STUDIES IN NON-LINEAR OPTICS

THESIS PRESENTED FOR THE DEGREE OF
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BY

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Abstract

A general method of approach to resonant non-linear optical phenomena involving travelling waves has been developed. Maxwell's equations are solved for the electric field in a spherical Fabry-Perot type optical resonator, which encloses uniaxial arisotropic media. The specific case of propagation perpendicular to the optic axis is considered but the theory can be extended to cover the general case including double refraction. In the presence of more than one optical field of this form, when the medium enclosed in the resonator is non-linear, by expanding the polarization in terms of the electric field in the normal way coupled mode equations are obtained for amplitudes of the eigenmodes concerned.

This general formalism is then used to examine resonant second harmonic generation in the small conversion approximation from a fundamental beam in the lowest order 0-0 mode. Analytical solutions are obtained in three limiting cases, weak focussing, strong focussing and the focus removed infinitely from the non-linear medium. The general case is solved numerically. From the results the values of the variable parameters can be obtained which give the maximum output in any given mode. Graphs are presented giving the output variation in 0-0 and 0-2 modes with focus position, phase matching, focussing and spot size. From there it can be seen that the optimum focussing for the 0-0 mode occurs at $\frac{l}{z_o} = 5.65$ (l crystal length, zo one half the confocal parameter).

Secondly degenerate parametric amplification between two lowest order modes is examined under the approximation that the pump beam is undepleted. An analytic solution is obtained for the single pass amplification in the weak focussing limit and preliminary results of numerical computations for the general case are given. From these results a value for the optimum threshold condition is calculated.

CONTENTS

Chapter 1

1.1 Introduction

The term "non-linear optics" is used in this thesis to describe processes in which energy exchange occurs between a number of optical fields at different frequencies. The discovery and development of lasers made available for the first time extremely high electric fields 100Kv/cm at optical frequencies which although still small compared with atomic field strengths 10^5Kv/cm make the classical assumption of a linear response of the material to the field untrue. When the non-linearities are included in the theory they not only modify the linear effects but also give rise to completely new non-linear effects.

The first observation of these effects was made by Franken et al (1) in 1961 using a ruby laser focussed into a quartz crystal. The emergent light was found to contain radiation not only at 0.6943 microns, the ruby laser wavelength, but also at 0.3472 microns the second harmonic wavelength. Since then many other non-linear processes have been observed.2, 3, 4, 5.

If a dielectric medium is subjected to an optical electric field E which is small it sets up a polarization in the medium that is linear in the field.

(1.1) $P = \chi^{(1)} E$ (ignoring the vector nature of the electric field)

At higher field strengths we have to consider the right hand side of

1.1 as the first term in a power series expansion of P in terms of E so that in general (1.2) $P = \chi^{(1)}E + \chi^{(2)}E^2 + \chi^{(3)}E^3 + \ldots\ldots\ldots\ldots$ The non linear optical effects arise from the higher order terms of this series.

We shall consider only non linear effects which arise from the second term, specifically second harmonic generation and parametric amplification.

1.2 Non Linear Oscillator Model

The classical theory of linear dispersion (6, 7.) giving rise to eq.. 1.1 is based on a model due to Lorentz that pictures each electron in a dielectric medium as being held in its equilibrium position by a harmonic restoring force. When an electric field is applied to the dielectric the electron moves according to the equation of motion.

1.3

$$\frac{d^2r}{dt^2} - 2\gamma\frac{dr}{dt} + \omega_0^2 r = -\frac{e}{m}E$$

Where r is the displacement of the electron from its equilibrium position, m its mass, e its charge, ω_0 the natural frequency of its motion and γ a damping parameter.

From this equation the polarization density of the medium can be calculated.

1.4 $P = Ner$, N electron density. To extend this to produce non linear effects we consider that if the electron has a large enough displacement the restoring force is no longer linear and can be written in the form

1.5 $F = -\omega_0^2 r + \lambda r^2 + \ldots\ldots\ldots\ldots$ (small)

and the displacement of the electron is now governed by the equation.

1.6 $$\frac{d^2r}{dt^2} + 2\gamma\frac{dr}{dt} + \omega_0^2 r - \lambda r^2 = \frac{-eE}{m}$$

The solution of which for an electric field of the form

1.7 $$\bar{E} = Ee^{-i\omega_1 t} + Ee^{i\omega_1 t}$$

is

1.8
$$r = \frac{-e}{m}\,\frac{E(\omega_1)\quad e^{-i\omega_1 t}}{(\omega_0^2 - 2i\gamma\omega_1 - \omega_1^2)} + \text{complex conjugate}$$

$$+ \frac{e^2}{m^2}\lambda E^2(\omega_1)\frac{e^{-2i\omega_1 t}}{(\omega_0^2 - 4i\gamma\omega_1 - 4\omega_1^2)(\omega_0^2 - 2i\gamma\omega_1 - \omega_1^2)^2}$$

$$+ \frac{e^2}{m^2}\lambda E(\omega_1)E^*(\omega_1)\frac{1}{\omega_0^2(\omega_0^2 + 2i\gamma\omega_1 - \omega_1^2)(\omega_0^2 + 2i\gamma\omega_1 - \omega_1^2)}$$

+ complex conjugate + ...

It can be seen that we have new terms over the solution to the linear equation which are quite distinct from the linear terms. The first will give rise to a polarization oscillating at the first harmonic frequency of the applied electric field and the second a constant polarization proportional to the magnitude of the applied electric field.

1.3 The Susceptibility Tensors.

(a) Definition. Since we will be dealing with optical fields which are monochromatic or nearly monochromatic it is more convenient to work with the Fourier transform of the electric field.

$$1.9 \quad \underset{\sim}{E}(\underset{\sim}{r},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\ \underset{\sim}{E}(\underset{\sim}{r},t)\ \exp(i\omega t)$$

than with the electric field itself.

From 1.9 it follows that

$\underset{\sim}{E}(\underset{\sim}{r},\omega) = \underset{\sim}{E}^{*}(\underset{\sim}{r},-\omega)$ since $\underset{\sim}{E}(\underset{\sim}{r},t)$ is real.

The Fourier transform of the monochromatic field defined in equation 1.7 is given by

$$1.10 \quad \underset{\sim}{E}(\omega) = \underset{\sim}{E}^{\omega}\,\delta(\omega_1-\omega) + E^{\omega *}\,\delta(\omega_1+\omega)$$

The polarization response of the medium in terms of the Fourier transforms of the electric field is given by:

(1) for the linear response

$$1.11 \quad P_i^{(1)}(t) = \int_{-\infty}^{\infty} d\omega\ X_{ij}(\omega)\ E_j(\omega)\ \exp(i\omega t)$$

which defines the linear susceptibility tensor.

$X_{ij}(\omega)$

for the second order response:

$$1.12 \quad P_i^{(2)}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega_1\ d\omega_2\ X_{ijk}(\omega_1,\omega_2)\ E_j(\omega_1)\ E_k(\omega_2)$$

defining the 2nd order susceptibility tensor.

Since $P_i(t)$ is real we have

$$1.13 \quad X_{ij}(\omega) = X^{*}_{ij}(-\omega)$$

$$1.14 \quad X_{ijk}(\omega_1, \omega_2) = X^{*}_{ijk}(-\omega_1, -\omega_2)$$

The order of writing the electric fields in equation 1.12 has no significance, therefore

$$1.15 \quad X_{ijk}(\omega_1, \omega_2) = X_{ijk}(\omega_2, \omega_1)$$

The Fourier transform of the polarization is defined as

$$1.16 \quad P_i(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_i(t) \exp(i\omega t)\, dt$$

Substituting for $P_i(t)$ from 1.12 and carrying out two of the integrations we have

$$1.17 \quad P_i^{①}(\omega) = X_{ij}(\omega)\, E_j(\omega)$$ and substituting ω^1 for ω_1

$$1.18 \quad P_i^{②}(\omega) = \int_{-\infty}^{\infty} d\omega^1\, X_{ijk}(\omega^1, \omega-\omega^1)\, E_j(\omega^1)\, E_k(\omega-\omega^1)$$

To show that this corresponds with the normal definition of the non linear susceptibility consider the case of sum mixing of two monochromatic fields.

$$1.19 \quad \underset{\sim}{E}(\omega_1) = \tfrac{1}{2}\underset{\sim}{E}^{\omega_1}\delta(\omega-\omega_1) + \tfrac{1}{2}E^{*\omega_1}\delta(\omega+\omega_1)$$

$$1.20 \quad \underset{\sim}{E}(\omega_2) = \tfrac{1}{2}\underset{\sim}{E}^{\omega_2}\delta(\omega-\omega_2) + \tfrac{1}{2}E^{*\omega_2}\delta(\omega+\omega_2)$$

Substituting into 1.18 and carrying out the integration picking out the coefficients of $\delta(\omega-\omega_1+\omega_2)$ and $\delta(\omega+\omega_1+\omega_2)$

1.21 $P_i^{(2)}(\omega_3) = \left(\frac{1}{4}X_{ijk}(\omega_1, \omega_2)E_j^{\omega_1}E_k^{\omega_2} + \frac{1}{4}X_{ijk}(\omega_2, \omega_1)E_j^{\omega_2}E_k^{\omega_1}\right) \times$

$\delta(\omega - \omega_1 + \omega_2) + \left(\frac{1}{4}X^*_{ijk}(\omega_1, \omega_2)E_j^{*\omega_1}E_k^{*\omega_2} + \frac{1}{4}X^*_{ijk}(\omega_2, \omega_1)E_j^{*\omega_2}E_k^{*\omega_1}\right) \times$

$\delta(\omega + \omega_1 + \omega_2)$

where $\omega_3 = \omega_1 + \omega_2$.

using the intrinsic symmetry of Xijk and changing the dummy suffices.

1.22 $P_i^{(2)}(\omega_3) = \frac{1}{2}X_{ijk}(\omega_1, \omega_2)E_j^{\omega_1}E_k^{\omega_2}\ \delta(\omega - \omega_1 + \omega_2)$

$+ \frac{1}{2}X^*_{ijk}(\omega_1, \omega_2)\ E_j^{*\omega_1}\ E_k^{*\omega_2}\ \delta(\omega + \omega_1 + \omega_2)$

which is just the form for monochromatic polarisation at frequency

$$\omega_3 = \omega_1 + \omega_2$$

and we can write

1.23 $P_i^{\omega_3} = X_{ijk}(\omega_1, \omega_2)\ E_j^{\omega_1}E_k^{\omega_2}$

1.24 (a) $P_i^{(2)}(\omega_3) = \frac{1}{2}P_i^{\omega_3}\ \delta(\omega - \omega_3) + \frac{1}{2}P_i^{*\omega_3}\ \delta(\omega + \omega_3)$

For second harmonic generation

$$\underset{\sim}{E}(\omega_1) = \frac{1}{2}\underset{\sim}{E}^{\omega_1}\ \delta(\omega_1 - \omega) + \frac{1}{2}E^{*\omega_1}\ \delta(\omega_1 + \omega)$$

so the polarization at the harmonic frequency is

$P_i(2\omega_1) = \frac{1}{4}X_{ijk}(\omega_1, \omega_1)\ E_j(\omega_1)\ E_k(\omega_1)\ \delta(\omega - 2\omega_1)$

$+ \frac{1}{4}X_{ijk}\ E_j^*(\omega_1)\ E_k(\omega_1)\ \delta(\omega + 2\omega_1)$

so that

1.24(b) $P^{2\omega_1} = \frac{1}{2}X_{ijk}(\omega_1, \omega_1)\ E_j^{\omega_1}\ E_k^{\omega_1}$

This definition is the same as Bloembergen, 9, Robinson, 11, but will differ from that of Kleinman, 12, when the d_{im} tensor is defined in

$$A_{ij} = -\delta_{ij}$$

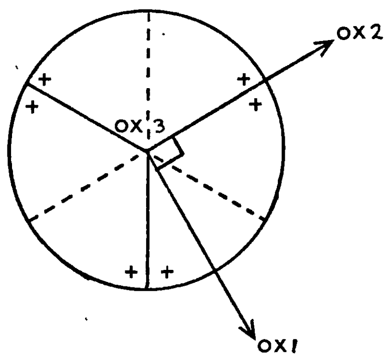
if the medium has inversion symmetry the susceptibility tensors must be invariant under this transformation.

$$X_{ijk}(\omega_1, \omega_2) = (-1)^3 X_{ijk}(\omega_1, \omega_2) \quad \text{and therefore}$$

$$X_{ijk} = 0$$

so the second order tensor vanishes identically.

In general the process of finding all the restrictions is rather tedious so we will consider one special case that of Lithuim Niobate Li Nb O_3. for reasons which will become apparent later. Lithuim Niobate has point group symmetry 3m, (C_{3v}), a threefold axis of symmetry with three reflection planes passing through the rotation axis.



The diagram represents a project of a sphere with points marked on the top surface at the positions marked by the = signs. The axes are taken as shown with the Ox, axis being the axis of threefold symmetry vertical through the centre.

The rotation transformation is given by the matrix equation 1.36.

b. Spatial Symmetry.

The susceptibility tensors transform between different co-ordinate systems according to the usual transformation laws for polar tensors. If we transform the tensor with a symmetry transformation of the medium the new tensor must be identical with the old one. Hence this restricts the values of the elements of the tensor. A full description of symmetry groups is given in the International Tables for X-ray Crystallography, 13, and Nye, 12.

The transformation laws are easily derived. Consider two co-ordinate systems which are related to each other by a rotation. The co-ordinates of a point with respect to the first set of axes will be related to those of the second set by an orthogonal linear transformation.

1.25 $$X_i^{1} = A_{ij} X_j$$

The electric field and polarization being both polar vectors transform in the same way as the co-ordinates.

1.26 $$P_i^{1}(t) = A_{ij} P_j(t)$$

$$E_i^{1}(\omega) = A_{ij} E_j(\omega)$$

Thus using 1.11, 1.12, the susceptibility tensors transform as

1.27 $$X^{1}_{ij}(\omega) = A_{ir} A_{js} X_{rs}(\omega)$$

$$X^{1}_{ijk}(\omega_1, \omega_2) = A_{ir} A_{js} A_{kt} X_{rst}(\omega_1, \omega_2)$$

As a general example we consider the inversion transformation

1.28
$$A = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ -\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and } A^2$$

and the reflection transformations

1.29
$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

AR and A^2R

Consider the effect of the reflection on $X_{ijk} \ (\omega_1, \omega_2)$

$$X_{221} (\omega_1, \omega_2) = R_{2i} \ R_{2j} \ R_{1k} \ X_{ijk} \ (\omega_1, \omega_2)$$

$$= R_{2i} \ R_{2j} \cdot - X_{ij1} \ (\omega_1, \omega_2)$$

$$= -X_{221} \ (\omega_1, \omega_2)$$

1.30 $\therefore \ X_{221} \ (\omega_1, \omega_2) \equiv 0$

Similarly any element which has an odd number of subscripts equal to one is identically zero.

This reduces the two tensors to

1.31
$$X^{①} = \begin{pmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & X_{23} \\ 0 & X_{32} & X_{33} \end{pmatrix}$$

1.32 $$X^{②} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & X_{131} & X_{113} & X_{112} & X_{121} \\ X_{211} & X_{222} & X_{223} & X_{233} & X_{232} & 0 & 0 & 0 & 0 \\ X_{311} & X_{322} & X_{333} & X_{323} & X_{332} & 0 & 0 & 0 & 0 \end{pmatrix}$$

applying the rotation transformation these reduce further to

1.33 $$X^{①} = \begin{pmatrix} X_{11} & 0 & 0 \\ 0 & X_{11} & 0 \\ 0 & 0 & X_{33} \end{pmatrix}$$

and

1.34 $$X^{②} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & X_{131} & X_{113} & -X_{222} & -X_{222} \\ -X_{222} & X_{222} & 0 & X_{113} & X_{131} & 0 & 0 & 0 & 0 \\ X_{311} & X_{311} & X_{333} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for the case of second harmonic generation we are dealing with the tensor $X_{ijk}(\omega_1, \omega_1)$ which from equation 1.15 is symmetric in j and k. This symmetry reduces the number of independant elements of $X^{②}$ still further since $X_{113} = X_{131}$ and we can write the tensor in a reduced form similar to that used for piezo electric tensors (33).

Define $d_{im} = X_{ijk}(\omega_1, \omega_1)$ when $j = k$.

1.35 $$d_{im} = \tfrac{1}{2}(X_{ijk}(\omega_1, \omega_1) + X_{ikj}(\omega_1, \omega_1)) \qquad j \neq k$$

when m is given by

m =	1	2	3	4	5	6
when jk =	11	22	33	23	31	12

Thus for crystals with symmetry point group 3m as Lithuim Niobate

$$1.36 \quad d_{im} = \begin{pmatrix} 0 & 0 & 0 & 0 & d_{15} & -d_{22} \\ -d_{22} & d_{22} & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{pmatrix}$$

The 3 x 6 matrix d_{im} operates on a column vector $(E\ E)_m$ where

$$(\underset{\sim}{E}\,\underset{\sim}{E})_1 = E_1^2, \quad (\underset{\sim}{E}\,\underset{\sim}{E})_2 = E_2^2, \quad (\underset{\sim}{E}\,\underset{\sim}{E})_3 = E_2^2,$$

$$(\underset{\sim}{E}\,\underset{\sim}{E})_4 = E_2E_3, \quad (\underset{\sim}{E}\,\underset{\sim}{E})_5 = E_1\,E_3 \text{ and } (\underset{\sim}{E}\,\underset{\sim}{E})_6 = E_1\,E_2.$$

c. <u>Kleinman's Symmetry Condition</u>

In addition to these symmetry restrictions there are other restrictions which apply when the non linear polarization is of electronic origin rather than ionic and the crystal is non absorbing for all the optical fields concerned in the interaction.

So the power loss P of the field $\underset{\sim}{E}\,(\underset{\sim}{r},t)$ is zero. These restrictions were first put forward by Kleinman, 12, and the argument given here follows that given by him, 15.

The average power loss in time 2T is given by

$$1.37 \quad P = -\frac{1}{2T}\int_{-T}^{T} dt \int d^3r \; E_i\,(r,t)\cdot \frac{d}{dt}\; P_i\;(r,t).$$

and must be zero under our approximations. Replacing $\underset{\sim}{E}$ and $\underset{\sim}{P}$ by their Fourier transforms,

$$P = \frac{i\pi}{T}\int d^3r \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\, d\omega^1\; E_i^*\;(r,\omega)\;\; P_i\;\;(r,\omega^1)\,\omega^1\,\delta_T(\omega-\omega^1)$$

where

as $T \to \infty$ $\quad \frac{\delta_T}{T}(\Omega) \to \delta(\Omega)$

$\therefore$ for T large enough

$$P = i \int d^3r \int_{-\infty}^{\infty} d\omega \; E_i^* (\underset{\sim}{r}, \omega) \; P_i (\underset{\sim}{r}, \omega) \; \omega$$

substituting from 1.18.

$$1.38 \quad P = i \int d^3r \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega^1 \; E_i^* (\underset{\sim}{r}, \omega) \; X_{ijk} (\omega^1, \omega - \omega^1)$$
$$\times \; E_j (\underset{\sim}{r}, \omega^1) \; E_k (\underset{\sim}{r}, \omega - \omega^1) \; \omega \; .$$

writing $\omega = \omega^1 + (\omega - \omega^1)$ Since

X is symmetric in the last two subscripts, We see that the integral with ω^1 is equal to that with $\omega - \omega^1$.

so

$$P = 2i \int d^3r \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega \; d\omega^1 \; E_i^* (\underset{\sim}{r}, \omega) \; X_{ijk} (\omega^1, \omega - \omega^1)$$
$$\times \; E_j (\underset{\sim}{r}, \omega^1) \; E_k (\underset{\sim}{r}, \omega - \omega^1)$$

using $\underset{\sim}{E} (\underset{\sim}{r}, \omega) = \underset{\sim}{E}^* (\underset{\sim}{r}, -\omega)$ and re-arranging we have

$$1.39 \quad P = -2i \int d^3r \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega \, d\omega^1 \; E_i^* (\underset{\sim}{r}, \omega) \; X_{jik} (-\omega, \omega - \omega^1)$$
$$\times \; E_j (\underset{\sim}{r}, \omega^1) \; E_k (\underset{\sim}{r}, \omega - \omega^1)$$

but since $P = 0$

comparing with 1.38, they are both equal to zero, therefore we have

$$^{(\omega^1)}X_{jik} (-\omega, \omega - \omega^1) = {}^{(-\omega)}X_{ijk} (\omega^1, \omega - \omega^1)$$

or writing X with its three associated frequencies.

1.40 $X_{jik}(\omega_2, \omega_3, \omega_1) = X_{ijk}(\omega_3, \omega_2, \omega_1)$

Armstrong et al, 16, and others have carried out Quantum Mechanical calculations for the form of the X' s under the conditions we have stated and found this symmetry and also the further symmetry.

1.41 $X_{ijk}(\omega_3, \omega_2, \omega_1) = X_{kji}(\omega_1, \omega_2, \omega_3)$ is predicted.

Ward and Franken, 19, have discussed these symmetries in the presence of dispersion and absorption and concluded that they are relatively insensitive to small amounts of dispersion and absorption. The measurements that have been carried out seem to verify these predictions. Several tables of values appear in the literature, e.g. Robinson, 11, and and Yariv, 20, (Yariv defines his values as Kleinman, 15, et al, Robinson, 11, as Bloembergen et all, 9) 1.4.

