University of London

PROBLEMS ON INTERSTELLAR AND INTERGALACTIC MATTER

A Thesis Submitted by

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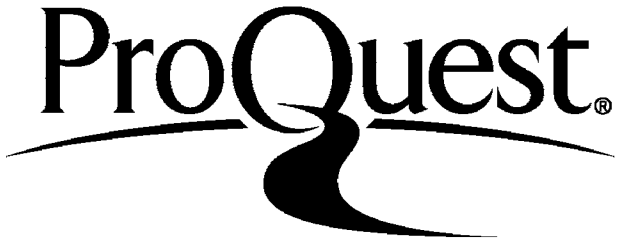

Abstract

The work reported in this thesis falls conveniently into two parts. The first two chapters deal with interstellar while the final two chapters deal with intergalactic material.

The problem of the formation of molecules (principally H_2) at the surfaces of interstellar dust grains is considered in Chapter I. It is shown that the amounts of H_2 formed are sufficient to maintain the interstellar gas at 100 ^{o}K. The results obtained for CH, CH^{+} show that the mechanism considered does not lead to absurd results under the conditions assumed. The importance of H_2 at high densities is also discussed.

Chapter II deals with a theory of star formation in a medium composed of randomly moving cloudlets or floccules. An outline is given of a recent paper by McCrea who shows that this idea leads to a satisfactory account of the formation of the solar system and removes the angular momentum difficulty of other star formation theories. The collisions of floccules are considered in detail and a theory of the gravitational capture of the fragments of collision is outlined.

The expansion of a fully ionised gas into a

vacuum is discussed in Chapter III. This problem is connected with a recent theory of galaxy formation. The expansion of a monatomic gas and Lagrange's Ballistic problem are first considered. The latter leads to a new model for a freely expanding gas. It is then shown that, if the fully ionised gas remains neutral, by suitably defining the sound speed the equations for a fully ionised gas may be reduced to those for a monatomic gas.

Finally in Chapter IV, Hoyle's theory of the origin of the angular velocities of galaxies is reviewed. A specific model based on this theory is discussed. It is shown that some features of galaxy rotation may be accounted for in terms of this theory.

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I should like to thank Professor W.H. McCrea for suggesting these problems to me, for the benefit of his continued interest and advice and for many interesting and stimulating discussions.

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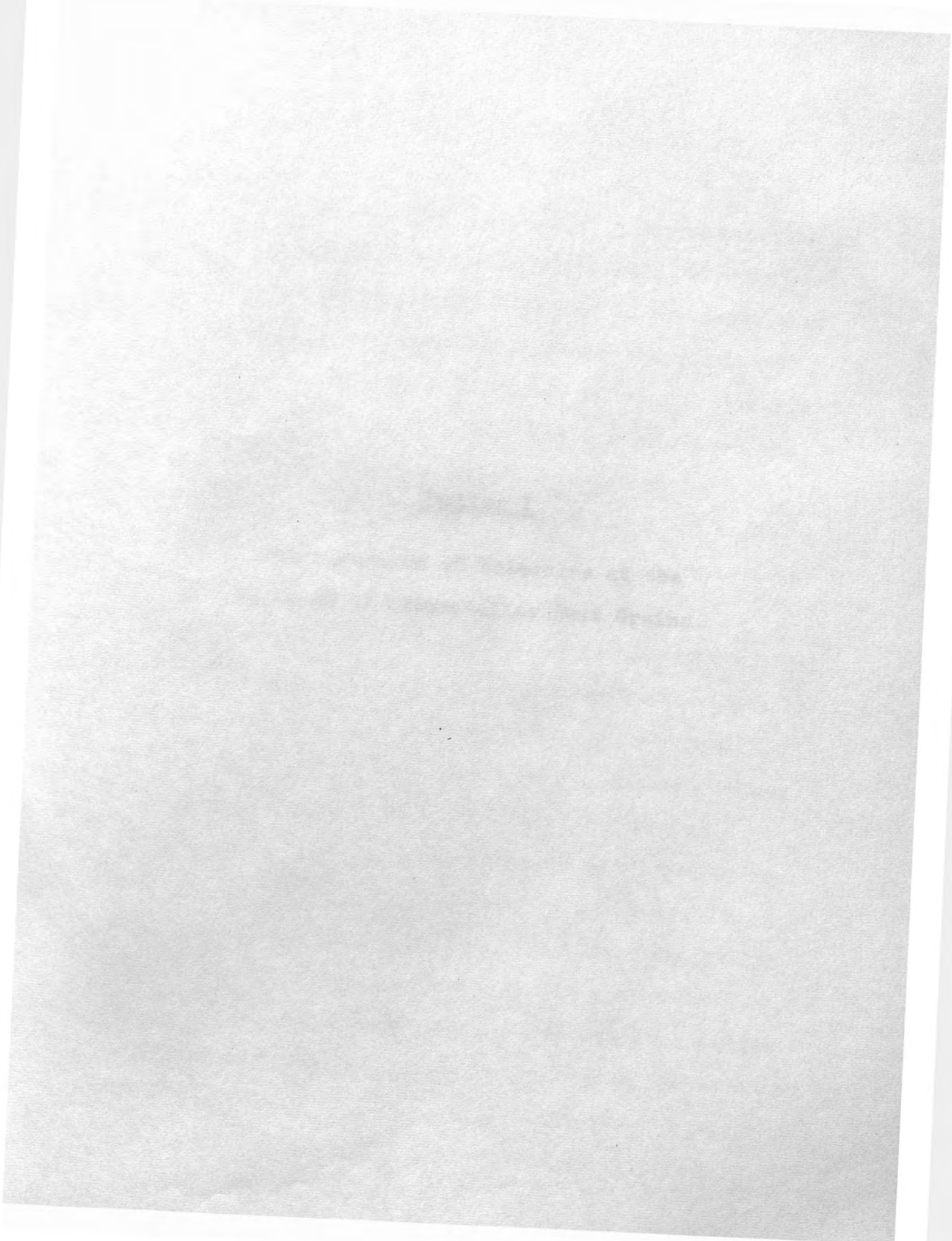


Chapter I

The Formation of Molecules at the Surfaces of Interstellar Dust Grains.

Introduction

Since interstellar material is the material out of which stars are ultimately formed, it is of interest to know the composition of the material over a wide range of temperatures and densities. Interstellar material exists in clouds of mean radius 8 pc. and density 10 H. atoms cm^{-3}. The space between the clouds has a particle density of less than 0.1 H. atoms cm^{-3}. The clouds have a line of sight random velocity of 7 $km.sec^{-1}$. The values have been taken from Allen (1955).

The interstellar material is composed mainly of atomic hydrogen with the addition of atoms such as Na, Ca, K, etc., molecules CH, CH^+, CN, and ions O II, O III, N II, Ca II etc, as well as electrons. Since atomic hydrogen is a poor radiator at kinetic temperatures below 10^4 ^oK, the cooling of the interstellar gas is done by the other atoms, molecules, ions and electrons. In order to maintain the gas at the observed temperature ($\sim$ 100 ^oK) some molecular hydrogen must be postulated. The molecular hydrogen can radiate through quadrupole transitions between the rotational levels of the ground vibrational state. Unfortunately the amounts of molecular hydrogen

present in the interstellar medium are unknown and assumptions must be made about its density. Assuming that H_2 constitutes ten percent of the particle density Spitzer (1948, 1949) and Spitzer and Savedoff (1950) have shown that the temperature of interstellar material is about 60 ^{o}K. More recent estimates taking a one percent mixture of H and H_2 suggest that the mean temperature of 100 ^{o}K. should be taken for the gas.

However, the composition of the interstellar material may alter on compression and it is of interest to investigate the mechanisms which lead to the formation of molecules and in particular to the formation of H_2. Molecular hydrogen if it existed in large quantities would enable the interstellar gas to cool quickly and this property would have important consequencies for the theory of stellar formation.

With this general aim, the question of the formation of molecular hydrogen was investigated in this chapter. Associated with the problem, the formation of other hydride molecules was also considered.

Section 1. Previous work on the formation of diatomic molecules.

Bates (1951) has considered the formation of certain diatomic molecules by radiative association. In this process the atoms approach each other along some potential energy curve and make a transition to a stable potential curve by the emission of a photon. The process is summarised by the equation,

$$A + B \rightarrow AB + h\nu$$

where A, B denote the atomic species, and $h\nu$ denotes the emitted photon. Bates computed the rate coefficient γ for the process where γ is defined by

$$\frac{dn(AB)}{dt} = \gamma n(A) n(B)$$

where n denotes the particle densities of the atomic or molecular species indicated. The molecules studied by Bates were CH, CH^+, N_2^+, H_2^+. The values of γ which he obtained for these molecules are given in Table I.

Table I

Molecule	Rate Coefficient ($cm^3\ sec^{-1}$)
CH	2.10^{-18} $\left[\frac{1}{9} C \left\{ {}^3P_0 + 3{}^3P_1 + 5{}^3P_2 \right\}\right]$
	6.10^{-18} $\left[C \left\{ {}^3P_0 \right\}\right]$
CH^+	2.10^{-18} $\left[\frac{1}{3} C^+ \left\{ {}^2P_{1/2} + 2{}^2P_{3/2} \right\}\right]$
	0 $\left[C^+ \left\{ {}^2P_{1/2} \right\}\right]$
N_2^+	3.10^{-17}
H_2^+	$1.6.\ 10^{-18}$

The values of the rate coefficients are for a kinetic temperature of 100 ^oK. except H_2^+ which is the value at 500 ^oK, (the rate coefficient in the case of H_2^+ increases with temperature). In the table two values are given for the rate coefficients for the formation of CH, CH^+. The ground state of carbon has three levels and one value for γ has been computed assuming that the carbon can be in any of these levels.

The second case which is considered to be more nearly realised under interstellar conditions is that all the carbon is in the lowest level of the ground state. A similar argument applies to singly ionised carbon which has two levels composing the ground state.

The table shows that radiative association is a very slow process. The rate coefficient for the formation of N_2^+ is the largest while the other rate coefficients are an order of magnitude smaller.

Bates & Spitzer (1951) have applied the results for the radiative association of CH, CH^+ to investigate the equilibrium of these molecules in interstellar space. They assumed that the interstellar radiation field could be represented by the radiation emitted by a black body at 10^4 °K, diluted by a factor of 3.10^{-15} at moderate energies ($\sim$ 4 e.V.), 10^{-14} at high energies ($\sim$ 10 e.V.) and by a factor of 10^{-16} beyond the Lyman limit. They also assumed that the gas temperature was 100 °K.

In Table II are listed the reactions considered relevant by Bates and Spitzer for the equilibrium of these molecules.

Table II

Reaction	Rate Coefficient Designation	Value
(i) $C + H \rightarrow CH + h\nu$	γ_1	2.10^{-18} to 6.10^{-18} cm^3 sec^{-1}
(ii) $C^+ + H \rightarrow CH^+ + h\nu$	γ_2	2.10^{-18} to 0 cm^3 sec^{-1}
(iii) $CH + h\nu \rightarrow CH^+ + e$	β_1	8.10^{-12} sec^{-1}
(iv) $CH^+ + e \rightarrow CH + h\nu$	α_1	7.10^{-12} sec^{-1}
(v) $CH^+ + e \rightarrow CH' + h\nu$, $CH' \rightarrow CH + h\nu$	α_1	7.10^{-12} sec^{-1}
(v) $CH^+ + e \rightarrow CH' + h\nu$, $CH' \rightarrow C + H$	α_2	Unknown
(vi) $CH^+ + e \rightarrow C + H$	α_2	Unknown
(vii) $CH + h\nu \rightarrow C + H$	β_2	$1.5.10^{-11}$ sec^{-1}
(viii) $CH^+ + h\nu \rightarrow C^+ + H$	β_3	5.10^{-13} sec^{-1}

Of the values of the rate coefficients given in Table II the most uncertain are those for photo dissociation. The values adopted by Bates and Spitzer are those estimated by Kramers and ter Haar (1946). Using Table II Bates and Spitzer consider the

equilibrium in four separate cases,

(a) $\gamma_2 << \gamma_1$; $\alpha_2 << \alpha_1$

(b) $\gamma_2 << \gamma_1$; $\alpha_2 >> \alpha_1$

(c) $\gamma_2 \simeq \gamma_1$; $\alpha_2 << \alpha_1$

(d) $\gamma_2 \simeq \gamma_1$; $\alpha_2 >> \alpha_1$.

The results of their equilibrium calculations for a cloud whose density is 20 H-atoms cm^{-3} have been summarised in Table III.

Table III

Case	n(CH) (cm^{-3})	n(CH^+) (cm^{-3})
(a)	2.10^{-10}	3.10^{-9}
(b)	2.10^{-10}	9.10^{-11}
(c)	2.10^{-9}	5.10^{-7}
(d)	7.10^{-11}	2.10^{-11}

Bates and Spitzer also computed the values of n(CH)/n(H) and n(CH^+)/n(H) from the observations on the spectral lines CH λ 4300, CH^+ λ 4232 respectively

obtaining,

$$\frac{n(CH)}{n(H)} = 1.6.\ 10^{-8}$$

$$\frac{n(CH^+)}{n(H)} = 8.\ 10^{-9}$$

In the computation of these values Bates and Spitzer used the theoretically determined value (Herzberg 1955a) for the oscillator strength for the line CH λ 4300. The oscillator strength for the line CH^+ λ 4232 is taken to be twice that for the CH line since the orbital degeneracy factor is two for the CH^+ line, but unity for the CH line. However, experimental work by Dunham (1940) suggests that the oscillator strength for the CH line may be larger than the theoretical value by a factor of thirty. If this change were made in the oscillator strengths for both molecules the above ratios would be depressed by a factor of thirty.

Since the hydrogen atom density rarely exceeds 20 atoms cm^{-3} there is a discrepancy between the calculations and the observations. Cases (a), (b) yield much too low values for n(CH) and $n(CH^+)$ and in addition the ratios of n(CH) and $n(CH^+)$ from the theory are too small. Case (c) leads to a rather better estimation of

the amount of CH present but grossly overestimates the amount of CH^+ while case (d) fails completely.

From these results Bates and Spitzer estimate that if theory and observation are to be brought into agreement then the rate coefficients must be increased by a factor in the range 500 to 2000. If this is not acceptable interstellar clouds of density 10^3 H. atoms cm^{-3} must be postulated. The situation is not very much improved if the experimental value is adopted for the oscillator strength since there would still be a discrepancy of a factor between 15 and 60 in the results.

In order to circumvent this discrepancy Bates and Spitzer suggest that the grains may be responsible for the observed molecular density though no definite mechanism is examined in detail. However, they suggest that the clouds of CH^+ observed near late B type stars may result from the sublimation of CH_4 from the grains. The CH_4 is degredated to CH by photo-dissociation and finally photo-ionised. The sublimation, degredation and ionisation take place in the vicinity of the star. This theory gives a result consistent with the observations of CH^+ near late B stars adopting the theoretical estimates for the oscillator strength. However, this theory does not account for the observations on CH. The theory also must assign some chemical

structure to the grains and it also requires the destruction of the grains.

Unfortunately no rate coefficient is available for the production of H_2 by radiative association but Herzberg (1955b) considers the transition probability, for the process of molecule formation in this way, to be so small that the formation of molecular hydrogen by radiative association would be very slow. Herzberg at the same time also suggested that chemical exchange reactions might be a possible source of molecular hydrogen. He suggested reactions of the following type may be involved,

$$CH + H \longrightarrow H_2 + C$$

$$NH + H \longrightarrow H_2 + N$$

These reactions would have to be very rapid in order to offset the low densities of the molecules involved. In the case of CH a factor of 10^8 would be required before the reaction could proceed at the same rate as a reaction involving only H-atoms which had the same rate coefficient. However, the rates of these reactions are unknown and it is not clear even if this type of process is exothermic or not.

At the present time the only interstellar

molecules whose formation has been treated in detail are CH, CH^+. Radiative association does not appear able to account for the observed amounts of these molecules and the sublimation theory is only satisfactory in explaining the presence of CH^+ in the neighbourhood of late B type stars. Chemical exchange reactions may offer a solution in the case of the formation of H_2, but even for these the starting molecules must be present in sufficient quantity. Consequently a further type of mechanism must be sought and a different mechanism is the production of molecules at the surfaces of interstellar grains.

Section 2. The formation of molecules at grain surfaces.

The idea that molecules may be formed at the surfaces of interstellar grains has been suggested several times (e.g. the extensive discussion by van de Hulst (1949) and the more recent shorter discussion by Kahn (1955)). Kahn, using the results of Spitzer (1949), found that only 0.5 percent of the hydrogen present in an interstellar gas cloud need be in the molecular form to maintain the gas at a kinetic temperature of 100 ^{o}K. Assuming that the probability of two hydrogen atoms striking a grain ultimately leaving as a molecule is about unity, Kahn showed that if this were the only formation process and photo-dissociation were the only removal process, then molecular hydrogen would form about 0.25 percent of the total particle population.

The idea of the formation of molecules on interstellar dust grains may be extended to cover not only the formation of H_2 but the formation of any diatomic molecule in which hydrogen is one component. Consequently the theory proposed here takes account, principally, of the formation of H_2, but also of the formation of such molecules as CH, NH, OH.

The interstellar grains are surrounded by gas

which is composed, in the main, of hydrogen atoms. These hydrogen atoms collide with the grain and some of the hydrogen atoms on striking the grain may remain on the grain surface. The precise nature of the forces holding the atomic hydrogen onto the surfaces will not be investigated here, but since atomic hydrogen is very reactive chemically, it may be imagined that the atoms enter into a loose chemical bond with the grain surface. In his discussion of the problem van de Hulst (1949) discusses the experimental evidence for the accommodation of atoms on surfaces at low temperatures. Unfortunately, there is no experimental evidence on atomic hydrogen, but the evidence from experiments with other atoms suggests that nearly every atom striking the surface will remain bound to it. Table IV which has been taken from van de Hulst's review shows how the accommodation coefficient a varies with temperature for the adsorption of various gases on glass. The accommodation coefficient is defined to be the probability that an atom or molecule striking the surface is captured by the surface. The maximum value of a is therefore unity.

Table IV

He on glass		H_2 on glass		Ne on glass		N_2 on glass	
T(^oK)	a	T(^oK)	a	T(^oK)	a	T(^oK)	a
273.10	0.336	273.10	0.283	273.10	0.670	273.10	0.855
77.20	0.383	70.10	0.555	90.11	0.803	86.58	1.041
17.85	0.569	17.87	0.984	17.87	1.056	70.08	1.005
13.80	0.611	14.31	1.053	13.80	1.036		
12.10	0.666						

From Table IV it is clear that at the low temperatures of the grain surfaces ($\sim$ 20^oK) the hydrogen atoms striking the grain will have a good chance of being permanently captured.

We further suppose that any hydrogen atom which strikes the grain remains fixed on the grain surface and is not free to wander over the surface. The question of mobility in the surface layer will be discussed later. The hydrogen atoms on the grain surface are exposed to further impacts with fresh atoms from the gas. The atoms of the surface layer on the grain will be assumed to be hydrogen atoms only, but the atoms colliding with this surface layer will be mainly atomic hydrogen though other atoms such as carbon, nitrogen and oxygen may also be assumed present. This last type of collision may

lead to the formation of a molecule. By forming a molecule the nature of the bond holding the atom to the surface is altered, with the result that the molecule may be returned to the interstellar gas. The energy released by the process of molecule formation is taken up by the grain and radiated away.

A grain whose surface is originally "clean", i.e. free from adsorbed hydrogen atoms, will become coated with a complete layer of hydrogen atoms in a time $\tau_{c.l.}$ given by

$$\tau_{c.l.} = \frac{1}{n(H)\,v(H)\,A},$$

where $v(H)$ is the mean velocity of the hydrogen atoms and A is the area occupied by a hydrogen atom on the grain surface. It will be assumed that a complete layer one atom thick is formed when the surface is covered with hydrogen atoms which are in contact with each other. If the radius of the hydrogen atom is σ then the area effectively occupied by the hydrogen atom will be $4\sigma^2$ (not $\pi\sigma^2$). This is the value adopted for A. Taking $n(H)$ to be 10 atoms cm^{-3}, the kinetic temperature to be 100 ^{o}K. and σ to be 10^{-8} cm, we obtain

$$\tau_{c.l.} = 6.25.\ 10^{9}\ \text{sec.} \quad \text{or} \quad 2.10^{2}\ \text{yr.}$$

This time is short compared with 10^6 yr. the time scale usually adopted for the process of star formation. Consequently the grain may be assumed to be coated with a complete layer, one atom thick, of hydrogen atoms. The consequences of the layer being greater than one atom thick will be discussed later.

Once a molecule has been formed at the surface of the grain the type of bond holding the molecule to the grain will be weaker than the bond holding the atom. For molecules the energy required to break a chemisorption bond is of the order of 1 e.V. (e.g. Brunaurer ., 1945) and the molecule formation process releases about 4 e.V. per molecule so that sufficient energy is available to release a molecule on formation.

Furthermore in the case of molecular hydrogen van de Hulst (1949) has shown that a layer of H_2 10^{-5} cm. thick would evaporate in 10^{-5} sec. under interstellar conditions. This can be regarded as evidence that a molecule of hydrogen (and we shall assume this to be true for any molecule) ultimately leaves the grain after its formation.

Several experiments have been performed in the laboratory on the recombination of hydrogen and oxygen atoms. This type of experiment was pioneered by Smith (1943) who studied the recombination of hydrogen at

various surfaces. The experiments were repeated for metallic surfaces using more refined techniques by Wood and Wise (1958). They found that the recombination coefficient (the fraction of hydrogen atoms striking the surface that come off as molecules) was in the range 0.1 to 0.25 for most metals. Aluminium was outstanding in giving a value of 10^{-3}. The recombination coefficient was also determined for Pyrex glass and was found to be $7.5. 10^{-4}$.

Experiments by Linnett and Marsden (1956) and Greaves and Linnett (1958) gave recombination coefficients in the range 10^{-3} to 10^{-1} for metallic surfaces and 10^{-5} to 10^{-2} for non-metallic surfaces. Linnett and Marsden also investigated the variation of the recombination coefficient with temperature on oxide surfaces in the temperature range 293 ^{o}K. to 673 ^{o}K. and a substantial change of several orders of magnitude was found in the rate coefficient in this range. The rate coefficient was found to increase with increasing temperature, but at both high and low temperatures it tended to a constant value and in some cases a minimum value seemed to be obtained near room temperature. However, there do not seem to be any experiments in which the recombination coefficient was measured at temperatures substantially below room temperature.

The experimental evidence, though tenuous, indicates that reactions of the type considered can take place in so far as hydrogen and oxygen are concerned. The calculations of van de Hulst show that molecules once formed will be evaporated back into the interstellar medium and our own calculation coupled with those of van de Hulst indicates that a complete monatomic layer of hydrogen could form quickly on the surface of an interstellar grain.

Section 3. The mathematical theory and numerical results.

The theory as outlined in Section 2 will now be developed mathematically. In the first instance the theory will be presented for the formation of molecular hydrogen. The modifications required to take account of the formation of other molecules will be made at the end.

(a) Notation

For the hydrogen gas we write,

m_H = atomic mass,

n_1 = number of free hydrogen atoms per unit volume,

n_2 = number of hydrogen molecules per unit volume,

$n_0 = n_1 + 2n_2$ = total number of free and combined atoms per unit volume,

T = kinetic temperature, assuming a Maxwellian distribution of velocities

We define the quantity μ ($\simeq$ atomic weight) by

$\mu n_0 m_H$ = total mass of interstellar material per unit volume.

For the grains we write,

N = number of grains per unit volume,

r_1 = mean radius of a grain,

$A = 4\pi N r_2^2$ = total surface area of the grains per unit volume,

$V = \frac{4\pi}{3} N r_3^3$ = total volume of the grains per unit volume,

ρV = total mass of grains per unit volume,

$\delta = \rho V (\mu n_0 m_H)^{-1}$ = fraction of total material in the form of dust.

We define β, the rate coefficient for the formation of H_2, by

$\beta n_0 n_1$ = number of H_2 molecules formed per unit volume per unit time,

and we define w to be the probability that a hydrogen atom impinging on the grain will ultimately form a molecule and return to the interstellar gas.

(b) The derivation of an expression for β.

Let the number of hydrogen atoms per unit volume having velocity v be $n(v)$. Therefore the number of collision per grain per unit time may be written

$$n(v) A v. \qquad (1)$$

Averaging (1) over a Maxwellian distribution of velocities we obtain,

$$n_1 A \left\{\frac{m_H}{2\pi kT}\right\}^{3/2} \int_0^\infty\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \exp\left\{-\frac{m_H}{2kT}(u^2+v^2+w^2)\right\} du\,dv\,dw$$

$$= n_1 A \left\{\frac{kT}{2\pi m_H}\right\}^{1/2} \qquad (2)$$

where u, v, w are the velocity components and k is Boltzmann's constant. In writing down equations (1) and (2) we have neglected any random velocity which the grains may possess. However, the grains are so massive compared with the atoms that little error is introduced by this assumption.

Since ϖ is the probability that a hydrogen atom striking the grain leaves as part of an molecule atom, then $\frac{1}{2}\varpi$ is the probability that any given collision between an atom and a grain results in the formation of a molecule. From equation (2) the rate of formation of hydrogen molecules is

$$\beta n_0 n_1 = \frac{1}{2}\varpi n_1 A \left\{\frac{kT}{2\pi m_H}\right\}^{1/2}$$

$$= \frac{3}{2}\left\{\frac{kT m_H}{2\pi}\right\}^{1/2} \frac{\varpi \mu \delta}{\rho} \frac{r_2^2}{r_3^3} n_0 n_1 . \qquad (3)$$

(c) The properties of the grains

Oort and van de Hulst (1946) have developed a theory of grain formation and have computed a distribution function $N(r)dr$ for the number of grains having radii in the range $r,\ r+dr$. This distribution function has a complicated character but over the significant range of r it tends to the form

$$N(r) = C\exp\left\{-\left(\frac{r}{a}\right)^p\right\} \qquad (4)$$

where $C,\ a,\ p$ are constants.

An integral of the form

$$\int_0^\infty r^q \exp\left\{-\left(\frac{r}{a}\right)^p\right\} dr$$

can be transformed by making the substitution ,

$$\left(\frac{r}{a}\right)^p = t$$

into the form

$$K\int_0^\infty t^\nu \exp(-t)\,dt \qquad (5)$$

where $K = \frac{a^{q+1}}{p}$ and $\nu = \frac{q+1-p}{p}$.

Equation (5) is then a standard form which may be

integrated to give

$$\int_0^\infty r^q \exp\left\{-\left(\frac{r}{a}\right)^p\right\} dr = \frac{a^{q+1}}{p}\Gamma\left(\frac{q+1}{p}\right). \qquad (6)$$

Equation (6) may now be applied to determine the mean radius r_1 and the values of r_2^2 and r_3^3,

$$r_1 = \frac{\int_0^\infty r N(r)\,dr}{\int_0^\infty N(r)\,dr} = \frac{\frac{a^2}{p}\Gamma(2p^{-1})}{\frac{a}{p}\Gamma(p^{-1})} = a\frac{\Gamma(2p^{-1})}{\Gamma(p^{-1})}, \qquad (7)$$

$$r_2^2 = \frac{\int_0^\infty r^2 N(r)\,dr}{\int_0^\infty N(r)\,dr} = \frac{\frac{a^3}{p}\Gamma(3p^{-1})}{\frac{a}{p}\Gamma(p^{-1})} = a^2\frac{\Gamma(3p^{-1})}{\Gamma(p^{-1})}, \qquad (8)$$

$$r_3^3 = \frac{\int_0^\infty r^3 N(r)\,dr}{\int_0^\infty N(r)\,dr} = \frac{\frac{a^4}{p}\Gamma(4p^{-1})}{\frac{a}{p}\Gamma(p^{-1})} = a^3\frac{\Gamma(4p^{-1})}{\Gamma(p^{-1})}. \qquad (9)$$

Therefore

$$\frac{r_2^2}{r_3^3} = \frac{\kappa(p)}{r_1} \qquad (10a)$$

where

$$\kappa(p) = \frac{\Gamma(2p^{-1})\Gamma(3p^{-1})}{\Gamma(p^{-1})\Gamma(4p^{-1})}. \qquad (10b)$$

Using this result for τ_2^2/τ_3^3 we can eliminate it from equation (3) and obtain,

$$\beta n_0 n_1 = \frac{3}{2}\left\{\frac{k m_H T}{2\pi}\right\}^{1/2} \frac{\varpi \mu \delta}{\rho \tau_1} K(p)\, n_0 n_1 \qquad (11)$$

Inserting numerical values for the constants k, m_H we may write

$$\beta = 9 \cdot 1 \cdot 10^{-15} \frac{\varpi \mu \delta}{(10^5 \tau_1)\rho}\left(\frac{T}{100}\right)^{1/2} K(p) \text{ cm}^3 \text{ sec}^{-1} \qquad (12)$$

If ϖ does not vary or varies only slowly with temperature, the rate coefficient varies as the square root of the temperature. The rate coefficient also varies inversely as the mean radius of the grains.

Equation (12) must be ammended if account is to be taken of the formation of other molecules of the type XH. The expression for β is changed in two ways.

(i) Assuming that all the atoms in an interstellar cloud are in thermal equilibrium the velocity of the atomic species X must be decreased by a factor $M_X^{-\frac{1}{2}}$ where M_X is the atomic weight of the species X.

