As we have mentioned previously, Tutte and others [34] have shown that given g greater than or equal to 3 there exists a finite trivalent graph with girth at least g - we start by showing there is a Cayley graph with these properties. The argument is similar to that used by Evans [16] to show that given $k \geqq 2$, $g \geqq 3$ there is an embeddable g-net of valency k.

First we need two lemmas.

Lemma

Let G be a group. If N_1 and N_2 are normal subgroups of finite index in G, then the intersection of N_1 and N_2 is also a normal subgroup of finite index in G.

Proof

By the Second Isomorphism Theorem the quotient group N_1N_2/N is is isomorphic to $N_2/N_1 \cap N_2$. Now N_1N_2 is a subgroup of G and N_1 is of finite index in G so N_1N_2/N_1 must be finite. Also the order of $N_2/N_1 \cap N_2$ is the same as the order of N_1N_2/N_1 so $N_2/N_1 \cap N_2$ is finite. But N_2 is of finite index in G so so $G/N_1 \cap N_2$ is finite.

Lemma

Let G be the free product of a finite number of cyclic groups. Then G is residually finite, that is given any non-identity element g in G there is a normal subgroup N_g of finite index in G that does not contain g.

Proof

This was proved first by Gruenberg [20]. The neatest proof is in a paper by Baumslag and Tretkoff [3].

Theorem 3.1

If n is an integer larger than 2, there is a finite group whose Cayley graph is trivalent and has girth at least n.

Proof

Let $G = < R_1, R_2, R_3 | R_1^2 = R_2^2 = R_3^2 = 1_G >$, where 1_G is the identity element of G. Then by the Lemma G is residually finite. Hence given g a non-identity element of G we can find N_g a normal subgroup of finite index in G not containing g.

We now use the set of generators $\{R_1, R_2, R_3\}$ to construct A a Cayley graph of G, and we denote the vertex in V(A) corresponding to the element g of G by v_g. A is in fact the infinite trivalent tree.

Let $S = \{\gamma | \gamma \in G, 0 < d_A(V_1, V_\gamma) < n\}$, that is the set of words in G of length less than n. S is finite.

Now let $N = \bigcap_{\gamma \in S} N_\gamma$. Then N is of finite index by the Lemma.

Let Γ be the Cayley graph of quotient group G/N using $\{NR_1, NR_2, NR_3\}$ as the generating set. We claim Γ has girth at least n.

For suppose there is a cycle of length m in Γ where m is strictly less than n. Then

$$Nw_1\, Nw_2 \ldots Nw_m = N \quad \text{for some} \quad w_i \text{ in } \{R_1, R_2, R_3\}$$

so $\quad Nw_1 w_2 w_3 \ldots w_m = N \quad$ since $\quad Ng = gN \quad$ for all g in G

and $\quad w_1 w_2 \ldots w_m$ is in N.

But $w_1 w_2 \ldots w_m$ is in S since $m < n$, and thus $w_1 w_2 \ldots w_m$ is not in $N_{w_1 \ldots w_m}$ and cannot be in N. Hence there can be no cycles of length less than n in Γ and Γ must have girth at least n.//

If the subgroups referred to in the above proof as Ng are chosen more carefully we can ensure that the Cayley graph Γ is not just vertextransitive but also edgetransitive.

Corollary 3.2

Given $n \geq 3$, there exists a finite trivalent graph that is symmetric and has girth at least n.

Proof

Again let $G = \langle R_1, R_2, R_3 \mid R_1^2 = R_2^2 = R_3^2 = 1_G \rangle$, and let A be the Cayley graph constructed from G using $\{R_1, R_2, R_3\}$ as the set of generators. As in the previous proof we let $S = \{\gamma \mid \gamma \in G,$
$0 < d_A(v_\gamma, v_{1_G}) < n\}$

Given $g = R_{i_1} R_{i_2} R_{i_3} \ldots R_{i_m}$ with $i_j \in \{1,2,3\}$ $1 \leq j \leq m$, define

$\Pi g = R_{\Pi i_1} R_{\Pi i_2} \ldots R_{\Pi i_m}$ where Π represents the permutation (123). Π is clearly an automorphism of G.

Now given γ we choose $N\gamma$ such that $N\gamma$ is a normal subgroup of finite index in G not containing γ such that

$$N\gamma = \langle R_1, R_2, R_3 \mid R_1^2 = R_2^2 = R_3^2 = W_1 = \ldots = W_r = 1_G \rangle \quad \text{if and only if}$$

$$N_{\Pi\gamma} = \langle R_1, R_2, R_3 \mid R_1^2 = R_2^2 = R_3^2 = \Pi W_1 = \ldots = \Pi W_r = 1_G \rangle .$$

since $\gamma \in S$ if and only if $\Pi\gamma \in S$ the image of $N = \bigcap_{\gamma \in S} N\gamma$ under Π will be N.

Let Γ be the Cayley graph of G/N using $\{NR_1, NR_2, NR_3\}$ as the set of generators. Then Γ has girth at least n; it is just required to show that Γ is edgetransitive.

Suppose Ng_1 is adjacent to Ng_2 in Γ. Then $g_1 = ng_2r$ for some n in N and some r in $\{R_1, R_2, R_3\}$. This means

$$\begin{aligned} \Pi g_1 &= \Pi(n\, g_2\, r) \\ &= \Pi n\; \Pi g_2\; \Pi r \\ &= n'\; \Pi g_2\; \Pi r \quad \text{where } n' \in N \text{ and} \end{aligned}$$

Πr must be in $\{R_1, R_2, R_3\}$ and so $N\Pi g_1$ is adjacent to $N\Pi g_2$. Hence Π represents an automorphism of Γ and it stablizes the vertex corresponding to the identity element while cyclically permuting its adjacent vertices. Since Γ is a Cayley graph and all Cayley graphs are vertextransitive [5] Γ must be symmetric. //

The same results can be obtained for k-valent graphs where $k > 3$ by similar methods. As we shall see we can also construct trivalent

2-arctransitive graphs in this way. We use the Lower Central Series of the group $G = < R_1, R_2, R_3 | R_1^2 = R_2^2 = R_3^2 = 1_G >$.

Definitions

Given x, y elements in a group G, we write the commutator $x^{-1}y^{-1}xy$ as (x,y). For subgroups A, B of G the notation (A,B) will mean the group generated by all (a,b) with $a \in A$, $b \in B$. If $G_0 = G$ and $G_{N+1} = (G, G_N)$ for $N \geq 1$, the series

$$G = G_0 \geq G_1 \geq G_2 \ldots$$

is called the Lower Central Series of G. If g is a member G_i but not a member of G_{i+1} we say g is a commutator of weight i. We have that G_i is a normal subgroup of G for all i.

Let $G = < R_1, R_2, R_3 | R_1^2 = R_2^2 = R_3^2 = 1_G >$, and let G_i denote (G, G_{i-1}) where $G_o = G$, that is the ith term in the Lower Central Series of G. Let Γ_i correspond to the Cayley graph of the quotient group G/G_i using $\{G_iR_1, G_iR_2, G_iR_3\}$ as the generating set.

Theorem

The girth of Γ_i increases unboundedly.

Proof

Mal'cev [27] has shown that G is an N-group, that is the infinite intersection $\bigcap_{i>1} G_i$ is the identity element in G. This means that given an element g of G, there exists r_g such that g is not in G_i for all i greater than r_g.

As in the proof of Theorem 3.1, we let A be the Cayley graph of G using generating set $\{R_1, R_2, R_3\}$ and define $S_n = \{\gamma | \gamma \in G,\ 0 < d_A(v_1, v_\gamma) < n\}$ where v_g represents the vertex in $V(A)$ corresponding to the group element g in G.

Now let r be the largest value of r_γ for all γ in S_n. Then the girth of Γ_r must be at least n by a similar argument is that used in Theorem 3.1. //

The graphs Γ_r are finite. Gaglione [19] has shown that G_i/G_{i+1} is elementary Abelian of order 2^{λ_n} where

$$\lambda_n = {}^1/_n \sum_{\substack{k|n \\ k>1}} \mu({}^n/_k)\ (k\alpha_k) \tag{1}$$

where μ is the Mobius function, and

$$\alpha_n = {}^{-1}/_{n!} \frac{d^n}{dx^n} [\ln[1 - 2x^3 - 3x^2]]\Big|_{x=0}. \tag{2}$$

From these formulae we can calculate the order of G/G_i, and hence we find the number of vertices in Γ_i is given by

$$2^L \text{ where } L = \sum_1^i \lambda_n .$$

The first nontrivial graph in the sequence Γ_i is Γ_2 which is the cube. This has 8 vertices and girth 4. Γ_3 has 64 vertices and girth 8 [17], while Γ_4 has 2^{11} vertices and has been computed to have girth 14.

Theorem 3.4

The girth of Γ_i is less than i^2 if $i \geq 3$.

Proof

First we need a result concerning the weight of a commutator. If u is in G_i and v is in G_j, then (u,v) is in G_{i+j} [21], and the words (u,v) and (u^{-1},v) correspond to cycles in Γ_{i+j}.

Choose u and v such that the lengths of the words u and v correspond to the girths of Γ_i and Γ_j respectively, and (u,v) is not the identity element.

Let $u = R_a \ldots R_b$; $R_a \neq R_b$ since $R_a u R_a$ is in Γ_i and would be of shorter length than u if R_a were the same as R_b. Similarly let $v = R_c .. R_d$ where R_c is different from R_d.

Then $(u,v) = R_b .. R_a R_d \ldots R_c R_a \ldots R_b R_c \ldots R_d$. If there is no cancellation in (u,v), that is $R_b \neq R_c$, $R_a \neq R_c$ and $R_a \neq R_d$, there must be a cancellation in

$$(u^{-1},v) = R_a \ldots R_b R_d \ldots R_c R_b \ldots R_a R_c \ldots R_d \quad \text{since} \quad R_b = R_d.$$

Hence if $g(\Gamma_i)$ represents the girth of the graph Γ_i

$$g(\Gamma_{i+j}) \leq 2(g(\Gamma_i) + g(\Gamma_j)) - 2$$

so

$$g(\Gamma_{2n}) \leq 4g(\Gamma_n) - 2$$

and

$$g(\Gamma_{2n+1}) \leq 2(g(\Gamma_n) + g(\Gamma_{n+1})) - 2.$$

Suppose $g(\Gamma_i) \leq i^2$ whenever $2 \leq i < n$.

Now if $n = 2i$ $\quad g(\Gamma_n) \leq 4(g(\Gamma_i)) - 2 \leq n^2 - 2 < n^2$

and if $n = 2i+1$ $\quad g(\Gamma_n) \leq 2(g(\Gamma_i) + g(\Gamma_{i+1})) - 2 \leq (2i+1)^2 - 1 < n^2$

But $g(\Gamma_2) = 4$ and $g(\Gamma_3) = 8$ so by induction

$g(\Gamma_i) < i^2$ whenever i is greater than 2.//

We now turn our attention to the values c_i if the number of vertices in Γ_i or $|V(\Gamma_i)|$ is taken to be $2^{c_i g(\Gamma_i)}$. Recall that λ_i is given in equation (1) and α_i is given in equation (2), but α_i is alternatively seen to be the coefficient of x^n in the infinite sum

$$A = 6(x^2 + x) \sum_{i=0}^{\infty} (2x^3 + 3x^2)^i .$$

Since $(1 - 3x^2 - 2x^3) = (2x - 1)(x + 1)^2$ and the nearest zero to the origin is $x = \frac{1}{2}$, the radius of convergence of $\Sigma\alpha_n x^n$ is $\frac{1}{2}$ and consequently as $n \to \infty$ $\frac{\alpha_{n+1}}{\alpha_n} \to 2$.