Propagation of E.M. radiation in a Dielectric medium

The propagation of electromagnetic radiation in a dielectric whose magnetic polarization is negligable is described by solution of Maxwell's equations.

$$\nabla \times \underset{\sim}{E} = -\frac{1}{C}\frac{dH}{dt}$$

1.42 $$\nabla \times \underset{\sim}{H} = \frac{1}{C}\frac{dD}{dt}$$

where $\underset{\sim}{D} = \underset{\sim}{E} + 4\pi \underset{\sim}{P}$

Eliminating the magnetic field $\underset{\sim}{H}$ from these equations in the usual way produces the equation

$$\nabla \times (\nabla \times \underset{\sim}{E}) + \frac{1}{c^2}\frac{d^2 E}{dt^2} = \frac{-4\pi}{c^2}\frac{d^2 P}{dt^2}$$

for the electric field.

Where $\underset{\sim}{E}$ and $\underset{\sim}{P}$ are functions of $\underset{\sim}{r}$ and t. Taking the Fourier transform of this equation it produces

$$1.44 \quad \nabla \times (\nabla \times \underset{\sim}{E}(\underset{\sim}{r}, \omega) - \frac{\omega^2}{c^2} \underset{\sim}{E}(\underset{\sim}{r}, \omega) = \frac{+4\pi\omega^2}{c^2} \underset{\sim}{P}(\underset{\sim}{r}, \omega)$$

under the same conditions of convergence as equation 1.9 which are discussed by Butcher, 22, $\underset{\sim}{P}(\underset{\sim}{r}, \omega)$ as we have seen can be expanded in terms of the electric field $\underset{\sim}{E}$ and the susceptibility tensors.

a. Medium with linear response

The polarization in this case is given by

$$1.45 \quad \underset{\sim}{P}(\underset{\sim}{r}, \omega) = \chi^{(1)}(\omega)\, \underset{\sim}{E}(\underset{\sim}{r}, \omega)$$

substituting into 1.44 and dropping the explicit frequency label, which is an unnecessary complication in the linear regime, we have

1.46

$$\text{a.} \quad \nabla \times (\nabla \times \underset{\sim}{E}) - \frac{\omega^2}{c^2} \epsilon \cdot \underset{\sim}{E} = 0$$

where

$$\text{b.} \quad \epsilon = 1 + 4\pi\chi^{(1)}$$

To investigate the propagation of a plane wave in this medium we write

$$1.47 \quad \underset{\sim}{E} = E_0 \, \underset{\sim}{a} \exp\left(\frac{i\omega}{c} \eta \, \underset{\sim}{n} \cdot \underset{\sim}{r}\right)$$

where $\underset{\sim}{r}$ is the position vector, $\underset{\sim}{n}$ the direction of propagation and η the refractive index, $\underset{\sim}{a}$ the direction of the polarization which are to be obtained. Substituting 1.47 into 1.46 we obtain Fresnel's equation

$$1.48 \quad \eta^2 (\underset{\sim}{n} \times (\underset{\sim}{n} \times \underset{\sim}{a})) + \epsilon \cdot \underset{\sim}{a} = 0$$

At this point we assume that ϵ is real, i.e. the medium is non-absorbing. Equation 1.48 represents a system of simultaneous homogenous linear equations for $\underset{\sim}{a} = (a_1, a_2, a_3)$. The equations are simplified by choosing axes along the principal axes of the dielectric tensor, they will be the same as those of $X^{(1)}(\omega)$ from 1.46(b). Applying the consistency condition gives a quadratic equation for η^2. For an isotropic medium where

$$\epsilon = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}$$

The equation for η^2 reduces to

$(\eta^2 - \epsilon)^2 = 0$ substituting this into 1.48 we have $\underset{\sim}{a} \cdot \underset{\sim}{n} = 0$ determining $\underset{\sim}{a}$.

In a uniaxial medium, such as the previous example, lithuim niobate, the dielectric tensor has the form of equation 1.33

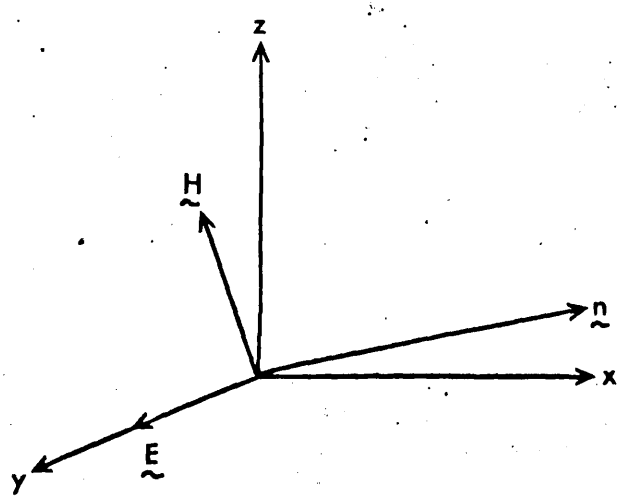
$$\epsilon = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix}$$

The orientation of the x and y principal axes is arbitrary and it is convenient to choose them so that $\underset{\sim}{n}$ lies in the Oxz plane and we can put

$$n_x = \sin\theta, \; n_y = 0, \; n_z = \cos\theta$$

where θ is the angle between the z axis and the direction of propagation. In this case the equation for η^2 reduces to

1·1



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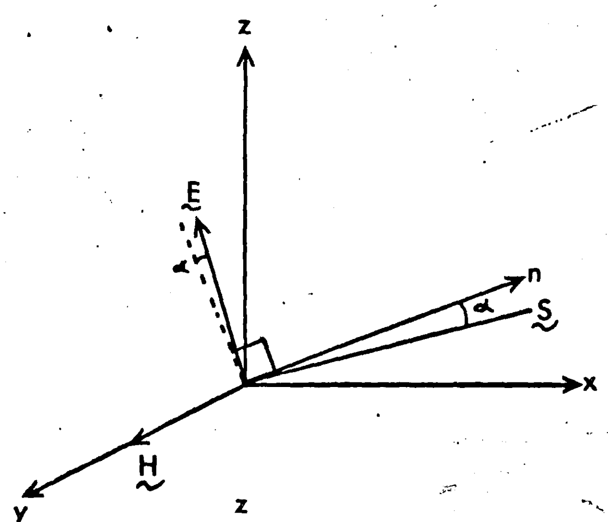

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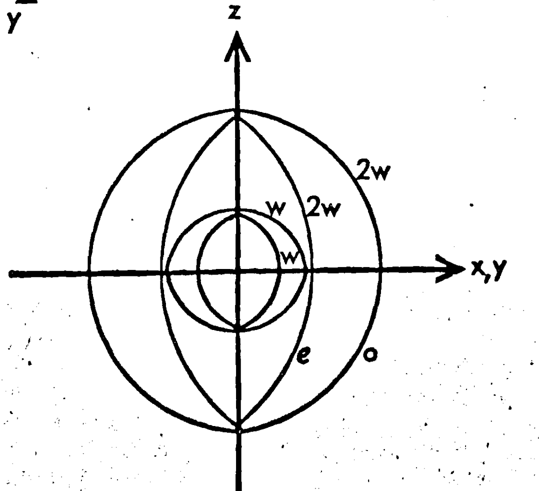

1.49 $(\eta^2 - \epsilon_1)(\eta^2(\epsilon_2\cos^2\theta + \epsilon_1\sin^2\theta) - \epsilon_1\epsilon_2) = 0$

Thus for one solution $\eta^2 = \epsilon_1$ which is independent of the direction of propagation. For this solution from 1.48

1.50 $\underset{\sim}{a} = (0, 1, 0)$

i.e. $\underset{\sim}{a}$ is orthogonal to both the direction of propagation and the z axis as in figure 1.1.

The second solution has refrective index given by.

1.51 $$\frac{1}{\eta^2} = \frac{\cos^2\theta}{\epsilon_2} + \frac{\sin^2\theta}{\epsilon_1}$$

This is the extraordinary wave for which the refractive depends on the direction of propagation. The solution for $\underset{\sim}{a}$ has $a_x = 0$ and

1.52 $$\frac{a_z}{a_y} = \frac{\epsilon_1 - \eta^2\cos^2\theta}{\eta^2\sin\theta \cos\theta}$$

So that $\underset{\sim}{a}$ lies in the plane of 0_z and $\underset{\sim}{n}$ and is inclined to $\underset{\sim}{n}$ at an angle of $\frac{\pi}{2} - \alpha$ where

1.53 $$\tan\alpha = \tfrac{1}{2}\eta^2\left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right)\sin 2\theta$$

The energy flux due to the wave is given by

1.54 $$\underset{\sim}{S} = \frac{C}{4\pi}\underset{\sim}{E} \times \underset{\sim}{H}$$

from 1.42 $\underset{\sim}{H}$ is parallel to $\underset{\sim}{n} \times \underset{\sim}{a}$ so that S lies in the plane of $\underset{\sim}{n}$ and the z axis but inclined to $\underset{\sim}{n}$ at the angle α given by 1.53. That is the energy of the extraordinary wave propagates at an angle, α, to the direction of phase propagation of the wave (see figure 1.2). Figure 1.3 represents the refractive index as distance from the origin of both ordinary and extraordinary waves with the direction of propagation for a negative uniaxial crystal.

b. Absorption

The imaginary part of ϵ accounts for absorption. Since for any medium of interest absorption will be small we can consider it as a small perturbation on the previous theory. Let the perturbed refractive index be N and polarization vector be $\underset{\sim}{b}$. So Fresnel's equation becomes

1.55 $N^2 (\underset{\sim}{n} \times (\underset{\sim}{n} \times \underset{\sim}{b}) + \epsilon^1 \cdot \underset{\sim}{b} + i\, \epsilon^{11} \cdot \underset{\sim}{b} = 0$

where we have written

$$\epsilon = \epsilon^1 + i\, \epsilon^{11}$$

Taking the scalar product with $\underset{\sim}{a}$, the old polarization vector, and using the symmetry of ϵ^1 (i.e. $\underset{\sim}{a} \cdot \epsilon^1 \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \epsilon^1 \cdot \underset{\sim}{a}$) we can write

1.56 $N^2 (\underset{\sim}{a} \cdot \underset{\sim}{n} \times (\underset{\sim}{n} \times \underset{\sim}{b})) - \eta^2 \underset{\sim}{b} \cdot (\underset{\sim}{n} \times (\underset{\sim}{n} \times \underset{\sim}{a})) + i\, \underset{\sim}{a} \cdot \epsilon^{11} \cdot \underset{\sim}{b}$
$= 0$

rearranging

1.57 $$N^2 = \eta^2 + \frac{i\, \underset{\sim}{a} \cdot \epsilon^{11} \cdot \underset{\sim}{b}}{(\underset{\sim}{n} \times \underset{\sim}{a})(\underset{\sim}{n} \times \underset{\sim}{b})}$$

assuming $\underset{\sim}{a}$ is very nearly equal to $\underset{\sim}{b}$ and ϵ^{11} is small we can write using the binomial expansion

$$N \doteq \eta + iK$$

1.58 where

$$K = \frac{a \cdot \epsilon^{11} \cdot a}{2\, \eta\, (\underset{\sim}{n} \times \underset{\sim}{a})^2}$$

Substituting this into the original form for the wave 1.47, we have

1.58 a. $\underset{\sim}{E} = E_o \underset{\sim}{a} \exp(i \frac{\omega}{c} \eta \underset{\sim}{n} \cdot \underset{\sim}{r}) \exp(- \frac{\omega \underset{\sim}{a} \cdot \epsilon^{11} \cdot \underset{\sim}{a}}{2c \eta (\underset{\sim}{n} \times \underset{\sim}{a})^2} \underset{\sim}{n} \cdot \underset{\sim}{r})$

so K is the extinction coefficient of the wave.

c. <u>Medium with Quadratic Response</u>

Including the quadratic response of the polarization to the electric field into Maxwell's equations 1.42 we can write equation 1.46 more generally as

1.59

$$\nabla \times (\nabla \times \underset{\sim}{E}(\underset{\sim}{r},\omega)) - \frac{\omega^2}{c^2} \epsilon(\omega) \underset{\sim}{E}(\underset{\sim}{r},\omega) = \frac{4\pi \omega^2}{c^2} \underset{\sim}{P}^{(2)}(\underset{\sim}{r},\omega)$$

The solution to this equation for infinite plane waves has been studies by many authors, 16, 17, 18, 21. We will consider here the main points.

We assume that none of the frequencies involved approach the singulatities of the second order susceptibility tensor. This tensor is then real and obeys Kleinman's symmetry conditions and $\underset{\sim}{P}^{(2)}$ is small and can be regarded as a perturbation on the linear equation which we have seen has solutions of the form

$$\underset{\sim}{E} = \underset{\sim}{a} \exp (i \frac{\omega}{c} \eta \underset{\sim}{n} \cdot \underset{\sim}{r})$$

We write the perturbed electric field as , 22,

1.60 $\underset{\sim}{E} = \left(A(\varsigma) \underset{\sim}{a} + \underset{\sim}{c}(\varsigma)\right) \exp (i\frac{\omega}{c} \eta \varsigma)$

where $\varsigma = \underset{\sim}{n} \cdot \underset{\sim}{r}$ a co-ordinate in the direction of propagation. $A(\varsigma)$ is a slowly varying amplitude factor, $\underset{\sim}{c}(\varsigma)$ is a slowly varying correction term which arises from the components of $P^{(2)}$ orthogonal to $\underset{\sim}{a}$.

Substituting this into equation 1.59 and using Fresnel's equation

1.48 and some vector manipulation, 22, we obtain the equation

1.61 $$\frac{d}{d\varsigma} A(\varsigma) = \frac{4\pi i \omega}{c\,\eta(\omega)} \frac{\underset{\sim}{a}\cdot P^{(2)}(\omega)}{(\underset{\sim}{n} \times \underset{\sim}{a})^2} \exp\left(i \frac{\omega}{c} \eta(\omega) \varsigma\right)$$

for the amplitude variation of the field at frequency ω.

For the simplest case when $A(\varsigma)$ represents the amplitude of the second harmonic of a field at frequency ω_1 (= $\omega/2$). the polarization takes the form.

1.62 $$\underset{\sim}{a} \cdot \underset{\sim}{P}^{(2)}(\omega) = \frac{E_o^{\,2}}{2} a_i \, X_{ijk}(\omega_1, \omega_1) \, b_j \, b_k$$

$$\times \exp\left(i \frac{2\omega_1}{c} \eta(\omega_1) \varsigma \right)$$

The driving fundamental field being of the form

$$\underset{\sim}{E}^{\omega_1} = E_o \, \underset{\sim}{b} \exp\left(i \frac{\omega_1}{c} \eta(\omega_1) \varsigma \right)$$

Therefore

1.63 $$\frac{d}{d\varsigma} A(\varsigma) \propto E_o^{\,2} \exp(i \Delta k \varsigma)$$ where

1.64 $$\Delta k = \frac{2\omega_1}{c} \left(\eta(\omega_1) - \eta(2\omega_1) \right)$$

The solution of this equation may be written down immediately if we assume that the second harmonic field is small enough not to affect the amplitude of the fundamental field (i.e. E_o is constant), then

1.65 $$A(\varsigma) \propto \frac{\sin\left(\frac{\Delta k \varsigma}{2}\right)}{\Delta k/2}$$

We see that $A(\varsigma)$ oscillates sinusoidally when $\Delta k \neq 0$ and grows linearly with ρ when $\Delta k = 0$. Most transparent optical materials exhibit normal dispersion. Thus the condition of perfect phase matching $\Delta k = 0$ cannot be achieved in an isotropic medium or if both waves are the same type in a uniaxial or biaxial medium. However as first pointed out by Giordmaine, 51 and Terhune, 52, the directional dependence of the extraordinary refractive index can be employed to balance out the effects of dispersion in some materials. For potassium dihydrogen phosphate one of the first materials which proved suitable a second harmonic extraordinary wave has the same refractive index as its fundamental ordinary wave when they propagate at an angle of 50° to the optic axis for a fundamental wavelength of 0.6943 microns. Then the polarization vectors can be written $\underset{\sim}{a} = (0, 1, 0)$, $\underset{\sim}{b} = (\cos 50, 0, -\sin 50)$ neglecting the small correction term. We see from equation 1.53 that the energy contained in the second harmonic wave propagates at an angle α to that of the fundamental wave so the interaction between the two waves will cease after some characteristic length. This is known as the aperture effect. This drawback can be overcome if phase matching could be achieved in a direction perpendicular to the optic axis when $\alpha = 0$. This case has also the added advantage that phase matching is not as critical since now the refractive index curves touch rather than intersect. The refractive index of lithium niobate is temperature dependent and obeys the Sellmeier equations, 23,

$$1.66 \qquad \eta_0^2 = 4.19130 + \frac{1.173 \times 10^5 + 1.65 \times 10^{-2} T^2}{\lambda^2 - (2.12 \times 10^2 + 2.7 \times 10^{-5} T^2)^2}$$

$$- 2.78 \times 10^{-8} \lambda^2$$

$$1.67 \quad \eta_e^2 = 4.5567 + 2.605 \times 10^{-7} T^2 + \frac{0.970 \times 10^5 + 2.70 \times 10^{-2} T^2}{\lambda^2 - (2.01 \times 10^2 + 5.4 \times 10^{-5} T^2)^2} - 2.24 \times 10^{-8} \lambda^2$$

λ wavelength 10^{-7} cms. (i.e. nms.)

T Temperature $^{\circ}K$

for wavelength between 0.4 and 4.0 microns and phase matching can be achieved in a perpendicular direction to the optic axis for a variety of wavelengths (24, 26).

The polarization vectors for this case are $\underset{\sim}{a} = (0, 1, 0)$, $\underset{\sim}{b} = (0, 0, 1)$ and

$$\tfrac{1}{2} a_i X_{ijk} (\omega_1, \omega_1) b_j b_k = \tfrac{1}{2} d_{31} \quad \text{from } 1.36.$$

The process involving this tensor element dominates since the others are not phase matched.

For parametric effects when the medium is subjected to two optical fields at frequencies ω_1 and ω_2 of the form

$$E^{\omega_1} = E_{01} \underset{\sim}{b}_1 \exp \left(i \frac{\omega_1}{C} \eta(\omega_1) \zeta\right)$$

$$E^{\omega_3} = E_{03} \underset{\sim}{b}_3 \exp \left(i \frac{\omega_3}{C} \eta(\omega_3) \zeta\right)$$

$$\omega_3 > \omega_1 > 0$$

the quadratic polarization then contains the difference frequency term

$$\underset{\sim}{a} \cdot \underset{\sim}{P}^{(2)}(\omega_2) = E_{03} E^*_{01} a_i X_{ijk}(\omega_3, -\omega_1) b_{3j} b_{1k}$$
$$\times \exp\left(i \varsigma/c \, (\omega_3 \eta(\omega_3) - \omega_1 \eta(\omega_1)) \right)$$

from equation 1.23

Again the interaction will depend on phase matching term

$$\Delta k = \frac{1}{c} (\omega_3 \eta(\omega_3) - \omega_1 \eta(\omega_1) - \omega_2 \eta(\omega_2))$$

and the difference frequency will only be exited to a significant amount if $\Delta k = 0$.

Attempts at optical parametric amplification have been made using Lithium Niobate 25, for the degenerate case $\omega_1 = \omega_2$. In this case phase matching occurs if the pump wave at ω_3 is extraordinary i.e. $\underset{\sim}{b}_3 = (0, 0, 1)$ and the subharmonic wave is ordinary $\underset{\sim}{a} = \underset{\sim}{b}_1 = (0, 1, 0)$ then from 1.36

$$a_i X_{ijk}(\omega_3, -\omega_1) b_{3j} b_{1k} = d_{15}$$

as long as $X_{131} = X_{113}$ which is true under our assumptions that ω_3 is much less than the frequencies of the electronic transitions.

Enhancement of Nonlinear Effects

The first and most obvious way to enhance the effect is to use a focussed beam. Several authors, 27-31, 53, 54 have treated second harmonic generation by focussed beams of light; they considered the focussed beam as a sum of plane wave components. Kleinman and Boyd, 15, have treated this in great detail for the case of a gaussian fundamental beam focussed into a nonlinear crystal. They have studied the variation and form of the output with focus position, focussing, length of crystal, phase matching both theoretically and experimentally and their results are similar to those obtained here.

The second method of enhancing the effect is to place the nonlinear crystal inside a resonator of its own firstly outside the laser cavity ultimately inside. Ashkin, Boyd and Dziedzic, 32, have applied their work on focussed beams to the first case but this method of treating resonant second harmonic generation is rather clumsy and cannot be applied to the general case. We will study this problem here by solving Maxwell's equations for the field in the resonator containing the nonlinear crystal. By basing the study on the modes of the resonator we can avoid the inherent difficulties of the plane wave approach for solving the general case when there is more than one mode present.

This theory can then be applied with minor changes to the theory of parametric amplification and oscillation.

Chapter 2

The Modes of the Optical Resonator

1 Introduction

The open resonator was first proposed as a resonant cavity for lasers by Dicke, 34, and Shawlow and Townes, 35, and Prokhurov, 36, because even at optical frequencies, when the cavity dimensions must be much greater than the wavelength of the contained radiation, they only support relatively few psuedo*-eigen modes of oscillation. Only waves which propagate along the axis of the resonator will be supported; any others will have very high losses. Fox and Li, 37, by making self consistent field calculations based on Huygen's principle showed that discrete modes existed both for the plane parallel mirror resonator and the spherical mirror resonator and indicated the field distribution over the mirrors and the losses of the modes. Boyd and Gordon, 38, Boyd and Kogelnik, 39, solved these equations for the spherical mirror resonator and gave an expression for the field in the resonator. I shall give a brief outline of their method to show the way in which the solutions arise and then go on to derive them as solutions to Maxwell's equations for the resonator, firstly for the case when the resonator is filled with an isotropic medium, secondly a uniaxial medium.