(ii) The value of ϖ must be altered. Since the atom X on striking the grain finds an almost

complete layer of hydrogen atoms on the surface, the maximum value that can be taken by ϖ is unity, i.e. the chance that any given atom of species X colliding with the surface of the grain, comes off as part of a molecule is unity.

Taking account of these changes, the rate coefficient β_{xH} for the production of the molecule XH is

$$\beta_{xH} = 1{\cdot}82.10^{-14} \frac{\varpi_x \mu \delta}{(10^5 t)\rho M_x^{1/2}} \left(\frac{T}{100}\right)^{1/2} K(p) \text{ cm}^3 \text{ sec}^{-1}, \quad (13)$$

where ϖ_x denotes the probability that the molecule XH is formed.

(d) The numerical values

Since the remaining parameters in equations (12) and (13) are not precisely defined we will choose values for them defining a set of favourable circumstances as set out below.

$\varpi = 1.0$.

The experimental values for ϖ as determined for the formation of H_2 by Wood and Wise (1958) for metallic surfaces lay in the range 0.1 to 0.25.

The work by Smith (1943) on the formation of H_2 on non metallic surfaces such as K_2SiO_3, K_2CO_3 also shows that $\varpi \geq 0.1$. For substances such as quartz and glass the value is much lower being of the order of 10^{-4}. However, it is unlikely that a surface similar to that possessed by quartz or glass would be present under interstellar conditions.

Furthermore we have considered only surfaces which are smooth and have an immobile monatomic layer of H atoms on them. If the grain is a loose aggregation with a porous structure rather than a compact body, then the chance that an atom may encounter an active site (i.e. a hydrogen atom) on the surface will be increased. If there is any mobility within the layer i.e. the hydrogen atoms are able to move around within the layer the chance of encounters will be increased. Mobility will be much more likely if the grain is covered by several layers of hydrogen atoms. Since the binding of the outer layers will be much weaker than the binding of the innermost layer, it is unlikely that the hydrogen on the grain surface will be present in more than one layer. Such a condition if it did exist would enhance the production of molecular hydrogen.

In order to give the process the best chance of working we choose the value of ϖ to be unity.

This choice is further supported by the conclusions of de Boer reported by Kahn (1955). This value is also adopted for ϖ_x .

$2.5 < p < 3.0 : \quad K(p) = 0.55$

The results obtained by Oort and van de Hulst (1946) on the distribution function for grain radii are given in Table 2 of their paper. Taking a distribution law of the form of equation (4) we find that their results are best fitted by choosing $2.5 < p < 3.0$. The distribution law curve of Oort and van de Hulst and the curves for $p = 2.5$, $p = 3.0$ are drawn in Figure 1. The approximate curves are drawn so that they agree with the curve of Oort and van de Hulst when $r = a$.

If $p = 2.5$, $K(p) \doteq 0.54$ while if $p = 3.0$, $K(p) \doteq 0.56$ so that a sufficiently good approximation will be obtained if $K(p)$ is taken to be 0.55 without specifying p more closely since $K(p)$ is not very sensitive to changes in p . Although r_1 still depends on p we do not require any closer specification of p since the value of r_1 may be taken directly from the results of Oort and van de Hulst.

$r_1 = 10^{-5}$ cm.

The mean radius r_1 quoted by Oort and van de

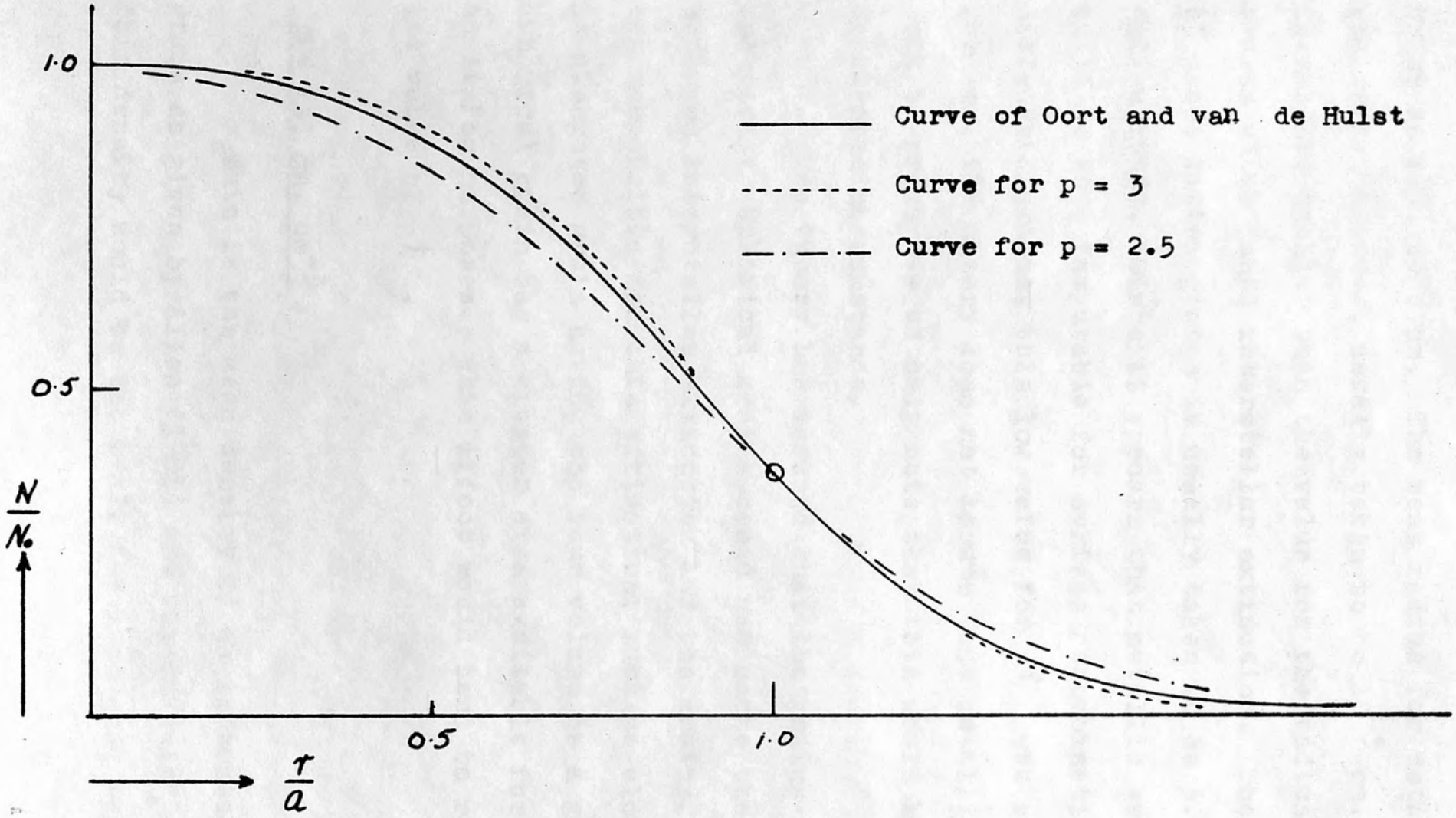

1·0
0·5
N/N₀
0·5
1·0
r/a
Curve of Oort and van de Hulst
Curve for p = 3
Curve for p = 2.5

Figure 1.

Hulst is 1.5. 10^{-5} cm. The mean radius for metallic grains is, however, usually taken to be 10^{-5} cm. These values are smaller than the value for the radius of the grains which cause interstellar extinction. The radius of these latter grains is usually taken to be 3.10^{-5} cm. (Allen 1955). Since it appears that metallic grains would be more favourable for surface recombination than dielectric grains, this low value for t_1 was adopted. However, the theory does not assume that metallic grains must be present - we only note that this would be a favourable circumstance.

The theory has assumed that the grains are spherical. Spherical grains could not cause the observed interstellar extinction and the grains, if they are responsible for this extinction, must be elongated. An elongated grain having the same volume as a given spherical grain has a greater area available for collision purposes. This effect would tend to reduce the value of t_1 .

$\rho = 1.1$ gm. cm^{-3}

This is the mean density of an interstellar grain as given by Allen (1955) and van de Hulst (1949). This density would be too small for a compact metallic

grain, but the grain may be a porous structure, so that this value may still be appropriate even if the density of the material of the grain is somewhat larger.

$\delta = 10^{-2}$.

The mean density of grains in space near the galactic plane is given by Allen (1955) to be $1.4\ 10^{-26}$ gm. cm^{-3}. The mean density of the interstellar gas is 10^{-24} gm. cm^{-3}. This would suggest a value for δ of $1.4\ 10^{-2}$. The value given by Allen is not likely to be exceeded since there is almost certainly insufficient heavy atoms to give a much higher density. Some estimates of the density of the interstellar grains are appreciably less than this and the above value was adopted for δ so limiting ourselves to regions in which grains are relatively plentiful.

$\mu = 1.3$.

This value for the mean molecular weight assumes that the elements are distributed in the interstellar gas in the same way as they are in stars. Thus the abundance of the elements will be given by the cosmic abundance and the mean atomic weight will be 1.3 (Allen, 1955).

$T = 100\ ^{o}K$.

The mean temperature of the interstellar gas in HI regions is of the order of 100 ^{o}K. A small change in T will not lead to a large change in β since β depends on $T^{1/2}$.

The parameters chosen as listed above are regarded as being favourable for the production of molecules at the surfaces of grains. Of these the parameter ϖ is the most adjustable and has been assumed to have its maximum value. Accordingly the calculations will be sensitive to any changes which may be made in ϖ by the results of further theoretical or experimental investigations. The other parameters are typical for HI regions and are not likely to vary appreciably.

(e) The numerical values of the rate coefficients.

Using the values obtained in (d) above, we can evaluate equations (12) and (13). The values obtained are given in Table V.

Table V

Molecule Produced	Rate Coefficient (cm^3 sec^{-1})
H_2	$5.9.\ 10^{-17}$
CH or CH^+	$3.4.\ 10^{-17}$
NH	$3.2.\ 10^{-17}$
OH	$3.0.\ 10^{-17}$

The results obtained in Table V do not represent upper bounds for the rate coefficients since the conditions for which the rate coefficients were calculated, while favourable, are not the most favourable. It may happen that there are many more grains of small radius than predicted by the distribution function or the grains may have a surface activity greater than has been assumed.

Furthermore, it would not be meaningful to give a lower bound for β . However, if the laboratory experiments may be taken as a guide the activity of the surface should not be reduced below the value assumed by much more than a factor of 10^{-5}. Since the behaviour of surfaces at low temperatures is not well known, any assessment of a lower bound would be of little value.

This inability of the present theory to give

upper and lower bounds for the rate coefficients is an inescapable but unsatisfactory feature of the theory. We shall later show, however, that the favourable conditions do not lead to absurd predictions about the state of the interstellar gas.

Section 4. Collisions between interstellar clouds.

The results obtained in the last section were obtained for normal undisturbed regions of the interstellar gas. However, interstellar clouds collide with one another. The mean random velocity of interstellar clouds is 7 km. sec^{-1} (Allen, 1955) so that any given cloud makes a collision approximately every 10^7 yr. Since the mean thermal speed of the atoms in the gas cloud is about 1 km. sec^{-1} the clouds are moving at supersonic speeds and on collision a shock wave will move into the clouds from the interface.

Under these conditions the density and temperature of the region between the shock waves is higher than in the normal undisturbed cloud. In this type of region therefore enhanced molecule formation may take place. In order to investigate the conditions which may be found in such a region we use the analysis developed by McCrea (1956) for dealing with shock waves.

Let a subscript 1 denote the supersonic side of the shock and subscript 2 denote the subsonic side. If p_i, ρ_i, v_i denote pressure, density and velocity respectively where $i = 1, 2$ then, assuming isentropic flow, the conditions for conservation of momentum, mass

and energy through the shock are,

momentum $$p_2 + \rho_2 v_2^2 = p_1 + \rho_1 v_1^2 , \qquad (14)$$

mass $$\rho_2 v_2 = \rho_1 v_1 , \qquad (15)$$

energy $$\frac{1}{2} v_2^2 + \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} = \frac{1}{2} v_1^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} , \qquad (16)$$

where γ is the ratio of the specific heats.

It is convenient to introduce the Mach number u_i defined by

$$u_i = \frac{v_i}{c_i} \qquad (17)$$

where $$c_i^2 = \frac{\gamma p_i}{\rho_i} \qquad (18)$$

is the sound speed. Introducing u_i into equations (14) to (16) gives

$$(1 + \gamma u_2^2) \frac{c_2}{u_2} = (1 + \gamma u_1^2) \frac{c_1}{u_1} \qquad (19)$$

and using equations (15) and (17) in (14)

$$\rho_2 c_2 u_2 = \rho_1 c_1 u_1 \qquad (20)$$

and $$\left\{2 + (\gamma-1)u_2^2\right\}c_2^2 = \left\{2 + (\gamma-1)u_1^2\right\}c_1^2 . \qquad (21)$$

Eliminating c_1/c_2 between equations (19) and (21) gives

$$2\gamma u_1^2 u_2^2 - (\gamma-1)(u_1^2 + u_2^2) - 2 = 0 . \qquad (22)$$

If u_1 is very much greater than u_2 equation (22) may be written

$$2\gamma u_1^2 u_2^2 - (\gamma-1)u_1^2 = 0 ,$$

$$\text{or} \qquad u_2^2 = \frac{\gamma-1}{\gamma+1} . \qquad (23)$$

Since the kinetic temperature T_i of the gas is related to the sound speed by the relation

$$c_i^2 = \frac{\gamma R T_i}{\mu} , \qquad (24)$$

where R is the gas constant and μ is the mean molecular weight, we have

$$\frac{T_2}{T_1} = \left(\frac{c_2}{c_1}\right)^2 = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} u_1^2 , \qquad (25)$$

using equation (24). Equation (20) gives

$$\frac{\rho_2}{\rho_1} = \frac{c_1 u_1}{c_2 u_2} = \frac{\gamma+1}{\gamma-1} . \qquad (26)$$

Equation (26) gives the maximum compression that can occur behind such a shock.

If two similar clouds collide with relative velocity $2v$ then

$$v = c_1 u_1 - c_2 u_2 \qquad (27)$$

from the definition of u_i . Using equations (20) and (26) in equation (27) we have

$$c_1 u_1 = \tfrac{1}{2}(\gamma + 1) v . \qquad (28)$$

If one of the clouds has thickness d in the direction of motion then the time required for the shock wave to traverse the cloud is

$$t(d) = \frac{d}{c_1 u_1} = \frac{2d}{(\gamma+1) v} . \qquad (29)$$

In deriving these results we have assumed that the gas between the shock waves has no means of losing energy. However, if the material could cool rapidly, say by radiation, to some temperature T_2' then since u_1 is very much greater than unity we may write the density ρ_2' corresponding to the temperature T_2' in the approximate form

$$\rho_2' T_2' = \rho_2 T_2 \qquad (30)$$

i.e. cooling takes place under conditions of

approximately constant pressure.

To apply these results obtained by McCrea to the collision of two interstellar clouds let us take $\mu = 1.3$, $\gamma = \frac{5}{3}$, $T_1 = 100\ ^{o}K$, $d = 16$ pc. and $v = 7$ km. sec^{-1}. Then equation (26) gives,

$$\rho_2 = 4\rho_1 , \tag{31}$$

and equation (25) gives,

$$\left(\frac{T_2}{T_1}\right)^{1/2} = \frac{c_2}{c_1} = 0{\cdot}56\, u_1 \tag{32}$$

and

$$c_1 = 1{\cdot}07.10^5 \left(\frac{T}{100}\right)^{1/2} \text{ cm. sec}^{-1} \tag{33}$$

Since $c_1 \sim 1$ km. sec^{-1}, $u = \frac{28}{3}$ so that

$$\left(\frac{T_2}{T_1}\right)^{1/2} = 5{\cdot}23 , \tag{34}$$

whence $$t(d) \simeq 2.10^6 \text{ yr.} \tag{35}$$

Since the reaction rates in which we are interested are proportional to the square of the density and to the square root of the temperature then under these conditions the rate of production of

molecules would be enhanced by a factor of $4^2 \times 5.23 \simeq 84$. This enhancement would be effective over a period of the order of 10^6 yr. This type of enhancement would take place in cloud collisions provided the collisions were not sufficiently violent to disrupt the grains.

Section 5. The equilibrium between atoms and molecules in interstellar space.

This section is divided into three parts. In the first part some characteristic times are derived from general considerations while in the second part the equilibrium between molecular and atomic hydrogen is considered both with and without a radiation field. The third part discusses the equilibrium of CH and CH^+.

(a) Some characteristic times.

If we consider only the formation of H_2 molecules then we can write down the rate at which H atoms disappear. This is

$$\frac{dn_1}{dt} = -2\beta n_0 n_1 \qquad (36)$$

The factor of two is required since two hydrogen atoms disappear when a molecule of hydrogen is formed. Integration of equation (36) gives,

$$n_1 = n_0 \exp(-2\beta n_0 t) \qquad (37)$$

where $n_1 = n_0$ at time $t = 0$. The time $\tau(H)$ required to convert a fraction $(1 - e^{-1})$ of the atoms into molecules is therefore

$$\tau(H) = (2\beta n_0)^{-1} \qquad (38)$$

We adopt $\tau(H)$ as the characteristic time for H_2 formation.

If we assume that hydrogen molecules can only be removed by photo-dissociation then the rate of molecule removal is

$$\frac{dn_2}{dt} = -\alpha n_2 \tag{39}$$

where α is the chance that an H_2 molecule will be photo-dissociated in unit time. Integrating (39) we have that

$$n_2 = {}_0n_2 \exp(-\alpha t) \tag{40}$$

where ${}_0n_2$ is the original number of H_2 molecules present. The characteristic time for molecule removal is therefore

$$\tau(H_2) = \alpha^{-1} \tag{41}$$

If molecule formation and dissociation are in a steady state then the rates of formation and removal are equal so that

$$\beta n_0 n_1 = \alpha n_2 ,$$

or

$$\frac{n_1}{2\tau(H)} = \frac{n_2}{\tau(H_2)} \tag{42}$$

If there is not a steady state then,

$$\frac{dn_1}{dt} = -2\beta n_0 n_1 + 2\alpha n_2 = -(2\beta n_0 + \alpha) n_1 + \alpha n_0 . \quad (43)$$

The characteristic time for approach to the steady state is

$$\tau(S.T.) = (2\beta n_0 + \alpha)^{-1} \quad (44)$$

From the form of $\tau(S.T.)$ it is clear that it is not very different from the shorter of the times $\tau(H), \tau(H_2)$.

In the case of CH, CH^+ these results would need minor modification since only one carbon atom goes into a molecule. The equilibrium of CH, CH^+ involves processes other than photo-dissociation causing their removal so that the times discussed above for the case of hydrogen do not have much meaning in the case of CH, CH^+.

(b) The equilibrium of hydrogen.

The equilibrium between the atomic and molecular forms of hydrogen will be examined in two stages. In the first stage, it will be assumed that the interstellar gas is in thermal equilibrium in the absence of a radiation field. The second stage of the

investigation will consider the equilibrium when a radiation field is present assuming a single formation process and a single removal process.

For a gas in thermal equilibrium in the absence of a radiation field the equilibrium between the molecular species AB and the atomic species A, B may be expressed in terms of the dissociation constant K' defined by

$$K' = \frac{n(A)\, n(B)}{n(AB)} . \qquad (45)$$

Fowler and Guggenheim (1949) give the following expression for K',

$$K' = \sigma_{AB} \frac{g_A g_B}{g_{AB}} \left\{ \frac{2\pi M}{h^2} \right\}^{3/2} (kT)^{1/2} \frac{e^{-\frac{D}{kT}}}{q(T)} \frac{h^2}{8\pi^2 I} \qquad (46)$$

where M is the reduced mass of the molecule,

D is the dissociation energy of the molecule,

I is the moment of inertia of the molecule,

k is Boltzmann's constant,

h is Planck's constant,

T is the kinetic temperature,

g_A, g_B are the electronic weights for the normal states of the atoms,

g_{AB} is the electronic weight of the lowest molecular state,

σ_{AB} is the symmetry number being 2 for A, B identical and unity otherwise,

and $q(T)$ is the partition function.

If the energy of the lowest vibrational level is small compared with the dissociation energy we can write

$$q(T) = (1 - e^{-S})^{-1}, \qquad (47)$$

where $S = h\nu/kT$, ν being the frequency of the lowest vibrational level. Hence

$$K' = \sigma_{AB} \frac{g_A g_B}{g_{AB}} \left\{ \frac{2\pi M}{h^2} \right\}^{3/2} \frac{h^2 (kT)^{1/2}}{8\pi^2 I} (1 - e^{-S})\, e^{-\frac{D}{kT}}, \qquad (48)$$

which is the expression given by Aller (1953) apart from a factor of kT required to convert K' to K the dissociation constant in terms of partial pressures and the symmetry factor σ_{AB}. Expressing M, I in atomic units, D in electron volts, writing $I = M\dagger^2$ where $\dagger$ is the mean interatomic distance and taking logarithms to the base 10 we obtain

$$\log_{10} K' = \log_{10}\left\{ \frac{\sigma_{AB} g_A g_B}{g_{AB}} \right\} + \log_{10}\left\{ \frac{(2\pi)^{3/2} h^2 k^{1/2}}{8\pi^2 h^3} \right\} + \tfrac{1}{2}\log_{10} M + \tfrac{1}{2}\log_{10} T$$

$$- 2\log_{10}\dagger + \log_{10}(1 - e^{-S}) + \log_{10}\left\{ e^{-\frac{D}{kT}} \right\}, \qquad (49)$$

i,e.

$$\log K' = -\frac{5040D}{T} + \tfrac{1}{2}\log T + \tfrac{1}{2}\log M - 2\log t + \log(1-e^{-S})$$

$$+ \log\left\{\sigma_{AB}\frac{g_A g_B}{g_{AB}}\right\} + 21.66 \quad . \qquad (50)$$

For hydrogen the numerical values taken were

D = 4.476 e.V., M = 0.5041, t = 0.7416 A°.

and $S = \frac{hc\nu/c}{kT} = \frac{6352}{T}$ where ν/c = 4395 cm^{-1}.

so that

$$\log K' = -\frac{22,559}{T} + \tfrac{1}{2}\log T + 21.55 + \log(1-e^{-\frac{6352}{T}}) \quad . (51)$$

The last term in equation (51) is not very effective until $T \simeq 10^3$ °K. At interstellar temperatures T = 100 °K so that $\log K'$ = - 203. Therefore under interstellar conditions K' is very small so that amounts of atomic hydrogen present in an interstellar gas cloud in the absence of a radiation field will be negligible. However, the observations on interstellar hydrogen show that it is almost all in the atomic form. Therefore photo-dissociation by the interstellar radiation field must be very rapid with a slow recombination process in order to maintain the hydrogen in the atomic form. The recombination

mechanism considered in this chapter is such a slow process.

We can consider the equilibrium of atomic and molecular hydrogen by supposing that molecular hydrogen is formed only by the mechanism discussed in this chapter and is removed only by photo-dissociation. The rate of photo-dissociation has been given by Kahn to be 1.1. 10^{-14} sec^{-1} using Dunham's results (1939) on the interstellar radiation field or using the more recent results of Lambrecht (1955) the rate of photo-dissociation is 3.3. 10^{-14} sec^{-1}. Equating the rates of formation and removal,

$$5.9.10^{-17} n_0 n_1 = \begin{cases} 1.1. \\ 3.3. \end{cases} 10^{-14} n_2 \qquad (52)$$

using our previous notation. Taking n_0 to be 10 particles cm^{-3} the molecular hydrogen has a density of 0.48 molecules cm^{-3} using Dunham's results or 0.17 molecules cm^{-3} using Lambrecht's results.

Using the results obtained in part (a) of this section, we can evaluate $\tau(H)$, $\tau(S.T.)$ assuming a constant radiation field at all densities. The results are given in Table VI.

Table VI

n_o	$\tau(H)$ (sec.)	$\tau(S.T.)$ (sec.)
1	$8.5.\ 10^{15}$	$3.0.\ 10^{13}$
10	$8.5.\ 10^{14}$	$2.9.\ 10^{13}$
10^2	$8.5.\ 10^{13}$	$2.2.\ 10^{13}$
10^3	$8.5.\ 10^{12}$	$6.6.\ 10^{12}$
10^4	$8.5.\ 10^{11}$	$8.2.\ 10^{11}$

From the table it is clear that as the density increases the rate of conversion of hydrogen from the atomic to the molecular form increases. At a density of 10^4 H atoms cm^{-3} $\tau(H) \simeq 3.10^4$ yr. which is a short time compared with that of star formation.

The condition given in equation (42) for a steady state may also be expressed as

$$\frac{n_1}{n_o} = \frac{\alpha}{\alpha + 2\beta n_o} \quad \text{or} \quad \frac{n_2}{n_o} = \frac{\beta n_o}{\alpha + 2\beta n_o} \qquad (53)$$

so that we can estimate the fraction of hydrogen in the form of free atoms and molecules. From the form of the second expression in equation (53) it is clear that $n_2 \to \frac{1}{2} n_o$ as n_o becomes large. The values of n, n_2 have been computed for various values of n_o for various temperatures assuming the radiation field

which was discussed by Lambrecht (1955). The results are given in Table VII.

Table VII

	n_o	1		10		10^2		10^3		∞
T		n_1	n_2	n_1	n_2	n_1	n_2	n_1	n_2	$\alpha/2\beta$
50		0.998	0.001	9.751	0.124	79.8	10.1	282	359	393
100		0.996	0.002	9.65	0.173	73.6	13.2	218	391	280
500		0.992	0.004	9.27	0.365	56.0	22.0	112	444	127

The table shows that if the density becomes very large then whatever the density the number of free atoms can never exceed the limiting value $\alpha/2\beta$ since the remainder of the hydrogen will be in the molecular form. This value $\alpha/2\beta$ is a maximum value too in view of the fact that the radiation field has been assumed constant at all densities. This however will not be true and as the density increases the value of α will decrease for the interior of the cloud. The value of n_1 for $n_o = \infty$ therefore represents the maximum number of free hydrogen atoms. It is interesting to note that observations have never demanded that the hydrogen atom density should exceed 100 atoms cm^{-3}. The present work suggests that the formation of molecular hydrogen tends

to take place in order to maintain this limit. If this suggestion is valid regions of high density in the interstellar material may occur, but their presence will not be revealed by the measurement of the hydrogen atom densities. However, without further investigation of all the relevant factors this explanation must remain tentative and of application only to those regions in which grains are also present.

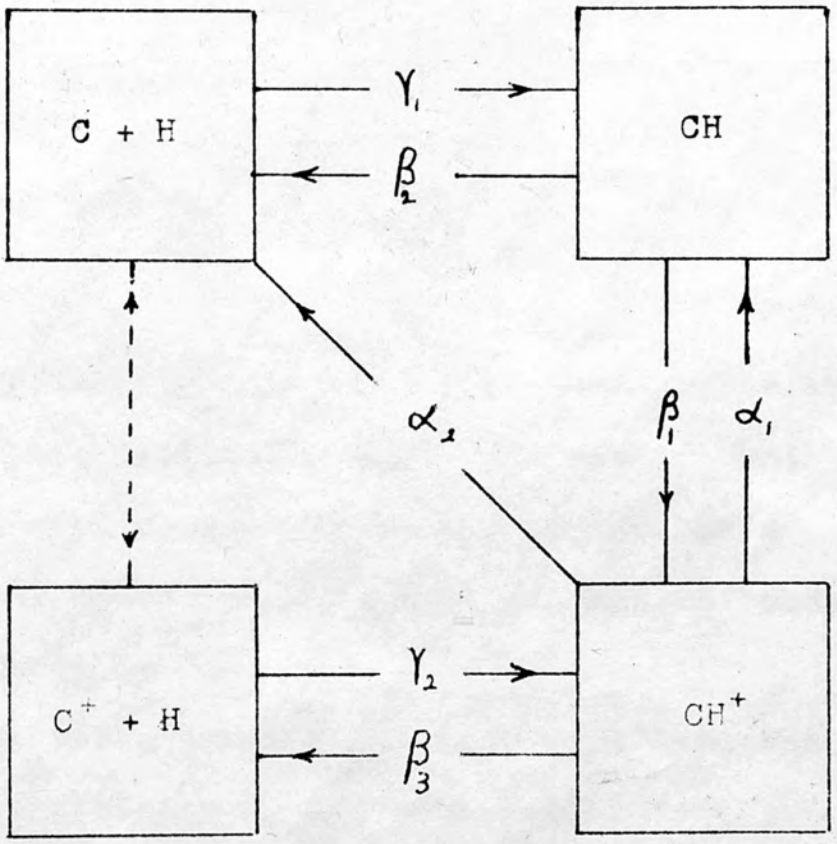
(c) The equilibrium of CH, CH^+.

The reactions listed in Table II may be used to determine the equilibrium of CH, CH^+ except that we take $\gamma_1 = \gamma_2 = 3.4. 10^{-17}$ cm.^{3} sec.$^{-1}$. We also adopt the same radiation field as Bates and Spitzer. The equilibrium of CH, CH^+ is shown diagrammatically in Figure 2. Retaining γ_1, γ_2 separately for the present we can write the equilibrium equations as

$$CH: \quad \gamma_1 n(C) n(H) + \alpha_1 n(CH^+) n(e) = (\beta_1 + \beta_2) n(CH), \quad (54)$$

$$CH^+: \quad \gamma_2 n(C^+) n(H) + \beta_1 n(CH) = (\alpha_1 + \alpha_2) n(CH^+) n(e) + \beta_3 n(CH^+) \quad (55)$$

where $n(e)$ denotes the electron number density and using the notation of Table II. Solving this pair of



◄ – – ► Equilibrium of C, C^+.