But α_n is much the largest term in the sum

$$1/_n \sum_{k|n} \mu(n/_k)\, k\, \alpha_k$$

so λ_{n+1}/λ_n also tends to 2 as n tends to infinity.

Thus as $n \to \infty$ the number of vertices in Γ_i tends to 2^{2^i} and so as $g(\Gamma_i) < i^2$ and $|V(\Gamma_i)| = 2^{c_i g(\Gamma_i)}$ c_i tends to infinity as i tends to infinity.

Theorem 3.5

The graphs in the family $\{\Gamma_i\}$ are all s-arctransitive, where $s \geq 2$.

Proof

Given $G = R_{i_1} R_{i_2} \ldots R_{i_m}$ with $i_j \in \{1,2,3\}$ $1 \leq j \leq m$, define

$\Pi g = R_{\Pi i_1} R_{\Pi i_2} \cdots R_{\Pi i_m}$ where Π represents an element of the symmetric group of permutations on the set {1,2,3}. Π is an automorphism of G, and the image of G_i under Π is still G_i. We follow the proof of Corollary 3.2 and find that Π corresponds to an automorphism of the graph Γ_i fixing the vertex corresponding to the identity element. Because there are six permutations on 3 letters, the order of the stablizer of a vertex is at least 6 and the graph Γ_i must be at least 2-arctransitive. //

Additive Excess

Until now in this chapter we have been viewing excess as a multiplicative function of g. We now show that although c(g) may tend to $\frac{1}{2}$ as g becomes large, the additive excess, the actual number of extra vertices required as the girth increases, is unbounded. In this section not only trivalent graphs will be considered but also graphs in which every vertex has degree k, or k-valent graphs. Biggs has shown that for each odd integer k the excess $e_{T,k}(g)$ of a vertextransitive graph with valency k and girth g is unbounded as a function of g [7]. It will now be shown this is true for all even integers k as well.

Let G be a vertextransitive graph of girth $g = 2r+1$ and valency k, and let $\Delta_i(v)$ denote the set of vertices at distance i from a given vertex v. Because there are no cycles of length less than g

$$|\Delta_i(v)| = k(k-1)^{i-1} \qquad i \leq r.$$

The number of cycles of length g through v is equal to the number of edges in E(G) which join two members of $\Delta_r(v)$, and as G is vertextransitive this number is a constant x independent of v. Let J denote the number of edges from a vertex of $\Delta_r(v)$ to one in $\Delta_{r+1}(v)$. The excess of G is given by $\left|\bigcup_{s>r} \Delta_s(v)\right|$, the number of vertices at distance greater than r from vertex v, and will be denoted by e.

Lemma

$0 < k(k-1)^{(g-1)/2} - 2x \leqq k\,e.$

Proof

Each vertex in $\Delta_r(v)$ is adjacent to one vertex in $\Delta_{r-1}(v)$ and $k-1$ other ones so that

$$2x + J = (k-1)\,|\Delta_r(v)|.$$

But $|\Delta_{r+1}(v)| \leqq e$, and each vertex in $\Delta_{r+1}(v)$ has valency k, so we have $0 \leqq J \leqq ke$. Putting $|\Delta_r(v)| = k(k-1)^{(g-3)/2}$ gives the required result. //

Theorem 3.6

For each integer $k \geqq 3$, there is an infinite sequence of values of g such that the excess of any vertextransitive graph with valency k and girth g satisfies $e > \sqrt{g/k}$.

Proof

Firstly if k is odd, Biggs has shown $e > g/k$ for all g in an infinite set of primes S_k.

Now if k is even there is an odd prime p dividing $(k-1)$. Let $g = p^{2m}$, where m is a positive integer. Let the number of cycles of length g in G be N and let the number of vertices $|V(G)|$ be n.

Each of the n vertices is contained in X g-cycles, so $nX = Ng$, and g must divide nX. But $g = p^{2m}$ so either

$$X \equiv 0 \pmod{p^m} \quad \text{or} \quad n \equiv 0 \pmod{p^m}.$$

Suppose first $X \equiv 0 \pmod{p^m}$.

Then $J = k(k-1)^{(g-1)}/2 - 2X \equiv 0 \pmod{p^m}$.

But $J > 0$ since G is connected, so $J \geqq p^m$.

From the lemma we have $ke \geqq p^m$, so $e \geqq \sqrt{g}/_k$.

Next suppose $n \equiv 0 \pmod{p^m}$.

$$\text{Now} \quad n = \Big|\bigcup_{s\geqq 0} \Delta_s(v)\Big|$$

$$= \Big|\bigcup_{s=0} \Delta_s(v)\Big| \quad + e$$

$$= 1 + \sum_{s=1}^{r} k(k-1)^{s-1} \quad + e$$

$$= \frac{k}{k-2}\,\{(k-1)^{\frac{1}{2}(g-1)}-1\} + (1+e).$$

Hence $(k-2)n = k\{(k-1)^{\frac{1}{2}(g-1)}-1\} + (e+1)(k-2)$.

But $n \equiv 0 \pmod{p^m}$ and $(k-1)^{\frac{1}{2}(g-1)} \equiv 0 \pmod{p^m}$ since p divides $k-1$ so

$$0 \equiv -k + (e+1)(k-2) \pmod{p^m}.$$

Hence $$e \equiv \frac{2}{(k-2)} \pmod{p^m}.$$

Bannai and Ito have shown $e > 1$ for $k = 4$ [2],

so

$$e \geqq \frac{2 + p^m}{k - 2} > p^m/k .$$

Thus $$e > \sqrt{g}/k . //$$

Chapter 4

Trivalent Cayley Cages

This chapter is devoted to the problem of finding the smallest Cayley graphs of a given girth. It is true that for some small values of the girth the "Cayley Cages" are of similar order to the ordinary cages, but there is no general result of this kind. Since Cayley graphs are all vertextransitive the results in Chapter 3 concerning the excess of vertextransitive graphs apply.

The (3,k) Cayley Cages $k \leq 9$

We start by noting that the (3,4) cage $K_{3,3}$ is the Cayley graph of the group S_3 using the three involutions as generating set. The Heawood graph, the unique (3,6) cage is also a Cayley graph, the group being a subgroup of the group of linear transformations of the field with seven elements isomorphic to the dihedral group of order 14, the generating set being the three involutions {1-x, 2-x, 4-x}. Examining C_{10} and D_{10} shows that the unique (3,5) cage the Petersen graph is not a Cayley graph and indeed the (3,5) Cayley cage has considerably more than 10 vertices.

Theorem 4.1

Trivalent Cayley graphs of girth 5 have at least 50 vertices.

Proof

Let Γ be the smallest trivalent graph of girth 5 which is also a Cayley graph, and let it be the Cayley group G with generating set Ω. We have that G has more than ten elements.

Consider the cycles of length 5 in the graph. Each such cycle corresponds to an identity word $W_1W_2W_3W_4W_5$ in the generators of G. Suppose $W_i \neq W_{i+1}$ for some i. Then the five words $W_1W_2W_3W_4W_5$, $W_2W_3W_4W_5W_1, \ldots, W_5W_1W_2W_3W_4$ must all represent different cycles through a given vertex in the Cayley graph.

Let $\Gamma_r(x)$ denote the set of vertices at distance r from a given vertex x, and let the subgraph $\Gamma_r(x)$ have vertex-set and edge-set

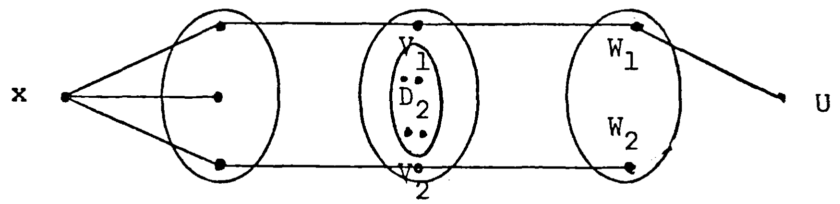
$$V(\Gamma_r(x)) = \bigcup_{i=0}^{r} \Delta_r(x); \; E(\Gamma_r(x)) = \{(v,w) \mid (v,w) \in E(\Gamma)\}$$

There cannot be six edges from $\Delta_2(x)$ to $\Delta_2(x)$, or Γ would be the Petersen graph, so there must be exactly 5 edges between vertices in $\Delta_2(x)$.

Now we show x is not a <u>cutvertex</u>; this means the graph remains connected when the vertex x is removed. Every 5-cycle through x must also pass through 2 members of $\Delta_1(x)$, and there are at most 2 5-cycles passing through x and two given members of $\Delta_1(x)$. Hence there is a path of length 3 between any two members of $\Delta_1(x)$ not caontaining x. Thus x is not a cutvertex. We also have that through any 2-arc there is at least one 5-cycle.

There are six vertices in $\Delta_2(x)$. Since there are six edges from vertices in $\Delta_2(x)$ to vertices in $\Delta_1(x)$ and five edges from $\Delta_2(x)$ to $\Delta_2(x)$ there must be exactly 2 edges from $\Delta_2(x)$ to $\Delta_3(x)$. Suppose these 2 edges have a coincident end in $\Delta_2(x)$ vertex V say. Then there can be no path from $\Delta_3(x)$ to the vertex x which does not pass through V and V must be a cutvertex. But x is not a cutvertex so the vertextransitivity of Γ is contradicted. Hence e_1 and e_2 have distinct ends in $\Delta_2(x)$, and similarly they have distinct ends in $\Delta_2(x)$.

Let the edges e_1, e_2 be (V_1, W_1) and (V_2, W_2) where V_1, V_2 are in $\Delta_2(x)$ and W_1, W_2 are in $\Delta_3(x)$. Hence $\Delta_3(x) = \{W_1, W_2\}$. W_1 has at most one neighbour in $\Delta_3(x)$, and exactly one neighbour V_1 in $\Delta_2(x)$, so there is a vertex U say in $\Delta_4(x)$ joined to W_1.



x $\Delta_1(x)$ $\Delta_2(x)$ $\Delta_3(x)$

There is a 5-cycle C through the 2-arc (V_1, W_1, U). Any path from V_1 to W_1 not containing e_1 must contain e_2 since e_2 is the only other edge connecting $\Delta_2(x)$ to $\Delta_2(x)$. Hence e_2 is also in C. C contains but 5 edges so (U, W_2) and (V_1, V_2) must also be in $E(\Gamma)$. Now consider the subgraph D_2 whose vertexset is $\Delta_2(x) \setminus \{V_1, V_2\}$. This is a graph with 4 vertices and 4 edges, which must contain either a 3-cycle or a 4-cycle contradicting the girth.

Hence the only word that could possibly represent a cycle of length 5 in a graph of girth 5 is S^5 for some generator S.