* psuedo in that there is a larger possibility that a photon will stay in the mode for a long time but not indefinitely, 40.

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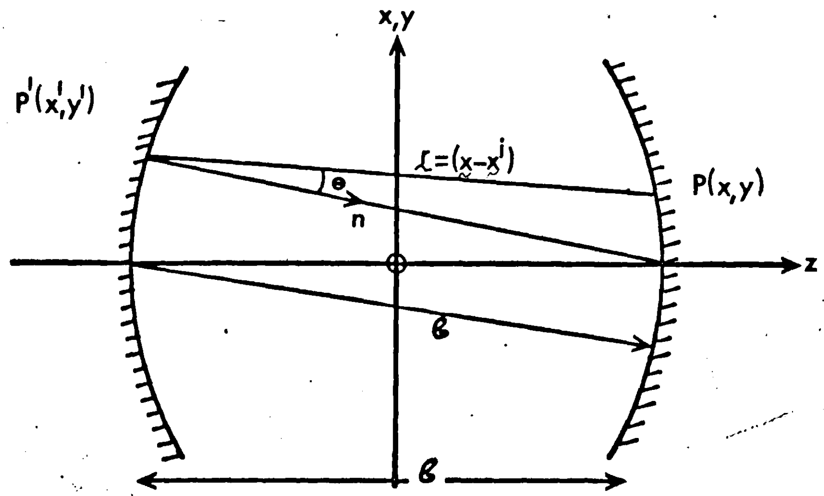

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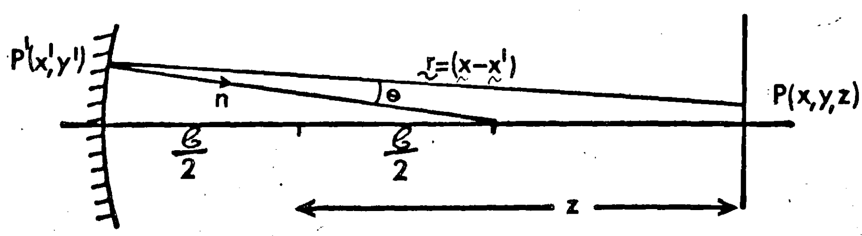

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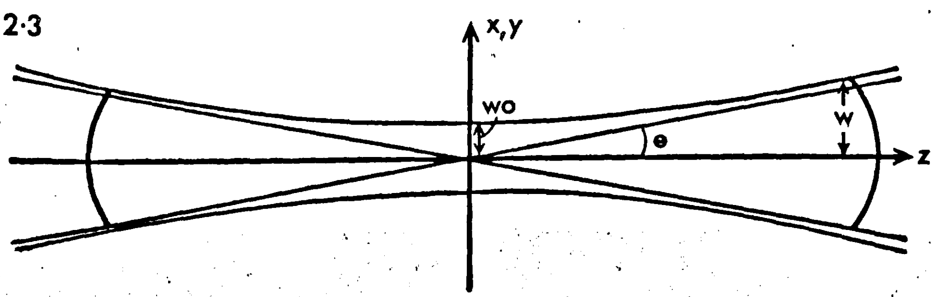


2 Derivation

a. Mirror Field Distribution

We consider the simplest case a confocal resonator (see figure 2.1). Two spherical mirrors, radius of curvature b, separated by a distance b. Because of the axial symmetry there will be a plane polarized solution and we can use Kirchoff's diffraction theory, 50, to give the field $E(\underset{\sim}{x})$ at the point $P(\underset{\sim}{x})$ due to the distribution $E(\underset{\sim}{x}^1)$ over the first mirror surface.

2.1 $$E(\underset{\sim}{x}) = \frac{1}{4\pi} \iint_S \left(E \frac{\partial \mathcal{U}}{\partial n} - \mathcal{U} \frac{\partial E}{\partial n} \right) ds$$

where $\mathcal{U} = \dfrac{e^{-ikr}}{r}$ $\qquad r = |(\underset{\sim}{x} - \underset{\sim}{x}^1)|$

and $\frac{\partial}{\partial n}$ represents differentiation along the outward normal to the mirror surface S. Since $\frac{1}{r} << k$ at optical frequencies and

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial r} \cos \theta$$

we can write 2.2 $\frac{\partial}{\partial n} \left(\frac{e^{-ikr}}{r} \right) = ik\cos\theta \, \frac{e^{-ikr}}{r}$ and to the first order,

2.3 $$\frac{\partial}{\partial n} (E(\underset{\sim}{x}^1)) = -ikE(\underset{\sim}{x}^1)$$

substituting into 2.1 we have

2.4 $$E(\underset{\sim}{x}) = \frac{ik}{4\pi} \iint_S E(\underset{\sim}{x}^1) \frac{e^{-ikr}}{r} (1 + \cos\theta) \, ds$$

for the resonator θ never departs very far from zero and S is a square mirror of side a.

By symmetry if $E(\underset{\sim}{x}^1)$ is an eigen mode of the resonator the field distribution will reproduce itself on the second mirror apart from some loss factor. In the usual fashion solving for eigen modes we assume

$$E(\underset{\sim}{x}^1) = E_o\, f_m(x^1)\, g_n(y^1)$$

and so the field

$$E(\underset{\sim}{x}) = E_1\, f_m(x^1)\, g_n(y^1)$$

where

$$E_1 = \sigma_m\, \sigma_n\, E_o$$

$\sigma_m\, \sigma_n$ being the eigen value of the equation. Therefore

$$\sigma_m\, \sigma_n\, f_m(x)\, g_n(y) = \int_{-a}^{a}\int_{-a}^{a} \frac{ik}{2\pi r}\, e^{-ikr}\, f_m(x^1) g_n(y^1) dx^1\, dy^1 \qquad 2.7$$

from the diagram 2.1 to the first order, r is given by

$$r = b\left(1 - \frac{xx^1 + yy^1}{b^2} + \dots\dots\dots\dots\right)$$

substituting this into 2.7 and separating the integrals

$$\sigma_m\, \sigma_n\, f_m(x)\, g_n(y) = \frac{ike^{-ikb}}{2\pi b} \int_{-a}^{a} e^{\left\{\frac{ikxx^1}{b^2}\right\}} f_m(x^1)\, dx^1$$

$$\times \int_{-a}^{a} e^{\left\{\frac{ikyy^1}{b^2}\right\}} g_n(y^1)\, dy^1$$

in terms of dimensionless variables

$$c = \frac{a^2 k}{b} = \frac{2\pi a^2}{b\lambda}, \quad X = \frac{x\sqrt{c}}{a} \quad Y = \frac{y\sqrt{c}}{a}$$

$$F_m(X) = f_m(x) \quad G_n(Y) = g_n(y)$$

$$2.9 \quad \sigma_m \sigma_n F_m(X) G_n(Y) = \frac{ie^{-ikb}}{2\pi} \int_{-\sqrt{c}}^{\sqrt{c}} F_m(X^1) e^{iXX^1} dX^1 \times \int_{-\sqrt{c}}^{\sqrt{c}} G_n(Y) e^{iYY^1} dY^1$$

The solution to this equation will be the product of the solutions to two equations of the form

$$2.10 \quad F_m(X) = \sqrt{\frac{i}{2\pi}} \frac{e^{\frac{ikb}{2}}}{\sigma_m} \int_{-\sqrt{c}}^{\sqrt{c}} F_m(X^1) e^{iXX^1} dX^1$$

These are given by, 41, the angular and radial wave functions in prolate spheroidal co-ordinates. The field given by these functions falls away quite rapidly from the axis so that we can approximate 2.10 by taking the limits of the integration to infinity and then the solution is given by

$$2.12 \quad F_m(X) = C H_m(X) e^{-\frac{1}{2}X^2}$$

where $H_m(X)$ is the Hermite polynomial of degree m, C a constant. Under this approximation the field distribution over each mirror will be given by

$$E = E_o H_m(X) H_n(Y) \exp -\tfrac{1}{2}(X^2 + Y^2)$$

$$2.13 \quad E = E_o H_m\left(x\sqrt{\tfrac{k}{b}}\right) H_n\left(y\sqrt{\tfrac{k}{b}}\right) \exp -\frac{k}{2b}(x^2 + y^2)$$

(for a two dimensional resonator we can apply the same theory to arrive at

$$E = E_o H_m\left(x\sqrt{\tfrac{k}{b}}\right) \exp\left(-\frac{k}{2b} x^2\right)$$

b. Travelling wave in the resonator

Using this solution for the field distribution over the mirror we can now calculate from equation 2.4 the form of the travelling wave which is produced in the cavity. If we assume one of the mirrors is partially transmitting this will also be the form of the wave outside the cavity.

For the simplest case of the lowest order mode, in the approximation of infinite mirrors the defraction integral is

$$2.14 \quad E\,(x,y,z) = -E_o \frac{ik}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp - \frac{k}{2b}(x^{12} + y^{12})\, \frac{e^{-ikr}}{r}\, dx^1\, dy^1$$

To the first approximation from diagram 2.2

$$2.15 \quad r = \xi - \frac{W^{12}}{2b} + \frac{W^2}{2\xi} + \frac{W^{1^2}}{2\xi} - \frac{(xx^1 + yy^1)}{\xi}$$

where $\xi = z + \frac{b}{2}$, $W^2 = x^2 + y^2$, $W^{12} = x^{12} + y^{12}$

substituting into 2.14.

$$2.16 \quad E(x,y) = \frac{E_o ik}{2\pi\xi} \exp\left\{ -\frac{ikw^2}{2\xi} - ik\xi \right\}$$

$$x \int_{-\infty}^{\infty} \exp\left\{ -\left(\frac{k}{2b} - \frac{ik}{2b} + \frac{ik}{2\xi}\right) x^2 + ik\, xx^1 \right\} dx^1$$

$$x \int_{-\infty}^{\infty} \exp\left\{ -\left(\frac{k}{2b} - \frac{ik}{2b} + \frac{ik}{2\xi}\right) y^2 + ikyy1 \right\} dy^1$$

taking the first of the integrals, I_x

$$2.17 \quad I_x = \exp\left\{ \frac{-kx^2}{2\xi^2 \left(\frac{1-i}{b} + \frac{i}{\xi}\right)} \right\} \int_{-\infty}^{\infty} \exp\left\{ x^1 - \left[x^1 \left(\frac{k}{2}\left(\frac{1-i}{b} + \right.\right.\right.\right.$$

$$\frac{i}{\xi}))^{\frac{1}{2}} - \frac{ikx}{2\xi\left(\frac{k}{2}\left(\frac{1-i}{b} + \frac{i}{\xi}\right)\right)^{\frac{1}{2}}}\Bigg]^{2}\Bigg\} dx^{1}$$

changing the variable of integration to

2.18 $$\eta = x^{1}\left(\frac{k}{2}\left(\frac{1-i}{b} + \frac{i}{\xi}\right)\right)^{\frac{1}{2}} - \frac{ikx}{2\xi\left(\frac{k}{2}\left(\frac{1-i}{b} + \frac{i}{\xi}\right)\right)^{\frac{1}{2}}}$$

we have

2.19 $$I_x = \frac{1}{\frac{k}{2}\left(\frac{1-i}{b} + \frac{i}{\xi}\right)} \exp\left\{\frac{-kx^2}{2\xi^2\left(\frac{1-i}{b} + \frac{i}{\xi}\right)}\right\} \int_C \exp - \eta^2 d\eta$$

Where C is a straight line in the complex η plane. This integral and also the integral which we obtain for the higher mode case is treated in Appendix I. Using the result

2.20 $$I_x = \frac{\sqrt{\pi}}{\frac{k}{2}\left(\frac{1-i}{b} + \frac{i}{\xi}\right)} \exp\left\{\frac{-kx^2}{2\xi^2\left(\frac{1-i}{b} + \frac{i}{\xi}\right)}\right\}$$

note that the expression for E (x, y, z) is given as the product of two two-dimensional terms, so the two dimensional result follows immediately. The field E (x, y, z) is now given by

2.21 $$E(x, y, z) = \frac{E_{oi}}{\xi\left(\frac{1-i}{b} + \frac{1}{\xi}\right)} \exp\left\{\frac{-kw^2}{2\xi^2\left(\frac{1-i}{b} + \frac{i}{\xi}\right)} - \frac{ikw^2}{2\xi} - ik\xi\right\}$$

Simplifying and substituting for $z = \xi - b/2$

$$2.22 \quad E\ (x,\ y,\ z) = \frac{E_o\ (1 + i)}{(1 - \frac{2iz}{b})} \exp\left\{ \frac{-k\ (x^2 + y^2)}{b\ (1 - \frac{2iz}{b})} - ik\ (z + \frac{b}{2}) \right\}$$

Redefining E_o to absorb the constants

$$2.23 \quad E\ (x,\ y,\ z) = \frac{E_o}{\left(1 - \frac{2iz}{b}\right)} \exp\left\{ \frac{-k(x^2 + y^2)}{b(1 - \frac{2iz}{b})} - ikz \right\}$$

Separating the real and imaginary parts this can be written alternatively as

$$2.24 \quad E\ (x,\ y,\ z) = \frac{E_o}{(1 + \frac{4z^2}{b^2})^{\frac{1}{2}}} \exp\left\{ \frac{-k\ (x^2 + y^2)}{b\ (1 + \frac{4z^2}{b^2})} - i\left(\frac{2k\ (x^2 + y^2)}{b^2\left(1 + \frac{4z^2}{b^2}\right)} + kz - \tan^{-1}\left(\frac{2z}{b}\right) \right) \right\}$$

We see from this equation for fixed z the field falls off from the axis of the resonator as

$$\exp\left\{ \frac{-k\ (x^2 + y^2)}{b\ (1 + \frac{4z^2}{b^2})} \right\}$$

We define the spot size W of the mode to be the radius where the field has fallen to $\frac{1}{e}$ of its value on the axis.

2.25 Therefore $W^2 = \frac{b}{k}\left(1 + \frac{4z^2}{b^2} \right)$

which can be written $W^2 = W_o^{\ 2} \left(1 + \frac{4z^2}{b^2} \right)$

defining $W_o = \left(\frac{b}{k} \right)^{\frac{1}{2}}$ the minimum spot size. We can write

$$\frac{2z}{b} = \frac{2z}{kW_o^2} = \frac{\lambda z}{\pi W_o^2}$$ where λ is the wavelength and

defining Z_o and ξ

2.26 $$\frac{b}{2} = Z_o \quad \text{and} \quad \frac{Z}{Z_o} = \frac{2Z}{b} = \xi$$

These definitions will be adhered to throughout this thesis from this point on.

The beam spread is defined by (see Figure 1.3)

2.27 $$\Theta = \lim_{Z \to \infty} \left\{ \operatorname{Tan}^{-1} \left(\frac{W}{Z} \right) \right\} = \operatorname{Tan}^{-1} \left(\frac{Wo}{Zo} \right)$$

using these definitions we can write the general travelling wave form for the $n\,m^{th}$ mode as

2.28 $$E_{n\,m}(x, y, z) = E_{n\,m}^{o} \frac{(1 + i\xi)^{\frac{n+m}{2}}}{(1 - i\xi)^{\frac{m+n+1}{2}}}$$

$$\times\ H_m\left(\frac{\sqrt{2}x}{W}\right) H_n\left(\frac{\sqrt{2}y}{W}\right) \times \exp\left\{ \frac{-(x^2 + y^2)}{W^2(1 - i\xi)} - ikz \right\}$$

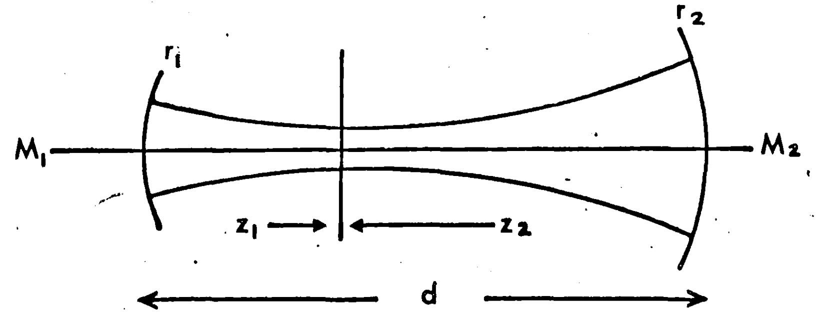
or as (39).

2.29 $$E_{n\,m}(x, y, z) = E_{n\,m}^{o} \frac{W_o}{W} H_m\left(\frac{\sqrt{2}x}{W}\right) H_n\left(\frac{\sqrt{2}y}{W}\right)$$

$$\times \exp\left\{ \frac{-(x^2 + y^2)}{W^2} - i\left(\frac{(x^2 + y^2)\xi}{W^2} - (m + n + 1)\tan^{-1}(\xi) + kz \right) \right\}$$

note that these can be split (except for the exp (-ikz) term) to give the two dimensional result.

2·4


r1
r2
M1
M2
z1
z2
d

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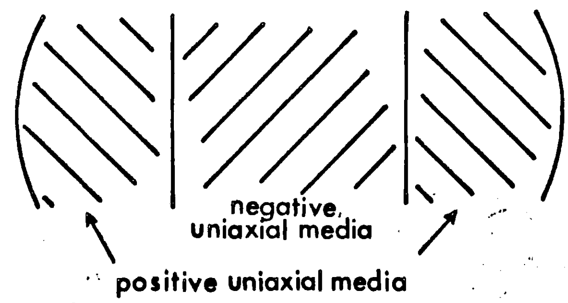

negative, uniaxial media
positive uniaxial media

$$2.30 \quad E_m(x, z) = E_o \frac{(1 + i\xi)^{m/2}}{(1 - i\xi)^{\frac{m+1}{2}}} H_m\left(\frac{\sqrt{2}x}{W}\right) \exp\left\{\frac{-x^2}{W_o^2(1 - i\xi)} - ikz\right\}$$

c. <u>The general spherical mirror resonator</u>

Equation 2.28 will also represent the field inside a resonator with arbitrary mirror spacing as long as the mirrors lie on surfaces of constant phase of the electric field. The surfaces of constant phase of the field represented by the equation 2.21 are given by

$$\frac{(x^2 + y^2)}{W^2}\xi + kz - (m + n + 1)\tan^{-1}\xi$$

$$= \text{constant} = kz^1 .$$

whereas z^1 is the point at which the surface intersects the z axis. Neglecting the term $(m + n + 1)\tan^{-1}\xi$, since it is small compared with kz, and re-arranging this gives:

$$x^2 + y^2 = (z^1 - z)\ 2z\ (1 + \frac{1}{\xi}2)$$

which represents a spherical surface, radius of curvature

$$R = Z(1 + \frac{1}{\xi}2) =$$

$$2.31 \quad \text{i.e.} \quad R = Z\left(1 + \frac{Z_o^2}{Z^2}\right)$$

Thus if we are given two mirrors radius of curvature r_1 r_2 respectively, placed a distance apart d, distances Z_1, Z_2 from the beam waist (see figure 2.4) then the beam parameters can be calculated from 2.31 [9].

$$\frac{kW_o^2}{2} = Z_o^{\ 2} = \frac{d\,(r_1 - d)(r_2 - d)(r_1 + r_2 - d)}{(r_1 + r_2 - 2d)^2}$$

from this expression we can obtain the values of r_1, r_2, d which support resonance. For a more detailed treatment see Bhawalker, 42.

d. <u>Resonant Frequency</u>

The phases of the wave at the two mirrors relative to the origin are (from equation 2.29)

$$\phi_1 = -kz_1 + (m + n + 1)\tan^{-1}\xi_1$$

$$\phi_2 = kz_2 - (m + n + 1)\tan^{-1}\xi_2$$

Therefore the phase change between the mirrors is

$$\phi_1 - \phi_2 = kd - \tan^{-1}\xi_2 - \tan^{-1}\xi_1$$

2.32 $$\phi_1 - \phi_2 = kd - (m + n + 1)\tan^{-1}\left[\left\{\frac{d\,(r_1 + r_2 - d)}{(r_1 - d)(r_2 - d)}\right\}^{\frac{1}{2}}\right]$$

using equation 2.31. For resonance $\phi_1 - \phi_2$ must be an integer multiple of π . Therefore

2.33 $$q\pi = kd - (m + n + 1)\tan^{-1}(\psi)$$

ψ is given by 2.32 and is a function of the cavity parameters only. In general q will be large and the resonator will support a number of longtitudinal modes the wave number separation of which is given by

$$\Delta k = k_{q+1} - k_q = \frac{\pi}{d}$$

Equation 2.33 also shows that the transverse modes will have a wave number separation of

$$\Delta k = k_{m+1} - k_m = \frac{\tan^{-1} \psi}{d}$$

3. <u>Solution of Maxwell's Equations for the Resonator</u>

(a) <u>Lossless Isotropic Medium</u>

Maxwell's equations for a homogeneous, isotropic, linear and non-magnetic dielectric reduce to

$$2.34 \quad \nabla^2 \underset{\sim}{E} (\underset{\sim}{r}, t) - \frac{\epsilon}{c^2} \frac{\partial^2 \underset{\sim}{E}(\underset{\sim}{r}, t)}{\partial t^2} = 0$$

Taking the Fourier transform we have

$$2.35 \quad \nabla^2 \underset{\sim}{E} (\underset{\sim}{r}, \omega) + \frac{\epsilon \omega^2}{c^2} E(\underset{\sim}{r}, \omega) = 0$$

Considering a plane polarised wave travelling in the z direction we can write.

