PROBLEMS IN RELATIVISTIC COSMOLOGY

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ELIZABETH ANN HILTON, B.Sc.

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Department of Mathematics
Royal Holloway College
(University of London)
Englefield Green, Surrey.

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ABSTRACT

This thesis is concerned with the study of two inter-related topics: the existence and properties of horizons, boundaries and barriers in cosmological models and the information that an observer in a certain model may, in principle, gain about his universe. Accordingly, it deals first with concepts of uncertainty and indeterminacy in cosmology (Chapter I).

In Chapter II, sections (i), (ii), (iii), (vi) and (vii) introduce and summarise earlier work on the subject of event and particle horizons in homogeneous and isotropic world-models; the remainder of the chapter discusses certain features and develops various problems which arise from this, during the course of which is introduced the new notion of the degenerate (invariant) horizon. Chapter III is concerned with the "Milne-type" boundary and discusses the boundary of distance by parallax. It is shown that the boundary in Milne's model is a degenerate particle horizon.

The behaviour of observables in the neighbourhood of an event horizon or a particle horizon is examined for five expanding model universes of the Robertson-Walker type, and for their duals, obtained by time-

reversal; the results are demonstrated diagrammatically (Chapter IV). This examination paves the way for, and finds application in, Chapter V which investigates information theory in cosmological models and studies in particular the rate of flow, and hence the rate of loss, of information in the models considered; the influence of the existence or otherwise of horizons is explicitly demonstrated.

The two remaining chapters (VI, VII) investigate the nature of the singularity at $r = 2m$ in the Schwarzschild space-time, in the Finkelstein space-time obtained by transformation from Schwarzschild's metric, and also in the space-time obtained by time-reversal of the Finkelstein metric. By studying the amount of information which an observer in the region $r > 2m$ may in principle receive from $r \leqslant 2m$ from light signals or probes, it is shown (Chapter VI) that the surface $r = 2m$ in the Schwarzschild space-time is a barrier which is a degenerate event horizon for such an observer. Chapter VI concludes by considering Darwin's work on the manifestation of the singularity in the presence of a neighbouring star and of a star-field background.

In Chapter VII, it is shown that the barrier at $r = 2m$ in the Finkelstein space-time is not a degenerate event horizon for the observer in $r > 2m$, but that, in contrast, it is a degenerate event horizon in the time-reversed case. This topic is completed by an investigation of the transformations concerned to discover to what extent they are valid and to demonstrate how this difference arises.

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CHAPTER I: UNCERTAINTY AND INDETERMINACY IN COSMOLOGY

(i) Introduction to Cosmology

It is the role of science to obtain and organise communicable information of the external, physical world. One branch of physical investigation into the external world is cosmology, which is concerned with the study of the structure, evolution and the properties and features of the universe as a single, whole unit; its aim is thus to elucidate the physical principles underlying the phenomena which occur on the largest scale.

To this end, cosmologists, taking as their subject matter for investigation the totality of physical objects and events which are significant on the largest scale, consider model universes in which local irregularities are smoothed out. Thus the foundation of cosmological models is an underlying substratum comprising fundamental particles (F.Ps.), that is, those particles which travel with the average motion of matter in their own, sufficiently large, neighbourhood. In identifying a model with the actual universe, F.Ps. are considered to be representations of the largest gravitationally bound groups existing in the universe, usually supposed to be clusters of galaxies.

For the sake of comparison with the observed properties of our own universe and for its intrinsic interest, investigation is made into the physical aspect of various model universes as it will appear to observers in the particular models under consideration; attention is usually confined to those observers who are attached to F.Ps., called fundamental observers (F.Os.) and forming a continuous 3-parameter set of equivalent observers, one passing through each point of space at any given instant of time.

The metrical properties of 4-dimensional space-time are assumed to be characterised by the interval s between any two events, this quantity being connected with the space-time co-ordinates x^μ by the invariant expression

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \tag{1.1}$$

where we have summation over μ and ν (which take the values 1, 2, 3, 4) and the $g_{\mu\nu}$ are functions of the x^μ. Application of the general theory of relativity would impose the condition that the metric coefficients $g_{\mu\nu}$ are related to the distribution of matter and energy in accordance with Einstein's (1916) field equations.

Under the name of the cosmological principle in one of its forms, the assumption is made in cosmological theory that on the large scale there is spherical symmetry around every F.P., so that every F.O. obtains the same picture of the universe as every other F.O. The necessity for this is clear if we wish to claim that a set of observations is typical rather than singular and so may form the basis for forming or rejecting a theory; its simplifying properties have also commended its adoption. Astronomical observations indicate that as regards the large scale features of our universe we have no reason for rejecting the hypothesis. The assumption of ideal spatial isotropy leads to the existence of a cosmic time and by Schur's theorem (Eisenhart, 1926) implies spatial homogeneity.

If, in addition, co-moving co-ordinates are adopted, so that according to any particular F.O. each F.P. has permanent spatial co-ordinates, then it has been shown by Robertson (1935) and independently by Walker (1937) and by Tolman (1934) that the cosmological metric has the form

$$ds^2 = c^2dt^2 - \frac{R^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2)}{\left(1 + \frac{kr^2}{4}\right)^2} \qquad (1.2)$$

This line-element has the following properties:

(i) t represents cosmic time

(ii) $R(t)$ is a disposable function of t

(iii) k is a constant which may take the values $-1, 0, +1$, $k/R^2(t)$ being the curvature of the 3-space t = constant at the epoch t

(iv) r is a co-moving radial co-ordinate such that $r = 0$, the spatial origin, may be identified with any F.O.

(v) θ, ϕ are the usual angular measurements made at $r = 0$

(vi) geodesics of the metric are given by r, θ, ϕ = constant; the world lines of F.Ps. are geodesics

(vii) null geodesics are obtained by setting $ds = 0$; the paths of light rays in the model are represented by null geodesics

We adopt the convention that locally within the model the speed of light is a constant which is to be identified with the constant c in the metric. Hence there are inter-related time and distance scales, the existence of a finite upper bound to the speed with which any causal influence may be propagated implying that the remoter the part of the universe that is observed, the earlier is the stage in its history at which it is seen.

One basic method for distinguishing the type of universe we inhabit is by forming such model universes and performing experiments to enable us to discriminate between them. It is recognised that the Robertson-Walker metric, given by (1.2), could give only a very rough approximation to our observed universe, neglecting as it does the concentration of matter into stars, galaxies and clusters of galaxies, characteristics which must ultimately be accounted for. Mayall, Scott and Shane (1960) point out that the smoothing out of concentrations to an ideally homogeneous state forms the basis for cosmologies which are wholly deterministic and they express the view that the fact that observation is concerned only with concentrations calls for a modification of the theory to make it indeterministic: they submit that if the galaxies were considered as the realisation of a discrete stochastic process, stationary with respect to the three spatial co-ordinates and stationary or otherwise with respect to time, then the postulate that galaxies occur in clusters would establish a direct relation between theory and the facts of observation.

However, it is widely regarded (Bondi, 1960a, McVittie 1937a) that being based on the cosmological principle, the Robertson-Walker metric provides at present the most satisfactory background for the comparison of theory with observation.

It is thus one of the tasks of cosmology to develop relations between observable quantities for the space-times represented by (1.2); then, by appeal to observation to evaluate the two unknowns $R(t)$, k or, rather, to reject those values of $R(t)$, k which are incompatible with the data obtained from that part of the universe which is accessible to observation. Since this is necessarily a very limited part of the whole system, we emphasise that it is reasonable to expect no unique answer to any cosmological question concerning the universe as a whole but rather a range of possibilities.

In comparing certain relations between the theoretical parameters $R(t)$, k with quantities actually observed, it is essential that we identify entities in theory and observation which are in fact equivalent. This may be ensured by adopting the operational technique, in which entities must be defined by a description of the type of experiment required for their measurement; otherwise we may well be employing concepts which are meaningless or ambiguous. For example, the concept of distance between two objects in a Robertson-Walker space-time is ambiguous unless the method of measurement is specified; different procedures, such as determination by means of apparent size, apparent luminosity, parallax, rigid measuring rods or radar methods, give rise to different expressions in terms of the theoretical parameters. Formulae for various distances referred to the Robertson-Walker metric have been given by McCrea (1934-5).

One of the most significant of the observed phenomena is the displacement of the spectral lines of the galaxies towards the red end of the spectrum. An analysis of the space-times given by (1.2) shows that they can give a representation of this observed red-shift. Consider a particle emitting a train of light waves from (r_1, θ, ϕ) which are received by the observer at the origin $r = 0$. Since $ds = 0$ for the motion of light, (1.2) gives for its radial motion

$$c\,dt = \pm \frac{R(t)\,dr}{(1 + kr^2/4)} \qquad (1.3)$$

where the + sign is to be chosen for motion of the light away from the observer, and the - sign for motion towards the observer. Taking the - sign, we obtain upon integration

$$c\int_{t_1}^{t_0} \frac{dt}{R(t)} = \int_0^{r_1} \frac{dr}{(1+kr^2/4)} \qquad (1.4)$$

for light emitted at epoch t_1 and received at epoch t_0. Suppose that light emitted from the same particle during the time interval dt_1 is received during the time dt_0. Then similarly we have

$$c\int_{t_1+dt_1}^{t_0+dt_0} \frac{dt}{R(t)} = \int_0^{r_1} \frac{dr}{(1+kr^2/4)} \qquad (1.5)$$

since r is a co-moving co-ordinate.

Subtracting (1.4) from (1.5) it follows that

$$\frac{dt_0}{dt_1} = \frac{R(t_0)}{R(t_1)} \qquad (1.6)$$

so that dt_0 and dt_1 are unequal in general unless $R(t)$ = constant. If $\lambda, \lambda+d\lambda$ are the locally observed wavelengths at emission and reception respectively, we have

$$\lambda = c\,dt_1 \quad ; \quad \lambda + d\lambda = c\,dt_0 \qquad (1.7)$$

so that, by (1.6), writing $\frac{d\lambda}{\lambda} \equiv z$,

$$1+z = \frac{R(t_0)}{R(t_1)} \qquad (1.8)$$

an expression which is independent of the wavelength.

It follows that light is always reddened on reception provided that $R(t_0) > R(t_1)$. Since $t_0 > t_1$, this implies that to account for the observed shift towards longer wavelengths, we must have $R(t)$ an increasing function of t. Hence the red-shift phenomenon may be accounted for as a Doppler effect due to the expansion of the universe. We note, however, that while observation gives us reason for considering the universe as expanding, no mechanism has yet been provided to ensure expansion rather than contraction; the metric (1.2), in the absence of a further criterion, is equally suitable for the description of a contracting universe, upon taking $R(t)$ as a decreasing function of t. Alternative explanations of the observed red-shift and observational tests for discriminating between recessional and non-recessional hypotheses have been considered by Whitrow (1954) and by Davidson (1958).

Accepting the Doppler interpretation of red-shift, (1.8) shows us that the larger the light travel time t_0-t_1 from source to observer, the larger is the corresponding value of z and because of the finite and locally constant speed of light the more remote is the region of the universe in which the source is situated at emission. The associated velocity of recession of the source is clearly dependent on which definition of distance is adopted.

Unlike distance, the spectral displacement has an unambiguous meaning and being measurable, it is a highly important quantity where observational tests are concerned. Other observables considered are the apparent magnitudes of sources, numbers of sources, the angular extension of sources and their distribution. It is by deriving the theoretical relations between such quantities and comparing them with those obtained by experiment that observational tests for the discrimination between world models are made.

(ii) Concepts of Uncertainty and Indeterminacy

It is well known (Bondi, 1960b) that in cosmology the inaccuracy of observations increases with the distance of the observed object; not only are errors then larger, particularly with regard to the estimation of magnitudes, but the measurements are far more difficult to perform and are fewer in number, so that observations of the more distant objects are necessarily less detailed and accurate than those of our own astronomical neighbourhood. At any particular time our knowledge is restricted by such limitations; these may, however, being problems of practice rather than principle, be overcome by subsequent improvements of technique and the application of further observational programmes. In the present work we shall confine our attention to ideas of uncertainty and indeterminacy in cosmology which are a matter of principle.

Clearly, the amount of information to be obtained from any experiment will be closely related to the accuracy of the experimental set-up. An uncertainty will be introduced at the observer due to the experimental errors which the modern view maintains are inevitable and may not be entirely eliminated. Although there is no exact limit to the accuracy, the smaller the error Δx the higher the cost of the observation in terms of energy, in the limit the high cost of complete accuracy making it unattainable. The impossibility, in principle, of measuring any quantity with unlimited accuracy implies a fundamental limitation on the possibilities of observation which is formulated in precise terms by quantum conditions and information theory (Brillouin, 1962). The amount of information to be obtained from an observation will depend on the ratio of the uncertainty that remains after the observation is made (Δx) to the original uncertainty corresponding to the whole field of observation (range of x) and it may not be defined unless the accuracy of the experiment and the field of observation are explicitly stated. In an optical instrument, for example, this necessitates consideration of the ratio of resolving power to the total field aperture of the instrument.

This modern view is in contrast to that of classical theory which ignores the fundamental role and importance of experimental errors, assuming that they may always, in principle, be made as small as one desires and hence generating the type of causality known as determinism. The fundamental impossibility of attaining infinite accuracy in any experiment implies a corresponding lack of absolute certainty in all physical laws and destroys any possibility of proving scientifically the validity of determinism. The experimental errors, because of their inevitable nature, should be recognised by and included in theory since in any case they form an essential part of our knowledge of the world around us.

In this way an inherent uncertainty enters observational cosmology on the human scale since, in the final analysis, the observer may never divorce himself completely from the measuring instruments and recording process which define the conditions of the experiment and in fact terminate the phenomenon under consideration; by the operational viewpoint this uncertainty, being inherent, must be recognised whether the considerations be of a theoretical or a practical nature.

Cosmology also admits a second type of fundamental uncertainty which was formulated by McCrea (1960) and is occasioned by the finite speed of light. It follows upon postulating that every part of the universe interacts with every other part. Then, assuming that such interactions are propagated with the speed of light, at any epoch an observer is seeing the rest of the universe in the state in which it is influencing his own part, so that in principle he may predict the immediate future of his own neighbourhood. But he will not be able to predict the immediate future of any remote part of the universe in the same way; for this depends on the influences, propagated with the speed of light, of all other parts of the universe, and because of the non-static nature of the universe the observer may know nothing from observation about the state of such parts when they influenced the observed state of the particular region under consideration. If the universe is regarded as an unbounded system, this limitation may not even be overcome by a sufficiently long period of observation.

Under the assumed conditions, it follows that in principle the observer may not predict the behaviour of a remote part of the universe with as much assurance as that of parts nearer to him. McCrea maintains that the unavoidable uncertainty in any prediction concerning the immediate future of particles at a particular distance is measured by $(1 + z)$ where z is the red-shift for that distance; this feature should be incorporated into any satisfactory cosmological theory conforming with the stated conditions.

The work of Florides and McCrea (1959) has demonstrated that the two basic types of cosmological theory (the evolutionary type in which matter is conserved and the steady state type in which the same picture is presented to any F.O. at all epochs, so necessitating the creation of matter) have an observable difference which is measured by the factor $(1 + z)$. The uncertainty principle outlined above does not itself restrict the exactness of any particular measurement and it does not follow on this account that the distinction between the evolutionary and steady state theories cannot be detected as a matter of principle, so losing its significance. Rather, the author sees the situation as being as follows: we may in principle obtain some knowledge of the universe by means of observation and experiment, but the subsequent passage of time destroys some of this knowledge; for our knowledge is thereafter restricted by the uncertainty principle which is concerned essentially with the impossibility of predicting the future with arbitrary certainty, whatever the accuracy of known initial conditions. Knowledge of the system obtained by observation may well be used for calculating the situation for times previous to that of the experiment; but this is of a purely speculative nature since no such knowledge or resulting theory may be used as an initial condition for predicting with certainty the future progress of the system and thus the theory cannot immediately be subjected to a final experimental disproof or verification. It is clearly doubtful in the circumstances whether in any case such a calculation regarding the past history of the system may be accorded any physical significance.

It is important here to emphasize that we do not maintain that no theory is subject to disproof according to certain reasonable criteria and we may cite the discrimination against a static or a contracting universe upon what we feel at the present time to be reasonable grounds. Nor do we take the view that a theory which it is impossible to discriminate against at some particular epoch will necessarily remain invulnerable to disproof at all subsequent epochs; for cosmological theories in their present form make definite statements about expected future observations and, without further detailed research into the inherent uncertainties involved in the observations themselves, it may not be precluded in principle that a comprehensive scheme of observations lasting over a period of time significant on the cosmological scale might produce results in direct conflict with those of the theory under consideration. Thus the uncertainty considerations do not restrict entirely our possible knowledge of the universe; they merely limit in principle any assurance we might wish to have of the final or fundamental significance of such knowledge and preclude its use as an initial condition for the prediction, according to any theory, of the future. From this standpoint cosmology turns out to be literally a question of waiting and seeing.

The viewpoint expressed here is found to be, in general, in accord with the criticism by Balázs and Paál (1961) of McCrea's (1960) communication. While accepting McCrea's principle of uncertainty to within a factor $(1 + z)$, we disagree that (in view of the fact that the evolutionary and steady state cosmologies have an observable difference measured by the same factor) it implies that the difference between the models cannot be detected as a matter of principle. This does not preclude the possibility that on rather different grounds the distinction between the two types of universe may be meaningless.

We may note here that quantum limitations may also have application in cosmology as pointed out by Peres (1960) and Peres and Rosen (1960). Peres considers a universe containing only one body which he calls the Rotator; according to the general theory of relativity, we may consider this body to have an absolute rotation. Quantum effects become significant for any body with very small mass, in such a way that the angular momentum L and angular position θ become uncertain according to

$$\Delta L . \Delta \theta \sim \hbar$$

Peres claims that such effects would cause an uncertainty in the gravitational field of a small Rotator, leading to an uncertainty in the metric which he shows is of the same order as the components of the metric themselves at very large distances from the Rotator. The metric is thus not well defined at very large distances and Peres concludes that it should be completely undetermined in an empty universe. He further infers, from the large uncertainty in the gravitational field at large distances from the Rotator, that a finite distribution of matter can determine only to within a finite distance which frames of reference are inertial. This is the introduction of a fundamental limitation into Mach's principle, which states that the distribution of matter in the universe should be the cause of the inertia of any body embedded in it. Although the idea has received no rigorous treatment, the consideration of a universe containing only one body being not only unphysical, but especially unsuitable for a discussion of Mach's principle which is concerned with mutual action between bodies, it would seem to merit further attention.

Quantum limitations on the measurement of gravitational fields are considered by Peres and Rosen (1960) who obtain uncertainty relations for the average values of some Christoffel symbols measured in two domains, by analogy between the gravitational field in the weak, quasi-static case and the quantised electromagnetic field. Moreover, they show that there exists a limitation to the accuracy with which the average value of a single one of the Christoffel symbols can be measured. These results of Peres and of Peres and Rosen are considered by the authors to provide arguments in support of the view that the gravitational field should be quantised.

Wigner (1957) has further pointed out quantum limitations of the concepts of general relativity, including the limits on accuracy in the measurements of the curvature of space by means of a clock and a mirror. As regards measurement by clocks, Wigner shows that inherent uncertainties arise both in the accuracy of the clock itself and in the accuracy of conversion to space-like measures.

In such ways may quantum mechanics, concerned with microscopic phenomena, find application in the macroscopic world of cosmology and general relativity. But we may note in this connection that Bertotti (1960), taking a different standpoint, has emphasised, because of its probabilistic nature, the possible inadequacy and unsatisfactory logical viewpoint of quantum mechanics, in view of the unavoidable uniqueness of the world and of any particular observation of it.

It follows from the investigations of this section that our possible knowledge of the universe, as we regard it at present, is limited to some extent both by inherent uncertainties in observation and by indeterminacy regarding predictions for the future. It is then of importance to enquire whether or not that information which may in principle reach us will necessarily be relevant to the whole of the universe defined as the totality of events and physical objects which are significant on the largest scale.

The answer to such a question turns out to be in the negative, being bound up with the existence in some cosmological models of what are known as horizons. Horizons are boundaries which separate things which are observable in principle from things which are not observable in principle, where in this context "things" may refer either to events or to F.Ps. Clearly, by definition, investigation into things beyond a horizon implies the unavoidable and complete abandonment of operational techniques. It follows immediately that the whole topic of horizons and boundaries in cosmological models is closely connected with the concepts of indeterminacy and uncertainty; horizons and boundaries play an essential part in determining and limiting the amount of information that may in principle be obtained from the universe. Apart from its inherent interest, therefore, the obvious fundamental importance of this topic with regard to our possible knowledge of the universe as a whole commends it to further detailed study.

This will be undertaken in the present work, where, as well as introducing the well-known event horizons and particle horizons and discussing their significance for cosmology, we shall also investigate the boundary in Milne's cosmological model and the nature of the singular surfaces in the space-times characterised by Schwarzschild's metric and by Finkelstein's metric; the connection between these two will also be studied. We shall consider too observables and other variables in cosmological models with particular reference to their behaviour in the neighbourhood of a horizon or boundary when such exists; these investigations will both pave the way for, and find application in, the part of the thesis which is concerned with the application to cosmology of information theory itself.

CHAPTER II: THE EVENT AND PARTICLE HORIZONS

(i) Introduction and preliminaries

Various paradoxes and misconceptions that arose with regard to the properties of visual horizons in world models, as witnessed by the correspondence between Whitrow (1953, 1954 a,b), Bondi and Gold (1954), Pirani (1954), Gold (1955), the writer of an article [Nature, 175, 68, 382 (1955)] and Hoyle (1955), revealed the need for a clarification of the subject. A systematic treatment was accordingly given by Rindler (1956) who in effect defined a horizon as "a frontier between a non-empty class of things which are observable in principle and a non-empty class of similar things which are unobservable in principle," distinguishing two quite different types of horizon which he has termed event horizons (E.Hs.) and particle horizons (P.Hs.). Although he made no attempt on philosophical aspects of the subject, Rindler's treatment was in other respects comprehensive and we here give a brief account of his results regarding the E.H. and the P.H.

The analysis is based exclusively on the Robertson-Walker form of the line-element applying to

all homogeneous and isotropic world models.
Because of the existence of a cosmic time in these models, it is possible to define the proper distance ℓ between the origin-observer O situated permanently at $r = 0$ and any F.P. A with $r = r_1$ (say) at any instant t as the sum of the infinitesimal distance measurements taken at t by a chain of F.Os. situated along the space-like geodesic joining A to O. Putting dt = 0 in (1.2) and integrating, we get

$$\ell = R(t)\int_0^{r_1} \frac{dr}{(1+kr^2/4)} \qquad (2.1)$$

Adopting Rindler's notation we may for mathematical convenience define an alternative co-moving radial co-ordinate by

$$\sigma(r) \stackrel{def.}{=} \int_0^{r} \frac{dr}{(1+kr^2/4)} \qquad (2.2)$$

where $\sigma(r)$ can take all values. This is evidently so when $k = 0$ or -1; when $k = +1$, σ appears to have a pole at π: in this case we define co-ordinates beyond this pole by treating it as an auxiliary origin, evaluating the new co-ordinate from there and adding π. This may be done as many times as necessary and is possible since it follows from our above remarks that σ is an additive co-ordinate. On any line of sight for the closed universe with $k = +1$, the co-ordinates

$\sigma + 2n\pi$ for $n = 0,1,2\ldots$ are taken to represent the same particle.

Combining (2.1) and (2.2) we obtain

$$\ell = R(t)\sigma(r_1) \tag{2.3}$$

which is the equation of motion of the F.P. with $r = r_1$ (r_1 being a constant for that particle).

For the equation of motion of a photon emitted at $t = t_1$ from A towards O, we have by integrating (1.3) and multiplying throughout by R(t)

$$R(t)\int_{r_1}^{r} \frac{dr}{(1+kr^2/4)} = R(t)\int_{t}^{t_1} \frac{c\,dt}{R(t)} \tag{2.4}$$

i.e. $$\ell = R(t)\sigma(r) = R(t)\left\{\sigma(r_1) - \int_{t_1}^{t} \frac{c\,dt}{R(t)}\right\} \tag{2.5}$$

(ii) <u>The Event Horizon (E.H)</u>

Inspection of (2.5) shows that if $\int^{\infty} \frac{dt}{R(t)}$ is convergent to a finite limit, then there exists on any line of sight at any given time t_0 a critical particle with

$$\sigma_{crit.} = \int_{t_0}^{\infty} \frac{c\,dt}{R(t)} \tag{2.6}$$

such that a photon emitted from $\sigma_{crit.}$ at $t = t_0$ will reach O ($\ell = 0$) in the infinite future. The position on any particular line of sight of this critical particle at t_0 is an event horizon point at that instant, for photons emitted at t_0 from all farther particles (having $\sigma > \sigma_{crit.}$) will never reach 0 whereas all those emitted from nearer particles (with $\sigma < \sigma_{crit.}$) will reach 0 after some finite time. The aggregate of all such points is an E.H., dividing as it does the class of events which are in principle at some time observable to 0 from the non-empty class of those which are forever unobservable to 0.

By (2.6), discarding the particularising suffix, we see that the equation of motion of the E.H. is given by

$$\ell_{E.H.} = R(t)\int_t^{\infty}\frac{c\,dt}{R(t)} \qquad (2.7)$$

Comparison with (2.5) for all possible lines of sight shows that the E.H. is in fact a closed light front travelling towards the origin-observer 0, because for any particular line of sight (2.5), representing the motion of a photon towards 0, reduces to (2.7) upon setting $\sigma(\tau_1) = \int_{t_1}^{\infty}\frac{c\,dt}{R(t)}$. Depending on the form of the function R(t), the proper radius of the E.H. surface, given by (2.7) for some particular instant, may or may not change with time. Allowing t its full range, we may regard the E.H. as a hypersurface in space-time.

Consider now any particular F.P. A which at some time is within O's E.H. when such exists. A will similarly have its own E.H. at proper distance from A given by (2.7). Thus on the line of sight from O to A we have, in order of increasing proper distance from O: the F.P. A, O's horizon point O_H and A's horizon point A_H where $OO_H = AA_H$, both distances being given by (2.7). Thus O_H, a photon travelling towards O via A lies within A's E.H. and therefore will reach A with the speed of light in A's temporal experience. This is tantamount to saying that A, once inside O's E.H., must eventually in finite time reach O's E.H. and pass beyond it at the speed of light as measured locally.

However, the particle A will forever, in principle, be in O's view for as we have seen O will always be receiving that light which was emitted by A up to the time A reaches O's E.H.; the light emitted on O's E.H. will be received by O an infinite time later in O's experience so that the event of A's crossing O's E.H. is the last event in A's history that O may observe; no light emitted by A when A is beyond O's E.H. may ever be received by O, although at any such instant O may observe A by light previously emitted from within O's E.H. This usefully serves to demonstrate that the E.H. is concerned exclusively with events, and in no sense separates particles which are visible from those which are not.

(iii) The Particle Horizon (P.H.)

Consider light emitted from a F.P. at the earliest moment in its history; referred to the Robertson-Walker models this will be at $t = 0$ or $t = -\infty$, depending on the values of t for which $R(t)$ is defined, but since the analysis is similar for the two cases we perform it explicitly for the first, more usual, alternative.

If $\int_0 \frac{dt}{R(t)}$ is convergent to a finite limit reference to (2.5) shows that at any given time t_0 all those particles for which $\sigma > \int_0^{t_0} \frac{c\,dt}{R(t)}$ have not yet been observable at the origin by O(ℓ has not yet vanished), whereas all others have. Hence the surface

$$\sigma = \int_0^{t_0} \frac{c\,dt}{R(t)} \stackrel{\text{def.}}{=} \phi(t_0) \qquad (2.8)$$

will divide the class of those particles that have been observable by O at or before t_0 from the non-empty class of those which have not; it is thus a P.H. for the origin-observer O and is the cross-section at t_0 of the hypersurface $\sigma = \phi(t)$ which Rindler calls O's space-time P.H.

When $\phi(t)$ exists, it is necessarily an increasing function of t since R(t) is finite and positive, so that by (2.8), since σ is constant for each particular F.P., more and more particles become visible to 0 as time goes on. All particles $(\sigma \leq \infty)$ will become visible eventually if $\phi(t) \to \infty$ as $t \to \infty$; but if the model admits an E.H. as well as a P.H., so that $\int_0^\infty \frac{c\,dt}{R(t)}$ is convergent, then those particles for which $\sigma > \int_0^\infty \frac{c\,dt}{R(t)}$ will clearly never become observable to 0. However, as before, once seen a particle remains forever visible.

Each F.P. that becomes visible to an observer 0 does so first by the light emitted at the earliest event in the particle's history and when 0 has a P.H. this is received a non-zero time after emission, as inspection of (2.5) will show. For those models postulating a unique creation event in the finite past so that R(0) = 0, this evidently implies that F.Ps. were shot off with initial speeds exceeding that of light (otherwise they would have been visible to 0, in the limit, at the creation event itself). It follows that F.Ps. are not visible to 0 before some particular instant because to begin with they lie outside 0's creation-light-cone.

This is further verified by comparison of the equation of O's space-time P.H.

$$\ell_{P.H.} = R(t)\int_0^t \frac{c\,dt}{R(t)} \qquad (2.9)$$

which follows from (2.8), with that of O's creation-light-cone. For the particle characterised by $\sigma(\tau) = a$, the creation-light-cone is given by

$$\ell_{C.L.C.} = R(t)\left\{a \pm \int_0^t \frac{c\,dt}{R(t)}\right\} \qquad (2.10)$$

and is obtained in the same way as (2.5) retaining both signs in (1.3) and setting $t_1 = 0$. For O, $a = 0$ so that (2.10) reduces to (2.9) showing that the space-time particle horizon of any observer is the boundary of his creation-light-cone.

It follows that we may identify the observer's P.H. with the light front emitted by him at t = 0 (or $t = -\infty$ as the case may be) which is diverging from him; F.Ps. entering the P.H. then evidently do so at the speed of light as measured locally.

(2.10) also shows that when $\int_0 \frac{dt}{R(t)}$ is divergent, there is a unique light cone at the creation event, the same for all F.Ps.; whereas when the integral converges each F.P. has a different creation-light-cone. In this case, two F.Ps. first become visible to each other at the instant each enters the creation-light-cone of the other; the typical situation

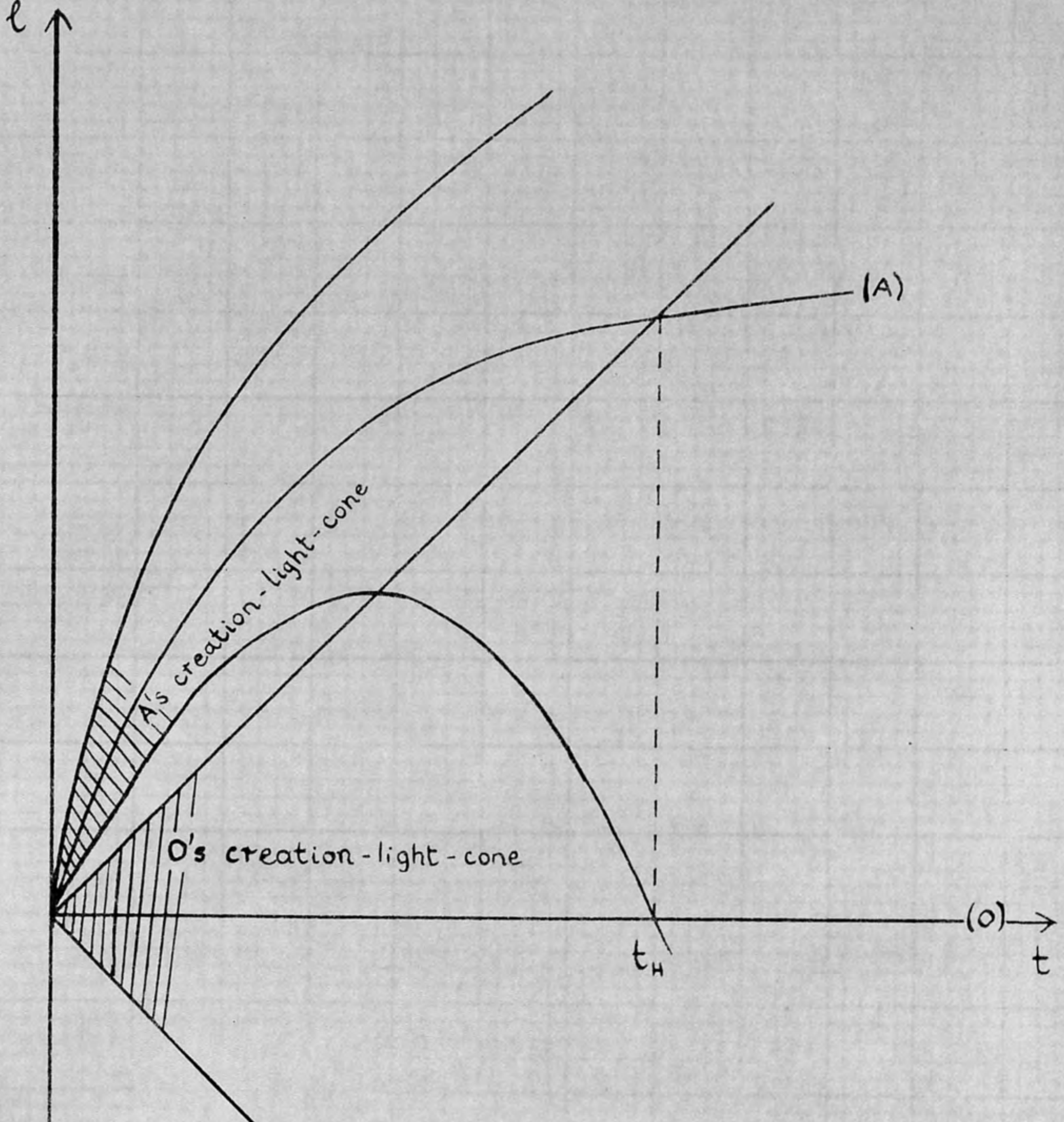


Fig. 2·1

is illustrated in Fig.2.1, where one F.P. is taken to be the origin-observer.

(iv) <u>The sufficiency of the conditions for the existence of an E.H. or a P.H.</u>

The results of the last two sections were obtained first by Rindler (1956) who, in addition, claimed without demonstration that the convergence of $\int^{\infty} \frac{dt}{R(t)}$ is both a necessary and a sufficient condition for the existence of an E.H. in a Robertson-Walker model, while, similarly the convergence of $\int_{0\,(-\infty)} \frac{dt}{R(t)}$ is a necessary and sufficient condition for the existence of a P.H. We consider that these questions require further consideration and we investigate first the sufficiency of the above conditions.

For the case of an E.H., we have demonstrated that if $\int^{\infty} \frac{dt}{R(t)}$ is convergent to a finite limit then at any instant, on any given line of sight, there exists a critical particle dividing the class of observable events from the non-empty class of unobservable events, these two classes of events forming together the totality of events which occur at that instant on that line of sight. If the universe is flat ($k=0$) or has a hyperbolic geometry ($k=-1$), then there does not exist any alternative route by which light emitted at any of the events which are not observable along that particular line of sight might reach the observer. This class of events are truly unobservable and evidently when $k=0$ or $k=-1$ the convergence of $\int^{\infty} \frac{dt}{R(t)}$ is a condition which is sufficient for the existence of E.Hs. in the model under consideration.

However when $k=+1$ the universe is closed and in principle it is possible at certain times in the model's evolution for a photon to make one or more complete circuits of the universe, so that while this may still happen all events are observable; moreover, even when photons may no longer make complete circuits, a light ray emitted at an event unobservable directly along the particular line of sight may conceivably reach the observer from the opposite direction, having first travelled round the universe via O's antipode. The convergence of $\int^{\infty} \frac{dt}{R(t)}$ is in this case clearly not sufficient for the existence in the model of an E.H. at all instants; we require also $t > T$ (say), where T is such that a photon emitted after that instant may no longer make a complete circuit of the universe, and in addition, to ensure that events unobservable from this one direction are not observable from the opposite direction, we require too that O's antipode has finally and forever crossed O's "horizon" (i.e. as defined solely by the convergence of the integral). Only then will there be a domain of events completely unobservable to O.

The possibility of this situation was fully realised by Rindler (1956) who explicitly demonstrated the above results and showed moreover that the instant T is that at which each F.P. crosses its own "horizon" for the last time. (We remember that for models with $k=+1$ F.Ps. with $\sigma+2n\pi$; $n=0,1,2\ldots$ are taken to be identical). Nevertheless he has claimed (1956a) that with complete generality (i.e. for all k , for any given time and for any given line of vision) the convergence of $\int^{\infty}\frac{dt}{R(t)}$ is a sufficient condition for the existence in a Robertson-Walker model of an E.H., stating that events occurring beyond this "horizon" are "evidently for ever outside the possible powers of observation of the origin observer".

Accordingly, upholding Rindler's definition of an E.H., we find the generality of his claim unjustified; for when $k = +1$, the class of unobservable events is empty for $t \leqslant T$ and when $T < t \leqslant \tau$, where τ is the instant at which O's antipode crosses O's "horizon" for the last time, the unobservability is not a matter of principle but of practice; only for $t > \tau$ does there exist a non-empty class of events which are in principle unobservable to O. Since the definition is satisfactory and in any case more fundamental, it would appear necessary to amend the condition for sufficiency regarding the existence of an E.H. when $k = +1$, demanding $t > \tau$ as well as the convergence of $\int^{\infty} \frac{dt}{R(t)}$.

A similar problem arises for the case $k = +1$ with regard to the sufficiency of the condition for the existence of a P.H. Examination shows that if $\int_0 \frac{dt}{R(t)}$ is convergent, then only when $t < t_H$ (say), where t_H is the instant at which (disregarding the creation instant itself) the F.O. first enters his own creation-light-cone, does there exist for this observer a non-empty class of particles. This situation is illustrated by Fig. 2.1 where O, A are now representations of the same F.P. at $\sigma = 0$, $\sigma = 2\pi$ respectively. By the time t_H, all other F.Ps. have become observable to the origin-observer and thereafter the non-observable class of particles is empty.

The sufficient condition for the existence of a P.H. at some given time t is therefore not only, as Rindler claims, the convergence of $\int_0 \frac{dt}{R(t)}$ whatever the value of k but, in addition, when $k = +1$ the requirement that $t < t_H$.

(v) <u>The necessity of the conditions for the existence of an E.H. or a P.H.; the degeneracy of horizons.</u>

If $\int^{\infty} \frac{dt}{R(t)}$ is divergent to infinity, then inspection of (2.5) reveals that on any particular line of sight and for any particular instant t_1, photons emitted towards O at t_1 from any F.P. ($\sigma \leq \infty$) on that line of sight will eventually reach O; it follows that all events are in principle observable to O so that the model has no E.H., the class of non-observable events being empty. The convergence of the integral is therefore a necessary condition for the existence of an E.H. in a Robertson-Walker model.

We have seen that when a Robertson-Walker model admits E.Hs., each F.O. will have an E.H. distinct from that of every other F.O. and that a feature of an E.H. when it exists is that it is situated at finite distance (according to some suitable definition) from the observer, in order that account may be taken of the existence of events beyond the horizon. We note further from (2.7) that the effect of the non-existence of an E.H. in a Robertson-Walker model, due to the divergence of the integral, is to send to infinity the quantity corresponding to the proper distance from the observer of his E.H., so that, in the limit of this happening, the surface given by (2.7) is the same for all observers.

We shall later show whether these features are essential to the existence or otherwise in any model of an E.H. as defined or whether they have arisen through investigation of models only of the Robertson-Walker type. In the meantime, we pave the way for such considerations by introducing the new concept of a surface which for a particular observer O divides all events into two classes: the non-empty class of events in principle observable to O and the empty class of events which are not observable to O. We define such a surface to be a DEGENERATE event horizon, provided that the surface ~~empty region~~ actually and irreducibly occupies a non-vanishing part of the space-time manifold.

~~Taking proper distance as our variable we see that no degenerate E.H. is admitted in a Robertson-Walker model, in view of the added condition.~~

With regard to P.Hs. in the Robertson-Walker models, we have already seen that the divergence of $\int_{0(-\infty)} \frac{dt}{R(t)}$ implies that there is a unique light cone at the creation event which is invariant i.e. the same for all F.Ps., so that all F.Ps. first become visible to each other at the creation instant. It follows that for all t, the class of non-observable particles is empty so that there exists no P.H. in the model according to the definition. The convergence of $\int_{0(-\infty)} \frac{dt}{R(t)}$ is therefore a necessary condition for the existence of a P.H.

The distinction between the two situations of existence and non-existence of P.Hs. lies in the fact that in the latter case the class of particles lying outside the observer's creation-light-cone is empty. We now define a DEGENERATE particle horizon as a surface which at a given instant divides all F.Ps. into two classes: a non-empty class of particles which have already been observable to a particular F.O. by that instant and an empty class of particles which have not already been observable to him; again the ~~empty region~~ surface must occupy a non-vanishing part of the space-time manifold.

~~We then find that all Robertson-Walker models for which~~ $\cancel{\int_0 \frac{dt}{R(t)}}$ ~~diverges admit an invariant degenerate P.H. which is the unique creation-light-cone~~.

The concept of the degeneracy of horizons, in the sense that the class of non-observables is empty, will find applications in later sections, when we shall also demonstrate and interpret a connection between this concept and that of the invariance of horizons.

(vi) Horizons for non-fundamental observers.

So far we have considered only those observers remaining attached to a F.P. We now relax this restriction and for the sake of completeness quote briefly Rindler's (1956) conclusions regarding horizons for non-fundamental observers.

(a) An observer may in principle reach any pre-assigned event within his forecone, so that all events whose forecones intersect his own become observable to him.

(b) For models admitting E.Hs. but no P.Hs., any observer may in principle be present at any one pre-assigned event (by detaching himself soon enough from his F.P.). Two pre-assigned events will not in general be observable (or attainable) to any one observer, so that the observer's horizon may never be completely abolished.

(c) For models admitting both E.Hs. and P.Hs., then for any observer originally attached to a F.P., there exists a class of events completely unobservable to him however he travels through space. We then speak of an ABSOLUTE horizon.

(vii) Rindler's α -horizon.

Rindler (1960) has considered also a particular case of a further and different generalisation of the event horizon concept. Proposing differential equations of motion for a test particle in curved space-time, Rindler solved them in detail for the de Sitter space-time which is relevant to the steady-state theory of cosmology and is characterised in terms of the Robertson-Walker line-element by $k = 0$, $R(t) = e^{Ht}$ where H is a constant known as Hubble's constant. We suppose that the particle has a given available acceleration α. Then Rindler has demonstrated the following points for the steady-state model.

(a) A particle moving radially with uniform proper acceleration ultimately moves with constant relative velocity through the substratum.

(b) There exists a critical first F.P. on its line of motion which it will never overtake.

(c) A light signal emitted at or after a certain critical time from the origin will not catch up with the uniformly accelerating particle which was originally released from rest at the origin.

(d) If a particle with a given available acceleration α passes beyond a certain distance (the α-horizon) it can no longer return to its place of origin.

Allowing $\alpha \to \infty$, it is found that the motion of the particle becomes geodesic motion with the constant velocity of light, so that the α-horizon becomes the usual event horizon, from beyond which light is unable to reach the origin. Thus Rindler's generalisation of the E.H. concept in the steady-state model consists in considering the reception of non-fundamental particles as well as light rays, rather than in considering horizons for non-fundamental observers.

(viii) Comparison and contrast of the E.H. and the P.H.; considerations of time.

The E.H. and the P.H. which might exist for a F.O. in a cosmological model of the Robertson-Walker type have been identified with light fronts which respectively converge on the observer and diverge from him. (It should be noted that by equations (2.7) and (2.9) the proper distance of horizon from observer may however remain constant). Because there exists a finite upper bound to the speed with which causal influences may be propagated, equal to the speed of light, we see that both E.Hs. and P.Hs. are semi-permeable membranes allowing the passage of causal influences in one direction only. These directions differ in the two cases, being towards the observer in the case of a P.H., but away from him in the case of an E.H.; we emphasise here that we use "towards" and "away from" as corresponding respectively to the use of "divergence" and "convergence" when speaking of the light fronts and that no implication is intended as regards measurement by proper distance.