Thus Γ has 'Type II'. Let $G = \langle R,S \rangle$ where $R^2 = S^5 = 1$ be the group whose Cayley graph is Γ. Suppose the subgroup generated by S, $\langle S \rangle$ is normal in G. Then RS^aR is in $\langle S \rangle$ for all values of a and hence $\langle R,S \rangle$ has ten elements. But Γ has more than 10 vertices so $\langle S \rangle$ is not normal in G.

Sylow's Theorems state that if the order of a finite group H is $p^r m$, where p is a prime not dividing m, then all subgroups of H of order p^r are conjugate, and the number of them is congruent to 1 modulo p and divides to order of H. Since R is of order 2 and S is order 5 the order of G must be divisible by 10. By applying Sylow's Theorems we find that any subgroup of order 5 of a group of order 20 or 40 must be normal, and Coxeter and Moser [12] have shown there are no groups of order 30 with 6 Sylow 5 subgroups so again any subgroup of order 5 a group of order 30 must be normal. Hence the order of G is at least 50.

We find that if G is given by the presentation

$$G = \langle R,\ S \mid R^2 = S^5 = (RS)^2(RS^{-1})^2 = 1 \rangle$$

G is of order 50, and the Cayley graph of G using $\{R,S\}$ as generating set is indeed trivalent and of girth 5. //

Corollary

There are no edgetransitive trivalent Cayley graphs of girth 5, nor are there any Cayley graphs of girth 5 of Type I.

Proof

We have already shown that all Cayley graphs which are trivalent and have girth 5 are Type II. Suppose Γ is the Cayley graph of the group G with generating set $\{R, S\}$ where $R^2 = S^5 = 1_G$. Then there is a 5-cycle through any edge labelled S but there is no 5-cycle through an edge labelled R, and hence Γ cannot be edgetransitive.//

Before examining the trivalent Cayley cages with girth greater than 5, we need a result involving dihedral groups. The dihedral group D_{2n} is the group of symmetries for the regular n-gon. Let G be a dihedral group of order $2n$ and let G' be the cyclic subgroup of G of order n. Let Ω be a generating set of G chosen such that the resulting Cayley graph of G is trivalent.

Lemma 4.2

If Γ is the Cayley graph of G with generating set Ω the girth of Γ is less than or equal to 6.

Proof

Suppose Γ is Type I. Then Ω consists of 3 involutions $\{R,S,T\}$ say, if none of R,S,T are in G', then the product RST is not in G', and $(RST)^2 = 1$ and the graph contains a 6-cycle. At least one member of Ω is not in G', R say, so if S is in G' $(RS)^2 = 1$ giving a cycle of length 4.

If however Γ is Type II, then Ω consists of one involution R say and an element of order > 2 S say and we have $RSRS^{-1} = 1$ and Γ contains a 4-cycle.

Hence the girth of Γ is at most 6. //

Theorem 4.3

The smallest Cayley graph with degree 3 and girth 7 has 30 vertices.

Proof

First let Γ be the Cayley graph of $C_5 \times D_6$ with the generators A,B represented by the permutations

$$A = (1\ 2), \quad B = (1\ 2\ 3)(4\ 5\ 6\ 7\ 8).$$

The shortest identity word in A and B is $ABAB^4$ and Γ has 30 vertices and girth 7.

The unique (3,7) cage, the McGee graph has 24 vertices and is not vertextransitive and consequently not Cayley. There are only 3 nonAbelian groups which have 26 or 28 elements [12]. Two of these are dihedral groups whose Cayley graphs must have girth less than 7 by the Lemma. The third group is the dicyclic group Q_{14} which contains only one involution, and whose Sylow 7 subgroup is normal. These two properties ensure no generating set may be chosen from this group to give a trivalent Cayley graph that is connected.//

We now examine the cases where the girth is 8 or 9.
First various possibilities have to be eliminated.

Lemma 4.4

The girth of a trivalent Cayley graph on 36 vertices is less than 8.

Proof

We separate the trivalent Cayley graphs into 2 classes. Suppose G is a group with 36 elements, and let Γ be the Cayley graph of G using S as generating set.

Suppose Γ is Type II. Then $S = \{x,y\}$ where x is of order n and n is greater than 2, and y is an involution. Now either the resultant Cayley graph has girth less than 8 or $n \geq 8$, so we consider the possible values of n where $n > 8$. Let X denote the subgroup generated by x.

a) $|X| = 18$ Then X is normal in G being of index 2. Hence $yxy = x^a$ for some a. From this we have

$$x^{a^2} = (yxy)^a = yx^a y = x.$$

So $a^2 \equiv 1 \pmod{18}$. There are only two solutions to this $a \equiv \pm 1 \pmod{18}$, and thus $yxyx^{-a}$ is a word of length 4.

b) $|X| = 12$ If X is normal $\langle y,x \rangle$ is a subgroup of order 24 which is not possible. Hence X is not normal, so there are 3 right cosets of X X, Xy and Xyx. $Xy \neq Xyx$ so $Xyx \neq Xyx^2$ and thus $Xyx^2 = Xy$ and yx^2y is in X. But only two elements x^2 and x^{-2} in X are of order 6 so either yx^2yx^2 or yx^2yx^{-2} is an identity word and the graph contains a 6-cycle.

c) $|X| = 9$ If X is normal, $<x,y>$ contains only 18 elements. Hence X is not normal. Let $z = yxy$. The cosets Xz^i $(0 \leq i \leq 8)$ are not all distinct (since $|G| = 36$), so z^j belongs to X for some j, $2 \leq j \leq 4$, and $X \cap <z>$ is a nontrivial proper subgroup of $<z>$. Hence $X \cap <z> = <z^3>$. Thus $z^3 = yx^3y$ belongs to X and must be either x^3 or x^{-3}.

If $yx^3y = x^3$, y commutes with x^3 and yx^3 is of order 6. Since G contains 4 Sylow-3-subgroups and 3 cyclic groups of order 6 (conjugates of $<yx^3>$), counting the elements of G we find the distinct elements xyx^{-1}, $x^{-1}yx$, y must all lie in the unique Sylow-2-subgroup of order 4 and $xyx^{-2}yxy = 1$ giving a word of length 7.

On the other hand, if $yx^3y = x^{-3}$, one of $(yx)^3$, $(yx)^3x^{-3}$ or $x^{-3}(xy)^3$ is the identity and again we have a word of length less than 8 corresponding to the identity.

Now suppose Γ is Type I. Then $S = \{x,y,z\}$ where x,y,z are all of order 2. Either the girth of Γ is less than 8 or each of the products xy, yz, zx are of order greater than 4. Suppose this is the case and that without loss of generality the product xy is of the highest order among them. We now consider the possible order of A the subgroup of G generated by xy.

a) $|A| = 18$. Then G is a dihedral group and the girth of Γ is at most 6 by Lemma 4.2.

b) $|A| = 9$. Then $\langle x,y\rangle$ is of index 2 and normal in G. Hence either $(xyz)^2$ or $yxzxyz$ is the identity and Γ has girth at most G.

c) $|A| = 6$. Suppose Γ is of girth 8.
Let M denote $\langle x,y\rangle$. M cannot be normal in G since $|\langle x,y,z\rangle|$ is not 24. Hence either zxz or zyz is not in M. Suppose zxz is not in M. Then M has 3 cosets M, Mz, Mzx. If $Mzx = Mzy$, then $zyxz$ is in M and we have either $(zyx)^2$ or $zyxzxy$ is the identity and Γ contains a 6-cycle.
So let zyz be in M. $(zy)^2$ is of order 3 and consequently $(zy)^2 = (yx)^2$. (If $(zy)^2 = (xy)^2$ Γ contains $zyzxyx$ a 6-cycle).
Now consider $N = \langle z,x\rangle$. Similarly we have that exactly one of yzy and yxy is in N. But $z(yzy)x = yxy$ so yzy is in N if and only if yxy is in N. Thus we have a contradiction and Γ is of girth less than 8.

Hence Cayley graphs on 36 vertices have girth less than 8 . //

Lemma 4.5

The girth of a trivalent graph of order 40 is less than 8 and the girth of a trivalent Cayley graph of order 56 is less than or equal to 8 .

Proof

Let G be agroup with 8p elements with p=5 or p=7. We know from Sylow's Theorems G contains a normal subgroup of order p or if p=7 a normal subgroup of order 8. (In this particular case any Cayley graph of G must be disconnected if Γ is of Type I since all involutions lie in the unique Sylow 2 subgroup and of girth less than 8 if it is of Type II since the only elements outside the Sylow 2 subgroup are of order 7). This normal subgroup is unique and cyclic. Let S be a generating set of G such that the resulting Cayley graph Γ is trivalent.

Suppose Γ is of Type II. Then S consists of an involution y and an element x of order greater than 2. We now consider the possible orders of X the subgroup generated by x.

a) $|X| = 4p$ Then X is of index 2 in G and is consequently normal in G. Hence $yxy = x^a$ for some a. From this we have

$$x^{a^2} = (yxy)^a = yx^ay = x .$$

Hence $a^2 \equiv 1$ (modulo 4p), so $a \equiv \pm 1$ (modulo 2p) and $2a \equiv \pm 2$ (modulo 4p). Thus either $(yx^2)^2$ or yx^2yx^{-2} is the identity and Γ contains a 6-cycle.

b) $|X| = 2p$ x^2 is of order p so the subgroup generated by x^2 is normal and cyclic . Again we find either $(yx^2)^2$ or yx^2yx^{-2} is the identity and Γ contains a 6-cycle.

c) $|X| = 8$ X is now a cyclic Sylow-2-subgroup of G. y is not in X so y must be inside a distinct cyclic subgroup of order 8 generated by z, say. Thus $y = z^4$, and since y does not lie in X no power of z can lie in X and the cosets Xz^i ($0 \leq i \leq 7$) must all be distinct. Thus we get a contradiction on the order of G, and deduce that there are no groups of order 40 or 56 which are generated by an involution and an element of order 8.

If the order of X is less than 8 Γ contains a cycle of length less than 8.

Suppose Γ is Type I, and S consists of 3 involutions x,y,z. We look at the order of the product xy, and by similar arguments to those used above, find the conjugate of xy by z is either xy or yx if xy is of order 4p, 2p or p, and Γ contains a 6-cycle. Hence either the girth of Γ is less than 8 or $(xy)^4 = (yz)^4 = (xz)^4 = 1$.

Suppose p=5 and $(xy)^4 = (yz)^4 = (xz)^4 = 1$. Then G contains 5 Sylow 2 subgroups H_1, H_2, H_3, H_4, H_5 each isomorphic to D_8. Let $H_1 = \langle x,y\rangle$, $H_2 = \langle y,z\rangle$ and $H_3 = \langle z,x\rangle$. Each of these is selfnormalizing. Now $xH_1x = H_1$ and $xH_2x = H_3$ and xH_2x is not

H_2 but $x(xH_2x)x = H_2$, so one of H_4 and H_5 contains x, say H_4. Similarly z is in exactly one of H_4 and H_5 and so is y. But if y or z are in H_4, $H_1 = H_4$ or $H_3 = H_4$ and if y and z are in H_5 $H_5 = H_2$. Hence we have a contradiction and one of the previous cases must occur.

