Hamiltonian paths, containing a given path or collection of arcs, in close to regular multipartite tournaments

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Abstract

A tournament is an orientation of a complete graph, and in general a multipartite or c-partite tournament is an orientation of a complete c-partite graph. If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x = y$). If $i_g(D) = 0$, then D is called regular.

Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If P is a directed path of length q in the c-partite tournament D such that

$$|V(D)| \geq 2i_g(D) + 3q + 2|V_c| + |V_{c-1}| - 2,$$

then we prove in this paper that there exists a Hamiltonian path in D, starting with the path P. Examples will show that this condition is best possible. As an application of this theorem, we prove that each arc of a regular multipartite tournament is contained in a Hamiltonian path. Some related results are also presented.

Keywords: Multipartite tournaments; Hamiltonian path; Regular multipartite tournaments

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1. Terminology

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. Multipartite tournaments are well studied (see e.g., Bang-Jensen and Gutin [2], Guo [3], Gutin [4], Volkmann [10], and Yeo [13]).

We shall assume that the reader is familiar with standard terminology on directed graphs (see, e.g., Bang-Jensen and Gutin [2]). In this paper all digraphs are finite without loops or multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we say that x *dominates* y. If X and Y are two disjoint subsets of $V(D)$ or subdigraphs of D such that there is no arc from Y to X, then we write $X \Rightarrow Y$. By $d_D(X,Y) = d(X,Y)$ we denote the number of arcs from X to Y, i.e., $d(X,Y) = |\{xy \in E(D) : x \in X, y \in Y\}|$.

The *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. For a vertex set X of D, we define $D\langle X\rangle$ as the subdigraph induced by X. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are the *outdegree* and the *indegree* of x, respectively. The *minimum outdegree* and the *minimum indegree* of D are denoted by $\delta^+ = \delta^+(D)$ and $\delta^- = \delta^-(D)$, respectively.

The *global irregularity* of a digraph D is defined by

$$i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$$

over all vertices x and y of D (including $x = y$), and the *local irregularity* by

$$i_l(D) = \max|d^+(x) - d^-(x)|$$

over all vertices x of D. If $i_g(D) = 0$, then D is *regular*.

By a *cycle* or *path* we mean a directed cycle or directed path. A path of length m is an *m-path*. A cycle or path in a digraph D is *Hamiltonian* if it includes all the vertices of D. A set $X \subseteq V(D)$ of vertices is *independent* if the induced subdigraph $D\langle X\rangle$ has no arcs. The *independence number* $\alpha(D) = \alpha$ is the maximum size among the independent sets of vertices of D. A *cycle-factor* of a digraph D is a spanning subdigraph consisting of disjoint cycles. A set of arcs A is called *path-extendible* if no arcs in A have the same head or tail, and the arcs in A don't induce any cycles (i.e., $D\langle A\rangle$ is acyclic and has $\delta^-, \delta^+ \leq 1$). If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$, then we define by $\gamma(D) = |V_1|$.

2. Preliminary Results

The following results play an important role in our investigations.

Lemma 2.1. If D is a multipartite tournament, then for every $x \in V(D)$

$$\frac{|V(D)| - \alpha(D) - i_g(D)}{2} \leq d^+(x), d^-(x) \leq \frac{|V(D)| - \gamma(D) + i_g(D)}{2}.$$

Proof. Let $x \in V(D)$ and let V_i be the partite set of D, which x belongs to. Assume, without loss of generality, that $d^+(x) \geq d^-(x)$. We note that $d^+(x)+d^-(x) = |V(D)|-|V_i|$ and $d^+(x) - d^-(x) \leq i_g(D)$. Adding these two inequalities, we obtain

$$2d^+(x) \leq i_g(D) + |V(D)| - |V_i| \leq |V(D)| + i_g(D) - \gamma(D).$$

We can analogously show that $2d^-(x) \geq |V(D)| - \alpha(D) - i_g(D)$, which completes the proof of the lemma. □

Lemma 2.2. If D is a digraph, $X \subseteq V(D)$ and $D' = D - X$, then $i_g(D') \leq i_g(D) + |X|$.