This use appears justified when we take into account the following considerations: with a P.H., at any particular instant information suddenly becomes available to the observer O about events which have previously occurred at the F.P. which at that instant crosses O's P.H., all such information being unavailable up to that time; the class of particles unknowable to the observer up to time t decreases in number as t increases. In contrast, with an E.H., information regarding events occurring at a particular F.P. after a certain instant is made forever unavailable to the F.O.; the class of events unknowable to the observer at any time t increases in number as t increases. In this sense, the region beyond a P.H. may well be regarded as a source of information, whereas that beyond an E.H. will correspondingly be seen as a sink of information.

The above formulation demonstrates the difference in the roles that time plays in the two cases; evidently phenomena concerning the P.H. are connected with the observer via his backward light cone, those concerning the E.H. being connected via the forward light cone; an unobservable particle in the former case belongs to that category for all time before a certain instant, whereas in the latter case, the unobservable events belong to that category forever after a certain instant, in each case the critical instant being that at which the F.P. under observation crosses the horizon concerned.

In this connection we observe that horizons represent fundamental limitations with respect to time in the description of the universe by any F.O. and that no actual spatial boundary is connected with the existence of horizons. Whitrow (1954b, 1961) has emphasised this aspect, calling the E.H. a "time-horizon" since according to the origin-observer time on his E.H. appears to "stand still" and observation of an event occurring on his E.H. will take up all the time available to the observer. Thus the E.H. is an important possible limitation to the whole concept of cosmic time.

A consideration of those Robertson-Walker models which have $R(t) \propto t^n$, with $n > 0$ for expanding models, illustrates a further difference between the two types of horizon. Of this class of models, those with $n > 1$, implying an increasing rate of expansion, satisfy the necessary condition for the existence of E.Hs., but not that for P.Hs., while the reverse is true for those models with $n < 1$ and therefore a decreasing rate of expansion. The uniformly expanding model universe with $n=1$ admits neither type of horizon. Clearly the rate of expansion of the universe is an essential factor in determining the existence of otherwise or horizons. An E.H. may exist only if the rate of expansion is great enough for some photons moving towards the observer never to reach him; a P.H. can occur only if the initial rate of expansion exceeds the speed of light. In the latter case, F.Ps. will subsequently be able to enter the observer's creation-light-cone only if the rate of expansion thereafter decreases. Should it afterwards increase again in a suitable manner, the model may admit both types of horizon.

In spite of the difference with regard to the role of time in the two cases of the existence of an E.H. and of a P.H., one feature remains common to both situations: a F.P. once visible remains forever visible and the phenomenon of the disappearance of a F.P. is unknown, at least in an expanding universe. To obtain insight into this whole problem of time and horizons, it is instructive to consider the corresponding situations in those contracting models obtained by time-reversal from expanding models i.e. to any expanding model we assign a contracting dual by replacing t by -t.

Rindler (1956) has shown that by this procedure an E.H. is transformed into a P.H. and vice versa. Consider first an expanding model with an E.H., for which t extends over the whole range $[-\infty, \infty]$. From (2.7) the equation of the E.H. is given by

$$\ell = R(t)\int_t^{\infty} \frac{c\,dt}{R(t)} \qquad (2.11)$$

If $t \rightarrow -t$, we get

$$\ell = R(-t)\int_{-\infty}^{t} \frac{c\,dt}{R(t)} \qquad (2.12)$$

Reference to (2.9) shows that this model has a P.H., its equation being given by (2.12). Conversely, the P.H. is similarly transformed into an E.H.

Suppose a model has a point-creation in the finite past, then this creation event transforms into a point-annihilation event in the finite future upon time-reversal. If the model admits a P.H. for a given observer, this will be transformed into an E.H. in the sense that events occurring beyond it will not be observed in the finite time left to the observer before annihilation. The above assertion applies in all cases, with similar suitable modifications.

We see that the phenomenon of the disappearance of a F.P. from the observer's view cannot possibly occur, even on time-reversal. Consequently, we argue that this feature depends not on the direction of time, which appears to govern whether a horizon is an E.H. or a P.H., but on some property that remains unaltered upon reversing the direction of time viz. the fact that an observation is made always by means of incoming light.

(ix) <u>Implications of the existence or the concept of E.Hs. and P.Hs.</u>

In a universe which admits a P.H. for each F.O. and in which F.Ps. form a discrete set, as in our own universe, the sudden appearance of a particular F.P. over his horizon is an occurrence completely and in principle unpredictable by the observer. We may suppose that the observer is familiar with the phenomenon of such an appearance through previous observations and even that with the aid of a far-sighted and comprehensive theory he may adequately explain it; these factors, together with other observational data, would enable the observer to predict on a statistical basis the rate at which further F.Ps. might be expected to suddenly enter his vision having previously been unobservable. (In fact, the converse situation has been discussed by McVittie (1937 b): namely, by measuring the rate of the sudden appearances, to determine the rate of expansion of the universe on the basis of a Robertson-Walker cosmological model. McCrea (1934-5) has also discussed the P.H. in the context of empirical determinations.) But in no case may any one particular such appearance be predictable by the observer, since prior information, i.e. initial conditions appertaining to the particular F.P., is in principle completely unavailable to the observer. Hence the existence of P.Hs. in a universe implies a fundamental indeterminacy with regard to the observation of particular F.Ps. in that universe, which is resolvable only on a statistical basis.

A different situation arises regarding the existence of E.Hs. in a universe. We have seen that events occurring at a particular F.P. after a certain instant are forever unobservable to a particular F.O. O, but that nevertheless the F.P. is always in O's view, its proper history up to that critical instant appearing to O in a more and more dilated form as time goes on and ultimately taking up the infinity of time available to O. We must reject as meaningless the "reality" in the experience of O of the unknowable extra-empirical events which occur beyond O's E.H. Nevertheless, the same events occurring after the critical instant of a F.P. crossing O's E.H. may undoubtedly, according to our theory, be assigned a reality in the experience of an observer A associated with that F.P. It follows that the existence of E.Hs. in a universe implies for any one observer the impossibility in principle of comprehensive knowledge about the universe, when this is defined as the aggregate of all processes occurring on a large enough scale in space and time.

Scientific investigation by an observer O is confined to those processes occurring on a large scale which are at some time in O's history physically linked with him. When no E.H. exists in a model, (so that the class of non-observable events is empty) the set of all such processes is invariant in the sense that it is the same for all F.Os.; but when a model admits E.Hs. the set contains different members for each F.O. This set may be compared with the set of F.Ps. visible at a given time; this is invariant when no P.H. exists, so that the class of particles which are non-observable at that instant is empty, but varies according to the observer when the model admits P.Hs. These remarks together with the definitions of degenerate horizons given in Chapter II (v), demonstrate a connection between degeneracy and invariance.

(x) Implications for Mach's principle.

These considerations have tackled to some extent the logical status of those particles or events which are beyond a horizon; an observer may consider scientifically only those things which are causally linked with him.

For consistency, it is required that all causal influences possess an upper bound to their speed of propagation, given by the speed of light, and this will include the gravitational influence of one body upon another. Accordingly, although the mechanism for gravitational interaction between bodies is not understood, it is usually supposed that the influence travels with the speed of light and that in an expanding universe its effect is subject to attenuation by the same Doppler factor due to the mutual recession.

In accordance with the ideas of Mach (1893), inertia as well as gravitation should depend upon mutual interaction between all the bodies of the universe, in such a way that the magnitude of the inertia of any body is determined by the masses of the rest of the universe and by their distribution; thus there would be a causal connection between the motion of the stars and the state of the local inertial frame. This statement gives expression to what is known as Mach's principle, which may thus account for the apparent coincidence of the local, dynamically determined, frame of reference and the frame given by the "fixed" stars. It is then to be expected that the influence of distant bodies would predominate; that the inertia of a body would increase with the amount of ponderable matter in its neighbourhood; and that the further away from all other matter that a body was removed, the more would its inertia be reduced.

Mach's principle, for which no mathematical formulation has been found, may well be consistent with the general theory of relativity for the field equations

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \qquad (2.13)$$

where κ, Λ are constants ($\Lambda \geqslant 0$) and the tensor $R_{\mu\nu}$ is constructed from the $g_{\mu\nu}$, unify the inertial field defined by the $g_{\mu\nu}$ and the gravitational field represented by $T_{\mu\nu}$, implying an influence of masses on the inertia of other bodies; but being only differential equations, it may be necessary (depending on the value of Λ), although it may not be sufficient, for boundary conditions to be specified before the $g_{\mu\nu}$ are completely determined. In any case, whatever the value of Λ, it was shown by de Sitter (1917) that the equations (2.13) do not satisfy the essential condition that it should be impossible to determine the $g_{\mu\nu}$ when $T_{\mu\nu} = 0$ throughout space-time, expressing the idea that there should be no inertia in the absence of matter. It is generally deduced that Mach's principle itself is therefore not fully incorporated in Einstein's theory; however, Davidson (1957) has put forward arguments to the contrary.

The interest for the present work lies in the fact that for any model which may be characterised by the Robertson-Walker metric Davidson has related σ , the cosmological density of gravitational mass, to the proper distance to the "horizon" by means of the field equations (taking $\Lambda = 0$). Where ρ is the average inertial density and p is the pressure

$$\sigma = \rho + \frac{3p}{c^2} \tag{2.14}$$

for the isotropic cosmological models; then application of the field equations with $\Lambda = 0$ to the metric (1.2) gives

$$4\pi G\sigma = -\frac{3\ddot{R}}{R} \qquad (2.15)$$

where a dot denotes differentiation with respect to time. σ is then relatedto the proper distance to a horizon through the function $R(t)$.

Davidson, however, has not connected σ with either the E.H. or the P.H. as defined by Rindler, both of which are the only boundaries of physical significance in any model in which they exist. Instead he has taken the horizon of any model to be at that distance, measured in a suitable way (which he has taken to be by proper distance) at which the velocity of matter relative to the space origin equals the velocity of light. (We have seen that for an E.H. and a P.H. matter crosses the horizon with the speed of light as measured locally, not from the space origin.) That the two concepts are not equivalent may be demonstrated by means of a counter example.

We have for the proper distance from r = 0 to the F.P. with $r = r_1$ at time t

$$\ell = R(t)\,\sigma(r_1) \qquad (2.16)$$

(σ written as a function of r will consistently represent a co-moving radial co-ordinate, not to be confused with the density of gravitational mass).

Therefore the F.P's radial velocity is given by

$$\dot{\ell} = \frac{\dot{R}}{R}\ell \qquad (2.17)$$

where $R = R(t)$.

According to Davidson, the proper distance to his "horizon", which we shall call $\mathfrak{R}$, is given by the value of ℓ when $|\dot{\ell}| = c$. Thus from (2.17)

$$\mathfrak{R} = \pm \frac{c\dot{R}}{R} \quad \text{according as} \quad \dot{R} \gtrless 0 \qquad (2.18)$$

(i.e. according as we have an expanding or a contracting model). For the class of models with $R(t) = t^n$, $\dot{R} = nt^{n-1}$ and

$$\mathfrak{R} = \pm \frac{ct}{n} \quad \text{according as} \quad n \gtrless 0 \qquad (2.19)$$

But by (2.7) and (2.9) respectively, we have for this class of models (taking the expanding case $n > 0$) the proper distance to an E.H. ($\ell_{E.H.}$) and to a P.H. ($\ell_{P.H.}$) at time t given by

$$\ell_{E.H.} = \frac{ct}{n-1} \qquad \text{for} \quad n > 1 \qquad (2.20)$$

$$\ell_{P.H.} = \frac{ct}{1-n} \qquad \text{for} \quad n < 1 \qquad (2.21)$$

both expressions formally being infinite for $n \leqslant 1$, $n \geqslant 1$ respectively. Comparison with (2.19) for $n > 0$ shows that the surface characterised by $\mathfrak{R}$ does not represent any horizon, as defined by Rindler, in this class of models; this is the case in general since proper distance, a parameter in Rindler's work, is raised to unwarranted physical significance by

Davidson by identifying a certain rate of change of proper distance with the velocity of light. (It is easily verified, however, that for the steady-state model given by $R(t) = e^{t/T}$; $T = \text{constant}$, $\mathcal{R}$ does happen to coincide with the proper distance to the E.H. in the model, obtained from (2.7), both being given by cT.).

We must therefore reject as incorrect Davidson's (1957) general statement that "for an observer at the origin matter which goes beyond this distance ($\mathcal{R}$) virtually ceases to exist"; for we have already shown that only particles which at time t lie beyond an E.H. or a P.H. are effectively non-existent as regards their possible influence on the origin-observer, where for the case of an E.H. t is relevant to the time of emission and in the case of a P.H. to the time of reception; when an E.H. exists we expect the influence of one F.P. A on another F.P. O to be more and more attenuated, being impossible altogether after the critical instant at which A crosses O's E.H.

Using (2.15), (2.17) and (2.18), Davidson's analysis shows that for $\dot{R} > 0$

$$G\sigma\mathcal{R}^2 = -\frac{3c^2}{4\pi}\left(1 - \frac{\dot{\mathcal{R}}}{c}\right) \qquad (2.22)$$

whereas for $\dot{R} < 0$

$$G\sigma\mathcal{R}^2 = -\frac{3c^2}{4\pi}\left(1 + \frac{\dot{\mathcal{R}}}{c}\right)$$

giving his relations between σ and the distance to the "horizon" at time t. These would imply that for $\dot{R} > 0$, $\sigma \gtrless 0$ according as $\dot{\mathcal{R}} \gtrless c$; for $\dot{R} < 0$, $\sigma \gtrless 0$ according as $-\dot{\mathcal{R}} \gtrless c$. Davidson states for $\dot{R} > 0$ that if $\dot{\mathcal{R}} > c$ matter is entering the region bounded by the defined "horizon" and if $\dot{\mathcal{R}} < c$ matter is passing beyond this "horizon". Taking again the class of models given by $R(t) = t^n$ $(n > 0)$, (2.19) shows that $\dot{\mathcal{R}} \gtrless c$ according as $n \gtrless 1$, that is according as there exists, respectively, a P.H. or an E.H. in the model. Since a P.H. is associated with matter entering the sphere of influence, and an E.H. is, in a sense, associated with matter leaving the sphere of influence according to Rindler's ideas, Davidson's interpretation in this respect happens not to be false.

We must note that in talking of a P.H. (when $n < 1$) we refer to the influence of the rest of the universe on the origin-observer as experienced at time t, via the observer's backward light-cone (for $n > 1$ there exists no P.H. in these models): whereas in talking of an E.H. (when $n > 1$) we refer to the potential influence as emitted at time t of a F.P. on the rest of the universe, that is, on all those F.Ps. which at that time are within the particle's E.H., this influence being propagated within the particle's forward light cone at the event of emission. We limit ourselves in the present work to the explicit consideration only of the cases admitting the existence of a horizon.

For $R(t) = t^n \ (n > 0)$,

$$\frac{\ddot{R}}{R} = \frac{n(n-1)}{t^2}\,; \quad n \neq 1 \tag{2.23}$$

and by (2.20), (2.21) the equation

$$\ell_H = \pm \frac{ct}{(n-1)} \tag{2.24}$$

gives the proper distance to an E.H. or to a P.H. according as $n > 1$ (taking the + sign) or $n < 1$ (taking the − sign). Thus

$$\frac{\ddot{R}}{R} = \pm \frac{n}{(n-1)} \cdot \frac{c^2}{\ell_H^2} \tag{2.25}$$

so that, by (2.15), our analogue, in terms of Rindler's concept of horizons, of Davidson's equations in terms of $\mathcal{R}$ (i.e. (2.22)) is given by

$$G\sigma = -\frac{3c^2}{4\pi} \cdot \frac{n}{(n-1)} \cdot \frac{(\pm 1)}{\ell_H^2} \tag{2.26}$$

Differentiating (2.17) with respect to t, we get

$$\ddot{\ell} = \frac{\dot{R}}{R}\dot{\ell} + \ell\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2}\right)$$
$$= \frac{\ddot{R}}{R}\ell \tag{2.27}$$

upon using (2.17) to eliminate $\dot{\ell}$. By (2.15) this implies that for all models of the Robertson-Walker class

$$\ddot{l} = -\tfrac{4}{3}\pi G \sigma l \qquad (2.28)$$

a Newtonian type equation relating gravitational force and proper distance at time t. This is a general result which we have derived independently of any particular horizon concept; it is therefore not surprising that Davidson has arrived at the same equation by using, instead of (2.26), the equations (2.22) which being in terms of $\mathcal{R}$ we consider to be of mistaken importance. Because of the generality of (2.28), the only important difference that is made to Davidson's work by application of the correct horizon concept arises in the evaluation of the potential Φ.

Consider, with Davidson, the gravitational work done by the field when a particle of unit mass is moved from its actual position at time t_0 to the horizon and therefore beyond the influence of the origin. According to Davidson this will be

$$\Phi_l = -\tfrac{4}{3}\pi G \int_l^{\mathcal{R}} \sigma l \, dl$$

which we must replace by

$$\Phi_l = -\tfrac{4}{3}\pi G \int_l^{l_H} \sigma l \, dl \qquad (2.29)$$

Evaluating this, by (2.28), as

$$\Phi_l = \int_l^{l_H} \ddot{l} \, dl \qquad (2.30)$$

we get $\Phi_\ell = \frac{1}{2}\left[\dot{\ell}^2\right]_\ell^{\ell_H}$ (2.31)

For the class of models with $R(t) = t^n$ $(n > 0)$, (2.17) shows that

$$\dot{\ell} = \frac{n\ell}{t} \qquad (2.32)$$

so that, using (2.24)

$$\dot{\ell}^2 = \frac{c^2 n^2}{(n-1)^2} \cdot \frac{\ell^2}{\ell_H^2} \qquad (2.33)$$

Thus

$$\Phi_\ell = \frac{c^2 n^2}{2(n-1)^2}\left(1 - \frac{\ell^2}{\ell_H^2}\right) \qquad (2.34)$$

gives the potential at proper distance ℓ at time t, where for $n > 1$, $\ell_H = \ell_{E.H.}$ (influence propagated at t into the future) and for $n < 1$, $\ell_H = \ell_{P.H.}$ (influence from the past received at time t). Φ_ℓ is clearly model dependent, even when we allow $\ell \to 0$ to obtain the potential at the origin, Φ_0, which is given by

$$\Phi_0 = \frac{c^2 n^2}{2(n-1)^2} \qquad (2.35)$$

for this class of models.

This is in contrast to the results of Davidson, who obtains, analagous to our equation (2.34)

$$\Phi_\ell = \frac{c^2}{2}\left(1 - \frac{\ell^2}{R^2}\right)$$

so that $\Phi_0 = c^2/2$ irrespective of which Robertson-Walker model is under consideration. (In all such models any inertial frame is Galilean in the neighbourhood of the space origin.) Having defined Φ by the equation $\Phi = \frac{1}{2}g_{44}$, Davidson has shown that it is precisely this value $c^2/2$ which arises for the static potential at the origin of a Galilean frame of reference, which has the metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \qquad (2.36)$$

Davidson therefore considers that his tentative analysis leading to the same value for Φ_0 provides a physical identification, in a natural way, of the potential Φ_0 arising from (2.36). Φ_0 is interpreted as "the gravitational potential of all the matter in the universe apparent to an observer at the origin and having influence there". It is not clear why this should be expected to be either constant with respect to time in a non-static universe or model independent and, indeed, following Davidson's procedure correcting only for the horizon concept, we have shown by (2.35) that Φ_0 is dependent on the cosmological model under consideration and does not coincide with the value arising from (2.36).

Because, in a steady-state model, the value of $\mathfrak{R}$ happens to coincide with that of $l_{E.H.}$, Φ_l, given by

$$\Phi_l = \frac{c^2}{2}\left(1 - \frac{l^2}{l_{E.H.}^2}\right) \qquad (2.37)$$

for influences in the forward light cone, does lead to $\Phi_0 = c^2/2$ upon allowing $l \rightarrow 0$. In view of the fact that this is not a general result, as demonstrated by our counter-example, it is no general justification for the physical interpretation of Φ_0.

It is interesting, however, to note that unlike other Robertson-Walker models, the steady-state model allows the expression of its metric in a static form which is due to de Sitter, and we may speculate that the coincidence in the values of Φ_0 and the constancy of Φ_0, interpreted as the gravitational potential due to all matter in the universe, arise at least in part because of the stationary character of the model. (For this interpretation of Φ_0, because of the connection only with the forward light cone, we require to go backwards in time; alternatively, we may make the interpretation that Φ_0 measures, in some way, the gravitational influence of the origin observer on the rest of the universe.)

In any other expanding model, intuitive ideas in accordance with Mach's principle suggest that we should expect any effect to be attenuated as time goes on; no such feature of attenuation is considered in Davidson's procedure, which on this account, as well as regards the horizon concept, therefore appears to be unsatisfactory. Although further research may well show that inertial mass can be associated with the influence of the whole universe, in accordance with Mach's principle, we find that this has not been satisfactorily demonstrated by Davidson, as claimed. Unfortunately, further investigation of this topic is beyond the scope of the present work.

CHAPTER III: THE MILNE-TYPE BOUNDARY

(i) The boundary in Milne's model

Although reference has been made to a boundary in Milne's model (Milne, 1948; Rindler, 1956; Bondi, 1960c) there seems to have been no clear definition laid down. Accordingly, an ambiguous situation persists, especially when reference is made (Rindler, 1956) to other cosmological models possessing a "Milne-type of boundary", so called since it is exemplified in Milne's model; for we find that there exist many possible definitions of a Milne-type boundary according to whether certain features of the boundary in Milne's model are considered essential or incidental.

Milne's model satisfies the postulates of homogeneity and isotropy; it may therefore be described by a Robertson-Walker metric. Referred to the form (1.2), it has been shown by Kermack and McCrea (1933) that for Milne's model $R(t) = ct$; $k = -1$ i.e.

$$ds^2 = c^2 dt^2 - (ct)^2 \left\{ \frac{dx^2 + dy^2 + dz^2}{(1 - \frac{1}{4}r^2)} \right\}; \; r^2 = x^2 + y^2 + z^2 \quad (3.1)$$

We note that there exists neither P.H. nor E.H. in the model according to the definitions and criteria laid down by Rindler as in the previous chapter.

Any F.O. in this model may consider himself to be situated at the centre of a spherically symmetric system whose boundary expands with the speed of light. At the initial creation instant F.Ps. would have been shot off with various velocities $\lesssim c$ relative to any particular F.O., afterwards to continue to move away radially from that F.O. with the same constant velocity. According to any F.O., F.Ps. are distributed in such a way that their density increases with distance from this central observer, tending to infinity at the boundary of the expanding sphere. (Bondi, 1960c). The boundary itself is not occupied by F.Ps. since the velocity of light is not attained; that is, the F.Ps. form an open set, so that there exists no "farthest" F.P., and the boundary may thus, in the first place, be considered as the locus of limiting points at which the density tends to infinity.

The boundary may also be considered as the locus of a light front, corresponding to the light wave emitted at the creation instant; since the model admits no P.H., all F.Ps. are always visible and the creation-light-cone is invariant, the same for all observers.

Transforming (3.1) according to the equations

$$\left.\begin{aligned} \bar{t} &= \frac{t(1+\frac{1}{4}r^2)}{(1-\frac{1}{4}r^2)} \quad ; \quad \bar{x} = \frac{xt}{(1-\frac{1}{4}r^2)} \; ; \\ \bar{y} &= \frac{yt}{(1-\frac{1}{4}r^2)} \quad ; \quad \bar{z} = \frac{zt}{(1-\frac{1}{4}r^2)} \end{aligned}\right\} \qquad (3.2)$$

gives

$$ds^2 = c^2 d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2 \qquad (3.3)$$

Bondi (1960c) has pointed out that as well as ds the quantity

$$X = c^2\bar{t}^2 - \bar{x}^2 - \bar{y}^2 - \bar{z}^2 \qquad (3.4)$$

is invariant under Lorentz-transformation and he states that "the surface $X = 0$ represents the invariant border of the universe which is advancing at the speed of light". This 3-dimensional surface is the creation-light-cone of all observers and is the boundary which the Milne model exemplifies. Since this model possesses no P.H., this unique creation-light-cone is a surface which encloses the 4-dimensional region which contains the world lines of all physically significant objects existing in the model. It separates F.Ps. into two classes, one of which is empty; thus we have proved that the boundary in Milne's model is a degenerate P.H.

(ii) Features of the Milne boundary.

The boundary in Milne's model has three important features:

(a) it is a creation-light-cone

(b) it is invariant

(c) it encloses the region containing all physically significant objects

We here define a Milne-type boundary as one possessing all the above features; hence a Milne-type boundary is to be identified with a degenerate P.H.

Any Robertson-Walker model not admitting P.Hs. with $k \neq +1$ will necessarily have a boundary with all the features manifested by this boundary in Milne's model. The necessary and sufficient condition for there to exist a Milne-type boundary (degenerate P.H.) in any given model with $k \neq +1$ is that the integral $\int_0 \frac{dt}{R(t)}$ (or $\int_{-\infty} \frac{dt}{R(t)}$ for those models for which the definition of $R(t)$ extends to negatively unbounded values of t) should not converge to a finite limit. For example, all models of the class $R(t) = ct^n$ $(n > 1)$ $k \neq +1$ satisfy this condition and therefore each has a Milne-type boundary; those with $n < 1$ do not have a Milne-type boundary.

Had we discarded as inessential the feature of invariance of the boundary, we would have, for a F.O. in any model, a creation-light-cone boundary which is the invariant degenerate P.H. in models for which $\int_0 \frac{dt}{R(t)}$ diverges and is the observer's P.H. when such exists; in the former case this boundary encloses the region containing all physically significant objects, satisfying c), but otherwise it does not.

Alternatively, upholding only feature c), we would have as boundary that surface enclosing the region containing all physically significant objects. This is identical with the degenerate P.H. when such exists; if, however, a model admits P.Hs. this boundary evidently lies beyond any observer's creation-light-cone (hence being of little significance) corresponding formally to the envelope of the creation-light-cones of all F.Ps.

These two further possibilities for a "Milne-type boundary" have been dismissed by our laying down in this section of a definition; an ambiguous situation no longer persists.

(iii) <u>The boundary of distance by parallax.</u>

The boundary in Milne's model corresponds at

The existence of a finite upper bound $\bar{P}$ is, according to Rindler, an essential feature of what he calls a Milne-type of boundary, the surface given by $\bar{P}$ being the boundary concerned. We shall show that the existence of $\bar{P}$ is, in fact, incidental. Characterising the situation in Milne's model, he states (1956b): "All F.Ps. are visible at all times and there exists at any time t a finite upper bound $\bar{P}(t)$ to the apparent distances by parallax of these particles". Rindler does not make it clear in what way the feature that all F.Ps. are visible at all times (i.e. the non-existence of P.Hs. in the model) is relevant to the existence of this "Milne-type boundary".

If it is taken to be a necessary condition for the existence of a boundary of the Milne-type, then as we have shown, it is also sufficient and consideration of a finite upper bound to P is unnecessary, the boundary already having been identified with a degenerate P.H. If it is not a necessary condition, Rindler is considering a boundary of apparent distance by parallax (P.B.) which we shall now investigate.

any instant to the locus of the light wave emitted at the creation instant, enclosing within it all F.Ps. Rindler (1956) has considered the boundary in the model, at any instant t and for any F.O., to be the unattained upper bound of the distances of F.Ps. from that observer, which appears as a sphere round the F.O. which expands with the velocity of light.

Let us for the moment adopt with Rindler apparent distance by parallax, P, as our distance variable. P is defined by the equation $P = \frac{\delta \ell}{\delta \theta}$ where $\delta \ell$ is unit length held at the observer O at r = 0 perpendicular to the line joining O to the event being observed at A ($r = r_1$ say) and $\delta\theta$ is the angle between the apparent directions of A as viewed from the ends of $\delta \ell$. McCrea (1934-5) has shown that, referred to the metric (1.2), P is given by

$$P = \frac{R(t_0)\, r_1}{1 - \frac{1}{4}k r_1^2 + \dot{R}(t_0)\, r_1/c} \tag{3.5}$$

In Milne's model, the upper bound $\overline{P}$ to the apparent distances by parallax of the F.Ps. from r = 0 at $t = t_0$ is given by

$$\overline{P} = \left. \frac{c t_0 r_1}{(1 + \frac{1}{2} r_1)^2} \right|_{r_1 = 2} = \frac{1}{2} c t_0$$

(iv) Properties of the boundary by parallax (P.B.)

Rindler has pointed out that both the class of models with $R(t) = ct^n (n>1), k=-1$ which admit E.Hs. and the class with $R(t) = ct^n (n<1), k=-1$ which admit P.Hs. and also the de Sitter model with $R(t) = e^{t/T}$, $k=0$ have a P.B. In fact it is easily verified that all of the currently important expanding relativistic cosmological models satisfying the metric (1.2) have a finite $\overline{P}$ for general constant t_0; the very definition of P seems to ensure this property. We may of course construct models in which P instantaneously assumes an infinite value; these will usually be oscillating models with the infinite value of $\overline{P}$ at the stationary instant. The only simple model having $\overline{P}$ infinite for general t appears to be a Special Relativity model, that having $k=0$, $R(t) = \text{constant}$.

We verify this statement by examining the conditions under which P can approach an infinite value for r_1 approaching its maximum, at $t = t_0$ say: the three cases $k = 0, \pm 1$ will be taken separately, remembering that for $k = 0, +1$ $r_{max.} = \infty$ but for $k=-1$ $r_{max.} = 2$ and confining our attention to expanding model universes for which $R(t)$ is an increasing function of t.

We have $$P = \frac{R(t_0) r_1}{1 - \frac{1}{4} k r_1^2 + \dot{R}(t_0) \frac{r_1}{c}}$$

(a) when $k = 0$

$$P = \frac{R(t_0)}{\frac{1}{\tau_1} + \dot{R}(t_0)/c}$$

A necessary and sufficient condition that $P \to \infty$ as $\tau_1 \to \infty$ for fixed but arbitrary t_0 is that $R(t_0)/\dot{R}(t_0)$ should be infinite for all t. This is satisfied only when $R = \text{constant}, \dot{R} = 0$.

(b) when $k = -1 \; (\tau_{max.} = 2)$

$$P = \frac{R(t_0)}{1 + \dot{R}(t_0)/c} \qquad \text{when } \tau_1 = 2$$

This will be infinite for general t when $\dot{R}(t_0) = -c$ but this represents a contracting universe.

(c) when $k = +1$

$$P = \frac{R(t_0)}{1/\tau_1 - \tau_1/4 + \dot{R}(t_0)/c}$$

This will be infinite if the denominator on the R.H.S. is zero when $\tau_1 = \infty$. This requires $\dot{R}(t_0)/c$ to be identically equal to $\frac{\tau_1}{4}$ for all τ_1, t_0. But these are independent and the condition can be fulfilled only if $\frac{\tau_1}{4} = \text{constant} = \frac{\dot{R}(t_0)}{c}$ which contradicts the fact that τ_1 may assume all values.

Thus our statement is proved. We see that all expanding models, whether or not they admit a degenerate P.H., that is a Milne-type boundary, have an upper bound for P ; hence the boundary of apparent distance by parallax has no unique claim to characterising the boundary in Milne's model, as does the degenerate P.H. In particular, we note that the one model ($R(t)$ = const, k = 0) which possesses no boundary by parallax admits a Milne-type boundary when we allow the range of t to extend to $-\infty$.

By these considerations we have demonstrated that in defining the type of boundary which is exemplified in Milne's model, the P.B. is a redundant concept when taken in conjunction with reference to a degenerate P.H. and that it is unsuitable altogether when taken alone.

(v) <u>Connection of distance by parallax with E.Hs. and P.Hs.</u>

We now enquire how the existence in a model of an E.H. or a P.H. for a F.O. affects his considerations of distance by parallax. For given constant t_0, P, given by (3.5), is a function only of r_1 in any particular model; whether or not $\overline{P}$, the value of P as $r_1 \rightarrow \infty$, is directly connected with the E.H. concept will depend on whether or not an infinite light-travel-time is associated with this value of r_1. Evidently there can be no connection between $\overline{P}$ and an E.H. in an evolutionary universe which has a creation instant a finite time before the instant t_0; for any event occurring on an E.H. if such exists will not have had time to manifest itself at the origin-observer. We must in this context therefore confine our attention to models which allow t to extend to negatively unbound values, such as the steady state model.

For this model, by (2.4)

$$r_1 = c\int_{t_1}^{t_0} e^{-t/T}\,dt = cT e^{-t_0/T}\left(e^{t_0-t_1/T} - 1\right)$$

For given finite t_0, r_1 is infinite when $t_0 - t_1 = \infty$ i.e. when $t_1 = -\infty$. The light-travel-time being infinite, emission occurred on the E.H. since such exists; but from (2.10) (replacing $t = 0$ by $t = -\infty$) we find that $t_1 = -\infty$ uniquely represents the invariant creation-light-cone so that the emission resulting at time t_0 in finite $\bar{P}$ occurred on the degenerate P.H. Substituting $r_1 = \infty$ into (3.5) we find for the steady state universe that $\bar{P} = cT$.

The property which is exemplified by the steady state model regarding $\bar{P}$ being associated with emission on the degenerate P.H. (when this exists) is found to be general. By (2.10) the equation of a degenerate

P.H. ($t_1 = 0$ or $-\infty$) is given by $t = 0$ or $t = -\infty$ as the case may be; by (1.4), since $\int_{0(-\infty)} \frac{dt}{R(t)}$ diverges to infinity, the associated value of r_1 is infinite (necessarily so when $k = 0$ or -1; for $k = +1$ considerations of Chapter II (i) must be applied); then, by definition, we have $P = \overline{P}$. Setting $r_1 = \infty$ in (3.5) yields a $\overline{P}$ value which is dependent both on the model under consideration and on the chosen t_0.

In those models admitting P.Hs., necessitating the convergence of $\int_{0(-\infty)} \frac{dt}{R(t)}$ to a finite limit, r_1 is found from (1.4) not to extend to a positively infinite value at given t_0. This may be illustrated by consideration of the class of models with $k = 0$, $R(t) = t^n$ $(n < 1)$. By (1.4)

$$r_1 = \frac{c}{(1-n)}\left[t_0^{\,1-n} - t_1^{\,1-n}\right] \qquad (3.6)$$

Applying the condition $0 \leqslant t_1 \leqslant t_0$, we get $0 \leqslant r_1 \leqslant \frac{ct_0^{\,1-n}}{(1-n)}$. At time t_0 only those F.Ps. with r_1 in this finite range have been seen by the origin-observer; those with r_1 outside this range ($r_1 > \frac{ct_0^{\,1-n}}{(1-n)}$ for the class considered) have not yet appeared over his P.H. In particular, the limit of r_1 tending to infinity, producing the boundary by parallax considered by Rindler, is beyond the P.H. and at finite time t_o possesses no physical significance whatever for the observer. The limit to P at t_o which is of observable significance is given by P when r_1 has its P.H. value, which for the above class of models is

$$\overline{P}_{P.H.} = ct_o \tag{3.7}$$

a value, it is worth noting, which is independent of n.

Thus Rindler's boundary by parallax is connected with the degenerate P.H. when such exists in that at any instant the value of the (unattained) upper bound $\overline{P}$ to the apparent distances by parallax is due (in the limit) to emission on the degenerate P.H. On the other hand, when a model admits a P.H. for a F.O., $\overline{P}$ is related at finite given t_o to the infinite value of r_1 and the F.Ps with r_1 approaching this value are at that instant beyond the F.O's P.H. and therefore as yet unobservable to him. It is then of no physical significance to him.

Remembering that $r_1 \to \infty$ corresponds to particles which are farther and farther from the observer, these remarks show that a P.B. is in any model that surface which encloses the region containing all physically significant objects. Hence by the results of section (ii) it describes the boundary in Milne's model only in respect of this feature and our conclusion of section (iv) is further emphasised.

In the sense that the distance corresponding to $r_1 = \infty$ may always be taken as infinite upon the adoption of a suitable alternative distance variable (e.g. proper distance l_0 which is given by $l_0 = R(t_0)\sigma(r_1)$ where $R(t_0)$ = constant for given t_0), the P.B. may in any case be transformed away; we must remember nevertheless that measurement of parallax is of operational significance.

To find the connection between $\overline{P}$ and an E.H. when such exists we must allow t_0 to vary over its whole range. Consider then a F.P. with r_1 = constant (K) pursuing its motion in a model which has $R(t) = t^n$, $k = 0$. We have, from (3.5)

$$P = \frac{t_0^n K}{1 + n t_0^{n-1} K/c} = \frac{K}{\frac{1}{t_0^n} + \frac{nK}{ct_0}}$$

P is found to increase steadily and without limit as $t_0 \to \infty$ for this class of models. When there exists an E.H. ($n > 1$), the E.H. value of P (i.e. that measured at $t_0 = \infty$) is infinite. When there exists a P.H. ($n < 1$), for any particular F.P. A the initial value of t_0 is non-zero so that for A the minimum observable value of P is non-zero and is measured at the instant when A is on the observer's P.H.; thereafter P increases steadily.

For the steady state model the picture is different in detail. We have

$$P = \frac{r_1}{\frac{1}{e^{t_0/T}} + r_1/cT}$$

so that P, while increasing steadily for any one particular F.P., tends to a finite limiting value equal to cT as $t_0 \to \infty$.

The necessary and sufficient conditions that P increases in any other model of the Robertson-Walker type is found by differentiating P with respect to t_0 for r_1 = constant and setting $\frac{dP}{dt_0} > 0$. This yields the single condition for the increase of P at t_0

$$\dot{R}(t_0)\left(1 - \frac{1}{4}kr_1^2 + \frac{\dot{R}(t_0)r_1}{c}\right) > \frac{r_1}{c}R(t_0)\ddot{R}(t_0)$$

CHAPTER IV: THE BEHAVIOUR OF OBSERVABLES IN THE NEIGHBOURHOOD OF AN EVENT HORIZON OR A PARTICLE HORIZON.

(i) The variables and models considered; the two approaches.

Any valid scientific knowledge to be obtained by an observer about his universe is inevitably concerned with the behaviour of observables. Being highly model dependent, they provide in principle a method for discriminating between various model universes. Up to the present apparently, no systematic study has been made of the precise connection between the existence or otherwise of horizons in a model and the behaviour of observables in that model. We consider that such an investigation would be worthwhile and instructive, both from the purely observational aspect and from the point of view of the significance of horizons.

To this end, we now consider the variation of red-shift z, apparent luminosity m and metric angular diameter θ in five model universes:

A: the steady state model with $R(t) = e^{t/T}$; $k = 0$ which requires the phenomenon of continual creation.

B: Page's evolutionary model with $R(t) = t^2$; $k = 0$; these two models admit E.Hs. but not P.Hs.

C: the Einstein-de Sitter model with $R(t) = t^{2/3}$; $k = 0$; which admits P.Hs. but not E.Hs.

D: the model with $R(t) = a\,(\cosh bt - 1)^{1/3}$; $k = 0$, where a, b are constants, which behaves like the Einstein-de Sitter model for small values of t and like the de Sitter model ($R \sim \exp\frac{bt}{3}$) for large values of t, admitting both E.Hs. and P.Hs.

E: Milne's model, $R(t) = t$; $k = -1$ which admits neither E.H. nor P.H.

For further insight, the contracting duals of these models, obtained by time-reversal, will also be studied.

In order that the connection of z, m, θ with the horizons shall be amply demonstrated, we consider them in terms of the parameters (which could be eliminated) t_0, t_1, l_0, l_1 where t_1 is the time of emission of the radiation which is received at the origin-observer O at time t_0 from a source with proper distance from O equal to l_1 at time of emission and l_0 at time of reception; $(t_0 - t_1)$

represents the light-travel-time. Of these parameters only t_0 is an observable.

Two approaches are open to us: the "snapshot" view, taking t_0 constant and considering an ensemble of sources at varying distances from O; and the "moving picture" view, which considers the progress of an individual source (r = constant) as time goes on. In the first case we make the usual assumption that the sources considered are similar, at least at the time when they emit the light by which they are seen at the epoch t_0; that is, they have the same absolute luminosity and the same proper diameter, no account being taken in this preliminary work of possible evolutionary effects. Similarly, we assume for the second approach that these features are constant throughout time for the F.P. under consideration.

It is recognised here that any such assumptions concerning the similarity of F.Ps. either in space or time will inevitably restrict the generality of the results and, even more important, that they are logically of doubtful scientific validity since they may well be, in principle, operationally unverifiable; in particular we still require a definition of "fixed proper distance". We consider that investigation under these conditions will nevertheless be instructive.

(ii) Some general results.

We first prove some results of a general nature, in so far as they have not previously been proved.

(a) In an expanding universe, the proper distance from 0 of a single F.P. always increases.
For by definition $\ell = R(t)\sigma(r)$ where $\sigma(r) =$ constant and $R(t)$ is an increasing function of t. Conversely, in a contracting universe, ℓ continually decreases.

(b) In an expanding universe, $1 \leqslant Z \leqslant \infty$ where
$$Z \overset{\text{def}}{=} 1 + z = \frac{R(t_0)}{R(t_1)}$$
Since $t_0 \geqslant t_1$ and $R(t)$ is an increasing function of t, the minimum value of Z is given for $t_0 = t_1$, viz. $Z = 1$; the maximum is given by $R(t_0) = \infty$ when $R(t_1)$ is finite, viz. $Z = \infty$.

(c) In an expanding model, a F.P. situated on an E.H. or on a P.H. has its red-shift infinite.
A particle on an E.H. has t_1 finite and t_0 infinite so that $R(t_0) = \infty$. Hence $Z = \infty$. In the neighbourhood of an E.H. Z must therefore be an increasing function of t_0.

A particle on a P.H. has its first radiation received at finite, non-zero t_0. Thus $R(t_1) = 0$ while $R(t_0)$ is finite so that $Z = \infty$. In the neighbourhood of the P.H. Z will be a decreasing function of t_o. In models admitting both E.Hs. and P.Hs. z for a single source must therefore decrease from ∞ on the P.H., reach a minimum and thereafter increase again to ∞ on the E.H.

(d) In a contracting universe, $0 \leqslant z \leqslant 1$, that is, we have a blue-shift. For $R(t)$ is always positive, so that $z \geqslant 0$ and $t_o \geqslant t_1$, so that $R(t_o) \leqslant R(t_1)$; thus $z \leqslant 1$.

(e) Rindler (1956) has shown that on time-reversal $z \rightarrow \frac{1}{z}$. Thus, from (b), in contracting models $z = 0$ on a P.H. or an E.H. $z \sim 1$ in the neighbourhood of the observer.

Hence in the neighbourhood of a P.H. z must be an increasing function of t_o and in the neighbourhood of an E.H. a decreasing function of t_o.

(f) In expanding models, the time for light to travel from an E.H. to O is infinite; when there exists a P.H. the time for light to travel from the creation event to O is non-zero.

(g) We prove now a result stated, but left unproved, by Metzner and Morrison (1959): if z is the spectral shift of any source A measured at O at a given instant t_o, then the proper distance from A to O's E.H. point H (on the line of sight from O to A) at the time of emission t_1 of the radiation received by O at t_0 is proportional to $\frac{1}{z}$.

Because of the existence of a cosmic time, we have σ an additive co-ordinate so that

$$\sigma_{OH} = \sigma_{OA} + \sigma_{AH}$$

Multiplying throughout by $R(t_1)$ we get

$$R(t_1)\sigma_{AH} = l_{AH(t_1)} = R(t_1)\left[\sigma_{OH} - \sigma_{OA}\right]$$

$$= R(t_1)\left[\int_{t_1}^{\infty}\frac{c\,dt}{R(t)} - \int_{t_1}^{t_0}\frac{c\,dt}{R(t)}\right]$$

the first integral being convergent when there exists an E.H. Thus

$$l_{AH}(t_1) = R(t_1)\int_{t_0}^{\infty}\frac{c\,dt}{R(t)}$$

$$= \frac{R(t_0)}{z}\int_{t_0}^{\infty}\frac{c\,dt}{R(t)}$$

$$\equiv \frac{f(t_0)}{z}$$ for constant t_0, which proves the result.

(iii) <u>The relevant equations and conditions.</u>

The following equations are relevant to our investigation.