Figure 2.

equations for $n(CH)$ and $n(CH^+)$ we obtain

$$n(CH) = n(H)\frac{\gamma_1\{(\alpha_1+\alpha_2)n(e)+\beta_3\}n(C)+\gamma_2\alpha_1 n(C^+)}{\beta_3(\beta_1+\beta_2)+\{\beta_1\alpha_2+\beta_2(\alpha_1+\alpha_2)\}n(e)}, \quad (56)$$

$$n(CH^+) = n(H)\frac{\gamma_1\beta_1 n(C)+\gamma_2(\beta_1+\beta_2)n(C^+)}{\beta_3(\beta_1+\beta_2)+\{\beta_1\alpha_2+\beta_2(\alpha_1+\alpha_2)\}n(e)}. \quad (57)$$

Bates and Spitzer (1951) found that they had to examine four cases (see Section 1) since the coefficient α_2 is unknown. The results of Bates and Spitzer's evaluation of equations (56) and (57) have already been presented in Table III.

The situation has not been much improved by the new determination of the rate coefficient since the new values are only increased by a factor in the range 5 - 15. It is possible that most of the formation of CH, CH^+ may take place under conditions of cloud collision so that a further gain of a factor of 80 may be obtained. There is also uncertainty as previously mentioned about the oscillator strength of the line measured experimentally. This could give a further gain by a factor of 30. If all these factors

worked together embarrassingly large amounts of the molecules could be formed.

Accordingly no attempt was made to reinvestigate the equilibrium of these molecules further beyond noting that it would be possible under conditions of cloud collision to enhance the rate of molecule formation by a factor in excess of that required by Bates and Spitzer to explain the observed amounts of CH, CH^+. Some modifications of the equilibrium will be suggested in the next section.

The equilibrium of NH, OH is not discussed since there is no information on the relevant reactions involved in their equilibrium.

Section 6. Discussion of the Results.

A. The results for CH, CH^+.

Bates and Spitzer (1951) showed that radiative association failed to account for the observed values by a factor in the range 500 to 2000 and the present results only give an increase by a factor of 5 to 15 in the rate coefficients. Consequently, the formation of CH and CH^+ at grain surfaces, while proceeding more rapidly than radiative association, does not proceed with sufficient rapidity in order to explain the observed amounts of CH, CH^+.

However, if collisions between clouds do occur, then under the conditions of increased density and temperature existing in the collision region a gain of about a factor of 80 may be obtained. Therefore under these conditions it would be possible to explain the observations. Under these conditions too the results of Bates and Spitzer would also account for the observations provided the experimentally determined oscillator strength was used. The use of the experimental oscillator strength in the results obtained in this chapter would lead to undesirably large amounts of the molecules formed.

The decay of CH, CH^+ takes place very rapidly with a decay time which is of the order of a few thousand years (using the rate coefficients listed in Table II). This means that any reaction which gives rise to these molecules must take place in situ. The decay time is too rapid to support any hypothesis which relies on the injection of molecules into the interstellar clouds from some region in which conditions are particularly favourable for molecule formation, e.g. regions of high density.

The present work offers a possible solution to this problem but if divorced from the possibility of cloud collisions the present theory of molecule formation is only a slight improvement on radiative association.

In our work we have assumed that carbon ions and carbon atoms exist in the interstellar gas, the balance between them being maintained by the radiation field. The ions or atoms on striking the grain are allowed only to form a hydride molecule. If the grain were charged it would have an effect on the carbon ions in two ways. The charge on the grain would alter the collision cross section for C^+. If the grain were positively charged the positive ions would be repelled so reducing the cross section for C^+ ions

whereas if the grain were negatively charged the C^+ ions would be attracted to the grain so increasing the effective cross section. The magnitude of the change in the cross section is uncertain since the amount of charge residing on the grain is not known with certainty as this depends on the structure of the surface of the grain. Spitzer and Savedoff (1950) have shown that unless the grain has a surface equivalent to a good photocell the charge is likely to be negative in regions not in the vicinity of early type stars. The degree of charging found by Spitzer and Savedoff is small being of the order of 0.01 volts at normal cloud densities. Van de Hulst (1949) gives the degree of charging as an order of magnitude larger but in the same sense i.e. negative.

The second effect is rather more subtle. The ions on striking the charged grain may neutralise before molecule formation can take place. This would result in the formation of CH only at the grain surface and any CH^+ which is formed will be due to the photo-ionisation of the CH molecule. This would then give an equilibrium situation of types (a), (b) considered by Bates and Spitzer. Since 1951 evidence has been accumulating (see for example Massey and Burhop, 1956) which shows that α_2 is very large and its value may

lie in the range 10^{-9} to 10^{-7}. Seaton (private communication) thinks that the larger value would be the more appropriate.

Since the effect of ions in neutralising the negative charge on the grains has been considered when assessing the resultant charge, it seems unlikely that the first effect will be very significant, but the second effect is likely to be very important. Therefore the equilibrium of CH, CH^+ has been recalculated assuming that CH only is produced at the grain surface and CH^+ results from the photo-ionisation of the CH. The value of $n(CH)/n(H)$ is 5.10^{-10} cm^{-3} which is again too small but would not be very far wrong if the experimental oscillator strength were adopted. However, the value of $n(CH^+)/n(H)$ is 7.10^{-12} cm^{-3} which is much too small. Furthermore the ratio of $n(CH)/n(CH^+)$ is also in disagreement with the observed value. In order to give this ratio the correct value α_2 would have to be 3.10^{-9} cm^3 sec^{-1} but even with this value the predicted values of $n(CH)/n(H)$ and $n(CH^+)/n(H)$ are still too small.

We shall not however pursue this topic further since the radiation field would also need review if a thorough investigation of the equilibrium was required. However, we are not here principally concerned with the

formation of interstellar CH and CH^+. We have considered them only because of the check that they offer when discussing the formation of H_2. The results show that since we have not predicted an excess of CH and CH^+ then the rate coefficient obtained for the formation of H_2 is probably not overestimated.

B. The results for H_2.

In order to examine the results we have obtained we shall compute the cooling produced in a cloud containing 10 H. atoms cm^{-3} and 0.02 H_2 molecules cm^{-3}. Spitzer (1949) has worked out the rate of cooling produced by molecular hydrogen assuming that the rotational levels of the ground vibrational state only are excited. Radiation is emitted by quadrupole transitions between the rotational levels. Spitzer examined two limiting types of emission when (a) the population of the rotational levels is given by a Maxwell-Boltzmann distribution function, (b) collisional de-excitation is negligible. We let E_1 denote the energy (in erg. sec^{-1}) emitted in case (a) and E_2 denote the energy (in erg. cm^3 sec^{-1}) emitted in case (b). Since the cases considered are limiting cases, Spitzer

assumes that the harmonic mean of these two rates gives an accurate representation of what is actually happening.

Recently the rate of cooling by molecular hydrogen has been reviewed by Seaton (1958). Using new cross sections determined by Takayanagi (1957) for the excitation of the vibrational levels of H_2, Seaton shows that if atomic hydrogen is very much in excess of molecular hydrogen the rate of cooling L must be written

$$L = \frac{10\,n(H)\,n(H_2)\,E_1 E_2}{10\,n(H)E_2 + E_1} \quad \text{erg. cm}^{-3}\,\text{sec}^{-1} \qquad (58)$$

However, if molecular hydrogen is the more abundant then the result obtained by Spitzer applies, namely

$$L = \frac{n^2(H_2)\,E_1 E_2}{n(H_2)E_2 + E_1} \quad \text{erg. cm}^{-3}\,\text{sec}^{-1} \qquad (59)$$

Using the tabulated results for E_1 and E_2 we can compute the variation of L (using equation (58)) with temperature for $n(H) = 10$, $n(H_2) = 0.02$. The results are given in Table VIII.

Table VIII

$\frac{5040}{T}$	E_1	E_2	L
1.0	$2.08.10^{-19}$	$1.7.10^{-23}$	$3.37.10^{-23}$
5.0	$8.52.10^{-22}$	$1.2.10^{-24}$	$2.10.10^{-24}$
10.0	$6.30.10^{-23}$	$2.6.10^{-25}$	$3.68.10^{-25}$
25.0	$1.02.10^{-24}$	$2.2.10^{-26}$	$1.39.10^{-26}$
50.0	$5.48.10^{-26}$	$1.5.10^{-27}$	$8.02.10^{-28}$
100.0	$3.35.10^{-28}$	$7.0.10^{-30}$	$4.51.10^{-30}$

Making use of the rates of cooling found in Table VIII we can estimate the rate of energy loss per hydrogen atom and compare this with the rate of energy gain per hydrogen atom. Spitzer (1954) supposed that the chief source of energy gain came from the cosmic rays and he estimated that the energy gain was 4.10^{-30} erg.sec.$^{-1}$ per H. atom. Curves are drawn for the gain and loss of energy in Figure 3. From the figure it is clear that the rate of energy loss equals the rate of energy gain at about 64 ^{o}K.

This calculation however assumes only one source of energy and one type of energy loss. Seaton (1955) and Kahn (1955) have considered the source of

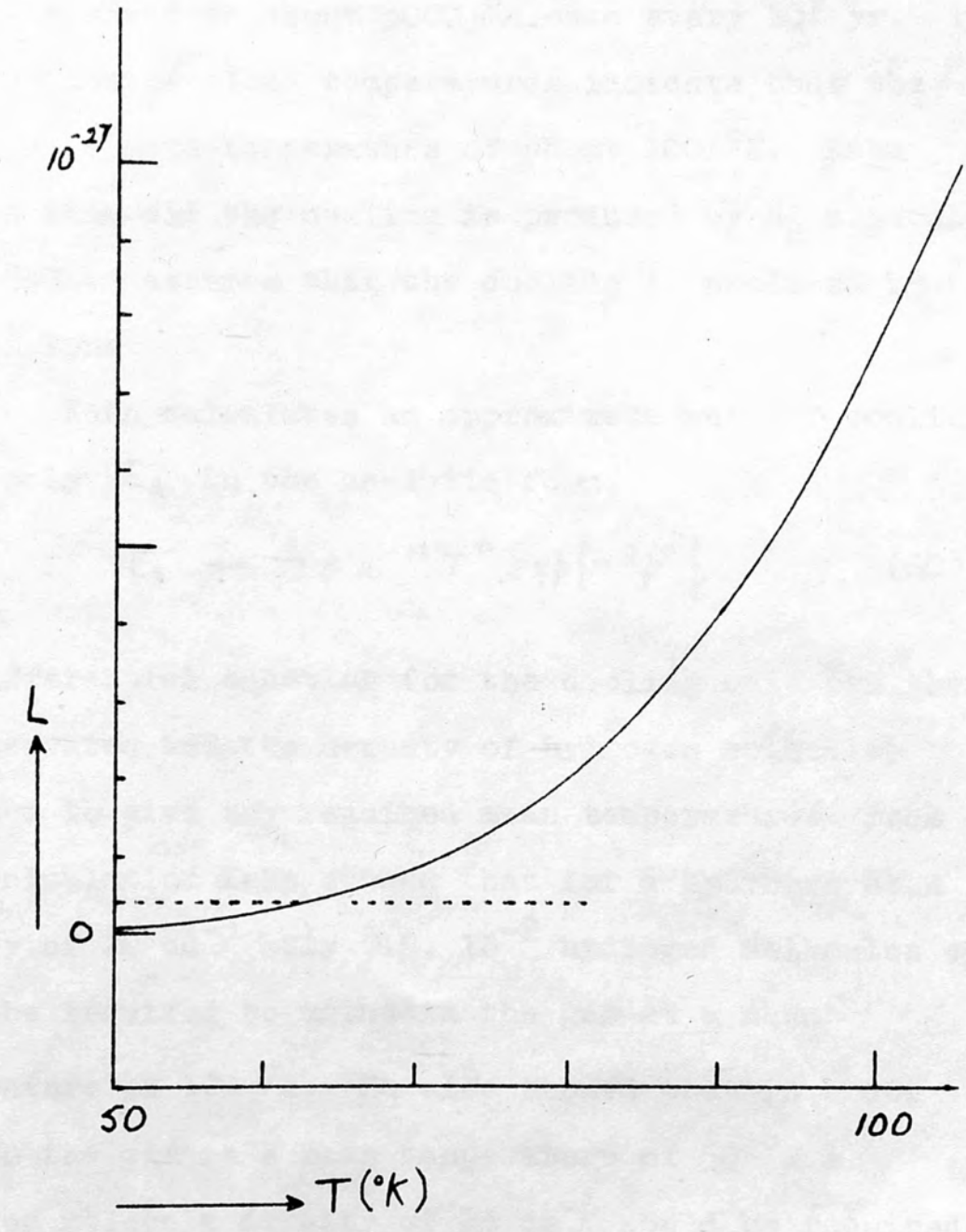


Figure 3.

cloud heating to be principally due to collisions between the clouds. They estimate that cloud collisions heat any given cloud to about 3000 ^{o}K. once every 10^{7} yr. The measurements of cloud temperatures indicate that the cloud has a mean temperature of about 100 ^{o}K. Kahn assumes that all the cooling is produced by H_2 molecules, while Seaton assumes that the cooling is produced by C^{+}, Si^{+} ions.

Kahn calculates an approximate rate of cooling using only E_2 in the analytic form,

$$E_2 = 2{\cdot}5.10^{-26} T^{1/2} \exp\left\{-\frac{520}{T}\right\} . \qquad (60)$$

The differential equation for the cooling rate can then be integrated and the density of hydrogen molecules computed to give any required mean temperature. From this calculation Kahn showed that for a hydrogen atom density of 20 cm^{-3} only 7.5. 10^{-2} hydrogen molecules cm^{-3} would be required to maintain the gas at a mean temperature of 100 ^{o}K. He also showed that in order to keep the gas at a mean temperature of 50 ^{o}K. a hydrogen molecule density of 23 cm^{-3} would be required.

Seaton showed that cooling by Si^{+}, C^{+} ions only could also give a mean temperature of 100 ^{o}K. However, we shall not consider this mechanism further.

The results obtained in Section 5 show that

more H_2 is present than is required to maintain the gas at 100 ^{o}K, but Kahn's calculations show that the degree of cooling produced will not lower the temperature much below 100 ^{o}K. Kahn, however, has used the unrevised result due to Spitzer and in consequence these values would need to be recomputed. However, an analytic expression cannot now be found for the rate of cooling.

The principal value of having H_2 present is that it provides an effective coolant for the interstellar gas. Atomic hydrogen is of no value as a coolant except under conditions in which an appreciable fraction of the hydrogen is ionised, i.e. at kinetic temperatures in excess of 10^4 ^{o}K. Molecular hydrogen is effective in keeping the temperature in the range 50^{o} - 100 ^{o}K, but below 50 ^{o}K. molecular hydrogen is also a poor radiator.

In the process of star formation the interstellar gas is compressed and we have shown that as the gas density increases the process of molecule formation takes place more and more rapidly until as has been shown in Table VII most of the hydrogen is in the molecular form. The radiation emitted by the molecules keeps the gas cool during the compression. The time required to cool the molecular gas from 5000 ^{o}K. will be

in the range 10^6 to 10^7 yr. This time is of the order of the time required for star formation.

Unfortunately molecular hydrogen cannot be observed experimentally with the techniques currently available. Regions of great density in interstellar space, as has been shown by this work, will not reveal their presence through observation of the atomic hydrogen. Consequently, we must ask if any other effects are to be expected that would enable us to detect these regions.

One method of detection would be the observation of other molecular lines in regions of high density. However, a high density region is likely to be of small extent and it is not surprising that no high density regions have been detected so far in this way. Furthermore the conditions of high density may introduce additional factors which would mask any means of detection, e.g. light from stars shining through the cloud may be so attenuated that any absorption lines would be undetectable.

However, the crucial test of this theory of H_2 formation is whether a layer of hydrogen atoms can form at the surface of a grain. Not only is this problem related to the properties of surfaces at very low temperatures, but it is related to the problem of

the formation of grains. Hydrogen is not thought to form a large part of the grains, but our theory demands that it forms a surface layer. Therefore a theory of grain formation would be required in which the grain could grow without taking up the surface layer except in so far as stable molecules of the type CH_4, NH_3, H_2O were concerned.

However, the theory as given here, although it predicts rather more hydrogen molecules than absolutely necessary to maintain the interstellar gas at 100 ^{o}K, does not predict such an excess of molecules that the temperature would be reduced much below 100 ^{o}K. Therefore problems in stellar structure should now consider the condensation of clouds of molecular hydrogen rather than atomic hydrogen.

C. Other Molecules.

In a table given by Dufay (1957) the only molecules which have been identified in interstellar space are CH^+, CH, CN. Molecules such as OH, NH whose rate coefficients have been predicted by this theory have not been observed. Due to this lack of observational material on OH, NH they cannot be discussed further with profit, but the case of CN is somewhat anomalous. If CN is formed at grain surfaces

then its reaction rate must be about 10^4 times faster than the rates calculated here since carbon and nitrogen have such a low abundance. Radiative association would be unlikely to give a rate coefficient larger than 10^{-17} cm.^{3} sec.$^{-1}$ so that yet another mechanism must be looked for to explain the production of CN. Two mechanisms are possible, namely the erosion of carbon and nitrogen from the grains and chemical exchange. The erosion mechanism can be eliminated since it would not proceed any faster than a surface reaction. A chemical exchange reaction could be of the type

$$CH + N \rightarrow CN + H$$

Assuming that such a reaction is possible and that every encounter of a CH molecule with a nitrogen atom leads to the formation of CN, then a rate coefficient of about 10^{-12} cm.^{3} sec.$^{-1}$ would be obtained at 100 ^{o}K. The number of molecules formed would be several orders of magnitude less than the number of CH molecules formed, but the removal processes may be less efficient. The mechanism would not constitute a serious loss of CH.

However, in view of the uncertainty involved here (the dissociation energy for CN is uncertain) and since this reaction is of a different type to that discussed in this chapter, calculations on the formation of CN have not been taken further.

Conclusion.

The mechanism whereby interstellar molecules are formed at the surfaces of interstellar grains, as outlined in this chapter, has shown that molecular hydrogen could be present in normal clouds in sufficient density to maintain the cloud at a kinetic temperature of 100 ^{o}K. The control calculation on the formation of CH shows that the rate of formation has not been overestimated, since the rate coefficient for CH production cannot account for the observed amounts of CH, unless it is assumed that CH is formed under conditions of cloud collision.

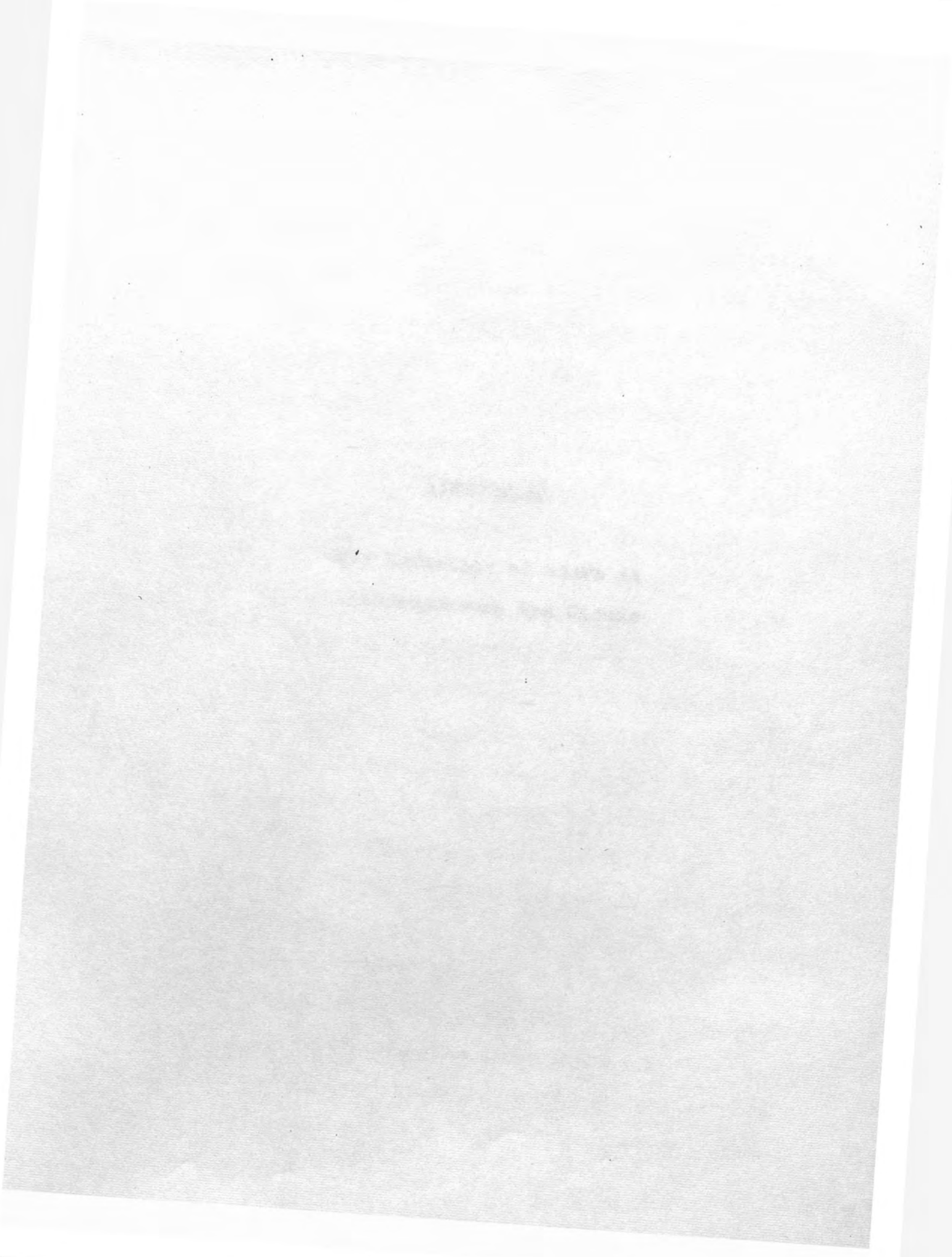
The production of H_2 has been shown to be very fast at densities in excess of normal cloud densities and if this type of reaction is at all possible, then the process of star formation must start by considering the compression of a cloud composed largely of molecular hydrogen. Such a cloud would be able to cool by radiation due to the quadrupole transitions between the rotational levels of the ground vibrational state of the molecule. Such a cooling mechanism would not be available for a cloud composed of atomic hydrogen.

However, many major problems remain to be

solved. The nature of the forces holding the hydrogen atom to the surface of the grain at low temperatures have not been specified in this treatment. Before the mechanism could be firmly established the nature of these forces would have to be investigated.

The equilibrium of even the best documented interstellar molecules CH, CH^+ is uncertain since the cross sections for some of the reactions involving these molecules are unknown. The radiation field too will have to be more precisely defined and a new calculation of the equilibrium densities of these molecules, based on recent estimates of the radiation field, would now seem appropriate.

The present work has put on a more mathematical basis the suggestion made by several authors in the past, that grains may be the principal seat of the largest part of interstellar chemistry. The results which we have obtained do not indicate that this type of reaction leads to serious disagreement with the experimental results.



Chapter II

The Formation of Stars in Inhomogeneous Gas Clouds.

Introduction

Most theories of star formation assume that the star condenses from a gas cloud whose density is uniform or varies only slowly. It is usually assumed that the cloud will contract when it has acquired sufficient mass for its own self gravitation to control any dissipative effects (e.g. the random thermal motion of the gas atoms). There are several objections to this type of process. In general, the interstellar gas clouds are not uniform but are in a chaotic state i.e. there are large scale variations of density within the cloud. Further, the criterion determining the onset of gravitational instability requires a cloud whose dimensions are larger than an average interstellar cloud. An average cloud contains about 400 solar masses so even if this cloud could contract on the large scale the problem of the fragmentation of small masses (of the order of a solar mass) still remains.

In the present chapter we consider gas clouds in which there are large scale departures from uniformity. These departures are idealised into small clouds called floccules which are able to move about independently (to a first approximation) within

the framework of the large cloud.

The collisions of the floccules are of interest since we suppose that these collisions are ultimately responsible for the production of stars. The collisions of the floccules can be approached from two different points of view. The first point of view still regards gravitational forces as being of primary importance. The collisions between the floccules serve to reduce their velocity to such a value that they may be captured gravitationally by a set of already bound floccules. In this way a set of bound floccules will be obtained and these may come together under their own self gravitation to form a star. However, it will be later shown that this process is itself unlikely to be responsible for star formation though it may act as a first step to the second approach.

The second approach has been developed in some detail by McCrea who considers the formation of an incipient condensation, say by the fortuitous collision of several floccules, to form a floccule of larger than average mass. Further floccules collide with this incipient condensation and adhere to it. The incipient condensation grows until it can pull

itself together by its own self gravitation. Further infall of floccules takes place by chance encounter, but as the condensation grows its gravitational attraction becomes more efficient in accreting floccules.

In this chapter we first discuss previous condensation theories and then discuss the ideas underlying the concept of a floccule. A discussion of the two floccule theories is then given.

Section 1. Previous work on condensation processes.

Most theories of star formation assume that the star is formed from the interstellar material. Since the mean density of the interstellar material is 10^{-24} gm.cm.$^{-3}$, or 10^{-23} gm.cm.$^{-3}$ in clouds and the mean density of the sun is 1.4 gm.cm.$^{-3}$ (Allen 1955), a compression of 10^{23} or 10^{24} must be obtained in order to convert the interstellar material into a star. Some of the past attempts to find such a compression by the uniform contraction of a uniform gas cloud will be briefly reviewed in this section.

Condensation by gravitational contraction.

A gas cloud if it is sufficiently massive may be able to contract under its own self gravitation. As infall proceeded the gravitational forces would always control the motion provided the random motions inside the cloud did not become of the order of the escape velocity. Consequently this type of process could give compression in a single stage. This problem was investigated by Jeans (1928) who showed that a cloud would become unstable if its linear dimensions

denoted by λ were given by

$$\lambda > \left(\frac{\pi}{G\rho_0}\frac{dp}{d\rho}\right)^{1/2} \qquad (1)$$

where ρ_0 is the uniform density, p is the pressure and G is the gravitational constant. Jeans derived this result, known as the Jean's Criterion, from a hydrodynamical argument assuming an infinite gas cloud of uniform density. He showed that if plane sound waves of wavelength λ were propagated in one direction (say along the x- axis), those sound waves whose wavelengths satisfied the inequality (1) had an amplitude which would increase exponentially with time. He therefore assumed that any region of linear dimensions of the order of λ would be unstable gravitationally against the propagation of small disturbances through the gas. The derivation of the inequality (1) has been given in vector form by Bonnor (1957) who re-discusses the Jean's criterion and its application. Since the treatment of Jeans and Bonnor becomes essentially one dimensional when the propagation of plane waves is introduced and since a uniform cloud of infinite extent is examined, the interpretation of λ is somewhat doubtful.

The problem of the instability of a finite mass of gas is more usefully approached by an

application of the Virial Theorem (e.g. see Chandrasekhar 1957). In order that the gas cloud should be unstable, the virial theorem shows that the gravitational energy of the cloud must exceed twice the thermal energy of the material contained within the cloud. For a cloud whose radius is λ and whose uniform density is ρ_0 the instability condition derived from the virial theorem may be expressed as

$$\lambda > \left(\frac{\pi}{G\rho_0}\frac{RT}{\mu}\right)^{1/2} \qquad (2)$$

where R is the gas constant, T is the kinetic temperature of the cloud material and μ is its mean molecular weight. The virial theorem approach is rather more flexible than the hydrodynamical approach since the effects of magnetic fields, rotation and external pressure may be included in it (e.g. see Chandrasekhar and Fermi 1953, Bel and Schatzman 1958, and McCrea 1957).

In order to find out the minimum value of λ for which instability would be possible, typical interstellar values for ρ_0 and T were inserted in inequality (2). The value found for λ is approximately 10^{20} cm. However, an average interstellar cloud has a diameter of 5.10^{19} cm. so that an average cloud is just stable against gravitational

contraction. Further an average cloud contains about 400 solar masses so that only massive larger than average clouds could contract under their own self gravitation. Even if such a cloud did contract, the problem of how further sub-condensations of about solar dimensions could form still remains.

Hoyle (1953) has proposed a heirarchical structure for the process of galaxy and star formation by gravitational contraction. Hoyle assumes that initially a mass of gas, of the order of a galaxy mass, can satisfy the inequality (2) and so start to condense. As the material contracts its density increases and there will ultimately come a stage when the Jeans criterion will apply to regions of the proto-galaxy whose radii are smaller than that of the proto-galaxy. These regions contract and the Jeans criterion can be applied to sub-regions and so on till massesof stellar size have been formed. Hoyle showed that the time scale for such a process is not very much longer than the time required for the initial condensation. Layzer (1958) has pointed out that such a separation will not occur since the main body of the cloud will be contracting faster than the fragment. The heirarchical model must therefore be regarded as suspect.

The gravitational instability of rotating gas clouds.

In order to retain the heirarchical model to some degree, it may be supposed that the gas cloud is initially rotating, so that at some stage of the contraction the rotational and gravitational forces may balance. This would remove the objection raised by Layzer and allow further sub-condensations to form.