Proof. Clearly, $\max\{d^+_{D'}(x), d^-_{D'}(x) : x \in V(D')\} \leq \max\{d^+_D(x), d^-_D(x) : x \in V(D)\}$. Furthermore, $\min\{d^+_{D'}(x), d^-_{D'}(x) : x \in V(D')\} \geq \min\{d^+_D(x), d^-_D(x) : x \in V(D)\} - |X|$, as deleting X vertices from D cannot make any degree drop by more than $|X|$. Now the lemma follows from the definition of i_g. □

Lemma 2.3. Let D b a regular multipartite tournament, and let $X \subset V(D)$ be non-empty. Then $d(X, V(D) - X) = d(V(D) - X, X)$.

Proof. As D is regular, $d^+(x) = d^-(x)$ for all $x \in V(D)$. This implies that there is an Eulerian tour in D, which must enter and leave X an equal number of times. This completes the proof. □

Lemma 2.4 (Tewes, Volkmann, Yeo [8]). Let D be a c-partite tournament with partite sets $V_1, V_2, \ldots, V_c$. Then $||V_i| - |V_j|| \leq 2i_g(D)$. (In particular, all partite sets have the same size in a regular multipartite tournament.)

Next we will prove a generalization of the famous First Theorem of Petersen [6] that a graph is 2-factorizable if and only if it is $2p$-regular.

Theorem 2.5. A digraph D is the union of cycle-factors if and only if it is r-regular for some integer $r \geq 1$.

Proof. Of course, for a digraph to be the union of r cycle-factors, it is necessary that it be r-regular.

Conversely, suppose that D is r-regular. Let $V(D) = \{x_1, x_2, \ldots, x_n\}$. We define the bipartite graph G with partite sets $U = \{u_1, u_2, \ldots, u_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$, where $E(G) = \{u_i w_j : x_i x_j \in E(D)\}$. The bipartite graph G is r-regular and so, by the Theorem of König [5], is 1-factorizable. Now every 1-factor of G corresponds to a cycle-factor of D and thus, D is the union of r cycle-factors. □

The following theorem, is one of the main results in [14].

Theorem 2.6 (Yeo [14]). Let D be a c-partite tournament with partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. If

$$i_g(D) \leq \frac{|V(D)| - 2|V_c| - |V_{c-1}| + 2}{2},$$

then D has a Hamiltonian cycle.

3. Hamiltonian paths, starting with a given path

Theorem 3.1. Let D be a c-partite tournament with partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$, and let P be a path of length q in D. If

$$|V(D)| \geq 2i_g(D) + 3q + 2|V_c| + |V_{c-1}| - 2,$$

then there exists a k-path, P_k, starting with the path P, for all $k = q, q+1, \ldots, |V(D)|-1$. (In particular, there is a Hamiltonian path containing P.)

Proof. Let $P = p_0p_1 \ldots p_q$ and let $D' = D - \{p_0, p_1, \ldots, p_{q-1}\}$. Now note that the following holds, where $V_1', V_2', \ldots, V_{c'}'$ are the partite sets in D', where $|V_1'| \leq |V_2'| \leq \ldots \leq |V_c'|$.

- $|V(D')| = |V(D)| - q$
- $i_g(D') \leq i_g(D) + q$, by Lemma 2.2
- $|V_{c'}'| \leq |V_c|$ and $|V_{c'-1}'| \leq |V_{c-1}|$

These properties imply together with the hypothesis

$$\begin{aligned} |V(D')| + q = |V(D)| &\geq 2i_g(D) + 3q + 2|V_c| + |V_{c-1}| - 2 \\ &\geq 2(i_g(D') - q) + 3q + 2|V_{c'}'| + |V_{c'-1}'| - 2, \end{aligned}$$

and this leads to

$$i_g(D') \leq \frac{|V(D')| - 2|V_{c'}'| - |V_{c'-1}'| + 2}{2}.$$

Therefore, according to Theorem 2.6, D' contains a Hamiltonian cycle $c_1c_2 \ldots c_{|V(D')|}c_1$, where, without loss of generality, $c_1 = p_q$. Hence, the path $p_0p_1 \ldots p_{q-1}c_1c_2 \ldots c_{k-q+1}$ is a k-path in D, starting with P, for all $k = q, q+1, \ldots, |V(D)| - 1$. □

In Section 6 below we will show that Theorem 3.1 is best possible.