2.36 $E = \mathcal{E}(x, y, z) \exp(-ikz)$ where the function $\mathcal{E}$ will represent the differences between the wave form we are considering and a plane wave. We assume $\mathcal{E}$ is a slowly varying function and

$$2.37 \quad \left| \frac{d^2 \mathcal{E}}{dz^2} \right| << k \left| \frac{d \mathcal{E}}{dz} \right| \quad \text{where} \quad k = \frac{\epsilon \omega^2}{c^2}$$

which is a reasonable assumption since k is large ($\sim 10^5$) at optical frequencies. Substituting 2.36 into 2.35 and using 2.37 we have

$$2.38 \quad \frac{\partial^2 \mathcal{E}}{\partial x^2} + \frac{\partial^2 \mathcal{E}}{\partial y^2} - 2ik \frac{\partial \mathcal{E}}{\partial z} = 0$$

This is the basic equation of the subsequent work which was first derived by Kogelnik and Li, 43. The presence of the resonator gives us the boundary conditions for the solution of 2.38. As we have already pointed out the boundary conditions are axially symmetric and so we can assume a plane polarized solution. Secondly they introduce an axis into the problem and so the solution will also have an axis. In the absence of these boundary conditions there will be no axis so that the solutions which we will derive cannot be regarded as solutions of Maxwell's equations in the absence of the resonator.

(i) Lowest-order mode

Following the derivation of the lowest order mode given by Kogelnik and Li, 43, we assume $\mathcal{E}$ can be written in the form

2.39 $$\mathcal{E} = \mathcal{E}_0 \exp -i \left(P + \frac{k r^2}{2q}\right)$$

where $r^2 = x^2 + y^2$,

P is a function only of z and represents a complex phase shift and q a function only of z which describes the variation of electric field away from the axis and the curvature of the phase front.

Substituting 2.39 into 2.38 we have

2.40 $$-\frac{2ik}{q} - \frac{k^2 r^2}{q^2} - 2k\frac{dP}{dz} - \frac{k^2 r^2}{2} \frac{d}{dz}\left(\frac{1}{q}\right) = 0$$

Equating coefficients of r

2.41 (a) $$\frac{dP}{dz} = \frac{-i}{q}$$ (b) $$\frac{dq}{dz} = 1$$

2.41 (b) can be integrated immediately to give

2.42 $$q = q_0 + z$$

guided by the previous work (eq. 2.29) we split $\frac{1}{q}$ into its real and imaginary parts

$$2.43 \quad \frac{1}{q} = \frac{1}{R} - \frac{i\,2}{kW^2}$$

R will be the radius of curvature of the phase front on the axis and w a measure of the decrease in field amplitude from the axis. The intensity distribution is gaussian at every cross section and W can be defined as the spot size or beam radius at a particular Z. We set up our axes at the point where $R = \infty$ and define the spot size at this point of the Z axis to be W_o. Therefore from 2.43 and 2.42

$$q_o = \frac{i\,kW_o^2}{2}$$

$$q = \frac{i\,k\,W_o^2}{2} + Z$$

which gives

$$2.44 \quad W^2 = W_o^2 \left(1 + \frac{4\,Z^2}{k^2 W_o^4}\right)$$

$$2.45 \quad \text{and} \quad R = Z\left(1 + \frac{k^2\,W_o^4}{4z^2}\right) = \frac{k^2\,W^2\,W_o^2}{4z}$$

This result corresponds exactly with our previous definitions.

Substituting for q into equation 2.41 (a)

$$\frac{dP}{dz} = \frac{-i}{z + \frac{ikW_o^2}{2}}$$

$$iP = \log\left(1 - \frac{i2z}{kW_o^2}\right)$$

2.46 $$= \log\left(1 + \frac{4Z^2}{k^2 W_o^4}\right) - \tan^{-1}\left(\frac{2Z}{kW_o^2}\right)$$

substituting back into equation 2.39

$$\mathcal{E} = \mathcal{E}_o \exp\left\{ - \log\left(\frac{1 + 4Z^2}{k^2 W_o^4}\right) + i \tan^{-1}\left(\frac{2Z}{kW_o^2}\right) - r^2\left(\frac{1}{W^2} + \frac{ik}{2R}\right) \right\}$$

2.47 $$\mathcal{E} = \mathcal{E}_o \frac{W_o}{W} \exp\left\{\frac{-r^2}{W^2} - \frac{ir^2 \, 2Z}{k W^2 W_o^2} + i \tan^{-1}\left(\frac{2Z}{kW_o^2}\right)\right\}$$

which corresponds with the lowest order mode of equation 2.29.

$$r^2 = x^2 + y^2 , \quad \frac{2Z}{kW_o^2} = \xi$$

(ii) Higher order modes

To derive the higher modes we have to introduce a more general form in the place of equation 2.39. Guided by experience we put

2.48 $$\mathcal{E} = \mathcal{E}_o f\left(\frac{\sqrt{2}x}{W}\right) g\left(\frac{\sqrt{2}y}{W}\right) \exp -i\left(P + \frac{kr^2}{2q}\right)$$

substituting this into equation 2.38 and dividing throughout by f g we have using 2.43

2.49 $$\frac{1}{f}\left(\frac{2}{W^2} f''\left(\frac{\sqrt{2}x}{W}\right) - \frac{4\sqrt{2}x}{W^3} f'\left(\frac{x\sqrt{2}}{W}\right)\right)$$

$$+ \frac{1}{g}\left(\frac{2}{W^2} g''\left(\frac{y\sqrt{2}}{W}\right) - \frac{4\sqrt{2}y}{W^3} g\left(\frac{y\sqrt{2}}{W}\right)\right) -$$

$$- \frac{2ik}{q} - \frac{2k\,dP}{dz} = 0$$

Separating off the variable x and then y and introducing the two constants of separation $-2m$, $-2n$ gives the equations

$$2.50 \quad f''(X) - 2X f'(X) + 2mf(X) = 0$$

$$g''(Y) - 2Yg'(Y) + 2ng(Y) = 0 \qquad \text{for f and g}$$

Where $X = \frac{\sqrt{2}x}{W}$ and $Y = \frac{\sqrt{2}y}{W}$. These are just the equations for the Hermite polynomials of degree m and n respectively so that

$$2.51 \quad f\left(\frac{\sqrt{2}x}{W}\right) = H_m\left(\frac{\sqrt{2}x}{W}\right)$$

$$g\left(\frac{\sqrt{2}y}{W}\right) = H_n\left(\frac{\sqrt{2}y}{W}\right)$$

The equation for P in this case is

$$\frac{dP}{dz} = \frac{-i}{q} - \frac{2(m+n)}{kW^2}$$

giving

$$2.52 \quad iP = \log\left(1 + \frac{4z^2}{k^2 W_o^4}\right) - i(m+n+1)\tan^{-1}\left(\frac{2z}{k W_o^2}\right)$$

Thus the general solution for the travelling wave mode in the cavity is given by

$$2.53 \quad \mathcal{E}_{mn} = \mathcal{E}_o \frac{W_o}{W} H_m\left(\frac{\sqrt{2}x}{W}\right) H_n\left(\frac{\sqrt{2}y}{W}\right) \exp\left\{\frac{-r^2}{W^2} - \frac{ir2z}{k W^2 W_o^2}\right.$$

$$\left. + i(m+n+1)\tan^{-1}\left(\frac{2z}{kW_o^2}\right)\right\}$$

changing the notation $r^2 = x^2 + y^2$, $\xi = \frac{2z}{k W_o^2}$

this corresponds to equation 2.29

(b) Validity of the approximation

We have approximated by assuming that

2.54 $$\left|\frac{d^2\mathcal{E}}{dz^2}\right| \ll \left|\frac{k\,d\mathcal{E}}{dz}\right|$$ in obtaining

equation 2.38. For self consistency our solution 2.53 must obey this condition. For the lowest order mode, calculating the derivatives and substituting into equation 2.54, gives

2.55 $$\left|\frac{1}{q^2}\left(2 - \frac{2\,ik\,r^2}{q} - \frac{k^2 r^4}{4q^2}\right)\right| \ll \left|\frac{k}{q}\left(-1 + i\,\frac{k r^2}{2q}\right)\right|$$

(i) at $r = 0$ we obtain

$$\left|\frac{2}{q}\right| \ll k$$

i.e. $$\frac{\lambda}{\pi} \ll |z + iz_o|$$

substituting for q from equation 2.42, $q_o = Z_o$. At worst this will give

$$\lambda \ll \pi z_o\text{, since } z_o = \frac{\pi W_o^2}{\lambda}$$

we have

2.56 $$\lambda \ll W_o$$

(ii) Finite r Substituting for $q = z + iz_o$ 2.55 becomes

2.57 $$\frac{2}{k(z^2 + z_o^2)}\left|\left(z^2 - z_o^2 + k r^2 z_o - \frac{k^2 r^4}{2}\right) - \right.$$

$$- i \ (kr^2 z - 2 \ zz_o) \Big| << \Big| - z + i \ (kr^2 - \frac{Zo}{2}) \Big|$$

This breaks down in two regions the first near $z = 0$ and $k r^2 - \frac{Z_o}{2} = 0$. The second if r is large enough. Region one is defined by $(z^2 + (r - W_o)^2) \ll \frac{1}{k^2}$. This is relatively unimportant since it is so small compared with the beam parameter.

Region two is defined by $r^2 \gg kW^2$. The magnitude of the field is proportional to $\exp(- r^2/W^2)$. So in this region this factor is much less than $\exp(-k)$ and the field is so small that this inconsistency can be neglected. Some error in the outer regions can be expected since we have approximated the finite mirrors of the resonator by infinite ones. Thus the only significant restriction is given by equation 2.56 the minimum s spot size must be much greater than the wavelength of the light of the beam.

(iii) Transverse Nature of the Polarization

Orienting the x and y axis so that the field in the resonator is polarized in the y direction we can write the divergence equation for the field as

2.58 $\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$ where E_z is the component of the field along the direction of propagation. We assume that

$$\frac{\partial E_z}{\partial z} \approx - ikE_z .$$

Taking the lowest order mode (equation 2.39) and calculating the derivative in the y direction we have

2.59 $\frac{\partial E_y}{\partial y} = - \frac{i k y}{q} E_y.$ so that from

equation 2.58 we can write.

$$\left| \frac{y}{q} E_y \right| = |E_z|$$

therefore as long as $|y| << |q|$

$|E_z| << |E_y|$ and the mode will be traversely polarized.

$$|q| = (z^2 + z_o^2)^{\frac{1}{2}} = \frac{z_o}{W_o} W_o \left(1 + \frac{z^2}{z_o^2}\right) = \frac{kW_o W}{2}$$

Therefore the condition reduces to

2.60 $|y| << \frac{k W_o W}{2}$

and again the ideal condition breaks down far from the axis when the field is small. So that over the region of interest it is a valid approximation to assume there is no component of the field in the direction of propagation.

(iv) <u>Normalization</u>

We assume as before we have a mode propagating along the axis of the resonator polarized in the y direction.

$\underset{\sim}{E} = (0, E, 0)$ to the approximation of the last section. From Maxwell's equations the magnetic field H will have the form

$\underset{\sim}{H} = (H, 0, 0)$

The Poynting vector $\underset{\sim}{S} = \frac{c}{4\pi} \underset{\sim}{E} \times \underset{\sim}{H}$

and therefore $\underset{\sim}{S} = \frac{c}{4\pi} (0, 0, -EH)$

In the usual way we write

$$E = \text{Re} \left\{ \mathcal{E} \exp -i (kz - \omega t) \right\}$$

2.61 $$H = \mathrm{Re}\left\{ \mathcal{H} \exp\left[-i\ (kz - \omega t)\right] \right\}$$

and using Maxwell's equations

$$\eta\, \mathcal{E} = \mathcal{H}$$

where η is the refractive index of the medium. Therefore the one non-zero component of the energy flux

2.62 $$S_z = \frac{c\eta}{4\pi} \left\{ \frac{\mathcal{E}^2}{4} \exp\left[-2i\ (kz - \omega t) \right] \right\} + \frac{\mathcal{E}\mathcal{E}^*}{2} + \frac{\mathcal{E}^{*2}}{4} \exp\left[2i\ (kz - \omega t) \right] \Bigg\}$$

taking the time average

2.63 $$\bar{S}_z = \frac{c\eta}{4\pi} \cdot \frac{\mathcal{E}\mathcal{E}^*}{2}$$

to calculate the total energy flow in the propagating mode we now integrate 2.63 over the cross section.

(i) In two dimensions $\mathcal{E}$ is given by 2.30 and

2.64 $$\bar{S}_{Total} = \frac{c}{8\pi} \int_{-\infty}^{\infty} \frac{|\mathcal{E}_o|^2}{(1 + \xi^2)^{\frac{1}{2}}} H_n^2 \left(\frac{\sqrt{2}x}{W} \right) \times \exp\left(\frac{-2x^2}{W^2} \right) dx$$

$$= \frac{c}{8\pi} \frac{Wo}{\sqrt{2}} |\mathcal{E}_0|^2 \int_{-\infty}^{\infty} H_n^2\ (\varsigma)\ \exp\ (-\varsigma^2)\ d\varsigma$$

where $\varsigma = \frac{\sqrt{2}x}{W}$

Carrying out the integration

2.65 $$\bar{S}_{Total} = \frac{c}{8\pi} \frac{W_o}{\sqrt{2}} \cdot 2^n\, n!\ \sqrt{\pi}\ |\mathcal{E}_o|^2$$

For unit energy flux

$$|\mathcal{E}_o|^2 = \frac{8\pi}{c} \quad \frac{1}{W_o\, n!\; 2^n} \sqrt{\frac{2}{\pi}}$$

It is convenient at this point to define a new normalised amplitude $\mathcal{E}_N$ so that

2.66 $$\overline{S}_T = \frac{c}{8\pi} |\mathcal{E}_n|^2$$

therefore

2.67 $$|\mathcal{E}_o| = |\mathcal{E}_N| \left(\frac{1}{W_o\, 2^n\, n!\;\; 1} \sqrt{\frac{2}{\pi}}\right)^{\frac{1}{2}}$$

and 2.30 becomes

2.68 $$\mathcal{E}_n = \mathcal{E}_N \left(\frac{1}{W_o\, 2^n\, n!} \sqrt{\frac{2}{\pi}}\right)^{\frac{1}{2}} \left(\frac{1 + i\xi}{1 - i\xi}\right)^{n/2} \frac{n+1}{2}$$

$$H_n \left(\frac{\sqrt{2}x}{W}\right) \exp\left\{\frac{-x^2}{W_o^2 \,(1 - i\xi)}\right\}$$

leaving out the exp $(-ikz)$ term

(ii) In three dimensions $\mathcal{E}$ is given by 2.28. Using this expression the resulting expression for $\overline{S}_{Total}$ contains a double integration which is separable into an integration over y and an integration over x each of the form of the integral in equation 2.64. Carrying out these integrations we have

2.69 $$\overline{S}_{Total} = \frac{c}{16\pi} \frac{W_o^2}{\;} |\mathcal{E}_o|^2 \; 2^n\, n!\, \sqrt{\pi} \;\; 2^m\, m!\, \sqrt{\pi}$$

Again we define a new normalized amplitude $\mathcal{E}_N$ where

$$|\mathcal{E}_o| = |\mathcal{E}_N| \frac{1}{W_o} \left(\frac{1}{2^{n+m}\, n!\, m!\, \pi}\right)^{\frac{1}{2}}$$

and the travelling wave mode is defined as

$$2.70 \quad \mathcal{E}_{mn} = \frac{\mathcal{E}_N}{W_o\,(2^{m+n}\,n!\,m!\,\pi)^{\frac{1}{2}}} \frac{(1+i\xi)^{\frac{m+n}{2}}}{(1-i\xi)^{\frac{m+n}{2}+1}}$$

$$H_N\left(\frac{\sqrt{2}x}{W}\right) H_m\left(\frac{\sqrt{2}y}{W}\right) \exp\left\{-\frac{(x^2+y^2)}{W_o^2\,(1-i\xi)}\right\}$$

This normalization is the most convenient and has been used also by Kogelnik, 44.

The energy flux now is equal to

$$\frac{c}{16\pi}\,/\mathcal{E}_N/^2 \times 10^{-7} \text{ watts}$$, and so

$$2.17 \quad /\mathcal{E}_N/ = \sqrt{\frac{16\pi\ \bar{S}_T \times 10^7}{c}} \quad , \quad \bar{S}_T \text{ in watts.}$$

Orthogonality

The normalised modes $(\mathcal{E}_N = 1)$ are orthogonal since for two dimensions

$$I = \int_{-\infty}^{\infty} \mathcal{E}_n^* \mathcal{E}_m \, dx$$

$$= \frac{1}{W_o 2^{\frac{n+m}{2}} (n!\ m!)^{\frac{1}{2}}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^{\frac{1}{2}}} H_n\left(\frac{\sqrt{2}x}{W}\right)$$

$$H_m\left(\frac{\sqrt{2}x}{W}\right) \exp\left(-\frac{2x^2}{W^2}\right) dx$$

$$= \frac{1}{\sqrt{2^{n+m}\,n!\,m!\,\pi}} \int_{-\infty}^{\infty} H_n(\zeta)\,H_m(\zeta)\,\exp(-\zeta^2)\,d\zeta$$

$$= \delta_{mn}$$

and for three dimensions

$$I = \int_{-\infty}^{\infty} \mathcal{E}_{n\ell}^{*} \, \mathcal{E}_{mk} \, dx \, dy$$

$$= \frac{1}{W_o^2 \, 2^{\frac{n+m+l+k}{2}} (n! \, m! \, l! \, k!)^{\frac{1}{2}}} \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^{\frac{1}{2}}}$$

$$\times \; H_n\left(\frac{\sqrt{2}x}{W}\right) H_m\left(\frac{\sqrt{2}x}{W}\right) dx \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^{\frac{1}{2}}} H_\ell\left(\frac{\sqrt{2}y}{W}\right)$$

$$\times \; H_k\left(\frac{\sqrt{2}y}{W}\right) dy = \delta_{mn} \, \delta_{k\ell}$$

Since the Hermite functions form a complete set over the interval $(-\infty, \infty)$ we can regard the resonator modes as forming a complete set in the cavity and therefore any travelling waveform in the cavity can be expanded as a sum of the modes.

The general solution of the differential equation 2.38 can be written, therefore as

$$\mathcal{E} = \sum_{n=0}^{\infty} A_n \, \mathcal{E}_n \quad \text{for two dimensions and}$$

$$\mathcal{E} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{nm} \, \mathcal{E}_{nm} \quad \text{for three dimensions}$$

where the A's are constants which for a given field can be determined in the normal way using the orthogonality of the $\mathcal{E}_n$, ($\mathcal{E}_{n,m}$).

4. Propagation in Anisotropic Media

(a) Two dimensions

The propagation of the lowest order two dimensional gaussian mode has been treated independently by Bhawalker, Gambling and Smith, 45, in some detail but here we are only interested in the particular case of propagation perpendicular to the optic axis. Bergstein and Zachos, 48, 49, have also studies propagation in an uniaxial anisotropic media using a diffraction theory approach.

We consider Maxwell's equations for an infinite uniaxial lossless medium enclosed in a resonator with the optic axis of the medium perpendicular to the axial plane of the resonator. We set up a system of co-ordinates so that the resonator is infinite in the y direction, the optic axis lies in the x direction and the z direction in the direction of propagation. These axes are principal axes of the dielectric tensor which can therefore be written.

2.72
$$\epsilon = \begin{pmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_z & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix}$$

The Fourier transforms of Maxwell's equations (e.g. 1.46) written out in full are

2.73 (a)
$$\frac{\partial^2 E_z}{\partial x \partial z} - \frac{\partial^2 E_x}{\partial z^2} - \frac{\omega^2}{c^2} \epsilon_x E_x = 0$$

(b)
$$-\frac{\partial^2 E_y}{\partial z^2} - \frac{\partial^2 E_y}{\partial x^2} - \frac{\omega^2}{c^2} \epsilon_z E_y = 0$$

$$(c) \quad \frac{\partial^2 E_x}{\partial z \partial x} - \frac{\partial^2 E_x}{\partial x^2} - \frac{\omega^2}{c^2} \epsilon_z E_z = 0$$

Note that the equations decouple. The x and z component describe the propagation of the extraordinary wave whose refraction index varies with direction of propagation and the y component describes the propagation of the ordinary wave. The y component has exactly the form of the isotropic equation and hence the foregoing theory applies without change.

Using the divergence equation

$$2.74 \quad \frac{\epsilon_x \partial E_x}{\partial x} + \epsilon_z \frac{\partial E_z}{\partial z} = 0$$

The x and z components can be separated, equation 2.73 (a) becoming

$$2.75 \quad \frac{\epsilon_x}{\epsilon_z} \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial z^2} + \frac{\omega^2}{c^2} \epsilon_x E_x = 0$$

We now consider an extraordinary wave propagating in the z direction. Since we can assume it is transverse it will be governed by this equation.