Hence the girth of a trivalent Cayley graph is less than 8 if it has 40 vertices and less than 9 if it has 56 // .

Lemma 4.6

The girth of a trivalent Cayley graph with 54 vertices is less than 9.

Proof.

There are only two nonAbelian groups of order 27 and in one of them A every element is of order 3 [12].

Suppose G is of order 54 and its Sylow 3 subgroup is H.

Suppose H is isomorphic to A. Let S be a generating set giving a trivalent Cayley graph. If any member of S is in H the Cayley graph contains a triangle; if not and G is of Type II the element in S of order greater than 2 must be of order 6 and the graph contains a 6-cycle, and if the graph is of Type I it also contains a 6-cycle since the product of any two generators is of order 3.

Now suppose H is Abelian. Suppose Γ, is the Cayley graph of G is Type I. None of the three involutions generating G lie in H, but their products pairwise must all lie in H and $xz . zy . zx . yz = (xyz)^2$ is an identity word. If on the other hand Γ is Type II with generating set {x,y} where x is not an involution, then either $yxy.x^{-1}.yx^{-1}y.x$

or $x^2.xy.x^{-2}.yx$ is an identity word depending on whether or not x lies in the subgroup H. Γ has girth at most 8 in this case.

The remaining possibility is that H is given by the presentation

$$< S,T \mid T^3 = T^{-1}STS^2 = 1 >$$

a group of order 27 containing 3 subgroups isomorphic to the cyclic group of order 9, whose centre Z is of order 3 and generated by the cube of any element of order 9.

Let Γ be a trivalent Cayley graph of G.

Suppose Γ is of Type I, and the generating set is given by $\{x,y,z\}$ a set of three involutions. If any of the products xy, yz, zx which all lie in H have order 3 Γ contains a 6-cycle so suppose the order of each of these products is 9. Let K be the dihedral group of order 18 generated by x,y. If K^z the conjugate of K by z is the same as K, $(xz)^2$ lies in K, x^z commutes with xy and $(xyz)^2 = 1$ so Γ has girth at most 6. Hence there are 3 subgroups of G all conjugate isomorphic to K each containing 9 involutions. Since these subgroups intersect in a cyclic subgroup of order 9, these 27 involutions are all distinct, and they must comprise all the elements in G not in H. But xyz is not in H so $(xyz)^2 = 1$ and Γ has girth at most 6.

Suppose now instead Γ is of Type II with generating set $\{x,y\}$ where x is not an involution and y is. If x is of order 9, x^3 lies in z and either yx^3yx^{-3} or yx^3yx^3 is the identity. If

x is of order 18, coset enumeration swiftly shows yx^3y lies in the subgroup generated by x and again yx^3yx^{-3} or yx^3yx^3 is the identity. The only other possible orders of x are less than 9, so Γ must contain a cycle of length less than 9.//

We are now in a position to establish the number of vertices in the smallest trivalent Cayley graphs of girth 8 and 9.

Theorem 4.7

The smallest trivalent Cayley graphs with girth 8 have 42 vertices.

Proof

The Tutte graph on 30 vertices is the unique (3,8) cage. Using the fact that this graph is bipartite and that there are 24 8-cycles through each vertex it is verifiable that this is not the Cayley graph of any of the three nonAbelian groups with 30 elements. Biggs and Ito have shown that excess 2 is not feasible in this instance, so there are no trivalent graphs of girth 8 with 32 vertices. The only nonAbelian groups of order 34 or 38 are dihedral so by Lemma 4.2 there are no trivalent Cayley graphs on 34 or 38 vertices with girth more than 6. Lemmas 4.4 and 4.5 rule out 36 and 40 respectively as possible orders for trivalent Cayley cages.

However, the group generated by the permutations

$A = (1\ 2)$, $B = (1\ 2\ 3)\ (4\ 5\ 6\ 7\ 8\ 9\ 10)$, or alternatively given by the presentation

$$G = \langle A, B \mid A^2 = B^{21} = ABAB^{-8} = 1 \rangle,$$

has a Cayley graph using $\{A,B\}$ as generating set with 42 vertices and girth 8. //

Theorem 4.8

The smallest trivalent Cayley graphs of girth 9 have 60 vertices.

Proof

As was mentioned in Chapter 2 McKay has shown that any trivalent graph of girth 9 has more than 52 vertices [28].
Lemmas 4.5 and 4.6 show that there are no trivalent Cayley graphs of girth 9 with 54 and 56 vertices and the only nonAbelian group of order 58 is dihedral. Hence the smallest trivalent Cayley graphs of girth 9 have 60 vertices. Two are known.

Firstly the icosahedral group with generating set of permutations

$$\{(1\ 2)\ (3\ 5),\ (1\ 3)\ (4\ 5),\ (1\ 4)\ (2\ 5)\}$$

has a Cayley graph with girth 9. This is known as Foster's graph [18].

Secondly the group with generating set of permutations

$$\{(1\ 2)\ (4\ 5),\ (1\ 2\ 3\ 4)\ (6\ 7\ 8)\}$$

has a Cayley graph of girth 9. This is an unpublished graph of Coxeter.

The Cayley graphs of $Aff(p^f)$

Given a finite field $GF(p^f)$ with p^f elements consider the group $Aff(p^f)$ of affine transformations of the form $x \mapsto ax+b$, where a,b are members of $GF(p^f)$ and a is nonzero.

This group is sharply 2-transitive and is of order $p^f(p^f-1)$.

Let R represent the transformation $x \mapsto -1-x$ and S represent the transformation $x \mapsto ax$ where a is a primitive element of $GF(p^f)$.

Theorem 4.9

R and S generate the entire group $Aff(p^f)$.

Proof

Take any element T of $Aff(p^f)$ where $Tx = mx+n$ with m nonzero. Since a is primitive $m=-a^i$ for some i. Also either $T = S^i$ or $n=-a^j$ for some j.

Then

$$\begin{aligned} mx+n &= mx - a^j \\ &= a^j(-1 + a^{-j}mx) \\ &= a^j(-1 - a^{i-j}\,x) \end{aligned}$$

so $Tx = S^j.R.S^{i-j}x$ and hence $Aff(p^f) = \langle R,S\rangle$.//

Hence the group with generating set $\{R,S\}$ has a trivalent Cayley graph with $p^f(p^f-1)$ vertices. We shall be interested in the girth of this graph. Should p^f be congruent to 3 (modulo 4) both R and S correspond to odd permutations of the elements of the field and the graph is bipartite.

Particular cases

The girths of all the graphs with $p \leq 23$ are given in Table 2. Certain of the graphs are of special interest.

i) If $p=11$ choose primitive root 7. Coxeter and Frucht [13] have shown the resultant Cayley graph which has girth 10 is 3-arctransitive.

ii) If $p = 17$ and primitive root 3 is chosen, the Cayley graph has girth 13. This is the smallest known graph which is trivalent and has girth 13; it has 272 vertices.

iii) If $p = 23$ and if 5,15 or 17 are chosen as primitive roots then the girth is 14. The group Aff(23) has the presentation

$$\mathrm{Aff}(23) = \langle A,B \mid A^2 = B^{22} = AB^5AB^4AB^2 \rangle$$

and A,B are equivalent to R,S when the primitive root is 17. The Cayley graph with $\{A,B\}$ as generating set is 4-arctransitive, as we shall see in Chapter 5.

Although the girth increases initially as the prime power increases there is an upper bound on the girth of a trivalent Cayley graph resulting from $\mathrm{Aff}(p^f)$.

Theorem 4.10

If G is isomorphic to $\mathrm{Aff}(p^f)$ and Γ is a Cayley graph of degree 3 resulting from G then the girth of Γ is less than or equal to 14.

Proof

The only involutions in G are of the form $x \mapsto a-x$ for some a. All the involutions are contained in the group $\{x \mapsto a \pm x\}$ which is a group of order $2p^f$. But Γ is the Cayley graph of $\mathrm{Aff}(p^f)$ and Γ is trivalent, so Γ is generated by some R,S where

$$Rx \mapsto c - x$$
$$Sx \mapsto ax + b$$

where a is not equal to 0 or -1.

Then

$$((RS)^2\, RS^{-2})\, x \equiv -(a(-1(a(-1\,(a^{-1}(a^{-1}x-a^{-1}b) - a^{-1}b)+c)+b)+c)+b)+c) = d-x$$

for some d.

Hence $((RS)^2RS^{-2})^2$ is the identity and Γ contains a cycle of length 14.//

Theorem 4.11

The smallest subgroup of a group of the form $Aff(p^f)$ to beget a trivalent Cayley graph of girth 14 is of order 406.

Proof

Let Γ be the Cayley graph of a group G a subgroup of $Aff(p^f)$ generated by

$$Rx \to c - x$$
$$Sx \to a^{-n}x$$

where a is a primitive root of p^f and $n > 1$. Let the order of S be m. Since S^m is the identity and the girth of Γ is 14 $m \geq 14$.

Now $nm \equiv -1$ (modulo p^f) so $p^f \geq mn + 1$.

$$\begin{aligned} \text{But } |V(\Gamma)| = |G| &\geq mp^f \\ &\geq 14(mn+1) \\ &\geq 14(14.2+1) \\ &\geq 406. \end{aligned}$$

This minimum may be attained if $p^f = 29$ and the generators

$$R : x \to 28 - x$$

$$S : x \to 4x$$

are chosen.//

Chapter 5

The Sextet Graphs

In this chapter we construct a family of highly transitive graphs for which it is conjectured that there is no upper bound for the girth. I have been unable to prove this conjecture but some partial results are given. The family is also of interest because it yields graphs which are in many cases the smallest known trivalent graphs with their particular girth. The girths and orders of the graphs known to have girth less than 32 are given in the Tables.

The Sextet construction

Let q be an odd prime power.
The projective line $PG(1,q)$ may be identified with the set $L = GF(q) \cup \{\infty\}$, where $GF(q)$ is a finite field with q elements.
A duet is an unordered pair of points $\{a,b\}$ on L and a quartet is an unordered pair of duets whose cross-ratio is -1.
Thus we shall write

$$\{a,b \mid c,d\} \text{ is a quartet } \iff \frac{(a-c)(b-d)}{(a-d)(b-c)} = -1$$

with the conventions about the element ∞ giving

$$\{\infty,a \mid b,c\} \text{ is a quartet } \iff \frac{(a-b)}{(a-c)} = -1.$$

A sextet $\{a,b \mid c,d \mid e,f\}$ is an unordered triple of duets such that each of $\{a,b \mid c,d\}$, $\{c,d \mid e,f\}$, $\{e,f \mid a,b\}$ is a quartet.

The group $PGL(2,q)$ of linear fractional transformations

$$t \mapsto \frac{at+b}{ct+d} \quad (a,b,c,d \in GF(q), \; ad - bc \neq 0)$$

acts sharply 3-transitively on L and its order is $q(q^2-1)$.

Lemma 5.1

The number of quartets is $\frac{1}{8}q(q^2-1)$. The number of sextets is $\frac{1}{24}q(q^2-1)$ if $q \equiv 1$ (modulo 4) and 0 if $q \equiv 3$ (modulo 4).