The introduction of rotation, as has been shown by Bel and Schatzman (1958), increases the diameter of the cloud which is just unstable gravitationally. Therefore rotation is at once aggravating the problem of the size of the cloud which must initially contract. However, a simple approach to this problem is not possible since an axis of rotation must be defined for the cloud. The problem may then be idealised to that of a rotating sphere of compressible fluid. This type of problem has been treated by Jeans (1928) who showed that the further evolution of the cloud would be as follows. As the sphere contracts under its own self-gravitation, the angular velocity will increase and the sphere will become deformed into an ellipsoid after passing through various spheroidal configurations. As the angular velocity increases, the equatorial velocities of

particles in the outer regions of the cloud will become larger than the escape velocity and material will stream away at the equatorial regions. No equilibrium configuration will be attained. A star will only be formed when the cloud has lost most of its material, so this process is very inefficient in its use of the interstellar material. Stars may form in the material ejected from the equatorial regions, but since the rate at which material is ejected cannot be determined we can derive no useful information from this line of attack.

The most damaging objection to the introduction of rotation is the very large equatorial angular velocity which a star formed in this way would have. Assuming that the cloud originally has a rotation of the same magnitude as the galactic rotation, then if angular momentum is conserved at all stages of the process, a star of solar dimensions will have a final equatorial velocity greater than the speed of light. If the cloud of diameter 5.10^{19} cm. has an initial angular velocity of 10^{-15} sec^{-1}, the equatorial velocity of a star whose radius is one solar radius, is 9.10^{12} $cm.sec^{-1}$. Such equatorial velocities are clearly absurd.

Therefore, if a rotational theory is to be

invoked, some additional mechanism must be introduced to dissipate the rotational energy. Various mechanisms have been proposed which will dissipate angular momentum. The angular momentum may be removed through the interaction of the surrounding interstellar gas with the star. This type of mechanism has been considered by von Weizsäcker (1947) and ter Haar (1949). This mechanism depends on the formation of stars surrounded by a large rotating envelope. The envelope acts on the central mass which will lose angular momentum to it. This angular momentum will be transported away as the envelope disperses. However ter Haar estimates that only 10^{-4} of the initial angular momentum can be removed in this way.

Magnetic braking has also been investigated by several authors (e.g. Lüst and Schlüter, 1955). A star is, in general, surrounded by an ionised region (i.e. its Strömgren sphere). If the star possesses a magnetic field, it will be magnetically coupled, through the ionised region, to the interstellar gas. The braking is produced by the coupling. The rate at which angular momentum is lost depends on the field assumed for the star and on the size of the ionised regions. Lüst and Schlüter have shown that if a star has insufficient angular momentum to produce

rotational instability, then a field of 100 gauss will remove this angular momentum in 10^6 yr. However, the star must lose a large amount of angular momentum before it can become rotationally stable. This does not find an explanation in terms of this theory since in the initial stages of the condensation the amount of ionisation produced will be small and so the coupling will be weak.

The theories based on the condensation of a uniform gas cloud under its own self gravitation fail in two ways. The Jeans criterion requires a large mass to contract and in order to obtain the condensation of masses of solar dimensions, additional hypotheses must be made. If rotation is introduced a difficulty arises in that the resulting star would have excess angular momentum. Mechanisms which dissipate this excess angular momentum, must be introduced. The fundamental assumption of all these theories is that the gas cloud contracts as a whole and is approximately uniform. However, observation shows that a gas cloud is non uniform and the consequences of supposing large inhomogeneities to exist will be examined in the following sections.

Section 2. The properties of floccules

In this section we shall discuss how floccules may be formed and how they interact with one another. The problem of defining a mean free path for floccule interactions will also be considered.

The formation of floccules.

Before the concept of floccules becomes of any value some compression of the interstellar gas will be required. The mechanism which causes this pressure may be considered to be similar to that proposed by Oort and Spitzer (1955) and Biermann and Schlüter (1954). This mechanism presupposes the existence of a hot O type star within a complex of interstellar clouds. This O type star is assumed to be suddenly born in the cloud complex since this leads to a simple model. The new star will ionise the hydrogen in its immediate vicinity. An ionisation front will proceed outwards from the star and will ionise the region within the radius of the Strömgren sphere for the star. The temperature inside the Strömgren sphere will be increased to about 10^4 °K. compared with 10^2 °K. for the temperature of the material outside this sphere.

Consequently the hot gas will tend to expand into the cold gas. A compressed region will form between the hot and cold gas masses and if this region can maintain itself at nearly the temperature of the cold gas, its density will be increased by a factor of 10^2. The acceleration of the compression front will cease due to the braking action of the surrounding interstellar gas.

This mechanism was developed primarily to explain the motions of the interstellar clouds, but we may adapt it for our purposes. Suppose the cloud complex contained several O stars each having a compression front. Even if the cloud complex initially has a uniform density the passage of the compression fronts through the gas would tend to break it up into a rather lumpy structure. These lumps are idealised into discrete floccules. Each floccule moves about independently of the others and the floccules move in a rather tenuous gas continuum, which serves as a background defining the parent cloud.

The above type of mechanism for floccule formation limits their usefulness to those regions in which pressure disturbances are moving randomly through the interstellar gas. Floccules do not, however, depend on the Oort-Spitzer, Biermann-Schlüter mechanism

for producing pressure variations, but they do require some mechanism which produces such variations. The problem of the formation of floccules needs more elucidation to see if some less drastic mechanism could produce them.

The permanence of floccules.

If the floccules are to have permanence, they must travel with speeds which are in excess of the thermal speeds of the gas atoms. This ensures that the density increase represented by a floccule is not rapidly smoothed out by the random motions of its constituent atoms and molecules. The floccules therefore move with supersonic speeds. The floccules must however not move so fast that they disperse the cloud which they form.

If the mean temperature of the interstellar gas is 100 ^{o}K. then the speed of sound in the gas is 1 km.sec^{-1}. The mean random velocity of an entire cloud in the line of sight is 7 km.sec^{-1} (Allen 1955). Therefore the floccules must have a velocity greater than 1 km.sec^{-1} and less than 7 km.sec^{-1}. An interstellar cloud collides on an average once in 10^{7} yr. so that the mean life of a cloud is of this

order. In order to give the cloud a lifetime of 10^7 yr. before its dispersal, a floccule velocity of about 2 km.sec.$^{-1}$ would be required. We shall therefore assume that the floccules move about with a mean velocity of 2 km.sec.$^{-1}$.

The interaction of floccules.

The interaction of floccules is described in different terms depending on the situation in which we are interested. In the models of star formation which we shall discuss later, it is convenient at some stages to regard the floccules as non-interacting gravitationally, while at other stages the gravitational interaction of the floccules is of primary importance.

In a cloud in which the floccules are moving about freely with a random velocity of 2 km.sec.$^{-1}$ we assume that the floccules do not interact gravitationally. The only way in which a floccule can become "aware" of the presence of another floccule is by colliding with it. This assumption may be justified if we consider the energy of the floccules compared with the energy of the gravitational field in which they are moving. The energy possessed by the field will be of the order of GM/R erg.gm.$^{-1}$ where G is the gravitational constant, M is the mass of a floccule and R is the mean separation of the floccules. Anticipating

a later result we may take $M = 9.10^{28}$ gm, $R = 10^{17}$ cm, the gravitational energy is 6.10^{5} erg. gm^{-1} or taking McCrea's values ($M = 2.10^{28}$gm., $R = 2.10^{13}$ cm.) the gravitational energy is $6.7.\ 10^{7}$ erg.gm^{-1}. The energy of random motion is 2.10^{10} erg.gm^{-1} so that the gravitational field is only at most about 0.33 percent of the random energy. Therefore the neglect of gravitational effects between floccules may be justified to a first approximation.

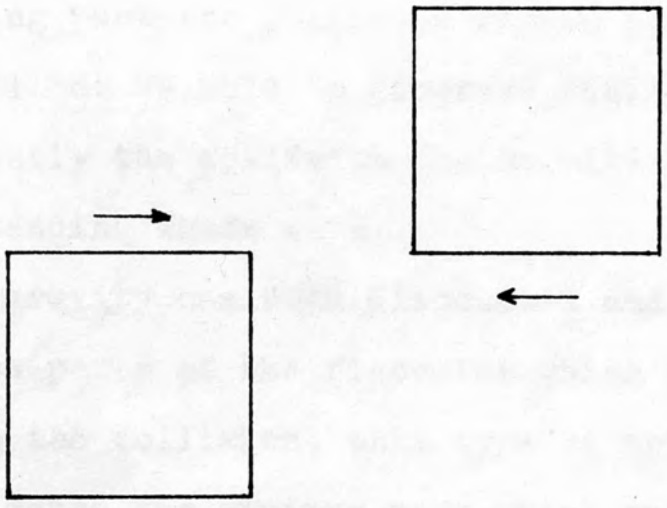
However, once collisions between the floccules have taken place the situation is altered. Some of the fragments of such a collision may have their velocities so reduced, that they cannot escape from some set of gravitationally bound floccules. Also some specially favoured floccule may grow so large that its gravitational effect may control the motions of the surrounding floccules. However, these effects are not operative during the early stages of any star forming process. Therefore we may regard floccules as not interacting gravitationally so long as they have not suffered a collision or come into the neighbourhood of a larger than average floccule. After a collision the question of whether any given fragment may be controlled by the gravitational field of another floccule or group of floccules must be re-investigated.

Collisions between floccules.

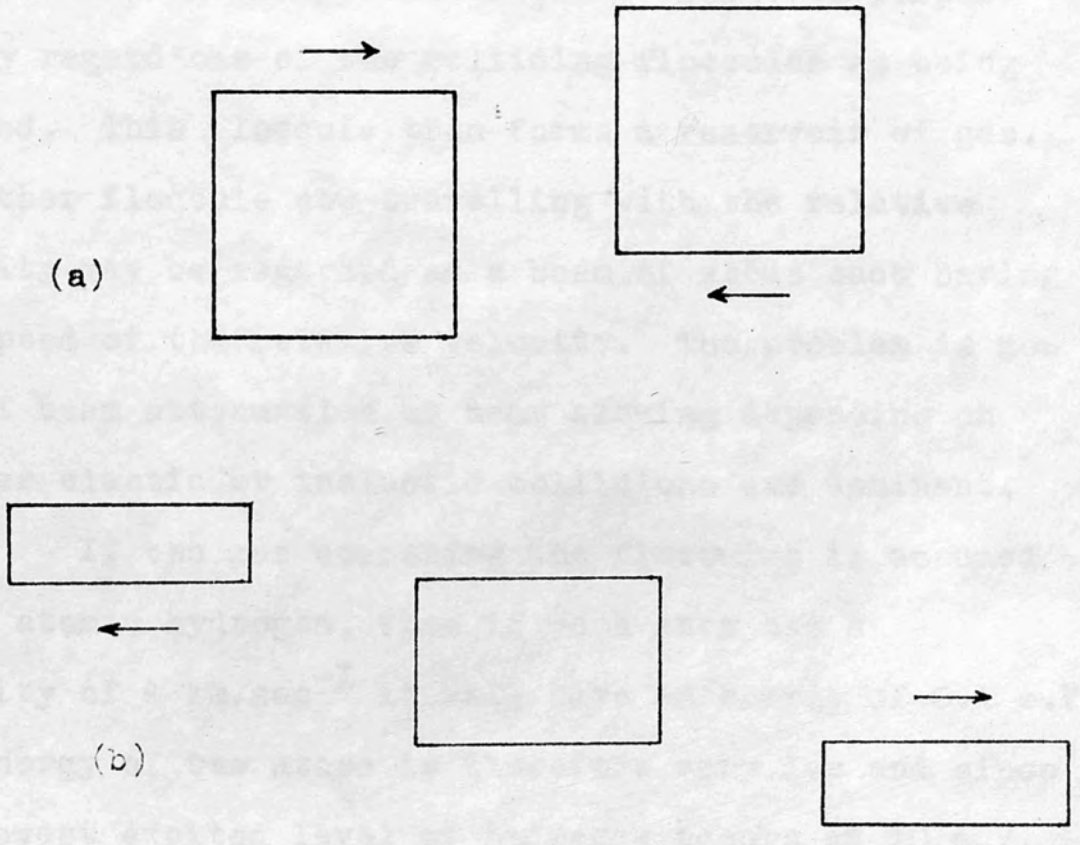
We adopt the following model to describe the collisions between floccules. If two floccules pass by one another and only graze, then no collision is assumed to have taken place. A collision between two floccules is assumed to concern only those regions of the floccules which physically "overlap". The parts of the floccules which do not overlap during the collision are assumed to be unaware that a collision has taken place and continue on their original paths. The model is represented diagrammatically in Figure 4. This model can only be an approximation to what actually happens, but its adoption can be justified on hydrodynamical and atomic grounds.

We consider now a collision of floccules which is not completely head on. In the region where the two floccules overlap shock waves will be generated since the floccules are travelling with supersonic speeds. Since gas will be pouring into the collision region the shock waves will tend to move out into the infalling gas. The situation described here is similar to that described in Section 4 of Chapter I.

Those regions of the floccules which do not actually overlap will not be aware of the collision



(i) no collision.



(ii) a collision - (a) before, (b) after.

Figure 4.

until reached by a shock wave. However, these regions will be streaming past the collision region and the shock waves will not be able to progress easily through them. Consequently the collision region will become bounded by a standing shock wave.

Since gravity has been discounted and since we only allow those parts of the floccules which overlap to take part in the collision, this type of collision model only estimates the minimum mass which can have its centre of mass velocity reduced by a collision.

It is perhaps more instructive to examine the model from an atomic point of view. For this purpose we may regard one of the colliding floccules as being stopped. This floccule then forms a reservoir of gas. The other floccule now travelling with the relative velocity may be regarded as a beam of atoms each having the speed of the relative velocity. The problem is now one of beam attenuation or beam slowing depending on whether elastic or inelastic collisions are dominant.

If the gas composing the floccules is assumed to be atomic hydrogen, then if each atom has a velocity of 4 km.sec^{-1} it will have an energy of 0.1 e.V. The energy of the atoms is therefore very low and since the lowest excited level of hydrogen occurs at 10 e.V. no inelastic collisions will take place. However, if

the floccules are composed of molecular hydrogen inelastic collisions will occur through the excitation of the rotational spectrum. We therefore examine the two cases to see if any substantial difference exists between them.

If atomic hydrogen is the principal atomic species present in the two floccules, the collisions between the atoms will be of the momentum transfer type. The cross sections for this type of collision have been recently computed by Dalgarno (1960) for atomic hydrogen in the energy range 0.1 to 100 e.V. The values found by Dalgarno are given in Table IX.

Table IX

Impact Energy (e.V.)	Cross section (πa_o^2)
0.1	47
1	37
10	29
10^2	22

The value of πa_o^2, the area of the first Bohr orbit, is $8.797. 10^{-17}$ cm^2.

Since large energy losses by momentum transfer are associated with scattering at large angles, an

incident beam will not appear to be slowed but attenuated, most of the particles being scattered out of the beam. Using these results we can determine the mean free path for this type of scattering. The mean free path ℓ is defined by

$$\ell = (nQ)^{-1}, \qquad (3)$$

where n is the density of particles in the scatterer and Q is the scattering cross section. Inserting the numerical values we can write down the mean free path as

$$\ell = 2.42.\ 10^{14}.\ n^{-1}\ \text{cm}. \qquad (4)$$

At normal cloud densities the mean free path will be of the order of $2.42.\ 10^{13}$ cm. which is small compared with the dimensions of the cloud. Under conditions of increased density, such as those envisaged in the floccule models to be discussed later, the mean free path will be very much smaller. If a compression of 5.10^{11} has been attained (i.e. the halfway stage in a complete compression of 10^{23}) the density of the gas will be 5.10^{12} particles cm^{-3} and the mean free path will be 48 cm.

Therefore since the mean free path for scattering is very much smaller than the dimensions of the floccules (which will never be less than 10^{12} cm.

at a compression of 5.10^{11}) the boundary region between the collision region and the undisturbed part of the floccule will be very small.

If molecules predominate inelastic collisions will occur as well as elastic collisions. If inelastic collisions are the most frequent type of encounter the beam will be slowed down, but will not appear to suffer much attenuation. Takayanagi (1957) has shown that if the gas is composed largely of hydrogen molecules and the temperature of the gas is less than 10^3 °K. the majority of the transitions excited by the collisions will be from the rotational level $J = 0$ to the level $J = 2$. Transitions from the level $J = 0$ to the level $J = 4$ will only be of importance when the temperature is in excess of 10^3 °K. The cross section for the transition $J = 0$ to $J = 4$ is estimated to be an order of magnitude smaller than the transition $J = 0$ to $J = 2$. At energies of 0.1 e.V. per molecule Takayanagi gives the cross section for the transition $J = 0$ to $J = 2$ to be $2.5.10^{-17}$ cm^2. Using this result in equation (3) the mean free path is

$$\ell = 4.10^{16} \cdot n^{-1} \text{ cm.} \qquad (5)$$

Again at all densities the mean free path is very much

smaller than the radius of the floccules. Therefore any molecule escaping from the collision region will not proceed far into the undisturbed parts of the floccule before it becomes slowed down.

Consequently the distinction between inelastic and elastic collisions is not very significant in these boundary regions, even though the cross sections differ by several orders of magnitude. The model we have proposed for floccule collisions is approximately correct, since the boundary layer between the collision region and the undisturbed cloud is of small extent compared with the size of a floccule. The model of floccule collisions, assuming that only those regions which physically overlap are involved in the collision, will be taken as a background when considering the collisions of average floccules.

The mean free path for floccules.

Since on collision the floccules coalesce the definition of a mean free path for floccule collision cannot be made by analogy with the kinetic theory of gases. Gas molecules may be treated more or less as rigid spheres, each atom or molecule preserving its overall identity during the collision. However, the same does not apply to the collision of floccules,

since part of the floccules coalesce on collision and part retains the original motion of the parent floccule, so that the identity of the floccules is not preserved.

Although a mean free path of the gas kinetic theory type cannot be defined for collisions between the floccules, it is usually possible to define a length which serves as a mean free path for most of the time involved in the process of star formation. In one model, which will be described later, the mean free path is assumed to be the radius of the cloud containing the floccules. This value is justified on the grounds that if the mean free path were larger than this few collisions would take place while if it were smaller too many collisions would take place. It means that each floccule on crossing the cloud will make one collision only. This has the advantage of taking into account that any floccule can make just one collision before loosing its identity.

However, such a definition of a mean free path does not take into account the variation of mean free path that will occur as the number of uncollided floccules decreases. The error introduced by this assumption of a constant mean free path only affects the rate at which the protostar will grow in the later

stages of its formation. In these stages other forces, such as gravitation, will probably be dominant so that any loss of efficiency in the collision process may be offset by increased efficiency in other processes.

The definition that the mean free path is of the same order as the dimensions of the parent cloud will be assumed and it will be shown that the results of this assumption do not lead to absurdity.

In the model proposed by McCrea the definition of a mean free path is much more difficult. He assumes that the material which is ultimately going to form the solar system resides within a sphere whose radius is that of the orbit of Neptune. Consequently the assumption of a mean free path of cloud dimensions is not legitimate. By analogy with the previous case we may assume that if the bulk of the material within the sphere must ultimately go into the star, the mean free path must be such that each floccule has at least one chance of making a collision before it leaves the sphere. Therefore the mean free path in this model has been taken to be the radius of the sphere. This definition is subject to the same limitations as previously discussed. The definition of the mean free path used by McCrea does not mean that only material originally present in the sphere can be included in

the solar system. Material from outside the sphere can enter it and collide with an original floccule. Similarly floccules can leave the sphere without making a collision. More will be said about this when discussing McCrea's model.

The properties of the floccules discussed in this section, though not an exhaustive survey, are designed to give a qualitative picture of the ideas underlying the concept of floccules. The ideas are vague at this stage, but we hope to make them a little more precise when discussing the actual models in the next two sections.

Section 3. The formation of a condensation by the gravitational binding of the products of a collision of two floccules.

In this model for star formation the collision of the floccules leads to the assembling of material which is gravitationallybound. This material as it falls together forms a protostar. We are interested in those floccule collisions which take place in such a way that most of the velocity of translation of the floccules is dissipated, i.e. the centre of mass velocity of the collision region is small. The parts of the floccules not taking part in the collision move away with their original velocity and are no longer considered.

The collision region or composite floccule may have a centre of mass velocity which is so small that it cannot escape from the gravitational field of a set of already gravitationally bound composite floccules. To start this process we may suppose that two floccules collide forming a composite floccule which for the sake of simplicity we may assume to be stationary with respect to a set of axes fixed at the centre of mass of the parent cloud. Somewhere in the gas cloud a further collision takes place resulting in a composite floccule

having a centre of mass velocity less than the velocity of escape from the original one. Further collisions then give rise to other composite floccules having less than the velocity of escape from this pair and so on till a set of gravitationally bound floccules has been built up. These floccules then fall together and form a protostar.

Initially the escape velocity will be very small so that any collision which is going to lead to a small centre of mass velocity must be nearly head on. In order to investigate the problem mathematically, we must idealise the collision of floccules still further. We suppose that the floccules may be treated as particles and that those parts of the floccules which interact can be represented by allowing the equivalent parts of the particles to adhere together. We shall assume that all the particles have the same mass m and that each collision forms a floccule of mass km where $0 < k < 2$.

The problem can now be investigated in two parts. We suppose in one case that the floccules all have the same distribution of velocities while in the second case we assume that the floccules have a Maxwellian distribution of velocities established amongst them.

The collision of floccules each having the same random velocity.

Suppose that in some Cartesian frame of reference attached to the centre of mass of the parent gas cloud, the two colliding floccules have velocity components (u, v, w), $(u' v', w')$ respectively. Since the floccules have the same random velocity we may write,

$$V^2 = u^2 + v^2 + w^2 = u'^2 + v'^2 + w'^2 . \qquad (6)$$

After the collision of the floccules the components of the velocity of the centre of mass of the composite floccule will be

$$\tfrac{1}{2}(u + u'), \ \tfrac{1}{2}(v + v'), \ \tfrac{1}{2}(w + w'),$$

assuming that each floccule contributes equally to the composite floccule. The velocity U of the composite floccule is given by

$$4U^2 = \{(u + u')^2 + (v + v')^2 + (w + w')^2\},$$

$$\text{i.e.} \quad 2U^2 = V^2 + (uu' + vv' + ww') . \qquad (7)$$

We now wish to consider only those collisions in which U

is less than a certain value C where C is small. For this to be true the floccules must approach nearly head on. Therefore only floccules which approach within a certain solid angle can have their velocity reduced to a low value.

Suppose the floccules approach along paths making an angle α with one another. Let the floccules have velocity vectors $\underset{\sim}{V}, \underset{\sim}{V}'$ respectively. Then since the floccules moving at these speeds have no appreciable gravitational interaction we can assume that the angle α is determined by the scalar product of the velocity vectors,

$$\underset{\sim}{V}.\underset{\sim}{V}' = -V^2\cos\alpha = \left\{u\underset{\sim}{i} + v\underset{\sim}{j} + w\underset{\sim}{k}\right\}.\left\{u'\underset{\sim}{i} + v'\underset{\sim}{j} + w'\underset{\sim}{k}\right\},$$

$$= uu' + vv' + ww' . \qquad (8)$$

where $\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ are unit vectors in the directions of the Cartesian axes. Using equation (8) in equation (7) we have

$$2U^2 = V^2 - V^2\cos\alpha = V^2(1 - \cos\alpha) . \qquad (9)$$

If $U \ll V$ we must have α very small. The maximum value that U can have is the escape velocity from

some original composite floccule. If the bound material within a sphere of radius R is uniformly distributed and has a total mass M, the escape velocity from a point inside the cloud distant $r < R$ from the centre is

$$C^2 = \frac{GM}{R^3}(3R^2 - r^2). \qquad (10)$$

This is a minimum value for the escape velocity since it concerns only the bound material. The other floccules will have an effect towards increasing even though they themselves are not bound. Therefore the maximum value of α is given by

$$\frac{2C^2}{V^2} = 1 - \cos\alpha . \qquad (11)$$

The corresponding solid angle $d\omega$ is given by

$$d\omega = 2\pi(1 - \cos\alpha),$$

so that the cone within which the velocity vectors must lie has a solid angle $d\omega$ and the fraction of all velocity vectors lying within this cone is

$$\frac{d\omega}{4\pi} = \frac{C^2}{V^2} . \qquad (12)$$

For a set of randomly moving floccules the fraction of

collisions leading to a velocity of the composite floccule which is less than the escape velocity C is $(C/V)^2$.

The following notation will be used,

ν = number of floccules per unit volume,

σ = radius of a floccule,

r = distance of a floccule from the centre of mass of the cloud,

ρ_0 = mean density of the cloud,

m = mass of an individual floccule.

The further simplifying assumption is made that the floccules are uniformly distributed throughout the parent cloud which is spherical in shape. Therefore the centre of the cloud is also its centre of mass.

The number of floccule collisions taking place per unit time per unit volume is

$$\pi \nu^2 \sigma^2 V.$$

The fraction of these collisions leading to a composite floccule having a velocity less than the escape velocity C is

$$\pi \nu^2 \sigma^2 V \frac{C^2}{V^2} = \pi \nu^2 \sigma^2 \frac{C^2}{V} . \qquad (13)$$

The number of collisions of this type taking place between the radii r, $r + dr$ is,

$$\pi \nu^2 \sigma^2 \frac{C^2}{V} 4\pi r^2 dr .$$

The number taking place in the volume between $r = 0$, $r = R$ is,

$$4\pi^2 \nu^2 \sigma^2 \frac{GM}{VR^3} \int_0^R (3R^2 - r^2) r^2 dr = \frac{16\pi^2}{5} \nu^2 \sigma^2 \frac{GM}{V} R^3 , \quad (14)$$

substituting from equation (10) for C^2. The rate at which mass is added to the already gravitationally bound material is

$$\frac{dM}{dt} = \frac{16\pi^2}{5} \nu^2 \sigma^2 \frac{GM}{V} R^2 km , \quad (15)$$

where km is the mass added per collision. Integration gives

$$M = M_0 \exp\left\{ \frac{16\pi^2}{5} km \nu^2 \sigma^2 \frac{GR^2}{V} t \right\} ,$$

$$= M_0 \exp\left\{ t/\tau \right\} , \quad (16)$$

where $$\frac{1}{\tau} = \frac{16\pi^2}{5} k \sigma^2 \nu G \rho_0 \frac{R^2}{V} , \quad (17)$$

and $$\nu m = \rho_0 .$$

The dimensions of the floccule may be removed from (17) by defining $(\pi\sigma^2\nu)^{-1}$ to be the mean free path for the collision of floccules by analogy with kinetic theory. Denoting the mean free path by λ we have,

$$\frac{1}{\tau} = \frac{16\pi k}{5} \frac{G\rho_0}{\lambda} \frac{R^2}{V} . \qquad (18)$$

The value of τ can now be computed using the values of ρ_0, R for an average gas cloud while values for λ, V may be assumed as outlined in Section 2.

In evaluating τ we shall assume that each collision of a pair of floccules gives a composite floccule, whose mass is the same as the original mass of a component floccule. This is an average value, since if the conditions were right a composite floccule of mass $2m$ could form and equally so could a floccule of mass very much less than m. Therefore we shall take $k = 1$. For the remaining values we shall take $V = 2$ km.sec^{-1}, $\rho_0 = 10^{-23}$ gm.cm^{-3}, $\lambda = R = 2.5.10^{19}$ cm. Inserting these values in equation (18) we find that τ is $3.8.10^7$ yrs. This is too large a value for τ which must be of the order of 10^6 yrs. If some compression is introduced we can decrease τ.

Increasing the density by a factor of 10^3 involves a decrease of 10 in the linear dimensions of the cloud. Therefore the introduction of such a compression decreases τ by a factor of 10^2 giving $\tau = 3.8.10^5$ yr.

The fact that some initial compression of the gas is necessary to obtain a value of τ which is of the correct order of magnitude is consistent with the assumption made in Section 2 that some compression is required in order to produce the floccules in the first place. If a time scale of $3.8.10^6$ yr. is allowed for the process of star formation then the mass collected in this time would be

$$M = M_0 \exp 10 .$$

If M is taken to be a solar mass the value of M_0 is 9.10^{28} gm. This value of 9.10^{28} gm. could be taken to be about the mass of a floccule. Therefore an average interstellar cloud which has been compressed by a factor of 10^3 would contain about 10^7 floccules. It is of interest to note that the mass of a floccule is an order of magnitude larger than the mass of the earth ($5.977. 10^{27}$ gm.).

The collision of floccules having a Maxwellian distribution of velocities.

This problem in the case of the kinetic theory of gases has already been solved by Jeans (1921). We shall adapt his results to apply to the present problem by taking a Maxwellian distribution of velocities among the floccules and making use of our definition of a collision. Jeans considered that if the two colliding particles just grazed each other this should count as a collision. In our case such an encounter would only lead to the dissipation of floccule material. We define our collision cross section in such a way that we only regard collisions in which the "edge" of one floccule would pass through the centre of the other, if this were possible. Jeans therefore defines his collision cross section by letting σ denote the diameter of the particle while we define σ to be the radius of the particle.