4. Hamiltonian paths, containing a given arc

The following theorem, is a slight reformulation (and weakening) of the main result in [12].

Theorem 4.1 (Yeo [12]). Let D be a multipartite tournament with a cycle-factor. Then there exists a cycle-factor $F = C_1 \cup C_2 \cup \ldots \cup C_p$ such that every vertex in C_i has an out-neighbor in C_{i+1}, for all $i = 1, 2, \ldots, p-1$, and either $d(C_1, F - V(C_1)) \geq 2 \times d(F - V(C_1), C_1)$ or $p = 1$.

The next result follows from a more general theorem by Amar and Manoussakis [1] and Wang [11].

Theorem 4.2 (Amar, Manoussakis [1], Wang [11]). Every arc of a regular bipartite tournament is contained in a Hamiltonian cycle.

Theorem 4.3. Every arc of a regular c-partite tournament D is contained in a Hamiltonian path of D.

Proof. If $c = 2$, then the desired result follows immediately from Theorem 4.2.

Let next $c \geq 4$. In view of Lemma 2.4, we see that all partite sets of D have the same size, namely $\alpha(D)$. Now the following holds, which by Theorem 3.1 completes the proof for $c \geq 4$.

$$2i_g(D) + 3 + 2\alpha(D) + \alpha(D) - 2 = 3\alpha(D) + 1 \leq |V(D)|$$

Finally, let $c = 3$ and let $e = uv$ be an arbitrary arc of D. According to Theorem 2.5, there exists a cycle-factor F, containing the arc $e = uv$. If F just contains one cycle, then we are done, so assume that this is not the case. Let C be the cycle of F containing e, and let $F' = F - V(C)$. By Theorem 4.1, we may assume that $F' = C_1 \cup C_2 \cup \ldots \cup C_p$, and that F' has the properties given in Theorem 4.1. As D is regular, Lemma 2.3 implies

$$d(C_1, F' - V(C_1)) + d(C_1, C) = d(F' - V(C_1), C_1) + d(C, C_1).$$

If any vertex except u has an arc to C_1, then, in view of Theorem 4.1, it is easy to find a Hamiltonian path containing e, by using this arc, and first picking up all vertices in C, then all vertices in C_1, then all in C_2, etc.

Thus, it remains the case that $V(C_1) \Rightarrow (V(C) - u)$. Since $|V(C)| \geq 3$ and D is a multipartite tournament, we conclude that $d(C_1, C) \geq |V(C_1)| \geq d(C, C_1)$.

If F' only contains one cycle, then, if $|V(C)| = 3$, then either $d(C_1, C) > d(C, C_1)$ or $d^+(u) = |V(C_1)| + 1 = |V(D)| - 2 \geq 4$ and $d^-(u) = 1$, giving rise to a contradiction in both cases. And if $|V(C)| > 3$, then $d(C_1, C) > d(C, C_1)$, a contradiction.

So assume that F' contains at least two cycles. It follows from Theorem 4.1 that $d(C_1, F' - V(C_1)) > d(F' - V(C_1), C_1)$ (as there are some arcs between C_1 and $F' - V(C_1)$). This is a contradiction against $d(C_1, F' - V(C_1)) + d(C_1, C) = d(F' - V(C_1), C_1) + d(C, C_1)$, and the proof is complete. □.

5. Hamiltonian paths, containing sets of arcs

The following theorem is proved in [15].

Theorem 5.1 (Yeo [15]). Let D be a multipartite tournament, and let x and y be different vertices in D. If $13i_g(D) + 11\alpha(D) - 18 < 5|V(D)|$, then there exists an (x, y)-path of length at most 5 in D.

For the main theorem of this section, we use the following lemma.

Lemma 5.2. Let D be a multipartite tournament, and let $A = \{a_1, a_2, \dots, a_k\}$ be a path-extendible set of arcs in D. If $13i_g(D) + 11\alpha(D) + 108k - 198 < 5|V(D)|$, then there exists a path P in D, containing all arcs of A, which has length at most $6k - 5$.

Proof. We will show how to construct a path P_q, containing q arcs of A, such that $|E(P_q)| \leq 6q - 5$, and if $D_q = D\langle A \cup E(P_q)\rangle$, then $\delta^-(D_q), \delta^+(D_q) \leq 1$ and D_q is acyclic (for $q = 1, 2, \dots, k$).