Proof

Clearly PGL(2,q) acts transitively on the duets so we need only consider a particular duet $\{0,\infty\}$. Now $\{0,\infty \mid x,y\}$ is a quartet if and only if $x+y=0$, so there are $\frac{1}{2}(q-1)$ quartets containing $\{0,\infty\}$. The number of quartets is

$$\frac{1}{2} \cdot \frac{1}{2}q(q+1) \cdot \frac{1}{2}(q-1) = \frac{1}{8}q(q^2-1).$$

Since the points $\{0, \infty, 1\}$ determine the unique quartet $\{0,\infty \mid 1,-1\}$ and PGL(2,q) acts 3-transitititively on L, it acts transitively on the quartets. The condition that $\{0,\infty \mid 1,-1 \mid u,v\}$ be a sextet are

$$u + v = 0, \; uv = 1,$$

so that u,v must be primitive fourth roots of unity i and -i. If $q \not\equiv 1$ (modulo 4) there are no solutions and consequently there are no sextets. If $q \equiv 1$ (modulo 4) there is a unique solution. Thus each quartet determines a unique sextet and each sextet arises from three quartets so that the number of sextets is $\frac{1}{24}q(q^2-1)$.

From now on we shall assume $q \equiv 1$ (modulo 4).

From Hirschfeld we have that an involution in PGL(2,q) is uniquely determined by two pairs of corresponding points, and that if the two pairs from a quartet, then the fixed points of the involution are the third pair in the unique sextet determined by the given quartet [23].

For example if the quartet is $Q = \{1,-1|i,-i\}$ the involution is $i_Q(t) = -t$ and the fixed points are $\{0, \infty\}$. The four points of Q may be split into two pairs in two other ways, $R = \{1,-i|-1,i\}$ and $S = \{1,i|-1,-i\}$ and the corresponding involutions are

$$i_R(t) = {}^{i}/_{t}, \quad i_S(t) = {}^{-i}/_{t} .$$

Solving formally to obtain the fixed points of i_R and i_S we see that we require a square root of i, that is an eighth root of unity. Now if $q \equiv 1$ (modulo 8), $q-1 = 8n$ and τ is a primitive element of GF(q) then $\tau^n = \sigma$ is an eighth root of unity and $\sigma^2 = i$. So in this case the fixed points of i_Q, i_R and i_S are $\{0,\infty\}$, $\{\sigma,-\sigma\}$, $\{\sigma^3,-\sigma^3\}$ and we remark that they form a sextet.

This remark is the basis for the construction of a cubic graph whose vertices are the sextets. We shall suppose that $q \equiv 1$ (modulo 8), and let σ denote an element of order 8 in GF(q). The sextet $\{a_1 a_2|b_1 b_2|c_1 c_2\}$ is adjacent to $\{a_1' a_2'|b_1' b_2'|c_1' c_2'\}$ if

a_1', a_2' are the fixed points of the involution determined by $b_1 b_2 ; c_1 c_2$

b_1', b_2' — $b_1 c_1 ; b_2 c_2$

c_1', c_2' — $b_1 c_2 ; b_2 c_1$.

In fact $\{a'_1, a'_2\}$ is the same as $\{a_1, a_2\}$. Thus there are three sextets adjacent to a given sextet, each having one duet in common with it. Furthermore it cannot be verified that the relation of adjacency is symmetric (since PGL(2,q) is transitive on the sextets we need only check one sextet). Thus we have a cubic graph $S(q)$ with $^1/24\ q(q^2-1)$ vertices.

In order to show that an element g of PGL(2,q) is an automorphism of $S(q)$ we remark that if θ_1, θ_2 are the fixed points of an involution j_Q then $g\theta_1$, $g\theta_2$ are the fixed points of $gj_Qg^{-1} = j_{gQ}$. Hence g preserves adjacency in $S(q)$ and the group PGL(2,q) acts as a group of automorphisms of $S(q)$.

The components and automorphisms of $S(p^f)$.

Now we come to consider the size of the components of $S(p^f)$. The component of $S(p^f)$ containing the sextet mentioned previously $k_o = \{0, \infty | 1, -1 | i, -i\}$ will be denoted by $S_o(p^f)$. We have already established that each element of $PGL(2,p^f)$ preserves adjacency and corresponds to an automorphism of $S(p^f)$.

Let $A : t \mapsto \dfrac{\sigma[t-1]}{[t+1]}$ and $B : t \mapsto \dfrac{\sigma[t+1]}{[t-1]}$, where

σ denotes an eighth root of unity in the field $GF(p^f)$.

Theorem 5.2

The automorphisms A,B are twin shunts of a 4-arc in $S(p^f)$.

Proof

We consider the actions of the first five powers of A and B on the sextet $k_{-1} = \{1, -1 \mid \frac{1+\sigma}{1-\sigma}, \frac{1-\sigma}{1+\sigma} \mid \frac{1+\sigma}{\sigma-1}, \frac{\sigma-1}{1+\sigma}\}$. $A^i k_{-1} = B^i k_{-1}$ for $0 \leq i \leq 4$ but $A^s k_{-1} = B^s k_{-1}$.

Hence A and B do correspond to twin 4-shunts.//

There is a theorem of Tutte [34] which states that given a connected graph G with automorphism group Aut (G) and two elements X,Y in Aut(G) which both act as shunts on an s-arc then $\langle X,Y \rangle$, the subgroup of Aut(G) generated by X,Y acts at least s-arctransitively on G. Hence $\langle A,B \rangle$ the subgroup of $PGL(2,p^f)$ generated by A,B acts at least 4-arctransitively on $S_o(p^f)$.

Theorem 5.3

$S_o(p^f)$ is isomorphic to $S_o(p^m)$ if f is greater than m and p is an odd prime, and $p^m \equiv 1$ (8).

Proof

Suppose σ_p an eighth root of unity in $GF(p^f)$ lies in a subfield $GF(p^m)$ of $GF(p^f)$. Then the elements $0, \infty, 1, -1, i, -i, \sigma_p, -\sigma_p$ where $i = \sigma^2$ must all lie in the subset $GF(p^m) \cup \{\infty\}$. As A,B generate a group that is vertextransitive on $S_o(p^f)$ and A,B are linear fractional transformations involving only powers of σ_p the elements of any sextet in the same component as k_o must also be in $GF(p^m) \cup \{\infty\}$, and $S_o(p^f)$ is isomorphic to $S_o(p^m)$.//

Corollary

$S_o(p^f)$ is isomorphic to $S_o(p^2)$ for all odd primes p with f greater than or equal to 2, and $S_o(p^f)$ is isomorphic to $S_o(p)$ if $p \equiv 1 \pmod 8$.

Proof

$p^2 \equiv 1$ (modulo 8) for all odd primes. //

From now on we will only be concerned with the family of graphs $S_o(p^2)$, where p is an odd prime. A,B will be considered as elements of $PGL(2,p^2)$ and G will denote $\langle A,B \rangle$ the subgroup of $PGL(2,p^2)$ generated by A,B. G acts vertextransitively on $S_o(p^2)$ and as G must be isomorphic to one of a small number of subgroups of $PGL(2,p^2)$ we have a way of calculating the order of $S_o(p^2)$. If $p \equiv 1 \pmod 8$, we need only consider $S_o(p)$.

First we consider the cases where $p \equiv 1$ or $7 \pmod 8$ when A,B are both within $PSL(2,p^2)$ the subgroup of $PGL(2,p^2)$ consisting of those linear fractional transformations

$$P : t \mapsto \frac{at+b}{ct+d}$$

where $ad - bc$ is a square in the field $GF(p^2)$.

The subgroups of $PSL(2,p^2)$ were found by Dickson and are listed in [24].

Lemma (Dickson)

The group $PSL(2,p^f)$ has the following subgroups:

1) Elementary Abelian p-groups

2) Cyclic groups

3) Dihedral groups

4) Groups isomorphic to A_4

5) Groups isomorphic to S_4

6) Groups isomorphic to A_5

7) Semidirect products of elementary abelian p-groups with cyclic groups

8) $PSL(2,p^m)$ with $m|f$ and $PGL(2,p^m)$ with $2m|f$.

We remark that there are no subgroups of $PSL(2,p^n)$ isomorphic to $S_4 \times Z_2$. It is also true that there are no such subgroups of $PGL(2,p^n)$. Since this group itself occurs as a subgroup of $PSL(2,p^{2n})$ [14]. So we have immediately:

Lemma 5.4

In all cases $G = \langle A,B \rangle$ acts 4-arctransitively on S(p).

Proof

We have seen that G acts transitively on the 4-arcs, so G must act either 4-arctransitively or 5-arctransitively. G is a subgroup of a PGL group and so it cannot contain the subgroups of type $S_4 \times Z_2$ required as the vertex-stabilizers in the 5-arctransitive case. Thus G acts 4-arctransitively. //

Recalling the remarks following the Dickson Lemma, we see that the determination of the order n of S(p) now depends on the order of G: we must have $n = |G|/24$.

Theorem 5.5

$G = \langle A,B \rangle$ is isomorphic to one of $PSL(2,p)$, $PGL(2,p)$, $PSL(2,p^2)$ or $PGL(2,p^2)$.

Proof

G contains the S_4 subgroup fixing the sextet k_o, and the element A which does not fix k_o. The only category of subgroup of $PSL(2,p^n)$ strictly containing an S_4 subgroup is category 8 by the Dickson Lemma.//

Theorem 5.6a

If $p \equiv 1 \pmod{16}$ $G \cong PSL(2,p)$.

Proof

A,B lie inside PGL(2,p) and have square determinants. Hence by Theorem 5.5 G must be isomorphic to PSL(2,p).

Theorem 5.6b

If $p \equiv 9 \pmod{16}$ $G \cong PGL(2,p)$.

Proof

The generators of the stabilizer of k_o are induced by matrices with square determinants and so they belong to $G \cap PSL(2,p)$. The element A^2 also belongs to $G \cap PSL(2,p)$ and it is not in the stabilizer of k_o so $G \cap PSL(2,p) \cong PSL(2,p)$. Since 4 contains the element A not in PSL(2,p) we must have $G \cong PGL(2,p)$. //

Theorem 5.6c

If $p \equiv 15 \pmod{16}$ $G \cong PSL(2,p)$.

Proof

Since $p^2 \equiv 1 \pmod{16}$ in this case we can choose a primitive 16th root of unity τ in $GF(p^2)$ and put $\sigma = \tau^2$. The matrix $A_o = (\tau\sqrt{2})^{-1}A$ induces the same automorphism as A, and it has the properties

$$\det A_o = 1, \quad A_o A_o^* = I,$$

where A_o^* is transpased conjugate of A_o with respect to the field automorphism $x \to x^p$ of $GF(p^2)$. In other words A_o belongs to the special unitary group $SU(2,p^2)$. The same is true for $B_o = (\tau\sqrt{2})^{-1}B$, and so $G = \langle A, B\rangle$ is a subgroup of $PSU(2,p^2)$. However it is known that $PSU(2,p^2)$ is isomorphic to $PSL(2,p)$. Hence by Theorem 5.5 G is isomorphic to $PSL(2,p)$. //

Theorem 5.6d

If $p \equiv 7 \pmod{16}$ $G \cong PGL(2,p)$.