Jeans shows that the number of collisions leading to a composite particle or floccule having a velocity vector in the range $\underset{\sim}{c}$, $\underset{\sim}{c} + d\underset{\sim}{c}$ for an initial relative velocity in the range V, $V + dV$ is

$$8h^3m^3\nu^2\sigma^2 \exp\left\{-hm\left(2c^2 + \tfrac{1}{2}V^2\right)\right\} c^2 V^3 d\underset{\sim}{c}\, dV \,, \qquad (19)$$

where $\hbar = \frac{3}{2m\bar{c}}$, $\bar{c}$ being the mean speed of the floccule. We now suppose that the speed of the composite floccule is small compared with the mean velocity of the original particles. In this case we can take $exp\{-2\hbar mc^2\}$ to be unity. Integrating over V we have the number of collisions per unit time per unit volume resulting in a centre of mass velocity less than c to be (dropping the vector notation),

$$c^3 \frac{8\nu^2\sigma^2}{\bar{c}^2} = \frac{8\nu^2\sigma^2}{\bar{c}^2}\left\{\frac{GM}{R^3}(3R^2 - r^2)\right\}^{3/2} \quad \text{cm.}^{-3}\ \text{sec.}^{-1} \quad (20)$$

using equation (10). This result is also subject to the same limitations as equation (14) introduced by our definition of c . The number of collisions occurring in the shell of material lying between the radii $r, r + dr$ is,

$$\frac{8\nu^2\sigma^2}{\bar{c}^2}\left\{\frac{GM}{R^3}(3R^2 - r^2)\right\}^{3/2} 4\pi r^2 dr . \quad (21)$$

Integrating over the whole cloud this becomes,

$$1{\cdot}25.10^2 \frac{\nu^2\sigma^2}{\bar{c}^2}\{GMR\}^{3/2} . \quad (22)$$

The rate at which mass is added is then

$$\frac{dM}{dt} = 1{\cdot}25.10^2 \frac{\nu^2\sigma^2}{\bar{c}^2}\{GMR\}^{3/2} km \quad \text{gm.sec.}^{-1}$$

or integrating (23)

$$2\left(\frac{1}{M_0^{1/2}} - \frac{1}{M^{1/2}}\right) = 1{\cdot}25k10^2 \frac{\nu^2\sigma^2}{\bar{c}^2} m\{GR\}^{3/2} t$$

where the mass is M_0 at time $t = 0$ and M at any later time. Equation (23) may be rearranged to give,

$$M = M_0\left\{1 - \frac{1{\cdot}25k10^2}{2} M_0^{1/2} \frac{\nu^2\sigma^2}{\bar{c}^2} m\{GR\}^{3/2} t\right\}^{-2},$$

$$= M_0\left\{1 - t/\tau\right\}, \qquad (24)$$

$$\text{where } \frac{1}{\tau} = \frac{1{\cdot}25k10^2}{2} \frac{\nu^2\sigma^2}{\bar{c}^2} m M_0^{1/2} \{GR\}^{3/2} \text{ sec}^{-1} \qquad (25)$$

Again introducing the mean density ρ_0 and the mean free path λ we have

$$\frac{1}{\tau} = \frac{1{\cdot}25k10^2}{2\pi} \frac{\rho_0 M_0^{1/2}}{\bar{c}^2 \lambda} \{GR\}^{3/2} \text{ sec}^{-1} \qquad (26)$$

The value of τ depends on the value chosen for M_0. This difficulty did not arise in the previous case. From equation (24) we see also that $M \longrightarrow \infty$ as $t \rightarrow \tau$ i.e. the mechanism can build up an infinite mass in a finite time. These points will be discussed later.

Using the same values for ρ_0, R, λ as taken

in the last calculation (assuming no compression) and taking $\bar{c}$ = 2 km.sec.$^{-1}$, k = 1, we have

$$\tau = 2{\cdot}34.10^{33} M_{\circ}^{-1/2} \text{ sec.} = 7{\cdot}40.10^{25} M_{\circ}^{-1/2} \text{ yr.} \qquad (27)$$

If $M_{\circ} = 9.10^{28}$ gm., $\tau = 2.46.\ 10^{11}$ yr. which is much too long a time for the process of star formation. Once again we must postulate some compression. If a compression of 10^3 is taken with $M_{\circ} = 9.10^{28}$ gm. the value of τ becomes $7.8.\ 10^8$ yr. This is still a long time for star formation and accordingly a somewhat greater compression of the order of 10^6 is required to give τ a value of the right order of magnitude.

The results depend upon the value chosen for $M_{\circ}$. If the floccules are rather more massive than has been assumed, a smaller initial compression will be required. The results also indicate that the Maxwellian case is less favourable than the case in which all the floccules had the same velocity. The results obtained here will be discussed in detail in Section 5, after consideration of the model proposed by McCrea.

Section 4. The formation of stars from an incipient condensation.

The conditions under which stars form are rather different in this model from those described in the last section. The model has been developed in some detail by McCrea (1960a, 1960b) so a brief outline only of this work will be given here. McCrea supposes that the interstellar gas cloud is compressed by an external pressure of a type which he has previously discussed (1957). Due to the compression the distributions of density and velocity within the cloud become highly chaotic and again the situation is idealised into a set of floccules in random non-interacting motions. The degree of compression required by McCrea is of the order of 5.10^{11} as opposed to compressions of the order of 10^3 to 10^6 which were required in the last section.

The mechanism proposed by McCrea depends on the formation of a larger than average floccule or incipient condensation. The incipient condensation may form through a favourable collision of two floccules leading to a slightly more massive floccule. Further encounters may in favourable circumstances lead to the addition of considerably more mass to the composite floccule than it loses in these encounters. Therefore in these favourable circumstances an incipient

condensation may grow. The incipient condensation is assumed to be larger than an average floccule, so that any collision between a floccule and the condensation leads to the capture of the floccule by the condensation. Many such incipient condensations will form throughout the gas cloud.

In the early stages of this process the gravitational effect of the incipient condensation is negligible so that the condensation grows by chance encounters with the floccules. However, once the incipient condensation has grown sufficiently large it will be able to contract under its own self gravitation. Thereafter gravitation assists the infall of further floccules onto the incipient condensation.

A fundamental assumption made by McCrea is that the angular momentum, possessed by the floccules in a certain region of space, will go into the condensation formed in that region, despite the fact that not all the floccules in that region participate in this condensation. Therefore angular momentum is not precisely conserved in the region considered due to the migration of floccules in and out of the region. However, the correlation between events in one region and events in a neighbouring region will be very weak for this type of chaotic motion. Consequently

if a meaningful mean free path can be defined then angular momentum can be approximately conserved within a region whose dimensions are a few times this mean free path, for a certain period of time.

The conservation of angular momentum can be rigorously satisfied by assuming that stars are not formed singly but in groups and floccules starting off from one region may ultimately coalesce with an incipient condensation of another region. Consequently while angular momentum need not be precisely conserved for any single star, the angular momentum is conserved whenever the angular momentum of the star group, about its mass centre, is considered.

If the floccules are initially considered as being non interacting gravitationally we may find the angular momentum possessed by a large group of floccules by applying the theory of random flights to their motion. The argument given here is slightly different to that used by McCrea.

Suppose we have N floccules within a certain radius R each of mass m and each having a random velocity V. In order to find the average angular momentum of each floccule due to its translational velocity, we move all the floccules to the mean radius and then average over all values of

the direction of the velocity vector. We assume that the floccules are uniformly distributed throughout the sphere of radius R. The mean value of the radius is given by

$$\frac{\int_0^R \frac{3N}{4\pi R^3} t^2 dt}{\int_0^R \frac{3N}{4\pi R^3} t^2 dt} = \frac{\int_0^R t^3 dt}{\int_0^R t^2 dt} = \frac{3}{4} R , \qquad (28)$$

where t denotes the distance of the floccules from the centre of the sphere of radius R. The angular momentum possessed by a floccule of mass m whose velocity vector makes an angle θ with the radius has angular momentum

$$m V \sin\theta . \frac{3}{4} R .$$

The average value of $\sin\theta$ in the range $0 \leq \theta \leq \pi/2$ is 0.5, so that the average value of the angular momentum possessed by a floccule is

$$\frac{3}{8} m V R . \qquad (29)$$

The angular momenta of the N floccules must now be summed. The magnitude of the angular momentum possessed by each floccule is the same for all floccules, but the orientations are different. The problem is now that of a random walk with steps of

equal length, but the directions of the steps are random. Chandrasekhar (1943) has shown that the probability of finding a particle after a number N of random steps of length ℓ between the points $\underset{\sim}{L}$, $\underset{\sim}{L} + d\underset{\sim}{L}$ is,

$$W(\underset{\sim}{L})d\underset{\sim}{L} = \frac{1}{\left(\frac{2\pi}{3}N\ell^2\right)^{3/2}} \exp\left\{-\frac{3|\underset{\sim}{L}|^2}{2N\ell^2}\right\} d\underset{\sim}{L} . \qquad (30)$$

In this particular case $\underset{\sim}{L}$ denotes the angular momentum. Applying this result we find that the mean value of the angular momentum is,

$$(A.M.)_{av} = \frac{\int_{\underset{\sim}{L}} W(\underset{\sim}{L})\, d\underset{\sim}{L}}{\int W(\underset{\sim}{L})\, d\underset{\sim}{L}} ,$$

$$= \frac{\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} L\left(\frac{1}{2\pi/3 N\ell^2}\right)^{3/2} \exp\left\{-\frac{3|\underset{\sim}{L}|^2}{2N\ell^2}\right\} L^2 \sin\theta\, d\theta\, d\varphi\, dL}{\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(\frac{1}{2\pi/3 N\ell^2}\right)^{3/2} \exp\left\{-\frac{3|\underset{\sim}{L}|^2}{2N\ell^2}\right\} L^2 \sin\theta\, d\theta\, d\varphi\, dL} ,$$

$$= \left(\frac{8}{3\pi}N\ell^2\right)^{1/2} = \left(\frac{3}{8\pi}\right)^{1/2} mVRN^{1/2} = \left(\frac{3}{8\pi}\right)^{1/2} MVRN^{-1/2} . \qquad (31)$$

where the length of the step ℓ is given by equation (29) and M is the total mass of the floccules contained within the sphere of radius R. The coefficient $(3/8\pi)^{1/2}$ in equation (31) has the numerical value 0.35. McCrea (1960b) obtains a value

of 0.5 for the coefficient. The difference is not significant. From equation (31) it is clear that if the mass contained within the sphere of radius R is maintained constant then sub-division into more and more floccules decreases the total angular momentum available.

McCrea specifies the problem more precisely and restricts the theory to the formation of the solar system. To do this he assumes that the floccules ultimately forming the solar system are contained within a region whose radius is the present radius of the solar system. This is taken to be the radius of the orbit of Neptune, since Pluto is now thought not to be a planet but an escaped satellite. The radius of this region which is also the mean free path for the floccules is 5.10^{14} cm. The angular momentum of the floccules residing within the sphere of this radius is taken to be the present angular momentum of the solar system, namely $1.7.10^{50}$ gm.cm^2 sec^{-1}, while the mass residing within this sphere is a solar mass.

Having obtained a value for the total angular momentum the number of floccules present can be derived from equation (31). Taking the velocity of the floccules to be 1 km.sec^{-1} McCrea finds that there will be 10^5 floccules within the sphere. Therefore each

floccule has a mass of 2.10^{28} gm. Having fixed all these values there are no parameters left arbitary. McCrea then showed that for the conditions assumed the incipient condensation became unstable against gravitational contraction after it had grown to a mass of some 20 floccule masses. Thereafter he treated the infall problem as one of the gravitational accretion of floccules.

A floccule, distant R from a gravitating mass $\mathcal{M}$ at some point O, is initially moving with speed V along a line whose perpendicular distance from O is q. At its nearest approach to O let U be the speed of the floccule and a its distance from O, then using the conservation of energy and momentum laws McCrea writes,

$$\left.\begin{aligned} U^2 - \frac{2G\mathcal{M}}{a} &= V^2 - \frac{2G\mathcal{M}}{R} \\ Ua &= Vq \end{aligned}\right\} \qquad (32)$$

McCrea neglects terms in a/R and shows that the impact parameter q may be written in the form

$$q \simeq \left\{a(a + 2G\mathcal{M}V^{-2})\right\}^{1/2} \qquad (33)$$

Evaluating q for $\mathcal{M}$ = 20 m, $a = (20)^{1/3}S$ where S is the radius of a floccule and for

$\mathcal{M} = 2.10^{33}$ gm, $a = 7.10^{10}$ cm. McCrea found

$$q_0 \simeq 4.5.\ 10^{12} \text{ cm.}, \tag{34a}$$

$$q_1 \simeq 43.\ 10^{12} \text{ cm.}, \tag{34b}$$

where the subscript zero refers to the first set of values and the subscript one refers to the second set. Thus the value of the impact parameter q does not vary by more than a factor of ten throughout the process. This is due to the fact that the decrease in radius in offset by the increase in mass. Taking the mean value of q to be $\frac{1}{2} q_1$ and since for any impact parameter q the mean value of q is $\frac{2}{3} q$ McCrea takes the angular momentum of any floccule entering the condensation to be $\frac{1}{3} m V q_1$. Therefore the most probable value of the resultant angular momentum of the sun is

$$\tfrac{1}{3} m V q_1 \left(\tfrac{2}{3} N\right)^{1/2} = \tfrac{2}{3}\left(\frac{G M_\odot R_\odot}{3N}\right)^{1/2} = 4.4.10^{48} \text{ gm. cm}^2 \text{ sec}^{-1}, \tag{35}$$

where a subscript $\odot$ denotes that the value of the quantity so marked has the solar value.

Although the theory presented by McCrea is quite simple this value of the angular momentum is only 4.6 times the actual angular momentum possessed by the

sun. This feature of the theory, the prediction of the solar angular momentum, is one of its principal successes.

McCrea further shows that the time required before the incipient condensation becomes self gravitating is of the order of 5000 yrs, while the time required for the whole process is almost certainly less than $1.7.10^6$ yr. and may be about 2.10^5 yr. This time scale is consistent with other estimates of star ages.

Having considered the formation of the sun, McCrea also considers the formation of the planets from the floccules, which have not fallen into the central condensation but remain trapped by its gravitational field. He estimates that there will be about 1000 floccules trapped and these will circulate about the central condensation in randomly orientated orbits. These orbits, through the collision of the floccules, will ultimately tend to settle down into a single plane. In the initial stages of the condensation of planetary material, it was supposed that the tidal effect of the sun was large so that the planets rotated always presenting the same face towards the sun. However, when the planets had contracted to their present size the tidal effect would be small and the rotation of

the planets would increase to their present values. The initial densities of the material from which the planets were formed could then be computed and these formed an orderly decreasing progression as one moved out from the earth. The value for Neptune was just an order of magnitude larger than the assumed density of a floccule.

A further feature of the theory indicates that the Roche limit for stability against the disruptive tidal action of the sun is placed about the orbit of Jupiter. Consequently the theory as developed by McCrea applies to the formation of the major planets which have been able to condense with their full quota of hydrogen, but does not apply to the "terrestrial" planets. The Asteroids are regarded as protoplanets which were at the crucial distance for disruption, the disruption occurring when they lost their hydrogen.

This theory developed by McCrea accounts very well for the angular momentum possessed by the sun and for the formation of the major planets. For such a simple theory, not elaborated in detail, it gives very good numerical agreement with the major parameters of the solar system. The theory leads naturally to the formation of a planetary system and it also requires that stars are formed in groups and clusters and not in isolation.

Section 5. Discussion of the results.

Two models of star formation have been considered in this chapter. For the purposes of this discussion the model of Section 3 will be called the gravitational model, while the model of Section 4 will be called McCrea's model.

The gravitational model.

The gravitational model has several defects. It assumes that material throughout the whole extent of the cloud becomes bound and it does not give any information on whether more than one system of bound material can be formed. If a Maxwellian distribution of velocities is assumed for the floccules, then the theory predicts that an infinite amount of material may become gravitationally bound in a finite time. This result is clearly absurd.

However, in considering the Maxwellian case the approximation was made that the centre of mass velocities of the composite floccules were small. This is certainly true initially, but it will not be true in the later stages of the process. In these later stages the exponential term $exp(-2\hbar mc^2)$ cannot be neglected and its re-introduction may prevent the

acquisition of an infinite mass. On physical grounds an infinite mass could not be obtained since the amount of available material would put a natural limit on the growth. There is another effect which applies not only to the Maxwellian case but also to the case in which the floccules are all assumed to have the same velocity. Once two floccules have collided to form a bound composite floccule we have assumed that this composite floccule suffers no further collision. This assumption is quite plausible from the definition of the mean free path. However, a small number of collisions will take place in which bound floccules may be disrupted by the collision. If such encounters took place the amount of bound material collected, as discussed in Section 3, would be a maximum.

The gravitational theory has considered the formation of a set of bound floccules throughout the whole cloud. However, it may happen that the mean free path is shorter than has been assumed. When discussing the mean free path it was mentioned that it should be of the same order as the cloud radius if collisions were not to take place too rapidly or too slowly. This is certainly a good guiding rule, but once having obtained a model based on this assumption it is of interest to note the results obtained by altering some of the basic assumptions.

The effect of shortening the mean free path would have the effect, as pointed out when describing McCrea's model, of dividing the cloud into a number of independent subregions whose dimensions were of the order of a few mean free paths. Therefore a set of condensations of gravitationally bound floccules could be obtained. If the mean free path were reduced by a factor of ten and the radius of the independent subregion was two mean free paths, then the density would need to be increased only by a factor of two to maintain the short time scale. Therefore by making a small change to the initial conditions we can obtain a set of gravitationally bound systems of floccules. However, the amount of material available to each such condensation has now diminished and the process may not be so effective.

This type of consideration suggests that the gravitational model is not very suitable for a discussion of star formation, but as will be suggested later it may have a place in a heirarchical floccule model.

The question of angular momentum has not been investigated for this model since its computation could not be carried out in any simple way such as discussed in McCrea's model. It is not therefore known

whether this model would resolve the angular momentum difficulty.

McCrea's model.

The model proposed by McCrea, although it uses a simple mathematical analysis, accounts in a most elegant fashion for the large scale features observed in the solar system. It depends however on whether the assumed starting conditions can in fact be realised or not. The theory demands that the inter-stellar gas should be at almost constant temperature during a large compression, but it is not known whether this would occur. It depends on the ability of the atomic hydrogen to convert itself into molecular hydrogen and on the ease with which the radiation from the molecular hydrogen can escape from the compressed cloud. The results of Chapter I suggest that there will be no difficulty in forming molecular hydrogen, but the absorption of the radiation from molecular hydrogen at various densities has not yet been investigated.

McCrea does not specify his pressure source precisely and some mechanism producing a compression of at least $2.3.10^{11}$ would need to be found. It is not clear just what degree of compression the mechanism

discussed by Oort and Spitzer (1955) and Biermann and Schlüter (1954) would produce. The initial stages of the floccule forming process therefore require more discussion.

McCrea has also assumed that the floccules have a mean speed of 1 km.sec^{-1}, but when discussing the origin of the planets he requires floccules having speeds of 10 km.sec^{-1}. The existence of these floccules is justified on the grounds that since the sun possesses very little of the angular momentum of the solar system, the floccules from which the planetary system forms must carry about as much of the total angular momentum as the low velocity floccules. However, these faster floccules would tend to disperse very rapidly. These faster floccules may not be present initially but may acquire these velocities at a later stage through gravitational interaction with other floccules. This sort of additional hypothesis introduces some indeterminancy into the calculations. The relaxation of a system of floccules in weak gravitational interaction was therefore examined very briefly.

The relaxation of a system of floccules.

The relaxation time for a system of floccules

may be found from the results obtained by Chandrasekhar (1942) for the relaxation of a star cluster. The relaxation time for a system of floccules of mass 9.10^{28} gm. in a cloud of radius 1 pc. is approximately 6.10^{5} yr. if all the material is in the form of floccules. For a system of floccules of mass 2.10^{28} gm. within a cloud whose radius is 5.10^{14} cm. this relaxation time is increased by a factor of 60. This last calculation ignores gravitational effects from floccules outside the cloud. If a value of 10^{6} yr. is taken for the relaxation time then it would just be possible in the time available to obtain some spread of the floccule velocities. However, the evaluation of a precise relaxation time is difficult since the number of floccules will be varying throughout the condensation process and the relaxation time depends on the floccule density.

This calculation also suggests that the Maxwellian distribution of velocities, assumed in one case of the gravitational model, is not particularly appropriate. Therefore the assumption that all the floccules have the same velocity initially is justifiable.

A hierarchical model for star formation.

The following is a tentative suggestion for a hierarchical model for star formation, whereby the gravitational model and McCrea's model may be combined.

The gravitational model is more efficient at low compressions than McCrea's model, requiring only a compression of 10^3 to 10^4 as opposed to a compression in excess of 10^{11} required by McCrea's model. However, the gravitational model would be too rapid at larger compressions.

The size of the floccules demanded by the gravitational model are somewhat larger than those needed by McCrea. Consequently, assuming that gravitational condensations may form in subregions of the cloud as previously mentioned in this section, we suggest that the regions of bound material formed as in the gravitational model may become the incipient condensations required by McCrea. The fragments of the floccules produced in the encounters may then become the floccules for McCrea's model. This model has the advantage that it suggests how gravitationally bound incipient condensations may be formed, but it has the disadvantage that while only requiring an initial compression of 10^3 to 10^4 the high densities

obtaining in McCrea's floccules must now be explained in some other way.

However, this suggestion is only tentative and before anything definite can be said on the initial stages of star formation more work will be required to investigate the properties of clouds of molecular hydrogen at various densities and on the compression of the gas clouds by external agencies.

Conclusion

This chapter has reviewed some of the problems which a theory of star formation will encounter. Gravitational instability was shown to lead to the condensation of a much larger mass than that usually associated with stars. Any endeavour which tried to improve the situation, such as introducing rotation, lead to the formation of further hypotheses in order to obtain agreement with the observations. In the case of rotation some form of braking had to be considered in order to reduce the angular momentum of the star to acceptable values.

However, the interstellar material is not uniform and this non uniformity was taken to the limiting case of supposing that a gas cloud was composed of small cloudlets or floccules which were in a state of random motion. The collisions of these floccules were examined and it was found that a collision could be regarded as the interaction of those parts of the floccules which physically overlapped, the remainder of the floccule being unaffected.

The effects of the collisions of these floccules were then investigated. The results obtained by McCrea were outlined and it was shown that an

acceptable theory of the formation of both the sun and the major planets could be obtained in a simple way. The theory predicted an angular momentum for the sun which was of the correct order of magnitude (as opposed to an angular momentum many orders of magnitude too large as predicted on the basis of uniform contraction theories) and gave results which were of the correct size for the planetary system. The theory, however, assumed a high degree of initial compression.

A second way in which condensations could grow was also given. In this mechanism the floccules were treated as rigid spherical gas particles and the number of collisions leading to a low final velocity after collision was determined. The final velocity was found by supposing the collision lead to a velocity less than the velocity of escape from some nearby stationary floccule. It was shown that if an initial compression of 10^3 was assumed such a system could collect up a solar mass in less than 10^6 yr. provided the mass of a typical floccule was 9.10^{28} gm.

However, the second mechanism had some disadvantages since it involved the whole cloud and not just part of the cloud. This disadvantage could be removed by re-defining the mean free path for floccule collisions. It was tentatively suggested

that the gravitational model, while perhaps an unsuitable description of the entire process of star formation, could be used as the first stage towards the realisation of McCrea's model.

The models discussed are at present rather sketchy since the processes involved are very complicated. The stages are separated out, but in practice one stage will lead on to the next continuously. The work of McCrea is an exciting starting point for future work in this field, since it achieves a single theory of the formation of the solar system without a host of additional hypotheses.

Chapter III

The Expansion of a Fully Ionised Gas into a vacuum.

Introduction

Hoyle has recently proposed a new theory of galaxy formation within the framework of steady state cosmology. Since steady state cosmological theory demands that matter is continually created, Hoyle supposes that the material is created in the form of neutrons. The neutrons decay radioactively into protons and electrons forming a gas of high kinetic temperature.

The kinetic temperatures are different for protons and electrons and it may be shown that at the densities assumed the gas will not be able to attain thermal equilibrium in the time available. The theory presented by Hoyle makes no clear distinction between the kinetic temperature of the electrons and the kinetic temperature of the protons, since he assumes thermal equilibrium at all times. The distinction between these two temperatures has several minor consequences and an important one.

The minor consequences will be dealt with as they arise. The important case arises out of Hoyle's discussions of the condensation process. He assumes that cool fully ionised gas is compressed by a hot fully ionised gas under the influence of a large

pressure gradient. The problem is to find how rapidly the cool gas is compressed. Hoyle assumes that the compression front will move with the speed of sound in the cool gas.

If the pressure differences are sufficiently great the motion of the hot gas may be idealised into the motion of a gas expanding into vacuum. Therefore in this Section we attempt to study the expansion of a fully ionised gas into a vacuum. We shall find that whereas we could not solve this problem in general, it could be solved for a particular case.

In investigating the expansion of a fully ionised gas we were lead to examine the previous solutions obtained for the expansion of a monatomic gas into a vacuum. By considering the fundamental theory governing such flow we are able to show that these solutions follow inevitably from the definition of the Riemann invariants. Lagrange's Ballistic Problem is also considered since it is mathematically simpler than the free expansion problem. Consideration of the ballistic problem allows us to investigate the expansion of a gas obeying van der Waal's equation of state and to postulate a model for free expansion.

We then deal with the expansion of the fully ionised gas under restricted conditions and show that

its equations of motion may be reduced to those for a monatomic gas.

From the equations of motion we can derive a sound speed for the fully ionised gas which shows that even though the gas has not attained thermal equilibrium the speed of sound is governed by the electron temperature. This result depends on there being a large disparity between the proton and electron temperatures.

Consequently in the problem considered by Hoyle the compression rate will be governed by the conditions prevailing in the hot gas provided the pressure gradients are very steep. The compression rate will be very rapid since the gas causing the compression is at a very high temperature.

Section 1. The theory of galaxy formation proposed by Hoyle.

Hoyle (1958) has put forward a new theory of galaxy formation within the framework of steady state cosmology. In order to maintain a steady state universe Hoyle suggested that the necessary new material is created in the form of neutrons. These neutrons decay radioactively to form protons and electrons. The intergalactic medium is therefore in the form of a fully ionised gas. The gas is very hot since the electrons released by neutron decay are very energetic. Hoyle takes the temperature of the gas to be 10^9 °K.

At the density of the intergalactic material predicted by the steady state theory (i.e. 10^{-29} gm.cm^{-3}) cooling by radiative free-free transitions is very slow. Hoyle therefore proposed that the intergalactic material had already suffered some compression and cooling and he takes as the basis of his discussion intergalactic material at a density of 10^{-27} gm.cm^{-3} and at a kinetic temperature of 10^7 °K. The initial cooling and compression will be further discussed later. Under these conditions the radiation from free-free transitions and free-bound transitions can

dispose of the kinetic energy of the protons and electrons in a Hubble time. The radiation rate used by Hoyle is,

$$8.7.10^{-4} T^{1/2} \left\{ 1 + \frac{3.85.10^5}{T} \right\} n \quad \text{etg. gm}^{-1} \text{sec}^{-1}, \qquad (1)$$

where the free-free radiation depends on $T^{1/2}$ and the free-bound radiation depends on $T^{-1/2}$, and T denotes the kinetic temperature and n denotes the number density of the electrons (equal to the number density of the protons). The energy possessed by the ionised hydrogen is the sum of the ionisation energy and the kinetic energy. This is

$$1.3.10^{13} + 2.5.10^{8} T \quad \text{etg. gm}^{-1} \qquad (2)$$

where thermal equilibrium has been assumed to exist between the electrons and the protons.

In order that the hydrogen should be able to radiate away the energy given in expression (2) in a time of 4.10^{17} sec. (the Hubble time) the ion number density n will be given approximately by

$$n \simeq 7.10^{-7} T^{1/2}. \qquad (3)$$

Equation (3) is obtained by multiplying expression (1) by the Hubble time and equating the product to expression (2) and neglecting the contribution from

free-bound transitions and the ionisation energy.

Hoyle makes the further assumption that the pressure equilibrium is maintained at all stages of the cooling process. This assumption is later waived when considering the collapse of the cool gas under the influence of gross pressure differences. This means that the pressure is determined by the hot gas and so has a value of 10^{-12} dynes cm^{-2}. The pressure equilibrium condition may then be expressed as

$$nT \simeq 5.10^{3}. \qquad (4)$$

Combination of equations (3) and (4) gives $T \simeq 10^{7}$ °K, $n \simeq 10^{-3}$ cm^{-3}. Therefore radiative cooling is important on the cosmological time scale at densities greater than 10^{-27} $gm.cm^{-3}$. At the lower density of 10^{-29} $gm.cm^{-3}$ the rate of cooling falls to ten percent of the value at the larger density, even when the temperature of the gas is increased by a factor of 10^{2}. As the density increases radiative cooling becomes of greater and greater importance. Hoyle shows that if pressure equilibrium is maintained the time scale for cooling behaves as $n^{-3/2}$ for $T > 3.10^{5}$ °K. and approximately as $n^{-5/2}$ for lower temperatures (these results may be deduced from (1) and (2)). Since n varies as R^{-3} where R is the radius of the cloud the cooling time varies respectively

as $R^{4.5}$ and $R^{7.5}$. Therefore as R diminishes, cooling rapidly increases. Hoyle supposes that this rapid cooling induces the formation of blobs of material and does not effect uniform condensation. The mechanism whereby this non uniform cooling is effected is not made explicit in Hoyle's paper, but we may suppose that this effect is due to the existence of initial density variations in the cooling cloud. Cooling will cease when the gas is no longer ionised, i.e. at about 10^4 °K. and the pressure criterion (equation (4)) shows that this occurs for a density of about one atom cm^{-3}. The condensed blobs at this stage of their evolution are identified by Hoyle with proto-galaxies.