P_1 clearly exists (e.g., let $P_1 = a_1$). Now assume that $1 < q \leq k$, and that P_{q-1} exists. Assume that $P_{q-1} = p_0 p_1 \dots p_t$ such that $t \leq 6(q-1) - 5$. If there is an arc from A that is not on P_{q-1} but has a vertex in common with P_{q-1}, then the arc must leave p_t or enter p_0, so we can just add such an arc. So assume that no arc from $A - E(P_{q-1})$ has a vertex in common with P_{q-1}.

Let $a_i = uv$ be an arc from A, which is not on P_{q-1}. Let $X = V(A) \cup V(P_q) - \{p_t, u\}$. Note that

$$|X| \leq [6(q-1) - 5 + 1] + [2(k - q + 1)] - 2 = 4q + 2k - 10 \leq 6k - 10.$$

Let $D^* = D - X$, and note that the following holds.

- $i_g(D^*) \leq i_g(D) + |X| \leq i_g(D) + 6k - 10$
- $\alpha(D^*) \leq \alpha(D)$
- $|V(D^*)| \geq |V(D)| - (6k - 10)$

These properties imply together with the hypothesis

$$\begin{aligned} 5|V(D^*)| &\geq 5(|V(D)| - 6k + 10) \\ &> 13i_g(D) + 11\alpha(D) + 108k - 198 - 30k + 50 \\ &\geq 13i_g(D^*) - 78k + 130 + 11\alpha(D^*) + 78k - 148 \\ &= 13i_g(D^*) + 11\alpha(D^*) - 18. \end{aligned}$$

In view of Theorem 5.1, there exists a (p_t, u)-path of length at most 5 in D^*. By adding this path and uv, to P_{q-1}, we obtain a new path P_q, with all the desired properties. This proves the lemma. □

Theorem 5.3. Let D be a c-partite tournament, and let $A = \{a_1, a_2, \dots, a_k\}$ be a path-extendible set of arcs in D. Let $V_1, V_2, \dots, V_c$ be the partite sets in D such that $|V_1| \leq |V_2| \leq \dots \leq |V_c|$.

(a) If

$$|V(D)| > \max\left\{\frac{13i_g(D) + 11|V_c| + 108k - 198}{5}, 2i_g(D) + 2|V_c| + |V_{c-1}| + 18k - 18\right\},$$

then there exists a Hamiltonian path in D, containing all arcs of A.

(b) If

$$|V(D)| > 2.6i_g(D) + 2.2|V_c| + 0.8|V_{c-1}| + 21.6(k-1),$$

then there exists a Hamiltonian path in D, containing all arcs of A.

Proof. (a) Lemma 5.2 implies that there exists a path P in D of length of most $6k-5$, containing all the arcs of A. Because of $|V(D)| \geq 2i_g(D)+2|V_c|+|V_{c-1}|+3(6k-5)-2$, Theorem 3.1 yields that there exists a Hamiltonian path containing P, which contains all the arcs of A. This proves (a).

To prove (b), we do the following. Note that the condition in (a) is equivalent to

$$|V(D)| > \max\{2.6i_g(D)+2.2|V_c|+21.6(k-1)-18, 2i_g(D)+2|V_c|+|V_{c-1}|+18(k-1)\}.$$

As $2.2|V_c|+0.8|V_{c-1}| \geq 2|V_c|+|V_{c-1}|$, we see that $|V(D)| > 2.6i_g(D)+2.2|V_c|+0.8|V_{c-1}|+21.6(k-1)$ implies the condition in (a), and so (b) is proved. □

6. Examples of multipartite tournaments

We will show below that there exist infinitely many examples where Theorem 3.1 cannot be improved. First however we need the following lemma, which can easily be proved using the famous Euler Theorem for undirected graphs (see e.g. Exercise 3.4 in [9] on page 79).

Lemma 6.1. Every graph G has an orientation D such that $i_l(D) \leq 1$.

We now construct infinitely many examples showing that Theorem 3.1 is best possible. Consider the following construction.

Let a, d and q be any positive integers and let b be a positive integer, large enough so all of the following holds.