For the light emitted at $t = t_1$ by a F.P. with $r = r_1$

$$\int_0^{r_1}\frac{dr}{1+\frac{kr^2}{4}} \equiv \sigma(r_1) = \int_{t_1}^{t_0}\frac{c\,dt}{R(t)} \qquad (4.1)$$

The proper distance at time t of that F.P. is given by

$$l_t = R(t)\,\sigma(r_1) \qquad (4.2)$$

We write $l_{t_0} \equiv l_0,\ l_{t_1} \equiv l_1$.

The spectral shift z is given by

$$1 + z \equiv Z = \frac{R(t_0)}{R(t_1)} \qquad (4.3)$$

We have for the apparent luminosity m of a source with luminosity at a distance of 10 parsecs M (supposed constant)

$$m = M.\frac{10^2}{L^2} \qquad (4.4)$$

by the relation which defines L, the luminosity distance of the source; McCrea (1934-5) has shown this to be given by

$$L = \frac{R^2(t_0)}{R(t_1)}\cdot\frac{r_1}{1+\frac{kr_1^2}{4}} \qquad (4.5)$$

If θ is the metric angular diameter of an object, treated as a sphere of proper diameter d, McCrea (1934-5) has shown that

$$\theta = \frac{d}{R(t_1)\,\sigma(r_1)} = \frac{d}{l_1} \qquad (4.6)$$

on the assumption that θ is small. We suppose that d remains constant.

The relations between these variables may be obtained for the particular models by substitution and the insertion of the appropriate values for R(t) and k, under the following conditions:

(i) $t_0 \geqslant 0$ for the expanding evolutionary models.

$t_0 \geqslant -\infty$ for the steady state model.

(ii) $t_1 \leqslant t_0$

(iii) $R(t_1) \leqslant R(t_0)$ for expanding models.

(iv) $R(t_1) \geqslant R(t_0)$ for contracting models.

For contracting models we have t lying in the range $[-\infty, \infty]$ for the dual steady state model, but in $[-\infty, 0]$ for the duals of the evolutionary models.

In addition, if we adopt the "snapshot" method, we have

I. t_0 = constant; $\sigma(r)$ varying

whereas for the "moving picture" method

II. $\sigma(r_1)$ = constant; t_0 varying

(iv) <u>The results</u>

Applying particular values of $R(t)$, k to equations (4.1) to (4.6), we obtain results relevant to the corresponding model universes. Results for the various cases are illustrated by diagrams which we emphasise are purely qualitative in character, ignoring considerations of scale; this applies throughout the present work. Because of the assumptions and the definitions employed in (4.1) to (4.6), the equations break down or are approximate in the neighbourhood of the origin-observer (in Chapter V (iii) we shall prove explicitly that a lower limit to L is given by unity and that $\mathcal{Z}$ must be cut-off at a value exceeding unity). In the present chapter we take account of this by allowing the full lines of the diagrams to be accompanied by dotted lines to indicate where the results are purely formal, that is, where they merely follow the equations, uncorrected for the true physical situation. Where the equations allow values which the conditions state are outside our domain of interest we use broken lines alone; arrows are inserted to indicate the direction of development with time in those diagrams representing the history of a single F.P.

The model with $R(t) = a(\cosh bt - 1)^{1/3}$ is studied only for the expanding case when t_0 varies and only by analogy with the steady state and Einstein-de Sitter models. Results for the contracting duals relate t_0, t_1, ℓ_1, ℓ_0, m, θ to the variable z; the same diagrams illustrate their behaviour both in the expanding models and in their contracting duals; no confusion should arise since the range of z differs completely in the two cases.

We have then the following results.

A: <u>The steady state model and its contracting dual</u>

<u>The expanding model, $R(t) = e^{t/T}$ (T const.); $k = 0$</u>

$$Z = e^{(t_0 - t_1)/T} \qquad (4.7)$$

$$\sigma(r_1) = cTe^{-t_0/T}\left(e^{(t_0-t_1)/T} - 1\right) = cTe^{-t_0/T}(Z-1) \qquad (4.8)$$

$$l_1 = cT\frac{(Z-1)}{Z} \;;\; Z = \left(1 - \frac{l_1}{cT}\right)^{-1} \qquad (4.9)$$

$$l_0 = cT(Z-1) \;;\; Z = \left(1 + \frac{l_0}{cT}\right) \qquad (4.10)$$

$$m = \frac{10^2 M}{(cT)^2} \cdot \frac{1}{Z^2(Z-1)^2} = 10^2 M \frac{\left(1-\frac{l_1}{cT}\right)^4}{l_1^2} = 10^2 M \cdot \frac{1}{l_0^2\left(1+\frac{l_0}{cT}\right)^2} \qquad (4.11)$$

$$\theta = \frac{d}{cT} \cdot \frac{Z}{(Z-1)} = \frac{d}{l_1} = \frac{d\left(1 + l_0/cT\right)}{l_0} \qquad (4.12)$$

<u>The contracting dual, $R(t) = e^{-t/T}$; $k = 0$</u>

$$e^{t_0/T} = \frac{A}{1-Z} \;;\; e^{t_1/T} = \frac{AZ}{1-Z} \quad \text{for a single source} \qquad (4.13)$$

$$l_1 = cT \cdot \frac{(1-Z)}{Z} \;;\; l_0 = cT(1-Z) \qquad (4.14)$$

$$m = \frac{10^2 M}{(cT)^2} \cdot \frac{1}{Z^2(1-Z)^2} \qquad (4.15)$$

$$\theta = \frac{d}{cT} \cdot \frac{Z}{1-Z} \qquad (4.16)$$

Results for the steady state model are illustrated by Fig. 4A. We note that the θ-l_1 relationship is, by definition, the same for all models.

Since relationships in general are epoch independent in a steady state model, they are formally the same whether we apply condition I or condition II, but we must admit the possibility that a single F.P. may be created at any epoch in the model's history, giving a certain lower limit to t_1. For a single F.P. in the expanding case we have $Z - 1 = Ae^{t_0/T}$; A = const.

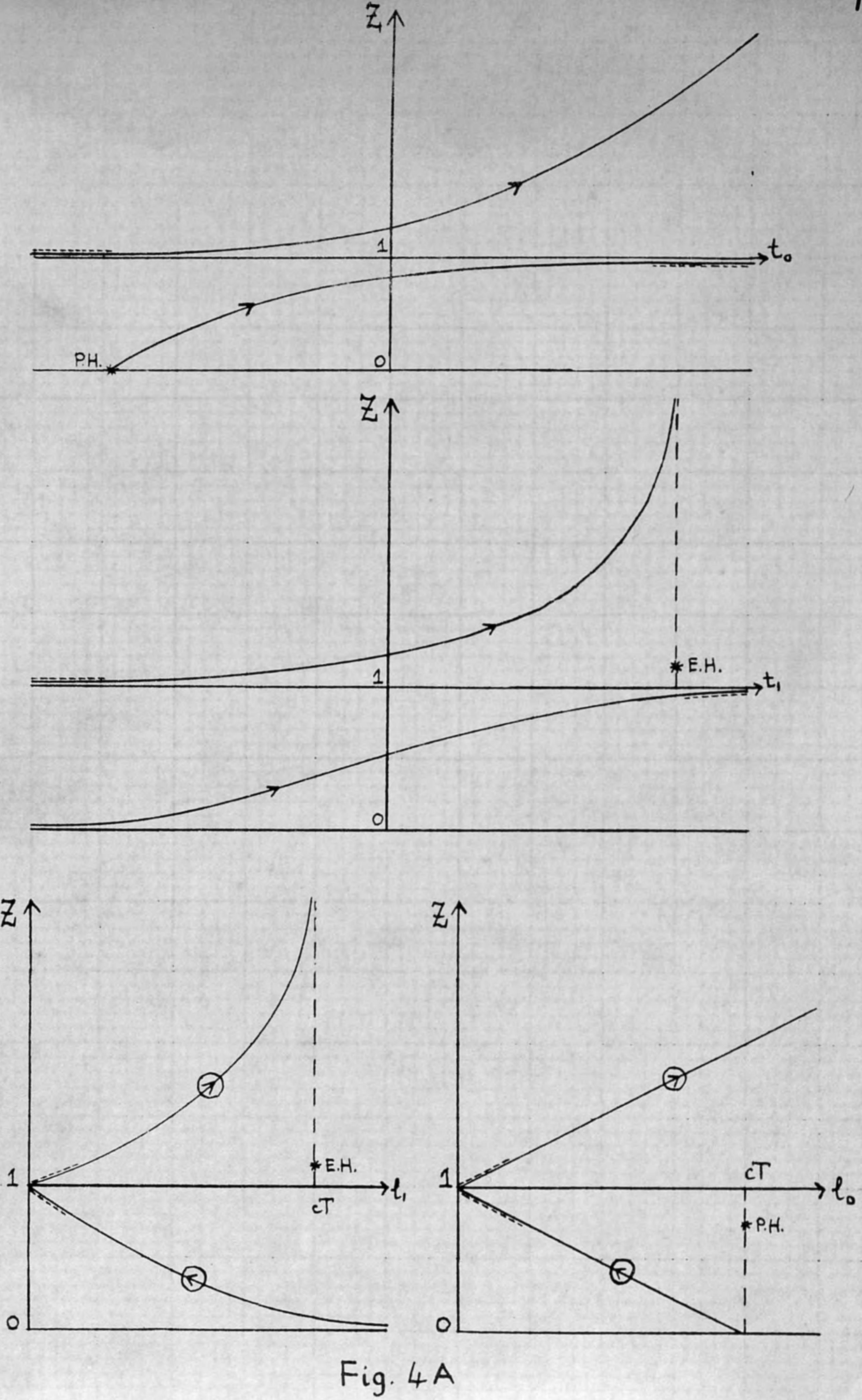


Fig. 4A

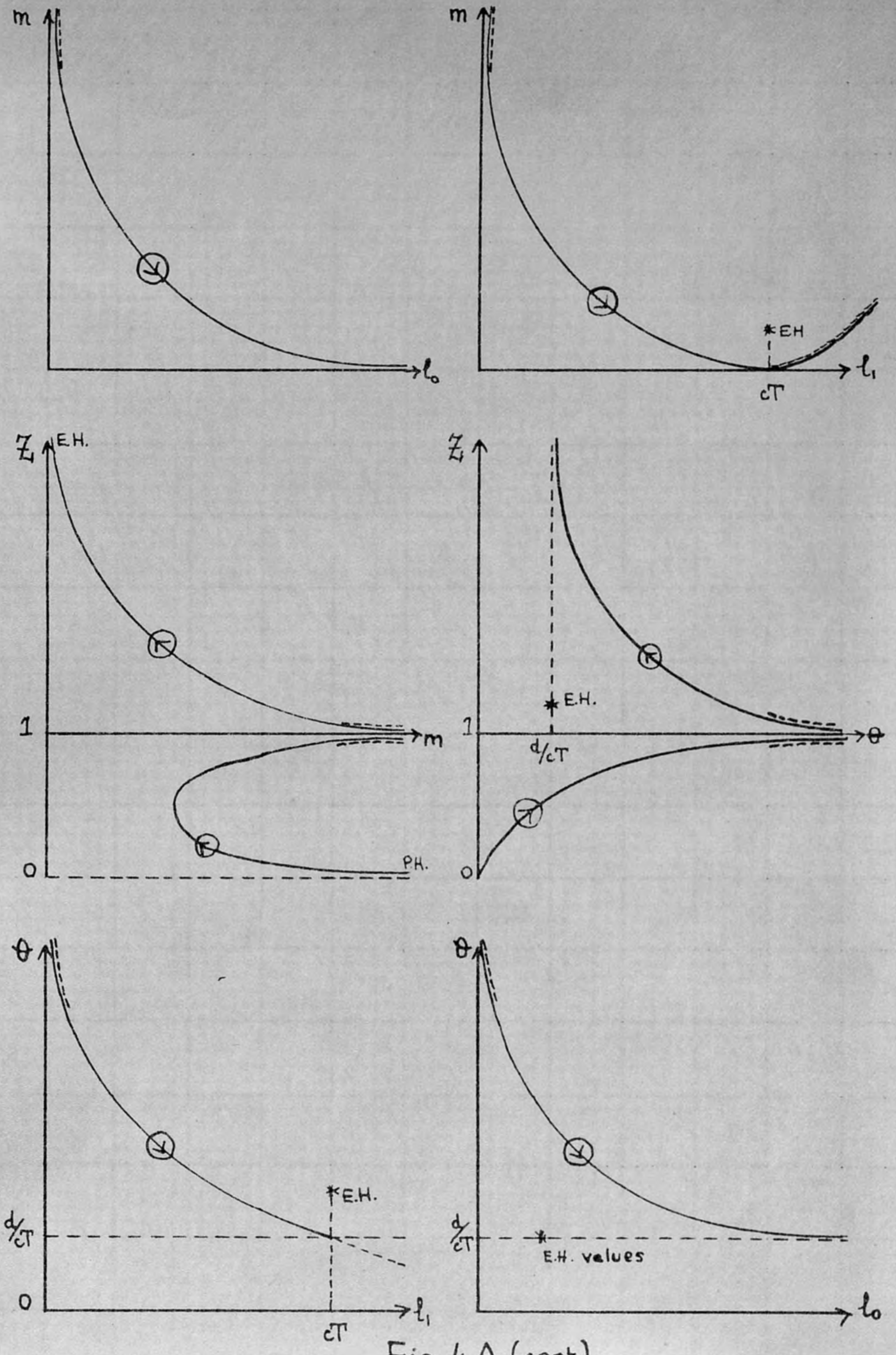

m
l_0
m
E.H
l_1
cT
Z_1
E.H.
1
m
0
P.H.
Z_1
E.H.
1
d/cT
θ
0
θ
E.H.
d/cT
0
l_1
cT
θ
d/cT
E.H. values
l_0

Fig. 4 A (cont.)

B: Page's model and its contracting dual.

The expanding model $R(t) = t^2$; $k = 0$

I. t_0 = constant

$$Z = \frac{t_0^2}{t_1^2} \tag{4.17}$$

$$\sigma(r_1) = c\left[\frac{1}{t_1} - \frac{1}{t_0}\right] = \frac{c}{t_0}\left[Z^{1/2} - 1\right] \tag{4.18}$$

$$\left.\begin{aligned} \ell_1 &= ct_1\left(1 - \frac{t_1}{t_0}\right) = \frac{ct_0}{Z^{1/2}}\left[1 - \frac{1}{Z^{1/2}}\right] \\ \text{i.e.} \quad Z &= \frac{ct_0}{2\ell_1}\left(1 \pm \sqrt{1 - \frac{4\ell_1}{ct_0}}\right) \end{aligned}\right\} \tag{4.19}$$

$$\left.\begin{aligned} \ell_0 &= ct_0\left[Z^{1/2} - 1\right] \\ \text{i.e.} \quad Z^{1/2} &= 1 + \frac{\ell_0}{ct_0} \end{aligned}\right\} \tag{4.20}$$

$$m = \frac{10^2 M}{(ct_0)^2} \cdot \frac{1}{Z^2(Z^{1/2} - 1)^2} \tag{4.21}$$

$$m = \frac{10^2 M}{(ct_0/2)^8} \cdot \frac{\ell_1^6}{\left(1 \pm \sqrt{1 - \frac{4\ell_1}{ct_0}}\right)^8} \tag{4.22}$$

$$m = 10^2 M \cdot \frac{1}{\ell_0^2 (1 + \ell_0/ct_0)^4} \tag{4.23}$$

$$\theta = \frac{d}{ct_0} \cdot \frac{Z}{(Z^{1/2} - 1)} = d\,\frac{(1 + \ell_0/ct_0)^2}{\ell_0} \tag{4.24}$$

II. $\sigma(r_1)$ = constant

By (4.18) $$ct_0 = \frac{Z^{1/2} - 1}{B} \tag{4.25}$$

where B is a positive constant. The above equations then become

$$\ell_1 = \frac{(Z^{1/2} - 1)^2}{BZ} \; ; \; Bct_0 = \frac{(B\ell_1)^{1/2}}{1 - (B\ell_1)^{1/2}} \tag{4.26}$$

$$\ell_0 = \frac{(Z^{1/2} - 1)^2}{B} \; ; \; Bct_0 = (B\ell_0)^{1/2} \tag{4.27}$$

$$m = 10^2 M B^2 \cdot \frac{1}{Z^2(Z^{1/2} - 1)^4} \tag{4.28}$$

$$m = \frac{10^2 M B^2}{(Bl_1)^2}\left[1 - (Bl_1)^{\frac{1}{2}}\right]^8 \tag{4.29}$$

$$m = 10^2 M B^2 \cdot \frac{1}{(Bl_0)^2\left[1+(Bl_0)^{\frac{1}{2}}\right]^4} \tag{4.30}$$

$$\theta = dB . \frac{Z}{(Z^{\frac{1}{2}}-1)^2} = dB . \frac{\left[1+(Bl_0)^{\frac{1}{2}}\right]^2}{(Bl_0)} \tag{4.31}$$

<u>The contracting dual $R(t) = (-t)^2$; $k = 0$</u>

I. <u>t_0 = constant</u>

$$l_1 = c|t_0|\frac{(1-Z^{\frac{1}{2}})}{Z} \quad ; \quad l_0 = c|t_0|(1-Z^{\frac{1}{2}}) \tag{4.32}$$

$$m = \frac{10^2 M}{(c|t_0|)^2} \cdot \frac{1}{Z^2(1-Z^{\frac{1}{2}})^2} \tag{4.33}$$

$$\theta = \frac{d}{c|t_0|} \cdot \frac{Z}{(1-Z^{\frac{1}{2}})} \tag{4.34}$$

II. <u>$\sigma(r_1)$ = constant</u>

$$c|t_0| = \frac{(1-Z^{\frac{1}{2}})}{B} \quad ; \quad c|t_1| = \frac{(1-Z^{\frac{1}{2}})}{BZ^{\frac{1}{2}}} \tag{4.35}$$

$$l_1 = \frac{(1-Z^{\frac{1}{2}})^2}{BZ} \quad ; \quad l_0 = \frac{(1-Z^{\frac{1}{2}})^2}{B} \tag{4.36}$$

$$m = 10^2 M B^2 \cdot \frac{1}{Z^2(1-Z^{\frac{1}{2}})^4} \tag{4.37}$$

$$\theta = dB . \frac{Z}{(1-Z^{\frac{1}{2}})^2} \tag{4.38}$$

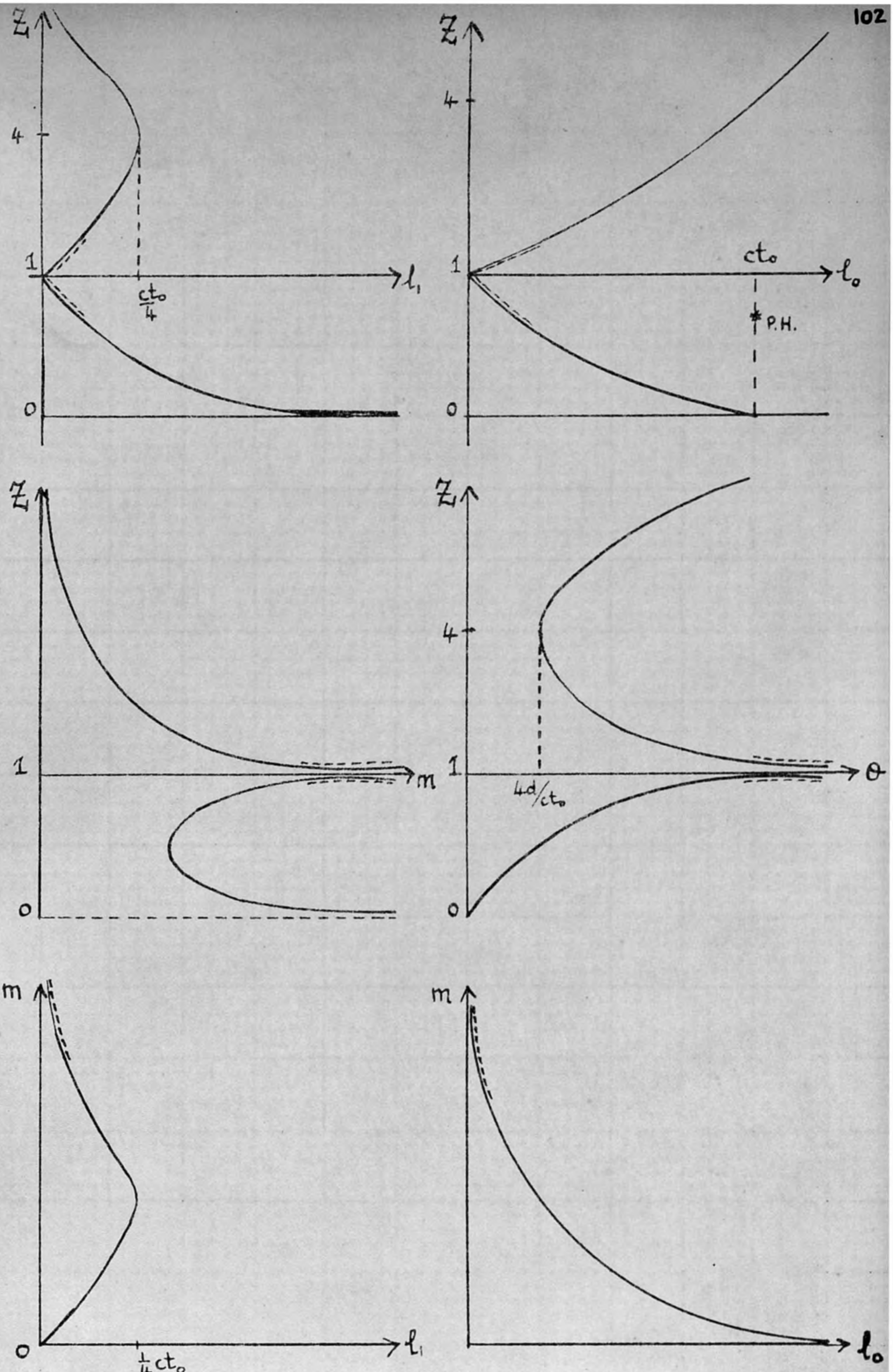

Z
4
1
0
$\frac{ct_0}{4}$
l_1
Z
4
1
0
ct_0
l_0
P.H.
Z
1
0
m
Z
4
1
0
$4d/ct_0$
θ
m
0
$\frac{1}{4}ct_0$
l_1
m
l_0

Fig. 4B I

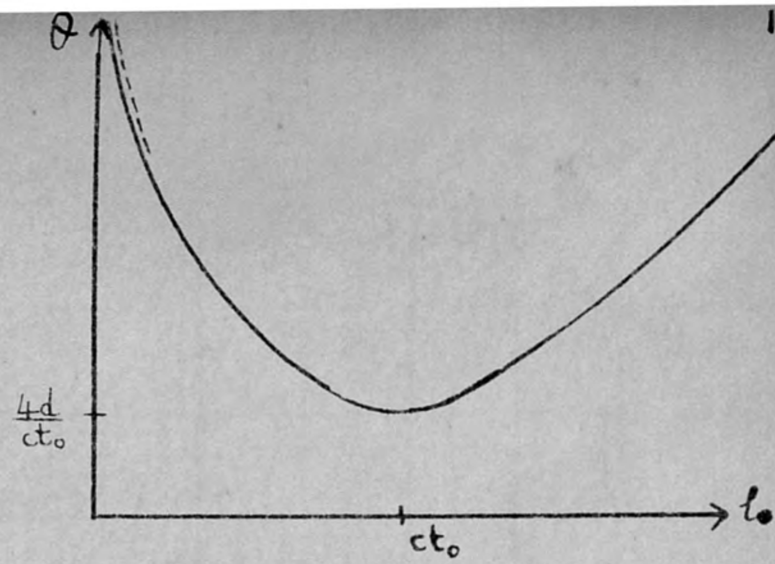


Fig. 4B I (cont.)

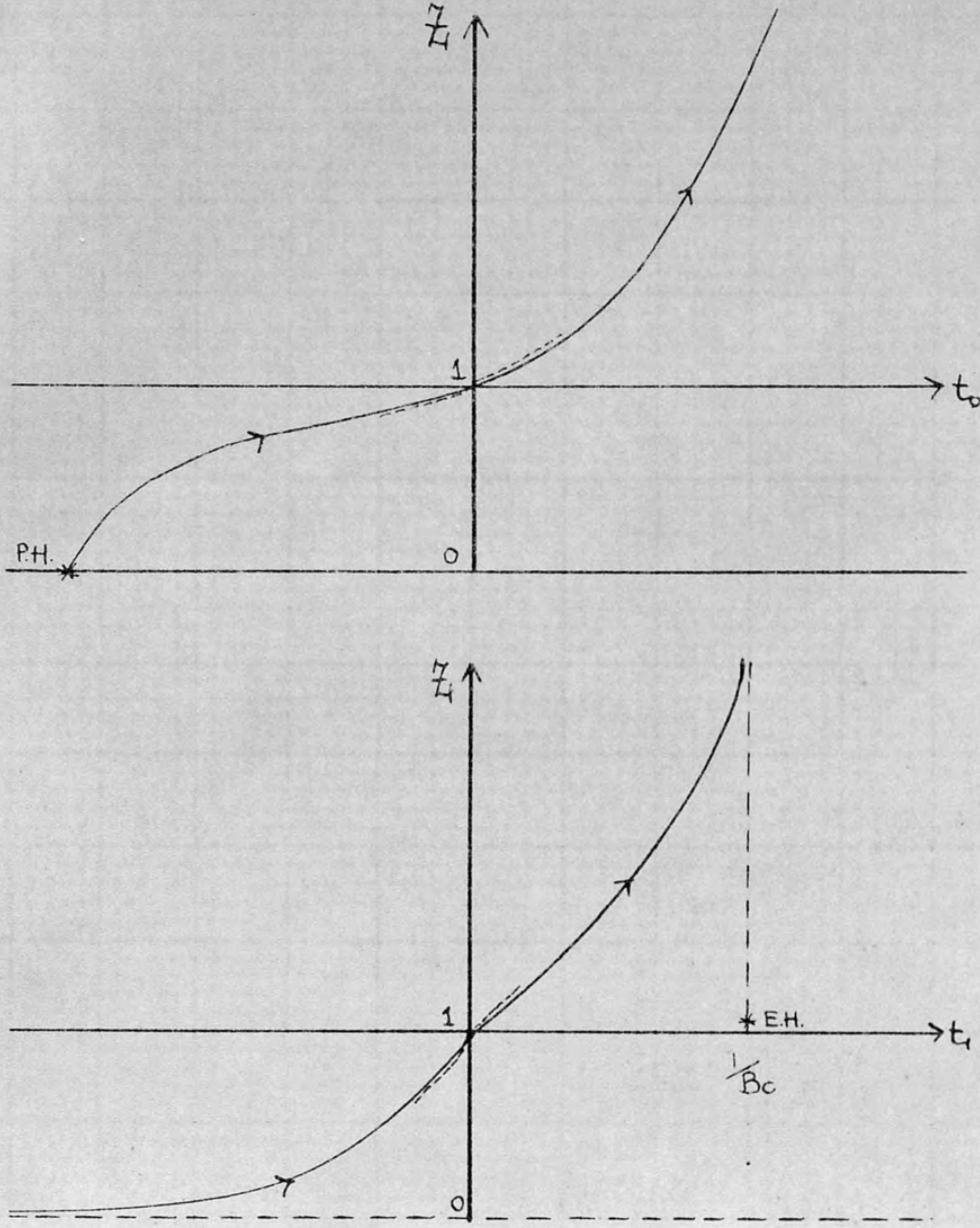


Fig. 4B II

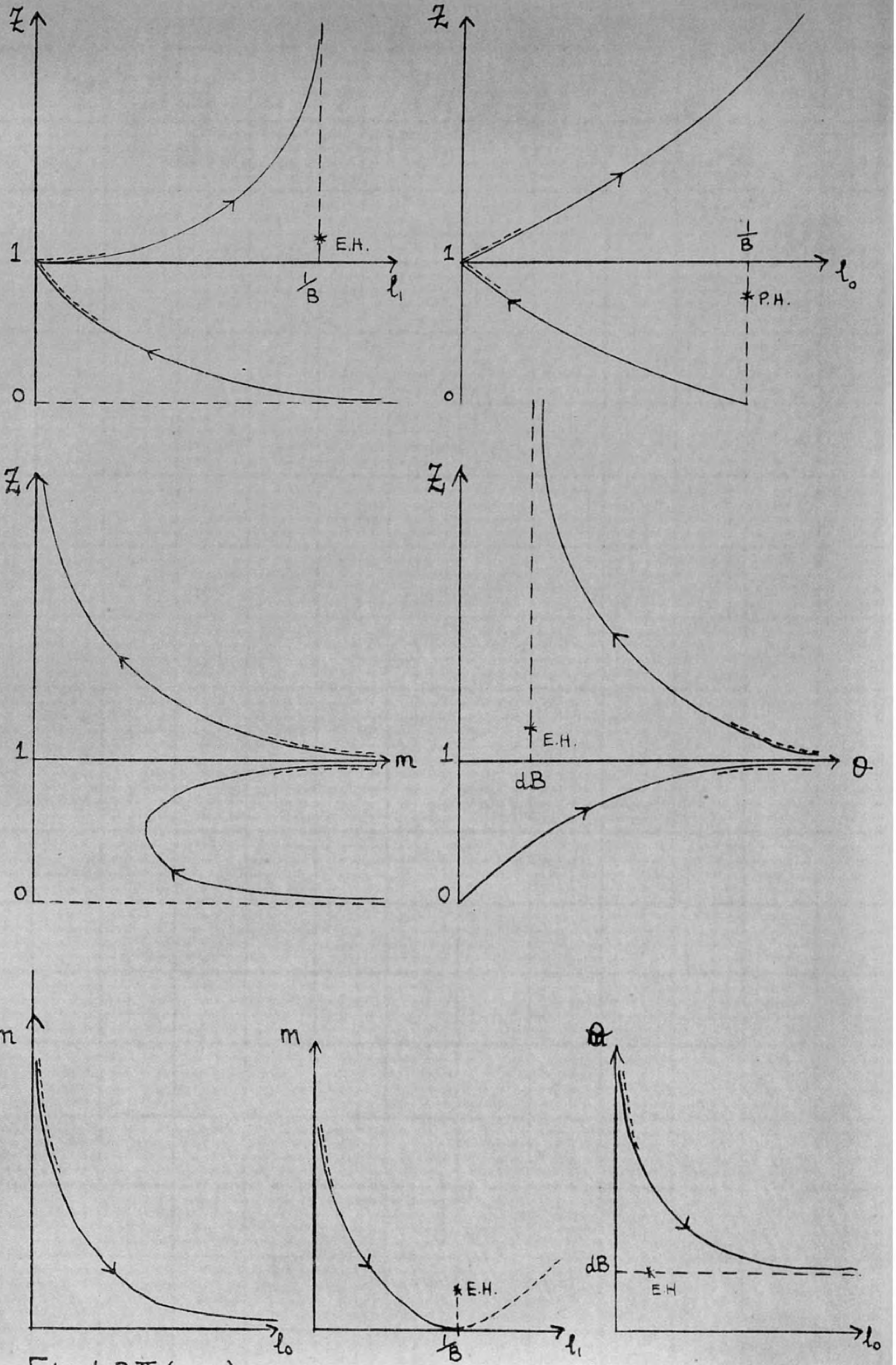


Fig. 4BII (cont.)

C: The Einstein-de Sitter model and its contracting dual

The expanding model, $R(t) = t^{2/3}$; $k = 0$

I. t_o = constant

$$Z = \frac{t_o^{2/3}}{t_1^{2/3}} \tag{4.39}$$

$$\sigma(r_1) = 3ct_o^{1/3}\left[1 - \frac{1}{Z^{1/2}}\right] \tag{4.40}$$

$$l_o = 3ct_o\left(1 - \frac{1}{Z^{1/2}}\right) \; ; \; Z = \frac{1}{(1 - l_o/3ct_o)^2} \tag{4.41}$$

$$l_1 = 3ct_o \frac{(Z^{1/2}-1)}{Z^{3/2}} \tag{4.42}$$

$$m = \frac{10^2 M}{(3ct_o)^2} \cdot \frac{1}{Z(Z^{1/2}-1)^2} \tag{4.43}$$

$$m = \frac{10^2 M}{(3ct_o)^2} \cdot \frac{\left(1 - \frac{l_o}{3ct_o}\right)^4}{l_o/3ct_o} \tag{4.44}$$

$$\theta = \frac{d}{3ct_o} \cdot \frac{Z^{3/2}}{(Z^{1/2}-1)} \tag{4.45}$$

$$\theta = \frac{d}{3ct_o} \cdot \frac{1}{l_o/3ct_o\,(1 - l_o/3ct_o)^2} \tag{4.46}$$

II $\sigma(r_1)$ = constant

$$t_o^{1/3}\left(1 - \frac{1}{Z^{1/2}}\right) = \text{const.} = C \tag{4.47}$$

$$Z = \frac{1}{(1 - C/t_o^{1/3})^2} = \left(1 + \frac{C}{t_1^{1/3}}\right)^2 \tag{4.48}$$

$$l_1 = \frac{3cC^3}{(Z^{1/2}-1)^2} \; ; \; Z = \left[1 + \left(\frac{3cC^3}{l_1}\right)^{1/2}\right]^2 \tag{4.49}$$

$$l_o = \frac{3cC^3 . Z}{(Z^{1/2}-1)^2} \; ; \; Z = \left[1 - \left(\frac{3cC^3}{l_o}\right)^{1/2}\right]^{-2} \tag{4.50}$$

$$m = \frac{10^2 M}{(3cC^3)^2} \cdot \frac{(Z^{1/2}-1)^4}{Z^4} \tag{4.51}$$

$$m = \frac{10^2 M}{(3cC^3)^2} \cdot \frac{1}{\left(\frac{l_1}{3cC^3}\right)^2\left[1 + \left(\frac{3cC^3}{l_1}\right)^{1/2}\right]^8} \tag{4.52}$$

$$m = \frac{10^2 M}{(3cC^3)^2} \cdot \frac{\left[1 - \left(\frac{3cC^3}{l_0}\right)^{1/2}\right]^4}{(l_0/3cC^3)^2} \tag{4.53}$$

$$\theta = \frac{d}{3cC^3}\left(Z^{1/2}-1\right)^2 \tag{4.54}$$

$$\theta = \frac{d}{3cC^3} \cdot \frac{1}{l_0/3cC^3\left[1 - \left(\frac{3cC^3}{l_0}\right)^{1/2}\right]^2} \tag{4.55}$$

<u>The contracting dual, $R(t) = (-t)^{2/3}$; $k = 0$</u>

I. <u>t_0 = constant</u>

$$l_1 = 3c|t_0| \frac{(1-Z^{1/2})}{Z^{3/2}} \; ; \; l_0 = 3c|t_0|\left(\frac{1}{Z^{1/2}} - 1\right) \tag{4.56}$$

$$m = \frac{10^2 M}{(3c|t_0|)^2} \cdot \frac{1}{Z(1-Z^{1/2})^2} \tag{4.57}$$

$$\theta = \frac{d}{3c|t_0|} \cdot \frac{Z^{3/2}}{(1-Z^{1/2})} \tag{4.58}$$

II. <u>$\sigma(r_1)$ = constant</u>

$$-t_0 = C^3 \frac{Z^{3/2}}{(1-Z^{1/2})^3} \; ; \; -t_1 = \frac{C^3}{(1-Z^{1/2})^3} \tag{4.59}$$

$$l_1 = \frac{3cC^3}{(1-Z^{1/2})^2} \; ; \; l_0 = \frac{3cC^3 Z}{(1-Z^{1/2})^2} \tag{4.60}$$

$$m = \frac{10^2 M}{(3cC^3)^2} \cdot \frac{(1-Z^{1/2})^4}{Z^4} \tag{4.61}$$

$$\theta = \frac{d}{3cC^3} \cdot (1-Z^{1/2})^2 \tag{4.62}$$

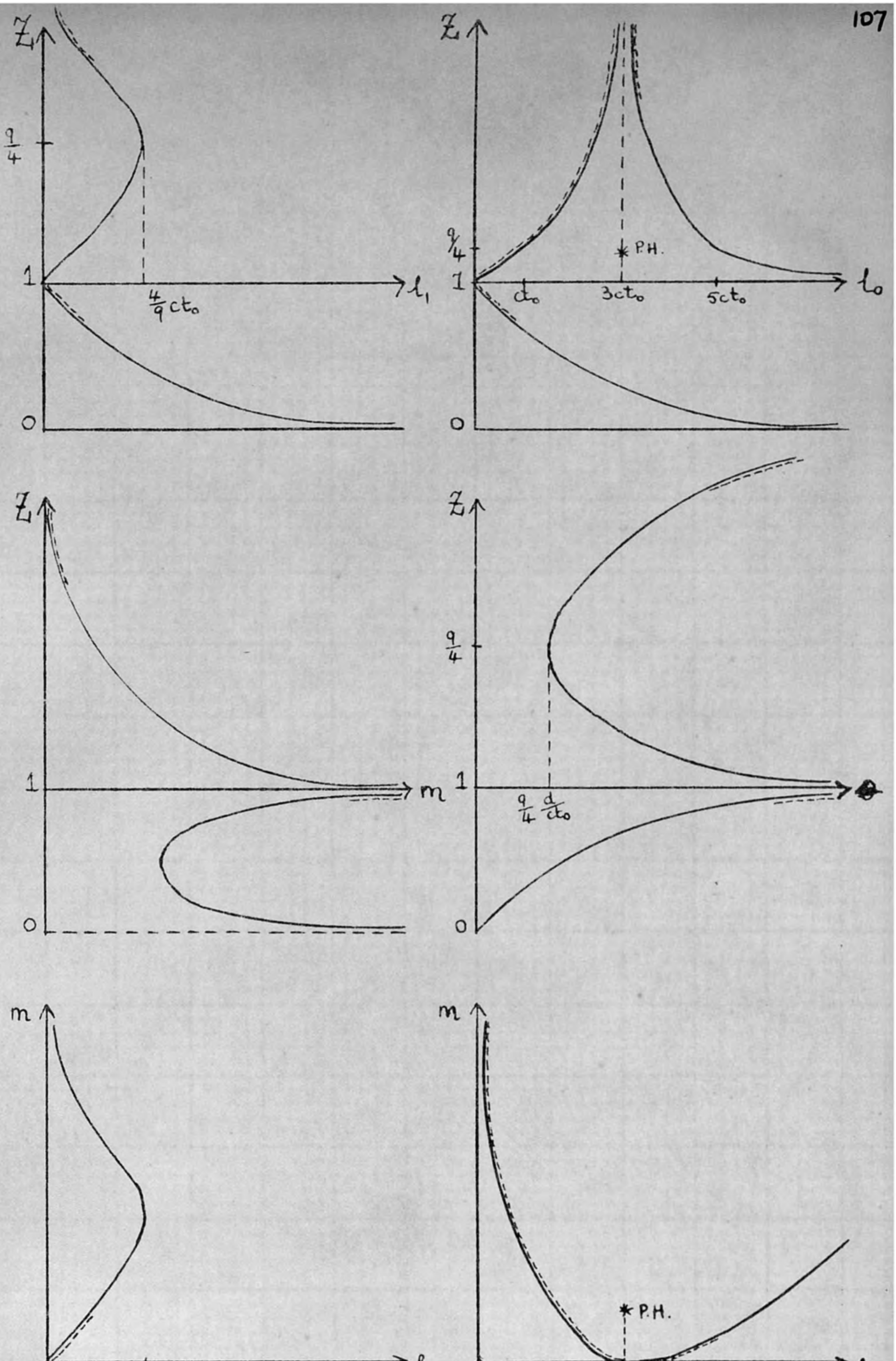


Fig. 4C I

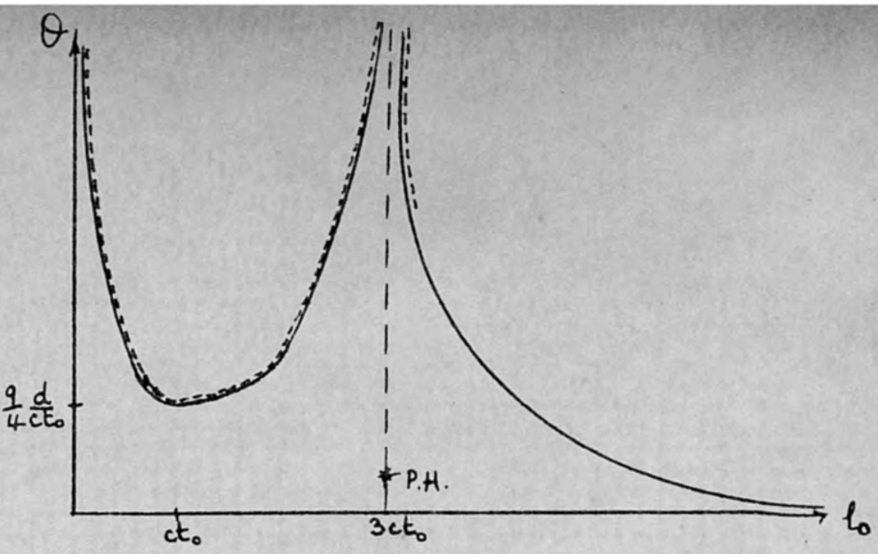
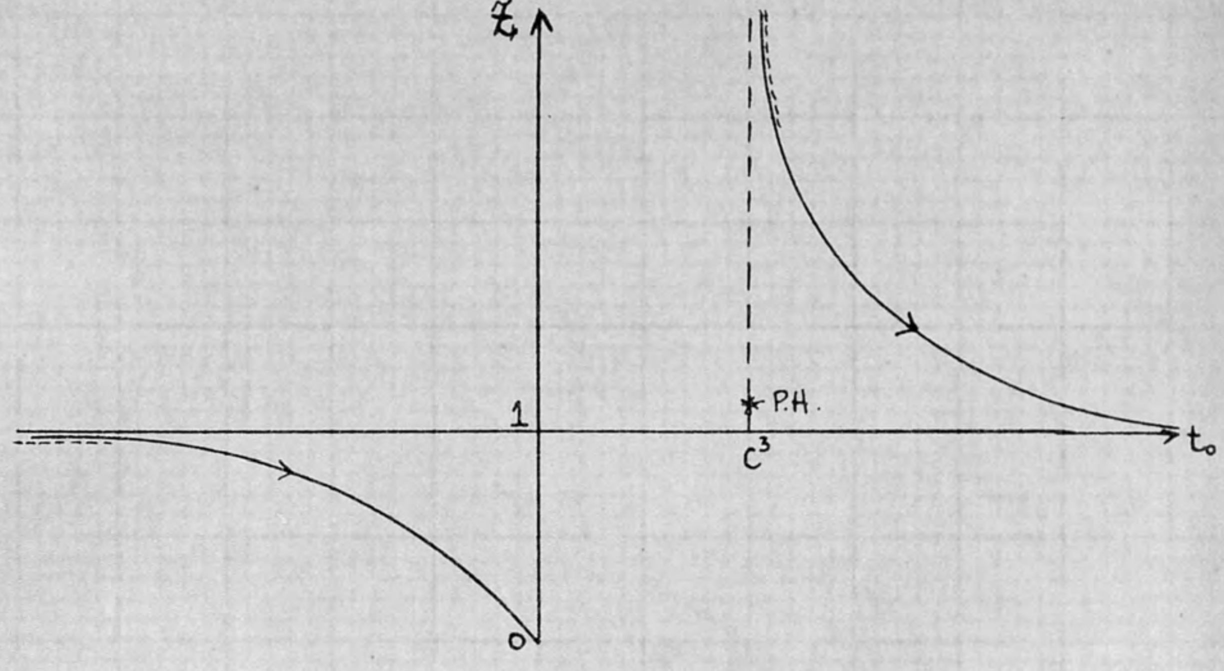


Fig. 4CI (cont.)