At a temperature of 10^7 °K. the speed of sound in a gas which is in thermal equilibrium is of the order of 500 $km.sec^{-1}$. Let us denote this sound speed by V . The condensation of the cloud will not take place with a speed less than V , so that the time for condensation is R/V . At a density of 10^{-27} $gm.cm^{-3}$ a galaxy mass is contained within a radius of 3.10^{23} cm. Therefore R/V is approximately 10^{16} sec. But as the time scale for cooling is 4.10^{17} sec., the cool gas will not permit itself to be compressed at this rate, so that at the

start of the condensation mechanism, compression and cooling will keep in step. When R has decreased to one third of its initial value the cooling time scale will be about 10^{15} sec. while the compression time scale is 2.10^{15} sec. At this stage compression begins to have difficulty in keeping up with cooling. The assumption of pressure equilibrium must now be abandoned since the condensation will be driven inwards by external pressure. During this stage appreciable dynamical motions will be built up in the compressed gas.

When pressure equilibrium has been restored Hoyle assumes that the condensed cloud may be regarded as a set of condensed blobs in random motion. The gas composing the blobs will have considerable random energy through the generation of dynamical motions during the condensation process. Hoyle takes the speed for these motions to be less than, but of the same order as, V the initial speed of sound in the gas cloud. (He assumes, in fact, that the speed is $\frac{1}{2} V$.) If the blobs are to be able to control this motion gravitationally their masses must be of the order of 2.10^{44} gm. This is of the order of a galaxy mass.

Since the time scale for cooling has been

chosen to be a Hubble time and allowing a further period of 4.10^{17} sec. for the initial cooling and compression, the mean density of material in the form of galaxies was found to be 10^{-31} gm.cm^{-3}. Hoyle computed this value assuming that the mean life of intergalactic material was $\frac{4}{3}\ 10^{17}$ sec. This value agrees well with the observed value.

The primary cooling mechanism by which the intergalactical material at a density of 10^{-29} gm.cm^{-3} and temperature of 10^{9} °K. is brought to the density and temperature necessary for the previously outlined mechanism for galaxy formation, is not at all clear. Hoyle rejects the idea that any substantial cooling is effected by the presence of already existing galaxies, since the mean density of material in the form of galaxies is one percent of the mean density of the intergalactic gas. He suggests that some form of highly efficient heat engine effect may operate, but the type of effect required is not made explicit. Some cooling of the gas is due to radiation loss and heat transfer to the existing galaxies, but the largest amount of the available energy is assumed to be converted into dynamical motions. Hoyle supposes that these dynamical motions may be used for the acceleration of the cosmic rays. He finds that he can

get agreement with the observed cosmic ray spectrum at high energies, provided half the cosmic rays present are regarded as newly injected into the intergalactic medium after acceleration by this mechanism.

Hoyle's theory of galaxy formation is open to several objections. He makes no clear distinction between the electron and proton temperatures. Using the data on neutron decay obtained by Robson (1951) we can find the kinetic temperatures which must be assigned to the protons and the electrons. Robson finds that the cut-off for the emitted electron's energy occurs at 782 ± 13 keV., while the maximum number of emitted electrons have an energy of approximately 400 keV. This mean energy was used in deriving the kinetic temperature for the electrons, while the cut-off energy was used to find the kinetic temperature for the protons. The emitted neutrino has been neglected and the problem treated as a two body recoil problem. The temperature to be associated with the electrons is $1.4.\ 10^{9}\ ^{o}K.$ and the temperature to be associated with the protons is $5.8.\ 10^{6}\ ^{o}K.$ The proton temperature is therefore very much less than the electron temperature and so most of the thermal energy is carried by the electrons. This would suggest that expression (2) should be altered to

$$1.3.10^{13} + 1.25.10^{8} T_i + 1.25.10^{8} T_e , \qquad (5)$$

where T_i is the proton temperature and T_e is the electron temperature, while T_e should replace T in expression (1).

However, Hoyle had assumed that the protons and electrons were in thermal equilibrium. The above results suggest that the equilibrium temperature for the gas would be 7.10^8 °K, but we must find the time required for the protons and electrons to come to equilibrium before Hoyle's assumption can be finally justified. We can use a result obtained by Spitzer (1956) to compute the time required for a fully ionised gas to come to equilibrium. The equilibrium times for a fixed value of the proton temperature ($5.8.\ 10^6$ °K) and various values of the electron temperature have been computed for the relevant density range. The values are given in Table X. The equilibrium time τ_E is given by

$$\tau_E = \frac{3 m_i m_e k^{3/2}}{8 (2\pi)^{1/2} n_i e^4 \log_e \Lambda} \left\{ \frac{T_e}{m_e} + \frac{T_i}{m_i} \right\}^{3/2}, \qquad (6)$$

where $$\Lambda = 9.72 . 10^2 \left\{ \frac{k^3 T_e^2}{\pi n_e e^6} \right\}^{1/2}, \qquad (7)$$

where m is the mass of the particle denoted by the subscript, e is the electron charge, k is Boltzmann's constant and the other terms have their

usual meanings. The values of $\log_e \Lambda$ are given in Table XI.

Table X

n_e \ T_e	10^9	10^8	10^7	10^6
10^{-5}	$1.89.\ 10^{19}$	$5.75.\ 10^{17}$	$2.12.\ 10^{16}$	$6.50.\ 10^{14}$
10^{-4}	$1.94.\ 10^{18}$	$5.92.\ 10^{16}$	$2.19.\ 10^{15}$	$6.72.\ 10^{13}$
10^{-3}	$2.00.\ 10^{17}$	$6.10.\ 10^{15}$	$2.26.\ 10^{14}$	$6.95.\ 10^{12}$
10^{-2}	$2.06.\ 10^{16}$	$6.30.\ 10^{14}$	$2.33.\ 10^{13}$	$7.20.\ 10^{11}$
10^{-1}	$2.12.\ 10^{15}$	$6.50.\ 10^{13}$	$2.42.\ 10^{12}$	$7.47.\ 10^{10}$
1	$2.19.\ 10^{14}$	$6.72.\ 10^{12}$	$2.50.\ 10^{11}$	$7.77.\ 10^{9}$

Table XI

n_e \ T_e	10^9	10^8	10^7	10^6
10^{-5}	42.38	40.08	37.78	35.47
10^{-4}	41.23	38.93	36.63	34.32
10^{-3}	40.08	37.78	35.47	33.17
10^{-2}	38.93	36.63	34.32	32.02
10^{-1}	37.78	35.47	33.17	30.87
1	36.63	34.32	32.02	29.73

The values of $\log_e \Lambda$ in Table XI extend a table given by Spitzer (1956) to lower densities. The value of τ_E from Table X shows, for regions in which the electron temperature is 10^9 °K. and density of 10^{-5} electrons cm^{-3}, that equilibrium will only be established after fifty Hubble times have elapsed. Therefore the intergalactic medium, if it is formed by the decay of neutrons, will not be in thermal equilibrium.

However, after the compression of the material to a density of 10^{-27} $gm.cm^{-3}$ and cooling to 10^7 °K. the electrons and protons will rapidly reach equilibrium in approximately 2.10^{14} sec. Consequently in treating this region expressions (5) and (2) are identical and Hoyle's assumption is valid.

The low density hot regions which are not in thermal equilibrium may be investigated further by finding out what period of time must elapse before a Maxwellian distribution of velocities can become established among the electrons and among the protons. Using a further result given by Spitzer (1956) the time for a Maxwellian distribution of velocities to become established is given by

$$\tau_M = \frac{m^{1/2}(3kT)^{3/2}}{5.912\pi n e^4 \log_e \Lambda} \quad . \qquad (8)$$

For the electrons τ_M has the value 2.10^{16} sec., so that the electrons may be regarded as having a Maxwellian distribution of velocities. The protons on the other hand require a time of $1.6.\ 10^{23}$ sec. to elapse before they can establish a Maxwellian distribution of velocities among themselves. The protons therefore will not have a Maxwellian distribution of velocities.

Since the hot gas is not in thermal equilibrium it is of interest to investigate the problem of how such a gas will compress a cool gas. The maximum speed with which the hot gas can compress the cool gas is the speed with which the hot gas can expand into a vacuum. This approximation is only a good one when there is a large difference in pressure between the hot gas and the cold one. Such a situation will occur during the middle stages of the compression process considered by Hoyle. This problem will be discussed in the following sections.

The rate of compression of a cool gas by a hot one is not the only problem which arises out of Hoyle's new theory of galaxy formation. By far the largest problem remaining unsolved is that of the primary cooling mechanism. The cooling produced by the material actually in galaxies will be negligible, but this type of mechanism cannot be ruled out, since

galaxies are grouped in clusters and it is not known how much gas exists between the galaxies in a cluster. If large amounts of gas are present in the clusters they may act as very efficient cooling agents. If the cluster were able to produce cooling and also was in random motion through the gas, it could tunnel out a region of cold gas. This cold gas will not necessarily be in pressure equilibrium with the surrounding hot gas. The pressure disequilibrium may then lead to the collapse of the cooler region. The idea here is similar to that proposed by Sciama (1955) except that pressure forces and not gravitational forces are involved.

A further line of approach in accordance with the ideas outlined in Chapter II would be to assume that the intergalactic medium is non-uniform. The regions which have a higher density than average will cool more rapidly and will tend to become compressed by the hot gas. It would be of interest to investigate what sort of variation of density would be required to establish such a system.

Hoyle did not consider intergalactic magnetic fields. If the intergalactic material is fully ionised, as has been suggested by Hoyle's theory, then any intergalactic magnetic field which may exist will

be of the greatest significance. The effect of these fields on non-uniform material would also be of interest.

However these problems, though important, lie outside the scope of this thesis and we shall confine our attention to discussing expansion of a fully ionised gas into a vacuum in order to see how rapidly pressure forces may compress the medium.

Section 2. The fundamental theory of expansion waves.

The fundamental theory of expansion waves for an unionised gas will be discussed here very briefly in order to clarify the various solutions of the problem of the expansion of a monatomic gas into a vacuum as discussed in Section 3. This section is a brief summary of chapter one and part of chapter three of a book by Courant and Friedrichs (1948). Their discussion of hyperbolic flow, while of fundamental importance to this work, is too long to be included here. Reference will be made to their discussion of hyperbolic flow when necessary.

In order to simplify the problem we shall assume that flow takes place in one direction only, namely along the x-axis. The motion of the gas is then defined by four equations, two of which are conservation laws, one is an equation of state, while the fourth indicates how changes of state are to be made. For a compressible fluid of density ρ flowing with speed u along the x-axis, the conservation laws and the equation of state may be

written as below

Conservation of Momentum: $$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0, \quad (9)$$

Conservation of Mass: $$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0, \quad (10)$$

Equation of State: $$p = f(\rho, S), \quad (11)$$

where p, S are the pressure and entropy of the fluid.

Changes of state in a flow problem may be made in three ways, (a) isothermally, (b) isentropically, (c) adiabatically. In this work we shall assume that all changes of state are made isentropically, so assuming that the fluid is inviscid. The fourth equation expressing that changes of state are made isentropically has the form

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = 0. \quad (12)$$

The speed of sound a for the gas is defined by

$$a^2 = \frac{\partial p}{\partial \rho}. \quad (13)$$

Using equations (11) and (13) it may be shown that

$$dp = a^2 d\rho + \frac{\partial f}{\partial S} dS. \quad (14)$$

Division of equation (14) by dt and use of equations (10) and (12) gives

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho a^2\frac{\partial u}{\partial x} = 0. \qquad (15)$$

Multiplication of equation (9) by a gives on addition to equation (15)

$$\frac{\partial p}{\partial t} + (u+a)\frac{\partial p}{\partial x} + \rho a\left\{(u+a)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right\} = 0, \qquad (16)$$

while on subtraction,

$$\frac{\partial p}{\partial t} + (u-a)\frac{\partial p}{\partial x} - \rho a\left\{(u-a)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right\} = 0. \qquad (17)$$

Courant and Friedrichs from the form of equations (12), (16) and (17) choose the following three directions in the x, t plane,

$$dx = u\,dt, \qquad (18a)$$

$$dx = (u+a)\,dt, \qquad (18b)$$

$$dx = (u-a)\,dt, \qquad (18c)$$

since the theory of hyperbolic flow discussed by them suggests that these directions are of fundamental importance. We shall denote these three directions in

the x, t plane by C_o, C_+, C_- respectively. Substitution of equation (18a) in equation (12), equation (18b) in equation (16) and equation (18c) in equation (17) shows that

$$dS = 0 \qquad \text{along } C_o, \qquad (19a)$$

$$dp = -\rho a\, du \qquad \text{along } C_+, \qquad (19b)$$

$$dp = +\rho a\, du \qquad \text{along } C_-. \qquad (19c)$$

Since the flow has been assumed to be isentropic equation (19a) adds no new information, but equations (19b), (19c) are called the Characteristic Equations since they are true along the Characteristic Directions C_+, C_- respectively.

Courant and Friedrichs from their discussion of hyperbolic flow show that the directions C_+, C_- are of fundamental importance in this type of flow problem. They show that C_+ or C_- represents, in the x, t plane, the path taken by the "head" of a "disturbance wave". A disturbance wave in this context may be thought of, for example, as the disturbance caused when a piston is suddenly pushed into a gas initially at rest: the head of such a disturbance wave separates the as yet undisturbed gas

from the disturbed gas. Equations (18b), (18c) show the velocity of the disturbance is $u - a$ along a C_- characteristic and $u + a$ along a C_+ characteristic. The velocity of the disturbance relative to the gas, whose flow speed is u, is $\mp a$ for each point on the characteristic. The head of a disturbance wave therefore travels with the speed of sound relative to the gas. Courant and Friedrichs use this argument to justify the identification of the characteristics with the paths of sound waves.

They now introduce two quantities $\uparrow$ and $\downarrow$ called the Riemann Invariants, which are defined to be the constants of integration obtained in the integration of equations (19b) and (19c) respectively

$$u + \int_{p_0}^{p} \frac{dp}{\rho a} = 2\uparrow , \tag{20}$$

$$u - \int_{p_0}^{p} \frac{dp}{\rho a} = -2\downarrow . \tag{21}$$

For a real gas it is assumed that the integrals in equations (20) and (21) are positive for a finite ρ and vanish as ρ vanishes. In order to evaluate these integrals we must assume a relation between p and ρ. The relation between p and ρ is taken to be the adiabatic law,

$$p = K\rho^{\gamma}, \tag{22}$$

where γ is the ratio of specific heats and K is a constant. Substituting for p in equations (20), (21) and performing the integration one obtains

$$2\uparrow = \frac{2a}{\gamma - 1} + u \,, \qquad (23)$$

$$2\downarrow = \frac{2a}{\gamma - 1} - u \,, \qquad (24)$$

assuming that ρ_0 tends to zero.

Since equations (23), (24) have been obtained by the integration of equations (19b), (19c) which relate the change of pressure and flow speed along a characteristic, it can be inferred that $\uparrow$ and $\downarrow$ are constant along a characteristic: $\uparrow$ is constant along a C_+ characteristic and $\downarrow$ along a C_- characteristic. By using the Riemann Invariants the motion at the head of a disturbance may be investigated.

Three types of flow may now be distinguished: (a) a constant state in which u, ρ remain constant, (b) a simple wave in which one or other of $\uparrow, \downarrow$ remain constant, (c) the general case in which neither $\uparrow$ nor $\downarrow$ remains constant. A rarefaction or expansion wave is a simple wave and this is the type of wave considered here.

The Riemann Invariants can be used to investigate what happens when a gas expands. Suppose

initially that a state of uniform density exists in the region $-\infty \leq x \leq 0$ with a boundary at $x = 0$. This boundary is taken to be a moveable frictionless piston. If the piston is made to recede from the gas (i.e. moving from left to right) a wave will move from the piston into the gas and only those gas particles, which have been reached by this wave front, will take part in the expansion. Let the piston start from rest and recede from the gas with a continually increasing speed u_p. The gas at the piston acquires the speed of the piston, so that the piston speed can replace the flow speed in the expression for the Riemann Invariant. The Riemann Invariant for a wave proceeding from right to left is $\uparrow$ if u_p is taken in the direction of increasing x.

If u_p is initially zero and a_0 is the initial sound speed in the uniform gas then,

$$2\uparrow = \frac{2a_0}{\gamma - 1},$$

initially and in general,

$$u_p = \frac{2}{\gamma - 1}(a_0 - a). \tag{25}$$

When the piston reaches a speed of $2a_0(\gamma - 1)^{-1}$, the sound speed in the gas at the piston becomes zero, implying that the gas has reached zero density at the

piston. This is known as the stage of complete rarefaction. If u_p exceeds $2a_0(\gamma - 1)^{-1}$ the piston would simply move away from the gas. For $\gamma = 5/3$ complete rarefaction would be obtained when the piston had acquired a speed of $3a_0$. The wave proceeding into the gas from the piston moves with a constant speed a_0, which is the speed of sound in the undisturbed gas.

The object of this summary of the fundamental theory of rarefaction waves is to show that a state of complete rarefaction demands that a speed of $2a_0(\gamma-1)^{-1}$ be associated with it. This is subject to the requirement that the flow is isentropic and the gas obeys the adiabatic law connecting the pressure and density. This interpretation enables us to understand what is happening in the case of a freely expanding gas which is discussed in the next section.

Section 3. The expansion of a monatomic gas into a vacuum.

The expansion of a monatomic gas into a vacuum is considered because this problem has been treated by several authors. These authors obtain a solution for the speed of expansion of the gas under different conditions. The value of considering this problem lies in the fact that the equations of motion for a fully ionised gas, under the conditions later assumed, can be reduced to the equations of motion for a monatomic gas.

The problem has been considered principally by Burgers (1946), McVittie (1950) and Copson (1950). We shall show that the results obtained by these authors are a consequence of the boundary conditions which they assume, namely that the density of the gas on the gas vacuum interface is zero. This is a state of complete rarefaction, which we have shown in Section 2 to imply an expansion speed of $2a_0(\gamma - 1)^{-1}$. We shall, however, start with a brief review of the work of previous authors who tackled this problem.

Previous work on the expansion of a monatomic gas.

Burgers (1946) considered the expansion of an initially uniform semi-infinite distribution of gas into a vacuum. Except at the initial instant, he assumed that the density of the gas at the gas vacuum interface was zero. He found that the gas expanded into vacuum with a constant speed of $2a_0(\gamma-1)^{-1}$ where a_0 is the sound speed in the initially undisturbed uniform gas.

McVittie (1950) attempted to improve on Burgers solution by introducing a model which removed the discontinuity in the density boundary condition at the initial instant. McVittie assumed that the semi-infinite gas had a non-uniform density distribution. The density of the gas declined to zero at the initial boundary. He found that the gas attained Burgers' speed of expansion after an infinite time, where a_0 is now defined to be the sound speed in the undisturbed gas in the interior of the gas distribution far from the expansion front. However, McVittie offers no evidence on how rapidly this final speed is attained.

Copson (1950) pointed out that the density distribution for the gas considered by McVittie, would

cause the gas to be in motion, since there would be pressure differences within the gas due to the non-uniform density. He therefore suggested a compromise model. He assumed that most of the semi-infinite gas had a uniform density distribution and that there was a boundary layer in which the gas density declined to zero at the gas vacuum interface. Copson showed that the speed of expansion $2a_0(\gamma-1)^{-1}$ (where a_0 is the speed of sound in the uniform undisturbed gas) was attained after a time, which depended on the thickness assumed for the boundary layer. After the elapse of this time the expansion proceeded with a constant velocity as found by Burgers. Copson's model has been used by most subsequent authors in discussing the expansion of a monatomic gas into a vacuum.

Copson (1953) has rediscussed this problem using a complex variable approach for a general value of γ. For the special case of $\gamma = 5/3$ he recovered his previous solution. Pack (1953a) reviewed the problem and showed that if the flow remained continuous, Burgers result must ultimately apply. He further showed that if discontinuities did arise, then an even greater final speed of expansion could result. Pack (1953b) also considered the expansion into vacuum

of a gas in which there was an initial inhomogeneous temperature distribution. He showed that the initial complex waves grew into a simple wave and the results were similar to those obtained from Copson's model.

Mackie (1953) extended this work to a finite region containing gas free to expand in opposite directions into a vacuum, when the density of the gas was inhomogeneous having a maximum at the centre of the distribution. The final speed of expansion was $2(\gamma-1)^{-1}$ times the initial sound speed at the centre of the distribution of gas.

Khare (1953, 1954a, 1954b) treated Copson's model in a new way not involving the Riemann Invariants. In his first paper he discussed Copson's model using the Riemann Invariants for any general γ, while in his second paper he showed that by introducing the velocity potential he could solve the equations of motion without recourse to the Riemann Invariants. His results from the second paper could be interpreted to show that the final speed of expansion was $2a_0(\gamma-1)^{-1}$. In his final paper Khare showed that viscosity could be taken into account if the velocity potential method of solution were adopted. However, he could only obtain a solution for $\gamma = 3$. In this case the effect of viscosity disappeared and the

expansion speed was a_0 .

The mathematical formulation of the problem.

We shall now give the mathematical formulation of the expansion of a monatomic gas as developed by McVittie (1950). The resulting equations will be solved for a uniform semi-infinite distribution of gas by a method also given by McVittie obtaining Burgers' (1946) solution to the problem.

The problem will be solved assuming the adiabatic law (equation (22)). The fundamental equations governing the flow are equations (9), (10) and (13). Substituting from equations (13) and (22) the conservation laws (equations (9), (10)) may be written in the form,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{2a}{\gamma-1}\frac{\partial a}{\partial x} = 0 , \qquad (26)$$

$$\frac{2}{\gamma-1}\frac{\partial a}{\partial t} + \frac{2u}{\gamma-1}\frac{\partial a}{\partial x} + a\frac{\partial u}{\partial x} = 0 . \qquad (27)$$

The Riemann Invariants are introduced in the form,

$$2\lambda = \frac{2a}{\gamma-1} + u , \qquad 2\mu = \frac{2a}{\gamma-1} - u .$$

Differentiation of the Riemann Invariants with respect

to x gives

$$2\frac{\partial f}{\partial x} = \frac{2}{\gamma-1}\frac{\partial a}{\partial x} + \frac{\partial u}{\partial x}, \qquad 2\frac{\partial s}{\partial x} = \frac{2}{\gamma-1}\frac{\partial a}{\partial x} - \frac{\partial u}{\partial x},$$

and differentiation with respect to t gives

$$2\frac{\partial f}{\partial t} = \frac{2}{\gamma-1}\frac{\partial a}{\partial t} + \frac{\partial u}{\partial t}, \qquad 2\frac{\partial s}{\partial t} = \frac{2}{\gamma-1}\frac{\partial a}{\partial t} - \frac{\partial u}{\partial t}.$$

Addition of the derivatives of f and s with respect to x gives

$$\frac{\partial f}{\partial x} + \frac{\partial s}{\partial x} = \frac{2}{\gamma-1}\frac{\partial a}{\partial x},$$

while subtraction gives

$$\frac{\partial f}{\partial x} - \frac{\partial s}{\partial x} = \frac{\partial u}{\partial x}.$$

Addition of the derivatives of f and s with respect to t gives

$$\frac{\partial f}{\partial t} + \frac{\partial s}{\partial t} = \frac{2}{\gamma-1}\frac{\partial a}{\partial t},$$

while subtraction gives

$$\frac{\partial f}{\partial t} - \frac{\partial s}{\partial t} = \frac{\partial u}{\partial t}.$$

Making these substitutions for the derivatives of a, u with respect to x, t in equations (26) and (27)

one obtains,

$$\frac{\partial \dagger}{\partial t} + (u + a)\frac{\partial \dagger}{\partial x} = 0 , \tag{28a}$$

$$\frac{\partial \diamond}{\partial t} + (u - a)\frac{\partial \diamond}{\partial x} = 0 . \tag{28b}$$

From the definition of $\dagger, \diamond$ it follows that

$$\dagger + \diamond = \frac{2a}{\gamma - 1} , \quad \dagger - \diamond = u .$$

Therefore $$u + a = \alpha \dagger + \beta \diamond$$

and $$u - a = -(\beta \dagger + \alpha \diamond)$$

where $$\alpha = \tfrac{1}{2}(\gamma + 1) , \quad \beta = \tfrac{1}{2}(\gamma - 3) .$$

Substitution of these results into equations (28) gives,

$$\frac{\partial \dagger}{\partial t} + (\alpha \dagger + \beta \diamond)\frac{\partial \dagger}{\partial x} = 0 , \tag{29a}$$

$$\frac{\partial \diamond}{\partial t} - (\alpha \diamond + \beta \dagger)\frac{\partial \diamond}{\partial x} = 0 . \tag{29b}$$

McVittie then assumes a result due to Riemann that if,

$$\frac{\partial \dagger}{\partial x}\frac{\partial \diamond}{\partial x} \neq 0 ,$$

then $\dagger, \diamond$ may be used as independent variables. Since,

$$d\dagger = \frac{\partial \dagger}{\partial t} dt + \frac{\partial \dagger}{\partial x} dx , \quad d\diamond = \frac{\partial \diamond}{\partial t} dt + \frac{\partial \diamond}{\partial x} dx ,$$

use of equation (29) gives,

$$d\tau = \frac{\partial \tau}{\partial x}\left\{ -(\alpha\tau + \beta\sigma)dt + dx \right\}$$

$$= \frac{\partial \tau}{\partial x}\left[d\left\{ x - (\alpha\tau + \beta\sigma)t \right\} + t(\alpha d\tau + \beta d\sigma) \right],$$

and similarly,

$$d\sigma = \frac{\partial \sigma}{\partial x}\left[d\left\{ x + (\beta\tau + \alpha\sigma)t \right\} - t(\beta d\tau + \alpha d\sigma) \right].$$

McVittie then defines functions X, Y by setting,

$$X = x - (\alpha\tau + \beta\sigma)t \,, \quad Y = x + (\alpha\sigma + \beta\tau)t, \tag{30}$$

and by regarding X, Y as functions of τ, σ obtains,

$$d\tau = \frac{\partial \tau}{\partial x}\left\{ \left(\frac{\partial X}{\partial \tau} + \alpha t\right)d\tau + \left(\frac{\partial X}{\partial \sigma} + \beta t\right)d\sigma \right\},$$

$$d\sigma = \frac{\partial \sigma}{\partial x}\left\{ \left(\frac{\partial Y}{\partial \tau} - \beta t\right)d\tau + \left(\frac{\partial Y}{\partial \sigma} - \alpha t\right)d\sigma \right\}.$$

But since τ, σ are independent variables the above equations give,

$$\frac{\partial X}{\partial \sigma} + \beta t = 0 \,, \quad \frac{\partial Y}{\partial \tau} - \beta t = 0 \,. \tag{31}$$

From equations (31) one obtains,

$$\frac{\partial X}{\partial \sigma} - \left(-\frac{\partial Y}{\partial \tau}\right) = 0 \,, \tag{32}$$

and so a function w of τ, σ can be defined by setting

$$dw = X d\tau - Y d\sigma \,.$$

Therefore,

$$-\frac{\partial w}{\partial s} = Y = x + (\alpha s + \beta t)t \,, \qquad (33a)$$

$$\frac{\partial w}{\partial t} = X = x - (\alpha t + \beta s)t \,. \qquad (33b)$$

Using equation (31) and (33),

$$-\frac{\partial^2 w}{\partial t\,\partial s} = \frac{\partial Y}{\partial t} = \beta t \,,$$

and $$Y - X = (\alpha + \beta)(t + s)t \,,$$

then $$\frac{\partial^2 w}{\partial t\,\partial s} + \frac{\beta}{(\alpha+\beta)(t+s)}\left\{\frac{\partial w}{\partial t} + \frac{\partial w}{\partial s}\right\} = 0 \qquad (34)$$

Equation (34) is the fundamental equation to be solved. McVittie and Copson solve this equation for the particular case in which $\gamma = 5/3$ when equations (34) takes the form,

$$\frac{\partial^2 w}{\partial t\,\partial s} + \frac{1}{(t+s)}\left\{\frac{\partial w}{\partial t} + \frac{\partial w}{\partial s}\right\} = 0 \,. \qquad (35)$$

Khare (1953, 1954a) solved equations (34) quite generally.

Equation (35) was solved by McVittie in the following way. Consider gas expanding along the **x** - axis as shown in Figure 5. Initially the gas vacuum interface was at $(B_1)_i$ while at some later

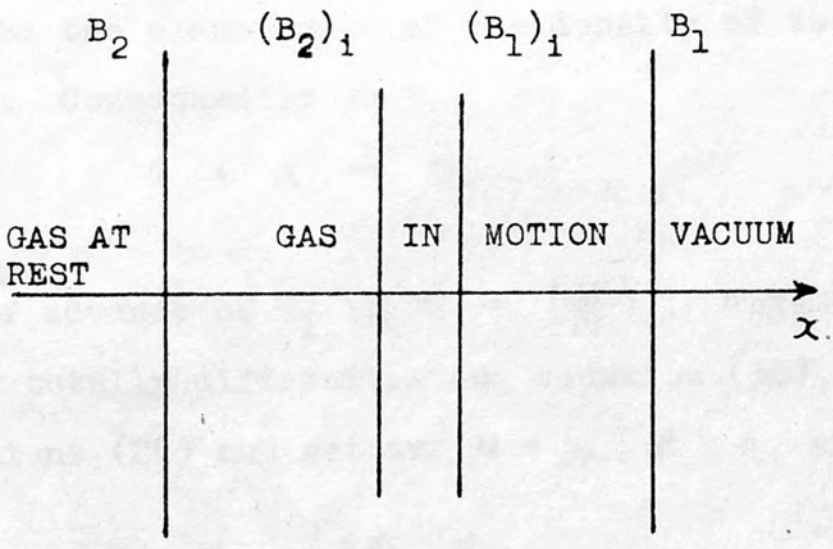


Figure 5.

time it is at B_1. The wave front moving into the gas at rest was initially at $(B_2)_i$ and at any later time is at B_2. For the present, the assumption that $(B_1)_i$, $(B_2)_i$ are at the same place will not be made nor will the origin of coordinates be taken at one or other of these interfaces.