(i): $d(b+2a-1)+q \leq d(3b+2a-3)+1$

(ii): $d(b+2a-1)+q \leq d(2b+a-1)+\lfloor \frac{q}{2} \rfloor$

(iii): $d(3b+2a-3)+1 \geq d(2b+a-1)+\lceil \frac{q}{2} \rceil$

(iv): $d(b+2a-1)+q \leq d(2b-1)+2ad+\lfloor \frac{q+1}{2} \rfloor$

(v): $d(3b+2a-3)+1 \geq d(2b-1)+2ad+\lceil \frac{q+1}{2} \rceil$

The multipartite tournament we are constructing is going to have vertex set $Y \cup Z \cup R_1 \cup R_2$, which will be defined as follows. Let $T_1, T_2, \ldots, T_{2b-1}$ be disjoint sets of vertices, such that $|T_i| = 2d$ for all $i = 1, 2, \ldots, 2b-1$ and partition T_i into T_i' and T_i'' such that $|T_i'| = |T_i''| = d$ for all $i = 1, 2, \ldots, 2b-1$. Now let $R_1 = T_1' \cup T_2' \cup \ldots \cup T_{2b-1}'$ and let $R_2 = T_1'' \cup T_2'' \cup \ldots \cup T_{2b-1}''$. Let Y be a set of $2ad+1$ vertices and let Z be a set of $2ad+q$ vertices.

The partite sets of D will be the vertices in Z (which will be partite sets of size one), as well as $T_1, T_2, \ldots, T_{2b-1}$ and Y. Keeping these sets independent and add arcs so that the following is true: $R_2 \Rightarrow Z \Rightarrow R_1 \Rightarrow Y \Rightarrow R_2$ and $R_1 \Rightarrow R_2$. Furthermore, let $i_l(D\langle Y \cup Z \rangle) \leq 1$, $i_l(D\langle R_1 \rangle) \leq 1$, and $i_l(D\langle R_2 \rangle) \leq 1$ by Lemma 6.1.

As $D\langle Z \rangle$ is a tournament, let $P = p_0p_1 \ldots p_q$ be any q-path in Z (such a path exists, since every tournament has a Hamiltonian path by Redei's Theorem [7]).

We will now show that $|V(D)| = 2i_g(D) + 3q + 2(2ad+1) + 2d - 3$, where $2ad+1$ is the size of the largest partite set in D, and $2d$ is the size of the next largest partite set in D. Furthermore, we will show that there is no Hamiltonian path starting with the path P in D, which shows the optimality of Theorem 3.1.

Assume that there is a Hamiltonian path starting with the path P in D. As there are $2ad-1$ vertices in $Z - V(P)$ and $2ad+1$ vertices in Y, there must be two vertices from Y on the Hamiltonian path, with no vertices from Z on the subpath between them. However, this is a contradiction as every path from one vertex in Y to another vertex in Y, must go through a vertex in Z. This proves that there is no Hamiltonian path starting with P, in D. So we will now show that $|V(D)| = 2i_g(D)+3q+2(2ad+1)+2d-3$.

Since $2b-1$ is odd and $|T_i'| = |T_i''| = d$ $(i = 1, 2, \ldots, 2b-1)$, we obtain that $i_l(D\langle R_1\rangle)$ and $i_l(D\langle R_2\rangle)$ must be even, which implies that $i_l(D\langle R_1\rangle) = 0$ and $i_l(D\langle R_2\rangle) = 0$. Let $\Delta^{\max} = d(3b+2a-3)+1$ and let $\Delta^{\min} = d(b+2a-1)+q$, and note that by (i) above, we have $\Delta^{\max} \geq \Delta^{\min}$. The following now implies that $\max\{d^+(x), d^-(x) : x \in V(D)\} = \Delta^{\max}$, and $\min\{d^+(x), d^-(x) : x \in V(D)\} = \Delta^{\min}$.

(a): If $v \in R_1$, then $d^+(v) = d(v,Y)+d(v,R_2)+d(v,R_1) = (2ad+1)+d(2b-2)+d(b-1) = \Delta^{\max}$ and $d^-(v) = d(Z,v) + d(R_1,v) = (2ad+q) + d(b-1) = \Delta^{\min}$.

(b): If $v \in R_2$, then $d^+(v) = \Delta^{\min}$ and $d^-(v) = \Delta^{\max}$ (analogously to part (a)).