Let $E_x = \mathcal{E} \exp(-ikz)$. where $\mathcal{E}$ is a slowly varying function of x and z and $k = \omega/c \sqrt{\epsilon_x}$. Substituting into equation 2.75 and using the previous approximation 2.37 we obtain

$$2.76 \quad \epsilon_x/\epsilon_z \frac{\partial^2 \mathcal{E}}{\partial x^2} - 2ik \frac{\partial \mathcal{E}}{\partial z} = 0$$

Making the scale change $x = (\epsilon_x / \epsilon_z)^{\frac{1}{2}} X$ we obtain the two-dimensional analogue of equation 2.38

2.77 $$\frac{\partial^2 \mathcal{E}}{\partial X^2} - 2\,ik\,\frac{\partial \mathcal{E}}{\partial z} = 0$$

which has solutions

$$\mathcal{E}_n = \mathcal{E}_o \frac{(1 + i\xi)^{n/2}}{(1 - i\xi)^{\frac{n}{2}+1}} H_n\left(\frac{\sqrt{2}\,X}{W^1}\right) \exp\left\{- \frac{X^2}{W_o^{1^2}(1-i\xi)}\right\}$$

Therefore a mode polarized as an extraordinary wave propagates as

2.78 $$\mathcal{E}_n = \mathcal{E}_o \frac{(1 + i\xi)^{n/2}}{(1 - i\xi)^{\frac{n+1}{2}}} H_n\left(\frac{\sqrt{2}x}{W^1}\left(\frac{\epsilon_z}{\epsilon_x}\right)^{\frac{1}{2}}\right) \exp\left\{- \frac{\frac{\epsilon_z}{\epsilon_x} x^2}{W_o (1 - i\xi)}\right\}$$

W_o^1 no longer represents the spot size of the beam, but we define

2.79 $W_o = \left(\frac{\epsilon_x}{\epsilon_z}\right)^{\frac{1}{2}} W_o^1$ and now W_o is the spot size as defined previously (equation 2.25).

also $$\xi = \frac{2z}{kW_o^{1\,2}} = \frac{2z}{k}\left(\frac{\epsilon_x}{\epsilon_z}\right) W_o^2$$

2.80 $$\xi = \frac{2z}{kW_o^2}\left(\frac{\epsilon_x}{\epsilon_z}\right)$$

so that the beam spread is altered to

2.81 $$\theta = \tan^{-1}\left\{\frac{2}{kW_o}\left(\frac{\epsilon_x}{\epsilon_z}\right)\right\}$$

The beam spread therefore is less for an extraordinary beam in a negative uniaxial medium, more for an extraordinary beam in a positive uniaxial medium than the beam spread for the ordinary beam.

The lines of constant phase of equation 2.78 are given by (c.f. section 2 (c)).

$$2.82 \quad x^2 = 2\,(z^1 - z)\; z\,\frac{\epsilon_x}{\epsilon_z}\left(1 - \frac{z_o^2}{z^2}\right)$$

As before these are circles, now of radius of curvature

$$2.83 \quad R = Z\,\frac{\epsilon_x}{\epsilon_z}\left(1 - \frac{z_o^2}{z^2}\right)$$

$$\text{where} \quad z_o = \frac{kW_o^2}{2}\left(\frac{\epsilon_z}{\epsilon_x}\right)$$

The spot size, confocal parameters etc. of a set of modes produced by two given mirrors can be calculated from this data as in section 2 (c) (see reference 45 for explicit formulae). Note that the spot sizes of the extraordinarily polarized modes will be different from the ordinary polarized ones for a given resonator.

The normalization of equation 2.78 can be carried out exactly as before and all other conclusions of the earlier work hold for this case.

Three Dimensions

Consider a resonator filled with uniaxial lossless dielectric with axes oriented as for the two dimensional case. Then equations 1.46 written out in full are

2.84

$$\text{(a)} \quad \frac{\partial^2 E_x}{\partial x \, \partial z} + \frac{\partial^2 E_y}{\partial x \, \partial y} - \frac{\partial^2 E_x}{\partial z^2} - \frac{\partial^2 E_x}{\partial y^2} - \frac{\omega^2}{c^2} \epsilon_x E_x = 0$$

$$\text{(b)} \quad \frac{\partial^2 E_x}{\partial y \, \partial x} + \frac{\partial^2 E_z}{\partial z \, \partial y} - \frac{\partial^2 E_y}{\partial x^2} - \frac{\partial^2 E_y}{\partial z^2} - \frac{\omega^2}{c^2} \epsilon_z E_y = 0$$

$$\text{(c)} \quad \frac{\partial^2 E_y}{\partial z \, \partial y} + \frac{\partial^2 E_x}{\partial z \, \partial x} - \frac{\partial^2 E_z}{\partial y^2} - \frac{\partial^2 E_z}{\partial x^2} - \frac{\omega^2}{c^2} \epsilon_z E_z = 0$$

and the divergence equation

$$\text{(d)} \quad \epsilon_x \frac{\partial E_x}{\partial z} + \epsilon_z \frac{\partial E_y}{\partial y} + \epsilon_z \frac{\partial E_z}{\partial z} = 0$$

Using the divergence equation the others can be reduced to

2.85

$$\text{(a)} \quad \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \frac{\omega^2}{c^2} \epsilon_x E_x = 0$$

$$\text{(b)} \quad \frac{\partial^2 E_x}{\partial y \, \partial x} \left(1 - \frac{\epsilon_x}{\epsilon_z}\right) - \nabla^2 E_y - \frac{\omega^2}{c^2} \epsilon_z E_y = 0$$

$$\text{(c)} \quad \frac{\partial^2 E_x}{\partial z \, \partial x} \left(1 - \frac{\epsilon_x}{\epsilon_z}\right) - \nabla^2 E_z - \frac{\omega^2}{c^2} \epsilon_z E_z = 0$$

We assume that the anisotropy is small i.e. $\left(1 - \frac{\epsilon_x}{\epsilon_z}\right) \ll 1$ and look for a solution representing an almost plane wave travelling in the z direction. We expect two solutions, extraordinary and ordinary waves, therefore we put

2.86

$$E_x = \mathcal{E}_x \exp(-i k_e z)$$

$$E_y = \mathcal{E}_y \exp(-i k_o z)$$

$$E_z = \mathcal{E}_x^{o} \exp(-i k_o z) + \mathcal{E}_z^{e} \exp(-ikez)$$

where $\mathcal{E}_x$, $\mathcal{E}_y$, $\mathcal{E}_z$ are slowly varying functions of x, y, z. Substituting these into the divergence equation 2.84 (d) and using the approximations

$$2.87 \quad \left|\frac{\partial \mathcal{E}_z^{o}}{\partial z}\right| << k_o \left|\mathcal{E}_z^{0}\right| \text{ and } \left|\frac{\partial \mathcal{E}_z^{e}}{\partial z}\right| << k_e \left|\mathcal{E}_z^{e}\right|$$

it becomes

$$2.88 \quad \left(\frac{\partial \mathcal{E}_y^{o}}{\partial y} - i k_o \mathcal{E}_z^{o}\right) \exp(-i k_o z) + \left(\frac{\epsilon_x}{\epsilon_z} \frac{\partial \mathcal{E}_x}{\partial x} - ik_e \mathcal{E}_z^{e}\right)$$

$$\times \exp(-i k_e z) = 0$$

This equation can be split into two equations if the change of the quantities in the brackets is negligable over a period of the exponential term

$$\exp\left\{-i(k_o - k_e) z\right\} \quad \text{which is } 2\pi / (k_o - k_e)$$

This condition will be true since we have assumed the waves to be nearly plane waves and that $\mathcal{E}_x$, $\mathcal{E}_y$, $\mathcal{E}_z$ are slowly varying compared with the $\exp(-i k_e z)$.

2.88 becomes therefore

2.89 (a) $$\frac{\partial \mathcal{E}_y}{\partial y} - i k_o \mathcal{E}_x^{o} = 0$$

(b) $$\frac{\epsilon_x}{\epsilon_z} \frac{\partial \mathcal{E}_x}{\partial z} - i k_e \mathcal{E}_z^{e} = 0$$

and these can be used with the substitution of 2.86 to reduce equations 2.85 to

2.90 (a) $$\frac{\epsilon_x}{\epsilon_z} \frac{\partial^2 \mathcal{E}_x}{\partial x^2} + \frac{\partial^2 \mathcal{E}_x}{\partial y^2} - 2 i k_e \frac{\partial \mathcal{E}_x}{\partial z} = 0$$

(b) $$\frac{\partial^2 \mathcal{E}_y}{\partial x^2} + \frac{\partial \mathcal{E}_y}{\partial y^2} - 2 i k_o \frac{\partial \mathcal{E}_y}{\partial z} = 0$$

Equation (b) corresponds to that found by, 46, for the more general case, including double refraction.

The third equation is of negligible order. So under these approximations the equations have split as in the two dimensional case, into one governing the ordinary component and the other the extraordinary. The ordinary component 2.90 (b) again propagates as for the isotropic case. Making a scale change in 2.90 (a)

$$x = \left(\frac{\epsilon_x}{\epsilon_x}\right)^{\frac{1}{2}} X$$

we can reduce it to the isotropic case and thus the solutions of 2.90 (a) will be

$$2.91 \quad \mathcal{E}_{x_{mn}} = \mathcal{E}_o \frac{(1 + i\xi)^{\frac{m+n}{2}}}{(1 - i\xi)^{\frac{m+n+2}{2}}} H_n\left(\frac{\sqrt{2}x}{W}\left(\frac{\epsilon_z}{\epsilon_x}\right)^{\frac{1}{2}}\right) H_m\left(\frac{\sqrt{2}y}{W}\right) \exp - \frac{\left(\frac{\epsilon_z}{\epsilon_x} x^2 + y^2\right)}{W_o^2 (1 - i\xi)}$$

We see from this that the beam is no longer circular, the curves of constant intensity for a given z are given by

$$2.92 \quad \frac{\epsilon_z}{\epsilon_y} X^2 + y^2 = W^2$$ which is an ellipse with semi axes W and $W\left(\frac{\epsilon_x}{\epsilon_z}\right)^{\frac{1}{2}}$.

The surface of constant phase which define the resonator surfaces are given by

$$\frac{\left(\frac{\epsilon_z}{\epsilon_x} x^2 + y^2\right)\xi}{W_o^2 (1 + \xi^2)} + kz = kz^1$$

(as for section 2.2 (c))

Rearranging

$$2.93 \quad \frac{\epsilon_z}{\epsilon_x} x^2 + y^2 = 2 (z^1 - z) \; Z \left(1 + \frac{z_o^2}{z^2}\right)$$

This is the surface of an ellipsoid, so that a resonator enclosing a uniaxial medium needs to have ellipsoidal mirrors to achieve resonance for the extraordinary polarized modes. In this situation the ordinary polarized modes will have greater loss due to the wrong curvature of the mirrors and so effectively the resonator will have only one set of modes. One possible way around the experimental difficulty of ellipsoidal mirrors in the situation when the non-linear properiies of a particular uniaxial medium are required would be to include also inside the resonator some medium of the opposite uniaxial nature to reproduce spherical wave fronts on the mirrors (see diagram 2.5). This would also enable the ordinary polarized modes to resonate.

The normalization of the mode will be changed by the factor ϵ_z / ϵ_x which has now appeared. Keeping the definition of

$\zeta = 2z / kW_o^2$ $\quad k = \eta_{ext} \frac{\omega^2}{c^2}$ the normalized form of $\mathcal{E}_{nm}$ is

$$2.94 \quad \mathcal{E}_{nm} = \frac{\mathcal{E}_n}{W_o \left(\frac{\epsilon_x}{\epsilon_z}\right)^{\frac{1}{4}} \left(2^{n+m-2} n!\, m!\, \pi\right)^{\frac{1}{2}}} \quad \frac{(1 + i\zeta)^{\frac{n+m}{2}}}{(1 - i\zeta)^{\frac{n+m+1}{2}}}$$

$$H_n \left(\frac{\sqrt{2}x}{W} \left(\frac{\epsilon_z}{\epsilon_x}\right)^{\frac{1}{2}}\right) H_m \left(\frac{\sqrt{2}y}{W}\right) \exp \left\{ - \frac{\left(x^2 \frac{\epsilon_z}{\epsilon_x} + y^2\right)}{W_o^2 (1 - i\zeta)} \right\}$$

<u>Effects of Absorption</u>

The derivation of equation 2.38 from equation 2.34 holds without change when ϵ is complex. The solution to equation 2.38 follows the same lines except that k is now complex since

$$k = \frac{\omega}{c} \epsilon^{\frac{1}{2}} = k_r - i\, k_{im}$$

Instead of equation 2.43 we define

$$2.95 \quad \frac{1}{q} = \frac{1}{R} - \frac{i\,2}{k_r W^2}$$

and hence

$$q = \frac{i\, k_r W_o^{\;2}}{2} + z \quad \text{and} \quad z_o = \frac{k_r W_o^{\;2}}{2}$$

The spot size W is given by $W^2 = W_o^{\;2} \left(1 + \frac{4z^2}{k_r^{\;2} W_o^4} \right.$

and the wavefront radius of curvature $R = z \left(1 + \frac{k_r^{\;2} W_o^{\;4}}{4z^2} \right)$

$$= \frac{k_r^{\;2} W^2 W_o^{\;2}}{4z}$$

The equation for the factor P of equation 2.39 for the general case remains unchanged

$$\frac{dP}{dz} = \frac{-i}{q} - \frac{2\ (m + n)}{kW^2}$$

but the solution for P, equation 2.52, is altered to

$$2.96 \quad iP = \log \left(\frac{1 + 4z^2}{k_r^{\;2} W_o^{\;4}} \right) - i \left(1 + \frac{k_r}{k} \right. (m + n) \tan^{-1} \left(\frac{2z}{k_r W_o^{\;2}} \right) \Bigg)$$

Thus the general solution for the travelling wave modes after some manipulation to bring out the imaginary part of k is given by (c.f. equation 2.53)

$$2.97 \quad \mathcal{E}_{mn} = \mathcal{E}_o \frac{W_o}{W} H_m \left(\frac{\sqrt{2}x}{W} \right) H_n \left(\frac{\sqrt{2}y}{W} \right) \exp \left\{ \frac{-r^2}{W^2} \; \frac{-2ir^2 z}{k_r W_o^{\;2} W^2} + i\ (m + n + 1) \tan^{-1} \left(\frac{2z}{k_r W_o^{\;2}} \right) \right\} f\ (k_{im})$$

where

$$f(k_{im}) = \exp\left\{ \frac{k_{im}}{k} \left(\frac{ir^2}{W^2} - \frac{2r^2 z}{k\, W^2 W_o^2} \right) + \frac{ik_{im}}{k} (m+n) \times \tan^{-1} \left(\frac{2z}{k_r W_o^2} \right) \right\}$$

The exponential term $f(k_{im})$ is due to the presence of absorption. When considering the electric field we will also have the term $\exp(-k_i z)$ arising from $\exp(-ikz)$. As we have remarked previously all materials of interest will necessarily have low absorption therefore

$$k_i << 1$$

This implies the very much stronger inequality at optical frequencies

$$k_i << k_r$$

All the terms in the second exponential depend on the factor k_{im}/k_r and hence can be neglected. This leaves only the term $\exp(-k_i z)$ which has been introduced by the absorption present.

Chapter 3

Derivation of the Coupled Mode Equation and Calculation of the Coupling Coefficients

1 Two dimensions

We consider now the propagation of the travelling wave modes which were studied in the last chapter in a resonator which contains a uniaxial medium with a non linear response. From Maxwell's equations for the media coupled equations governing the interchange of energy between the various modes of the resonators are derived. This provides a new direct method for studying the interaction of optical fields due to the non linear medium when the interaction is inside a resonator. Previously the theory of the non resonant effect has been adapted to the resonant case. The optical field at frequency is subject to equation 1.59.

$$3.1 \quad \nabla \times (\nabla \times \underset{\sim}{E}^{\omega}) - \frac{\omega^2}{c^2} \epsilon(\omega)\cdot\underset{\sim}{E}^{\omega} = \frac{4\pi\omega^2}{c^2} \underset{\sim}{P}^{(2)\omega}$$

and writing $\underset{\sim}{D} = \underset{\sim}{E} + 4\pi\underset{\sim}{P}^{(1)} + 4\pi P^{(2)}$

the divergence equation.

$$3.2 \qquad \nabla\cdot(\epsilon(\omega)\cdot\underset{\sim}{E}^{\omega}) = -4\pi\nabla\cdot P^{(2)\omega}$$

Setting up axes as before, we have the x axis along the optic axis of the uniaxial medium, the z axis in the direction of propagation and the yz plane the axial plane of the resonator. The x and y components of equation 3.1 are

$$3.3 \quad (a) \quad \frac{\partial^2 E_z^\omega}{\partial x \partial z} - \frac{\partial^2 E_x^\omega}{\partial x^2} - \frac{\omega^2}{c^2} \epsilon_x \, E_x^\omega = 4 \frac{\pi \omega^2}{c^2} P_x \textcircled{2}^\omega$$

$$(b) \quad \frac{\partial^2 E_y^\omega}{\partial z^2} - \frac{\partial^2 E_y^\omega}{\partial x^2} - \frac{\omega^2}{c^2} \epsilon_z \, E_y^\omega = 4\pi \frac{\omega^2}{c^2} P_y \textcircled{2}^\omega$$

and the divergence equation

$$3.4 \qquad \epsilon^\omega \frac{\partial E_x^\omega}{\partial x} + \epsilon_z^\omega \frac{\partial E_z^\omega}{\partial z} = 4\pi \nabla \cdot P \textcircled{2}^\omega$$

after substituting 3.4 into 3.3 (a) to eliminate $\frac{\partial E_z^\omega}{\partial z}$ 3.3 (a) becomes

$$3.5 \quad \frac{\epsilon_x}{\epsilon_z} \frac{\partial^2 E_x^\omega}{\partial x^2} + \frac{\partial^2 E_x^\omega}{\partial z^2} + \frac{\omega^2}{c^2} \epsilon_x \, E_x^\omega = \frac{-4\pi \omega^2}{c^2} P_x \textcircled{2}^\omega$$

The term $-4\pi \frac{\partial}{\partial x} \left(\nabla \cdot \underset{\sim}{P} \textcircled{2}^\omega \right)$ has been neglected for the reason that because $\underset{\sim}{P}$ arises from optical folds propagating in the z direction which vary slowly in the x,y plane the relation

$$3.6 \quad \left| \frac{\partial}{\partial x} \left(\nabla \cdot \underset{\sim}{P} \textcircled{2}^\omega \right) \right| \qquad \frac{4\pi \omega^2}{c^2} \left| P_x \textcircled{2}^\omega \right| \qquad \text{will hold.}$$

Hence we have the two equations 3.5 and 3.3 (b) governing the propagation of the extraordinary and ordinary waves respectively in the resonator. From here we can proceed as in the linear case putting

$$E_{x,\,y}^\omega = \mathcal{E}_{o,\,e}^\omega \quad \exp \left(-ik_{o,e} z \right)$$

substituting into 3.5 and 3.3 (b). This gives on approximating as before

3.7 (a) $$\frac{\partial^2 \mathcal{E}_o^\omega}{\partial x^2} - 2ik_o \frac{\partial \mathcal{E}_o}{\partial z} = \frac{-4\pi\omega^2}{c^2} P_y \,②^\omega \exp(ik_o z)$$

(b) $$\frac{\epsilon_x}{\epsilon_z}\frac{\partial^2 \mathcal{E}_e^\omega}{\partial x^2} - 2ik_e \frac{\partial^2 \mathcal{E}_e}{\partial z} = \frac{-4\pi\omega^2}{c^2} P_x \,②^\omega \exp(ik_e z)$$

We have seen that the general solution to the equation without the non linear polarization can be represented by an infinite sum over the resonator modes.

3.8 $$\mathcal{E} = \sum A_n \mathcal{E}_n$$ where the A_n are constants.

Since the second order polarization is "small" we can regard the solution to the non linear equations as a perturbation of this solution. In general the non linear term couples energy into or out of the wave as it propagates so we introduce 3.8 as a solution of the non linear equation allowing the A_n to be slowly varying functions of z. This coupled mode approach has been used in the study of systems with small non linearities. 55, 56,

let 3.9 $$\mathcal{E}_o^\omega(x,y,z) = \sum_{n=0}^{\infty} A_n^\omega(z)\, \mathcal{E}_{on}^\omega(x,y,z)$$

$$\mathcal{E}_e^\omega(x,y,z) = \sum_{n=0}^{\infty} B_n^\omega(z)\, \mathcal{E}_{ne}^\omega(x,y,z)$$

and substituting these into equation 3.7 we have

3.10 (a) $$\sum_{n=0}^{\infty} - 2ik_o \frac{dA_n}{dz} \mathcal{E}_{on}^\omega = \frac{-4\pi\omega^2}{c^2} P_y \,②^\omega \exp(ik_o z)$$

$$\text{(b)} \quad \sum_{n=0}^{\infty} - 2ik_e \frac{dB_n^{\omega}}{dz} \mathcal{E}_{en}^{\omega} = \frac{-4\pi\omega^2}{c^2} P_x^{(2)\omega} \exp(ik_e z)$$

Taking the scalar product with $\mathcal{E}_{om}^{\omega}$, $\mathcal{E}_{em}^{\omega}$ respectively we have

$$3.11 \quad \text{(a)} \quad \frac{dA_m^{\omega}}{dz} = \frac{-2\pi i\omega^2}{k_o c^2} \int_{-\infty}^{\infty} \mathcal{E}_{om}^{*\omega} P_y^{(2)\omega} \, dx \quad \exp(jk_o z)$$

$$\text{(b)} \quad \frac{dB_m^{\omega}}{dz} = \frac{-2\pi i\omega^2}{k_e c^2} \int_{-\infty}^{\infty} \mathcal{E}_{em}^{*\omega} P_x^{(2)\omega} \, dx \quad \exp(ik_e z)$$

These equations give us the rate of change of the m^{th} resonator mode ((a) ordinary (b) extraordinary) for the field at frequency ω due to the non linear polarization at that frequency.