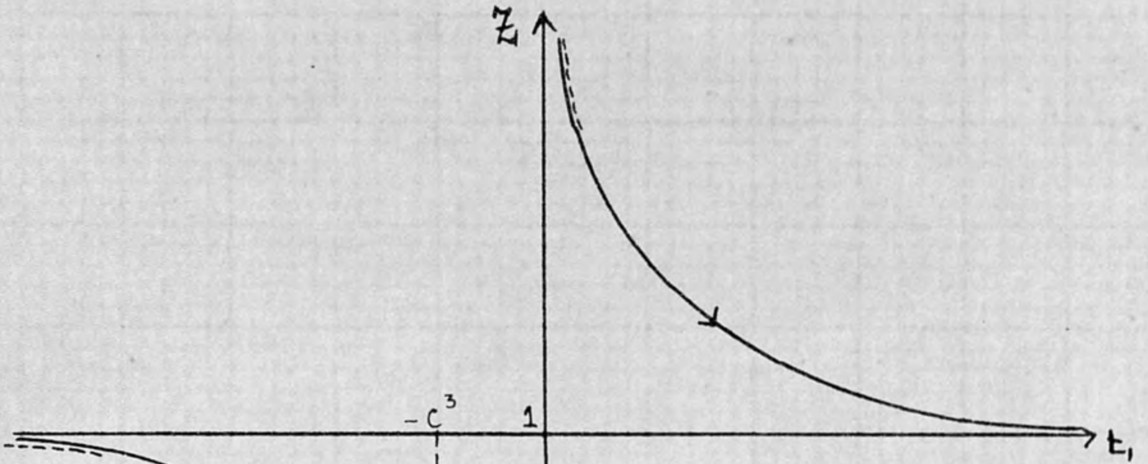


Fig. 4C II

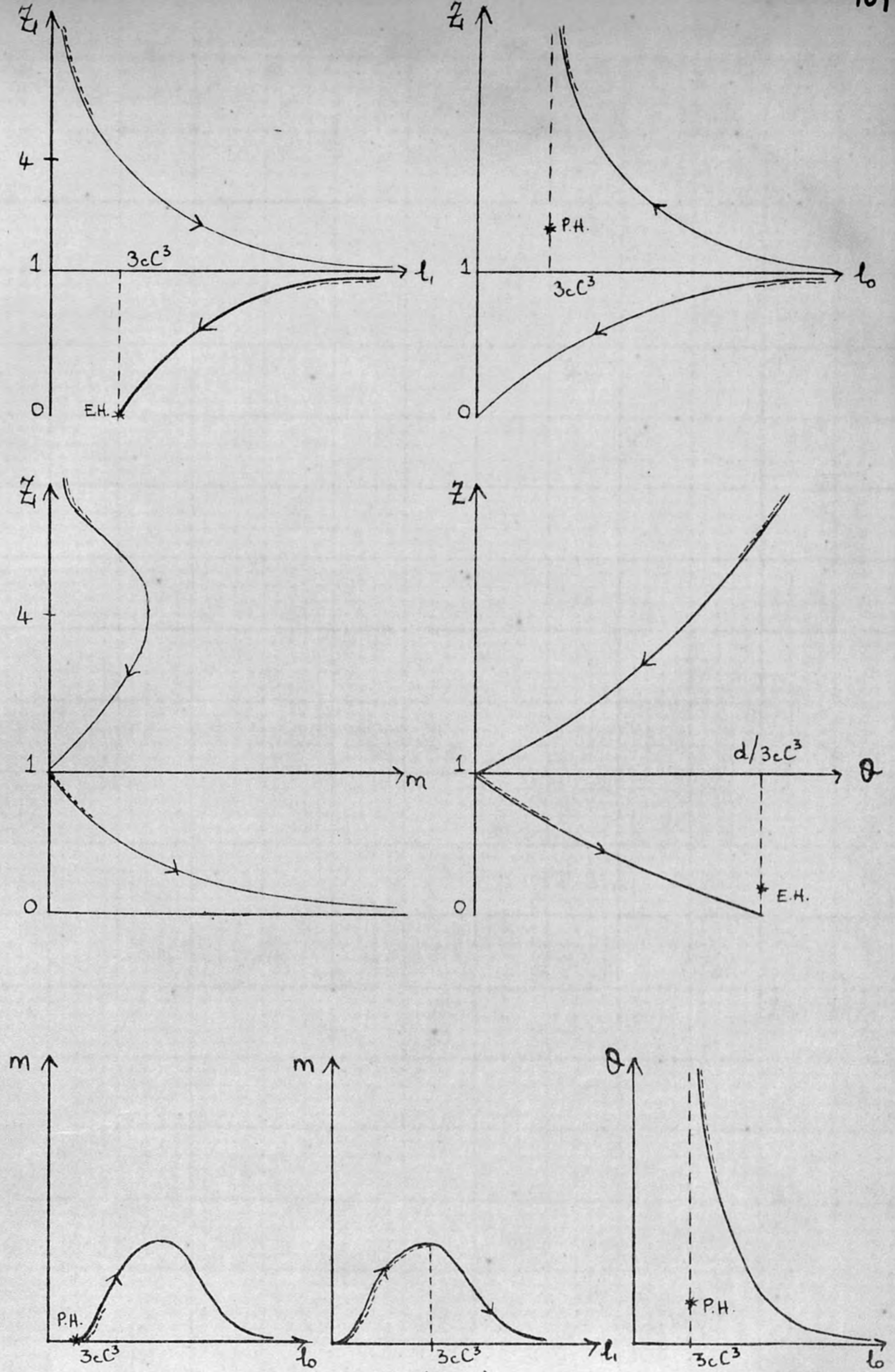


Fig. 4C II (cont.)

D: The expanding model with $R(t) = a(\cosh bt - 1)^{1/3}$; $k = 0$.

II. $\sigma(r_1)$ = constant

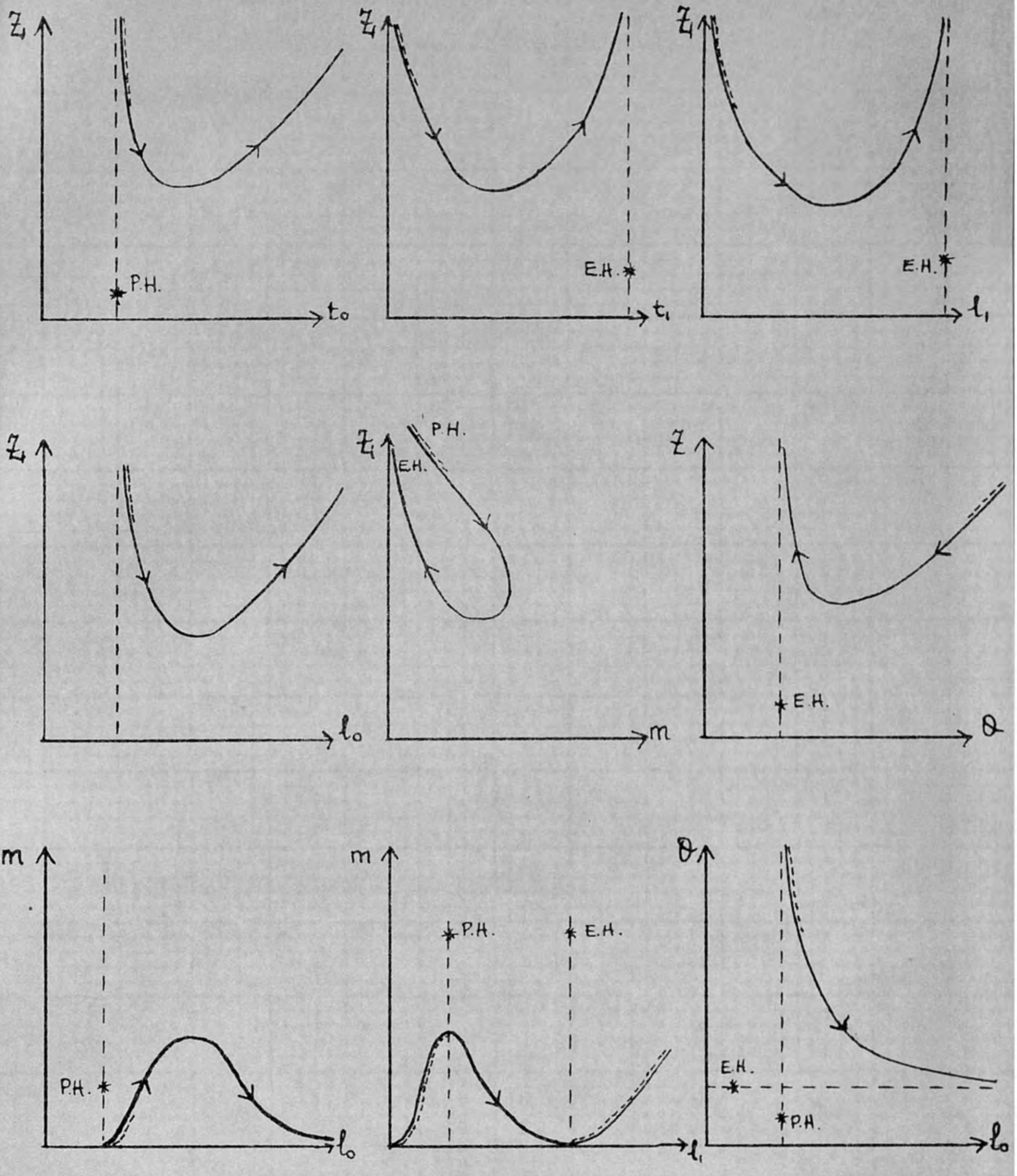


Fig. 4D II

E: Milne's model and its contracting dual

The expanding model, $R(t) = t; k = -1$ $(c = 1)$

I. t_0 = constant

$$\tilde{Z} = \frac{t_0}{t_1} \tag{4.63}$$

$$\sigma(r_1) = \log \frac{t_0}{t_1} = \log Z \tag{4.64}$$

$$l_1 = t_0 \frac{\log Z}{Z} \quad ; \quad l_0 = t_0 \log Z \tag{4.65}$$

$$m = \frac{10^2 M}{t_0^2} \cdot \frac{4}{(Z^2-1)^2} \tag{4.66}$$

$$m = \frac{10^2 M}{t_0^2} \cdot \frac{4}{(e^{2l_0/t_0} - 1)^2} \tag{4.67}$$

$$\theta = \frac{d}{t_0} \cdot \frac{Z}{\log Z} = \frac{d}{t_0} \frac{e^{l_0/t_0}}{l_0/t_0} \tag{4.68}$$

II. $\sigma(r_1)$ = constant

$$Z = \text{const.} \quad ; \quad t_0 = Z t_1 \tag{4.69}$$

$$m = \frac{10^2 M}{t_0^2} K_1 = \frac{10^2 M}{t_1^2} K_2 \tag{4.70}$$

$$m = \frac{10^2 M}{l_1^2} K_3 = \frac{10^2 M}{l_0^2} K_4 \tag{4.71}$$

$$\theta = \frac{d}{t_0} \cdot K_5 = \frac{d}{l_0} K_6 \tag{4.72}$$

$(K_1 \ldots K_6 = \text{const.})$

The contracting dual, $R(t) = -t; k = -1$

I. t_0 = constant

$$l_1 = |t_0| \frac{\log \frac{1}{Z}}{Z} \quad ; \quad l_0 = |t_0| \log \frac{1}{Z} \tag{4.73}$$

$$m = \frac{10^2 M}{|t_0|^2} \cdot \frac{4}{(1-Z^2)^2} \tag{4.74}$$

$$\theta = \frac{d}{|t d|} \cdot \frac{z}{\log \frac{1}{z}} \qquad (4.75)$$

II. <u>$\sigma(r_1)$ = constant</u>

$$z = \text{const.} \qquad (4.76)$$

$$m = \frac{10^2 M}{|t_0|^2} \cdot k_1 = \frac{10^2 M}{|t_1|^2} k_2 \qquad (4.77)$$

$$m = \frac{10^2 M}{l_1^2} k_3 = \frac{10^2 M}{l_0^2} k_4 \qquad (4.78)$$

$$\theta = \frac{k_5}{|t_0|} = \frac{k_6}{l_0} \qquad (4.79)$$

$(k_1 \ldots k_6 = \text{const.})$

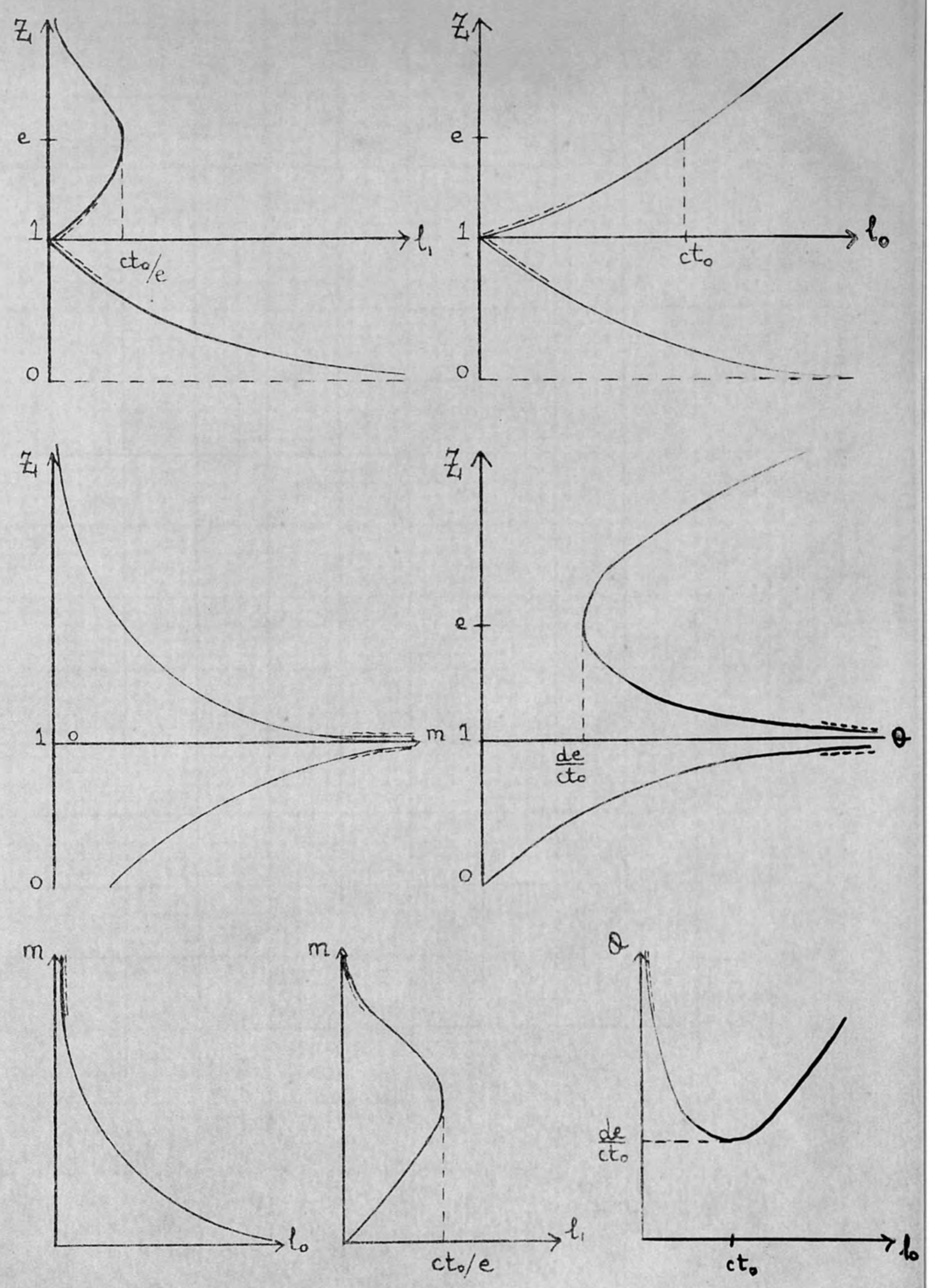

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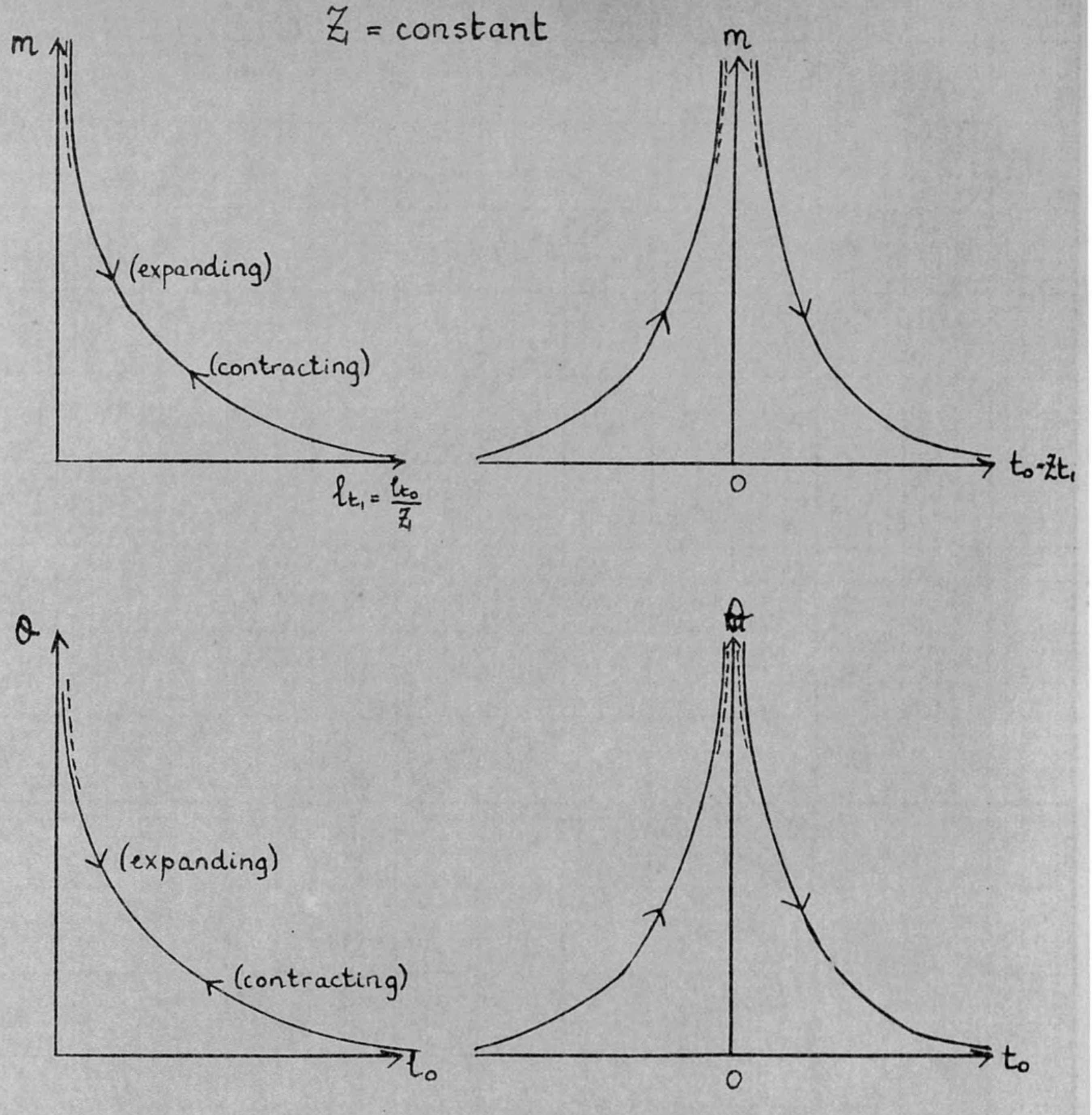
Fig. 4 E I



Z_1 = constant
m
(expanding)
(contracting)
$l_{t_1} = \frac{l_{t_0}}{Z_1}$
m
0
$t_0 = Z_1 t_1$
θ
(expanding)
(contracting)
l_0
θ
0
t_0

Fig. 4E II

(v) The existence of a minimum $\theta(z)$ and a maximum $\ell_1(z)$ and $\ell_1(m)$ for the expanding models (t_0 constant)

The behaviour of the variables in the neighbourhood of a horizon should be clear from an inspection of Figures 4A to 4E. They show also that for the expanding models with t_0 constant θ possesses a model-dependent minimum value in terms of z (no such feature is manifested in the contracting cases); since both θ and z are of observable significance, this feature provides us with a test for discriminating against world-models. If all our theoretical conditions regarding the similarity of sources etc. were known to be satisfied, or if any differences or evolutionary effects were accounted for, we could reject those models which manifested a minimum value of θ larger than any particular observed value. Such a test had already been discussed by Hoyle (1959), Davidson (1960) and Sandage (1961). Sandage has shown that a minimum occurs only where metric angular diameters are concerned, and that isophotal angular diameters, which measure the size of contours of equal surface brightness, pass through no minimum for given t_0 but approach zero as $z \to \infty$.

One striking feature deserves further comment, namely, the manifestation of an upper bound to ℓ_1 in terms of z and m for t_o constant. Since this feature is common to all expanding models (it is not manifested at all in the contracting cases) irrespective of whether or not they admit horizons, it is clearly not connected directly with the existence or otherwise of either an E.H. or a P.H.; this is further verified by the fact that in all models except the steady state model the maximum value of ℓ_1 is reached at a finite value of z, whereas horizons are associated exclusively with infinite z. In the steady state model it happens that the maximum value of ℓ_1 coincides with the constant proper distance cT to the E.H.; in Page's model, the maximum ℓ_1 is such that by the time of reception the F.P. under consideration is just crossing the E.H. i.e. $\ell_0 = \ell_{E.H.} = ct_o$. We find by examination of the Einstein-de Sitter model and Milne's model that the fact that the E.H. seems involved is accidental but that a common feature is that in these models the ℓ_0 which corresponds to the maximum value of ℓ_1 is given by the value ct_o. However, in the steady state model, the only model considered in which there is not a creation instant in the finite past, the value of ℓ_0 corresponding to $\ell_1 = cT$ is infinite ($z = \infty$).

Since the feature under discussion is common to all models and is unconnected with the P.H., it is not representative of a Milne-type boundary as we defined it in Chapter III, that is, a degenerate P.H.

Since $\sigma(r)$ may take all values, a fact demonstrated in Chapter II(i), ℓ, defined by the equation $\ell = R(t)\sigma(r)$ may also take all values for constant t; this is illustrated by the diagrams for ℓ_0 when t_0 is constant. It may then be seen that it is this definition of ℓ which is responsible for the apparently peculiar behaviour of ℓ_1 when t_0 is constant. For a given t_0, on any line of sight a particular value of t_1 characterises the emitting F.P. completely; but so does a certain value of $\sigma(r)$ which is constant throughout time for that particle. The more remote the particle considered, the smaller is t_1 (and hence $R(t_1)$), but the larger is the value of $\sigma(r)$. Thus, in the expression for ℓ_1 both $R(t_1)$ and $\sigma(r)$ vary, in such a way that the product equals zero both when $t_1 = 0$ and when $t_1 = t_0$ ($\sigma = 0$) and has a maximum value for $0 < t_1 < t_0$.

This results for the evolutionary models in the variables z, m having two values for each value of ℓ_1 and in the variable θ passing through a minimum in terms of both z and ℓ_0 : a comparison of the actual values involved at the minima shows that these effects are due to the same cause. In the steady state case the effect is different in detail: z and ℓ_1 are in 1-1 correspondence while to each value of m there formally correspond two values of ℓ_1 , one of which is greater than cT, the constant proper distance to the E.H.; θ has a non-zero minimum in terms of z which is approached asymptotically as $z \to \infty$, its value on the E.H.

Finally, we may note that for the steady state model it happens that for given t_0, ℓ_1 coincides with the distance by parallax P; for inserting $R(t) = e^{t/T}$, $k = 0$ in (3.5) and using (4.2) we get

$$P = \frac{R(t_0) r_1}{1 + \ell_0/cT}$$

$$= R(t_0) r_1 / z$$

by (4.10). Thus by (4.3), $P = R(t_1) r_1 = \ell_1$. Consequently the behaviour of P as well as ℓ_1 is represented in figures 4A, I and II.

CHAPTER V: INFORMATION THEORY IN COSMOLOGICAL MODELS

(i) Introduction.

Eddington (1953) has remarked that he believes that the second law of thermodynamics - the law that the entropy of an isolated, closed system can never diminish - holds the supreme position among the laws of nature. This law has been widely held to imply that the increase of entropy can continue for the universe as a whole only until the entropy has reached its maximum value so that the universe is "running down" and will stop functioning altogether when a state of complete uniformity has been reached. Although this conclusion is disputed (Lewis, 1930; Tolman, 1931; Whitrow, 1961a) the concept of entropy, first introduced by Clausius (1865), is undoubtedly of fundamental importance to our knowledge of the universe.

Entropy is a property of material systems which has a characteristic value for each state of the system. For any reversible process the change from one state of the system to another implies zero change in entropy, but there will be an increase of entropy when any irreversible process has occurred. The net entropy increase is a measure of the amount of energy that is converted into a form unavailable for doing mechanical work. This increase in the entropy of a system is associated with the change of the system from one state characterised by a certain degree of organisation to another state of lesser organisation, that is, greater randomness, complete randomness making stable equilibrium the state of maximum probability.

The study of statistical thermodynamics shows a close correlation between entropy and probability considerations, which leads to the use of entropy as a measure of the information transmitted during a communication process; for, in communication theory, the amount of information I given by a particular message is measured by the freedom of choice or number of alternatives available in constructing that message. Hence probability considerations play a major role in information; in fact I is proportional to the logarithm of the reciprocal prior probability of the alternative actually chosen.

Boltzmann in 1894 seems to have been the first to observe that entropy is related to "missing information" in that it is "related to the number of alternatives which remain possible to a physical system after all the macroscopically observable information concerning it has been recorded"

(Shannon and Weaver, 1949a). Lewis (1930) has shown that increase of entropy is associated with the process of a known situation becoming an unknown situation, this irreversible change being characterised by loss of information.

Any lack or loss of information about a particular situation is associated with the uncertainty H in our knowledge of that situation where H is given for the discrete case by the standard result (Brillouin, 1962; Shannon and Weaver, 1949b)

$$H(p_1 p_2 \ldots p_n) = -\sum_{k=1}^{n} p_k \log p_k \qquad (5.1)$$

where $p_1 \ldots p_n$ are the prior probabilities of n mutually exclusive events which exhaust all possibilities (i.e. $\sum_k p_k = 1$)

H is the entropy of the source which is a system with n states of a certain probability and it is interpreted as the rate (in binary digits or bits per symbol when the base of the logarithm is 2) at which the source generates information. We see that a completely predictable situation (having $p_k = 1$ for $k = i$ (say), $p_k = 0$ for all $k \neq i$) has H = 0, that is, there is no uncertainty and the entropy is zero: and for the case of complete uncertainty when each $p = \frac{1}{n}$, H achieves its maximum value log n.

For a discrete noiseless case, the capacity C of a channel is defined as the maximum rate of generation (in bits per second) of all those sources that may be connected directly to the channel. It can be shown (Shannon and Weaver, 1949c) that any given source of entropy H bits per symbol can be encoded for the channel and run at rates arbitrarily close to $\frac{C}{H}$ symbols per second.

A most important standard theorem, and one that will find direct application in our investigation, concerns the capacity C of a band-limited continuous channel subject to noise interference. We consider continuous signals emitted in a band of width W_1 cycles per second, the average power of emission being limited to S_1 and the channel being subject to noise of power N. Then if the noise is white thermal noise (that is, the noise itself is limited in frequency, the amplitudes of the various frequency constituents being subject to a normal Gaussian probability distribution), it can be shown (Shannon and Weaver, 1949d) that it is possible to transmit information at the rate given by

$$C_1 = W_1 \log\left(1 + \frac{S_1}{N}\right) \qquad (5.2)$$

with arbitrarily small frequency of errors: it is not possible by any encoding to send information at a higher rate with an arbitrarily low frequency of error. Similarly, the maximum rateof reception of information is given by

$$C_0 = W_0 \log\left(1 + \frac{S_0}{N}\right) \tag{5.3}$$

where the suffix o refers to the values at reception by an observer O at the origin of spatial co-ordinates, the suffix $_1$ referring to emission. We assume that N is constant.

For reception of information emitted by a source in the universe which is at a sufficiently great distance to make cosmological considerations important we have by a standard result

$$W_0 = \frac{W_1}{1+z} \cong \frac{W_1}{z} \tag{5.4}$$

where z is the red-shift of the source as measured at the epoch of reception. The power S is usually considered to be attenuated according to an inverse square law of luminosity distance L, which McCrea (1934-5) has shown is given by

$$L = \frac{r\,R(t_0)}{1+kr^2/4} \cdot \frac{R(t_0)}{R(t_1)} \tag{5.5}$$

in terms of the metric (1.2). Thus S_0 is given by

$$S_0 = \frac{S_1}{L^2} \tag{5.6}$$

and the rate of reception of information, C_0, is seen to be model-dependent.

(ii) The work of Metzner and Morrison.

A brief analysis of the flow of information in cosmological models was attempted by Metzner and Morrison (1959) choosing the steady state model as an illustrative example. Although they claim that results for various models are discussed in relation to their visual horizons, if any, the work is limited in that the particle horizon is given only passing attention; indeed, the notions of event horizon and particle horizon are often confused, as when the authors make the incorrect statement that "sources actually on the event horizon at the time of emission can only be 'seen' by the light they emitted when they were created". Moreover, little explicit attention was paid to the flow of information from single sources, that is, allowing t_0, the time of reception of information, to vary.

Metzner and Morrison showed that for the expanding Robertson-Walker models considered (t_0 constant), the total energy flux for all sources with red-shift up to Z_1 (given in our notation by $\int_1^{Z} S(Z)N(Z)dZ$ where $N(Z)dZ$ is the number of sources with Doppler ratios between Z and $Z+dZ$) assumed the form constant x $(1 - Z^{-n})$ where n, a positive constant, is dependent on the particular model under consideration. The total rate of reception of information from all sources with Doppler ratios from $z_0 (\geqslant 1)$ up to z given by $\int_{z_0}^{z} C(z)\,N(z)\,dz$ was shown to have the form constant x $(z_0^{-m} - z^{-m})$ where m is a positive, model-dependent constant; this is true to within an additive constant depending on the values of S_1, N in (5.3) and (5.6), which was neglected by Metzner and Morrison. Allowing z to tend to infinity, we see that both the total energy flux and the total information rate from all sources in a particular channel must be finite.

Assuming the constancy of the quantities we have called W_1, S_1, N and therefore by (5.2) the constancy of C_1, Metzner and Morrison considered too the variation of C_0 with z for the case t_0 = constant that is, for the distribution of distinct but similar sources, as observed at the given time t_0 by the origin-observer O. Due to the definition and use of luminosity distance L and since the authors took no account of the finite size of the source in the neighbourhood of the observer, the unsatisfactory situation arose that for a source in an expanding universe with parts of its surface area coincident with the observer ($L = 0$) the rate of reception of information from that source is given by an infinite value rather than by a finite value approximately equal to the rate of emission C_1 of that information. It may be shown (Cherry, 1957) that in fact, if noise is included, the information rate can never be infinite.

(iii) The lower limit to luminosity distance and red-shift.

We shall show now that taking into account that any F.P. will not be a point source, a fact which is indeed explicitly used in the derivation of luminosity distance L, will result in its being unnecessary to consider $\mathfrak{z} = 1$ or values of $L < 1$. Consider a set of similar F.Ps., each of which is supposed to be ideally spherical with constant unit proper radius. The origin-observer O will be on the surface of one such F.P., which has surface area A_0 (say) at time t_0. Suppose that with respect to O the r-co-ordinate of the centre P_i of another F.P. is r_i ($r_i \neq 0$), the surface area of the pseudo-sphere centred at P_i being given by A_i at time t_i. Deriving the expression (5.5) for the luminosity distance L of P_i from O, McVittie (1956) has shown that the ratio of the outward flux of energy through unit area of A to the average rate at which it flows onto unit area of A_0 is given by L^2. As the two areas used for the calculation come into contact, in the limit this ratio must necessarily, by definition, approach the value unity; at such an instant the quantity $(t_0 - t_i)$, where t_i is the time of emission from P_i and t_0 is the time of reception at O, in the surface A_0, will be small because of the proximity of P_i and O, so that z given by $\frac{R(t_0)}{R(t_i)}$ will be slightly greater than unity in an expanding universe. As P_i recedes from O, $(t_0 - t_i)$ and therefore z and L, will increase without limit. Thus confining our attention to F.Ps. of finite size which are distinct from that associated with O, we have a minimum value of L, L_m say, given by unity and a minimum value of z, z_m say, such that $z_m = 1 + \varepsilon$, where ε is a small positive quantity. The cut-off at these values is essential; then by (5.3) and (5.6), C_0 always remains finite.

Similarly, in a contracting universe, the z value of a nearby source has a maximum value less than unity (since, as Rindler (1956) has pointed out, $t \rightarrow -t$ implies $z \rightarrow \frac{1}{z}$) and other quantities are correspondingly cut off.

Taking account of these properties in the present work, we shall continue the analysis commenced by Metzner and Morrison in 1959. In order that the connection between the flow of information and the existence of horizons may be amply demonstrated, we shall perform our analysis in terms of the parameters ℓ_1 (proper distance at time t_1 of emission) and ℓ_0 (proper distance at time t_0 of reception) for the expanding models and in terms of z for both expanding and contracting models. By using the results of Chapter IV, it is then possible to eliminate these parameters and so obtain the flow of information in terms of the various observables. We shall consider five model universes, as in Chapter IV, both for a single F.P. and for the case t_0 = constant.

(iv) The results

By (5.3) to (5.6) we have the rate of reception of information by the origin-observer O given by

$$C_0 = \frac{W_1}{z} \log\left(1 + \frac{S_1}{N} \cdot \frac{1}{L^2}\right) \tag{5.7}$$

where the rate of generation of the information by the source with red-shift z at O is given by (5.2). For simplicity and because evolutionary effects are unknown at present, we suppose that W_1, S_1 and N are constant throughout space and time. From (5.5), L is given by

$$L = \frac{R(t_0)\,r_1}{1 + kr_1^2/4} \cdot Z \tag{5.8}$$

For all those models with $k = 0$, this is equivalent to

$$L = l_0 Z \qquad (k = 0) \tag{5.9}$$

since by (2.2) $\sigma(r_1) = r_1$ when $k = 0$. We then have the following results in the various model universes.

A: <u>The steady state model and its contracting dual</u>

<u>The expanding model $R(t) = e^{t/T}$, $k = 0$</u>

From (4.9), (4.10) and (5.9)

$$L = cTZ(Z-1) = \frac{l_1}{(1 - l_1/cT)^2} = l_0\left(1 + \frac{l_0}{cT}\right) \tag{5.10}$$

and

$$C_0 = \frac{W_1}{Z} \log\left(1 + \frac{S_1}{N(cT)^2} \cdot \frac{1}{Z^2(Z-1)^2}\right) \tag{5.11}$$

$$C_0 = W_1\left(1 - \frac{l_1}{cT}\right) \log\left(1 + \frac{S_1}{N} \cdot \frac{(1 - l_1/cT)^4}{l_1^2}\right) \tag{5.12}$$

$$C_0 = \frac{W_1}{(1 + l_0/cT)} \log\left(1 + \frac{S_1}{N} \cdot \frac{1}{l_0^2 (1 + l_0/cT)^2}\right) \tag{5.13}$$

These relationships are formally the same both for t_0 = constant and $\sigma(r)$ = constant and are qualitatively illustrated by Fig. 5A, where the arrows indicate the direction of variation with increasing t_0 for the latter case. Metzner and Morrison have considered the $C_0 - Z$ relationship for the steady state model, showing that $C_0(Z) \rightarrow$ constant x Z^{-5} as $Z \rightarrow \infty$ and demonstrating plots of the equations for different values of $\frac{S_1}{N}$. However, none of the plots is cut off at a value of Z exceeding unity.

The contracting dual, $R(t) = e^{-t/T}$, $k = 0$

$$L = cT\,Z(1-Z) = \frac{l_1}{(1+l_1/cT)^2} = l_0\left(1-\frac{l_0}{cT}\right) \tag{5.14}$$

$$\left.\begin{aligned}
C_0 &= \frac{W_1}{Z}\log\left(1+\frac{S_1}{N(cT)^2}\cdot\frac{1}{Z^2(1-Z)^2}\right)\\
C_0 &= W_1\left(1+\frac{l_1}{cT}\right)\log\left(1+\frac{S_1}{N}\cdot\frac{\left(1+\frac{l_1}{cT}\right)^4}{l_1^2}\right)\\
C_0 &= \frac{W_1}{(1-l_0/cT)}\log\left(1+\frac{S_1}{N}\cdot\frac{1}{l_0^2(1-l_0/cT)^2}\right)
\end{aligned}\right\} \tag{5.15}$$

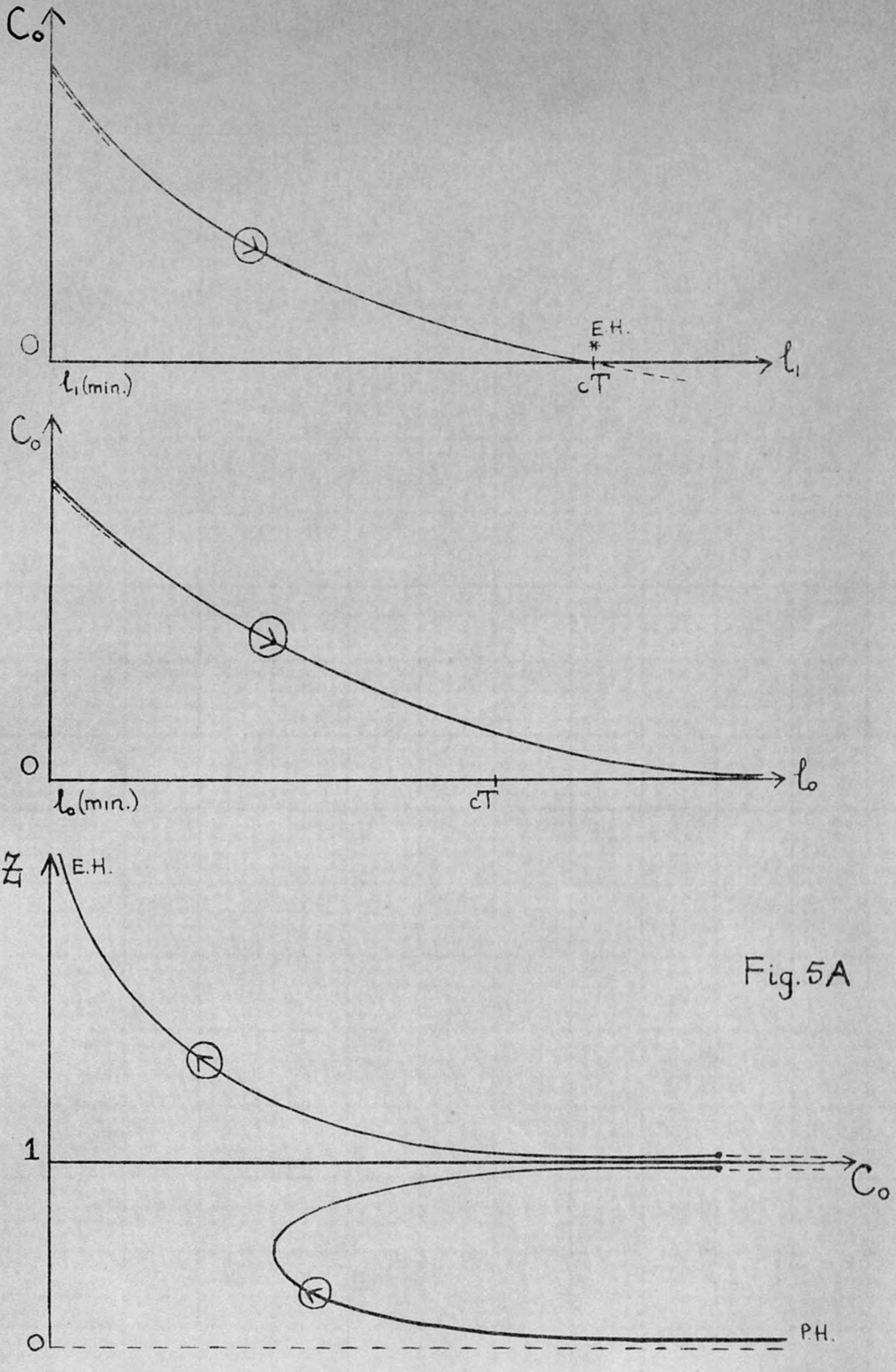


Fig. 5A

B: Page's model and its contracting dual

The expanding model $R(t) = t^2$, $k = 0$.