The interface B_1 advancing into the vacuum is defined to be the plane on which the density of the gas is zero. Consequently on B_1,

$$f_1 + g_1 = 0 . \tag{36}$$

The speed of advance of B_1 is $v_1 = \left(\frac{dx}{dt}\right)_1$. This is obtained by totally differentiating equation (36), using equations (28) and setting $u = u_1$, $a = 0$ whence

$$v_1 = u_1 \frac{\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}}{\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}} = u_1 . \tag{37}$$

The boundary therefore advances with the speed of the gas at the boundary.

On the front B_2 the boundary conditions are $u_2 = 0$, $a = -a_0$ whence

$$f_2 - g_2 = 0 , \tag{38a}$$

$$f_2 + g_2 = -\frac{2a_0}{\gamma - 1} . \tag{38b}$$

For a homogeneous distribution of gas a_0 is constant. Let B_2 be at x_2 at time t_2 and at $x_2 - dx_2$ at time $t_2 + dt_2$, then the speed of advance of B_2 is $v_2 = (-\frac{dx}{dt})_2$, which by total differentiation of the equations (38) and the application of the boundary conditions gives

$$(\frac{\partial \Lambda}{\partial x})_2 = 0 \quad , \tag{39a}$$

$$v_2 = -a_0 \quad . \tag{39b}$$

Hence B_2 is moving to the left with the speed of sound in the undisturbed gas. Equation (39a) indicates that Λ_2 is constant in the region $x_2 \leq x \leq (x_2)_i$.

The Riemann invariant Λ_2 is constant so that the subscript may now be dropped and the value of Λ may be found from the boundary conditions at B_2 giving,

$$\Lambda = -\frac{a}{\gamma - 1} \quad . \tag{40}$$

Using equation (40) in equation (36) gives,

$$v_1 = u_1 = 2\uparrow_1 = -2\Lambda_1 = \frac{2a_0}{\gamma - 1} \quad .$$

Therefore equation (29a) may be written,

$$\frac{\partial \uparrow}{\partial t} + (\alpha \uparrow - \frac{\beta a_0}{\gamma - 1})\frac{\partial \uparrow}{\partial x} = 0 \quad ,$$

and its solution equation (30a) may be written

$$x - t(\alpha \uparrow - \frac{\beta a_0}{\gamma - 1}) = f(\uparrow) \quad , \tag{41}$$

where $f(\tau)$ is an arbitary function of τ .
Substituting for τ, α and β ,

$$x - t\left\{\tfrac{1}{4}(\gamma+1)\left(\frac{2a}{\gamma-1} + u\right) - \frac{\gamma-3}{2(\gamma-1)}a_0\right\} = f\left(\frac{a}{\gamma-1} + \tfrac{1}{2}u\right), \quad (42)$$

$$\text{where} \qquad u = \frac{2}{\gamma-1}(a + a_0). \quad (43)$$

This solution (equations (42) and (43)) gives a complete description of the motion of the front advancing into a vacuum.

If we now follow Burgers (1946) and assume that $(B_2)_i$, $(B_1)_i$ are identical and are at the origin of x then since $x = 0$ when $t = 0$, $f(\tau) = 0$ so that solving for a and u we obtain,

$$u = \frac{2}{\gamma+1}\frac{x}{t} + \frac{2}{\gamma+1}a_0 , \quad (44)$$

$$a = \frac{\gamma-1}{\gamma+1}\frac{x}{t} - \frac{2}{\gamma+1}a_0 . \quad (45)$$

Equations (44) and (45) are Burgers' solution to the problem. The results show that the front advances into the vacuum with speed $2a_0(\gamma-1)^{-1}$ and that a wave moves back into the gas at rest with speed a_0 .

Using a similar approach McVittie solves the problem for a gas in which there is a non-uniform density distribution. Copson's (1950) approach is

also similar though his boundary conditions are more complicated. The results of their analyses have been discussed previously.

For a monatomic gas $\gamma = 5/3$ so that an originally uniform gas will expand into a vacuum with a speed of three times the sound speed in the undisturbed gas.

Discussion.

The result obtained from this analysis is a result of the boundary conditions assumed at the gas vacuum interface. The interface is defined to be the plane over which the gas density is zero. Such a boundary condition implies that a state of complete rarefaction exists at the interface. Therefore the speed of the interface must be $2a_0(\gamma - 1)^{-1}$ as has been already shown when discussing the Riemann Invariants in Section 2.

It is, however, unrealistic to assume that complete rarefaction has been attained at the interface at all stages of the motion. Consider an atomic model of the expanding gas. Suppose we have a semi-infinite uniform gas contained behind a barrier. Let all the atoms of the gas have the same random

velocity. When the barrier is removed those atoms near the boundary which happen to have a velocity component in the x-direction will move out into the vacuum. Immediately after the barrier has been removed the boundary between gas and vacuum will still be quite sharp, since dispersion of the atoms due to their random motion will not have had an appreciable effect. At a much later stage, assuming no interatomic collisions for simplicity, only those atoms which initially happen to be moving very closely parallel to the x-axis will constitute the leading edge, the remainder having dispersed. As one proceeds further from the initial boundary the number of atoms moving parallel to the x-axis will become fewer so that ultimately the approximation of zero density on the gas vacuum interface becomes a very accurate one.

While the work of Burgers tells us little about the start of an expansion of a semi-infinite gas into a vacuum, the atomic point of view suggests that some time after the removal of the barrier Burgers solution is an accurate description of what is happening. This is also the conclusion reached by Pack (1953b) on somewhat different grounds. The atomic model indicates that Copson's model is the most realistic of the three models considered. Consequently, we may assume

that a uniform gas ultimately expands into a vacuum with a speed $2a_0(\gamma-1)^{-1}$ where a_0 is the speed of sound in the undisturbed gas. This speed is attained when sufficient time has elapsed to make the approximation, that the density of the gas at the gas vacuum interface is zero, a good one and this time will depend on the specific problem considered.

Section 4. Lagrange's Ballistic Problem.

The solutions for the expansion of a monatomic gas into a vacuum obtained by McVittie and Copson require a complicated mathematical analysis for their evaluation. However, Lagrange's Ballistic Problem as considered by Love and Pidduck (1922) offers an easier mathematical analysis and their analysis is used first to discuss the expansion of a gas which obeys a van der Waal's equation of state and then to discuss a new model for the expansion of a gas.

The solution by Love and Pidduck to Lagrange's ballistic problem.

Love and Pidduck (1922) solved Lagrange's ballistic problem for a gas contained in a cylinder between two frictionless movable pistons. They considered the motion of the two pistons quite generally taking into account the effects of the collisions of the rarefaction waves with both other rarefaction waves and the pistons. We are concerned here only with one part of the problem, namely with the motion of a single piston without the additional complication of its collision with wave fronts.

The problem we wish to consider is similar to

that already discussed when considering the interpretation of the Riemann Invariants. Consider a semi-infinite distribution of gas contained in a cylinder closed at one end by a frictionless movable piston of mass M. In order to remove any difficulties about wall effects the cylinder may be considered as extending to infinity in all directions perpendicular to the direction of expansion, which is taken to be the x-axis described positively. Only unit area of the piston is considered, so that M is the mass of the piston per unit area. The system is sketched in Figure 6.

Suppose that the gas has pressure p_0 and density ρ_0 initially where p_0, ρ_0 are homogeneous throughout the gas. Let the piston be held at rest at $x = 0$, the gas occupying the region between $x = 0$ and $x = -\infty$. When the piston is released the gas pressure pushes the piston in the direction of x increasing. Consequently, the gas becomes rarefied and the process may be described in terms of simple waves as discussed in Section 2.

In the mathematical analysis which follows we use the method devised by Love and Pidduck. If, when the speed of the piston is v from left to right, the pressure at the piston is p, then the equation

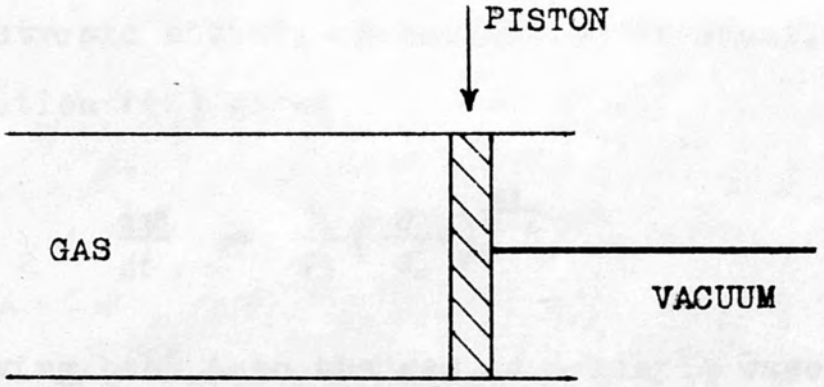


Figure 6.

of motion for the piston is,

$$M\frac{dv}{dt} = p \tag{46}$$

Defining σ to be $2a(\gamma-1)^{-1}$ then,

$$\frac{p}{p_0} = \left(\frac{\sigma}{\sigma_0}\right)^{\frac{2\gamma}{\gamma-1}}, \tag{47}$$

for the isentropic motion. Substitution of equation (47) in equation (46) gives

$$\frac{dv}{dt} = \frac{p_0}{M}\left(\frac{\sigma}{\sigma_0}\right)^{\frac{2\gamma}{\gamma-1}}. \tag{48}$$

The wave moving back into the gas is a simple wave moving from right to left, so that the Riemann Invariant $\uparrow$ is constant while $\downarrow$ varies. If a_0 is the sound speed in the undisturbed gas then the constancy of $\uparrow$ implies,

$$v + \frac{2a}{\gamma-1} = \frac{2a_0}{\gamma-1},$$

$$\text{or} \quad v + \sigma = \sigma_0 \tag{49}$$

where the piston speed v now replaces the flow speed. From equation (49) $dv = -d\sigma$ so that equation (48) becomes,

$$dt = -\frac{M}{p_0}\left(\frac{\sigma_0}{\sigma}\right)^{\frac{2\gamma}{\gamma-1}} d\sigma.$$

Integration of this equation subject to the boundary conditions $t = 0$, $\sigma = \sigma_0$; $t = t$, $\sigma = \sigma$ gives

$$t = \frac{M\sigma_0}{p_0}\frac{\gamma-1}{\gamma+1}\left\{\left(\frac{\sigma_0}{\sigma}\right)^{\frac{\gamma+1}{\gamma-1}} - 1\right\} . \qquad (50)$$

From equations (49) and (50) the velocity of expansion may be written,

$$v = \sigma_0 - \sigma = \sigma_0\left\{1 - \left(1 + \frac{\gamma+1}{2}\frac{p_0 t}{Ma_0}\right)^{-\frac{\gamma-1}{\gamma+1}}\right\},$$

$$= \frac{2a_0}{\gamma-1}\left\{1 - \left(1 + \frac{\gamma+1}{2}\frac{p_0 t}{Ma_0}\right)^{-\frac{\gamma-1}{\gamma+1}}\right\} . \qquad (51)$$

Equation (51) may be integrated further to give the distance travelled by the piston. Taking the boundary condition to be $x = 0$, $t = 0$ then at any later time,

$$x = \frac{2a_0}{\gamma-1}\left[t - \frac{Ma_0}{p_0}\left\{\left(1 + \frac{\gamma+1}{2}\frac{p_0 t}{Ma_0}\right)^{\frac{2}{\gamma+1}} - 1\right\}\right] \qquad (52)$$

Equations (51) and (52) were obtained by Love and Pidduck as a first step in their solution of Lagrange's ballistic problem. From equation (51) it would appear that v attains the value $2a_0(\gamma-1)^{-1}$ only after an infinite time. However, this is not particularly relevant since we are interested in how rapidly it approaches this limiting value. The rate at which v

attains its limiting value is determined by the initial conditions of pressure and density as well as by the mass of the piston.

Using equation (51) the density of the gas at the piston is given by

$$\rho = \rho_0 \left(1 + \frac{\gamma+1}{2} \frac{p_0 t}{M a_0}\right)^{-\frac{2}{\gamma+1}} . \qquad (53)$$

Equation (53) shows that ρ vanishes as $t \longrightarrow \infty$ i.e. the final speed of expansion is associated with the state of complete rarefaction, as would be expected from our general discussion in Section 2.

It is also clear from equation (51) that if the mass of the piston vanishes the speed of expansion is constant. Therefore this solution passes over into Burgers' (1946) solution for a piston of zero mass.

The motion of a piston pushed by a van der Waal's gas.

The method of solution devised by Love and Pidduck as outlined above can be modified to take account of a gas obeying van der Waal's equation of state. We have examined this case in order to see how rapidly a gas, approximating more nearly to a real gas, could push the piston.

One form of van der Waal's equation of state is,

$$(p + c\rho^2)(1 - b\rho) = knT , \qquad (54)$$

where ρ is the density, p is the pressure, k is Boltzmann's constant, n is the number of particles per unit volume, T is the absolute temperature and b, c are positive constants. The pressure P of the equivalent perfect gas is

$$P = p + c\rho^2 . \qquad (55)$$

Suppose the piston acquires a speed U when pushed by a perfect gas as in Lagrange's ballistic problem described earlier and let ΔU be the increment in speed due to the fact that the gas is not perfect. Then we may define the piston speed v to be $U + \Delta U$. As before the equation of motion of a piston of mass M per unit area is

$$M\frac{dv}{dt} = p , \qquad (56)$$

$$\text{or} \quad M\frac{d}{dt}(U + \Delta U) = P - c\rho^2 . \qquad (57)$$

Equation (57) can now be divided into two parts. By definition U is the speed acquired by the piston when pushed by a perfect gas so,

$$M\frac{dU}{dt} = P .$$

But this is simply equation (46) which has solution,

$$U = \frac{2a_0}{\gamma-1}\left\{1 - \left(1 + \frac{\gamma+1}{2}\frac{P_0 t}{Ma_0}\right)^{-\frac{\gamma-1}{\gamma+1}}\right\},$$

where P_0 is the initial pressure and γ, a_0 are defined for the perfect gas.

Removal of that part of equation (57) which concerns the perfect gas leaves us with

$$M\frac{d}{dt}(\Delta U) = -c\rho^2 \qquad (58)$$

Since c is a positive constant a deceleration of the piston will result from the departure of the gas from a perfect gas.

In order to solve equation (58) we must know the dependence of ρ on t. However, this is unknown and so some approximate method of solution must be found. As a first order approximation it will be assumed that ρ depends on t as for a perfect gas. This dependence is given by equation (53). Therefore taking the boundary condition to be $\Delta U = 0$, $t = 0$ we have

$$\Delta U = -\frac{c\rho_0^2}{M}\int_0^t \left(1 + \frac{\gamma+1}{2}\frac{P_0 t}{Ma_0}\right)^{-\frac{4}{\gamma+1}} dt,$$

$$= -\frac{2a_0}{\gamma-1}\frac{\gamma-1}{3-\gamma}\frac{c\rho_0^2}{P_0}\left\{1 - \left(1 + \frac{\gamma+1}{2}\frac{P_0 t}{Ma_0}\right)^{-\frac{3-\gamma}{\gamma+1}}\right\} \qquad (59)$$

Eliminating P_0 equation (59) may be written,

$$\Delta U = -\frac{2a_0}{\gamma-1}\left[\frac{\gamma-1}{3-\gamma}\frac{\gamma c\rho_0}{a_0^2}\left\{1-\left(1+\frac{\gamma+1}{2\gamma}\frac{\rho_0 a_0}{M}t\right)^{-\frac{3-\gamma}{\gamma+1}}\right\}\right]. \quad (60)$$

Therefore the speed of the piston is,

$$v = U + \Delta U = \frac{2a_0}{\gamma-1}\left[\left(1-\frac{\gamma(\gamma-1)}{3-\gamma}\frac{c\rho_0}{a_0^2}\right)-\left(1+\frac{\gamma+1}{2\gamma}\frac{\rho_0 a_0}{M}t\right)^{-\frac{\gamma-1}{\gamma+1}}\right.$$
$$\left.\cdot\left\{1-\frac{\gamma(\gamma-1)}{3-\gamma}\frac{c\rho_0}{a_0^2}\left(1+\frac{\gamma+1}{2\gamma}\frac{\rho_0 a_0}{M}t\right)^{\frac{2(\gamma-2)}{\gamma+1}}\right\}\right]. \quad (61)$$

From equation (61) it is clear that v can never attain the value $2a_0(\gamma-1)^{-1}$ provided $\gamma < 3$ which it is for all real gases. Consequently in a gas in which there is an interaction between the molecules of the gas, the final flow speed obtained by the leading wave front will be less than $2a_0(\gamma-1)^{-1}$ in the case of free flow. The amount by which it will be less will be small for a very tenuous gas such as exists under astrophysical conditions.

This type of approach to the problem of a freely expanding gas allows us to examine, in a simple if approximate way, how a real gas will behave, since the solution of Love and Pidduck transforms into Burgers' solution by letting M tend to zero.

A model for a freely expanding gas.

The method of solution developed for Lagrange's ballistic problem may be used in order to construct a simple model of a freely expanding gas. In this model we again suppose that the gas is perfect.

We suppose that the piston is replaced by part of the expanding gas, i.e. we suppose that the gas composing the leading edge of the expanding gas is being pushed by the remainder of the gas. The density of the gas in the leading edge is continually changing. The first stage in the problem is to decide how the density will change with the flow speed.

If we assume that the expansion is taking place as a simple wave, we may suppose that the conditions at the leading edge can be described by a single Riemann Invariant. If the simple wave is proceeding from right to left then $\uparrow$ is constant. We can derive a relation for the density from $\uparrow$ using the adiabatic law (equation (22)). The Riemann invariant $\uparrow$ is,

$$u + \frac{2a}{\gamma - 1} = \frac{2a_0}{\gamma - 1} .$$

Hence,

$$\rho = \rho_0 \left\{ 1 - \frac{\gamma - 1}{2} \frac{u}{a_0} \right\}^{\frac{2}{\gamma - 1}} , \qquad (62a)$$

$$p = p_0 \left\{ 1 - \frac{\gamma - 1}{2} \frac{u}{a_0} \right\}^{\frac{2\gamma}{\gamma - 1}} . \qquad (62b)$$

Suppose now that the leading unit volume of gas is being pushed along by the remainder of the gas. Then its equation of motion will be

$$\rho \frac{dv}{dt} = p \,, \tag{63}$$

where ρ is the density of the leading unit volume of gas and p is the pressure on it. In equation (63) v denotes the speed of the leading unit volume.

Substituting for ρ, p from equations (62) allows us to integrate the equation of motion,

$$\int_0^v \frac{dv}{\left\{1 - \frac{\gamma-1}{2}\frac{v}{a_0}\right\}^2} = \frac{p_0}{\rho_0}\int_0^t dt \,,$$

$$\text{i.e.} \quad \frac{v}{1 - \frac{\gamma-1}{2}\frac{v}{a_0}} = \frac{p_0 t}{\rho_0} = \frac{a_0^2 t}{\gamma}$$

$$\text{whence,} \quad v = \frac{2a_0}{\gamma-1}\left\{\frac{a_0 t}{\frac{2\gamma}{\gamma-1} + a_0 t}\right\} \,. \tag{64}$$

This result for v shows that since $2\gamma(\gamma-1)^{-1}$ is small (< 10 for $4/3 \leq \gamma \leq 3$) the final speed is attained very rapidly (less than 1 sec. if a_0 is 10^5 cm.sec^{-1} for example).

The model shows that if a_0 is large the final speed is rapidly attained. However, this result

is approximate since the situation will not exist in practice. We have assumed that the expansion may be described by a simple wave for which a Riemann invariant is constant and in the general free flow problem this will not be the case. It does represent a model which tries to take into account the fact that the density of the gas does not vanish immediately on the gas-vacuum interface. Its use lies in the fact that it shows that Burgers' approximation was a good one because the final flow speed is rapidly approached and it is conceivable that the density will vary in a manner not far different from that assumed.

Section 5. The expansion of a fully ionised gas into vacuum.

We shall consider a fully ionised gas which is composed of protons and electrons. The electrons and protons composing the gas will not be assumed to be in thermal equilibrium. The expansion of a fully ionised gas is a more complicated problem than the corresponding one already discussed for a monatomic gas. The complications arise from the fact that the protons and electrons can interact with each other and with external fields.

We shall attempt to find a solution to the problem for a set of special circumstances. These special conditions relate the problem to that discussed in Section 1 where the expansion of a fully ionised gas into a vacuum was regarded as the limiting case of the compression of a cool gas by a hot gas under gross pressure differences. The general problem of the free expansion of a fully ionised gas into vacuum will not be attempted.

The boundary region between the hot and cool gases.

To make the problem specific we shall consider

that the conditions discussed in Section 1 apply, namely that the hot gas has an electron temperature of 10^9 °K. while the cool gas has a temperature of 10^7 °K. Since electrons have a greater velocity than protons in general (this will be so unless the electron temperature is about a factor of 2000 less than the proton temperature), they will tend to move away from the protons at any free boundary. In the case of the boundary between the hot and cool gases more electrons from the hot gas will move into the cool gas than return from the cool gas into the hot gas. This initial movement on removal of the barrier between the two regions is caused in two ways. In the first place the hot electrons move faster than the cool electrons and in the second place they are pushed by the pressure difference between the hot and cool gas. This movement of charge from the hot region to the cool one cannot continue indefinitely, since it would result in large scale charge separation, the cool gas becoming negatively charged and the hot gas positively charged. This would lead to the build up of large scale electrical forces. We therefore make the assumption that neutrality is strictly maintained in both the hot and cool gases. These assumptions need not apply to the boundary layer however. We shall

describe a mechanism, which may operate within the boundary layer tending to maintain neutrality.

In the boundary layer we expect an excess of negative charge on the cool side and an excess of positive charge on the hot side of the layer. This space charge is assumed to limit itself when the negatively charged region can produce a field which will prevent the further escape of electrons from the hot gas. On this view the negative charge is built up by the escape of electrons from the hot gas. The negative space charge region is assisted in maintaining itself by the change of resistivity encountered on crossing the boundary. Electrons crossing the boundary from the hot gas to the cool gas find that the electrical resistance of the cool gas is about 1000 times greater than the electrical resistance of the hot gas. This follows since the resistivity of a fully ionised gas has been shown to depend on $T_e^{3/2}$ (e.g. see Spitzer (1956)) where T_e is the electron temperature. Therefore electrons crossing from the hot gas to the cool one are slowed down not only by the electric field due to charge separation, but also by meeting an increased resistivity. This resistivity effect would not apply to a freely expanding fully ionised gas.

It may be objected that such a space charge region would decay rapidly as would be deduced from the relaxation time for the decay of space charge, calculated from Maxwell's equations. However, the space charge with which we are concerned is not a static phenomenon. Due to the fact that the hot region is compressing the cool region the boundary separating the two regions will be continually advancing. Therefore, the space charge region in the boundary layer will be continually replenished by fresh electrons from the hot gas. We can however determine approximately the amount of negative charge which would exist in the boundary region if the input of charge from the hot gas was exactly balanced by the rate at which the space charge decayed. If N denotes the amount of charge continuously present in the space charge region then the rate at which charge decays is,

$$\frac{4\pi}{K}\sigma c N , \qquad (65)$$

from Maxwell's equations, where σ is the conductivity of the gas, c is the velocity of light and K is the dielectric constant. The rate at which charge is entering the region is,

$$\frac{1}{4} n \bar{c} . \qquad (66)$$

where n is the density of the electrons in the hot gas and $\bar{c}$ is their mean thermal speed. In actual fact $\bar{c}$ should be the speed of the electrons relative to the compression front, but since the speed of advance of the compression front is slow compared with the speed of the electrons the error is negligible. Equating expressions (65) and (66) for a unit volume we have,

$$N = \frac{K n \bar{c}}{16 \pi \sigma c}, \qquad (67)$$

for the density of the negative space charge. The conductivity may be calculated from the expression for the resistivity given by Spitzer (1956) which is

$$\eta = 6{\cdot}53.10^{12} \frac{\log_e \Lambda}{T_e^{1/2}} \quad \text{e.m.u.,} \qquad (68)$$

where η is the resistivity and Λ has been defined in Section 1. Assuming that the relevant electron temperature is 10^9 °K., then making use of Table XI we have that

$$\eta = 8{\cdot}75 \quad \text{e.m.u.,}$$

so that the conductivity is $1.45.\ 10^{-1}$ e.m.u. Assuming that n is 10^{-5} electrons cm^{-3} the space charge density is $1.44.\ 10^{-6}$ electrons cm^{-3} if K is unity. This represents the number of excess electrons

permanently in the space charge region. Although we speak of a permanent space charge region we do not imply that it is always composed of the same electrons.

We must now investigate the extent of the space charge region, which would be required to prevent loss of electrons from the hot gas. In the calculation which follows we shall use the above value for the space charge density and we note here that a lower value for the conductivity (corresponding to a lower temperature) would increase the space charge density. Therefore the size of the region which we shall compute will be its maximum value.

The Size of the space charge region.

In order to determine the approximate size of the space charge region we shall make use of an argument used by Spitzer (1956). We suppose that the flow of charge takes place along the x -axis. We want to know the extent of a region composed of negative charges which will produce a field, such that further negative charges entering the field will be stopped by it.

In this calculation we shall suppose that the only charge which is effective in stopping the electrons is the space charge, i.e. we now neglect any

effects from the electrons and protons of the cool gas. We further suppose that the density of the space charge is uniform. Then from Poisson's equation the electrostatic potential V of the negatively charged region is given by,

$$\frac{d^2V}{dx^2} = 4\pi ec, \tag{69}$$

where N is the density of the space charge determined from equation (67). Integration of equation (69) assuming that the electric field vanishes for $x = o$, (this will be only approximately satisfied in practice) gives,

$$V - V_o = 2\pi Nec x^2, \tag{70}$$

where $V = V_o$ at $x = o$. Therefore in crossing a region whose width is x, a change of $2\pi Nec x^2$ occurs in the electrostatic potential. For an electron moving along the x-axis into the negative region equation (70) measures the loss of energy suffered by the electron in moving distance x. We shall find the distance h which the electron must travel before it loses all its kinetic energy. We find this distance rather than that required for the electron to lose its kinetic energy in the x-direction, since collisions may randomise the motion of the escaping

electron. Therefore the value of h is found by equating the mean kinetic energy of the electrons with the energy lost in moving a distance h,

$$\tfrac{3}{2} k T_e = 2\pi N e^2 h^2$$

$$\text{or} \quad h = \left\{ \frac{3kT_e}{4\pi N e^2} \right\}^{1/2} = 11{\cdot}95 \left(\frac{T_e}{N} \right)^{1/2}. \qquad (71)$$

Using a similar argument Spitzer (1956) deduced a slightly different value for h. h is called the Debye Shielding Distance and Spitzer showed that while its precise applicability for fully ionised gases is obscure, it represents the distance over which a fully ionised gas can depart appreciably from neutrality.

We interpret the value of h as defined by equation (71) to be the extent of the negative space charge region separating the hot from the cool gas. Taking $T_e = 10^9$ °K. and $N = 1.44. 10^{-6}$ electrons cm.$^{-3}$ the value of h is $3.15. 10^8$ cm. This is a small distance compared with other lengths of interest in the problem of galaxy formation.

This discussion has shown that if the decay of space charge is offset by replenishment with fresh charge only a small departure from neutrality would be required over a small region, in order to preserve

neutrality in the hot gas.

The calculations are only approximate, but they do serve as an order of magnitude indication of the relevant quantities. The amount of space charge required is only about 10 percent of the electron density in the hot gas in addition to the electron density already present. The assumption that rigorous neutrality is preserved in the hot gas may therefore be justified in a tentative manner on these grounds.

This justification only applies to the model we have considered. We have assumed that the hot gas is pushing a cooler gas and we have developed a model of the boundary which leans on this assumption, since the cool gas is invoked to assist the prevention of electron escape from the hot gas. The conditions which should apply to the free expansion of a fully ionised gas into a vacuum are not clear, but we expect that the conditions at the boundary will not be very far different from those discussed. Their precise evaluation will however be much more complicated.

The expansion of a fully ionised gas assuming neutrality.

We shall now consider the expansion of a fully

ionised gas into a vacuum assuming that there is no charge separation. The maintainance of neutrality may be regarded as being a result of the formation and continuous renewal of a space charge region ahead of the expanding gas.

Since it is assumed that the fully ionised gas is neutral and no magnetic fields are present the problem may be treated as a hydrodynamical problem for a binary gas mixture. However, the effect of the electric field due to the space charge will be retained in the equation expressing momentum conservation.

In our treatment of the problem we assume that there is no magnetic field present so that we may write the equations of conservation of momentum for the protons and electrons as,

$$n_i m_i \left\{ \frac{\partial \underset{\sim}{v}_i}{\partial t} + \underset{\sim}{v}_i \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_i \right\} = \frac{n_i e}{c} \underset{\sim}{E} - \underset{\sim}{\nabla} p_i + \underset{\sim}{P}_{ie} , \qquad (72)$$

$$n_e m_e \left\{ \frac{\partial \underset{\sim}{v}_e}{\partial t} + \underset{\sim}{v}_e \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_e \right\} = -\frac{n_e e}{c} \underset{\sim}{E} - \underset{\sim}{\nabla} p_e + \underset{\sim}{P}_{ei} . \qquad (73)$$

following Spitzer (1956) where the subscript i denotes the protons and a subscript e denotes the electrons, m is the mass of a particle, $\underset{\sim}{E}$ is the electric field, $\underset{\sim}{v}$ is the flow speed, p is the pressure and $\underset{\sim}{P}$ is a term governing the proton-

electron interaction. $\underset{\sim}{P}_{ie}$, $\underset{\sim}{P}_{ei}$ are equal and opposite.