(a) <u>Second Harmonic Generation</u>

(1) Coupled Mode Equation:- For second harmonic generation we have to consider two optical fields, frequencies ω and 2ω Since the ordinary wave at ω will be near phase matching to the extraordinary at 2ω we need only consider these two components.

From section 1.4 (c) we have

$$P_y^{(2)\omega} = d_{15} \quad E_y^{*\omega} \quad E_x^{2\omega}$$

$$P_x^{(2)2\omega} = \tfrac{1}{2} d_{31} \quad E_y^{\omega} \quad E_y^{\omega}$$

In terms of the resonator modes.

$$3.12 \quad P_y^{(2)\,\omega} = d_{15} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A_j^{*\,\omega}\, B_k^{2\omega}\, \mathcal{E}_j^{*\,\omega}\, \mathcal{E}_k^{2\omega}$$

$$\exp\left(-i\,(k_2 - k_1)\, z\right)$$

$$P_x^{(2)\,2\omega} = \tfrac{1}{2}\, d_{31} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A_j^{\omega}\, A_k^{\omega}\, \mathcal{E}_j^{\omega}\, \mathcal{E}_k^{\omega} \quad \exp\left(-2ik_1 z\right)$$

dropping the explicit o, e subscript and substituting these equations into the appropriate equations from 3.11 we have

$$3.13 \quad (a) \quad \frac{dA_m^{\omega}}{dz} = \frac{-2\pi i\, \omega^2}{k_1 c^2}\, d_{15}\, A_j^{*\omega}\, B_k^{2\omega} \int_{-\infty}^{\infty} \mathcal{E}_m^{*\omega}\, \mathcal{E}_j^{*\omega}\, \mathcal{E}_k^{2\omega}\, dx$$

$$\times \exp\left(i\,(k_e - 2k_o)\, z\right)$$

$$(b) \quad \frac{dB_m^{2\omega}}{dz} = \frac{4\pi i\, \omega^2}{k_2 c^2}\, d_{31}\, A_j^{\omega}\, A_k^{\omega} \int_{-\infty}^{\infty} \mathcal{E}_m^{*\,2\omega}\, \mathcal{E}_j^{\omega}\, \mathcal{E}_k^{\omega}\, dx$$

$$\times \exp\left(+i\,(k_2 - 2k_1)\, z\right)$$

which can be written

$$3.14 \quad (a) \quad \frac{dA_m^{\omega}}{dz} = C_{mjk}^{\omega}\, A_j^{*\omega}\, B_k^{2\omega}$$

$$(d) \quad \frac{dB_m^{2\omega}}{dz} = C_{jkm}^{2\omega}\, A_j^{\omega}\, A_k^{\omega}$$

where the coupling coefficients are defined by 3.13 (a) (b)

(ii) Energy Conservation relation

For a lossless medium there exists a useful relation between the C^{ω}_{mjk} and $C^{2\omega}_{mjk}$

The total energy of the system is represented by

$\sum_{n=0}^{\infty}(A_n^{*\omega} \cdot A_n^{\omega} + B_n^{*2\omega} \cdot B_n^{2\omega})$ and this must be constant. Thus differenting this expression for the energy we have

$$3.15 \qquad \frac{dA_n^{\omega}}{dz} A_n^{*\omega} + \frac{dB_n^{*2\omega}}{dz} B_n + \text{complex conjugate} = 0$$

substituting from 3.14

$$C^{\omega}_{njk} A_j^{*\omega} A_n^{*\omega} B_k^{2\omega} + C^{*2\omega}_{jkn} A_j^{*\omega} A_k^{*\omega} B_n^{2\omega} + \text{complex conjugate} = 0$$

interchanging dummy suffices we have

$$(C^{\omega}_{njk} + C^{*2\omega}_{njk}) A_j^{*\omega} A_n^{*\omega} B_k^{2\omega} + \text{complex conjugate} = 0$$

This relation must be true for all values of the A_j's and B_j's hence

$$3.16 \qquad C^{\omega}_{njk} = -C^{*2\omega}_{njk}$$

This is an important result which corresponds to Klienman's symmetry condition.

(iii) Calculation of the coefficients $C_{njk}^{2\omega}$

From 3.13 (b) the $C_{njk}^{2\omega}$ depend on the integral

$$3.17 \qquad I_{jkm} = \int_{-\infty}^{\infty} \mathcal{E}_m^{*\,2\omega} \; \mathcal{E}_j^{\omega} \; \mathcal{E}_k^{\omega} \; dx$$

which on substituting for the resonator mode functions introducing the subscript 1 , 2 for the fundamental and harmonic quantities respectively is essentially the integral.

$$3.18 \qquad I_{jkm} = \int_{-\infty}^{\infty} H_m\left(\frac{\sqrt{2}x}{W_2}\right) H_j\left(\frac{\sqrt{2}x}{W_1}\right) H_k\left(\frac{\sqrt{2}x}{W_1}\right)$$

$$\exp\left\{\frac{-x^2}{W_{o2}^2\,(1 + i\,\xi_2)} \quad \frac{-2x^2}{W_{o1}^2\,(1 - i\xi)}\right\} dx$$

where the coefficients will be given by

$$3.19 \qquad C_{jkm}^{2\omega} = \frac{-4i\,\omega^2\, d_{31}\,(2\pi)^{\frac{1}{4}}\exp(i\Delta kx)}{k_2 c^2\,(2^{m+j+k}\, m!\, j!\, k!\, W_{o1}{}^2\, W_{o2})^{\frac{1}{2}}} \quad \times$$

$$\frac{(1 - i\,\xi_2)^{m/2}\,(1 + i\,\xi_1)^{\frac{j+k}{2}}}{(1 + i\,\xi_2)^{\frac{m+1}{2}}\,(1 - i\,\xi_1)^{\frac{j+k+1}{2}}} \times I_{jkm} \quad \text{where } \Delta k = k_2 - 2k_1$$

$$3.20 \qquad \frac{\sqrt{2}x}{W_1} = u \qquad \alpha = \frac{W_1}{\sqrt{2}\,W_2}$$

3.18 becomes

3.21 $$I_{jkm} = \frac{W_1}{\sqrt{2}} \int_{-\infty}^{\infty} H_m(\sqrt{2}\alpha u)\; H_j(u)\; H_k(u)$$

$$\times \exp\left(-u^2\left(\alpha^2(1 - i\xi_2) + 1 + i\xi_1\right)\right)\, du$$

This can be evaluated using the generating function for the Hermite polynomials

3.22 $$\exp(2sx - s^2) = \sum_{n=1}^{\infty} H_n \frac{(x)\, s^n}{n!}$$

I_{jkm} will be the coefficient of $t^m s^j p^k$ multiplied by $j!\,k!\,m!$ in the expansion of the expression in powers of t, s and p

$$I = \frac{W_1}{\sqrt{2}} \exp(2\sqrt{2}\alpha ut - t^2 + 2us - s^2 + 2pu - p^2$$

$$- b^2u^2)\, du$$

where $b^2 = \alpha^2(1 - i\xi_2) + (1 + i\xi_1)$

rearranging

3.23 $$I = \frac{W_1}{\sqrt{2}} \int_{-\infty}^{\infty} \exp\left(-b^2u^2 + 2u(\sqrt{2}\alpha t + s + p) - b^2 - s^2 - p^2\right)\, du$$

To evaluate this integral let $v = ub - \frac{(\sqrt{2}\alpha t + s + p)}{b}$

and it reduces to

$$3.24 \quad I = \frac{W_1}{\sqrt{2}b} \exp \left(\frac{(\sqrt{2}\,\alpha t + s + p)^2}{b^2} - t^2 - s^2 - p^2 \right) \int_C (\exp(-v^2))\, dv$$

using the result of appendix 1

$$3.25 \qquad \int_C (\exp(-v^2))\, dv = \sqrt{\pi}$$

Expanding the exponential term in 3.24 gives

$$\exp \left(\frac{(\sqrt{2}\,\alpha t + s + p)^2}{b^2} - t^2 - s^2 - p^2 \right) = \sum_{n=0}^{\infty} \frac{\left(\frac{(\sqrt{2}\,\alpha t + s + p)^2}{b^2} - t^2 - s^2 - u^2 \right)^n}{n!}$$

expanding further using the binomial theorem

$$3.26 \quad = \sum_{n=0}^{\infty} \sum_{\bar{m}=0}^{\infty} \frac{(-1)^{n-m}}{n!} \binom{n}{m} \left(\frac{\sqrt{2}\,\alpha t + s + p}{b} \right)^{2\bar{m}} (t^2 + s^2 + u^2)^{n-\bar{m}}$$

and again using the binomial theorem

$$3.27 \quad I = \frac{W}{b} \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \sum_{\bar{m}=0}^{n} \sum_{r=0}^{2\bar{m}} \sum_{q=0}^{n-\bar{m}} \sum_{l=0}^{r} \sum_{\bar{l}=0}^{q} \frac{(-1)^{n-m}}{n!}$$

$$\binom{n}{m} \binom{2\bar{m}}{r} \binom{n-\bar{m}}{q} \binom{r}{l} \binom{q}{\bar{l}} \times b^{-2\bar{m}} (\sqrt{2}\alpha)^{2\bar{m}-r} p^{l+2\bar{l}}$$

$$s^{(r-l-2q-2\bar{l})}\, t^{(2n-r-2q)}$$

We can rewrite this in a formal way in order to pick out the required coefficient putting $l + 2\bar{l} = k$

$$3.28 \qquad r + 2q - (l + 2\bar{l}) = j$$

$$2n - (r + 2q) = m$$

rearranging equation 3.27

$$3.29 \qquad r = m + j - 2q$$

$$l = m - 2\bar{l}$$

$$n = \frac{m + j + k}{2}$$

n is an integer, therefore for a non-zero coefficient $C^{2\omega}_{mjk}$, $m + j + k$ must be an even integer. This just expresses the symmetry conditions

odd	plus	even fundamental	mode	gives	odd	S.H.	mode
odd	"	odd "	"	"	even	"	"
even	"	even "	"	"	even	"	"

Picking out the required coefficients we have

$$3.30 \qquad I_{jkm} = \frac{W_1}{b}\sqrt{\frac{\pi}{2}}\,\frac{m!\,j!\,k!}{\left(\frac{m+j+k}{2}\right)!}\sum_{\bar{m}=0}^{\frac{m+j+k}{2}}\ \sum_{q=0}^{\frac{m+j+k-\bar{m}}{2}}\ \sum_{\bar{l}=0}^{q_1}$$

$$(-1)^{\frac{m+j+k-\bar{m}}{2}} \times \binom{\frac{m+j+k}{2}}{\bar{m}}\binom{2\bar{m}}{m+j-2q}\binom{\frac{m+j+k-m}{2}}{q}\binom{m+j-2q}{m-2\bar{l}}\binom{q}{\bar{l}}$$

$$b^{-2\bar{m}}\,(\sqrt{2}\alpha)^{2\bar{m}-m-j+2q}$$

Thus the coefficients can be evaluated analytically although the final expression is rather complicated in general. As we would expect the low-order coefficients are much simpler in form eg

3.31 $$I_{ooo} = \frac{W_1}{b}\sqrt{\frac{\pi}{2}}$$

$$I_{o2o} = \frac{W_1}{b}\sqrt{\frac{\pi}{2}}\left(\frac{1}{b} - 1\right)$$

$$I_{1o1} = \frac{W_1}{b}\sqrt{\frac{\pi}{2}}\quad\frac{1}{2b}$$

The most important set which we will examine more closely are those resulting from the coupling between the lowest order gaussian fundamental mode and the 2nth second harmonic mode.

3.32 $$I_{oo2n} = \frac{W_1}{b}\sqrt{\frac{\pi}{2}}\left(\frac{2\alpha^2}{b^2} - 1\right)^n \frac{(2n)!}{n!}$$

From 3.21 and 2.25 we have $\alpha = \frac{W_1}{\sqrt{2}W_2} = \frac{W_{o1}}{\sqrt{2}W_{o2}}\frac{(1+\xi_1^2)^{\frac{1}{2}}}{(1+\xi_2^2)^{\frac{1}{2}}}$

and defining $W = \frac{W_{o1}}{\sqrt{2}\,W_{o2}}$

3.33 $$b^2 = W^2\frac{(1+\xi_1^2)}{(1+\xi_2^2)}(1-i\,\xi_2) + (1+i\,\xi_1)$$

$$= \frac{(1+i\,\xi_1)}{(1+i\,\xi_2)}(W^2 + 1 - i\,(W^2\xi_1 - \xi_2)$$

and therefore

$$\frac{2\alpha^2}{b^2} - 1 = W^2 \frac{(1-i\xi_1)(1+i\xi_2) - (1+\xi_2^2)}{(1-i\xi_2)(W^2+1 - i(W^2\xi_1 - \xi_2))}$$

$$= \frac{(1+i\xi_2)}{(1-i\xi_2)} \frac{(W^2-1 - i(W^2\xi_1 - \xi_2))}{(W^2+1 - i(W^2\xi_1 - \xi_2))}$$

Substituting these expressions into that for I_{oo2n} and then the resulting expression into equation 3.19 for $C^{2\omega}_{jkm}$ we have

3.35 $$C^{2\omega}_{oo2n} = \frac{-4i\,\omega^2 d_{31}}{k_2\, c^2\, W_{o2}^{\frac{1}{2}}} \frac{\pi^{\frac{3}{4}}}{2^{\frac{1}{4}}} \frac{((2n)!)^{\frac{1}{2}}}{n!\,2^n} \frac{1}{(1-i\,\xi_1)^{\frac{1}{2}}} \exp(i\Delta kz)$$

$$\times \frac{(W^2-1-i(W^2\xi_1 - \xi_2))^n}{((W^2+1) - i(W^2\xi_1 - \xi_2))^{n+\frac{1}{2}}}$$

The factor $\frac{((2n)!)^{\frac{1}{2}}}{n!\,2^n}$ has values

n	o	1	2	3	4	6	10	etc
f	1	0.7071	0.6724	0.5590	0.5229	0.4749	0.4197	

and so decreases with n reducing the coupling between the lowest order fundamental mode and 2nth order harmonic mode with increasing n.

Thus far we have only considered the coupling case when the second harmonic resonator and the fundamental resonator are perfectly aligned and the foci of the beams are at the same position. There will be a reduction of the coupling (i) if the foci are any distance apart (ii) if the axial planes are parallel but a distance apart and (iii) if the axial planes are inclined at a small angle to each other.

(i) The evaluation of the $C^{2\omega}_{oo2n}$ is unaffected by this modification. Considering the fundamental mode to be focussed at the origin and the nth harmonic mode to be focussed at $z=f$, ξ_1 remains unchanged but

$$3.36 \quad \xi_2 = 2\frac{(z-f)}{k_2 W_{o2}^2}$$

and this introduces the extra term

$-2f/k_2 W^2_{o2}$ into equation 3.35.

(ii) If the axial plane of the fundamental jth mode is removed from that of the harmonic modes by a distance v of the harmonic modes it will be described by the equation

$$3.37 \quad \mathcal{E}^{\omega}_j = \left(\frac{1}{W_{o1}\, 2^j\, j!}\sqrt{\frac{2}{\pi}}\right)^{\frac{1}{2}} \frac{(1+i\xi_1)^{j/2}}{(1-i\xi_1)^{j+\frac{1}{2}}} H_j\left(\frac{\sqrt{2}\,(x-v)}{W_1}\right)$$

$$x \exp\left(-\frac{(x-v)^2}{W^2_{o1}\,(1-i\xi_1)}\right)$$

Substituting this form into equation 3.17 equation 3.22 becomes

$$3.38 \quad I_{mjk} = \frac{W_1}{\sqrt{2}} \int_{-\infty}^{\infty} H_m(\sqrt{2}\alpha u)\, H_j(u-\varsigma)\, H_k(u-\varsigma)\, \exp\left\{-u^2\alpha^2(1-i\,\xi_2) -(u-\varsigma)^2(1+i\,\xi_1)\right\} du$$

3.38 (b) where $\varsigma = \frac{\sqrt{2}\,v}{W_1}$

using the generating functions as before 3.23 becomes

$$3.39 \quad I = \frac{W_1}{\sqrt{2}} \int_{-\infty}^{\infty} \exp\left\{-u^2b^2+2u\,(\sqrt{2}\alpha t+s+p+\varsigma(1+i\,\xi_1)) -t^2-s^2-p^2\right\} du$$

putting $V = u\,b - \frac{(\sqrt{2}\,\alpha t + s + p + \varsigma(1+i\xi_1))}{b}$

and carrying out the integration (appendix I) we have

$$3.40 \quad I = \frac{W_1}{b}\sqrt{\frac{\pi}{2}}\, \exp\left\{ -\varsigma^2(1+i\,\xi_1)\left(1+\left(\frac{1+i\,\xi_1)}{b}\right)\right) + \frac{(\sqrt{2}\,\alpha t+s+p)^2}{b} -t^2-s^2-p^2+2\left(\frac{\sqrt{2}\alpha t+s+p)\varsigma}{b}\right)\right\}$$

Thus the separation of the axial planes has introduced two extra factors into equation

3.24 The first is

$$\exp\left\{-\varsigma^2(1+i\,\xi_1)\left(1-\frac{(1+i\,\xi_1)}{b}\right)\right\}$$

substituting for b this becomes

$$3.41 \qquad \exp\left\{ \frac{-\xi^2 W^2 (1+\xi_1^2)}{((W^2+1)^2 + (W^2\xi_1 - \xi_2)^2)} ((W^2+1) - i(\xi_1 W^2 - \xi_2)) \right\}$$

The real part of the expression which forms the argument of the exponential function is always less than zero and as this factor will modify each coefficient $C^{2\omega}_{jkm}$, they will all decrease with ξ as

$\exp(-\Theta\xi^2)$,

$$3.42 \qquad \text{Where } \Theta = W^2 \frac{(1+\xi_1^2)(W^2+1)}{(W^2+1)^2 + (\xi_1 W^2 - \xi_2)^2} > 0$$

Thus the coupling between any two modes has an overall decrease with increasing separation but of course there may be local increase.

The second factor

$$3.43 \qquad \exp\left\{ \frac{2\sqrt{2}\,\alpha t + s + p_1)}{b} \right\}$$ has as the argument of the

exponential function an expression which depends on an odd power of t, s and p and therefore it will break the symmetry conditions contained in equations 1.29. Consider as a specific example I_{oo1} which is zero under equation 1.29 for the coaxial case. I_{oo1} is the coefficient of t in the expansion of equation 3.40.

$$3.44 \qquad I_{oo1} = \frac{W_1}{b}\sqrt{\frac{\pi}{2}} \exp\left\{ \frac{-\xi^2 W^2 (1+\xi^2)}{(W^2+1-i(W^2\xi_1 - \xi_2))} \right\} \frac{2\sqrt{\alpha}\,\xi}{b}$$

so that substituting for ξ from 3.38 (b)

$$\left|C^{2\omega}_{001}\right| \propto \left|\frac{v}{W_{o1}}\right| \quad \exp\left(\frac{-2\Theta v^2}{W_{of}^{\;2}}\right)$$

Thus $C^{2\omega}_{001}$ is zero at $v = o$ as expected and increases with increasing separation , v , to a maximum when $v = \frac{W_{of}}{2\sqrt{\Theta}}$ and thereafter decreases asymtotically approaching zero. Since Θ depends on z this does not immediately give the separation of the axes which will maximize the coupling in a given crystal except for the case when $z/zo \ll 1$

(iii) For the case when the axial plane of the fundamental resonator is tilted at a small angle ,Θ,to the axial plane of the second harmonic resonator the equation representing the jth fundamental mode is

3.45 $$\mathcal{E}_j = \left(\frac{1}{W_{o1}^{\;2}\, 2^j\, j!}\sqrt{\frac{2}{\pi}}\right)^{\frac{1}{2}} \frac{(1 + i\,\xi_1)^{\frac{j}{2}}}{(1 - i\,\xi_2)^{\frac{j+1}{2}}}\; H_j\left(\frac{\sqrt{2}\,x}{W_1}\right) \times$$

$$\exp\left\{\left(\frac{-x^2}{W_{o1}^{\;2}\,(1 - i\,\xi_1)}\right) - ik\Theta x\right\}$$

if the z axis lies in the axial plane of the second harmonic resonator. The extra term depending on Θ arises from the propagation term exp (-ikz) of the electric field; all the other correction terms will be negligible in comparison with this one since $k \gg 1$

Using this equation, equation 3.23 becomes

3.46 $$I = \frac{W_1}{\sqrt{2}}\int_{-\infty}^{\infty} \exp\left(-u^2 b^2 + 2u\left(\sqrt{2}\alpha t + s + p - i\,\frac{k\Theta W_1}{\sqrt{2}}\right) - t^2 - s^2 - p\right) du$$

Carrying out the integration as before equation 3.24 becomes

$$3.47 \quad I = \frac{W_1}{b}\sqrt{\frac{\pi}{2}} \exp\left\{ \frac{(\sqrt{2}\alpha t + s + p)^2}{b^2} \quad \frac{-2ik\theta W_1}{\sqrt{2}} (\sqrt{2}\alpha t + s + p) \quad \frac{-k^2\theta^2 \; W_1^{\;2}}{2} \quad -t^2 - s^2 - p^2 \right\}$$

As in the previous case the tilt angle has introduced two new factors. The first being $\exp\left(\frac{-k^2 W_1 \theta^2}{2}\right)$ affecting all the coefficients, the second breaking the symmetry condition of equations 3.29

from 3.47

$$3.48 \quad I_{001} = \frac{-W_1}{b}\sqrt{\frac{\pi}{2}} \exp\left\{\frac{-k^2 W_1^2 \theta^2}{2}\right\} 2ik\theta W_1 \alpha$$

This has exactly the form of the previous coupling when the axes of the resonators were separated, increasing to a maximum and thereafter decreasing and asymtotically approaching zero. The maximum in this case occurs at an angle

$$\theta = \frac{1}{kW_1}$$ this again

depends on z and so will not give the maximum coupling condition immediately.