Proof

In this case we cannot normalize A so that it is both special and unitary - this is because $\tau^{p+1} = \tau^8 = -1$ when $p \equiv 7 \pmod{16}$, whereas $\tau^{p+1} = \tau^{16} = 1$ when $p \equiv 15 \pmod{16}$. So we must proceed rather differently.

Let G_o denote the stablizer of k_o and let $K = \langle G_o, A^2B^2\rangle$. G_o is generated by the elements $A^{1-r}B^rA^{-1}$ $(1 \leq r \leq 4)$, or by the transformations $t \to 1/t$, $t \to it$, $t \to (1-t)/(1+t)$. We can choose matrices representing these transformations as follows:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}, \quad \begin{pmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

which all belong to $SU(2,p^2)$. The matrices $A_o^2 = (2\sigma)^{-1}A^2$ and $B_o^2 = (-2\sigma)^{-1}B^2$ induce the same automorphisms as A^2 and B^2 respectively and both belong to $SU(2,p^2)$. Thus as before we have $K \cong PSU(2,p^2) \cong PSL(2,p$

Now for each generator $A^{1-r}\ B^rA^{-1}$, A^2, B^2 of K the result of conjugating by A or B is also in K. Since $AB^{-1} \varepsilon K$ we must have $AK = BK = KB = KA$. It follows that there are just two cosets of K in H, so from Theorem 5.5 $G \cong PGL(2,p)$. //

It must be remarked that when $p \equiv 7$ or $15 \pmod{16}$ the group G is not a "canonical" subgroup $PGL(2,p)$ or $PSL(2,p)$ of $PGL(2,p^2)$; the coefficients of the generators do not lie in $GF(p)$.

Case $p \equiv 3$ or $5 \pmod 8$

<u>Lemma 5.6c</u>

G is isomorphic to $PGL(2,p^2)$.

<u>Proof</u>

In this case $p^2 \equiv 9 \pmod{16}$ so σ is not a square in the field $GF(p^2)$. Both -1 and 2 are squares however, so neither A nor B is a member of $PSL(2,p^2)$. In a finite field the product of two nonsquare elements is always a square. Hence the product of two elements of $PGL(2,p^2)$ outside $PSL(2,p^2)$ must always lie in $PSL(2,p^2)$. Thus if G_o is the intersection of G and $PSL(2,p^2)$, G_o must contain AB^{-1}, A^2B^{-2}, A^3B^{-3} and A^4B^{-4} the elements generating the stabilizer of the vertex k_o. G_o lies inside $PSL(2,p^2)$ so we may now apply the Dickson Lemma. Since G_o contains a vertex-stabilizer

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}, \quad \begin{pmatrix} -i/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

which all belong to $SU(2,p^2)$. The matrices $A_o^2 = (2\sigma)^{-1}A^2$ and $B_o^2 = (-2\sigma)^{-1}B^2$ induce the same automorphisms as A^2 and B^2 respectively and both belong to $SU(2,p^2)$. Thus as before we have $K \cong PSU(2,p^2) \cong PSL(2,p$

Now for each generator $A^{1-r}\ B^r A^{-1}$, A^2, B^2 of K the result of conjugating by A or B is also in K. Since $AB^{-1} \in K$ we must have $AK = BK = KB = KA$. It follows that there are just two cosets of K in H, so from Theorem 5.5 $G \cong PGL(2,p)$. //

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Case $p \equiv 3$ or $5 \pmod 8$

Lemma 5.6c

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Proof

In this case $p^2 \equiv 9 \pmod{16}$ so σ is not a square in the field $GF(p^2)$. Both -1 and 2 are squares however, so neither A nor B is a member of $PSL(2,p^2)$. In a finite field the product of two nonsquare elements is always a square. Hence the product of two elements of $PGL(2,p^2)$ outside $PSL(2,p^2)$ must always lie in $PSL(2,p^2)$. Thus if G_o is the intersection of G and $PSL(2,p^2)$, G_o must contain AB^{-1}, A^2B^{-2}, A^3B^{-3} and A^4B^{-4} the elements generating the stabilizer of the vertex k_o. G_o lies inside $PSL(2,p^2)$ so we may now apply the Dickson Lemma. Since G_o contains a vertex-stabilizer which is isomorphic to S_4 and a further element A^2 which is not of order 2, G_o must be isomorphic to either PSL(2,p), PGL(2,p) or PSL$(2,p^2)$.

All elements of PGL(2,p) and PSL(2,p) have order dividing one of p-1, p,p+1 [24]. We now show G_o is isomorphic to PSL$(2,p^2)$ by showing that the element A^2 of G_o cannot be a member of any subgroup isomorphic to PGL(2,p) or PSL(2,p).

The eigenvalues of the matrix $\Phi^{-1}(A^2)$ lie in the field GF(p^4) and have order dividing p^4-1. The order of these eigenvalues must divide the order of A^2. Hence if A^2 lies in a subgroup isomorphic to either PGL(2,p) or PSL(2,p) the eigenvalues λ_1, λ_2 of $\Phi^{-1}(A^2)$ must have order dividing p-1 or p+1.

$$\Phi^{-1}(A^2) = \begin{pmatrix} \frac{\sigma-1}{2} & \frac{-(\sigma+1)}{2} \\ \frac{\sigma+1}{2\sigma} & \frac{1-\sigma}{2\sigma} \end{pmatrix}$$

and the characteristic equation of this matrix is given by

$$\lambda^2 - \left[\frac{(\sigma-1)}{2\sigma}\right]\lambda + 1 = 0.$$

We now use the identity $(\sigma-1)^2 = \sigma(\sqrt{2}-2)$ to obtain

$$\lambda_1 = \frac{\sqrt{2}-2+\sqrt{(-4\sqrt{2}-10)}}{4}$$

and $$\lambda_2 = \frac{\sqrt{2}-2-\sqrt{(-4\sqrt{2}-10)}}{4}.$$

Each element of GF(p^2) may be expressed in the form $a+b\sqrt{2}$ for some a,b in GF(p), since $\sqrt{2}$ is contained in GF(p^2) but not in GF(p). $(\sigma+\sigma^{-1})^2 = 2$ so $\sqrt{2} = \sigma+\sigma^{-1}$. Hence

$$(\sqrt{2})^p = (\sigma + \sigma^{-1})^p$$
$$= \sigma^3 + \sigma^{-3}$$
$$= \sigma^4(\sigma^{-1} + \sigma)$$
$$= -\sqrt{2} .$$

Thus $\qquad (a + b\sqrt{2})^p = a - b\sqrt{2} \qquad$ if $a,b \in GF(p)$.

Suppose λ_1 has order dividing p-1 or p+1. Because λ_1 is in $GF(p^2)$, and members of GF(p) a,b may be chosen such that

$$\lambda_1 = \frac{\sqrt{2} - 2 + a + b\sqrt{2}}{4}$$

and $$\lambda_2 = \frac{\sqrt{2} - 2 - a - b\sqrt{2}}{4} .$$

$\lambda_1^p = \lambda_1$ or λ_1^{-1}, and $\lambda_1^{-1} = \lambda_2$.

But

$$\lambda_1^p = \frac{-\sqrt{2} - 2 + a - b\sqrt{2}}{4} .$$

Immediately $\lambda_1^p \neq \lambda_2$. Also there can be no value of a satisfying $(a - \sqrt{2})^2 = -4\sqrt{2} - 10$ in $GF(p^2)$ so $(b+1) \neq 0$ and $\lambda_1^p \neq \lambda_1$. Hence the order of λ_1 does not divide p-1 or p+1.

Hence A^2 lies outside all subgroups of $PSL(2,p^2)$ isomorphic to PSL(2,p) or PGL(2,p) and the group G_o must be $PSL(2,p^2)$.

G strictly contains G_o and is a subgroup of $PGL(2,p^2)$ and consequently must be isomorphic to $PGL(2,p^2)$. //

The 5-arctransitive Cases

There are further automorphisms of $GF(p^2)$ under which sextets are preserved, which are not contained in $PGL(2,p^2)$. The group $P\Gamma L(2,p^2)$ is constructed by adjoining the field automorphism $\phi: x \to x^p$ of $GF(p^2)$. We need to find the values of p for which ϕ induces a new automorphism of $So(p^2)$.

Theorem 5.7

The group $P\Gamma L(2,p^2)$ acts transitively on $S_o(p^2)$ if $p \equiv 3$ or 5 (mod 8).

Proof

Let $p \equiv 3$ or 5 (mod 8), and σ denote an eighth root of unity in $GF(p^2)$. Now the sextets

$$k_{-1} = \{\sigma,-\sigma \mid 0, \infty \mid \sigma^3, -\sigma^3\}$$
$$k_o = \{0,\infty \mid i, -i \mid 1, -1\}$$
$$k_1 = \{i,-i \mid 1+\sqrt{2}, 1-\sqrt{2} \mid -1+\sqrt{2}, -1-\sqrt{2}\}$$
$$k_2 = \{1+\sqrt{2}, 1-\sqrt{2} \mid 3(i\sqrt{2}-1)^{-1}, (1-i\sqrt{2})^{-1} \mid -3(1+i\sqrt{2})^{-1}, (1+i\sqrt{2})^{-1}\}$$

constitute a 3-arc. We now use the fact that $(a+b)^p = a^p+b^p$ in a field of characteristic p to establish that this 3-arc is fixed by the field automorphism ϕ.

Two adjacent sextets have one duet in common. If a duet D is fixed by an automorphism α, then 2 adjacent sextets containing D are either both fixed by α or both moved. It is easily verified that the three duets $\{0,\infty\}$, $\{i, -i\}$, $\{1+\sqrt{2}, 1-\sqrt{2}\}$ are all fixed by ϕ and consequently so are the sextets k_{-1}, k_0, k_1, k_2. The duet $\{\sigma,-\sigma\}$ is fixed if $p \equiv 5$ (mod 8) but not if $p \equiv 3$ (mod 8); however the reverse is true for the duet $\{3(i\sqrt{2} - 1)^{-1}, (1 - i\sqrt{2})^{-1}\}$ which is fixed if $p \equiv 3 \pmod 8$ but not if $p \equiv 5 \pmod 8$. Hence if $p \equiv 3$ or $5 \pmod 8$ the automorphism ϕ is nontrivial and it fixes a four-arc (containing the 3-arc $k_{-1}k_0k_1k_2$ and one other sextet) and thus $S_o(p^2)$ is 5-arctransitive. //

If $p \equiv 1 \pmod 8$ then $S_o(p^2) = S_o(p)$, and ϕ acts trivially on $S_o(p)$ since it fixes the subfield $GF(p)$ of $GF(p^2)$. If $p \equiv 7 \pmod 8$ the automorphism of $S_o(p^2)$ induced by ϕ is the same as that induced by $\gamma : x \to {}^{-1}/x$ and so $P\Gamma L(2,p^2)$ induces a group of automorphisms acting 4-arctransitively on $S_o(p^2)$. Hence the only family of 5-arctransitive sextet graphs is $\{S_o(p^2) \mid p \equiv 3 \text{ or } 5 \pmod 8\}$

The girth of $S_o(p)$.

Now we attempt to find the girth of $S_o(p)$ by examining the constitution of the cycles in the terms of the shunts. The girth of $S_o(p^2)$ has been calculated for various values of p and tabulated in Tables 31 - 35 in the Appendix - here we are interested in the effect on the girth as p becomes large. It is believed the girth tends to infinity.