I. t_o = constant

By (4.19), (4.20) and (5.9)

$$\left.\begin{aligned} L &= ct_o Z(Z^{1/2}-1) \\ &= l_1\left(\frac{ct_o}{2l_1}\right)^4\left[1 \pm \sqrt{1-4l_1/ct_o}\right] \\ &= l_o\left(1+\frac{l_o}{ct_o}\right)^2 \end{aligned}\right\} \tag{5.16}$$

so that

$$C_o = \frac{W_1}{Z}\log\left(1+\frac{S_1}{N(ct_o)^2}\cdot\frac{1}{Z^2(Z^{1/2}-1)^2}\right) \tag{5.17}$$

$$C_o = 4W_1\left(\frac{l_1}{ct_o}\right)^2\frac{1}{\left(1\pm\sqrt{1-4l_1/ct_o}\right)^2}\log\left(1+\frac{S_1}{N}\cdot\frac{l_1^6}{(ct_o/2)^8\left(1\pm\sqrt{1-4l_1/ct_o}\right)^8}\right) \tag{5.18}$$

$$C_o = \frac{W_1}{\left(1+\frac{l_o}{ct_o}\right)^2}\log\left(1+\frac{S_1}{N}\cdot\frac{1}{l_o^2\left(1+\frac{l_o}{ct_o}\right)^4}\right) \tag{5.19}$$

II $\sigma(r_1)$ = constant

Substituting (4.25), (4.26) and (4.27) into (5.17), (5.18) and (5.19) respectively we get

$$C_0 = \frac{4W_1(Bl_1)\left[1-(Bl_1)^{\frac{1}{2}}\right]^2}{\left(1 \pm \sqrt{1-4(Bl_1)^{\frac{1}{2}}\left[1-(Bl_1)^{\frac{1}{2}}\right]}\right)^2} \log\left(1 + \frac{S_1}{N}\frac{2^8 B^2 (Bl_1)^2\left[1-(Bl_1)^{\frac{1}{2}}\right]^8}{\left(1 \pm \sqrt{1-4(Bl_1)^{\frac{1}{2}}\left[1-(Bl_1)^{\frac{1}{2}}\right]}\right)^8}\right) \quad (5.20)$$

$$C_0 = \frac{W_1}{Z} \log\left(1 + \frac{S_1}{N} \cdot B^2 \frac{1}{Z^2(Z^{\frac{1}{2}}-1)^4}\right) \quad (5.21)$$

$$C_0 = \frac{W_1}{\left[1+(Bl_0)^{\frac{1}{2}}\right]^2} \log\left(1 + \frac{S_1}{N} \cdot \frac{1}{l_0^2\left[1+(Bl_0)^{\frac{1}{2}}\right]^4}\right) \quad (5.22)$$

The contracting dual, $R(t) = (-t)^2$, $k = 0$

I. t_0 = constant

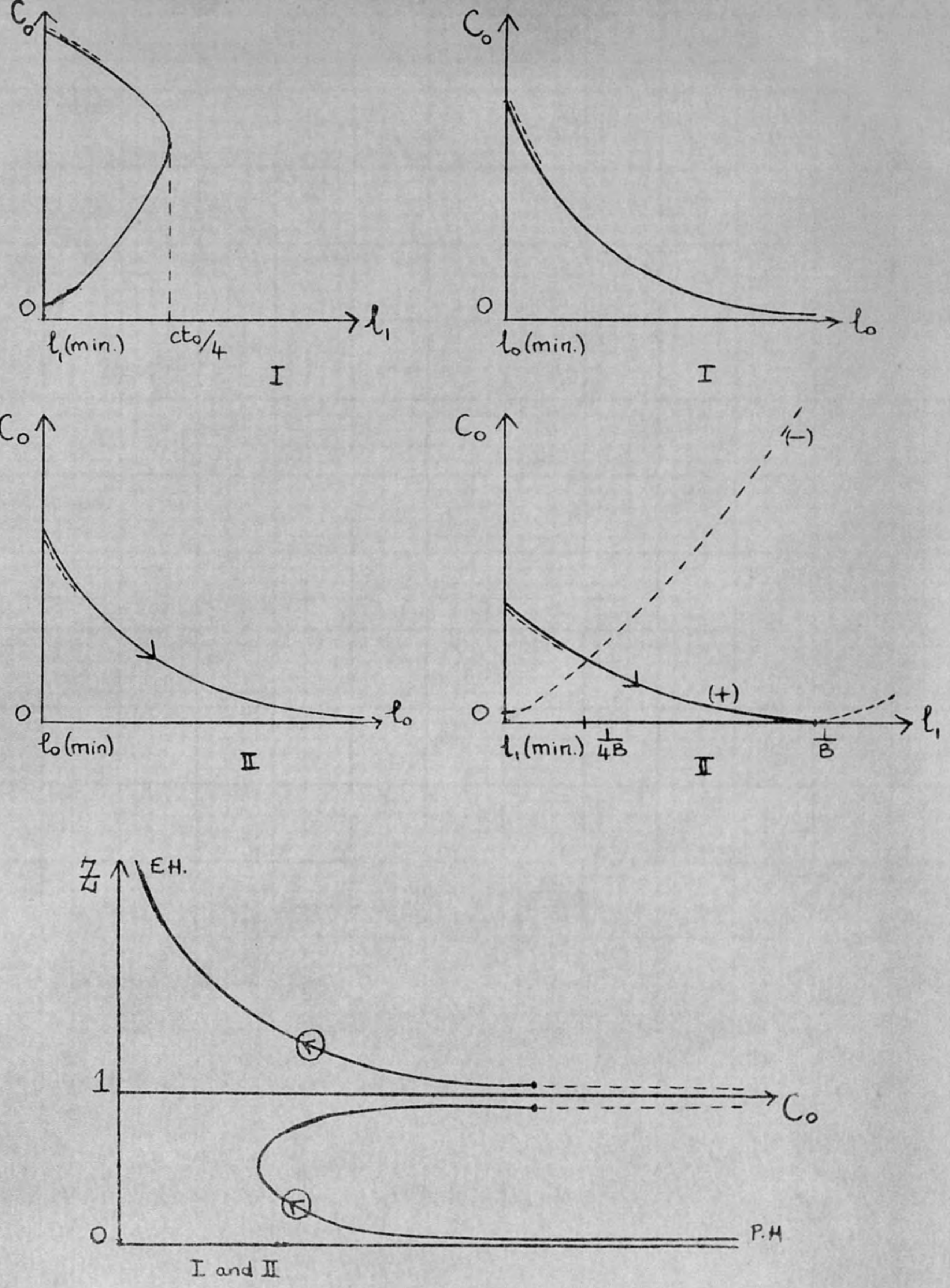
$$L = c|t_0|\, Z\left(1 - Z^{\frac{1}{2}}\right) \quad (5.23)$$

so that

$$C_0 = \frac{W_1}{Z} \log\left(1 + \frac{S_1}{N(c|t_0|)^2} \cdot \frac{1}{Z^2(1-Z^{\frac{1}{2}})^2}\right) \quad (5.24)$$

II $\sigma(r_1)$ = constant

$$C_0 = \frac{W_1}{Z} \log\left(1 + \frac{S_1 B^2}{N} \cdot \frac{1}{Z^2(1-Z^{\frac{1}{2}})^4}\right) \quad (5.25)$$



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I and II

Fig. 5B

C: The Einstein-de Sitter model and its contracting dual

The expanding model, $R(t)=t^{2/3}$, $k = 0$

I t_0 = constant

From (4.41), (4.42) and (5.9)

$$L = 3ct_0 \, . \, z^{1/2}(z^{1/2}-1) \quad = \quad \frac{l_0}{\left(1-\frac{l_0}{3ct_0}\right)^2} \tag{5.26}$$

so that

$$C_0 = \frac{W_1}{z} \log\left(1+ \frac{S_1}{N(3ct_0)^2} \cdot \frac{1}{z\,(z^{1/2}-1)^2}\right) \tag{5.27}$$

where z is given in terms of l_1 by

$$\frac{l_1}{3ct_0} = \frac{(z^{1/2}-1)}{z^{3/2}}$$

and

$$C_0 = W_1\left(1-\frac{l_0}{3ct_0}\right)^2 \log\left(1+\frac{S_1}{N}\cdot\frac{\left(1-\frac{l_0}{3ct_0}\right)^4}{l_0^2}\right) \tag{5.28}$$

II $\sigma(r_1)$ = constant

We have in addition equation (4.47) so that substituting for t_0 in (5.26) we have

$$L = \mathcal{C}_1 \frac{z^2}{(z^{1/2}-1)^2} \tag{5.29}$$

where $\mathcal{C}_1$ is a constant characterising the F.P.

By (4.49) and (4.50) we get also

$$L = \ell_1 \left[1 + \left(\frac{\ell_1}{\ell_1}\right)^{\frac{1}{2}}\right]^4 \tag{5.30}$$

and
$$L = \frac{\ell_0}{\left[1 - \left(\frac{\ell_1}{\ell_0}\right)^{\frac{1}{2}}\right]^2} \tag{5.31}$$

Thus
$$C_0 = \frac{W_1}{Z} \log\left(1 + \frac{S_1}{N\,\ell_1^{\,2}} \cdot \frac{(Z^{\frac{1}{2}} - 1)^4}{Z^4}\right) \tag{5.32}$$

$$C_0 = \frac{W_1}{\left[1 + \left(\frac{\ell_1}{\ell_1}\right)^{\frac{1}{2}}\right]^2} \log\left(1 + \frac{S_1}{N} \cdot \frac{1}{\ell_1^2\left[1 + \left(\frac{\ell_1}{\ell_1}\right)^{\frac{1}{2}}\right]^8}\right) \tag{5.33}$$

$$C_0 = W_1\left[1 - \left(\frac{\ell_1}{\ell_0}\right)^{\frac{1}{2}}\right]^2 \log\left(1 + \frac{S_1}{N} \cdot \frac{\left[1 - \left(\frac{\ell_1}{\ell_0}\right)^{\frac{1}{2}}\right]^4}{\ell_0^2}\right) \tag{5.34}$$

The contracting dual, $R(t) = (-t)^{2/3}$, $k = 0$

I. t_0 = constant

$$L = 3c|t_0|\,Z^{\frac{1}{2}}(1 - Z^{\frac{1}{2}}) \tag{5.35}$$

$$C_0 = \frac{W_1}{Z} \log\left(1 + \frac{S_1}{N(3c|t_0|)^2} \cdot \frac{1}{Z(1 - Z^{\frac{1}{2}})^2}\right) \tag{5.36}$$

II $\sigma(r_1)$ = constant

$$C_0 = \frac{W_1}{Z} \log\left(1 + \frac{S_1}{N\,\ell_2} \cdot \frac{(1 - Z^{\frac{1}{2}})^4}{Z^4}\right) \tag{5.37}$$

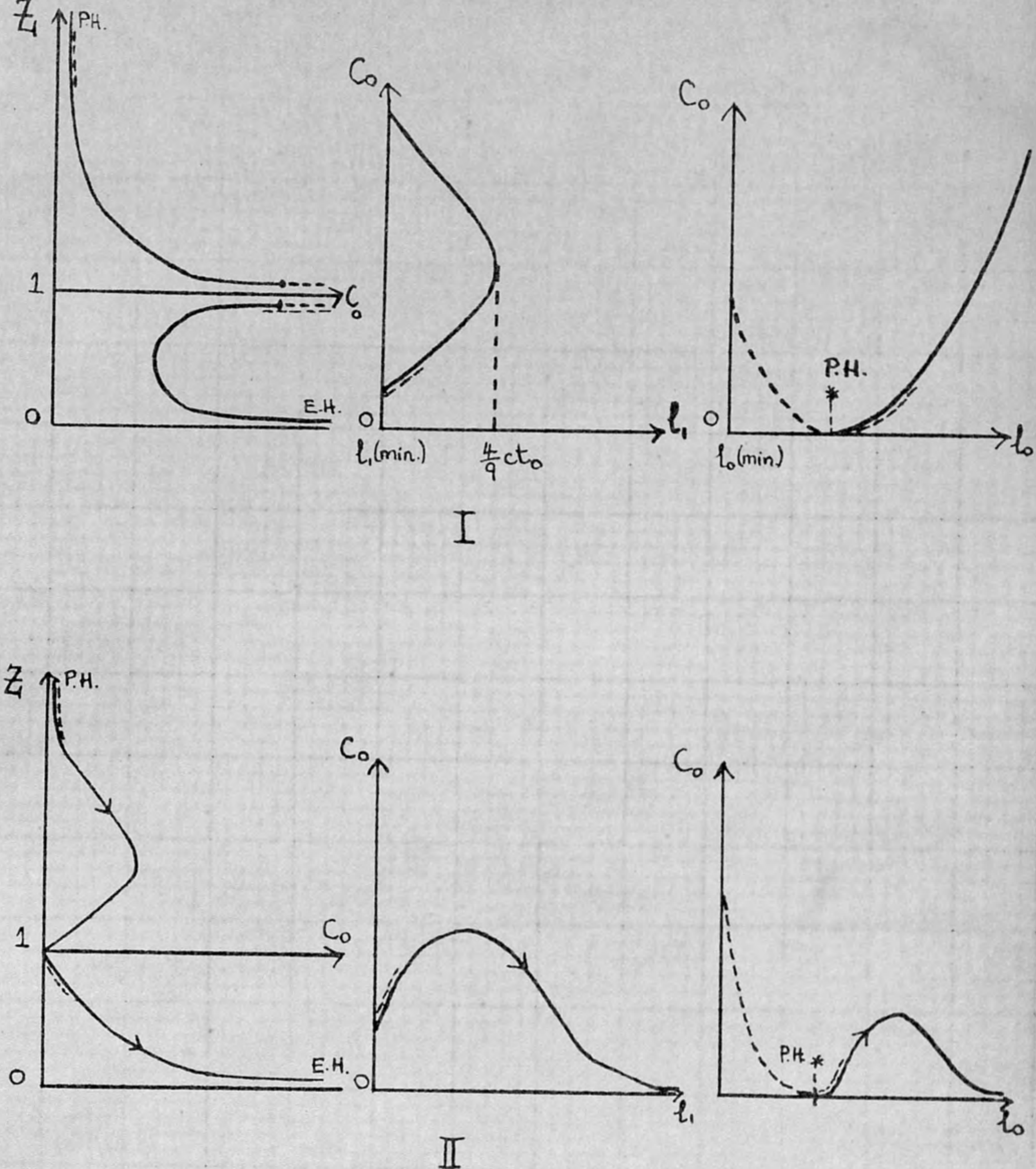

Z1
P.H.
1
C0
0
E.H.
C0
0
l1(min.)
4/9 ct0
l1
C0
P.H.
0
l0(min)
l0
I
Z1
P.H.
1
C0
0
E.H.
C0
0
l1
C0
P.H.
l0
II

Fig. 5C

D: The expanding model with $R(t) = a(\cosh bt - 1)^{1/3}$, $k = 0$

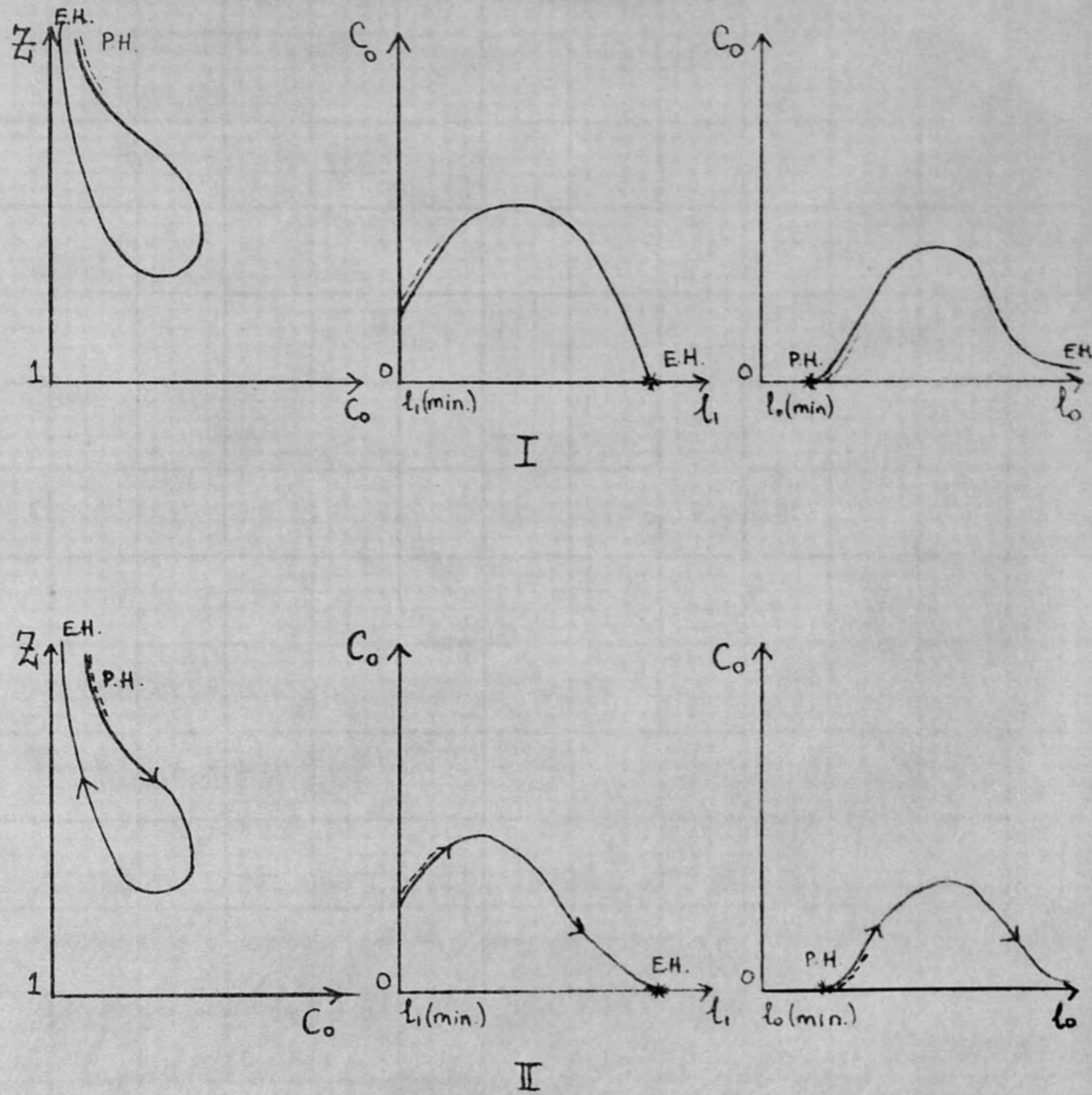


Fig. 5D

E: Milne's model and its contracting dual

The expanding model, $R(t) = t, k = -1$ $(c = 1)$

I t_o = constant

We now have from (5.8)

$$L = \frac{R(t_o)r_1}{1 - r_1^2/4} \cdot Z \tag{5.38}$$

which by (2.4) gives

$$L = \frac{t_o}{2}(Z^2 - 1) \tag{5.39}$$

Thus in terms of Z

$$C_o = \frac{W_1}{Z} \log\left(1 + \frac{S_1}{N}\frac{4}{t_o^2}\cdot\frac{1}{(Z^2-1)^2}\right) \tag{5.40}$$

Combining this with equations (4.65), namely

$$l_1 = t_o \frac{\log Z}{Z} \tag{5.41}$$

$$l_o = t_o \log Z \tag{5.42}$$

we may obtain C_o in terms of l_1 and l_o.

II $\sigma(r_1)$ = constant

By (4.64) Z is constant and by (5.41) and (5.42) l_1 and l_o increase linearly with t_o. By (5.40), C_o decreases steadily as l_1, l_o increase to infinity.

The contracting dual, $R(t) = (-t), k = -1$ $(c = 1)$

I t_o = constant

$$L = \frac{c|t_o|}{2}(1 - Z^2) \tag{5.44}$$

$$C_o = \frac{W_1}{Z}\log\left(1 + \frac{S_1}{N}\frac{4}{(|t_o|)^2}\cdot\frac{1}{(1-Z^2)^2}\right) \tag{5.45}$$

II $\sigma(r_1)$ = constant

C_o is given by (5.45) where now Z = constant and t_o varies.

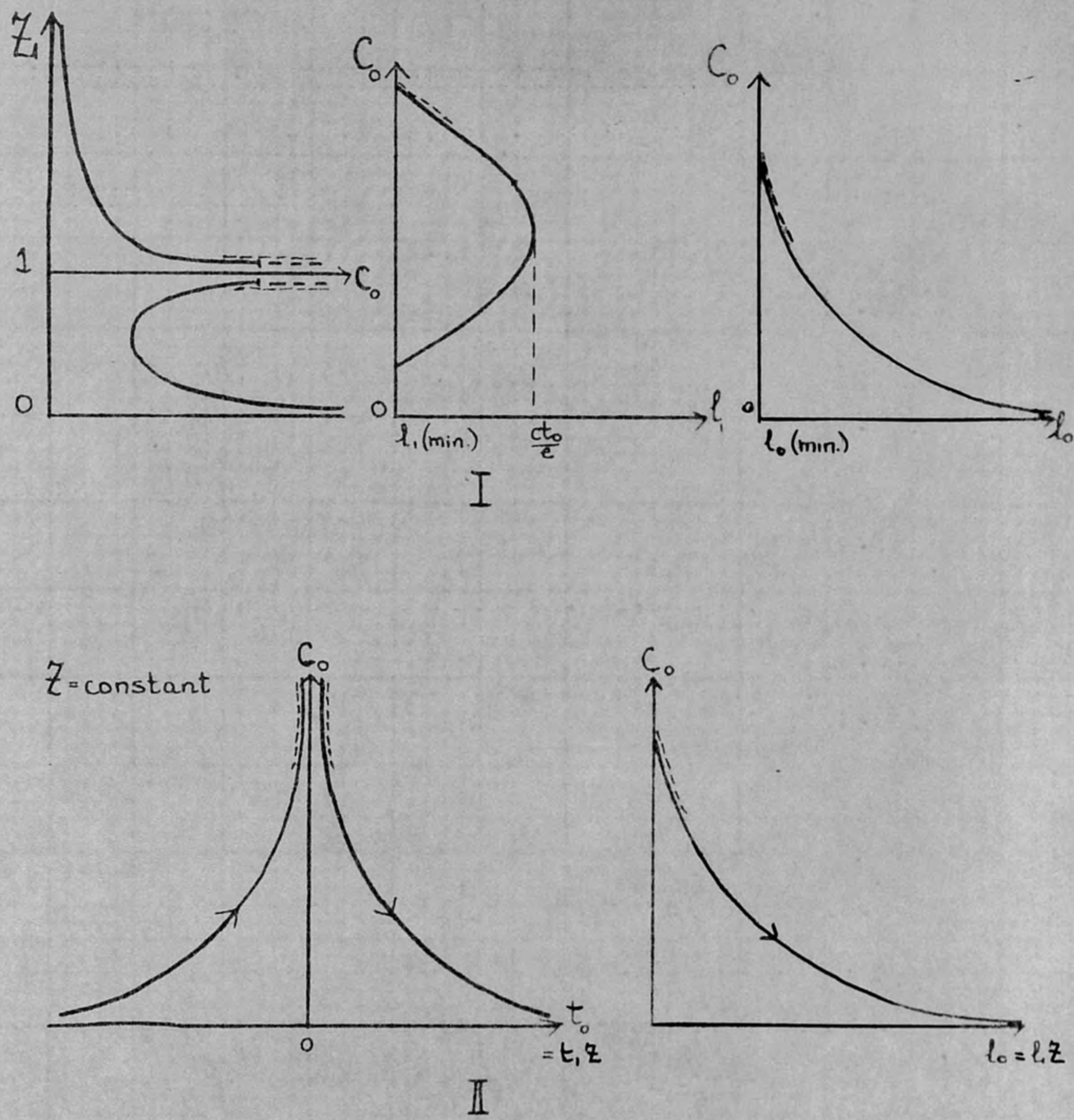

Z₁
1
0
C₀
C₀
0
l₁(min.)
ct₀/e
l₁
C₀
0
l₀(min.)
l₀
I
Z = constant
C₀
t₀ = t₁Z
0
C₀
l₀ = l₁Z
II

Fig. 5E

(v) The rate of reception and the rate of loss of information.

Figures 5A to 5E illustrate the behaviour of the rate of flow of information in the neighbourhood of an E.H. or a P.H. when such exist in the various cosmological models considered.

Comparison of these diagrams with those of Chapter IV shows, with the single exception of Milne's dual, the qualitative similarity of the behaviour of $C_o(z)$ and $m(z)$ in the contracting models, both for t_o constant and for a single source. There exists a minimum for both m and C_o in terms of z in those models admitting a P.H., these quantities approaching an infinite value at $z = 0$, which occurs on the horizon. In the contracting model we have considered which admits E.Hs., the dual Einstein-de Sitter model, this minimum occurs in the same way only for the case t_o = constant; for a single source both m and C_o decrease steadily and tend asymptotically to infinity on the E.H. as $z \to 0$.

The existence of a minimum $m(z)$ has been noted by Metzner and Morrison (1959a) for the contracting dual of the steady state model for the case of a single particle only. They state that as z goes from one to zero $C_o(z)$ decreases to a minimum and then increases again to infinity; we have demonstrated, however, that when there exists a P.H. in a contracting model, such as the steady state dual, the time direction of z is in the opposite sense, increasing initially from zero and tending to the value one if there exists no E.H., but decreasing again to zero if there exists an E.H.; hence it would be more correct to say that in the steady state dual $C_0(z)$ for a single source decreases from infinity to a minimum value, thereafter increasing again to a value which will be finite because of the cut-off of z at a value less than unity.

We see also from the diagrams that while the qualitative similarity between $m(z)$ and $C_0(z)$ holds too for the expanding cases, there is then no minimum for these quantities as in the contracting cases; when an expanding model admits P.Hs both m and C_0 manifest a maximum value in terms of z for single sources. Otherwise C_0 decreases steadily as z increases. It is seen that without exception in expanding models $C_0 = 0$ $(z = \infty)$ on an E.H. or a P.H., but that in contracting models, $C_0 = \infty$ $(z = 0)$ on a horizon; for a single source in the former case, $C_0 \rightarrow 0$ as the source recedes, having initially started with a finite value $(\sim C_1)$ unless there exists a P.H.

in the model, in which case C_0 is initially zero.

Taking new axes at $C = C_1$ with the positive direction downwards, it is interesting to note that the diagrams also indicate the rate of loss of information which is given by $(C_1 - C_0)$, where we have taken C_1 to be constant, either in space or time as the case may be. To the extent to which the rate of loss of information may be identified with the rate of increase of uncertainty, we see from (5.7) that this latter quantity depends both on $\frac{1}{Z}$ and on $\frac{1}{L^2}$ (which may be expressed in terms of Z). $(C_1 - C_0)$ is always positive in expanding models and the loss is complete for that information emitted by a particle when on the observer's E.H. $(C_0 = 0)$ and for that received by the observer at the instant when the source is on his P.H.; although formally the expression may have a non-imaginary value different from C_1 for that information emitted beyond a horizon (in terms of l_1 for a single particle when there exists an E.H. or in terms of l_0 for t_0 constant when there exists a P.H.) as witnessed by the relevant diagrams, we have seen already that in the case of an E.H. this information is never manifested at the observer and that in the case of a P.H. it is manifested at the observer only after the particle concerned has crossed the horizon.

In contracting models, using the same definition for C results in an apparent gain in information ($(C_0 - C_1)$ is negative) over and above that emitted; correspondingly, the apparent luminosity tends to an infinite value. The logical status of this "additional" information is plainly dependent on the associated uncertainty, a concept which does not appear to have found expression or formulation with regard to contracting universes. The whole question is evidently bound up with considerations of entropy, the "arrow of time" (Gold, 1962) and the thermodynamic foundations of contracting models and would necessitate a reconsideration of Olber's paradox regarding the density of radiation in the model universe.

Since, in the contracting models based on the Robertson-Walker line-element, the bulk of matter is relatively near to the origin-observer at the epoch of observation customarily considered and the nearer the matter the less is its apparent luminosity, it may well be possible to avoid an infinite radiation density at the observer in the contracting cases, especially if it is considered unjustified or illegitimate in any case to extrapolate out to vast distances; the possibility will depend critically on the thermodynamical considerations. Bondi (1960d) has forcefully pointed out that conditions during contraction would be almost unimaginably different from the present and that "the serious difficulties connected with Olber's paradox and the nature of thermodynamics in the contracting phase do not seem to have been examined, let alone resolved".

CHAPTER VI: THE BARRIER IN THE SCHWARZSCHILD SPACE-TIME

(i) Introduction

Einstein's general theory of relativity attempts to express Mach's (1893) view that the geometry of space-time should be causally determined by the distribution of the energy and matter that it contains. In this theory, gravitating matter is represented by the symmetric stress-energy tensor $T^{\mu\nu}$ which describes the density, momentum, energy and pressure of matter. $T^{\mu\nu}$ is such that its covariant derivative vanishes; it is expressed in terms of the symmetric metric tensor $g^{\mu\nu}$ by means of the gravitational field equations

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = -\kappa T^{\mu\nu} \qquad (6.1)$$

where $G^{\mu\nu}$ is the Einstein tensor which satisfies the conservation condition as a geometrical identity and κ is a universal constant, chosen so that in the limiting case of small masses and densities (6.1) yields the Newtonian law of gravitation.

Λ, also a universal constant, known as the cosmological constant, was introduced by Einstein in the belief that for positive Λ (6.1) had no solution for $T^{\mu\nu} = 0$ so that Mach's principle was fully incorporated into the theory. de Sitter (1917) showed, however, that a solution for empty space exists. Hence the inclusion of Λ in the field equations is of doubtful theoretical importance; physically, observation indicates that, in any case, it must be negligibly small for our solar system (Bondi, 1960e ; Tolman, 1934a). We shall throughout set $\Lambda = 0$.

For the case in which the number of variables is reduced by imposing conditions of spherical symmetry, Schwarzschild (1916) found a rigorous solution of the field equations of general relativity. He showed that the line-element holding in the empty space outside a spherical distribution of matter of mass M (here supposed positive) whose centre is located at r = 0 is given by

$$ds^2 = c^2\left(1-\frac{\alpha}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{\alpha}{r}\right)} - r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right) \quad (6.2)$$

where α is a constant of integration which must characterise the mass of the matter which creates the gravitational field. Reduction to Newtonian theory shows that we may identify α with the constant $\frac{2GM}{c^2}$ where G is the Newtonian constant of gravitation. Putting

$$\alpha \equiv \frac{2GM}{c^2} = 2m \quad (6.3)$$

(6.2) becomes

$$ds^2 = c^2\left(1-\frac{2m}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2m}{r}\right)} - r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right) \quad (6.4)$$

The field equations hold everywhere in the space-time except on $r = 2m$ and at $r = 0$. On the spherical surface $r = 2m$, some of the spatial components become infinite and the g_{tt} component vanishes. It is evident that at $r = 2m$ there is a singularity at which space-like and time-like co-ordinates change character. The singular region $0 \leqslant r \leqslant 2m$ reduces to the point singularity at $r = 0$ only when $m = 0$, but then the gravitational field disappears altogether, reducing (6.4) to the Minkowski metric.

In view of the great significance of the metric (6.4) for the discussion of the three so-called crucial tests of general relativity and since the metric (6.4) is valid only in the region exterior to the matter causing the field, where the energy tensor is zero, it is generally concluded that the radius r_0 of a mass M must be larger than 2m; for then all events occurring in the region $r \leqslant 2m$ take place inside the material under consideration and (6.4) has no singularity in the domain of its validity.

This assumption has so far seemed to be borne out in nature, for it has never been found that matter is concentrated enough to permit the Schwarzschild singularity to occur in empty space;

the physical radius of a star has always appeared to be very much larger than $\frac{2G}{c^2}$ times its mass, so that there is no singular region.

Einstein (1939) has attempted to show that the property that the region $r < 2m$ is embedded in a star's material is a theoretical necessity. For the particular case of a spherically symmetric system of mass points (whose individual singularities were neglected and whose paths were explicitly chosen) he succeeded in showing that they could not be so concentrated that the field manifested a Schwarzschild singularity; for the particles on the outside of the system would begin to move with the velocity of light before the critical density of the system was attained. It may be that this result can be extended to cover more general cases.

At any rate, whether or not such a singularity is manifested in nature, we may investigate mathematically the surface $r = 2m$ around a mass of radius $r_o < 2m$ by considering the behaviour of particles in the model; we emphasise that since the manifold with $r > r_o$ is empty of matter we consistently refer only to test particles.

Many authors (Robertson, 1939; Lemaitre, (1949);

Synge, (1950); Finkelstein, (1958); Fronsdal, (1959); Kruskal, (1960); Fuller and Wheeler, (1962)) are convinced that at least part of the singular character of the surface $r = 2m$ must be attributed the co-ordinate system being used to describe the field, for the reason pointed out by Finkelstein and Fronsdal that neither the Petrov curvature scalars nor the equations for the geodesics show a singular behaviour at $r = 2m$. Because of this, most of these authors at present appear to hold the view that it is possible for a small test particle travelling from $r > 2m$ in the Schwarzschild space-time to pass through the surface $r = 2m$ and reach the origin in finite proper time (s - time); the origin is presumably taken to be $r = 0$ although r has then a time-like character. We shall examine this contention more closely and reach different conclusions: yet we shall see that the whole range of t is certainly exhausted in the region $r > 2m$ alone; the co-ordinate system of (6.4) is, in fact, incomplete, covering only the region $2m < r \leqslant \infty$, whereas the Schwarzschild manifold exists for all $r > 0$. Thus it is impossible to describe the region $r < 2m$ in terms of t. An observer in $r > 2m$ who measures in t - time would therefore consider the surface $r = 2m$ to be a barrier and should it be possible for him to be at all aware of the region $r < 2m$ he would consider it as belonging to a different universe. As Dirac (1962) has pointed out, such an observer will not consider the region $r < 2m$ a physical space, because to send a signal inside and get it out again would require an infinite time; for the observer in $r > 2m$ measuring in t - time "the Schwarzschild radius provides a sort of natural boundary to space".

(ii) Removal of the singularity; introduction of the Finkelstein space-time.

Many attempts have been made to remove the singularity at $r = 2m$. Avoiding the use of co-ordinates by embedding space-time in a pseudo-Euclidean space of six dimensions, Fronsdal (1959) has made a completion of the manifold defined by (6.4) for $r > 2m$ so that all geodesics can be described in one picture. Kruskal (1960) has presented a simple transformation to different co-ordinates by means of which the singularity at $r = 2m$ is removed and the singularity - free space described by $r > 2m$ is maximally extended. Both Lemaitre (1933) and Synge (1950) have concluded that the singularity at $r = 2m$ is not essential; considering geometrical representations of space-time, Synge has removed the singularity by making a continuation of space-time. Both Rylov (1961) and Graves and Brill (1960) have presented transformations to co-ordinate systems for which both time-like and space-like co-ordinates retain their characters everywhere.

Finkelstein (1958) has shown that the singularity at r = 2m in the coefficient of dr^2 in (6.4) is due only to the choice of co-ordinates. Using units such that $\alpha = 1$, he describes the space-time in a form corresponding to

$$ds^2 = c^2\left(1-\frac{1}{\bar{r}}\right)d\bar{t}^2 + \frac{2}{\bar{r}}\,c\,d\bar{t}\,d\bar{r} - \left(1+\frac{1}{\bar{r}}\right)d\bar{r}^2 - \bar{r}^2\left(d\bar{\theta}^2+\sin^2\bar{\theta}\,d\bar{\phi}^2\right) \quad (6.5)$$

Finkelstein demonstrates that transforming (6.5) according to the equations

$$t = \bar{t} + \ln(\bar{r}-1) \;;\; r = \bar{r}$$
$$\theta = \bar{\theta} \;;\; \phi = \bar{\phi}$$

for the case $\bar{r} > 1$ yields the Schwarzschild metric for $r > 1$. In order that we may legitimately discuss the region $\bar{r} < 1$ we shall here transform according to the equation

$$t = \bar{t} + \ln|\bar{r}-1| \;;$$

since only derivatives are involved in the transformation this will result in precisely the same equation (6.5) for the case $\bar{r} < 1$.

We see from (6.5) that there is no singularity in the spatial part of the metric; however, g_{tt} still changes character at $\bar{r} = 1$, t being a space-like co-ordinate for $\bar{r} < 1$. Although the metric itself is singular only at $\bar{r} = 0$, the surface $\bar{r} = 1$ is everywhere tangent to the null-cone. For the Finkelstein metric too we have an incomplete co-ordinate system, as Fronsdal (1959) has pointed out. The spherical surface $\bar{r} = 1$ is a barrier still.

In view of the apparent possibility of removing the singularity at $r = 2m$ in the Schwarzschild metric by mathematical means, we may well enquire whether the barrier at $r = 2m$ in (6.4) and the barrier at $\bar{r} = 1$ in (6.5) have any physical significance whatever. We shall show that, within their respective space-times, the existence of these barriers is not due to the incomplete co-ordinate systems used to describe the space-times, but that the surfaces $r = 2m$, $\bar{r} = 1$ are indeed barriers in a physical sense. Moreover, we shall show that the surfaces $r = 2m$, $\bar{r} = 1$ differ remarkably in character.

(iii) The geodesic equations.

It has been deduced by many authors (Lemaitre, (1933); Einstein, (1939); Synge, (1950); Robertson, (1939); Darwin, (1961)) that a test particle falling freely from a finite distance r ($>$ 2m) towards the "origin" r = 0 will arrive at the singularity r = 2m in finite proper time (s-time) whereas measuring in co-ordinate time (t-time) it takes an infinite time for the particle to reach r = 2m. Let us examine these points by considering, independently, the equations of motion of particles and light rays. Adopting Darwin's (1961) notation, we shall reserve the use of the co-ordinate t to describe the motion of a particle, using τ for the motion of light rays. We now choose units such that 2m = 1 and c = 1. Then we have

$$ds^2 = \left(1-\frac{1}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{1}{r}\right)} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \qquad (6.6)$$

remembering that the field equations are not satisfied for r = 1.

The equations of motion of a free particle in the plane $\theta = \frac{\pi}{2}$ in the gravitational field (6.6) are obtained from the geodesic equations

$$\frac{d^2x^\sigma}{ds^2} + \left\{ {\sigma \atop \mu\nu} \right\} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \qquad (6.7)$$

and may be written (Tolman, 1934b)

$$\left(1-\frac{1}{r}\right)\left(\frac{dt}{ds}\right)^2 - \frac{1}{\left(1-\frac{1}{r}\right)}\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2 = 1 \qquad (6.8)$$

$$r^2\frac{d\phi}{ds} = p \qquad (6.9)$$

$$\frac{dt}{ds} = \frac{c}{\left(1-\frac{1}{r}\right)} \qquad (6.10)$$

where p, C are constants of integration. The fourth geodesic equation shows that a particle originally moving in the plane $\theta = \frac{\pi}{2}$ will continue to do so throughout its motion. We note that $\frac{dt}{ds}$ is positive for $r > 1$ only if $C > 0$, but that if we should require t and s to increase together in the region $r < 1$, C must be negative. We do not consider the equation (6.10) valid for r = 1 and we shall disallow this case, treating it throughout only as a limiting case: we maintain the condition that the metric coefficients and their determinant should be non-singular.

To begin with we confine our attention to purely radial motion so that $d\phi = 0$. Then (6.6) and (6.8) both reduce to

$$ds^2 = \left(1 - \frac{1}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{1}{r}\right)} \tag{6.11}$$

while p = 0 in (6.9). Substituting from (6.10) into (6.11) gives

$$\frac{dr}{ds} = \pm\left(C^2 - 1 + \frac{1}{r}\right)^{1/2} \tag{6.12}$$

so that
$$\frac{dr}{dt} = \pm\frac{\left(C^2 - 1 + \frac{1}{r}\right)^{1/2}\left(1 - \frac{1}{r}\right)}{C} \tag{6.13}$$

where we must take the positive sign for outgoing particles (r increasing) and the negative sign for ingoing particles (r decreasing).

(iv) The s - r relationship.

Integration of (6.12) then yields the following s - r relationships for the three cases $|C|>1$, $|C|=1$, $|C|<1$.

$|C|>1$:

$$\pm s(r)=\frac{1}{K^3}\left[Kr^{1/2}(K^2r+1)^{1/2}-\frac{1}{2}\ln\frac{(K^2r+1)^{1/2}+Kr^{1/2}}{(K^2r+1)^{1/2}-Kr^{1/2}}\right]+\text{const.} \quad (6.14)$$

where $K=(C^2-1)^{1/2}$

$|C| = 1$:

$$\pm s(r)=\frac{2}{3}r^{3/2}+\text{const.} \quad (6.15)$$

$|C|<1$:

$$\pm s(r)=\frac{1}{k^3}\left[-kr^{1/2}(1-k^2r)^{1/2}+\arcsin\left(kr^{1/2}\right)\right]+\text{const.} \quad (6.16)$$

where $k=(1-C^2)^{1/2}$

Reference to (6.12) shows that the cases $|C|>1$, $|C|=1$, $|C|<1$ correspond respectively to finite, zero and imaginary velocity of the test particle at infinity; for $|C|<1$ the upper bound of r is given, from (6.16), by $\frac{1}{k^2}\left(=\frac{1}{1-C^2}\right)$.

(6.15) demonstrates immediately that it seems possible for a particle starting in the region $r>1$ and travelling radially towards the barrier to approach infinitesimally near to, and in the limit to reach, $r = 1$ in a finite proper time. This is clearly the case too for those particles having $C>1$ or $C<1$, for the equations (6.14) and (6.16) are mathematically continuous through $r = 1$

right down to $r = 0$: we must, however, not forget that the failure of (6.10) at $r = 1$ and the fact that r changes character through $r = 1$ impose physical considerations on the equations. It is the mathematical continuity of the geodesic equations (6.14) to (6.16) which has led many authors to deduce that a test particle reaching $r = 1$ will actually penetrate that surface and eventually inevitably reach the "origin" $r = 0$, results quoted by, among others, Synge (1950), Graves and Brill (1960) and Kruskal (1960). We take the view that the whole of a geodesic is not meaningful when there exists a singular point along its length. Consideration of (6.10) to (6.13) for the cases $r > 1$ and $r < 1$ shows that although we may have infinitely many geodesics apparently passing through a particular point in $r = 1$, there exists no real correlation (except by a purely mathematical convention) between those approaching the point from the region $r > 1$ and those entering $r < 1$ from this point (Cf. Fronsdal (1959)). We believe this physical interpretation from mathematical continuity is not justified. Adopting throughout the procedure of considering only the results of possible observations which in principle could be made in the model, we shall later reconsider this point in the light of our findings.

(v) The t-r relationship.

The t-r relationships for a test particle are obtained by integration of (6.13) yielding:

$|c|>1$:

$$\pm t(r) = \frac{(K^2+1)^{\frac{1}{2}}}{K^3}\left[(K^2r+1)^{\frac{1}{2}}Kr^{\frac{1}{2}} + \frac{(2K^2-1)}{2}\ln\frac{(K^2r+1)^{\frac{1}{2}}+Kr^{\frac{1}{2}}}{(K^2r+1)^{\frac{1}{2}}-Kr^{\frac{1}{2}}}\right]$$

$$+\ln\left|\frac{K^2r^{\frac{1}{2}}+1-(K^2+1)^{\frac{1}{2}}(K^2r+1)^{\frac{1}{2}}}{(K^2r+1)^{\frac{1}{2}}-(K^2+1)^{\frac{1}{2}}}\right|$$

$$-\ln\left|\frac{K^2r^{\frac{1}{2}}+1+(K^2+1)^{\frac{1}{2}}(K^2r+1)^{\frac{1}{2}}}{(K^2r+1)^{\frac{1}{2}}+(K^2+1)^{\frac{1}{2}}}\right| + \text{const.} \qquad (6.17)$$

where $K^2 = c^2-1$

$|c|=1$:

$$\pm t(r) = \frac{1}{c}\left[\frac{2}{3}r^{3/2} + 2\sqrt{r} + \ln\frac{|\sqrt{r}-1|}{\sqrt{r}+1}\right] + \text{const.} \qquad (6.18)$$

$|c|<1$:

$$\pm t(r) = \frac{(1-k^2)^{\frac{1}{2}}}{k^3}\left[-(1-k^2r)^{\frac{1}{2}}kr^{\frac{1}{2}} + (1+2k^2)\arcsin(1-k^2r)^{\frac{1}{2}}\right]$$

$$-\ln\left|\frac{k^2r^{\frac{1}{2}}+1-(1-k^2)^{\frac{1}{2}}(1-k^2r)^{\frac{1}{2}}}{(1-k^2r)^{\frac{1}{2}}-(1-k^2)^{\frac{1}{2}}}\right|$$

$$+\ln\left|\frac{k^2r^{\frac{1}{2}}+1+(1-k^2)^{\frac{1}{2}}(1-k^2r)^{\frac{1}{2}}}{(1-k^2r)^{\frac{1}{2}}+(1-k^2)^{\frac{1}{2}}}\right| + \text{const.} \qquad (6.19)$$

where $k^2 = 1 - C^2$.

Examination of these equations shows that those particles starting in the region $r > 1$ and travelling towards the origin will take an infinite co-ordinate time to reach the barrier at $r = 1$. This is clear from (6.18) for the case $C = 1$. It holds too for the cases $C > 1$ and $C < 1$, for in (6.17) and (6.19) respectively the second term on the right hand side may be shown to tend to $-\infty$ as $r \to 1$, all other terms remaining finite; taking the minus sign on the left hand side for particles travelling towards the barrier from $r > 1$, we see that the co-ordinate travel time from finite r (> 1) to the barrier at $r = 1$ is infinite.

In the same way, any particles travelling in the outwards direction at $r = 1$ will take an infinite co-ordinate time to penetrate any finite distance into the region $r > 1$, although, as we have seen from (6.14) to (6.16), they may apparently leave the null surface $r = 1$ in finite proper time.

(vi) <u>The radial and non-radial motion of light rays.</u>

To investigate the motion of light rays in the Schwarzschild space-time, we put $ds = 0$ which gives, for radial motion, from (6.11)

$$\frac{dr}{d\tau} = \pm\left(1 - \frac{1}{r}\right) \qquad (6.20)$$

Comparison with (6.13) shows that this equation is obtained from (6.13) by allowing $C \to \infty$. Integrating (6.20) we have

$$\pm \tau = r + \ln|r-1| + \text{const.} \qquad (6.21)$$

In the limit, where radial motion is concerned, it thus takes an infinite co-ordinate time for light rays too to reach the barrier at r = 1 from finite $r > 1$ or to leave the surface r = 1 in an outward direction, even though the proper time for such journeys is necessarily zero. It is clear that both particles and light rays which are emitted, in the limit, from the surface r = 1 in a radially outwards direction take an infinite co-ordinate time to penetrate any finite distance into the region $r > 1$, and that they will never reach any observer A in the region $r > 1$ measuring in t-time and so situated at very large r (R, say). That is, no information from $r \leq 1$ on the radial line of sight may be carried to the specified observer A by particles or light rays travelling radially from those points. Thus on the radial line of sight, only events occurring in $r > 1$ are observable to A and this applies to all possible radial lines of sight.

Examination of equations analogous to (6.8) to (6.10) for light rays will show us whether signals from anywhere on the surface r = 1 may reach A by any non-radial orbits. For light rays , ds = 0 and the constants C and p become infinite but their ratio may be replaced by a finite constant (Darwin, 1959). Thus $\frac{p}{C} = \ell$, so that if τ is taken as the independent variable we have for the non-radial motion of light rays in the plane $\theta = \frac{\pi}{2}$

$$\frac{1}{\left(1-\frac{1}{r}\right)}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 - \left(1-\frac{1}{r}\right) = 0 \tag{6.22}$$

and
$$\frac{d\phi}{d\tau} = \frac{\ell\left(1-\frac{1}{r}\right)}{r^2} \tag{6.23}$$

Substitution of (6.23) into (6.22) gives

$$\frac{dr}{d\tau} = \pm\left(1-\frac{1}{r}\right)\left[1-\frac{\ell^2}{r^2}\left(1-\frac{1}{r}\right)\right]^{\frac{1}{2}} \tag{6.24}$$

Comparison of (6.24) and (6.20) shows that radial motion is given by the case $\ell = 0$.