(2) Parametric Amplification

(i) Coupled Mode Equation

We have now three optical fields $\omega_1, \omega_2, \omega_3$ where $\omega_3 = \omega_1 + \omega_2$.

We consider the case of parametric amplification when the pump field, ω_3 propagates as an extraordinary wave and the signal, ω_1 and idler ω_2 fields propagate as ordinary waves. This being the configuration appropriate to a negative uniaxial crystal. The theory can be used for positive unaxial crystals when the opposite configuration is appropriate by using the appropriate mode function. The relevant second order polarization terms are for this case therefore

$$P_y^{(2)\omega_1} = d_{15}\, E_y^{*\omega_2}\, E_x^{\omega_3}$$

$$P_y^{(2)\omega_2} = d_{15}\, E_y^{*\omega_1}\, E_x^{\omega_3}$$

$$P_x^{(2)\omega_3} = d_{31}\, E_y^{\omega_1}\, E_y^{\omega_2}$$

expanding the fields in terms of the resonator modes

$$3.50 \quad P_y^{(2)\omega_1} = d_{15}\, A_j^{*\omega_2}\, B_k^{\omega_3}\, \mathcal{E}_j^{*\omega_2}\, \mathcal{E}_k^{\omega_3} \exp\left\{-i\,(k_3 - k_2)\, z\right\}$$

$$P_y^{(2)\omega_2} = d_{15}\, A_j^{*\omega_1}\, B_k^{\omega_3}\, \mathcal{E}_j^{*\omega_2}\, \mathcal{E}_k^{\omega_3} \exp\left\{-i\,(k_3 - k_1)\, z\right\}$$

$$P_x^{(2)\omega_3} = d_{31}\, A_j^{\omega_1}\, A_k^{\omega_2}\, \mathcal{E}_j^{\omega_1}\, \mathcal{E}_k^{\omega_2} \exp\left\{i\,(k_1 + k_2)\, z\right\}$$

Substituting these relations into the appropriate equations from

3.11 (a) and (b) we arrive at the coupled mode equations for the parametric case

3.51
$$\frac{dA_m^{\omega_1}}{dz} = C_{mjk}^{\omega_1} A_j^{*\omega_2} B_k^{\omega_3}$$

$$\frac{dA_m^{\omega_2}}{dz} = C_{jmk}^{\omega_2} A_j^{*\omega_1} B_k^{\omega_3}$$

$$\frac{dB_m^{\omega_3}}{dz} = C_{jkm}^{\omega_3} A_j^{\omega_1} A_k^{\omega_2}$$

where the coupling coefficients are defined in a similar way to the previous case of second harmonic generation.

For example

3.52
$$C_{mjk}^{\omega_1} = \frac{-2\pi i \omega^2}{k_1 c^2} d_{15} \exp\left\{-i\Delta kx\right\} \int_{-\infty}^{\infty} \mathcal{E}_m^{*\omega_1} \mathcal{E}_j^{*\omega_2} \mathcal{E}_k^{\omega_2} dx$$

where $\Delta k = k_3 - k_1 - k_2$

For a lossless medium we can establish a relationship between the coupling coefficients as in the previous case

$$\sum_n (A_n^{\omega_1} A_n^{\omega_1} + A_n^{\omega_2} A_n^{\omega_2} + B_n^{\omega_3} B_n^{\omega_3})$$ is the total energy present in the system and is constant. By the previous argument used we have

$$3.53 \qquad C^{\omega_1}_{njk} + C^{\omega_2}_{njk} = -C^{*\omega_3}_{njk}$$

Note that this result does reduce to 3.16 if $\omega_1 = \omega_2$ because of the factor $\frac{1}{2}$ in front of equation 3.12 (b) produced by the $\omega_1 = \omega_2$ symmetry.

(ii) Calculation of the coefficients $C^{\omega_1}_{njk}$

The calculation will follow precisely the lines of the calculation for the second harmonic generation but we have now the added complication of three sets of beam parameters. $C^{\omega_1}_{jkm}$ will depend on an integral of the form

$$3.54 \qquad I_{jkm} = \int_{-\infty}^{\infty} \mathcal{E}^{*\,\omega_1}_j \; \mathcal{E}^{*\,\omega_2}_k \; \mathcal{E}^{\omega_3}_m \, dx$$

substituting for the resonator modes, introducing subscripts 1, 2, 3 to denote the terms arising from the modes at ω_1, ω_2, ω_3 respectively the integral will reduce to

$$3.55 \qquad I_{jkm} = \int_{-\infty}^{\infty} H_j \left(\frac{\sqrt{2}x}{W_1} \right) H_k \left(\frac{\sqrt{2}x}{W_2} \right) H_m \left(\frac{\sqrt{2}x}{W_3} \right) \exp \left\{ \frac{-x^2}{W_{o1}^2 (1+i\,\xi_1)} - \frac{x^2}{W_{o2}^2 (1+i\,\xi_2)} - \frac{x^2}{W_{o3}^2 (1-i\,\xi_3)} \right\}$$

where $$C^{\omega_1}_{jkm} = \frac{-2i\,\omega_1^{\,2}\, d_{15}\, \exp\,(i\,\Delta kx)\,(2\pi)^{\frac{1}{4}}}{k_1\, c^2\,(2^{m+j+k}\, m!\,j!\,k!\, W_{o1}\, W_{o2}\, W_{o3})^{\frac{1}{2}}}\; \frac{(1-i\,\xi_1)^{\frac{j}{2}}}{(1+i\,\xi_1)^{\frac{j+1}{2}}}\; \frac{(1-i\,\xi_2)^{\frac{k}{2}}}{(1+i\,\xi_2)^{\frac{k+1}{2}}}$$

3.56(a)
$$\frac{(1+i\,\xi_3)^{\frac{m}{2}}}{(1-i\,\xi_3)^{\frac{m+1}{2}}} \quad x \quad I_{jkm}$$

making the substitutions $\frac{\sqrt{2}x}{W_3} = u, \quad \frac{\sqrt{2}x}{W_1} = \alpha_1 u, \quad \frac{\sqrt{2}x}{W_2} = \alpha_2\, u$

where $\alpha_1 = \frac{W_3}{W_1}, \quad \alpha_2 = \frac{W_3}{W_2}$

the integral becomes

3.56(c)
$$I_{jkm} = \frac{W_3}{\sqrt{2}} \int_{-\infty}^{\infty} H_j\,(\alpha_1 u)\; H_k\,(\alpha_2 u)\; H_m\,(u)\;\exp\left\{ -u^2\left(\alpha_1^{\,2}(1-i\,\xi_1) + \alpha_2^2\,(1-i\,\xi_2) + (1+i\,\xi_3)\right)\right\} du$$

using the generating function of the Hermite polynomials the integral generating the I_{jkm} is

3.57
$$I = \frac{W_3}{\sqrt{2}} \int_{-\infty}^{\infty} \exp\left\{ -b^2u^2 + 2u\,(\alpha_1 t + \alpha_2 s + p) - t^2 - s^2 - p^2 \right\} du$$

where $b^2 = \alpha_1^{\,2}\,(1-i\,\xi_1) + \alpha_2^{\,2}\,(1-i\,\xi_2) + (1+i\,\xi_3)$

which becomes, carrying out the integration

$$3.58 \quad I = \frac{W_3}{b}\sqrt{\frac{\pi}{2}} \; \exp\left\{ \frac{(\alpha_1 t + \alpha_2 s + p)^2}{b^2} - t^2 - s^2 - p^2 \right\}$$

If we expand this expression the I_{mjk} can be picked out as before and the symmetry condition (3.39) $m + j + k$ = even integer, holds for this case. A few of the low-order coefficients are

3.59
$$I_{000} = \frac{W_3}{b}\sqrt{\frac{\pi}{2}}$$

$$I_{110} = \frac{W_3}{b}\sqrt{\frac{\pi}{2}} \quad 2\frac{\alpha_1\alpha_2}{b^2}$$

$$I_{101} = \frac{W_3}{b}\sqrt{\frac{\pi}{2}} \quad 2\frac{\alpha_1}{b^2}$$

etc.

We will now evaluate $C^{\omega_1}_{000}$ in detail as an example, substituting for b and α_1, α_2 we have

3.60
$$I_{000} = \sqrt{\frac{\pi}{2}} \; \frac{W_{o3}}{\left(\frac{\overline{W}_1}{(1+i\xi_1)} + \frac{\overline{W}_2}{(1+i\xi_2)} + \frac{1}{(1-i\xi_3)} \right)}$$

where we have defined $\overline{W}_1 = \frac{W_{o3}}{W_{o1}}$, the ratio of the pump and signal spot sizes, $\overline{W}_2 = \frac{W_{o3}}{W_{o2}}$ the ratio of the pump and idler spot sizes, and substituting this into equation 3.56(a) we arrive at the expression for $C^{\omega_1}_{000}$

$$3.61 \quad C^{\omega_1}_{000} = \frac{-2i\,\omega_1^{\,2}\, d_{15}(2\pi)^{\frac{1}{4}}}{k_1 c^2 \; W_{03}^{\frac{1}{2}}} \; \bar{W}_1^{\frac{1}{2}} \; \bar{W}_2^{\frac{1}{2}} \; \exp\,(-i\Delta kx)\sqrt{\frac{\pi}{2}}$$

$$\times \; \frac{1}{\left(\bar{W}_1\ (1+i\,\xi_2)\ (1-i\,\xi_3) + \bar{W}_2\ (1+i\,\xi_1)\ (1-i\,\xi_3)+(1+i\,\xi_1)(1+i\,\xi_2)\right)^{\frac{1}{2}}}$$

This is the result for the perfect case of coincident focus positions and axial planes. As before we can evaluate the result for the various combinations of non-coincident focus positions and axial planes of the three resonators. The method and results will in general be the same as for the case of second harmonic generation which has been considered.

(2) <u>Three Dimensions</u>

Under the approximation considered in chapter II the derivation of the coupled mode equations followed the same lines as for the two dimensional case. The relevant equations derived from Maxwell's equations are (c.f. equation 2.84)

$$3.62 \quad \frac{\partial^2 E_x^{\omega}}{\partial y\,\partial x} + \frac{\partial^2 E_y^{\omega}}{\partial z\,\partial y} - \frac{\partial^2 E_y^{\omega}}{\partial x^2} - \frac{\partial^2 E_y^{\omega}}{\partial z^2} - \frac{\omega^2}{c^2}\,\epsilon_z\, E_y^{\omega} = \frac{4\pi\omega^2}{c^2}\, P_y^{(2)\,\omega}$$

$$\frac{\partial^2 E_x^{\omega}}{\partial x\,\partial z} + \frac{\partial^2 E_x^{\omega}}{\partial x\,\partial y} - \frac{\partial^2 E_x^{\omega}}{\partial z^2} - \frac{\partial^2 E_x^{\omega}}{\partial y^2} - \frac{\omega^2}{c^2}\,\epsilon_x\, E_x^{\omega} = \frac{4\pi\,\omega^2}{c^2}\, P_x^{(2)\,\omega}$$

and the divergence equation

$$\epsilon_x \frac{\partial E_x}{\partial x} + \epsilon_z \frac{\partial E_y}{\partial y} + \epsilon_z \frac{\partial E_z}{\partial z} = -4\pi\, \nabla\cdot P^{(2)\,\omega}$$

Using the approximations which were formulated in chapter II for dealing with the linearized versions of these equations they can be reduced to the equations

$$3.63 \quad \frac{\partial^2 E_y^\omega}{\partial x^2} + \frac{\partial^2 E_y^\omega}{\partial y^2} + \frac{\partial^2 E_y^\omega}{\partial z^2} + \frac{\omega^2}{c^2} \epsilon_z E_y^\omega = \frac{-4\pi\omega^2}{c^2} P_y^{\textcircled{2}\omega}$$

$$\frac{\epsilon_x}{\epsilon_z} \frac{\partial^2 E_x^\omega}{\partial x^2} + \frac{\partial^2 E_x^\omega}{\partial y^2} + \frac{\partial^2 E_x^\omega}{\partial z^2} + \frac{\omega^2}{c^2} \epsilon_x E_x^\omega = \frac{-4\pi\omega^2}{c^2} P_x^{\textcircled{2}\omega}$$

for the electric fields of the ordinary and extraordinary waves respectively. Making the substitution $E = \mathcal{E} \exp(-ikz)$ as before reduces these equations to the equations

$$3.64 \quad \frac{\partial^2 \mathcal{E}_o^\omega}{\partial x^2} + \frac{\partial^2 \mathcal{E}_o^\omega}{\partial y^2} - 2ik_o \frac{\partial \mathcal{E}_o}{\partial z} = \frac{-4\pi\omega^2}{c^2} P_y^{\textcircled{2}\omega} \exp(ik_o z)$$

$$\frac{\epsilon_x}{\epsilon_z} \frac{\partial^2 \mathcal{E}_e^\omega}{\partial x^2} + \frac{\partial^2 \mathcal{E}_e^\omega}{\partial y^2} - 2ik_e \frac{\partial \mathcal{E}_e^\omega}{\partial z} = \frac{-4\pi\omega^2}{c^2} P_x^{\textcircled{2}\omega} \exp(ik_e z)$$

for the quantities $\mathcal{E}_{o,e}$. If we now express $\mathcal{E}_{o,e}$ as a sum over the relevant cavity modes allowing the coefficients to vary with z as before and substitute the resulting expressions into equations 3.6 we have

$$3.65 \quad (a) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} -2ik_o \frac{dA_{nm}^\omega}{\partial z} \mathcal{E}_{onm}^\omega = \frac{-4\pi\omega^2}{c^2} P_y^{\textcircled{2}\omega} \exp(ik_o z)$$

$$\text{(b)} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} -2ik_e \frac{dB_{nm}^{\omega}}{dz} \mathcal{E}_{enm}^{\omega} = \frac{-4\pi\omega^2}{c^2} P_x^{②\omega} \exp(ik_e z)$$

If we now use the orthogonality of the mode functions we can arrive at the expressions for the rate of change of the mode amptitudes with z as

$$3.66 \quad \frac{dA_{nm}^{\omega}}{dz} = \frac{-2\pi i\omega^2}{k_o c^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{E}_{onm}^{*\omega} P_y^{②\omega} \, dx\, dy \;\; \exp(ik_o z)$$

$$\frac{dB_{nm}}{dz} = \frac{-2\pi i\omega^2}{k_e c^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{E}_{emn}^{*\omega} P_x^{②\omega} \, dx\, dy \;\; \exp(ik_e z)$$

If we now apply these results to the theory of second harmonic generation, the expansion of the polarization in terms of the resonator modes can be written

$$3.67 \quad P_y^{②\omega} = d_{15} \; A_{jk}^{*\omega} \; B_{rs}^{2\omega} \; \mathcal{E}_{jk}^{*\omega} \; \mathcal{E}_{rs}^{2\omega} \exp(-i(k_2-k_1)\, z)$$

$$P_x^{2\,2} = \tfrac{1}{2} d_{31} \; A_{jk}^{\omega} \; A_{rs}^{\omega} \; \mathcal{E}_{jk}^{\omega} \; \mathcal{E}_{rs}^{2\omega} \exp(i(k_2-k_1)\, z)$$

Substituting these expansions into equations 3.66 produces the simultaneous coupled mode equations

$$3.68 \quad \frac{dA_{nm}^{\omega}}{dz} = C_{nmjkrs}^{\omega} \; A_{jk}^{*\omega} \; B_{rs}^{2\omega}$$

$$\frac{dB^{\omega}_{nm}}{dz} = C^{2\omega}_{jkrsnm} \; A^{\omega}_{jk} \; A^{\omega}_{rs}$$

where for the lossless case we have again the connecting relation

$$3.69 \qquad C^{\omega}_{nmjkrs} = - C^{*2\omega}_{nmjkrs}$$

between the two coefficients. In three dimension coefficients $C^{2\omega}$ depend on the double integral

$$3.70 \qquad I^{1}_{nmjkrs} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{E}^{\omega}_{jk} \; \mathcal{E}^{\omega}_{rs} \; \mathcal{E}^{*2\omega}_{nm} \; dx \, dy$$

which on substituting for the resonator modes from equation 2.91 and 2.28 splits into an integral over x, I_x and one over y, I_y. I_y has exactly the form of equation 3.18

$$3.71 \qquad I_y = \int_{-\infty}^{\infty} H_m\left(\frac{\sqrt{2}y}{W_2}\right) H_k\left(\frac{\sqrt{2}y}{W_1}\right) H_s\left(\frac{\sqrt{2}y}{W_1}\right) \exp\left\{\frac{-y^2}{W_{o2}^2(1+i\xi_2)} \frac{-2y^2}{W_{o1}^2(1-i\xi_1)}\right\} dy$$

whereas I_x is modified by the ϵ_z/ϵ_x factor (equation 2.91)

$$3.72 \qquad I_x = \int_{-\infty}^{\infty} H_n\left(\frac{\sqrt{2}x}{W}\frac{\epsilon_z}{\epsilon_x}\right) H_r\left(\frac{\sqrt{2}x}{W_1}\right) H_j\left(\frac{\sqrt{2}x}{W_1}\right) \exp\left\{\frac{-x^2}{W_{o2}^2(1+i\xi_2)}\frac{\epsilon_z}{\epsilon_x} \frac{-2x^2}{W_{o1}^2(1-i\xi_1)}\right\}$$

The coupling coefficient is given by the equation

$$3.73 \quad C^{2\omega}_{jkrsnm} = \frac{-4i\,\omega^2 d_{31}\, \exp\,(i\,\Delta kx)\,(2\pi)^{\frac{1}{4}}}{k_2 c^2\,(2^{m+j+k+n+r+s}\, m!\, j!\, k!\, n!\, r!\, s!)^{\frac{1}{2}}}$$

$$\times \frac{(1-i\,\xi_2)^{\frac{m+n}{2}}\,(1+i\,\xi_1)^{\frac{j+k+r+s}{2}}}{(1+i\,\xi_2)^{\frac{m+n+2}{2}}\,(1-i\,\xi_1)^{\frac{j+k+r+s+2}{2}}} \quad \left(\frac{\epsilon_z}{\epsilon_x}\right)^{\frac{1}{4}} \quad \frac{I_x\, I_y}{W_{o1}^2\, W_{o2}}$$

The integrals can be evaluated exactly as before and all the results of the previous theory apply to each integral. It is thus a simple matter to generalize the two dimensional results to three dimensions. The only difficulty being that the expressions are very long and tedious to write out. In the same way as has been done for second harmonic generation the expressions for three dimensional parametric amplification can be evaluated.

(3) Effect of absorption

It has been shown (section 2.5) that the presence of absorption in the medium only effects the term $\exp\,(-ikz)$ of the resonator mode functions. Hence the results of this chapter will be unaffected by the presence of absorption except that k will now be complex. This introduces x loss term $\exp\,(k_{2(im)} - 2k_{1(im)})z)$ into equations 3.35,

Chapter 4

Second Harmonic Generation in the small conversion approximation

4.1 The simplest problem to which the formalism developed in the last chapter can be applied is the problem of resonant second harmonic generation from a travelling wave gaussian beam, 32. In the small conversion approximation there is no reaction back on the fundamental and therefore only the original lowest order (gaussian) mode need be considered of the fundamental beam. We first study the second Harmonic generated in the limiting cases which can be solved analytically and then go on to study the general case numerically. We show that the theory that has been set up gives results consistent with those of Kleinman and Boyd, 15, for the non resonant case.

Consider a suitable resonator containing a uniaxial non linear dielectric orientated with its optic axis perpendicular to the axis of the resonator (fig. 4.1). It is assumed that the mirrors of this cavity are transparent to the fundamental laser frequency and that the alignment of the fundamental beam and the cavity is perfect. The theory can be extended immediately to cover the cases of imperfect alignment as indicated in the last chapter since the fundamental only travels in one direction through the non linear dielectric, second harmonic will only be produced in that direction. In the small conversion approximation the second harmonic produced in a single pass of the non linear dielectric is not dependent on the value of the second harmonic at the starting face of the dielectric and the problem splits into two parts. Firstly the calculation of the various second harmonic mode amplitudes produced by a single pass of the dielectric, assuming an initial value zero, and secondly using these results to calculate the actual mode amplitudes produced in the resonator and hence the output of the resonator.