Since we are assuming that no separation of the charges occurs, the electrons and protons will have the same flow speed which we denote by $\underset{\sim}{u}$. We consider only flow of the gas parallel to the x-axis. If the gas is flowing parallel to the x- axis then the electric field $\underset{\sim}{E}$ will be parallel to the x- axis by symmetry if $\underset{\sim}{E}$ is assumed to arise in the space charge layer, which we have supposed necessary to maintain neutrality. The equations (72) and (73) then become,

$$n_i m_i \left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right\} = \frac{n_i e}{c} E - \frac{\partial p_i}{\partial x} + P_{ie}, \qquad (74)$$

$$n_e m_e \left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right\} = -\frac{n_e e}{c} E - \frac{\partial p_e}{\partial x} + P_{ei}. \qquad (75)$$

The equation of continuity for the electrons is,

$$\frac{\partial n_e}{\partial t} = -\frac{\partial}{\partial x}(n_e u), \qquad (76)$$

with a similar equation for the protons (in which n_i replaces n_e).

Adding equations (74) and (75) we obtain

$$(n_i m_i + n_e m_e)\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right\} = -\frac{\partial}{\partial x}(p_i + p_e), \qquad (77)$$

since P_{ie} and P_{ei} are equal and opposite and using the result that $n_i = n_e$. Subtracting equations (74) and (75) we obtain

$$(n_i m_i - n_e m_e)\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right\} = -\frac{\partial}{\partial x}(p_i - p_e) + 2\frac{n_e e}{c}E + 2P_{ie} \quad (78)$$

We now make the further assumption that the proton gas and the electron gas each separately obey the adiabatic law. This may be justified since the interaction P_{ie} between the protons and the electrons is so very weak under the conditions assumed. This is equivalent to assuming that the gas expands in the same way as a gas containing two atomic species, which do interact in any way. The presence of the external field is assumed not to affect this assumption. Therefore we may write,

$$p_i = K_i \rho_i^{\gamma_i} = (K_i m_i^{\gamma_i}) n_i^{\gamma_i} \; ; \quad a_i^2 = K_i \gamma_i \rho_i^{\gamma_i - 1}, \quad (79)$$

$$p_e = K_e \rho_e^{\gamma_e} = (K_e m_e^{\gamma_e}) n_e^{\gamma_e} \; ; \quad a_e^2 = K_e \gamma_e \rho_e^{\gamma_e - 1}, \quad (80)$$

where K, γ are constants and a is the sound speed.

Using equations (79) and (80) in equation (77) we obtain

$$(\rho_i + \rho_e)\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right\} = -\frac{\partial}{\partial x}\left(K_i \rho_i^{\gamma_i} + K_e \rho_e^{\gamma_e}\right),$$

$$\text{or } \left(\frac{\rho_i}{\rho_e} + 1\right)\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right\} = -\frac{2a_i}{\gamma_i - 1}\frac{\partial a_i}{\partial x}\frac{\rho_i}{\rho_e} - \frac{2a_e}{\gamma_e - 1}\frac{\partial a_e}{\partial x}. \quad (81)$$

Since the gas is neutral $n_i = n_e$ and we may write

$$\frac{\rho_i}{\rho_e} = \frac{m_i}{m_e} = P, \quad \frac{p_i}{p_e} = Q, \quad \frac{a_i}{a_e} = R. \tag{82}$$

Using equation (82) in equation (81) we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{2a_e}{\gamma_e - 1}\frac{\partial a_e}{\partial x}\left\{\frac{\gamma_e - 1}{\gamma_i - 1}R^2 P + 1\right\}(1+P)^{-1}. \tag{83}$$

We now define a new quantity a such that,

$$a = a_e\left\{\frac{\frac{\gamma_e - 1}{\gamma_i - 1}R^2 P + 1}{1 + P}\right\}^{1/2}. \tag{84}$$

Substituting for a_e in equation (83) we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{2a}{\gamma_e - 1}\frac{\partial a}{\partial x} = 0. \tag{85}$$

Substituting for n_e in equation (76) in terms of a_e gives

$$\frac{2}{a_e(\gamma_e - 1)}\left\{\frac{\partial a_e}{\partial t} + u\frac{\partial a_e}{\partial x}\right\} = -\frac{\partial u}{\partial x},$$

and replacing a_e by a gives

$$\frac{2}{\gamma_e - 1}\frac{\partial a}{\partial t} + \frac{2u}{\gamma_e - 1}\frac{\partial a}{\partial x} + a\frac{\partial u}{\partial x} = 0. \tag{86}$$

If equations (85) and (86) are compared with

equations (26), (27) for a monatomic gas, it is clear that the equations for a fully ionised gas may be reduced to those for a monatomic gas provided the gas remains neutral.

The problem of the expansion of a fully ionised gas into a vacuum has now been solved, since we have reduced the equations of motion to those for the expansion of a monatomic gas. The final speed of expansion will be $2a_0(\gamma_e-1)^{-1}$ where a_0 is some initial value of a. We therefore identify a with the speed of sound in a fully ionised gas.

It is more usual to express a in terms of the kinetic temperature. We must therefore evaluate P and R,

$$P = \frac{\rho_i}{\rho_e} = \frac{m_i}{m_e},$$

$$R^2 = \frac{a_i^2}{a_e^2} = \frac{\gamma_i T_i m_e}{\gamma_e T_e m_i}.$$

Therefore

$$a^2 = k\frac{\gamma_e T_e + \gamma_i T_i \frac{\gamma_e - 1}{\gamma_i - 1}}{m_i + m_e} \quad (87)$$

In the case in which we are interested $T_e >> T_i$, $m_i >> m_e$ so that

$$a^2 = \frac{\gamma_e k T_e}{m_i} \quad (88)$$

In this case the sound speed is such that the electron

temperature must be associated with the protons. Therefore in a fully ionised gas which is far from thermal equilibrium the sound speed must be defined so that the electron temperature is associated with the protons. This result is not unexpected, since the speed of the sound wave generated by positive ion oscillations will also be given by equation (88) under similar conditions, as was shown by Spitzer (1956).

It may be shown from equation (78) using equation (77) to eliminate the flow speed that the electric field is given by

$$n E \simeq -\frac{m_i}{m_i + m_e} \frac{\partial}{\partial x} p_e , \qquad (89)$$

where $n = n_i = n_e$ and p_{ie} has been neglected. Spitzer (1956) interprets this result by supposing that the electric field present must almost cancel the electron pressure gradient and transmits to the protons the force associated with the electron pressure gradient. Returning to our space charge hypothesis this means that the space charge field acts as a brake on the electrons and as an accelerator for the protons.

The values for γ_e, γ_i .

Since the proton-electron interaction is very

weak the period between collisions will be very long. The pressure is non-isotropic so that the motion is essentially one dimensional as far as velocity space is concerned. This argument applies both to the electrons and to the protons. The specific heat of a gas may be defined in terms of the number of degrees of freedom possessed by the gas. Let $\mathcal{J}$ be the number of degrees of freedom possessed by the gas then,

$$\gamma = \frac{2 + \mathcal{J}}{\mathcal{J}} \qquad (90)$$

For a fully ionised gas there are no internal degrees of freedom to consider. Since the interaction between the protons and electrons is weak and the motion is one dimensional $\mathcal{J} = 1$ so that $\gamma_e = \gamma_i = 3$ for an ionised gas.

If randomising collisions did occur the value of $\mathcal{J}$ would be three giving $\gamma_e = \gamma_i = 5/3$ as for a monatomic gas. However, under the conditions assumed in Section 1 randomising collisions are unlikely. We therefore conclude that an initially uniform semi-infinite fully ionised gas would expand into a vacuum with the initial sound speed.

Discussion.

The work in this section implies that the hot gas would compress the cool gas, of Hoyle's (1958) model for galaxy formation, at speeds which would be nearly that of the sound speed in the hot gas. This would mean that some stages of the compression under gross pressure differences would proceed more rapidly than has been assumed by Hoyle with the consequence that larger dynamical motions would be built up in the gas than has been supposed by Hoyle. However, before this problem can be definitely settled the problem of what happens when one fully ionised gas is compressing another would have to be solved for a variety of circumstances. The case discussed here is only the limiting case for gross pressure differences.

The problem discussed in this section does not attempt to include the solution for the free expansion of a fully ionised gas. The present problem confines its attention to the expansion of the gas, when the gas is assumed to be neutral at all times.

A mechanism is suggested whereby the gas may be maintained neutral by supposing that a space charge region exists ahead of the expanding front. While recognising that this space charge region is

continually being renewed since the front is always moving into it, we can show by a quasi-static model of the region that it does not demand much excess charge to effectively prevent further electrons from leaving the gas. The space charge region was also shown to be small compared with other dimensions of interest.

The problem of the freely expanding fully ionised gas in principle should be similar to this. However, it is by no means clear if such a space charge region could exist. Electrons might escape from the space charge region and run away from the expanding cloud. A much more detailed analysis of the problem would be required, both from the atomic point of view and from the magnetohydrodynamical point of view. It may even be that the atomic point of view is misleading.

No attempt has been made here to include magnetic fields and the expansion has been supposed to take place in some region free of magnetic fields. The presence of these fields would act as a slowing down agent tending to drag the charged particles back. This clearly would be of the greatest importance in Hoyle's model where magnetic forces may act as a means of dissipating excess energy.

Summary

This chapter has reviewed the fundamental equations for expansion waves and considered the previous solutions obtained for the expansion of a monatomic gas into a vacuum. It was shown that the final speed of expansion is associated with the complete rarefaction of the gas.

Consideration of Lagrange's ballistic problem showed that not only could a solution for the free expansion of the gas be obtained with a simpler analysis, but the effects of the departure of the gas from being a perfect gas could also be investigated. An approximate model of the free expansion of a gas was then investigated and it was shown that the final speed of expansion is approached very closely a short time after the start of the motion.

The expansion of a fully ionised gas was discussed assuming that the gas remained neutral. A mechanism was suggested whereby the gas, under suitable conditions, could be maintained neutral. Under these conditions it was shown that the problem of the expansion of a fully ionised gas reduced to that of a monatomic gas already discussed. In deriving the equations of motion for the fully ionised gas an

expression was found for the speed of sound in the gas. This expression showed that if the electron temperature greatly exceeded the proton temperature the former characterised the speed of sound in the fully ionised gas.

Chapter IV

The Rotation of Galaxies.

Introduction.

In this chapter the rotation of galaxies is considered in an elementary fashion. It is supposed that the rotation of the galaxy is produced by the action of a non-uniform gravitational field on an irregular condensation which ultimately forms the galaxy.

This problem has been treated by Hoyle in a general way and we first review his work. We then examine a specific model in order to make Hoyle's discussion more definite. We considered a gas cloud which is shaped like a dumb-bell in which the two masses are equal and each contains half the mass of the resulting galaxy.

The results of the calculations based on this model account for the broad features of the observed results. However, the values obtained are maximum values and any dissipative process will tend to diminish the values obtained for the angular velocities. The results indicate that the angular velocity of a galaxy is dependent on the gravitational field in which it was formed. However, the results do not predict any variation of angular velocity with the mass of the proto galaxy to the order of the approximation to which they were taken. Furthermore no account

can be taken of the recently observed high rotational speed of the central nucleus of our own galaxy.

Section 1. Hoyle's theory for galaxy rotation.

Hoyle (1951) suggested that the angular momentum of a galaxy could be accounted for by supposing that the galaxy condensed from an initially irregular cloud situated in a non-uniform gravitational field. As the cloud is in a non-uniform field it would be acted upon by a couple, which would produce the angular motion. In order to apply this idea Hoyle assumed that the initial stages of the condensation occupied most of the time available for condensation, the final stages taking place comparatively rapidly.

Hoyle takes the condensation time to be $\{3/4\pi G\rho\}^{1/2}$ where ρ is the initial density of the cloud and G is the gravitational constant. This is the time for contraction under Newtonian forces. Hoyle assumes that the cloud remains irregular for approximately the whole of this time. An origin of coordinates is taken at the centre of mass of the irregular cloud and then principal axes are defined for the cloud. The cloud will have moments of inertia A, B, C respectively about these axes and in general these moments of inertia will not be the same. Hoyle supposes that the non-uniform gravitational field is

produced by the surrounding pre-existing galaxies and that for the purposes of mathematical treatment the field may be supposed due to a single point mass M distant d from the centre of mass of the cloud. The value of d is supposed to be very much larger than the initial radius R of the cloud. The components of the couple acting on the cloud are,

$$-\frac{3MG}{d^3}\left\{(B-C)mn\ ,\ (C-A)n\ell\ ,\ (A-B)\ell n\right\}\ , \qquad (1)$$

where ℓ, m, n are the direction cosines of the line joining M to the mass centre of the cloud. The result expressed in (1) has been derived by Milne (1948) in a very compact vector form.

From the form of (1) it is clear that the couple vanishes when $A = B = C$ i.e. when the gas cloud has become spherical in shape. Therefore the moments of inertia must be computed at the start of the condensation process when the cloud is non-spherical. The magnitude of the angular velocity produced by this couple, Hoyle takes to be given by the product of the condensation time and the largest couple component, which is denoted by $3MGx\,d^{-3}$. The angular velocity is therefore,

$$\frac{3MGx}{d^3}\left(\frac{3}{4\pi G\rho}\right)^{1/2}. \qquad (2)$$

This result is obtained by integrating the usual angular equation of motion assuming that the couple and moment of inertia remain constant throughout.

Once the radius of the condensation has become very much smaller than the initial radius R the angular momentum will become essentially constant, since the couple acting on the condensation will have almost vanished. However, as the condensation decreases in radius the angular velocity will increase. Hoyle therefore makes the assumption that the condensing cloud acquires most of its angular momentum when the cloud still has its initial dimensions and thereafter angular momentum is conserved. If the final radius of the condensed cloud is r the final angular velocity Ω possessed by the cloud will be

$$\Omega = \frac{3MGx}{d^3}\left(\frac{3}{4\pi G\rho}\right)^{1/2}\left(\frac{R}{r}\right)^2 , \qquad (3)$$

since the angular velocity increases as the inverse square of the radius.

Hoyle then makes the additional assumption that in the final state of the condensed cloud the angular and gravitational energies will be of the same order, so that

$$r\Omega^2 = \frac{mG}{r^2} , \qquad (4)$$

where m is the mass of the condensation. But,

$$m = \frac{4\pi}{3} R^3 \rho \quad . \tag{5}$$

Eliminating R/t from equation (3) by means of equations (4) and (5), the expression for the angular velocity becomes,

$$\Omega \simeq \left(\frac{d^3}{3MGx}\right)^{3} \left(\frac{4\pi G\rho}{3}\right)^{1/2} \quad . \tag{6}$$

In order to find a value for Ω Hoyle has to find some suitable value for the gravitational field producing the couple. Instead of using the nearest neighbour approximation for M and d, he notes that the field galaxies in general have peculiar velocities of the order of 200 km.sec^{-1}. These peculiar velocities he assigns to the effects of large scale irregularities in the distribution of the intergalactic material. Consequently, by setting the gravitational energy per unit mass possessed by a mass M equal to the square of the peculiar velocity. Hoyle eliminates M and obtains

$$\Omega \simeq 10^{104} (\rho G)^{1/2} x^{-3} \quad . \tag{7}$$

Hoyle then considered some numerical consequencies of

equation (7). He chooses the value of χ to lie in the range $0{\cdot}10 \leq \chi \leq 0{\cdot}33$ and two types of density region were considered. In the first type of region the density is taken to be 10^{-27} gm.cm^{-3}, while in the second type of region the gas density is taken to be 10^{-25} gm.cm^{-3}. The latter value is assumed to be more appropriate for galaxies forming in large clusters. For the low density region,

$$\Omega \simeq 10^{-16}\,\chi^{-3}\ \text{sec}^{-1}, \qquad (8a)$$

while in the high density region,

$$\Omega \simeq 10^{-15}\,\chi^{-3}\ \text{sec}^{-1} \qquad (8b)$$

If χ has a value 0.33 equation (8a) gives an angular velocity of approximately 3.10^{-15} sec^{-1}. The angular velocity of our own galaxy is about 10^{-15}sec^{-1}.

From equation (3) it can also be concluded that if two galaxies have comparable masses, the galaxy which has a low angular velocity has a large radius and the galaxy which has a large angular velocity has a small radius. Further, equations (8a) and (8b) together suggest that cluster galaxies should have systemmatically higher angular velocities than field galaxies. Hence cluster galaxies should be more condensed than field galaxies.

The values obtained by Hoyle for the angular

velocities are so very near the observed values that it was thought desirable to examine Hoyle's hypothesis by choosing a particular model for the initial irregular cloud. Consequently any discussion of Hoyle's hypothesis is deferred until Section 3.

Section 2. The dumb-bell model for galaxy rotation.

We now develop the ideas put forward by Hoyle in terms of a specific model. We suppose that the irregular cloud may be regarded as a dumb-bell. The mass of the dumb-bell is assumed to be divided into two equal parts which are separated by a constant distance $2l$. The mathematical treatment of the problem will not take into account the gravitational interaction of the two masses comprising the dumb-bell, though the effect of decreasing the separation of the two masses will be discussed. For the sake of simplicity it will be assumed that the masses composing the dumb-bell and the mass producing the non-uniform gravitational field are all in the same plane.

This problem is similar to one recently discussed by Synge (1959). Synge considered the motion of a pendulum attached to a satellite moving in orbit around the earth. Our present problem differs from that discussed by Synge, since the dumb-bell is not necessarily moving in a closed orbit and we are concerned here only with the motion of the dumb-bell about its centre of mass.

The motion of the dumb-bell about its centre of mass.

The system considered is sketched in Figure 7. The non-uniform field is assumed to be produced by a point source of mass M distant R from the centre of mass G of the dumb-bell. The dumb-bell is composed of two equal masses m separated by a constant distance $2l$. The centre of mass G of the dumb-bell is therefore midway between the two masses m. The angle between the line joining the masses of the dumb-bell and MG is θ.

The mass M attracts the masses m, but since in general one of the masses m will be nearer the mass M than the other, there will be a resultant moment about G. The resultant couple about G anticlockwise is,

$$-\frac{GMml\sin\theta}{R^3}\left\{\left(1-2\frac{l}{R}\cos\theta+\frac{l^2}{R^2}\right)^{-3/2}-\left(1+2\frac{l}{R}\cos\theta+\frac{l^2}{R^2}\right)^{-3/2}\right\}. \quad (9)$$

If terms in l^2/R^2 are neglected in the expansion of equation (9) the equation of motion for the motion of the dumb-bell about the mass centre G becomes,

$$2m\frac{d}{dt}(l^2\dot{\theta})+\frac{3GMml^2\sin 2\theta}{R^3}=0. \quad (10)$$

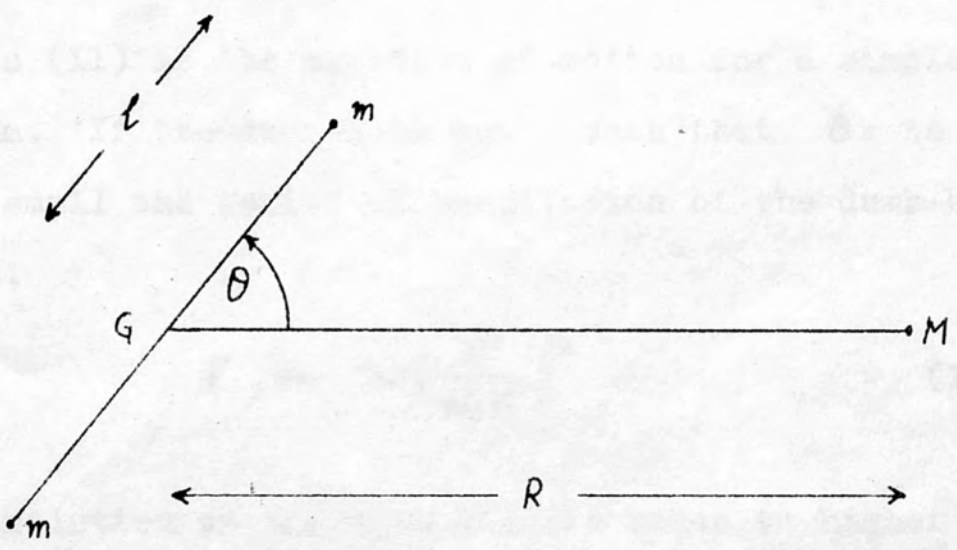


Figure 7.

The equation would also hold if the masses m attracted one another - ℓ would then vary with time. We are assuming however, that ℓ is constant so that equation (10) may be written,

$$\ddot{\theta} + \frac{3GM \sin 2\theta}{2R^3} = 0 . \qquad (11)$$

Equation (11) is the equation of motion for a simple pendulum. If the dumb-bell moves such that θ is always small the period of oscillation of the dumb-bell will be,

$$T = 2\pi \left(\frac{R^3}{3GM}\right)^{1/2} . \qquad (12)$$

If the solution of equation (11) is taken to higher orders of approximation the value of the period will be increased. Equation (11) shows that the period of the dumb-bell depends only on the parameters M and R determining the external field. In fact in the expansion of equation (9) the terms in ℓ^2/R^2 cancel, so that our approximation only neglects terms in ℓ^3/R^3 and higher powers. This suggests that if the rotation of the galaxies is a property of the gravitational field during the initial stages of their condensation, then the periods of rotation of the galaxies ought not to be very dependent upon the mass of the actual galaxy.

It would also indicate that a distinction could be made between galaxies that were formed under crowded conditions and those that were not.

Solving equation (11) for small θ we obtain

$$\theta = A\sin\left(\frac{3GM}{R^3}\right)^{1/2}t + B\cos\left(\frac{3GM}{R^3}\right)^{1/2}t \quad , \qquad (13)$$

where A, B are constants. Assuming that $\theta = \pi/2$ for $t = 0$ and $\theta = 0$ for $t = \frac{1}{4}T$ then A is zero and B is equal to $\pi/2$. Hence the maximum angular velocity that can be generated under these circumstances is

$$\omega = \frac{\pi}{2}\left(\frac{3GM}{R^3}\right)^{1/2} = \frac{\pi^2}{T} \quad . \qquad (14)$$

This value of ω has been derived from equation (13), which is true only for small angles, while A, B have been determined for finite angles. However, the error introduced is not very large. The calculations can therefore only be correct to within an order of magnitude. If the galaxy is to acquire maximum angular momentum we must assume that the dumb-bell remains rigid until the quarter period has elapsed and then rapidly becomes spherical.

However, in practice the cloud will be condensing and so ℓ will be continually decreasing. The couple acting on the dumb-bell will therefore

diminish as the collapse proceeds. Furthermore, since the period of a pendulum which moves through a finite angle is larger than that for a pendulum executing small oscillations, the angular velocity ω is the maximum that the galaxy could obtain.

The boundary conditions used in evaluating A, B assume $\theta = \pi/2$ at $t = 0$. If this were the case the dumb-bell would not move, so we assume that a small perturbation causes a small change in θ so allowing the non-uniform field to act.

We therefore assume for the rest of this discussion that the dumb-bell acquires the maximum angular velocity ω given in equation (14) before the gravitational collapse of the dumb-bell occurs. During the collapse we shall assume that the angular momentum is conserved.

The final angular velocity attained.

We shall use a slightly different argument to that used by Hoyle. The time required for two masses m initially separated by a distance 2ℓ to fall together from rest under their own self gravitation is,

$$\tau = \left(\frac{\pi^2 \ell^3}{2Gm}\right)^{1/2}. \qquad (15)$$

Assuming that the time for infall is of the same order as $\frac{1}{4}T$ we can compute a value for ℓ . This assumption means that the initial rate of infall is slow, but the final stages occur quickly. If ℓ is the initial value of the radius of the condensation and ℓ' is the radius of the resulting galaxy, the maximum angular velocity which the galaxy can attain is,

$$\Omega = \omega\left(\frac{\ell}{\ell'}\right)^2 . \qquad (16)$$

In order to compute values for Ω we shall deal with two cases. In the first case we shall assume that only the nearest neighbouring galaxy produces the couple on the condensation. Allen (1955) gives the mean mass of a galaxy as 10^{44} gm. and we shall suppose that the average diameter of a galaxy is 5.10^{22} cm, while the mean intergalaxy distance is 5.10^{23} cm. Using these values we find that $\frac{1}{4}T = \tau$ is $1.24.10^{17}$ sec. and ℓ is $2.2.10^{23}$ cm. Therefore Ω is $3.8.10^{-16}$ sec^{-1}.

The second model is applicable only to a cluster of galaxies. We suppose that a galaxy forms in a cluster of pre-existing galaxies. Allen (1955) states that a cluster has a mean diameter of $4.8.10^{24}$ cm. and contains on average 200 galaxies. Assuming that the galaxy forms at about 2.10^{24} cm. from the centre of the cluster, then we may suppose that all the galaxies

within this radius affect the rotation of the condensing galaxy. Using these values $\frac{1}{4}T = \tau$ is $0.9.\ 10^{17}$ sec. and ℓ is $1.79.\ 10^{23}$ cm. so that the final value of Ω is $2.4.\ 10^{-16}$ sec^{-1}.

Of these two approaches the nearest neighbour approach gives a larger value for Ω, but both approaches give values which are lower than the value for our own galaxy. However, they approach the values found by Kerr and de Vaucouleurs (1955) for the Magellanic Clouds (5.10^{-16} sec^{-1} for the Large Cloud, 3.10^{-16} sec^{-1} for the Small Cloud). The second approach has the advantage that the value of M can be increased by adding intergalactic material to the cluster so increasing the value of Ω.

The results obtained in this section are smaller than the results obtained by Hoyle. We have not needed to make the assumption that the rotational energy is of the same order as the gravitational energy of the galaxy.

The value of Ω could be increased if the nearest neighbour galaxy were of more than average mass or if the cluster contained large amounts of intergalactic material.

Section 3. Discussion

The hypothesis suggested by Hoyle requires that the condensing cloud which will form the galaxy must rotate as a rigid body at all densities. It is not known whether a gas cloud can rotate in this manner. This is the crucial test which would decide whether galaxies would acquire their angular momentum in this way or not.

However, there are other properties of a galaxy which may indicate whether or not the mechanism is valid at least for some parts of the galaxy. From Hoyle's analysis of the problem and from the mathematical treatment of a specific model, it is clear that the actual mass and shape of the condensing cloud has very little influence, within fairly broad limits, on the actual angular velocity which the galaxy will ultimately have. The major part of the angular velocity is determined by the gravitational field in which the galaxy is situated. This should have detectable consequences.

The angular velocities possessed by our own galaxy, M31 and M33 lie in the range 10^{-15} to 4.10^{-15} sec^{-1}, while the Magellanic Clouds have angular velocities 5.10^{-16}, 3.10^{-16} sec^{-1}. The range of angular velocities

is about a single order of magnitude, while the range of masses for the same galaxies covers several orders of magnitude. This would suggest that the mass of the galaxy plays little part in determining its angular velocity.

Furthermore, the hypothesis suggested by Hoyle would predict that galaxies, which were formed in regions in which the density of galaxies was greater than average, would have large angular velocities. Galaxies which were formed in regions of low galaxy density would have small angular velocities. This may be the explanation of the observation by de Vaucouleurs (1958) that as one passes along the galaxy classification sequence from E7.SO to Sc the period of rotation increases. If it is assumed that the galaxy ages are given by this sequence with E7 galaxies younger than Sa galaxies and so on, then the older galaxies (e.g. the Sc galaxies) being formed in less crowded conditions would rotate more slowly than the younger galaxies.

However, the discussion in Sections 1 and 2 of this chapter is very approximate. The contraction of the cloud is not allowed for in the mathematical analysis. It would be a very difficult problem to

treat with rigour, since it would involve an integration in which all the parameters governing the dimensions of the cloud would be varying continuously.

However, the approximation that the contraction is slow at first is a good one, since the speed of contraction depends on the square root of the distance through which contraction has already proceeded. The value predicted by the theory for the angular velocity is a maximum value and the calculations of Section 2 indicate that this mechanism may just fail to account for the observed velocities, except in clusters where the intergalactic matter may be used to increase the effective mass. The intergalactic material would need to constitute almost the entire mass of the cluster, while the galaxies would only have to contribute one percent to the cluster mass if an increase in the angular velocity of a factor of 10 is required. This is, however, about the correct proportion of free intergalactic material to galactic material.

Zwicky (1957) has published some diagrams showing the variation of angular velocity across a galaxy. The inner regions of a galaxy from these diagrams appear to rotate as a rigid body, but this is not maintained in the outer regions. The rotation of the outer parts of a galaxy can be better explained

in terms of Keplerian orbits. Therefore the theory outlined here cannot account for the rotational properties of the whole galaxy.

Recently there has been evidence to show that the central nucleus of the galaxy is rotating very rapidly. This is contrary to any uniform rotation hypothesis. Therefore, the hypothesis made by Hoyle, while appearing to account for some features of the rotation of galaxies, does not give a complete account. For a complete solution to the problem more investigation of the properties of a condensing cloud in a non-uniform gravitational field will be required.

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