(vii) <u>The reception of information by an observer stationary at large r.</u>

Some light rays will be projected so as never to approach r = 1 and these do not concern us since they may carry no information from r $\leq$ 1. It may well be that some light rays spiral in to r = 1, approaching the surface asymptotically so that ϕ does not tend to a limit as $r \to 1$, or they may have this motion in the reverse direction. If so, the co-ordinate travel time to or from $r = 1$ will necessarily be infinite and the observer A may gain no information from them about $r \leqslant 1$.

The final possibility is that ϕ tends to a limit as $r \to 1$, so that in the limit the inward travelling light ray actually meets the surface $r = 1$ and the light ray travelling in the outward direction actually starts from the surface. Suppose that the observer A lies on such a non-radial null geodesic and so is connected with the surface $r = 1$. What will be the light-travel-time, measuring in t-time, for photons travelling between $r = 1$ and the observer A?

In the neighbourhood of $r = 1$, (6.24) shows that $\pm \frac{dr}{d\tau}$ behaves like $(1 - \frac{1}{r})$, so that we have the same kind of singularity as we found in (6.20). Integrating for the neighbourhood of $r = 1$ only and allowing $r \to 1$ similarly yields that the co-ordinate light-travel-time between $r = 1$ and the observer A will be infinite. This is so whatever the limiting value of ϕ, if it exists, and holds a fortiori for the motion of particles too. Hence our specified observer A may receive information from nowhere in the whole region $r \leqslant 1$.

(viii) The barrier as an event horizon for A.

To obtain further insight into this situation suppose that light signals are sent out towards the stationary observer A, situated at large r (= R) and so measuring in t-time, by any test particle B falling freely in radial or non-radial motion towards the barrier. (6.21) and (6.24) show that as the r co-ordinate of emission of the signals from B decreases, the light-travel-time from B to A of the signals increases and tends to infinity as $r \rightarrow 1$, that is, as B approaches the barrier. Thus, although a particle may evidently reach the barrier in its finite life-time, the observer A will need an infinite time to verify this fact. As received by A, the history of the particle B becomes more and more dilated as B approaches the barrier. According to the work of Darwin (1961), Kruskal (1960) and Fronsdal (1959) the test particle B, reaching the barrier at r = 1, must inevitably pass into the region $r < 1$. Whether or not this is so, no event occurring at B after reaching r = 1 may ever be seen by A, whatever the subsequent motion of B; the event of the particle B reaching the barrier is the last event in B's history which A may observe, because of the infinite time in A's experience that light emitted at this event will take to reach A. No signal from this event may reach A sooner than this, causality being preserved in the Schwarzschild space-time (Cf. Fuller and Wheeler (1962); Kruskal (1960)). Thus although B will be forever in A's view, since light emitted by B on his journey to the barrier will always be travelling towards A, if any events occur at B after his reaching the barrier, they may not be seen by A.

No events which occur inside the region $r \leqslant 1$ may ever be seen by an observer A who measures in t-time in the region $r > 1$; moreover, measuring in t-time, we know that particles such as B which have once reached $r = 1$ or any which may have originated in $r < 1$ may not afterwards penetrate into the region $r > 1$. We may deduce immediately that for the specified observer A, the surface $r = 1$ is an event horizon, for it separates all events into two classes: those which are in principle observable to A and those which are not. Should the class of non-observable events be empty, then the event horizon will be of a degenerate type: we shall return to this point later.

(ix) Radial motion of the observer; the spectral displacement of light received from a radially moving probe.

So far the conclusion has been demonstrated only for an observer (A) who is stationary at large r and so measuring in t-time. The question remains whether any possible motion of the observer in $r > 1$ would allow him to observe events which occurred within the region $r \leqslant 1$ and whether he may perform any experiments which would afford him knowledge of that region. Let us therefore investigate the reception of a signal, emitted by a particle at an r co-ordinate arbitrarily greater than unity, by an observer confined to the region $r > 1$ but allowed any motion which is possible in principle within that region; we shall consider first purely radial motion of both the emitting particle B and the receiving observer A.

Suppose B emits at time τ_i from $r = r_i$ a light wave of period $\Delta\tau_i$ and that this wave is received by A at $r = r_0 > r_i$ at a time $\tau = \tau_0$. The motion of the light wave is along the null geodesic joining B to A and is given by (6.20), taking the positive sign since r is increasing, viz.

$$\frac{dr}{d\tau} = \left(1 - \frac{1}{r}\right) \qquad (6.25)$$

During the time of emission of the wave $\Delta\tau_i$, B will have moved from $r = r_i$ to $r = r_i + \Delta r_i$ where Δr_i may be positive or negative, depending on whether B is receding from or approaching the barrier at $r = 1$. Suppose that the light wave which has period $\Delta\tau_i$ at emission arrives at A during a time interval $\Delta\tau_o$ and that during the reception of the wave A has moved from $r = r_o$ to $r = r_o + \Delta r_o$ where again Δr_o may be positive or negative according to the direction of motion of A.

From (6.25) we therefore have

$$\tau_o - \tau_i = \int_{r_i}^{r_o} \frac{dr}{(1-\frac{1}{r})} \tag{6.26}$$

and
$$(\tau_o + \Delta\tau_o) - (\tau_i + \Delta\tau_i) = \int_{r_i+\Delta r_i}^{r_o+\Delta r_o} \frac{dr}{(1-\frac{1}{r})} \tag{6.27}$$

Subtracting (6.26) from (6.27) we get

$$\Delta\tau_o - \Delta\tau_i = \left(\int_{r_i+\Delta r_i}^{r_o+\Delta r_o} - \int_{r_i}^{r_o} \right) \frac{dr}{(1-\frac{1}{r})} \tag{6.28}$$

$$= \Big[r + \ln|r-1| \Big]_{r_i+\Delta r_i}^{r_o+\Delta r_o} - \Big[r + \ln|r-1| \Big]_{r_i}^{r_o}$$

$$\Delta\tau_o - \Delta\tau_i = \Delta r_o - \Delta r_i + \ln\left| \frac{1 + \frac{\Delta r_o}{(r_o-1)}}{1 + \frac{\Delta r_i}{(r_i-1)}} \right| \tag{6.29}$$

For Δr_o , Δr_i small both $\frac{\Delta r_o}{r_o-1}$ and $\frac{\Delta r_i}{r_i-1}$ will be less than unity and we may expand the final term of

(6.29) in terms of these quantities. We get

$$\Delta\tau_o - \Delta\tau_i = \Delta r_o + \frac{\Delta r_o}{(r_o-1)} - \frac{1}{2}\left(\frac{\Delta r_o}{r_o-1}\right)^2 + \cdots$$
$$- \Delta r_i - \frac{\Delta r_i}{(r_i-1)} + \frac{1}{2}\left(\frac{\Delta r_i}{r_i-1}\right)^2 - \cdots$$

Neglecting squares and higher orders, we have to the first order

$$\Delta\tau_o - \Delta\tau_i = \frac{\Delta r_o}{1-\frac{1}{r_o}} - \frac{\Delta r_i}{1-\frac{1}{r_i}} \tag{6.30}$$

Relative to the co-ordinate system (t, r, θ, ϕ) the velocity of a particle is defined (McVittie, 1956a) to be the non-tensor quantity V where

$$V^2 = \frac{1}{(1-\frac{1}{r})}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 \tag{6.31}$$

so that for radial motion

$$V^2 = \frac{1}{(1-\frac{1}{r})}\left(\frac{dr}{dt}\right)^2 \tag{6.32}$$

where, from (6.13)

$$\frac{dr}{dt} = \pm\frac{(c^2-1+\frac{1}{r})^{\frac{1}{2}}(1-\frac{1}{r})}{c} \tag{6.33}$$

so that

$$V = \pm\frac{(c^2-1+\frac{1}{r})^{\frac{1}{2}}(1-\frac{1}{r})^{\frac{1}{2}}}{c} \tag{6.34}$$

From (6.32) we see that the velocity of B during the emission of the light wave is given by

$$V_B = \frac{1}{(1-\frac{1}{r_i})^{\frac{1}{2}}}\,\frac{\Delta r_i}{\Delta\tau_i} \tag{6.35}$$

and the velocity of A during the reception of the same wave is given by

$$V_A = \frac{1}{(1-\frac{1}{r_o})^{\frac{1}{2}}}\,\frac{\Delta r_o}{\Delta\tau_o} \tag{6.36}$$

where V_A , V_B may be positive or negative depending on the signs of Δr_o, Δr_i respectively. Substituting (6.35) and (6.36) into (6.30), we get

$$\Delta\tau_o = \Delta\tau_i \frac{\left(1 - \frac{V_B}{(1-\frac{1}{r_i})^{1/2}}\right)}{\left(1 - \frac{V_A}{(1-\frac{1}{r_o})^{1/2}}\right)} \tag{6.37}$$

which gives the relation between the period of the light wave on emission at B and its period on reception at A. We see from (6.37) that if both V_A and V_B are zero, the period of the light wave suffers no change in magnitude on its journey from B to A.

On arrival at A the light wave will be compared with a similar light wave to determine the shift in wavelength. Following McVittie (1956b) we define the term similar by the statement that two waves are similar if the interval Δ s between the beginning and ending of the emission of the wave is the same for each, this being an invariant condition. From (6.11) we have, for the wave emitted at B,

$$\Delta s_i^2 = \left(1-\frac{1}{r_i}\right)\Delta\tau_i^2 - \frac{\Delta r_i^2}{\left(1-\frac{1}{r_i}\right)} \tag{6.38}$$

which, using (6.35), may be written

$$\Delta s_i^2 = \Delta\tau_i^2\left(1-\frac{1}{r_i}\right)\left[1 - \frac{V_B^2}{\left(1-\frac{1}{r_i}\right)}\right] \tag{6.39}$$

Let us now consider a light wave at A of period $\delta\tau_0$, the events of beginning and ending of emission being (τ_0, r_0) and ($\tau_0+\delta\tau_0$, $r_0+\delta r_0$) respectively. By (6.11) the interval between these events is given by

$$\delta s_0^2 = \left(1-\frac{1}{r_0}\right)\delta\tau_0^2 - \frac{\delta r_0^2}{\left(1-\frac{1}{r_0}\right)} \tag{6.40}$$

Both events occur at the particle A, so that by (6.32).

$$\delta r_0^2 = \delta\tau_0^2\, V_A^2\left(1-\frac{1}{r_0}\right) \tag{6.41}$$

and (6.40) may be written

$$\delta s_0^2 = \delta\tau_0^2\left(1-\frac{1}{r_0}\right)\left[1-\frac{V_A^2}{\left(1-\frac{1}{r_0}\right)}\right] \tag{6.42}$$

For this wave to be similar to that emitted by B, we apply the condition:

$$\delta s_0 = \Delta s_i$$

Equating (6.39) and (6.42) yields

$$\frac{\Delta\tau_i^2}{\delta\tau_0^2} = \frac{\left(1-\frac{1}{r_0}\right)\left(1-\frac{V_A^2}{\left(1-\frac{1}{r_0}\right)}\right)}{\left(1-\frac{1}{r_i}\right)\left(1-\frac{V_B^2}{\left(1-\frac{1}{r_i}\right)}\right)} \tag{6.43}$$

Eliminating $\Delta\tau_i$ by means of (6.37) we get

$$\frac{\Delta\tau_0^2}{\delta\tau_0^2} = \frac{\left(1-\frac{1}{r_0}\right)\left(1-\frac{V_A^2}{\left(1-\frac{1}{r_0}\right)}\right)\left(1-\frac{V_B}{\left(1-\frac{1}{r_i}\right)^{1/2}}\right)^2}{\left(1-\frac{1}{r_i}\right)\left(1-\frac{V_B^2}{\left(1-\frac{1}{r_i}\right)}\right)\left(1-\frac{V_A}{\left(1-\frac{1}{r_0}\right)^{1/2}}\right)^2}$$

so that

$$\frac{\Delta\tau_0^2}{\delta\tau_0^2} = \frac{\left(1-\frac{1}{r_0}\right)\left(1+\frac{V_A}{\left(1-\frac{1}{r_0}\right)^{1/2}}\right)\left(1-\frac{V_B}{\left(1-\frac{1}{r_i}\right)^{1/2}}\right)}{\left(1-\frac{1}{r_i}\right)\left(1-\frac{V_A}{\left(1-\frac{1}{r_0}\right)^{1/2}}\right)\left(1+\frac{V_B}{\left(1-\frac{1}{r_i}\right)^{1/2}}\right)} \tag{6.44}$$

If $\lambda + d\lambda$ is the wavelength of the wave emitted from B when it reaches A, and λ is the wavelength of the similar wave at A, we have

$$\lambda + d\lambda = p\,\Delta\tau_0 \quad ; \quad \lambda = p\,\delta\tau_0$$

where p is the velocity of light at $r = r_0$. (6.44) then gives

$$1 + \frac{d\lambda}{\lambda} = \left[\frac{\left(1-\frac{1}{r_0}\right)\left(1+\frac{V_A}{\left(1-\frac{1}{r_0}\right)^{1/2}}\right)\left(1-\frac{V_B}{\left(1-\frac{1}{r_i}\right)^{1/2}}\right)}{\left(1-\frac{1}{r_i}\right)\left(1-\frac{V_A}{\left(1-\frac{1}{r_0}\right)^{1/2}}\right)\left(1+\frac{V_B}{\left(1-\frac{1}{r_i}\right)^{1/2}}\right)}\right]^{1/2} \tag{6.45}$$

We see that even if both A and B are at rest within the co-ordinate system, so that $V_A = V_B = 0$ there will be a shifting of the spectral lines; (6.45) includes displacements due to both gravitational and Doppler effects.

Recalling equation (6.34), we have

$$V_A = \pm \frac{\left(C_A^2 - 1 + \frac{1}{r_0}\right)^{1/2}\left(1-\frac{1}{r_0}\right)^{1/2}}{C_A} \tag{6.46}$$

and

$$V_B = \pm \frac{\left(C_B^2 - 1 + \frac{1}{r_i}\right)^{1/2}\left(1-\frac{1}{r_i}\right)^{1/2}}{C_B} \tag{6.47}$$

where, since we are considering the region $r > 1$ only, we have both C_A and $C_B > 0$, as demanded by (6.10).

Consider first the case when the observer B is approaching the barrier at $r = 1$; then V_B is negative and we have from (6.45), using (6.47),

$$1 + \frac{d\lambda}{\lambda} = \left[\mathcal{A}\,\frac{\left[1 + \frac{1}{C_B}\left(C_B^2 - 1 + \frac{1}{r_i}\right)^{1/2}\right]}{\left(1-\frac{1}{r_i}\right)\left[1 - \frac{1}{C_B}\left(C_B^2 - 1 + \frac{1}{r_i}\right)^{1/2}\right]}\right]^{1/2} \tag{6.48}$$

where

$$\mathcal{A} = \frac{\left(1-\frac{1}{r_0}\right)\left(1+\frac{V_A}{(1-\frac{1}{r_0})^{1/2}}\right)}{\left(1-\frac{V_A}{(1-\frac{1}{r_0})^{1/2}}\right)} \tag{6.49}$$

If the observer A, confined to the region $r > 1$, is fixed at some r_0 so that $V_A = 0$, $\mathcal{A} = (1 - \frac{1}{r_0}) > 0$; if A has some permitted motion within that region, then $\mathcal{A}$ has a positive value which may be greater than or less than unity depending on the sign and value of V_A at any instant. In all cases, the right hand side of (6.48) tends to ∞ as $r_i \to 1$ for any value of C_B; that is, the red-shift of the spectral lines received from B increases without limit as B approaches the barrier, and the observer A receives less and less of the information sent out by B; in the limit of B reaching $r = 1$, A can receive no information at all from B. We have thus demonstrated that no possible radial motion of the observer A within the region $r > 1$ will enable him to receive information about the region $r \leqslant 1$ from the inward travelling particle who sends back radial light signals, for the signals received will have no information content.

Should the particle B be travelling away from the barrier radially towards A, we take the positive sign in (6.47) and get

$$1 + \frac{d\lambda}{\lambda} = \left[\frac{\mathcal{A}\left\{1 - \frac{1}{C_B}\left(C_B^2 - 1 + \frac{1}{r_i}\right)^{1/2}\right\}}{\left(1-\frac{1}{r_i}\right)\left\{1 + \frac{1}{C_B}\left(C_B^2 - 1 + \frac{1}{r_i}\right)^{1/2}\right\}}\right]^{1/2} \tag{6.50}$$

For the limit of $d\lambda$ as $r_i \to 1$, we use l'Hôpital's rule which yields

$$\lim_{r_i \to 1} \left(1 + \frac{d\lambda}{\lambda}\right) = \frac{\mathcal{A}^{\frac{1}{2}}}{2C_B}$$

so that the limiting red-shift in this case has a finite value and on this score may carry information, in the limit, from $r = 1$.

(x) <u>The recovery of a radially moving probe B by a freely moving observer travelling radially towards the barrier.</u>

We now investigate whether or not it is possible for such signals from the outward moving B ever to reach any observer A in the region $r > 1$. Consider a probe B sent out by a freely moving observer A radially towards the barrier at $r = 1$ to a point $r = x > 1$, at which point the motion of B is reversed; to avoid dynamical questions about the reversal, we may suppose that a second object B^1 passes B at $r = x$ with reversed velocity, synchronising his clock with that of B in passing. We shall allow the observer A any permitted radial motion within the region $r > 1$ provided only that B approaches the barrier faster than A does so that the essential character of the experiment is maintained. This implies, from (6.13), that if A moves towards the barrier in the same direction as B relative to the co-ordinate system, we must have $C_B > C_A$.

Let A and B start together from r = R (say) at t = 0, s = 0. It is required to find at which r co-ordinate they meet again and the lapse of time in A's experience before this happens in the particular case when x is allowed to tend to 1 so that B may explore the barrier. Results for the various cases are illustrated by Figures 6.1 a, b, c.

We take first the case when initially A moves towards the barrier at r = 1, so that $r \leqslant R$.

(A) $\underline{C_B > 1}$

From (6.17) the time taken for B to travel from $r = R$ to $r = x < R$ and from r = x outwards again to r = r is given by

$$t_B(r) = \Big[L_B + M_B - N_B\Big]_x^R + \Big[L_B + M_B - N_B\Big]_x^r \qquad (6.51)$$

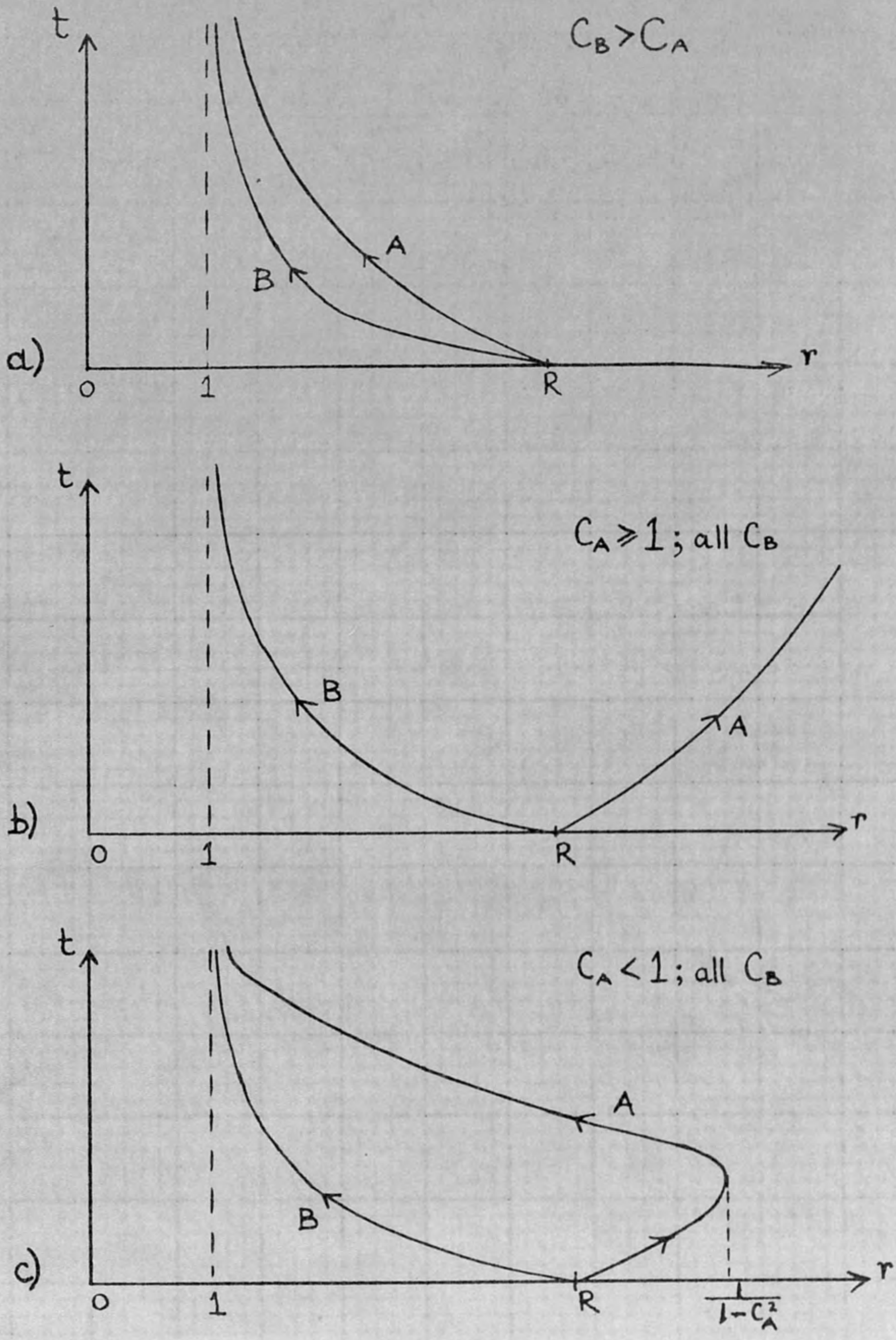
where we have written

$$L(r) = \frac{(K^2+1)^{\frac{1}{2}}}{K^3}\left[(K^2r+1)^{\frac{1}{2}}Kr^{\frac{1}{2}} + \frac{(2K^2-1)}{2}\ln\frac{(K^2r+1)^{\frac{1}{2}}+Kr^{\frac{1}{2}}}{(K^2r+1)^{\frac{1}{2}}-Kr^{\frac{1}{2}}}\right] \qquad (6.52)$$

$$M(r) = \ln\left|\frac{K^2r^{\frac{1}{2}}+1-(K^2+1)^{\frac{1}{2}}(K^2r+1)^{\frac{1}{2}}}{(K^2r+1)^{\frac{1}{2}}-(K^2+1)^{\frac{1}{2}}}\right| \qquad (6.53)$$

$$N(r) = \ln\left|\frac{K^2r^{\frac{1}{2}}+1+(K^2+1)^{\frac{1}{2}}(K^2r+1)^{\frac{1}{2}}}{(K^2r+1)^{\frac{1}{2}}+(K^2+1)^{\frac{1}{2}}}\right| \qquad (6.54)$$

Fig. 6·1


t
$C_B > C_A$
A
B
a)
0
1
R
r
t
$C_A \geq 1$; all C_B
B
A
b)
0
1
R
r
t
$C_A < 1$; all C_B
A
B
c)
0
1
R
$1-C_A^2$
r

and we recall $K^2 = C^2 - 1$. The suffix (A or B) refers throughout to K values for that particle and R, x, r indicate the limits between which these expressions are to be evaluated.

(a) $C_B > C_A > 1$

The time taken for A to travel between r = R and r = r < R is given, from (6.17), by

$$t_A(r) = \Big[L_A + M_A - N_A\Big]_r^R \qquad (6.55)$$

Setting $t_A(r) = t_B(r)$, we find by inspection that as $x \to 1$ we must have $r \to 1$ too; then the second term on the R.H.S. of (6.51) vanishes altogether and the remaining terms in L and N remain finite, whereas both $\Big[M_B\Big]_1^R$ in (6.51) and $\Big[M_A\Big]_1^R$ in (6.55) become infinite. Our solution for x = 1 is therefore given by r = 1; this is the only solution, for with x = 1 $t_B(r) = \infty$ for all $r \geqslant 1$ but $t_A(r) = \infty$ only for r = 1. Substituting in (6.14) we find the proper travel time before A and B meet again is given by

$$S = \frac{1}{K^3}\left[K r^{\frac{1}{2}}(K^2r+1)^{\frac{1}{2}} - \frac{1}{2}\ln\frac{(K^2r+1)^{\frac{1}{2}} + K r^{\frac{1}{2}}}{(K^2r+1)^{\frac{1}{2}} - K r^{\frac{1}{2}}}\right]_{r=1}^{r=R} \qquad (6.56)$$

an expression which holds for both A and B upon substitution of the corresponding value for C. We see that both s_A and s_B are finite, but have different values.

(b) $C_B > 1$; $C_A = 1$

By (6.18) the travel time for A is now given by

$$t_A(r) = \left[P \right]_r^R \tag{6.57}$$

where $$P(r) = \frac{2}{3} r^{3/2} + 2\sqrt{r} + \ln \frac{\sqrt{r}-1}{\sqrt{r}+1} \tag{6.58}$$

Equating this with (6.51) we again find that when $x = 1$ both equations are satisfied simultaneously only when $r = 1$; (6.57) clearly tends to infinity as $r \to 1$. In this case, the proper travel time for A is again finite and is given, from (6.15), by

$$s = \tfrac{2}{3}\left(R^{3/2} - 1\right) \tag{6.59}$$

with s_B given by (6.56).

(c) $C_B > 1$; $C_A < 1$

By (6.19), putting

$$Q(r) = \frac{(1-k^2)^{1/2}}{k^3}\left[-(1-k^2 r)^{1/2} k r^{1/2} + (1+2k^2) \arcsin (1-k^2 r)^{1/2} \right] \tag{6.60}$$

$$\mathcal{R}(r) = \ln \left| \frac{k^2 r^{1/2} + 1 - (1-k^2)^{1/2} (1-k^2 r)^{1/2}}{(1-k^2 r)^{1/2} - (1-k^2)^{1/2}} \right| \tag{6.61}$$

$$\mathcal{S}(r) = \ln \left| \frac{k^2 r^{1/2} + 1 + (1-k^2)^{1/2} (1-k^2 r)^{1/2}}{(1-k^2 r)^{1/2} + (1-k^2)^{1/2}} \right| \tag{6.62}$$

where $k^2 = 1 - C^2$, we have

$$t_A(r) = \Big[Q_A - \mathcal{R}_A + S_A \Big]_r^R \qquad (6.63)$$

provided for this case that $R \leq \frac{1}{k_A^2}$. Again, putting $x = 1$ in (6.51) we find the only solution for $t_A(r) = t_B(r)$ is given by $r = 1$; for in (6.63) both Q and S remain finite, while $\mathcal{R}(r)$ is infinite at $r = 1$. For $C_A < 1$, we have from (6.16)

$$S_A = \frac{1}{k_A^3}\Big[-k_A r^{\frac{1}{2}} (1 - k_A^2 r)^{\frac{1}{2}} + \arcsin\big(k_A r^{\frac{1}{2}}\big) \Big]_1^R \qquad (6.64)$$

with S_B given by (6.56).

(B) $\underline{C_B = 1; \ C_A < 1}$

For this case we equate

$$t_B(r) = \big[P_B\big]_x^R + \big[P_B\big]_x^r \qquad (6.65)$$

with $t_A(r)$ given by (6.63). Again, for $x = 1$, $r = 1$ is the required solution. Proper times before meeting are given by (6.64) for A and by (6.59) for B.

(C) $\underline{C_B < 1: \ C_A < 1; \ C_B > C_A}$

Travel times are now given by

$$t_B(r) = \Big[Q_B - \mathcal{R}_B + S_B \Big]_x^R + \Big[Q_B - \mathcal{R}_B + S_B \Big]_x^r \qquad (6.66)$$

and by (6.63) for the observer A. Equating (6.66) and (6.63) and allowing x to tend to 1, we find again that $r = 1$ is the only solution. Proper travel times for both A and B are given by (6.64) when the corresponding values are inserted.

Thus in all cases when the freely-moving observer A travels radially towards the barrier at $r = 1$, A is unable to explore the region $r < 1$ or the surface $r = 1$ while in the region $r > 1$ by means of sending in a probe; for he must himself reach $r = 1$ in order to regain the probe. That they may not coincide again at some $r > 1$ is seen from our equations above; for in all cases when $x = 1$, $t_B(r)$ is infinite and for $R > r > 1$ the corresponding $t_A(r)$ is always finite.

(xi) <u>The recovery of B by a freely moving observer travelling radially away from the barrier.</u>

Suppose now that the freely moving A initially travels radially outwards from the barrier, sending the probe B towards the barrier as before; since A, B travel initially in opposite directions we do not now have the restriction $C_B > C_A$. For this case however we must allow the r co-ordinate of A to be greater than R, the r co-ordinate of the starting point at $t = 0$, $s = 0$.

(A) $C_B > 1$

$t_B(r)$ is again given by (6.51) for this case.

(a) $\underline{C_A > 1}$

The time taken for A to travel between $r = R$ and $r = r > R$ is given, from (6.17), by

$$t_A(r) = \left[L_A + M_A - N_A\right]_R^r \tag{6.67}$$

Allowing x to tend to 1 in (6.51) we find, as before, that $t_B(r)$ has an infinite value for all $r \geqslant 1$. For $r > R > 1$, only when r is infinite will $t_A(r)$ be infinite; then in (6.67), $L_A(r)$ is infinite while $M_A(r)$ and $N_A(r)$ remain finite for all permitted k_A. Thus the only valid solution for $t_A(r) = t_B(r)$ is given by $r = \infty$ so that A and B will meet again only at infinity, after an infinite time has elapsed, by (6.14), in the experience of each of them.

(b) $\underline{C_A = 1}$

We have now

$$t_A(r) = \left[P\right]_R^r \tag{6.68}$$

where P is given by (6.58). Similarly, we find that A, B meet again only at infinity, if the turning point x of B is allowed to tend to 1, for $t_A(r)$ is infinite only when $r = \infty$. (6.15) shows that this occurs after an infinite time in A's experience, with s_B infinite too by (6.14).

(c) $\underline{C_A < 1}$

For $x = 1$, we again have, from (6.51), that $t_B(r)$ is infinite for all $r \geqslant 1$. For the case $C_A < 1$

we have the motion of A restricted by the condition $r \leq \frac{1}{k_A^2}$ where $k_A^2 = 1 - C_A^2$. The time of travel $\left[t_A\right]_R^{1/k_A^2}$ between $r = R$ and $r = \frac{1}{k_A^2}$ is given by

$$\left[t_A\right]_R^{\frac{1}{k_A^2}} = \left[Q_A - \mathcal{R}_A + S_A\right]_R^{r=\frac{1}{k_A^2}} \tag{6.69}$$

Taking principal values, we get upon evaluation

$$\left[t_A\right]_R^{\frac{1}{k_A^2}} = -\left\{Q_A(R) - \mathcal{R}_A(R) + S_A(R)\right\} \tag{6.70}$$

since the expression vanishes at the upper limit. This is finite and so B will not have rejoined A before A reaches $r = \frac{1}{k_A^2}$.

From (6.12) and (6.13), we see that both $\frac{dr}{ds}$ and $\frac{dr}{dt}$ are zero at $r = \frac{1}{k^2}$ and hence V is zero at this point by (6.32). Differentiation of these equations shows that a freely moving particle with $C < 1$ at $r = \frac{1}{k^2}$ will subsequently move in the direction of decreasing r. Thus the direction of motion of the freely moving observer A will be reversed at $r = \frac{1}{k_A^2}$ and A will start to approach the barrier at $r = 1$, its r co-ordinate continually decreasing.

Since we know by comparison of (6.70) and (6.51) that B will not have rejoined A before this reversal takes place, we have the travel time of A given by

$$t_A(r) = -\{Q_A(R) - \mathcal{R}_A(R) + S_A(R)\} + \left[Q_A - \mathcal{R}_A + S_A\right]_r^{1/k_A^2}$$
$$= \left[\mathcal{R} - Q - S\right](r) + \left[\mathcal{R} - Q - S\right](R) \qquad (6.71)$$

using (6.70), where $r < \frac{1}{k_A^2}$.

Equating $t_A(r)$ of (6.71) with the infinite $t_B(r)$ we find that the only possible solution is given by $r = 1$. Thus A and B meet again only at $r = 1$ after a finite lapse of time in A's experience found by evaluating (6.16) between the appropriate limits; s_B is again given by (6.56).

(B) $C_B = 1$

In the same way, it is easily seen that similar results will hold for the case $C_B = 1$: for $C_A \geq 1$, A and B meet again only at $r = \infty$ after an infinite lapse of time in the experience of both of them; for $C_A < 1$, they meet again at $r = 1$, when s_A, s_B will be finite but different.

(C) $C_B < 1$

The travel time for B is given from (6.19) by (6.66), viz.

$$t_B(r) = \left[Q_B - \mathcal{R}_B + S_B\right]_x^R + \left[Q_B - \mathcal{R}_B + S_B\right]_x^r$$

where $Q, \mathcal{R}, S$ are given by (6.60) to (6.62) and $R, r \leq \frac{1}{k_B^2}$.

(a) $\underline{C_A > 1}$:

For this case, (6.67) gives

$$t_A(r) = \Big[L_A + M_A - N_A\Big]_R^r$$

where $r > R > 1$, but from above $R \leq \frac{1}{k_B^2}$. If x is allowed to tend to 1, there exists no solution for r of the equation $t_A(r) = t_B(r)$ which satisfies the conditions upon r, viz. $1 < r \leq \frac{1}{k_B^2}$, for t_B is infinite while t_A remains finite in this range.

(c) $\underline{C_A = 1}$:

Now $t_A(r) = \Big[P\Big]_R^r$ for $r > R > 1$.

In the same way, there exists no solution for r of $t_A(r) = t_B(r)$ which satisfies the conditions.

(c) $\underline{C_A < 1}$

We have $t_B(r)$ given by (6.66) which is infinite for all r in the permitted range $1 \leq r \leq \frac{1}{k_B^2}$ when x = 1. $t_A(r)$ is given by (6.71) for $1 \leq r \leq \frac{1}{k_A^2}$, which is infinite only for r = 1, and so this is the required solution. If A, B are to start together at r = R, we must have R less than both $\frac{1}{k_A^2}$ and $\frac{1}{k_B^2}$.

A and B meet at r = 1 after a finite lapse of time in the experience of each of them, found in each case by evaluating (6.16) between the appropriate limits and inserting the correct k value.

We have thus demonstrated that the observer A, freely moving in a radial direction, may not be rejoined by the radially moving probe B at any point in the region $r > 1$ in his finite experience; for either A and B never rejoin, or they meet at $r = 1$, or they meet after an infinite proper time.

(xii) Extension to the case of light rays emitted from the barrier.

In particular, this will hold for the case $C_B = \infty$ corresponding to the propagation of a light ray, as inspection of (6.21) immediately shows. By (6.21) the signal emitted at $r = 1$ at finite t ($= 0$, say) will arrive at any $r = r_a > 1$ an infinite time later; in fact, we have the travel time given by

$$t_a = \Big[r + \ln|r-1| \Big]_{r=x}^{r=r_a} \tag{6.72}$$

where t_a is the time at which the signal arrives at $r = r_a$ and $t_a \to \infty$ as $x \to 1$. The radial motion of A is given by one of the equations (6.55), (6.57), (6.63), (6.67), (6.68), (6.71) under the stated conditions, where we take $r = R > 1$ at $t = 0$ where R is finite. For the signal emitted at $r = 1$ to reach the freely moving A, they must have the event (t_a, r_a, θ, ϕ) in common, where t_a is infinite.

Suppose that the conditions on θ and ϕ are satisfied: then our previous work demonstrates the following results (if R is greater than unity and finite only three possibilities arise for r_a when t_a for A is infinite):-

(a) an r_a common to both the signal and A does not exist

(b) $r_a = \infty$, in which case the lapse of time in A's experience before the signal reaches A is infinite

(c) $r_a = 1$; but then t_a for the signal is zero by (6.72) and the condition on t is not fulfilled.

If R should be infinite, then t_a for A is infinite for any finite value of r; conditions on both r_a and t_a will be satisfied for some finite $r_a > 1$. Again the lapse of time in A's experience before meeting the signal will be infinite. A final possibility is that conditions on θ or ϕ may not be satisfied simultaneously with those on t, r.

Under all circumstances it follows that whatever the radial motion of A the signal travelling radially outward from r = 1 (which it was shown in a previous section may possibly have information content, when only finitely red-shifted) may never reach A in $r > 1$ in his finite experience. Examination of (6.24) in the neighbourhood of r = 1 shows that a similar analysis will yield the same conclusions for signals emitted from r = 1 in non-radial motion; moreover, this applies for any plane through r = 0.

We may conclude immediately from the results now established that no observer in $r > 1$ able to move freely in any manner in radial motion may receive information from the region $r \leqslant 1$ in his own finite experience.

(xiii) <u>Extension to the case of a non-radially moving observer.</u>

A similar analysis leads us to believe that this conclusion may be extended to cover all freely moving observers in $r > 1$, with no restriction whatever on the observer's motion. It is sufficient for this purpose to establish now that no light ray emitted, in the limit, from r = 1 may reach a freely moving observer with arbitrary non-radial motion in $r > 1$ in that observer's finite experience.

Accordingly, consider the equations of motion of such an observer in any plane $\theta = \frac{\pi}{2}$ given by (6.8) to (6.10). Eliminating s and ϕ we get from (6.8)

$$\pm \frac{dr}{dt} = \left(1-\frac{1}{r}\right)\left[1- \frac{p^2}{C^2}\cdot\frac{1}{r^2}\left(1-\frac{1}{r}\right)\right]^{\frac{1}{2}} \qquad (6.73)$$

so that $$\pm \frac{dr}{ds} = C\left[1- \frac{p^2}{C^2}\cdot\frac{1}{r^2}\left(1-\frac{1}{r}\right)\right]^{\frac{1}{2}} \qquad (6.74)$$

and these equations apply too to the motion of light rays when $ds = 0$, $C = \infty$ and $\frac{p}{c}$ equals a finite constant, ℓ. According to (6.73), in the neighbourhood of $r = 1$ $\left|\frac{dt}{dr}\right|$ behaves like $\frac{1}{(1-\frac{1}{r})}$ so that, as before, all particles and light rays will, in the limit, take an infinite t - time to either reach $r = 1$ from finite $r > 1$ or to penetrate a finite distance into $r > 1$ from $r = 1$. In the neighbourhood of infinite r, $\left|\frac{dt}{dr}\right|$ will behave like unity by (6.73) so the co-ordinate travel time from finite r to $r = \infty$ of both particles and light rays will be infinite; moreover, for particles, (6.74) shows that in the neighbourhood of $r = \infty$, $\left|\frac{ds}{dr}\right|$ behaves like $\frac{1}{c}$ so that the proper travel time too for that journey would be infinite.

A more exact treatment for the case of non-radial motion is obtained by examining the integrated forms of equations (6.73) and (6.74) giving the orbits of particles in the model. Darwin (1959) has examined possible orbits that could be described round an attracting central mass, using methods first applied by Forsyth (1920) only to orbits of small eccentricity.

Suppose that a light ray is emitted from $r = x$

at $t = 0$, say, and that A is at any finite r (= R say) at that instant. For the signal to eventually reach A, they must have an event in common. If x is allowed to tend to 1 then t_a for both must tend to infinity if we maintain the condition $r_a > 1$. The considerations applied previously indicate that here too, in exactly the same way, this may occur, if at all in $r > 1$, only for $r_a = \infty$ when, as we have demonstrated, t_a for A is infinite. As before, if the signal does not reach A there is no problem and if $r_a = \infty$ the signal reaches A only after an infinite lapse of A's proper time.

Thus in the general case it seems that an observer A in $r > 1$ may never receive information from $r \leqslant 1$ in his own finite experience, and that no motion of his will ever enable him to do so.

(xiv) <u>The surface $r = 2m$ as a degenerate E.H.</u>

It follows that the hypersurface $r = 2m$ in the Schwarzschild space-time given by (6.4) is an event horizon, dividing events which are in principle observable to any fundamental observer in the region $r > 2m$ from events which are not; it is invariant in the sense that the same surface is an E.H. for all F.O's in $r > 2m$. It is a surface which is not permeable in the direction of r increasing to any causal influence whatsoever.

Since we have confined our attention to the consideration of experiments which might possibly be performed by a F.O. in $r > 2m$, we have not touched upon the question of whether or not the surface is actually permeable to particles or other causal influences in the direction of r decreasing. We have pointed out that particles may indeed reach the horizon in finite proper time, but that the continuity of the geodesic equations across this surface is due to mathematical convention; on each such geodesic there is a singular point at $r = 2m$ and a physical correlation between the two parts of the geodesic is not necessarily justifiable.

In this connection we should note that the invariance of the E.H. implies that the theory, for its own consistency, does not require events to occur in the region $r < 2m$, that is, beyond the E.H. This is in marked contrast to the situation in those models of the Robertson-Walker type which admit E,Hs.

In the latter, each F.O. has a different E.H. which depends on the state of his motion. In order that it remain true that at any cosmic instant any F.O. may see the same picture of the universe as any other F.O., it is necessary that events occur outside each E.H.; for if they did not, an observer X situated near the E.H. of an observer Y would not obtain the same picture as Y. In other words, for any F.O. A, an E.H. which occurs in a Robertson-Walker model divides events into two classes, these that are in principle observable and those that are not, and it is necessary that these two classes are <u>non-empty</u>.

In the Schwarzschild model, each F.O. in $r > 2m$ has the same E.H. and it is <u>not</u> necessary that the non-observable class of events is non-empty. Moreover, in a Robertson-Walker model, if X lies on Y's E.H. then Y lies on O's E.H.; this is not true in the Schwarzschild model. For these reasons, having already demonstrated in Chapter II (x) a connection between degeneracy and invariance, we consider the barrier at $r = 2m$ in the Schwarzschild space-time to be a <u>degenerate</u> E.H.

Bearing in mind the invariance of ds and in view of the fact that the existence of an E.H. depends on such features as the red-shift of light from $r = 2m$ and the lapse of proper time before the reception of the light, it is evident that the degenerate E.H. is of real physical significance in the model, dividing events which are in principle observable to the observer in $r > 2m$ from those which are not. We must conclude that the existence of this horizon in the Schwarzschild model is independent of the parameter t used to describe co-ordinate time. While acknowledging that in terms of t the manifold of the Schwarzschild space-time is incomplete, in view of our results we must reject the view of those authors quoted in the introductory section who maintain that the existence of a barrier at $r = 2m$ is due wholly or in part to the fact that t is a defective co-ordinate. No clock regraduation may remove the physical features associated with the horizon at $r = 2m$ in the Schwarzschild model, even though certain equations may thereby be made to appear continuous.

(xv) Darwin's problem.

Darwin (1961) has illustrated the features of the barrier in the Schwarzschild space-time given by (6.4) by considering an experiment which could be performed by two observers, one (A) placed at a great distance from the central mass at $r = 0$ and the other (B) travelling radially from A towards the mass, both being equipped with standard clocks. The experiment consists of each observer signalling to the other by means of sending out a light flash at each tick of his clock: each observer will count the number of flashes he receives from the other and will compare it with the number of ticks his own clock has made, all counts starting from the moment when B left A. B will be measuring in s-time and A in t-time.