Definition

We define a positive word of length n in x and y w(x,y) to be a string of n letters each of which is either x or y. Given a positive word in x and y w(x,y) and a semi-group H containing two elements u,v, the element of H $w_H(u,v)$ is obtained from w(x,y) by replacing each x and y in the string by u and v respectively and treating the string as a product in the semi-group H.

Suppose Γ is a cubic graph on which the group G acts s-arctransitively where $s \geq 2$. Let $p_o,p_1,\dots,p_s$ be an s-arc in Γ. Then there exist elements of A,B of G representing the twin shunts mapping $p_o,p_1,\dots,p_s$ onto its successors $p_1,\dots,p_s,p_{s+1}$ and $p_1,\dots,p_s,p'_{s+1}$.

Theorem 5.8

There is a one-to-one correspondence between the cycles through the s-arc $p_o,p_1,\dots,p_s$ and the positive words such that $W_G(A,B)$ is the identity in the group G.

Proof

Let $p_o,\ldots,p_s,\ldots,p_g = p_o$ be a cycle of length g in Γ and let $p_{g+k} = p_k$ for all nonnegative k. $p_1,\ldots,p_{i+s}$ is also an s-arc, so by the s-arctransitivity of G there is a unique element W_i of G such that $W_i p_a = p_{a+i}$ $0 \leqq a \leqq s$. For instance $W_o = 1_G$ the identity element of G. We now find possible expressions for W_{i+1} in terms of W_i, given $W_{i+1}p_a = p_{a+i+1}$ $0 \leqq a \leqq s$.

Now $W_iAp_a = W_iBp_a = W_ip_{a+1} = p_{i+a+1}$ $0 \leqq a \leqq s-1$. The vertices W_iAp_s and W_iBp_s are both adjacent to W_iAp_{s-1}, and $W_iAp_{s-1} = W_iBp_{s-1} = p_{s+i}$, but neither of them can be p_{s+i-1} since $p_{s+i-1} = W_iBp_{s-2} = W_iAp_{s-2}$. Hence one of W_iAp_s and W_iBp_s must be p_{s+i+1}, and so W_{i+1} is either W_iA or W_iB. Now using induction and the fact that $W_o = 1_G$ we have $W_i = C_1C_2\ldots C_i$ where C_j is either A or B for all j and consequently there is a positive word W(A,B) in A and B such that $W_i = W_g(A,B)$. But $w_g = w_o = 1_G$ so each cycle corresponds to a positive word W(A,B) in A and B such that $w_G(A,B) = 1_G$.

Conversely, suppose $W_G(A,B) = 1_G$ for some positive word, say $W_G(A,B) = C_1\ldots C_g = 1_G$ with C_j equal to either A or B for all j. Let $W_i = \prod_1^i C_j$ and $W_o = 1_G$. Now let $p_{i+s} = W_ip_s$, so $p_i = W_ip_o = W_{i-1}C_ip_o$. But $C_ip_o = p_1$ so $p_i = W_{i-1}p_1$ which is adjacent to $W_{i-1}p_o = p_{i-1}$. Hence $p_o,p_1\ldots p_g$ is a sequence of vertices with the property p_i is adjacent to p_{i+1} $0 \leqq i \leqq g-1$. Further if $p_{i-1} = p_{i+1}$ then $W_{i-1}p_o = W_{i-1}p_2$ which is not possible, so $p_o,\ldots,p_g$ must be a cycle ($p_g = p_o$ because $W_g = 1_G$) through the s-arc $p_o, \ldots, p_s$. //

Let $\alpha = \begin{pmatrix} x & -x \\ 1 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} x & x \\ 1 & -1 \end{pmatrix}$, elements in the ring R of 2×2 matrices whose entries are polynomials with integer coefficients. Let $W(X,Y)$ be a positive word of length n.

Then

$$W = W_R(\alpha,\beta) = \begin{pmatrix} a_w(x) & b_w(x) \\ c_w(x) & d_w(x) \end{pmatrix}$$

for some $a_w(x)$, $b_w(x)$, $c_w(x)$, $d_w(x)$ polynomials in x with integer coefficients. The leading coefficient of each of these polynomials is always ± 1, and $a_w(x)$ and $b_w(x)$ are of degree n while $c_w(x)$ and $d_w(x)$ are of degree $n-1$. This is easily verified by induction.

Given $p \equiv 1 \pmod 8$ there exists an element σ_p in $GF(p)$ of order 8. Let $\overline{f(x)}$ denote the polynomial $f(x)$ with coefficients reduced modulo p. We define the mapping ϕ_p from the set of positive words in α and β to the group $PGL(2,p)$ of linear fractional transformations of $GF(p)$ as follows. If $W = w_R(\alpha,\beta)$

$$\phi_p(W) : t \mapsto \frac{\overline{a_w(\sigma_p)}t + \overline{b_w(\sigma_p)}}{\overline{c_w(\sigma_p)}t + \overline{d_w(\sigma_p)}} .$$

From theorem 5.2 we deduce the following lemma.

<u>Lemma 5.9</u>

When α,β are considered as positive words in α and β $\phi_p(\alpha)$

and $\phi_p(\beta)$ are twin shunts of a four arc in S(p).

Let I_p denote the identity element of the groups PGL(2,p) and $W = w_R(\alpha,\beta)$ an element of R.

Theorem 5.10

If $\phi_p(W) = I_p$ for every p in an infinite set of primes P then $\phi_p(W) = I_p$ for all primes p.

Proof

Suppose $\phi_p(W) = I_p$ for some word W and prime p.

$$\phi_p(W) : t \mapsto \frac{\overline{a_w(\sigma_p)}t + \overline{b_w(\sigma_p)}}{\overline{c_w(\sigma_p)}t + \overline{d_w(\sigma_p)}} \text{, where } \sigma_p \text{ satisfies } \sigma_p^4 + 1 \equiv 0 \pmod{p}.$$

$\phi_p(W) = I_p$ implies $b_w(\sigma_p) = 0$, which in turn implies σ_p satisfies the equations $b_w(x) \equiv 0$ and $x^4+1 \equiv 0$ modulo p simultaneously. Then the polynomials $\overline{b_w(x)}$ and x^4+1 when considered as elements of the ring of polynomials with coefficients in $\mathbb{Z}_p$ have a common nonconstant factor. If $\overline{m} = m$ reduced mod p

$$b_w(x) = b_n x^n + b_{n-1}x^{n-1} + .. + b_1 x, \text{ implies } \overline{b_w(x)} = \overline{b}_n x^n + .. \overline{b}_1 x.$$

The resultant of $\overline{b_w(x)}$ and x^4+1 is the determinant of the $(n+4) \times (n+4)$ matrix

$$M_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & & & & \\ & 1 & 0 & 0 & 0 & 1 & & & \\ & & 1 & 0 & 0 & & & & \\ & & & 1 & & & & & \\ \vdots & & & & & & 0 & 0 & 1 \\ \bar{b}_n & \bar{b}_{n-1} & & \bar{b}_{n-2} & & & & & \\ & \bar{b}_n & & \bar{b}_{n-1} & & & & & \\ & & & \bar{b}_n & & & \bar{b}_1 & 0 & \\ & & & & & \bar{b}_n & & \bar{b}_1 & 0 \end{bmatrix} .$$

The resultant of two polynomials vanishes if and only if they have a common nonconstant factor [36]. Hence $\det(M_p) \equiv 0$ (modulo p).

If $M = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & & & & \\ & 1 & 0 & 0 & 0 & 1 & & & \\ & & 1 & 0 & 0 & & & & \\ & & & 1 & & & & & \\ & & & & & & 0 & 0 & 1 \\ b_n & b_{n-1} & & b_{n-2} & & & & & \\ & b_n & & b_{n-1} & & & & & \\ & & & b_n & & & b_1 & & \\ & & & & & b_n & & b_1 & 0 \end{bmatrix}$

then $\det(M) \equiv \det(M_p)$ (mod p). But $\det(M_p) \equiv 0$ (mod p) so p divides $\det(M)$. If $\det(M)$ is nonzero only a finite number of primes divide $\det(M)$ and consequently $\phi_p(W) = I_p$ for only a finite number of primes p.

If det $M = 0$ x^4+1 and $b(x)$ have a common nonconstant factor, and since x^4+1 is the minimal polynomial for each of its roots x^4+1 divides $b(x)$.

Taking the resultants of x^4+1 and $c(x)$, and of (x^4+1) and $a(x) - d(x)$, we obtain the result that either $\phi_p(W) = I_p$ for only a finite number of primes p

or

$$W = (x^4+1)K + g(x)I$$

where K is an element of R and I is the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $g(x)$ is a polynomial with integer coefficients.

In this case $\phi_p(W) = I_p$ for all primes p.//

We are now in a position to examine, firstly, the length of the shortest odd cycle in $S_o(p)$ and secondly the girth itself, as p tends to infinity.

Theorem 5.11

Given an odd number g there exists a prime p_g such that the length of the shortest odd cycle in the graph $S_o(p)$ is longer than g if p is greater than p_g.

Proof

We need the fact that if $p \equiv 9 \pmod{16}$ $S_o(p)$ is bipartite and contains no odd cycles. Let $W = w_R(\alpha,\beta)$ be a positive word in α and β of odd length. Then $\phi_p(W)$ does not correspond to a cycle in $S(p)$

if $p \equiv 9 \pmod{16}$. Thus by Theorem 5.10 $\phi_p(W)$ only corresponds to an identity word in $S(p)$ for a finite number of values of p and we can define $p(W)$ to be the largest prime with the property $\phi_p(W) = I_p$, the identity element in $PGL(2,p)$. Since there is only a finite number of positive words of a given length, if S_g is the set of positive words of odd length less than or equal to g, then all the odd cycles in $S(p)$ are of length greater than g if p is more than $p(W)$ for all W in S_g.//

Theorem 5.12

Either there exists a value g and a prime p_g such that if $p > p_g$ then the girth of $S(p)$ is g, or given g there exists q_g such that if $p > q_g$ then the girth of $S(p)$ is greater than g.

Proof

As in the previous proof we define S_n to be the set of positive words of length strictly less than n and $p(W)$ is defined for words W for which there exists a prime π such that $\phi_\pi(W) \neq I_\pi$ to be thelargest prime with the property $\phi_{p(w)}(W) = I_{p(w)}$, or 1 if there is no such prime. If ℓ is the length of the shortest word W such that $\phi_p(W) = I_p$ for all primes p, then if p is more than $p(W)$ for all W in S_ℓ the girth of $S(p)$ is g; if no such word exists then given $g > 0$ if p is more than $p(W)$ for all W in S_g the girth of $S(p)$ is at least g.//

Remarks

Hence we have constructed a family of highly transitive graphs for which it is conjectured there is no upper bound to the girth. The family also yields examples of graphs which in many cases are the smallest known trivalent graphs with a given girth. The girths and orders of the shunts of the graphs known to have girth less than 32 are given in the tables on pages 77f; we now take a closer look at some of them.