(c): If $v \in Y$, then $d^+(v) = d(2b-1) + d(v,Z)$ and $d^-(v) = d(2b-1) + d(Z,v)$. This implies (by (ii) and (iii)) that $\Delta^{\min} \leq d(2b-1) + \lfloor \frac{2ad+q}{2} \rfloor \leq d^+(v)$ and $d^-(v) \leq d(2b-1) + \lceil \frac{2ad+q}{2} \rceil \leq \Delta^{\max}$.

(d): If $v \in Z$, then (by (iv) and (v)) we obtain $\Delta^{\min} \leq d(2b-1) + \lfloor \frac{|Y|+|Z|-1}{2} \rfloor \leq d^+(v)$ and $d^-(v) \leq d(2b-1) + \lceil \frac{|Y|+|Z|-1}{2} \rceil \leq \Delta^{\max}$.

Therefore, $i_g(D) = \Delta^{\max} - \Delta^{\min} = d(2b-2)+1-q$. We note that $|V(D)| = 2d(2b-1)+(2ad+1)+(2ad+q) = d(4b+4a-2)+q+1$ and the size of the largest partite set in D is $2ad+1$ (namely Y), and the size of the second largest partite set is $2d$ (namely any of the T_i's). Now we obtain the following identity, which proves the optimality of Theorem 3.1

$$\begin{aligned} 2i_g(D) + 3q + 2(2ad+1) + 2d - 2 &= 2d(2b-2) + 2 - 2q + 3q + 4ad + 2 + 2d - 2 \\ &= d(4b+4a-2) + q + 2 \\ &= |V(D)| + 1 \end{aligned}$$

As a, d and q were chosen arbitrary, this gives us infinitely many examples on which Theorem 3.1 cannot be improved. Even for given a, d and q there are also infinitely many b values for which Theorem 3.1 cannot be improved. Note that we didn't have to let $D\langle Z\rangle$ be a tournament. We could let it be any multipartite tournament with partite sets smaller than or equal to $2d$.

Note that if we only need to find a Hamiltonian path that contains a given path P, not necessarily starting with P, then Theorem 3.1 still gives a good bound. This is the case, since if we let $|Y| = ad+2$ in the above construction, then we see that

$|V(D)|+5 = 2i_g(D)+3q+2(2ad+1)+2d-2$, and that there is no Hamiltonian path in D containing the q-path P. So Theorem 3.1 could be improved by at most a constant, namely four.

7. Open problems

Problem 7.1. Determine the optimum coefficients in Theorem 5.3.

This means the following. Let D be a c-partite tournament, where V_c denotes the largest partite set in D and V_{c-1} denotes the next largest partite set in D. Find the optimal coefficients a, b, c, d and e, such that $|V(D) > a \times i_g(D) + b \times |V_c| + c \times |V_{c-1}| + d \times (k-1) + e$ implies that there is a Hamiltonian path containing any set of k arcs in D, which are path-extendible.

Other possible open problems could be to find the optimal coefficients for Theorem 3.1, if we only require our paths to contain P, and not necessarily start with P. By our construction in Section 6, we see that we cannot improve Theorem 3.1 by more than a constant (which is at most 4).

It seems natural to ask about Hamiltonian cycles containing a given arc or path, or even a collection of arcs. It was shown in [15], that there exist infinitely many regular 3-partite tournaments, which do not have a Hamiltonian cycle through a given arc. However, in [15] it was proved that there are at most a finite number of c-partite tournaments, with $c \geq 4$, which do not have a Hamiltonian cycle containing a given arc. In fact the following stronger result was shown.

Theorem 7.2 (Yeo [15]) Let D be a c-partite tournament, with $c \geq 4$, and let P be a q-path in D, such that $|V(D)| > 220 i_g(D) + 221q + 6436$. Then there exists a Hamiltonian cycle in D, containing the path P.

The following conjecture was first stated by the first author, and again in [15].

Conjecture 7.3. Let D be a regular c-partite tournament with $c \geq 4$. Then every arc is contained in a Hamiltonian cycle.

Note that the following was proved in [12].

Theorem 7.4 (Yeo [12]) Every regular multipartite tournament has a Hamiltonian cycle.

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