Darwin shows that a radar experiment between A and B yields the well-known result that although it should take B only a short time to reach $r = 2m$, A requires an infinite time to verify this. Darwin's analysis also shows that the two flash-rates (i.e. the number of flashes received per tick of the observer's clock), which qualitatively measure rate of reception of information, are given by

$$F_A = C - \left(C^2 - 1 + \frac{2m}{r}\right)^{1/2}$$

$$F_B = \left[C + \left(C^2 - 1 + \frac{2m}{r}\right)^{1/2}\right]^{-1}$$

where C is the constant of integration introduced in

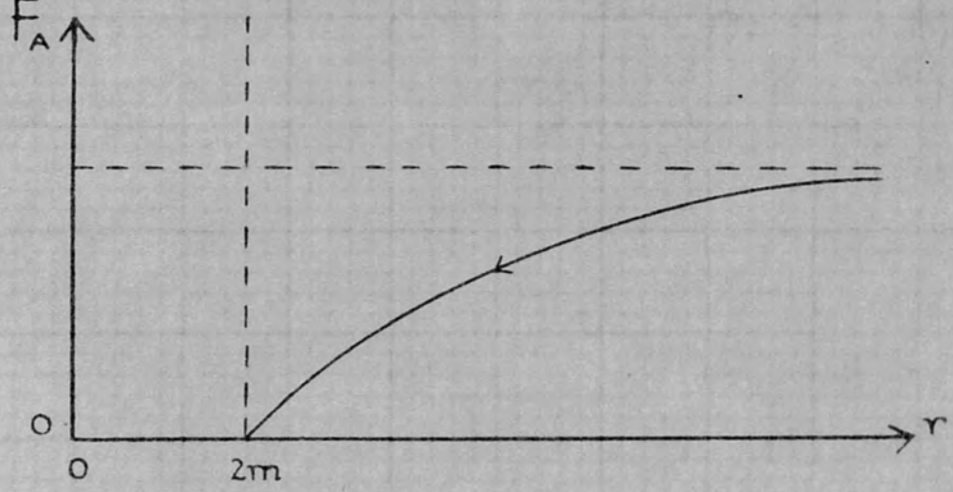


Fig 6.2

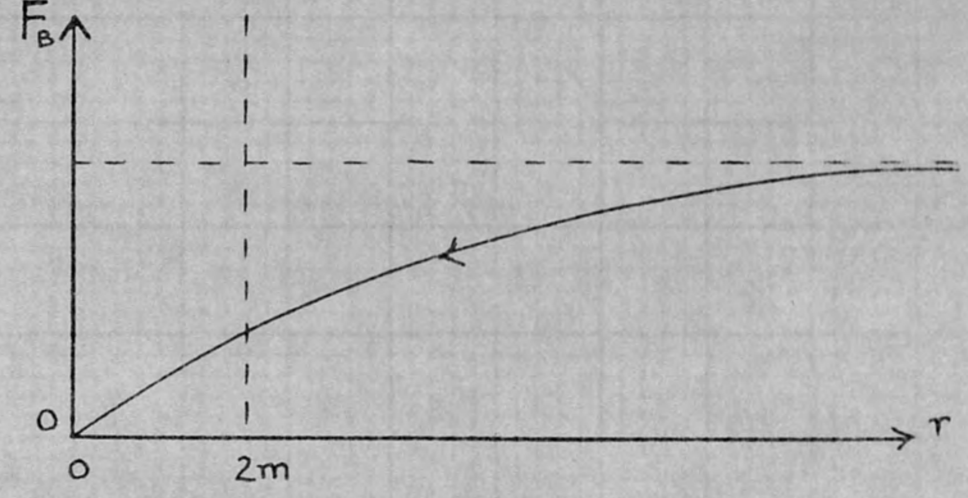


Fig 6.3

(6.7) and units have been chosen so that c, the velocity of light at infinity, is unity. For B, the flash-rate refers to the time when he is at r, but for A it refers to the past time when B was at r, the observation being made only later.

We may see from Figure 6.2 that as $r \rightarrow 2m$, $F_A \rightarrow 0$. Darwin believes that A would deduce from this that B had been in a critical situation at $r = 2m$. At the same point where B was supposed by A to be in a critical situation, B's flash-rate is finite and continuous, as may be seen from Figure 6.3; from F_B there seems to be no indication to B that $r = 2m$ is in any way peculiar.

Darwin finds these results "curious and unexpected": the fact that a light signal emitted at $r = 2m$ will take an infinite time to reach A suggests to Darwin that the use of radar appears not to be a very powerful method; he believes that the apparent contradiction illustrated by the behaviour of the flash-rates may well be attributable to a weakness in the flash-tick method and it seems to him that some new different experiment must be devised to throw some light on the matter.

The features of the surface $r = 2m$ which have been demonstrated and established in the previous sections lead us to disagree with Darwin's viewpoint. To begin with, even should it be possible for B, in principle, to penetrate the region $r < 2m$, the fact that no causal influence from $r \leq 2m$ may penetrate the region $r > 2m$ and reach any observer in that region makes it illegitimate for Darwin to suppose that a passenger on B could escape to $r > 2m$ by means of a rocket, say, just before B finally reaches the central mass, or that any second observer B^1 may pass B inside the region $r < 2m$ and subsequently enter $r > 2m$.

Moreover, we must remember that in Darwin's example the observer A at large r receives the zero flash-rate only after an infinite time and that B is always visible to A by the light emitted while B was in the region $r > 2m$; it seems incorrect to infer, as Darwin does, that A would believe from his observations of B that B had crashed at $r = 2m$.

The event horizon is an absolute feature of the Schwarzschild model in the sense that the same surface $r = 2m$ is an E.H. for all fundamental observers in the region $r > 2m$; in contrast to the situation in those of the Robertson-Walker models which allow E.Hs., no symmetric relationship exists between any two observers A and B such that if A lies in B's E.H. then B lies in A's E.H. Seen in this light it is not surprising that the rate of reception of flashes by A, F_A , shows a peculiarity when B is at $r = 2m$, whereas the flash-rate F_B does not. We maintain that, in principle, no experiment could be devised which would give results in contradiction to those achieved by the radar and flash-tick methods and that these methods are neither weak nor defective but reflect accurately the nature of the barrier at $r = 2m$ in the space-time under consideration; the realisation that this is a degenerate type of E.H. fully explains Darwin's otherwise curious results.

(xvi) Observation of a star behind the barrier.

If a Schwarzschild E.H. were present in empty space, no light from a sun within the barrier could penetrate into the outer region. The question arises as to whether an observer in this region could be aware of the existence of such an object and, if so, what features he would in fact observe. This may be investigated by considering such an object against the background of a star-field and in the presence of a star which passes slowly behind it and by examining the propagation of light rays from stars to observer in the neighbourhood of the object. Darwin (1959) has tackled this problem to some extent and for the sake of completeness we shall give a brief account of it here.

Regarding the propagation of light rays in this model, Darwin has shown that no light ray from infinity can escape capture unless its initial asymptotic distance is greater than $3\sqrt{3}$ m; in this limit, the orbit ends by approaching asymptotically towards a circle of radius 3m. We have shown that light rays may emerge radially from $r < 3m$, though not from $r < 2m$, and it is clear from Darwin's work and the fact that the dynamics in the region $r > 2m$ is reversible that light rays may in principle also spiral out from $r < 3m$.

Consider first what an observer would see of a star situated at a great distance $L_{\odot}$ from the sun (which is surrounded for the observer by an E.H.) in a plane through the observer's telescope and the centre (see Fig. 6.4). Rays of light whose equations have asymptotic distances at infinity less than $3\sqrt{3}$m will never reach the observer; other rays of light will pass from star to telescope, leaving the star at an angle χ_0 to the line joining star to sun. χ_0 is given by $\frac{l}{L_0}$ where l is the perpendicular

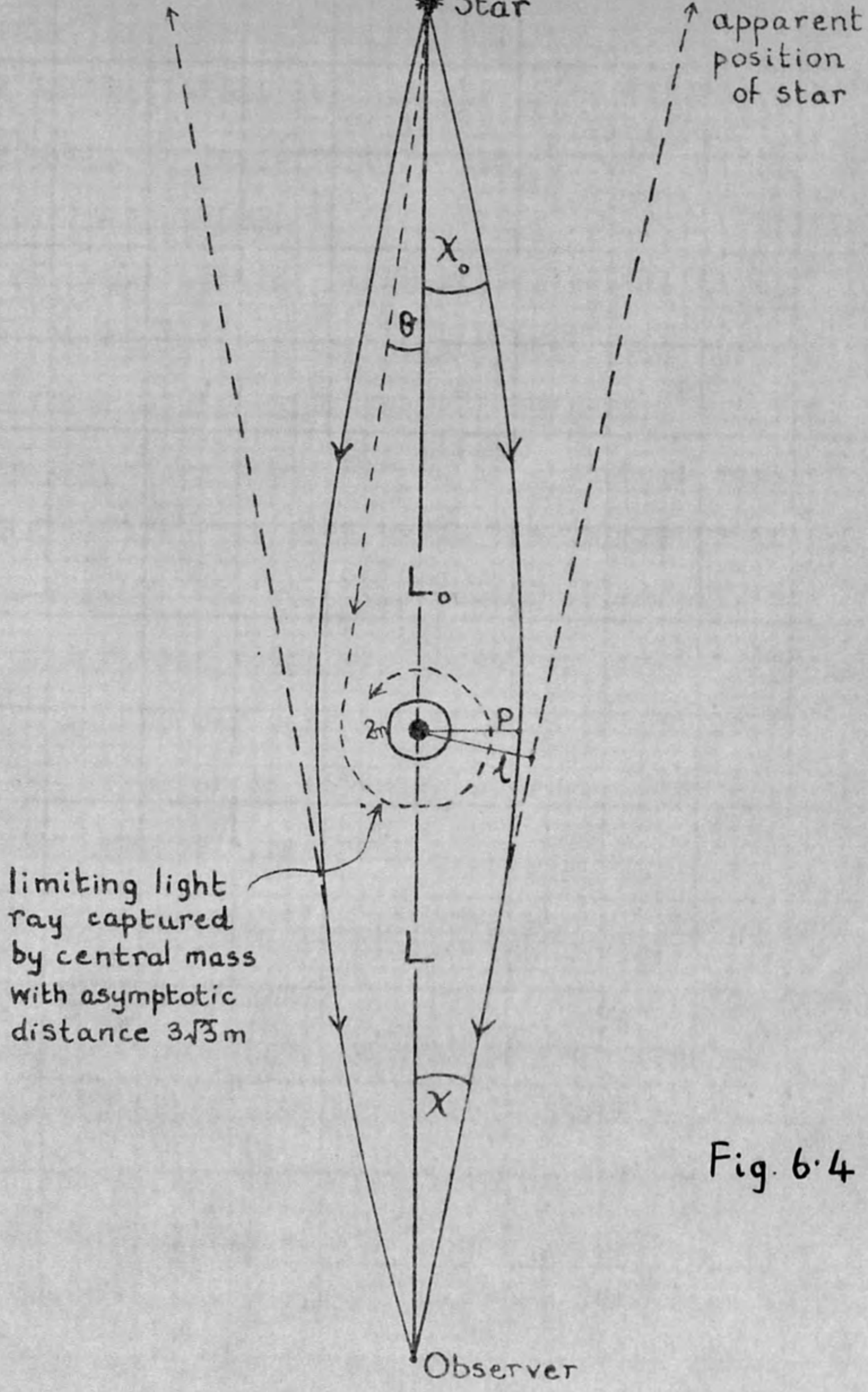

Star
apparent
position
of star
χ_0
θ
L_0
2m
p
l
limiting light
ray captured
by central mass
with asymptotic
distance $3\sqrt{3}m$
L
χ
Observer

Fig. 6·4

distance from the sun to the tangent of the ray's orbit at the telescope; to the approximation considered, x_0 is negligible and $l \sim P$, the perihelion distance.

The rays will approach the telescope at an angle $x = \frac{l}{L}$ to the direct line from the sun, where L is the distance between telescope and sun. Darwin has shown that $x = 2\sqrt{\frac{m}{L}}$: since rays of light from the star will reach the telescope in all planes through the sun, the distant star will be seen as a circle round the object of angular radius $2\sqrt{\frac{m}{L}}$. We may consider this as unambiguous when the observer is situated at very large L in the Galilean part of space.

(xvii) Observation of a star in the neighbourhood of the barrier.

Suppose now that observations are made on a star in the neighbourhood of the object and passing slowly behind it, say from right to left in the observer's field of vision (see Fig. 6.5). While still well to the right of it, some rays from the star will reach the telescope by a direct route while some of the rays advancing from the star will reach the telescope after passing round the object.

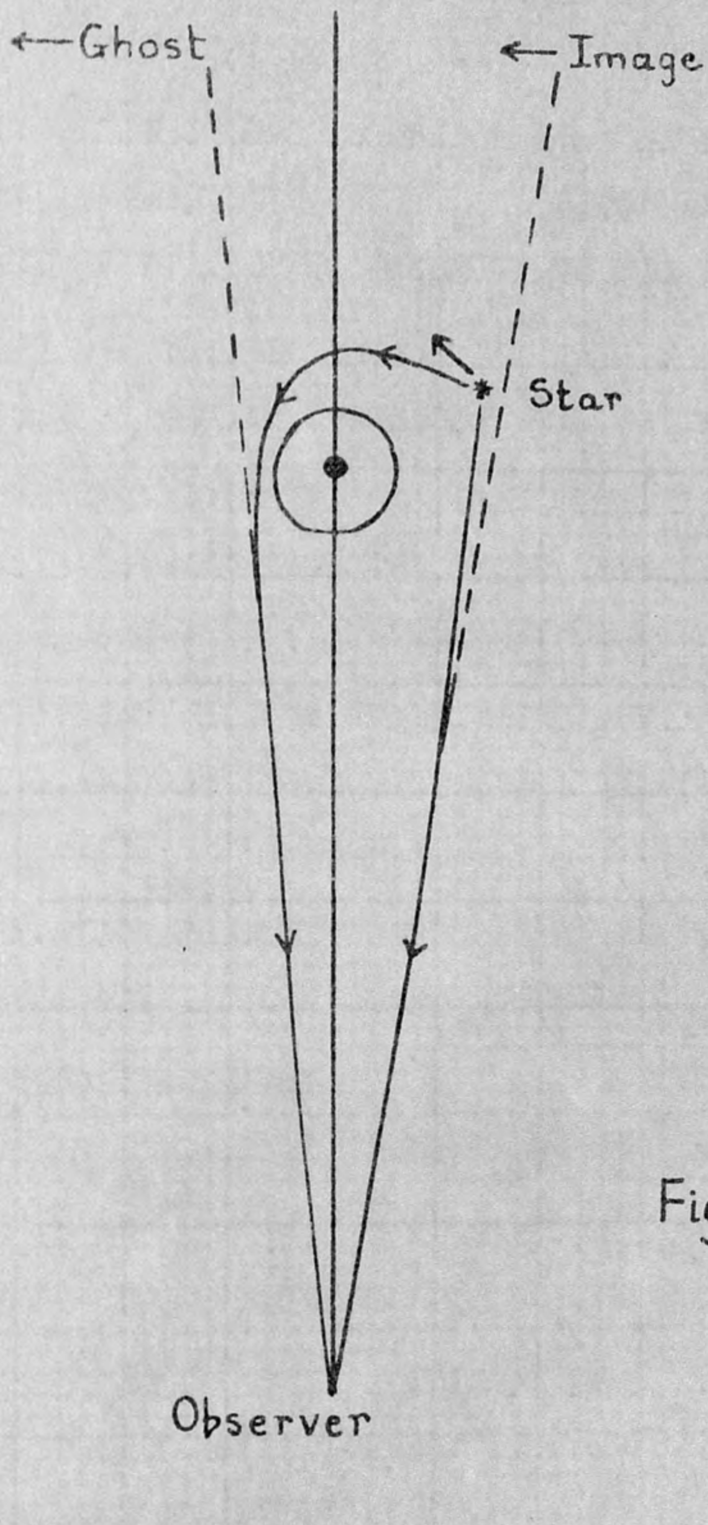


Fig. 6.5

The observer will therefore see an image of the star to the left of the object, called a "ghost" by Darwin, which will subsequently move outwards as the star progresses. As the star approaches the line of the sun, it will appear to lag behind its real position and an image of it will always be visible to the right even when the star has passed right behind the object because light rays will be reaching the telescope from the right having passed back round the object. This ghost to the right will move inwards and will be in view indefinitely, its apparent position eventually being only slightly greater than $3\sqrt{3}$m. As the star progresses round the object the ghost on the left will accelerate and gradually increase in brilliance and it will finally become the main star image.

If rays of light from the star may indeed pass completely round the object more than once before escaping to the telescope, as stated by Darwin, they would produce further ghosts on both sides increasingly nearer to $3\sqrt{3}$m. Darwin shows that the successive ghosts would crowd together more and more closely and with increasing feebleness the nearer that $3\sqrt{3}$m is approached and he takes an example to show that the effect of the crowding would counterbalance the increasing feebleness.

(xviii) Observation of the barrier against a star-field.

Finally, consider the object against the background of a roughly uniform distant star-field. According to a distant observer, the field will have a density which is uniform far away from the object resulting from the direct images only of the stars, since rays which pass round the horizon will not reach the observer at a large angle with the line joining observer to object. As the line of sight approaches the object, the density will increase for two reasons: the direct rays of light from stars in the sky behind the object will be deflected so that the stars appear to be situated at a greater angular distance from the object than is actually the case; also, "ghosts" of stars may then be in the observer's vision, due to rays of light which have passed from the stars round the horizon before reaching the observer's telescope. In general, the ghost in any particular plane through the object and the observer on one side of the line joining observer to object will be the secondary images of stars in that plane which are actually situated on the other side, including those stars which are to the side of and behind the observer with respect to the direction of the object.

Before we could say with certainty what picture the sky would present to the observer, further research would be required into various problems: it would be necessary to know whether rays of light may indeed pass more than once round the horizon before escaping to the observer; what the precise effect on the intensity of the emitted light would be and most important would be an investigation into the nature of the geometry of space in the neighbourhood of the observer. While the geometry is unambiguous when the observer may be considered to lie in the Galilean part of space, it is not necessarily so straightforward and is of increasing significance the nearer that the observer is to the object. An examination of the directions at the telescope of rays of light which are neighbouring upon emission from a star, which may be situated anywhere round the sky, and of the equations of neighbouring rays of light at the telescope will determine what images the observer would see in the sky and whether distant point sources would necessarily appear as point sources to the observer or whether orbits of neighbouring rays might be such as to produce a large image. Unfortunately, such problems are beyond the scope of the present investigation.

Consider, however, a distant observer. Darwin has shown that the ghosts will contribute a faint glow round a circle of radius $3\sqrt{3}m$ which will be of uniform density in the immediate neighbourhood of the circle. What will be seen in the area inside this circle? Darwin states that only the sun itself can contribute light to this area. We would not agree with this without qualifications. Firstly, we must be sure that the geometry in the neighbourhood of the observer is such that the area subtends a non-zero solid angle. If this is the case, then it is certainly clear that no light which does not originate in $r < 3m$ may appear in this region, but in view of our results we maintain that only if the radius r_0 of the sun were greater than 2m could it contribute light at all. Moreover, in principle, other sources of light such as test particles spiralling or travelling radially inwards to $r = 2m$ may well emit light to the observer which would be seen in this region. If we assume that $r_0 > 2m$ and do not consider other light sources in the area, we get Darwin's results: namely, that if the rays emerging from the sun followed straight lines only, there would appear a brilliant point of light surrounded by blackness; if the rays could emerge also in spiral orbits, a question which Darwin points out would call for deeper assumptions about the structure of the sun, then the sun would probably be seen as a blaze of light entirely filling the circle of radius $3\sqrt{3}$m.

Should the radius of the sun be less than 2m, its structure and indeed its very existence would be immaterial, since no observer could ever receive light from it. In the absence of light from any other source, the area within the radius $3\sqrt{3}$m would probably appear completely black; allowing other small sources, the whole area might appear filled with a glow or with discrete points of light, depending critically on the actual conditions fulfilled regarding the amount and nature of the emission.

It is interesting to consider what knowledge an observer with such an object in his vision might gain regarding its distance, size and other properties and to speculate what he might infer about its nature.

CHAPTER VII: THE BARRIER IN THE FINKELSTEIN SPACE-TIME.

(i) The Finkelstein metric and its time-reversal.

Let us consider now the metric obtained by Finkelstein (1958) from the Schwarzschild metric

$$ds^2 = \left(1-\frac{1}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{1}{r}\right)} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \qquad (7.1)$$

by the transformation, for $r > 1$,

$$t = \bar{t} + \ln(\bar{r}-1); \qquad (7.2)$$

$$r = \bar{r}\ ;\ \theta = \bar{\theta}\ ;\ \phi = \bar{\phi} \qquad (7.3)$$

Differentiation of (7.2) yields

$$dt = d\bar{t} + \frac{d\bar{r}}{\bar{r}-1} \qquad (7.4)$$

so that $dt^2 = d\bar{t}^2 + \dfrac{2d\bar{t}\,d\bar{r}}{(\bar{r}-1)} + \dfrac{d\bar{r}^2}{(\bar{r}-1)^2}$ (7.5)

Only (7.5) is actually involved in the transformation, so that an ambiguity of sign arises immediately upon taking its square root. Instead of (7.4) we obtain

$$\pm\, dt = d\bar{t} + \frac{d\bar{r}}{\bar{r}-1} \qquad (7.6)$$

Using (7.5) we obtain Finkelstein's line-element

$$ds^2 = \left(1-\frac{1}{\bar{r}}\right)d\bar{t}^2 + \frac{2}{\bar{r}}\,d\bar{t}\,d\bar{r} - \left(1+\frac{1}{\bar{r}}\right)d\bar{r}^2 - \bar{r}^2\left(d\bar{\theta}^2 + \sin^2\bar{\theta}\, d\bar{\phi}^2\right) \qquad (7.7)$$

for $\bar{r} > 1$.

For $r > 1$, we note that (7.1) is symmetric with respect to the time co-ordinate t, which is why the ambiguity of sign may arise in (7.6); it is clear, however, that (7.7) is not time-symmetric for $\bar{r} > 1$. Reversing the direction of time in (7.7) by setting $\bar{t} = -\bar{t}$, we obtain

$$ds^2 = \left(1-\frac{1}{\bar{r}}\right)d\bar{t}^2 - \frac{2d\bar{r}d\bar{t}}{\bar{r}} - \left(1+\frac{1}{\bar{r}}\right)d\bar{r}^2 - \bar{r}^2\left(d\bar{\theta}^2 + \sin^2\bar{\theta}\,d\bar{\phi}^2\right) \tag{7.9}$$

It is easily verified that this line element may be obtained for $\bar{r} > 1$ from the Schwarzschild line-element (7.1) with $r > 1$ by the transformation (7.3) together with

$$\pm\, dt = d\bar{t} - \frac{d\bar{r}}{\bar{r}-1} \tag{7.10}$$

It is interesting, and indeed significant, to note that comparison of (7.6) and (7.10) shows that time-reversal of (7.7) into (7.9) and vice versa is effected as much by changing the sign of $\frac{d\bar{r}}{\bar{r}-1}$ as by changing that of $d\bar{t}$, and that the sign of $\frac{d\bar{r}}{\bar{r}-1}$ changes as $\bar{r}$ passes through the value 1 if $d\bar{r}$ retains the same sign. We emphasise again that for reasons stated in the previous chapter we consider none of the equations valid for $\bar{r} = 1$ and we shall always treat this value only as a limiting case. However, it is possible to make a mathematical extension to cover the part of the manifold $\bar{r} < 1$ for each of the above cases.

Consider the equations (7.6) and (7.10) which together with (7.3) transform (7.1) for $r > 1$ into (7.7) and (7.9) respectively for $\bar{r} > 1$. One possibility is simply to consider (7.3), (7.6) and (7.10) to hold now for all $r \neq 1$; then the whole of (7.1) is transformed into the whole of (7.7) or (7.9) excepting only the surfaces r, $\bar{r} = 1$. The line-elements (7.7) and (7.9) in themselves appear to present no singularities except those at the respective origins, $\bar{r} = 0$, and they have been presented by Finkelstein as two distinct completions of the Schwarzschild exterior metric; we note however that in each the metric coefficient g_{tt} vanishes at $\bar{r} = 1$ and the manifolds are still incomplete. For this case, integration of the equations (7.6) and (7.10) yield respectively the alternative equations of transformation

$$\pm t = \bar{t} + \ln|\bar{r} - 1| \tag{7.11}$$

and

$$\pm t = \bar{t} - \ln|\bar{r} - 1| \tag{7.12}$$

for all $\bar{r} \neq 1$.

It is interesting to note in passing that for $\bar{r} > 1$, (7.6) and (7.10) may have been written in the equivalent forms

$$\pm\, dt = d\bar{t} + \frac{d\bar{r}}{|\bar{r} - 1|} \tag{7.13}$$

and

$$\pm\, dt = d\bar{t} - \frac{d\bar{r}}{|\bar{r} - 1|} \tag{7.14}$$

respectively and that another way of covering the region $\bar{r} < 1$ is to consider (7.13) and (7.14) valid for all $r \neq 1$.

Integrating (7.13) yields (7.11) for $\bar{r} > 1$ and (7.12) for $\bar{r} < 1$, so that transforming the Schwarzschild line-element (7.1) according to (7.3) and (7.13) we obtain a space-time (A) represented by the metric (7.7) for $\bar{r} > 1$ together with (7.9) for $\bar{r} < 1$. Similarly, but giving the opposite results, integration of (7.14) gives (7.12) for $\bar{r} > 1$ and (7.11) for $\bar{r} < 1$; transforming (7.1) according to (7.3) and (7.14), we obtain a space-time (B) represented by the metric (7.9) for $\bar{r} > 1$ combined with (7.7) for $\bar{r} < 1$.

The existence of (A) and (B) (both with the form of the metric discontinuous at $\bar{r} = 1$) serves to illustrate that if we take the view that none of the equations of transformation is valid for $\bar{r} = 1$, it is purely by arbitrary mathematical convention that we may consider the two regions $\bar{r} > 1$, $\bar{r} < 1$ in (7.7) and (7.9) to be joined into one manifold at $\bar{r} = 1$. The mathematical correlation is an arbitrary choice. Whether or not there exists a real physical criterion to distinguish between the possible ways of extending the manifold $\bar{r} > 1$ and, more fundamentally, whether or not there exists a physical criterion for evaluating the validity of transformations of the Schwarzschild metric into a different form are problems worthy of investigation. At present we adopt the procedure of considering the regions $\bar{r} > 1$ and $\bar{r} < 1$ as physically distinct in each of (7.7) and (7.9), even though we have apparent mathematical continuity across the surface $\bar{r} = 1$.

(ii) The geodesic equations of the Finkelstein metric; the $s - \bar{r}$, $\bar{t} - \bar{r}$ relationships for $\bar{r} > 1$.

(7.7) has been obtained from the Schwarzschild metric (7.1) by a transformation which does not remove the barrier at $r = 1$. Let us investigate the nature of the barrier from the point of view of an observer situated in the region $\bar{r} > 1$. We shall consider only radial motion, so that $d\bar{\theta} = d\bar{\phi} = 0$. (7.7) becomes

$$ds^2 = \left(1 - \frac{1}{\bar{r}}\right) d\bar{t}^2 + \frac{2}{\bar{r}} d\bar{t}\, d\bar{r} - \left(1 + \frac{1}{\bar{r}}\right) d\bar{r}^2 \qquad (7.15)$$

Applying the condition that particles move on geodesics, we get

$$\left(1 - \frac{1}{\bar{r}}\right) \frac{d\bar{t}}{ds} + \frac{1}{\bar{r}} \frac{d\bar{r}}{ds} = \text{const.} = \bar{C} \text{ (say)} \qquad (7.16)$$

Substituting into (7.15) gives

$$\frac{d\bar{r}}{ds} = \pm \left(\bar{C}^2 - 1 + \frac{1}{\bar{r}}\right)^{\frac{1}{2}} \qquad (7.17)$$

so that

$$\frac{d\bar{t}}{ds} = \frac{\bar{C} - (\pm) \frac{1}{\bar{r}} \left(\bar{C}^2 - 1 + \frac{1}{\bar{r}}\right)^{\frac{1}{2}}}{\left(1 - \frac{1}{\bar{r}}\right)} \qquad (7.18)$$

and

$$\frac{d\bar{r}}{d\bar{t}} = \pm \frac{\left(\bar{C}^2 - 1 + \frac{1}{\bar{r}}\right)^{\frac{1}{2}} \left(1 - \frac{1}{\bar{r}}\right)}{\bar{C} - (\pm) \frac{1}{\bar{r}} \left(\bar{C}^2 - 1 + \frac{1}{\bar{r}}\right)^{\frac{1}{2}}} \qquad (7.19)$$

With barred co-ordinates replacing unbarred throughout (7.17) is the same as equation (6.12) and upon integration yields equations (6.14), (6.15) and (6.16) respectively for the three cases $|\bar{c}| > 1$, $|\bar{c}| = 1$, $|\bar{c}| < 1$. We may therefore deduce immediately that test particles from $\bar{r} > 1$ may, in the limit, reach the barrier at $\bar{r} = 1$ in finite proper time, and indeed may penetrate from $\bar{r} = 1$ into $\bar{r} > 1$ in finite proper time.

(7.19) may be rewritten in the form

$$\pm\, d\bar{t} = \frac{\bar{c}\, d\bar{r}}{(\bar{c}^2 - 1 + \frac{1}{\bar{r}})^{\frac{1}{2}}(1 - \frac{1}{\bar{r}})} - (\pm)\, \frac{d\bar{r}}{\bar{r} - 1} \qquad (7.20)$$

Comparison of (7.20) and (6.13) shows that integration will give for the three cases $|\bar{c}| > 1$, $= 1$, < 1 the equations (6.17), (6.18) and (6.19) with barred co-ordinates replacing unbarred, each with the extra term $-(\pm)\ln|\bar{r} - 1|$ on its right hand side. That is, we have

$\underline{|\bar{c}| > 1}$:

$$\pm\,\bar{t}(\bar{r}) = \frac{(\bar{K}^2+1)^{\frac{1}{2}}}{\bar{K}^3}\left[(\bar{K}^2\bar{r}+1)^{\frac{1}{2}}\bar{K}\bar{r}^{\frac{1}{2}} + \frac{2\bar{K}^2-1}{2}\ln\frac{(\bar{K}^2\bar{r}+1)^{\frac{1}{2}} + \bar{K}\bar{r}^{\frac{1}{2}}}{(\bar{K}^2\bar{r}+1)^{\frac{1}{2}} - \bar{K}\bar{r}^{\frac{1}{2}}}\right]$$

$$+ \ln\left|\frac{\bar{K}^2\bar{r}+1 - (\bar{K}^2+1)^{\frac{1}{2}}(\bar{K}^2\bar{r}+1)^{\frac{1}{2}}}{(\bar{K}^2\bar{r}+1)^{\frac{1}{2}} - (\bar{K}^2+1)^{\frac{1}{2}}}\right|$$

$$- \ln\left|\frac{\bar{K}^2\bar{r}^{\frac{1}{2}}+1 + (\bar{K}^2+1)^{\frac{1}{2}}(\bar{K}^2\bar{r}+1)^{\frac{1}{2}}}{(\bar{K}^2\bar{r}+1)^{\frac{1}{2}} + (\bar{K}^2+1)^{\frac{1}{2}}}\right| - (\pm)\ln|\bar{r}-1| + \text{const.}$$

where $\bar{K} = (\bar{c}^2 - 1)^{\frac{1}{2}}$ (7.21)

$\bar{C} = +1$:

$$\pm \bar{t}(\bar{r}) = \tfrac{2}{3}\bar{r}^{3/2} + 2\sqrt{\bar{r}} + \ln\frac{|\sqrt{\bar{r}}-1|}{\sqrt{\bar{r}}+1} - (\pm)\ln|\bar{r}-1| + \text{const.} \qquad (7.22)$$

$\bar{C} = -1$:

$$\pm \bar{t}(\bar{r}) = -\tfrac{2}{3}\bar{r}^{3/2} - 2\sqrt{\bar{r}} - \ln\frac{|\sqrt{\bar{r}}-1|}{\sqrt{\bar{r}}+1} - (\pm)\ln|\bar{r}-1| + \text{const.} \qquad (7.23)$$

$|\bar{C}| < 1$: $\bar{C} \neq 0$

$$\pm \bar{t}(\bar{r}) = \frac{(1-\bar{k}^2)^{1/2}}{\bar{k}^3}\left[-(1-\bar{k}^2\bar{r})^{1/2}\bar{k}\bar{r}^{1/2} + (1+2\bar{k}^2)\arcsin(1-\bar{k}^2\bar{r})^{1/2}\right]$$
$$- \ln\left|\frac{\bar{k}^2\bar{r}^{1/2}+1-(1-\bar{k}^2)^{1/2}(1-\bar{k}^2\bar{r})^{1/2}}{(1-\bar{k}^2\bar{r})^{1/2}-(1-\bar{k}^2)^{1/2}}\right|$$
$$+ \ln\left|\frac{\bar{k}^2\bar{r}^{1/2}+1+(1-\bar{k}^2)^{1/2}(1-\bar{k}^2\bar{r})^{1/2}}{(1-\bar{k}^2\bar{r})^{1/2}+(1-\bar{k}^2)^{1/2}}\right| - (\pm)\ln|\bar{r}-1| + \text{const.} \qquad (7.24)$$

where $\bar{k} = (1 - \bar{C})^{1/2}$. To ensure that in the region $\bar{r} > 1$ $\bar{t}$ and s increase together, we must have $\bar{C} > 0$ by (7.18).

(7.22) shows that for the case $\bar{C} = 1$, particles will take an infinite co-ordinate time to reach $\bar{r} = 1$ but a finite co-ordinate time to penetrate from $\bar{r} = 1$ a finite distance into the region $\bar{r} > 1$: for taking the upper sign throughout for motion away from $\bar{r} = 1$, the ln terms combine to form $-2 \ln(\sqrt{\bar{r}} + 1)$, making the right hand side finite in the limit as $\bar{r} \to 1$, whereas, taking the lower (-) sign for motion towards $\bar{r} = 1$, they combine to form $2 \ln(\sqrt{\bar{r}} - 1)$ which tends to infinity as $\bar{r} \to 1$, other terms remaining finite. The same results hold for the cases $\bar{C} > 1$, $\bar{C} < 1$.

For $\bar{C} > 1$, the right hand side of (7.21) is composed of finite terms and

$$\ln\left|\frac{\bar{K}^2\bar{r}^{\frac{1}{2}} + 1 - (\bar{K}^2+1)^{\frac{1}{2}}(\bar{K}^2\bar{r}+1)^{\frac{1}{2}}}{(\bar{K}^2\bar{r}+1)^{\frac{1}{2}} - (\bar{K}^2+1)^{\frac{1}{2}}}\right| - (\pm)\ln|\bar{r}-1| \, ;$$

we have already seen that the first term of this expression tends to $-\infty$ as $\bar{r} \to 1$ and taking the lower sign in the second term reinforces this; if, however, we take the upper sign throughout, these terms combine to give an expression which, by l'Hôpital's rule, tends to a finite limit, $-\ln 2(1+\bar{K}^2)^{\frac{1}{2}}$, as $\bar{r} \to 1$. A similar process with (7.24) yields the same conclusions for the case $\bar{C} < 1$.

(iii) <u>The motion of light rays.</u>

Now putting $ds = 0$ in (7.7) for the motion of light rays in $\bar{r} > 1$, we have

$$\left(1 - \frac{1}{\bar{r}}\right)\left(\frac{d\bar{\tau}}{d\bar{r}}\right)^2 + \frac{2}{\bar{r}}\left(\frac{d\bar{\tau}}{d\bar{r}}\right) - \left(1 + \frac{1}{\bar{r}}\right) = 0 \qquad (7.25)$$

so that

$$\frac{d\bar{r}}{d\bar{\tau}} = \frac{(\bar{r}-1)}{(\pm\bar{r}-1)} \qquad (7.26)$$

where we must, as usual, take the + sign for motion in the direction of increasing r (outwards) and the - sign for motion in the direction of decreasing r (inwards).

Then $$\left(\frac{d\bar{r}}{d\bar{\tau}}\right)_{IN} = -\frac{(\bar{r}-1)}{(\bar{r}+1)} \tag{7.27}$$

and $$\left(\frac{d\bar{r}}{d\bar{\tau}}\right)_{OUT} = 1 \tag{7.28}$$

Integration gives

$$\bar{\tau}_{OUT} = \bar{r} + \text{const.} \tag{7.29}$$

$$-\bar{\tau}_{IN} = \bar{r} + 2\ln|\bar{r}-1| + \text{const.} \tag{7.30}$$

Inspection of (7.29) and (7.30) shows immediately that light rays, as well as particles, while taking an infinite co-ordinate time to reach $\bar{r} = 1$ from a finite $\bar{r} > 1$, may leave $\bar{r} = 1$ and reach a finite $\bar{r} > 1$ travelling in the opposite direction in finite co-ordinate time. Equations (6.14) to (6.16) show that therefore an observer in $\bar{r} > 1$ may receive outward travelling light rays emitted, in the limit, from $\bar{r} = 1$ in his own finite experience. Such rays may carry information to the observer provided the displacement of their spectral lines is not infinite.

(iv) <u>The displacement of spectral lines</u>.

Suppose that a light wave of period $\Delta\bar{\tau}_i$ were emitted from a particle B at $\bar{r} = \bar{r}_i$ at $\bar{\tau} = \bar{\tau}_i$

and that this wave is received by A at $\bar{r} = \bar{r}_o > \bar{r}_i$ at a time $\bar{\tau} = \bar{\tau}_o$ during an interval $\Delta\bar{\tau}_o$. Let the motions of B and A be such that B has moved from $\bar{r}_i$ to $\bar{r}_i + \Delta\bar{r}_i$ during the time of emission, whereas A moves from $\bar{r}_o$ to $\bar{r}_o + \Delta\bar{r}_o$ during the reception of the same wave.

From (7.29) we have

$$\bar{\tau}_o - \bar{\tau}_i = \bar{r}_o - \bar{r}_i \tag{7.31}$$

and $$(\bar{\tau}_o + \Delta\bar{\tau}_o) - (\bar{\tau}_i + \Delta\bar{\tau}_i) = (\bar{r}_o + \Delta\bar{r}_o) - (\bar{r}_i + \Delta\bar{r}_i) \tag{7.32}$$

so that $$\Delta\bar{\tau}_o - \Delta\bar{\tau}_i = \Delta\bar{r}_o - \Delta\bar{r}_i \tag{7.33}$$

Using (7.19) we have

$$(\pm)\frac{\Delta\bar{r}_i}{\Delta\bar{\tau}_i} = \frac{(\bar{C}_B^2 - 1 + \frac{1}{\bar{r}_i})^{\frac{1}{2}}(1 - \frac{1}{\bar{r}_i})}{\bar{C}_B - (\pm)\frac{1}{\bar{r}_i}(\bar{C}_B^2 - 1 + \frac{1}{\bar{r}_i})^{\frac{1}{2}}} \equiv X \tag{7.34}$$

and $$\pm\frac{\Delta\bar{r}_o}{\Delta\bar{\tau}_o} = \frac{(\bar{C}_A^2 - 1 + \frac{1}{\bar{r}})^{\frac{1}{2}}(1 - \frac{1}{\bar{r}_o})}{\bar{C}_A - (\pm)\frac{1}{\bar{r}_o}(\bar{C}_A^2 - 1 + \frac{1}{\bar{r}_o})^{\frac{1}{2}}} \equiv Y \tag{7.35}$$

Substituting (7.34) and (7.35) into (7.33) we get

$$\Delta\bar{\tau}_o\left[1 - (\pm)Y\right] = \Delta\bar{\tau}_i\left[1 - (\pm)X\right] \tag{7.36}$$

For the wave emitted at B

$$\Delta s_i^2 = \left(1 - \frac{1}{\bar{r}_i}\right)\Delta\bar{\tau}_i^2 + \frac{2}{\bar{r}_i}\Delta\bar{\tau}_i\Delta\bar{r}_i - \left(1 + \frac{1}{\bar{r}_i}\right)\Delta\bar{r}_i^2 \tag{7.37}$$

which, using (7.34), may be written

$$\Delta s_i^2 = \Delta\bar{\tau}_i^2\left[\left(1 - \frac{1}{\bar{r}_i}\right) \pm \frac{2X}{\bar{r}_i} - \left(1 + \frac{1}{\bar{r}_i}\right)X^2\right] \tag{7.38}$$

The interval corresponding to the beginning and ending of the emission of a wave at A is similarly given by

$$\delta s_o^2 = \delta\bar{\tau}_o^2\left[\left(1 - \frac{1}{\bar{r}_o}\right) \pm \frac{2Y}{\bar{r}_o} - \left(1 + \frac{1}{\bar{r}_o}\right)Y^2\right] \tag{7.39}$$

For this wave to be similar to that emitted by B we must have $\delta s_0 = \Delta s_i$. Equating (7.38) and (7.39) yields

$$\frac{\Delta \bar{\tau}_i^2}{\delta \bar{\tau}_0^2} = \frac{\left[(1-\frac{1}{\bar{r}_0}) \pm \frac{2Y}{\bar{r}_0} - (1+\frac{1}{\bar{r}_0})Y^2\right]}{\left[(1-\frac{1}{\bar{r}_i}) \pm 2X/\bar{r}_i - (1+\frac{1}{\bar{r}_i})X^2\right]} \equiv Z \qquad (7.40)$$

Eliminating $\Delta\bar{\tau}_i$ by means of (7.36) we get

$$\frac{\Delta \bar{\tau}_0^2}{\delta \bar{\tau}_0^2} = Z\frac{\left[1-(\pm)X\right]^2}{\left[1-(\pm)Y\right]^2} \qquad (7.41)$$

If $\lambda + d\lambda$ is the wavelength of the wave emitted from B on reaching A and λ is the wavelength of the similar wave emitted at A

$$\frac{\lambda+d\lambda}{\lambda} = \frac{\Delta\bar{\tau}_0}{\delta\bar{\tau}_0} = Z^{\frac{1}{2}}\frac{\left[1-(\pm)X\right]}{\left[1-(\pm)Y\right]} \qquad (7.42)$$

that is

$$1+\frac{d\lambda}{\lambda} = \frac{\left[1-(\pm)X\right]\left[(1-\frac{1}{\bar{r}_0}) \pm \frac{2Y}{\bar{r}_0} - (1+\frac{1}{\bar{r}_0})Y^2\right]^{\frac{1}{2}}}{\left[1-(\pm)Y\right]\left[(1-\frac{1}{\bar{r}_i}) \pm 2X/\bar{r}_i - (1+\frac{1}{\bar{r}_i})X^2\right]^{\frac{1}{2}}} \qquad (7.43)$$

where X, Y are given by (7.34), (7.35) respectively.

Suppose first that B is moving towards $\bar{r} = 1$: then we must take the - sign in front of X and the - sign in the expression for X. We see from (7.34) that as $\bar{r}_i \to 1$, $X \to 0$, so that $1+\frac{d\lambda}{\lambda} \to \infty$ since terms involving Y will remain finite and non-zero whatever the permitted motion of A. On the other hand, if B is moving in the outward direction we take the + signs where X is concerned; we evaluate X as $\bar{r}_i \to 1$ by means of l'Hôpital's rule and find that it tends to a finite limit $2\bar{C}_B^2(2\bar{C}_B^2+1)^{-1}$. Thus $1+\frac{d\lambda}{\lambda}$ tends to a finite limit equal to $\mathcal{Y}/2\bar{C}_B$ where $\mathcal{Y}$ represents the terms involving Y, depending on the motion of A.

This result, that light emitted from $\bar{r} = 1$ by a particle B travelling into the region $\bar{r} > 1$ can carry information to the observer A in $\bar{r} > 1$, together with that already established (namely, that in the limit particles and light rays from $\bar{r} = 1$ may reach an observer in $\bar{r} > 1$ in his own finite experience) implies immediately that the barrier at $\bar{r} = 1$ in (7.7) is NOT, in principle, an event horizon for the observer in $\bar{r} > 1$.

(v) The geodesic equations of the time-reversed Finkelstein metric; the $s - \bar{r}$, $\bar{t} - \bar{r}$ relationships.

For radial motion we have

$$ds^2 = \left(1-\frac{1}{\bar{r}}\right)d\bar{t}^2 - \frac{2}{\bar{r}}\,d\bar{r}\,d\bar{t} - \left(1+\frac{1}{\bar{r}}\right)d\bar{r}^2 \qquad (7.44)$$

The condition that particles move on a geodesic yields

$$\left(1-\frac{1}{\bar{r}}\right)\frac{d\bar{t}}{ds} - \frac{1}{\bar{r}}\frac{d\bar{r}}{ds} = \text{const.} = C \ (\text{say}) \qquad (7.45)$$

Substitution into (7.44) gives

$$\frac{d\bar{r}}{ds} = \pm\left(C^2-1+\frac{1}{\bar{r}}\right)^{\frac{1}{2}} \qquad (7.46)$$

and

$$\frac{d\bar{t}}{ds} = \frac{C \pm \frac{1}{\bar{r}}\left(C^2-1+\frac{1}{\bar{r}}\right)^{\frac{1}{2}}}{\left(1-\frac{1}{\bar{r}}\right)} \qquad (7.47)$$

so that

$$\frac{d\bar{r}}{d\bar{t}} = \pm\frac{\left(C^2-1+\frac{1}{\bar{r}}\right)^{\frac{1}{2}}\left(1-\frac{1}{\bar{r}}\right)}{C \pm \frac{1}{\bar{r}}\left(C^2-1+\frac{1}{\bar{r}}\right)^{\frac{1}{2}}} \qquad (7.48)$$

The expression for $\frac{d\bar{r}}{ds}$ is again unaltered; we therefore have test particles able, in the limit, to travel between finite $\bar{r} > 1$ and $\bar{r} = 1$ in either direction in finite proper time.