In the family of 5-arctransitive graphs, the simplest case $p \equiv 3$ yields the graph $S_o(9)$ which is Tutte's 8-cage [34]; its group is $Aut(S_6) \cong P\Gamma L(2,9)$. It has 30 vertices. The next graph is the family is S(25) with 650 vertices. This graph was found independently by R.M. Foster and J.H. Conway but it has not been published before. There are only five known 5-arctransitive graphs with less than 1000 vertices; one of the others is a 3-fold covering of $S_o(9)$.

No other graphs in the family $S(p^2)$, $p \equiv 3$ or 5 (mod 8) have been previously noticed, and it seems that it has not hitherto been recognized that an infinite family of 5-arctransitive graphs can be constructed in this way. The general idea of using octahedral (S_4) subgroups of PSL and PGL groups has been familiar, at least since the paper of Wong [38] in 1967.

The original motivation for this study was a question raised by Djokavic and Miller [15]. In our notation, they asked for a formula for the girth of the graphs $S_o(p^2)$ in the cases $p \equiv 1$ or 15 (mod 16). We have already seen that in these cases $S_o(p^2)$ has $1/48\ p(p^2-1)$ vertices and its automorphism group is isomorphic to PSL(2,p) and in fact also acts primitively on the graph. The girths of many of the sextet graphs have been computed but no general result has been found. There is, however, apparently no upper bound for the girth. Consequently the sextet graphs provide examples of cubic graphs with given girth g for many values of g for which no specific example is known except as a result of unwieldly general theorems. For example, the graph $S_o(313)$ has girth 30. It has $277,666 \simeq 2^{20}$ vertices, whereas previously it was known only that at least 2^{16} vertices are necessary and 2^{30} vertices are sufficient [34].

Of the sextet graphs whose automorphism group is isomorphic to PSL(2,p) Ito has shown [25] that only PSL(2,7) and PSL(2,23) can act 4-arctransitively on a Cayley graph so $S_o(49)$ and $S_o(529)$ are the only Cayley graphs in the family. $S_o(49)$ is the Heawood graph with 14 vertices which we have already seen is Cayley in Chapter 4. $S_o(529)$ has 506 vertices and is the Cayley graph of the group G with the presentation.

$$G = \langle R, S \mid R^2 = S^{22} = RS^5RS^2RS^4 \rangle .$$

We have already encountered this group as PG(1,23) with the generators

$$R : x \mapsto 22 - x, \quad S : x \mapsto 17x,$$

and in terms of the shunts the group is generated by

$$B \quad \text{and} \quad A^{3}B^{-3}A^{4}B^{-4}A^{3}B^{-3}A^{-1} .$$

This was established using "Cayley" a grouptheoretic computing package.

Appendix

Table 1

Table 1 is a tabulation of the results discussed in Chapter 2 and Chapter 4. $N(3,g)$ is taken to represent the order of the smallest known trivalent graph with girth g, and $N_C(3,g)$ the order of the smallest Cayley graph with these properties. If the value given is marked with an asterisk it is not known whether this figure represents the true minimum or not. Either the group attaining the known minimum is named or a reference to a previous chapter or another table is given. The 2-fold coverings mentioned are obtained as follows.

2-fold Coverings

Let G be a graph of order m with odd girth g and vertexset $V(G) = \{v_1, .. v_m\}$ and edgeset $E(G)$. Define $V'(G) = \{v_1', .. v_m'\}$. Now construct a new graph G' with vertexset $V(G') = V(G) \cup V'(G)$ and edgeset

$$E(G') = \{(V_a, V_b') \mid (V_a, V_b) \in E(G)\} .$$

This graph is bipartite and can contain no cycles of length g.

A graph with 6072 vertices and girth 17

Let G be the Cayley graph of the group PSL(2,23), with generating set {R,S} where

$$R : X \mapsto {}^{1}/_{X} \quad \text{and} \quad S : X \mapsto X + 2 \pmod{23}$$

acting on the set $GF(23) \cup \{\infty\}$ where $\infty + a = \infty$ and $-1/0 = \infty$, $-1/\infty = 0$.

Then it has been verified by computer that the girth of G is 17.

Table 2

Table 2 gives the girth g and diameter d of the Cayley graph of $Aff(p^f)$ with generating set $\langle R,S \rangle$ where

$$R : x \mapsto -1 - x \text{ and } S : x \mapsto ax \pmod p.$$

The arctransitivity of the graph is given in the column marked s and the number of vertices in that headed $|V(G)|$.

Tables 3.1 - 3.5

Tables 3.1 - 3.5 give the girth g of $S_o(p^2)$ for odd primes p. The constant c represents $g^{-1} \log_2 n$ where n gives the number of vertices in the graph. $|a|$ and $|b|$ represent the shunt orders and w_g gives the identity words of length g where known.

Table 4

Table 4 contains various details about the 32 known 60 vertex trivalent graphs of girth 9. N represents the number of 9 cycles in the graph and G its automorphism group. λ_{min} corresponds to the smallest eigenvalue of the adjacency matrix.

g	$n_o(3,g)$	$N(3,g)$	Graph	$N_c(3,g)$	Graph
3	4	4	K_4	4	K_4
4	6	6	$K_{3,3}$	6	$K_{3,3}$
5	10	10	Petersen	50	[C4]
6	14	14	Heawood	14	Heawood
7	22	24	McGee	30	[C4]
8	30	30	Tutte	42	[C4]
9	46	58*	[C2]	60	[C4]
10	62	70	Balaban & c.	100*	[11]
11	94	112*	[1]		
12	126	126	Benson		
13	190	272*	[T2]	272*	[T2]
14	254	406*	[C4]	406*	[C4]
15	382	620*	$S_o(31^2)$		
16	510	1240*	2 fold cov.		
17	766	6072*		6072*	
18	1022	12144*	2 fold Cov.	12144*	
19	1534				
20	2046	14910*	$S_o(71^2)$		
21	3070				
22	4094	16206*	S(73)		
25	12286	149768*	$S_o(193)$		
28	32766	527046*	S(223)		
30	65534	1227666*	S(313)		
32	131070	5892510*	S(521)		

TABLE 1

p	|V(G)|	a	g	d	s
7	42	3	6	6	1
11	110	2	10	7	0
		7	10	7	3
13	156	2	9	8	0
		6	9	8	0
17	272	3	13	8	0
		5	12	9	0
		10	11	8	0
		11	11	9	0
19	342	2	12	9	0
		3	12	9	0
		14	10	9	0
23	506	5	14	9	0
		7	12	10	0
		11	10	11	0
		15	14	10	0
		17	14	10	4
29	812	2	12	10	0
		3	14	11	0
		8	14	11	0
		14	10	13	0
		18	12	11	0
		19	12	11	0
31	930	3	12	12	0
		13	10	11	0
		17	14	11	0
		24	14	12	0

TABLE 2

SEXTET GRAPHS

$S(q)$ is defined for q a prime power $\equiv 1 \pmod 8$)

$|S(q)| = N = \frac{1}{24}q(q^2-1)$, and $PGL(2,q)$ acts 4-transitively

$S(q)$ has K components, all isomorphic, denoted by $S_o(q)$.

$|S_o(q)| = N_o = N/K$ and a group G_o acts S - arc transitively.

p (mod 16)	p^2 (mod 16)	S(p) K	S(p) S	S(p) G_o	$S(p^2)$ K	$S(p^2)$ S	$S(p^2)$ G_o
1	1	2	4	PSL(2,p)	$2p(p^2+1)$	4	PSL(2,p)
3	9	—	—	—	1	5	$P\Gamma L(2,p^2)$
5	9	—	—	—	1	5	$P\Gamma L(2,p^2)$
7	1	—	—	—	$p(p^2+1)$	4	PGL(2,p)
9	1	1	4	PGL(2,p)	$p(p^2+1)$	4	PGL(2,p)
11	9	—	—	—	1	5	$P\Gamma L(2,p^2)$
13	9	—	—	—	1	5	$P\Gamma L(2,p^2)$
15	1	—	—	—	$2p(p^2+1)$	4	PSL(2,p)

So we get five families of connected graphs:

F_1 (1) $S(p^2)$, $p \equiv 3,5 \pmod 8$, 5-transitive, bipartite.

F_2 (2) $S_o(p)$, $p \equiv 1 \pmod{16}$, 4-transitive, primitive.

F_3 (3) $S_o(p^2)$, $p \equiv 7 \pmod{16}$, 4-transitive, bipartite.

F_4 (4) $S(p)$, $p \equiv 9 \pmod{16}$, 4-transitive, bipartite.

F_5 (5) $S_o(p^2)$, $p \equiv 15 \pmod{16}$, 4-transitive, primitive.

TABLE 3.0

$p \equiv 3,5,11,13 \pmod{16}$

$n = \frac{1}{24} p^2(p^4 - 1)$, $G_o = P\Gamma L(2,p^2)$

p^2	g	c
3^2	8	.613
5^2	12	.779
11^2	20	.808
13^2	24	.734
19^2	28	.746

TABLE 3.1

This family contains 5-arctransitive graphs so the order of the shunts a,b and girth words are not relevant.

$p \equiv 7 \pmod{16}$

$n = \frac{1}{24}p(p^2-1), \quad G_o = PGL(2,p)$

p	g	c	$\|a\|$	$\|b\|$	
7	6	.634	6	8	a^6
23	14	.641	24	22	$(aba^4b)^2$
71	20	.693	70	72	
103	22	.703	104	104	$(a^4b^2a^4b)^2$
151	26	.659	152	152	
167	24	.733	168	166	

TABLE 3.3

$p \equiv 15 \pmod{16}$

$n = \frac{1}{48} p(p^2-1)$. $G_o = PSL(2,p^2)$

p	g	c	$\lvert a \rvert$	$\lvert b \rvert$	W_g
31	15	.618	15	16	a^{15}
47	15	.738	23	23	$(a^3b^2)^3$
79	13	1.025	13	20	a^{13}
127	21	.732	64	32	$(ab^2)^7$
191	19	.902	95	19	a^{19}
223	25	.712	111	111	$ab^5a^2ba^8ba^2b^5$
239	21	.862	119	119	$(a^2b^2ab^2)^3$
271	25	.746	27	135	$(a^3b^2)^5$

TABLE 3.5

GRAPH	λ_{min}	N	$\|G\|$
S	-2.61803	60	360
T1	-2.61803	60	120
T2	-2.73205	80	120
BB	-2.56155	72	48
PF	-2.61803	96	144
XA	-2.78165	84	24
XB	-2.78327	76	8
XC	-2.78686	74	4
XD	-2.78790	75	6
YB	-2.78327	76	8
YC	-2.78686	74	4
YD	-2.78804	75	2
YE	-2.78683	74	1
YF	-2.78790	75	3
YG	-2.78299	76	2
YH	-2.78165	84	6
BALA	-2.78816	75	2
BALB	-2.78419	72	4
BALC	-2.78165	84	8
PS1	-2.68909	76	4
PS2	-2.71199	80	1
H1	-2.68867	76	8
H2	-2.65527	80	10
H3	-2.77253	73	1
H4	-2.80734	72	4
H5	-2.78804	73	1
H6	-2.71397	78	1
H7	-2.80592	71	2
H8	-2.75372	74	1
H9	-2.77178	73	1
H10	-2.70076	75	1
H11	-2.78804	73	1

TABLE 4

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