(7.48) may be written

$$\pm\, d\bar{t} = \frac{C\, d\bar{r}}{(C^2 - 1 + \frac{1}{\bar{r}})^{1/2}(1 - \frac{1}{\bar{r}})} \pm \frac{d\bar{r}}{\bar{r} - 1} \qquad (7.49)$$

Comparison with (7.20) shows that for the three cases $|C| > 1$, $= 1$, < 1 we have upon integration equations (7.21) to (7.24) with the term $-(\pm)\ln|\bar{r}-1|$ replaced throughout by $+(\pm)\ln|\bar{r}-1|$. Taking the upper (+) sign on both sides for motion in the direction of increasing $\bar{r}$, we find immediately from previous work that in the limit test particles will take an infinite co-ordinate time to travel from $\bar{r} = 1$ to finite $\bar{r} > 1$; however, for motion in the opposite direction the travel time will be finite.

(vi) <u>The radial motion of light rays; existence of an E.H. point.</u>

The motion of light rays is given by

$$\left(1 - \frac{1}{\bar{r}}\right)\left(\frac{d\bar{\tau}}{d\bar{r}}\right)^2 - \frac{2}{\bar{r}}\, d\bar{\tau}\, d\bar{r} - \left(1 + \frac{1}{\bar{r}}\right) = 0 \qquad (7.50)$$

by putting $ds = 0$ in (7.44), so that

$$\frac{d\bar{r}}{d\bar{\tau}} = \frac{\bar{r} - 1}{(\pm\bar{r} + 1)} \qquad (7.51)$$

Taking the + sign for outward motion (increasing $\bar{r}$) and the − sign for inward motion, we get

$$\left(\frac{d\bar{r}}{d\bar{\tau}}\right)_{IN} = -1 \qquad (7.52)$$

and $$\left(\frac{d\bar{r}}{d\bar{\tau}}\right)_{OUT} = \frac{\bar{r}-1}{\bar{r}+1} \tag{7.53}$$

Integrating, we get respectively

$$\bar{\tau}_{IN} = -\bar{r} + \text{const.} \tag{7.54}$$

$$\bar{\tau}_{OUT} = \bar{r} + 2\ln|\bar{r}-1| + \text{const.} \tag{7.55}$$

It is clear from these equations that light rays, as well as particles, will take an infinite co-ordinate time to travel from $\bar{r} = 1$ to finite $\bar{r} > 1$, but travelling in the opposite direction they may make the same journey in finite co-ordinate time.

Suppose that an observer A stationary at large $\bar{r}$ and so measuring in $\bar{\tau}$-time observes the motion of a particle B travelling radially towards $\bar{r} = 1$. As B approaches the barrier at $\bar{r} = 1$, (7.55) shows that radial light signals from B will tend to take an infinite time to reach A; no events occurring at B at or beyond $\bar{r} = 1$ will be observable to A in his finite experience by these light rays. Moreover, no particle which has reached $\bar{r} = 1$ may ever subsequently meet A. The point $\bar{r} = 1$ on the radial line of sight is therefore an <u>event horizon point</u> for the specified observer A.

(vii) <u>Non-radial motion of light rays; existence of an E.H. for A.</u>

For non-radial motion of light rays we obtain the equations of motion by transforming (6.24) according to (7.10). The ambiguity of sign in (7.10) gives two possibilities: taking the + sign we get

$$\frac{d\bar{r}}{d\bar{\tau}} = \pm\frac{(1-\frac{1}{\bar{r}})\left[1-\frac{\ell^2}{\bar{r}^2}(1-\frac{1}{\bar{r}})\right]^{\frac{1}{2}}}{1 \pm \frac{1}{\bar{r}}\left[1-\frac{\ell^2}{\bar{r}^2}(1-\frac{1}{\bar{r}})\right]^{\frac{1}{2}}} \tag{7.56}$$

whereas the - sign gives

$$\frac{d\bar{r}}{d\bar{\tau}} = -\frac{(\pm)(1-\frac{1}{\bar{r}})\left[1-\frac{\ell^2}{\bar{r}^2}(1-\frac{1}{\bar{r}})\right]^{\frac{1}{2}}}{1-(\pm)\frac{1}{\bar{r}}\left[1-\frac{\ell^2}{\bar{r}^2}(1-\frac{1}{\bar{r}})\right]^{\frac{1}{2}}} \tag{7.57}$$

Reducing these to the case of radial motion by putting $\ell = 0$, (7.56) yields

$$\left(\frac{d\bar{r}}{d\bar{\tau}}\right)_{OUT} = \frac{\bar{r}-1}{\bar{r}+1} \tag{7.58}$$

and

$$\left(\frac{d\bar{r}}{d\bar{\tau}}\right)_{IN} = -1 \tag{7.59}$$

whereas (7.57) yields

$$\left(\frac{d\bar{r}}{d\bar{\tau}}\right)_{OUT} = -1 \tag{7.60}$$

and

$$\left(\frac{d\bar{r}}{d\bar{\tau}}\right)_{IN} = \frac{\bar{r}-1}{\bar{r}+1}$$

Comparison with (7.52) and (7.53) shows that we must choose the equation (7.56) to represent the non-radial motion of light rays. In the neighbourhood of $\bar{r} = 1$, (7.56) in fact reduces to one of (7.58), (7.59) and integrating for this neighbourhood and allowing $\bar{r}$ to tend to 1 we find that the co-ordinate travel time of a light ray travelling outwards from the surface $\bar{r} = 1$ to finite $\bar{r} > 1$ will always be infinite. Therefore for the specified observer A the whole surface $\bar{r} = 1$ will be an E.H.

(viii) Radial motion of the observer; the spectral displacement of light from a radially moving particle.

Let us now allow A to have any permitted radial motion in $\bar{r} > 1$; consider a light signal received by A from the particle B also in radial motion in $\bar{r} > 1$. Suppose B emits at time $\bar{\tau} = \bar{\tau}_i$ from $\bar{r} = \bar{r}_i$ a light wave of period $\Delta\bar{\tau}_i$, during which time B has moved from $\bar{r}_i$ to $\bar{r}_i + \Delta\bar{r}_i$. Let the light wave received by A have a period $\Delta\bar{\tau}_0$ during which time A has moved from $\bar{r} = \bar{r}_0 > \bar{r}_i$ at $\bar{\tau} = \bar{\tau}_0$ to $\bar{r}_0 + \Delta\bar{r}_0$ at $\bar{\tau}_0 + \Delta\bar{\tau}_0$. Then by (7.53) we have

$$\bar{\tau}_0 - \bar{\tau}_i = \int_{\bar{r}_i}^{\bar{r}_0} \frac{\bar{r}+1}{\bar{r}-1}\, d\bar{r} \tag{7.61}$$

and

$$(\bar{\tau}_0 + \Delta\bar{\tau}_0) - (\bar{\tau}_i + \Delta\bar{\tau}_i) = \int_{\bar{r}_i+\Delta\bar{r}_i}^{\bar{r}_0+\Delta\bar{r}_0} \frac{\bar{r}+1}{\bar{r}-1}\, d\bar{r} \tag{7.62}$$

Subtracting and using (7.55) we have

$$\Delta\bar{\tau}_0 - \Delta\bar{\tau}_i = \Big[\bar{r} + 2\ln|\bar{r}-1|\Big]_{\bar{r}_i+\Delta\bar{r}_i}^{\bar{r}_0+\Delta\bar{r}_0} - \Big[\bar{r} + 2\ln|\bar{r}-1|\Big]_{\bar{r}_i}^{\bar{r}_0}$$

$$= \Delta\bar{r}_0 + 2\ln\frac{\bar{r}_0+\Delta\bar{r}_0-1}{\bar{r}_0-1} - \Delta\bar{r}_i - 2\ln\frac{\bar{r}_i+\Delta\bar{r}_i-1}{\bar{r}_i-1}$$

$$= \Delta\bar{r}_0 - \Delta\bar{r}_i + 2\ln\frac{1+\dfrac{\Delta\bar{r}_0}{\bar{r}_0-1}}{1+\dfrac{\Delta\bar{r}_i}{\bar{r}_i-1}} \tag{7.63}$$

Expanding in terms of the small quantities $\frac{\Delta\bar{r}}{\bar{r}-1}$ and neglecting squares and higher orders, we obtain

$$\Delta\bar{\tau}_o - \Delta\bar{\tau}_i = \Delta\bar{r}_o + \frac{2\Delta\bar{r}_o}{\bar{r}_o-1} - \Delta\bar{r}_i - \frac{2\Delta\bar{r}_i}{\bar{r}_i-1} \tag{7.64}$$

$$= \Delta\bar{r}_o\left(\frac{\bar{r}_o+1}{\bar{r}_o-1}\right) - \Delta\bar{r}_i\left(\frac{\bar{r}_i+1}{\bar{r}_i-1}\right) \tag{7.65}$$

Using (7.48) we have

$$\pm\frac{\Delta\bar{r}_i}{\Delta\bar{\tau}_i} = \frac{\left(C_B^2-1+\frac{1}{\bar{r}_i}\right)^{1/2}\left(1-\frac{1}{\bar{r}_i}\right)}{C_B \pm \frac{1}{\bar{r}_i}\left(C_B^2-1+\frac{1}{\bar{r}_i}\right)^{1/2}} \equiv X' \tag{7.66}$$

and
$$\pm\frac{\Delta\bar{r}_o}{\Delta\bar{\tau}_o} = \frac{\left(C_A^2-1+\frac{1}{\bar{r}_o}\right)^{1/2}\left(1-\frac{1}{\bar{r}_o}\right)}{C_A \pm \frac{1}{\bar{r}_o}\left(C_A^2-1+\frac{1}{\bar{r}_o}\right)^{1/2}} \equiv Y' \tag{7.67}$$

Substituting these into (7.65), we get

$$\Delta\bar{\tau}_o\left[1-(\pm)Y'\left(\frac{\bar{r}_o+1}{\bar{r}_o-1}\right)\right] = \Delta\bar{\tau}_i\left[1-(\pm)X'\left(\frac{\bar{r}_i+1}{\bar{r}_i-1}\right)\right] \tag{7.68}$$

For the wave emitted at B

$$\Delta s_i^2 = \left(1-\frac{1}{\bar{r}_i}\right)\Delta\bar{\tau}_i^2 - \frac{2}{\bar{r}_i}\Delta\bar{r}_i\,\Delta\bar{\tau}_i - \left(1+\frac{1}{\bar{r}_i}\right)\Delta\bar{r}_i^2 \tag{7.69}$$

which, using (7.66), may be written

$$\Delta s_i^2 = \Delta\bar{\tau}_i^2\left[\left(1-\frac{1}{\bar{r}_i}\right)-(\pm)\frac{2}{\bar{r}_i}X' - \left(1+\frac{1}{\bar{r}_i}\right)X'^2\right] \tag{7.70}$$

Similarly, the interval corresponding to the beginning and ending of the emission of a wave at A is given by

$$\delta s_o^2 = \delta\bar{\tau}_o^2\left[\left(1-\frac{1}{\bar{r}_o}\right)-(\pm)\frac{2}{\bar{r}_o}Y' - \left(1+\frac{1}{\bar{r}_o}\right)Y'^2\right] \tag{7.71}$$

For this to be similar to that emitted by B we must have

$$\delta s_o = \Delta s_i$$

Equating (7.70) and (7.71) we get

$$\frac{\Delta\bar{\tau}_i^2}{\delta\bar{\tau}_o^2} = \frac{\left[\left(1-\frac{1}{\bar{r}_o}\right)-(\pm)\frac{2}{\bar{r}_o}Y' - \left(1+\frac{1}{\bar{r}_o}\right)Y'^2\right]}{\left[\left(1-\frac{1}{\bar{r}_i}\right)-(\pm)\frac{2}{\bar{r}_i}X' - \left(1+\frac{1}{\bar{r}_i}\right)X'^2\right]} \equiv Z' \tag{7.72}$$

Using (7.68) to eliminate $\Delta\bar{\tau}_i$ yields

$$\frac{\Delta\bar{\tau}_o^2}{\delta\bar{\tau}_o^2} = \frac{\left[1-(\pm)X^1\left(\frac{\bar{r}_i+1}{\bar{r}_i-1}\right)\right]^2}{\left[1-(\pm)Y^1\left(\frac{\bar{r}_o+1}{\bar{r}_o-1}\right)\right]^2}\cdot Z^1 \tag{7.73}$$

If $\lambda + d\lambda$ is the wavelength of the wave emitted by B upon reception at A and λ is the wavelength of the similar wave emitted at A.

$$\frac{\lambda+d\lambda}{\lambda} = \frac{\Delta\bar{\tau}_o}{\delta\bar{\tau}_o} = Z^{1\frac{1}{2}}\frac{\left[1-(\pm)X^1\left(\frac{\bar{r}_i+1}{\bar{r}_i-1}\right)\right]}{\left[1-(\pm)Y^1\left(\frac{\bar{r}_o+1}{\bar{r}_o-1}\right)\right]} \tag{7.74}$$

Suppose that B moves towards $\bar{r} = 1$ so that we must take the - sign in front of X^1 and in the expression for X^1. (7.66) shows that as $\bar{r}_i \to 1$, $X^1 \to \frac{1}{2}$ so that Z^1 tends to the finite limit $2\mathcal{Y}^1$ where $\mathcal{Y}^1$ represents the terms involving Y^1 in (7.72).

The term $X^1\frac{(\bar{r}_i+1)}{(\bar{r}_i-1)}$ is given by $\dfrac{(\bar{r}_i+1)\left(C_B^2-1+\frac{1}{\bar{r}_i}\right)^{\frac{1}{2}}}{C_B\bar{r}_i \pm \left(C_B^2-1+\frac{1}{\bar{r}_i}\right)^{\frac{1}{2}}}$

which tends to infinity as $\bar{r}_i \to 1$ upon taking the - sign. Since terms involving Y^1 in (7.74) remain finite and non-zero for any permitted motion of A, $1 + \frac{d\lambda}{\lambda} \to \infty$ as B approaches $\bar{r}_i = 1$.

If B is moving outwards from $\bar{r} = 1$, we take the + sign throughout; then $X^1 \to 0$ as $\bar{r}_i \to 1$ so that $Z^1 \to \infty$ as $\bar{r}_i \to 1$. The red-shift is given by

$$1 + \frac{d\lambda}{\lambda} = \mathcal{Y}^1\frac{\left[1-\dfrac{(\bar{r}_i+1)\left(C_B^2-1+\frac{1}{\bar{r}_i}\right)^{\frac{1}{2}}}{C_B\bar{r}_i+\left(C_B^2-1+\frac{1}{\bar{r}_i}\right)^{\frac{1}{2}}}\right]}{\left[\left(1-\frac{1}{\bar{r}_i}\right)-\frac{2X^1}{\bar{r}_i}-\left(1+\frac{1}{\bar{r}_i}\right)X^{12}\right]^{\frac{1}{2}}}$$

Simplifying,

$$1+\frac{d\lambda}{\lambda} = \frac{\mathcal{Y}'\,\bar{r}_i}{(\bar{r}_i-1)}\left[C_B-\left(C_B^2-1+\frac{1}{\bar{r}_i}\right)^{\frac{1}{2}}\right] \qquad (7.76)$$

Using l'Hôpital's rule, we find that

$\lim_{\bar{r}_i\to 1}\left(1+\frac{d\lambda}{\lambda}\right) = \frac{\mathcal{Y}'}{2C_B}$ which is finite.

We have thus established that a radially moving free observer in $\bar{r}>1$ may receive no information about $\bar{r}\leqslant 1$ from a particle moving from $\bar{r}>1$ in the direction of decreasing $\bar{r}$ and sending back radial light signals, for the light will be infinitely red-shifted upon reception at A; however, if light is received at A in A's finite experience from a particle moving in the direction of increasing $\bar{r}$, it will contain information, for the red-shift in this case is finite. We must examine whether or not light emitted at $\bar{r}=1$ may ever in fact reach a freely-moving observer A in $\bar{r}>1$ in his own finite experience.

(ix) <u>The ($\bar{t}$, $\bar{r}$) co-ordinates of reception of light emitted from the barrier.</u>

Suppose that a ray of light is emitted from a particle at $\bar{r}=\bar{x}$ at $\bar{t}=0$ (say). Let A be freely moving in the region $\bar{r}>1$, being at $\bar{r}=\bar{R}$ at $\bar{t}=0$ and suppose that the light ray reaches A at ($\bar{t}$, $\bar{r}$). (7.55) shows that the travel time for the ray between $\bar{r}=\bar{x}$ and $\bar{r}$ ($>\bar{x}$) is given by

$$\bar{\tau} = \Big[\bar{r}+2\ln(\bar{r}-1)\Big]_{\bar{x}}^{\bar{r}} \qquad (7.77)$$

Suppose first that A has $C > 1$. Then if A travels away from $\bar{r} = 1$, the travel time for A between $\bar{r} = \bar{R}$ and $\bar{r} > \bar{R}$ is given by

$$\bar{t}_{A(OUT)} = \Big[L + M - N\Big]_{\bar{R}}^{\bar{r}} + \Big[\ln(\bar{r}-1)\Big]_{\bar{R}}^{\bar{r}} \qquad (7.78)$$

whereas if A travels towards $\bar{r} = 1$, the travel time between $\bar{r} = \bar{R}$ and $\bar{r} < \bar{R}$ is given by

$$\bar{t}_{A(IN)} = \Big[L + M - N\Big]_{\bar{r}}^{\bar{R}} - \Big[\ln(\bar{r}-1)\Big]_{\bar{r}}^{\bar{R}} \qquad (7.79)$$

where L, M, N are given by (6.52), (6.53) and (6.54). At what $\bar{r}$ co-ordinate will the light meet A in the two above cases as we allow $\bar{x}$ to tend to 1?

Allowing $\bar{x}$ to tend to 1 in (7.77) we get an infinite value for $\bar{\tau}$ whatever the value $\bar{r}$, provided $\bar{r} \neq 1$; equating $\bar{\tau}$ of (7.77) with $\bar{t}_{OUT}$ of (7.78), only if $\bar{r}$ is infinite will the equation be satisfied for then for $\bar{t}_{OUT}$ terms in L and the final term are infinite, while terms due to M, N remain finite; for finite $\bar{r} \neq 1$, $\bar{t}_{OUT}$ will remain finite.

If we equate $\bar{\tau}$ of (7.77) with $\bar{t}_{IN}$ of (7.79) allowing $\bar{x}$ to tend to 1, we obtain

$$\Big[\bar{r} + 2\ln(\bar{r}-1)\Big]_1^{\bar{r}} = \Big[L + M - N\Big]_{\bar{r}}^{\bar{R}} - \Big[\ln(\bar{r}-1)\Big]_{\bar{r}}^{\bar{R}} \qquad (7.80)$$

$$-1 + \bar{r} + \ln(\bar{r}-1) - \Big[2\ln(\bar{r}-1)\Big]_{\bar{r}=1} = \Big[L + M - N\Big]_{\bar{r}}^{\bar{R}} - \ln(\bar{R}-1) \qquad (7.81)$$

Inspection of (7.81) shows that the L.H.S. is infinite for all $\bar{r} \geqslant 1$; the R.H.S. is finite for all finite $\bar{r} > 1$, but becomes infinite upon setting $\bar{r} = 1$, for $-[M]_{\bar{r}=1} = \infty$ and all other terms are finite. This solution for $\bar{r}$ does not, however, satisfy the condition on $\bar{\tau}$ as comparison of (7.77) and (7.79) shows and, in any case, means that A is no longer in the region being considered.

If A has C = 1, the travel times between $\bar{r} = \bar{R}$ and $\bar{r}$ for the outward and inward journeys are given respectively by

$$\bar{t}_{OUT} = \Big[P\Big]_{\bar{R}}^{\bar{r}} + \Big[\ln(\bar{r}-1)\Big]_{\bar{R}}^{\bar{r}} \tag{7.82}$$

and

$$\bar{t}_{IN} = \Big[P\Big]_{\bar{r}}^{\bar{R}} - \Big[\ln(\bar{r}-1)\Big]_{\bar{r}}^{\bar{R}} \tag{7.83}$$

where P is given by (6.58).

Equating $\bar{\tau}$ with $\bar{t}_{OUT}$ and allowing $\bar{x}$ to tend to 1 again yields the solution $\bar{r} = \infty$. For the inward motion we have

$$\Big[\bar{r} + 2\ln(\bar{r}-1)\Big]_1^{\bar{r}} = \Big[P\Big]_{\bar{r}}^{\bar{R}} - \Big[\ln(\bar{r}-1)\Big]_{\bar{r}}^{\bar{R}}$$

$$\bar{r} + \ln(\bar{r}-1) - \Big[\bar{r} + 2\ln(\bar{r}-1)\Big]_{\bar{r}=1} = \Big[P\Big]_{\bar{r}}^{\bar{R}} \tag{7.84}$$

The equation is satisfied only for $\bar{r} = 1$, when both the L.H.S. and the R.H.S. are infinite, but again this solution does not satisfy the condition on $\bar{\tau}$.

If A has $C < 1$, travel times are given by

$$\bar{t}_{OUT} = \Big[Q - \mathcal{R} + S\Big]_{\bar{R}}^{\bar{R} < \bar{r} < \frac{1}{\bar{R}^2}} + \Big[\ln(\bar{r}-1)\Big]_{\bar{R}}^{\bar{R} < \bar{r} < \frac{1}{\bar{R}^2}} \tag{7.85}$$

or by

$$\bar{t}_{OUT} = \Big[Q - \mathcal{R} + S\Big]_{\bar{R}}^{1/\bar{R}^2} + \Big[\ln(\bar{r}-1)\Big]_{\bar{R}}^{1/\bar{R}^2}$$
$$+ \Big[Q - \mathcal{R} + S\Big]_{\bar{r}}^{1/\bar{R}^2} - \Big[\ln(\bar{r}-1)\Big]_{\bar{r}}^{1/\bar{R}^2} \tag{7.86}$$

depending on whether or not A reaches its maximum distance $\bar{r} = \frac{1}{R^2}$ and turns back before meeting the light ray, and by

$$\bar{t}_{IN} = \left[Q - \mathcal{R} + S\right]_{\bar{r}}^{\bar{R}} - \left[\ln(\bar{r}-1)\right]_{\bar{r}}^{\bar{R}} \qquad (7.87)$$

Equating $\bar{\tau}$ with $\bar{t}_{OUT}$ of (7.85) yields no solution; for as $\bar{x} \to 1, \bar{\tau} \to \infty$ while $\bar{t}_{OUT}$ is finite for all $\bar{r}$ in the permitted range. We therefore equate $\bar{\tau}$ with $\bar{t}_{OUT}$ of (7.86). Then

$$\left[\bar{r} + 2\ln(\bar{r}-1)\right]_1^{\bar{r}} = \left[Q - \mathcal{R} + S\right]_{\bar{R}}^{1/R^2} + \left[Q - \mathcal{R} + S\right]_{\bar{r}}^{1/R^2} + \left[\ln(\bar{r}-1)\right]_{\bar{R}}^{\bar{r}}$$

$$\left[\bar{r} + \ln(\bar{r}-1)\right] - \left[\bar{r} + 2\ln(\bar{r}-1)\right]_{\bar{r}=1} = \left[Q - \mathcal{R} + S\right]_{\bar{R}}^{1/R^2} + \left[Q - \mathcal{R} + S\right]_{\bar{r}}^{1/R^2} - \ln(\bar{R}-1) \qquad (7.88)$$

A solution of (7.88) is given again by $\bar{r} = 1$, when both sides are infinite, which also satisfies $\bar{\tau} = \bar{t}_{IN}$ of (7.87) for the equation is (7.88) with the omission only of certain finite terms; but again this solution does not satisfy our conditions on $\bar{t}$. Applying the same procedure to the non-radial motion of light and remembering that the co-ordinate travel time from $\bar{r} = 1$ to finite $\bar{r} > 1$ is infinite, we may conclude that any radially moving free particle A may receive information from the surface $\bar{r} = 1$ or beyond only at $\bar{r} = \infty$, after an infinite lapse of time in his own experience, or at $\bar{r} = 1$ when A no longer belongs to the region $\bar{r} > 1$.

(x) Non-radial motion; the barrier as a degenerate E.H.

For non-radial motion, we apply the geodesic equations to the full metric (7.9) which will give us

$$\frac{d\bar{t}}{ds} = \frac{C + \frac{1}{\bar{r}}\frac{d\bar{r}}{ds}}{(1-\frac{1}{\bar{r}})} \tag{7.89}$$

$$\frac{d^2\bar{\theta}}{ds^2} + \frac{2}{\bar{r}}\frac{d\bar{\theta}}{ds}\frac{d\bar{r}}{ds} - \sin\bar{\theta}\cos\bar{\theta}\left(\frac{d\bar{\phi}}{ds}\right)^2 = 0 \tag{7.90}$$

$$\bar{r}^2\sin^2\bar{\theta}\,\frac{d\bar{\phi}}{ds} = \text{constant} = p \text{ (say)} \tag{7.91}$$

together with the line-element itself which provides one integral. If motion is originally in the plane $\bar{\theta} = \pi/2$ say, $\frac{d\bar{\theta}}{ds}$ and $\cos\bar{\theta}$ are both initially zero and by (7.90) are then permanently zero. The equations then reduce upon substitution to

$$\frac{d\bar{r}}{ds} = \pm\left[C^2 - 1 + \frac{1}{\bar{r}} - \frac{p^2}{\bar{r}^2}\left(1-\frac{1}{\bar{r}}\right)\right]^{1/2} \tag{7.92}$$

$$\frac{d\bar{t}}{ds} = \frac{1}{(1-\frac{1}{\bar{r}})}\left[C + \frac{1}{\bar{r}}\frac{d\bar{r}}{ds}\right] \tag{7.93}$$

$$\frac{d\bar{\phi}}{ds} = \frac{p}{\bar{r}^2} \tag{7.94}$$

so that

$$\frac{d\bar{r}}{d\bar{t}} = \pm\frac{\left(1-\frac{1}{\bar{r}}\right)\left[C^2-1+\frac{1}{\bar{r}} - \frac{p^2}{\bar{r}^2}\left(1-\frac{1}{\bar{r}}\right)\right]^{1/2}}{C \pm \frac{1}{\bar{r}}\left[C^2-1+\frac{1}{\bar{r}} - \frac{p^2}{\bar{r}^2}\left(1-\frac{1}{\bar{r}}\right)\right]^{1/2}} \tag{7.95}$$

In the neighbourhood of $\bar{r} = 1$, $\frac{d\bar{r}}{d\bar{t}}$ behaves like $\pm\frac{(1-\frac{1}{\bar{r}})}{(1\pm\frac{1}{\bar{r}})}$ according to (7.95) showing that for non-radial motion particles and light rays will in the limit take a finite co-ordinate time to reach $\bar{r} = 1$ from finite $\bar{r} > 1$ but an infinite co-ordinate time to penetrate a finite distance into $\bar{r} > 1$ from $\bar{r} = 1$. In the neighbourhood of infinite $\bar{r}$, $\frac{d\bar{r}}{d\bar{t}}$ behaves like $\pm\frac{1}{c}$ so that the co-ordinate travel time from finite $\bar{r}$ to $\bar{r} = \infty$ will be infinite; moreover, so will the proper travel time for that journey, since at infinity we have $\frac{ds}{d\bar{r}}$ behaving like $\pm\frac{1}{(c^2-1)^{1/2}}$.

Suppose that a light ray is emitted from $\bar{r} = \bar{x}$ at $\bar{t} = 0$ (say) and that A is at any finite $\bar{r}$ ($= \bar{R}$ say) at that instant. Suppose thatthe light signal reaches A at $\bar{t} = \bar{t}_a$. If we allow $\bar{x}$ to tend to 1, then the travel time to any $\bar{r} > 1$ is infinite for the signal. The travel time for A from finite $\bar{R}$ to $\bar{r}$ is infinite only if $\bar{r} = 1$ or $\bar{r} = \infty$ and the proper travel time in the latter case is infinite. Thus A may not receive the signal from $r = 1$ in $r > 1$ in his own finite experience and this result is independent of the motion of the light ray or the observer himself.

Therefore, the surface $\bar{r} = 1$ is an E.H. for all observers situated in the region $\bar{r} > 1$ in the space-time given by (7.9). It is of a degenerate type, being invariant in the sense that it is observer independent.

(xi) <u>The transformation between the Finkelstein metric and its time-reversal.</u>

We have shown that the transformation of (7.1) by the equation

$$\pm t = \bar{t}_1 - \ln|\bar{r}_1 - 1| \qquad (7.96)$$

taking $r = \bar{r}_1, \theta = \bar{\theta}, \phi = \bar{\phi}$ preserves the natureof the barrier in the resulting space-time given by (7.9) for an observer in $\bar{r} > 1$. However, the so-called "time-reversal" of this metric, given by Finkelstein (1958) and obtained from (7.1) by the transformation

$$\pm t = \bar{t}_2 + \ln|\bar{r}_2 - 1| \qquad (7.97)$$

together with $r = \bar{r}_2, \theta = \bar{\theta}, \phi = \bar{\phi}$ does not apparently preserve the nature of the barrier for an observer in $\bar{r} > 1$, as we have demonstrated.

The equations used directly in the transformation processes are given by differentiating (7.96) and (7.97) so that we have, respectively,

$$\pm dt = d\bar{t}_1 - \frac{d\bar{r}_1}{\bar{r}_1 - 1} \qquad (7.98)$$

and

$$\pm dt = d\bar{t}_2 + \frac{d\bar{r}_2}{\bar{r}_2 - 1} \qquad (7.99)$$

For (7.7) to be the time-reversal of (7.9) we take $\bar{t}_1 = -\bar{t}_2$ so that $d\bar{t}_1 = -d\bar{t}_2$. If we take the + sign in front of dt in one of (7.98), (7.99) and the - sign in the other, this implies that

$$\frac{d\bar{r}_1}{\bar{r}_1 - 1} = \frac{d\bar{r}_2}{\bar{r}_2 - 1} \qquad (7.100)$$

for the transformation between the two metrics. Integration gives

$$\ln|\bar{r}_1 - 1| = \ln|\bar{r}_2 - 1| + \text{const.}$$

$$\text{i.e.} \quad (\bar{r}_1 - 1) = \pm K(\bar{r}_2 - 1) \qquad (7.101)$$

where K is a positive constant. Then either

(i) $\bar{r}_1 = K\bar{r}_2 + (1 - K)$ and $\frac{d\bar{r}_1}{d\bar{t}_1} = -K\frac{d\bar{r}_2}{d\bar{t}_2}$

or

(ii) $\bar{r}_1 = -K\bar{r}_2 + (1 + K)$ and $\frac{d\bar{r}_1}{d\bar{t}_1} = +K\frac{d\bar{r}_2}{d\bar{t}_2}$

In both cases $\bar{r}_1 = 1$ and $\bar{r}_2 = 1$ are simultaneously true whatever the value of K, but to reduce the first alternative to that considered by Finkelstein we take K = 1 so that we have

$$\text{(i)} \quad \bar{r}_1 = \bar{r}_2 \text{ and } \frac{d\bar{r}_1}{d\bar{t}_1} = -\frac{d\bar{r}_2}{d\bar{t}_2} \qquad (7.102)$$

$$\text{and (ii)} \quad \bar{r}_1 = 2 - \bar{r}_2 \text{ and } \frac{d\bar{r}_1}{d\bar{t}_1} = +\frac{d\bar{r}_2}{d\bar{t}_2} \qquad (7.103)$$

(xii) <u>Retention of physical features in a transformation.</u>

Of course, transforming either of (7.7) or (7.9) according to the second alternative, $\bar{r}_1 = 2 - \bar{r}_2$; $\bar{t}_1 = -\bar{t}_2$, does not yield the other "time-reversed" metric; but it is our contention that some physical features associated with the metrics resulting from this transformation are incorporated in those obtained from either of (7.7) or (7.9) by using the first alternative.

For the two models described by ($\bar{t}_1$, $\bar{r}_1$) and ($\bar{t}_2$, $\bar{r}_2$) to have the same physical features (in other words to nullify the physical effects of time-reversal), (7.102) and (7.103) show us that we must either retain $\bar{r}_1 = \bar{r}_2$ and interchange in one model the "in" and "out" labels describing the direction of motion according to (i), or else retain the labelling throughout and consider the region $\bar{r}_1 > 1$ to correspond to the region $\bar{r}_2 < 1$ or vice versa according to (ii). In this connection we recall our previous remark that time-reversal is equivalent to changing the sign of $\frac{d\bar{r}}{\bar{r}-1}$ and that if the sign of dr is retained (i.e. the "in" and "out" labelling), then this is achieved by considering $r < 1$, rather than $r > 1$; performing the operation of changing the sign of $\frac{d\bar{r}}{\bar{r}-1}$ together with the operation $\bar{t}_1 \leftrightarrow -\bar{t}_2$ will transform the original physical situation into itself.

The existence or otherwise of an E.H. for an observer depends essentially on the travel time of light, or any causal influence propagated into the future, from the surface under consideration to the observer. The above considerations show clearly why, when both are obtained from the Schwarzschild metric, one of the Finkelstein space-times possesses an E.H. at $\bar{r} = 1$ for the observer in $\bar{r} > 1$ whereas the other does not; for in each case we have considered propagation of causal influences in the direction of increasing $\bar{t}$, in the region $\bar{r} > 1$ and where labelling of the direction of motion has been according to the usual convention. (7.103) shows that the physical effects associated, in the region $\bar{r} > 1$, with the barrier at $\bar{r} = 1$ are transformed to the region $\bar{r} < 1$ upon time-reversal, under these conditions.

In (7.98) and (7.99) we may alternatively have taken the same sign in each to obtain a transformation from (7.7) to (7.9). Then

$$d\bar{t}_1 = d\bar{t}_2 \quad \text{and} \quad \frac{d\bar{r}_1}{\bar{r}_1 - 1} = - \frac{d\bar{r}_2}{\bar{r}_2 - 1} \qquad (7.104)$$

Neglecting the constant of integration, we obtain

$$\ln|\bar{r}_1 - 1| = -\ln|\bar{r}_2 - 1|$$

so that $$(\bar{r}_1 - 1) = \pm (\bar{r}_2 - 1)^{-1}$$

Either (i) $\bar{r}_1 = \frac{\bar{r}_2}{\bar{r}_2 - 1}$ so that $\frac{d\bar{r}_1}{d\bar{t}_1} = \frac{-1}{(\bar{r}_2 - 1)^2} \frac{d\bar{r}_2}{d\bar{t}_2}$ (7.105)

or (ii) $\bar{r}_1 = \frac{\bar{r}_2 - 2}{\bar{r}_2 - 1}$ so that $\frac{d\bar{r}_1}{d\bar{t}_1} = \frac{1}{(\bar{r}_2 - 1)^2} \frac{d\bar{r}_2}{d\bar{t}_2}$ (7.106)

equations which are symmetrical in $\bar{r}_1$, $\bar{r}_2$.

In these cases the barrier at $\bar{r} = 1$ in one model is transported to $\bar{r} = \pm\infty$ in the other. In (i), for small positive ε, $\bar{r}_2 = 1 + \varepsilon$ implies $\bar{r}_1 = \frac{1+\varepsilon}{\varepsilon}$ which tends to $+\infty$ as $\varepsilon \to 0$ but $\bar{r}_2 = 1 - \varepsilon$ gives $\bar{r}_1 = \frac{1-\varepsilon}{-\varepsilon}$ which tends to $-\infty$ as $\varepsilon \to 0$. Similarly, in (ii), $\bar{r}_2 = 1 + \varepsilon$ gives $\bar{r}_1 = -\frac{(1-\varepsilon)}{\varepsilon} \to -\infty$ as $\varepsilon \to 0$ whereas $\bar{r}_2 = 1 - \varepsilon$ implies $\bar{r}_1 \to +\infty$ as $\varepsilon \to 0$. No continuity exists through the surface $\bar{r} = 1$. This case gives a further interesting insight into the nature and properties of the transformations considered.

(xiii) <u>The validity of the transformations from the Schwarzschild metric; choice of sign in the transformation</u>.

The question remains whether either of the two Finkelstein metrics is a valid transformation from the Schwarzschild metric. The invariance of physical features concerning the E.H. would suggest that the one obtained from (7.98) together with $r = \bar{r}_1$ has more claim to validity than that obtained by (7.99) and $r = \bar{r}_2$. Let us examine this point.

Consider first equation (7.98), taking the + sign, that is

$$dt = d\bar{t}_1 - \frac{dr}{r-1} \qquad (7.107)$$

The condition $\quad \dfrac{dr}{r-1} = \dfrac{d\bar{r}_1}{\bar{r}_1 - 1} \qquad (7.108)$

yields either $r = \bar{r}_1$ or $r = 2 - \bar{r}_1$ and substitution into (7.1) will give the metric (7.9) for $r = \bar{r}_1$, but a different form for ds^2 if we take $r = 2 - \bar{r}_1$. (7.107) and (7.108) give

$$\frac{dt}{dr} = \frac{1}{(r-1)}\left[\frac{d\bar{t}_1}{d\bar{r}_1}(\bar{r}_1 - 1) - 1\right] \qquad (7.109)$$

Take first $r > 1$ and $\frac{dt}{dr} > 0$. Then by (7.109)

$$\frac{d\bar{t}_1}{d\bar{r}_1} > \frac{1}{(\bar{r}_1 - 1)} > 0 \qquad \text{if } r = r_1 > 1$$

whereas if $r = 2 - r_1$ nothing may be said about the sign of $\frac{d\bar{t}_1}{d\bar{r}_1}$.

However, if $\frac{dt}{dr} < 0$ and $r > 1$ we have by (7.109) $\frac{d\bar{t}_1}{d\bar{r}_1} < \frac{1}{\bar{r}_1 - 1}$ and this has the same sign as $\frac{dt}{dr}$ for certain if we take the possibility $r = 2 - \bar{r}_1$ rather than $r = \bar{r}_1$.

These results lead us to realise that in considering transformations it has always been tacitly assumed that the "in" label implying motion into the future in the direction of decreasing r ($\frac{dr}{dt} < 0$) and the "out" label implying motion into the future in the direction of increasing r ($\frac{dr}{dt} > 0$) do not themselves require a transformation under any circumstances. We now maintain that on physical grounds the sign of $\frac{dr}{dt}$ should be preserved upon transformation if it is required just to obtain an alternative description of the original model rather than a model which is physically completely distinct.

Some authors would consider a model obtained without this condition applied to the transformation to be equivalent to the original; we say this can be legitimate only if the transformation embodies the correct physical criteria.

In the present case of the Finkelstein transformation given by (7.107) and $r = \bar{r}_1$ physical features concerned with outward motion in $r > 1$ have been carried into the new model unchanged since for $r = \bar{r}_1$, $\frac{dr}{dt} > 0$ implies $\frac{d\bar{r}_1}{d\bar{t}_1} > 0$ but physical features concerning inward motion have not since $r = \bar{r}_1$ does not necessarily imply that if $\frac{dr}{dt} < 0$ then $\frac{d\bar{r}_1}{d\bar{t}_1} < 0$, although in this case $r = 2 - \bar{r}_1$ does. Thus the Finkelstein metric given by (7.9) and obtained mathematically from (7.1) by the transformation (7.98) together with $r = \bar{r}_1$, embodies, in part, physical features of the original Schwarzschild model when the sign of $\frac{dr}{dt}$ is preserved, that is for outward motion; in part, it embodies physical features obtained from the model resulting from the transformation (7.98) together with $r = 2 - \bar{r}_1$ when this ensures that $\frac{dr}{dt}$ suffers no change in sign, that is, for inward motion.

We know that the existence or otherwise of an E.H.

is essentially concerned with the light travel time from the surface considered to the observer; where the surface is $r = 1$ and the observer is considered to lie in the region $r > 1$, this means that the <u>outward</u> motion is concerned so explaining why the existence of the E.H. is preserved in the present case.

For the time-reversed model, taking the + sign in (7.99), we have

$$dt = d\bar{t}_2 + \frac{dr}{r-1} \qquad (7.110)$$

$$\frac{dr}{r-1} = \frac{d\bar{r}_2}{\bar{r}_2 - 1} \qquad (7.111)$$

Then $r = \bar{r}_2$ or $r = 2 - \bar{r}_2$ and

$$\frac{dt}{dr} = \frac{1}{(r-1)}\left[\frac{d\bar{t}_2}{d\bar{r}_2}(\bar{r}_2 - 1) + 1\right] \qquad (7.112)$$

For $r > 1$ and $\frac{dt}{dr} > 0$ we have from (7.112)

$$\frac{d\bar{t}_2}{d\bar{r}_2} > -\frac{1}{(\bar{r}_2 - 1)}$$

and $r = \bar{r}_2$ does not necessarily imply that $\frac{d\bar{t}_2}{d\bar{r}_2} > 0$ whereas $r = 2 - \bar{r}_2$ does. Similarly, for $\frac{dt}{dr} < 0$, $r > 1$, (7.112) gives

$$\frac{d\bar{t}_2}{d\bar{r}_2} < -\frac{1}{(\bar{r}_2 - 1)}$$

and it is the case $r = \bar{r}_2$ which definitely maintains the same sign for $\frac{dt}{dr}$ whereas $r = 2 - \bar{r}_2$ does not. Thus features concerning <u>inward</u> motion are preserved, but those for outward motion are not when $r = \bar{r}_2$. Therefore the physical features, such as the existence of the E.H. which in the Schwarzschild model is concerned with outward motion in $r>1$ towards the observer, have been unwittingly transferred to $\bar{r}_2<1$ by means of $r = 2 - \bar{r}_1$. Alternatively, we may consider that the "in" and "out" labels together with the signs attached have been unwittingly interchanged.

Thus by neglecting physical criteria Finkelstein has been referring to a mixture of two different models in retaining both $r = \bar{r}_1$ and the labels. This shows as unrealistic his imposing the requirement on the gravitational field that it be invariant under the discrete group generated by $t \to \bar{t} = -t$; $r \to \bar{r} = r$; rather should we have the condition of invariance of the sign of $\frac{dr}{dt}$ under any transformation, which would explain away many of Finkelstein's otherwise curious results regarding past-future asymmetry.

The fact that we have considered only the region $r>1$ in the Schwarzschild space-time and that there is time symmetry in this region explains why to some extent the $\pm$ signs in the transformations (7.98), (7.99) have been arbitrary; but our demonstration that $r = \bar{r}$ cannot be strictly maintained under all conditions in the transformed metrics (7.7), (7.9) if we require these to be physically equivalent with the

Schwarzschild model (but that we must sometimes have $r = 2 - \bar{r}$ so that $r > 1$ implies $\bar{r} < 1$), together with the fact that we have considered only the regions $\bar{r} > 1$, shows why the results relevant to (7.7) and (7.9) depend on the choice of this sign. In particular, this was manifested in the previous section where it was necessary to compare (7.58) to (7.60) with (7.52), (7.53) in order to determine which sign to take in the transformation. This example further demonstrates our point about the role of the "in", "out" labelling; for while the + sign gave $\frac{d\bar{r}}{d\bar{t}}$, inward or outward, unchanged, the - sign gave $\frac{d\bar{r}}{d\bar{t}}_{\text{IN}} \longleftrightarrow \frac{d\bar{r}}{d\bar{t}}_{\text{OUT}}$.

Thus, while each of (7.7) and (7.9) does indeed represent a cosmological model, we maintain that it is illegitimate to claim that either has been obtained from the Schwarzschild model by a transformation which is valid physically; the two models are not alternative representations of the Schwarzschild space-time, as claimed by Finkelstein, but are connected with it by a purely mathematical correlation and are physically distinct from it.

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