Generalizing equation 3.35 to three dimensions the rate of change of the 2n 2m 'th second harmonic mode amplitude is given by the equation

4.1 $$\frac{dB^{2\omega}_{2n\;2m}}{dz} = C^{2\omega}_{2n\;2m} A^{\omega}_{o} A^{\omega}_{o} \quad \text{where}$$

4.2 $$C^{2\omega}_{2n\;2m} = \frac{4i\omega^2 d_{31} \;\; \pi^{5/4} \; (2n!\,2m!)^{\frac{1}{2}}}{k_2 c^2 \; e^{\frac{1}{2}} \; W_{o2} \;\; n!\,m!\,2^{n+m+\frac{1}{2}}} \; \frac{\exp\;(i\Delta k\, z\;)}{(1 - i\,\xi_1)}$$

$$\times \frac{\left(e^2W^2 - 1 - i\,(e^2W^2\xi_1 - \xi_2)\right)^n \; \left(W^2 - 1 - i(W^2\xi_1 - \xi_2)\right)^m}{\left(e^2W^2 + 1 - i\,(e^2W^2\xi_1 - \xi_2)\right)^{n+\frac{1}{2}} \left(W^2 + 1 - i\,(W^2\xi_1 - \xi_2)\right)^{m+\frac{1}{2}}}$$

where $\epsilon_x / \epsilon_z = e^2$.

Since $\xi_1 = \frac{2z}{k_1 W_{o1}^{\;2}}$, $\xi_2 = \frac{2z}{k_2 W_{o2}^{\;2}}$, $\Delta k = k_2 - 2k_1$

we have $W^2 \xi_1 - \xi_2 = W^2 \xi_1 \left(\frac{\Delta k}{k_2}\right)$

for $\Delta k < 10$ the term dependent on $\Delta k / k_2$ can be neglected

hence $W^2\xi_1 - \xi_2 = 0$

4.4 $$e^2 W^2 \xi_1 - \xi_2 = e^2 W^2 \xi_1 \left(1 - \frac{1}{e^2}\right)$$

defining $\epsilon = \frac{1}{e^2} - 1$ we can write

$$4.5 \qquad C^{2\omega}_{2n\ 2m} = \frac{\alpha_{n\ m}}{W_{o2}} \frac{\exp(i\Delta kz)}{(1 - i\xi_1)} \frac{(e^2W^2 - 1 + i\ e^2W^2\xi_1\epsilon)}{(e^2W^2 + 1 + i\ e^2W\xi_1\epsilon)^{n+\frac{1}{2}}}$$

$$\times \frac{(W^2 - 1)^m}{(W^2 + 1)^{m+\frac{1}{2}}}$$

for $\Delta k < 10$ where α_{nm} is a constant dependent on the mode number.

Since we have assumed A_o to be constant the amplitude of the nmth second harmonic mode at the exit face of the dielectric is given by

$$4.6 \qquad B^{2\omega}_{nm} = (A^{\omega}_o)^2 \int_{z_1}^{z_2} C^{2\omega}_{2n\ 2m}\ dz$$

where the entry face is given by $z = z_1$ the exit face $z = z_2$ (figure 4.1).

4.2 New Field approximation

We consider first the case where the dielectric lies in the near fold of both the second harmonic resonator and the fundamental beam i.e. where both

$$4.7 \qquad \xi_1 \ll 1 \quad \text{and} \quad \xi_2 \ll 1$$

This is the case treated by Ashkin, Boyd and Dziedzic but for the dielectric potassium Dihydrogen Phosphate, KDP, which does not phase match perpendicular to the optic axis and hence is somewhat less efficient than a material in which this can be done, e.g. Lithium Niobate. This corresponds to the weak focussing limit and can be treated analytically. Two important results are brought out by this case, first the variation of second harmonic output with the relative spot sizes of the two beams and second, the result that the normal phase matching condition $\Delta k = 0$ no longer holds.

a. Lowest order mode

To the first order in ξ_1, we can approximate

$$(1 - i\xi_1)^{-1} = \exp(i\xi_1)$$

$$(e^2W^2 + 1 + ie^2W^2\epsilon\,\xi_1)^{-\frac{1}{2}} = (e^2W^2 + 1)^{-\frac{1}{2}} \exp\left(\frac{-ie^2W^2\epsilon\,\xi_1}{2(e^2W^2+1)}\right)$$

Using these approximations equation 4.5 for the lowest order coupling reduces to

$$4.8 \quad C_{00}^{2\omega} = \alpha_{00} \exp\left\{\frac{i\left(\Delta kz + \xi_1\left(1 - \frac{e^2W\epsilon}{2(e^2W^2+1)}\right)\right)}{W_{02}\,(e^2W^2+1)^{\frac{1}{2}}\,(W^2+1)^{\frac{1}{2}}}\right\}$$

and the second harmonic output in the lowest order mode is given by

$$4.9 \quad B_{00}^{2\omega} = \frac{\alpha_{00}\, {A_0^{\omega}}^2}{W_{02}\,(e^2W^2+1)^{\frac{1}{2}}\,(W^2+1)^{\frac{1}{2}}} \int_{z_1}^{z_2} \exp\left\{ i\Delta k + \frac{1}{z_{01}}\left(\frac{1 - e^2W^2\epsilon}{2(e^2W^2+1)}\right) z \right\} dz$$

The variation of this output with the spot sizes of the laser beam and the second harmonic resonator is given to a good approximation by the function

$$4.10 \quad F(W) = \frac{1}{W_{02}\,(e^2W^2 + 1)^{\frac{1}{2}}\,(W^2 + 1)^{\frac{1}{2}}}$$

For constant W_{02} this increases with decreasing W (i.e. decreasing W_{01}). As the resonator mode functions are only valid as long as $W_{01} >> \lambda$ this just implies that for maximum output W_{01}, the laser spot size must be as small as possible as would be expected from physical considerations. For constant W_{01}, $F(W)$ has a maximum at

$$4.11 \quad W_{max} = \frac{1}{\sqrt{e'}} = \frac{\sqrt{\eta_o^{2\omega}}}{\sqrt{\eta_e^{2\omega}}}$$

(η refractive index)

The behaviour of $F(W)$ for fixed W_{01} is shown in graph 1 for $e^2 = 0.96$, (an approximate values for Lithuim Niobate) when it can be seen that $W_{max} = 1.02$.

The integral in equation 4.7 is of the same form as the integral which arises when considering Second Harmonic Generation from plane waves except that the phase matching condition is now given by

$$4.12 \quad \Delta k = \frac{-1}{Zo_1}\left(1 - \frac{e^2 W^2 \epsilon}{2(e^2W^2 + 1)}\right)$$

At the maximum value of $W = 1.02$

$$\Delta k = \frac{-0.99}{Z_{o1}} = \frac{-1.98}{k_1 W_{o1}^{\ 2}}$$

which is the limit of an infinite laser beam and Second Harmonic resonator tends to the usual $\Delta k = 0$ as expected but can be substantially different from this for the usual spot sizes $\sim 10^{-2}$ cms.

b. <u>Higher Order Modes</u>

Here we consider first the case when $eW - 1$ is not near zero then we can approximate

$$(e^2W^2 - 1 + i\, e^2\, W^2 \epsilon\, \xi_1\,)^n = (e^2W^2 - 1)^n \exp\left(\frac{i\, n\, e^2W^2\epsilon\, \xi_1}{(e^2W^2 - 1)}\right)$$

Using this and the previous approximations equation 4.5 becomes

$$4.13 \quad C^{2\omega}_{2n\ 2m} = \frac{\alpha_{nm}\, (W^2 - 1)^m\, (e^2W^2 - 1)^n}{W_{02}\, (W^2 + 1)^{m + \frac{1}{2}}\, (e^2W^2 + 1)^{n + \frac{1}{2}}}$$

$$x \exp\left\{ iz\, (\Delta k + \frac{1}{z_{01}}\, (1 + e^2W^2\epsilon \left(\frac{n}{(e^2W^2 - 1)} - \frac{(2n + 1)}{2(e^2W^2 + 1)}\right)\right\}$$

The variation with spot size is given again to a good approximation by the function

$$4.14 \quad F_{nm}\, (W) = \frac{(W^2 + 1)^m\, (e^2W^2 - 1)^n}{W_{02}\, (W^2 + 1)^{m + \frac{1}{2}}\, (e^2W^2 + 1)^{n + \frac{1}{2}}}$$

again if W_{02} is held constant this function increases with decreasing W_{01}, the laser spot size, as expected. For W_{01} held constant the function has two distinct maximum which for $e = 1$ occur at reciprocal points. The variation of a few of the lower order functions are plotted in graph I for $e = 0.96$. It can be seen that the value of w at which the maximum occurs increases with increasing mode numbers. This is brought out by the similar function which occurs in the two dimensional case

$$4.15 \quad F_n\, (W) = \frac{(W^2 - 1)^m}{W_{02}^{\frac{1}{2}}\, (W^2 + 1)^{n + \frac{1}{2}}}$$

which is simpler to handle and has its maxima at

$$4.16 \quad W_{max} = 4_n + 1 \pm \sqrt{(4n + 1)^2 - 1}$$

phase matching for the nth mode occurs when

$$4.17 \quad \Delta k = -\frac{1}{z_{01}} \left(1 + e^2W^2 \left(\frac{n}{(e^2W^2 - 1)} - \frac{(2n + 1)}{2(e^2W^2 + 1)} \right) \right)$$

Which is dependant on n the x - z plane mode number. Hence to some extent modes which are undesired in the output can be discriminated against by varying the relative spot size and the phase matching. For example putting $W = 1$ eliminates output in all the higher y, z plane modes without reducing noticeably the 0 - 0 output.

If $|eW - 1| << 1$ then all the coefficients except $C^{2\omega}_{2,\,2m}$ are of higher order in ζ_1, than the first and hence are negligible in the near field approximation. The $C^{2\omega}_{2,\,2m}$ are given by

$$4.18 \quad C^{2\omega}_{2,\,2m} = \alpha_{1m} \frac{(W^2 - 1)^m \; i\, e^2W^2 \epsilon\, \zeta_1}{W_{02}\,(W^2 + 1)^{m + \frac{1}{2}}\,(e^2W^2 + 1)^{n + \frac{1}{2}}}$$

$$\exp \left\{ iz \left(\Delta k + \frac{1}{z_{01}} \left(1 \frac{-(2n + 1)\, e^2W^2 \zeta_1}{2\,(e^2W^2 + 1)} \right) \right. \right.$$

Considering the results so far it can be seen that the region $|eW - 1| << 1$ will only be entered when it is the aim to maximise the output of the lowest order modes. From equations 4.16 and 4.6 we have

4.19 $$\left|\frac{C^{2\omega}_{20}}{C^{2\omega}_{00}}\right| = \left|\frac{e^2W^2\,\zeta_1\,\epsilon\,\alpha_{10}}{(e^2W^2+1)\,\alpha_{00}}\right|$$

which is approximately equal to $3 \times 10^{-2}\,|\zeta_1|$

For $|\zeta_1| << 1$ this will be negligible especially since if maximum coupling is required into the lowest order 00 mode than the 0 - 2 mode will neither be perfectly phase matched nor resonate perfectly in the cavity set for the 0 - 0 mode.

4.3 Far Field

The second limiting case which can be evaluated analytically when the non linear dielectric has in the far field of both the laser and the S.H. resonator, i.e. both.

$$\epsilon\,\zeta_1 >> 1 \quad \text{and} \quad \epsilon\,\zeta_2 >> 1 \quad \text{and} \quad l/z_1 << 1$$

Under these approximations the lowest order coefficient $C^{2\omega}_{00}$ can be written

4.20 $$C^{2\omega}_{00} = \frac{-\alpha_{00}}{W_{02}\,(W^2+1)^{\frac{1}{2}}W}\left(\frac{z^3{}_0}{-i\,e^2\,\epsilon\,z^3}\right)^{\frac{1}{2}} \exp\left(i\,\Delta\,k\,z\right)$$

and hence the S.H. output from a slab of dielectric of length $l = z_2 - z_1$ is proportional to

4.21 $$I = \int_{z_1}^{z_2} \frac{\exp\,(i\,\Delta\,k\,z)\,dz}{z^{3/2}}$$

$$= \frac{e^{i\,\Delta kz_1}}{z_1}\int_0^{l} \exp\,(i\,\Delta k\,x)\,dx \quad \text{to the first order}$$

in z/l. We can see the phase matching condition and the general form of the integral in this limiting case is just that which arises when considering the interaction of plane waves and hence the output will vary with the usual $\frac{\sin(\Delta k \ell / 2)}{\Delta k}$. This is as expected since far from the focus the beam tends to a plane wave.

4.4 Infinite Crystal

In the limit of an infinite crystal or equivalently the steep focussing limit $z_{01} \rightarrow 0$ the expression for the amplitude of the second harmonic output is from equation 4.5.

4.22 $$C^{2\omega}_{00} = \frac{\alpha_{00}}{W_{02}\,(W^2+1)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\exp(i\Delta k\, z)\, dz}{(1 - i\zeta_1)(e^2W^2 + 1 + ie^2W^2\epsilon\zeta_1)^{\frac{1}{2}}}$$

Changing the variable of integration and writing $\Delta k z_0 = y$, $\zeta = v$ the integral on which the output in this case depends is

4.23 $$I = \int_{-\infty}^{\infty} \frac{\exp(i\, y\, v)\, dv}{(1 - i\, v)(a + i\, b\, v)^{\frac{1}{2}}}$$

where $a = e^2W^2 + 1$ $b = e^2W^2\epsilon$

This integral can be evaluated under certain conditions by contour integration in the complex v plane. For $y = 0$ we consider the integral around the contour made up of a semicircle radius R in the lower half plane and the completing section of the real axis. In the limit $R \rightarrow \infty$ the integral around the semicircle tends to zero and the contour contains the simple pole of the integral at $v = -i$. Hence from Cauchy's theoram

$$4.24 \quad I = \frac{-2\pi \exp(y)}{(a+b)^{\frac{1}{2}}} \quad \text{for } y < 0$$

The integral has a branch point on the upper half plane at $V = ia$ and so cannot be integrated immediately when $\Delta k > 0$. However since b is small the limit $b \to 0$ will give an indication of the behaviour of the integral when $y > 0$. In this limit the integral is analytic in the upper half plane and hence integrating around a contour completed in the upper half plane we have by Cauchy's theoram.

$$4.25 \quad I = 0 \quad \text{for } y > 0 \quad b = 0$$

Thus we expect the integral to exhibit something of a discontinuity at $\Delta k = 0$. This general behaviour can be seen in graph 7 which records the variation of the integral I with Δk for $\ell/_{zo} = 100$. This shows clearly the discontinuity although it is modified by a periodic fine structure. The integral the limit $(b = 0)$ has been considered by Kleinman Ashkin and Boyd, 27, in a different context. They obtain a continuous analytic approximation for this fine structure.

4.5 General Formula (a) Lowest Order Mode

The S.H. output in the lowest order mode is given by the equation (from eq. 4.5)

$$4.26 \quad c_{00} = \frac{\alpha_{00} \; W \; z_{01}}{W_{01} \; (W^2+1)^{\frac{1}{2}}} \int_{\xi_1}^{\xi_2} \frac{\exp(i\,y\,\xi)\; d\xi}{(1 - i\xi)(a + i\,b\,\xi)^{\frac{1}{2}}}$$

where $a = e^2W^2 + 1$, $b = e^2W^2\epsilon$, $y = \Delta k\, z_0$ and so contains the mismatch parameter; $\xi_1 = z_1/z_{01}$, $\xi_2 = z_2/z_{01}$

and contains the length l of the non linear medium

$l = (\xi_2 - \xi_1) z_{01}$ and the position of the foci of the two beams (assumed coincident) $f = (\xi_2 + \xi_1) z_{01}/2$. The variation with the focussing parameter z_{01} of the fundamental beam for constant crystal length l can be obtained by rearranging 4.26 to the form

$$4.17 \quad c_{00} = \frac{\alpha_{00} W}{(W^2 + 1)^{\frac{1}{2}}} \sqrt{\frac{k_1 l}{2}} \frac{1}{\sqrt{(\xi_2 - \xi_1)}} \times \int_{\xi_1}^{\xi_2} \frac{\exp(i y \xi) \, d\xi}{(1 - i\xi)(a + i b \xi)^{\frac{1}{2}}}$$

and studying the variation of this form with ξ_2, ξ_1 for a given focus position.

Graphs 2, 3, 4, 6 and 7 show the second harmonic output variation with the phase matching parameter $\Delta k l/2$ for various relative crystal lengths $\xi_l = \frac{l}{z_{01}}$ with the focus at the centre of the crystal. Graph 2 shows the shortest length has the appearance expected from the new field limit consideration. The curve has the form $\sin(x)/x$ with the absolute maximum slightly off $\Delta k = 0$. Away from the absolute maximum all the curves exhibit a periodicity of π . (Cf. the plane wave interaction which is proportional to $\sin(\Delta k \, l/2)$ and thus has a period of π on the graph). As the length increases or equivalently the focussing increases (z_{01} decreases). The curves become assymetric. The optimum phase matching position moves away from $\Delta k = 0$ and the central peak moves out on the side $\Delta k < 0$ engulfing the smaller peaks (graph 4). This marks the increasing effect of the focussing terms in the denominator of the integral $(1 - i\xi)$, $(e^2W^2 + 1 - i\, e^2 \epsilon \xi W^2)^{\frac{1}{2}}$.

If the gaussian beam is considered as a sum of plane waves, propagating over a range of directions, the reason for the assymetry becomes apparent. When $\Delta k > 0$ the fundamental and S.H. refraction index surfaces (dgms. 1 - 3) no longer intersect for any direction of propagation hence none of the constituent plane waves is phase matched, but when $\Delta k < 0$ the index surfaces intersect and so there exist plane waves in the sum which are phase matched. The steep focussing or infinite crystal effect can be seen fully developed in graph 7 as has been previously pointed out. Note that the periodicity of the fine structure for $\Delta k < 0$ is twice that of that for $\Delta k > 0$. The width at half length of all the curves in the units of $\Delta k\, \ell / 2$ is approximately constant. All these curves are consistent with those of Kleinman and Boyd, 15. Graph 10 shows the variation with focussing at the optimum phase match angle and optimum relative spot size for three positions of the focus. With the focus at the centre the maximum occurs at $\ell/_{zo} = 5.65$ which is consistent with the result given by Kleinman and Boyd, 15, for free S.H.G. $\ell/_{zo} = 5.68$.

If we think of the free second harmonic generated as being made up of a number of confocal resonator modes, the higher order ones will contribute a small amount to the total and this would be sufficient to explain the difference between the two results. Since the higher modes maximise at a higher degree of focussing the direction of the difference is as expected.

With the forms of the fundamental beam at the entry face, or exit face, of the crystal the maximum occurs at $^{1}/_{zo} = 3.09$ as shown in the second curve. This figure would of course be the one required in a plano - concave resonator. The third curve represents the case when the focus is one half the crystal length outside the crystal. Here the curve has an early maximum and falls off sharply as the focal region withdraws from the crystal. Also shown on the same graph is the output in the 0.2 mode under the same conditions of phase matching and spot size as the corresponding curve for the 0 - 0 mode.

Graph 8 shown the variation of output with focus position for several lengths at optimum phase match and spot size. Graph 9 shows the variation in shape of the output curve with phase match angle as the focus moves away from the centre of the crystal. As expected from the consideration of the far field limit the optimum phase match position moves towards $\Delta k = 0$ and the shape of the curve approaches $\{\sin(\Delta k\, l/2) / (\Delta k\, l/z)\}$ as the focus moves away from the centre. A greater range of Δk is shown in graph 5 for the case when the focus is at the face of the crystal. It can be seen that the minima have moved up from zero and the curve has taken on much of the form of the steep focussing limit. These effects can be shown by expanding the integrand in powers of ϵ to be due to the increasing effect of the anisotropy as the focus moves from the centre.

0 - 2 Mode

The output in the 0 - 2 mode is given from equation 4.5 by

$$4.28 \quad C_{02} = \frac{\alpha_{02}}{W_{02}} \int_{z_1}^{z_2} \frac{\exp(i\,\Delta k\, z)\,(e^2 W^2 - 1 + 1\, e^2 W^2 \epsilon \varsigma_1)\, dz}{(1 - i\,\varsigma_1)(e^2 W^2 + 1 + i\, e^2 W^2 \epsilon \varsigma_1)^{3/2}}$$

This mode of the higher modes will have the greatest output when the various parameters are set for optimisation of the output in the 0 - 0 mode.

Graphs 11 and 12 show the output variation of this mode with phase matching for $l/_{zo} = 5.65$ the optimum value for the 0 - 0 mode. With the focus at the centre the output exhibits a double peak. This is present at all crystal lengths, the relative heights of the two peaks changing with length. As the focus moves away from the centre (graph 12) the second peak decreases in height and the curve takes on the form of those for the 0 - 0 mode. The positions of the twin peaks change little with increasing length (focussing) giving rise to the dip in the curve for the 0 - 2 mode, focus at the centre, on graph 10. This occurs as the position of optimum phase matching for the 0 - 0 mode moves through the region between the twin peaks of the 0 - z mode output.

Although the output variation with relative spot size W for the 0 - 0 mode has the form of graph 1 for the optimum length the curve for the output in the 0 - 2 mode has changed and become very assymetric about the central position (dotted curve as graph 1).

The relative outputs for the two modes are when the 0 - 0 mode is optimised.

$$\left|\frac{C_{00}}{C_{02}}\right| \approx 100 \text{ for the focus at the centre, and}$$

$$\left|\frac{C_{00}}{C_{02}}\right| \approx 50 \text{ for the focus at the entry/exit face.}$$

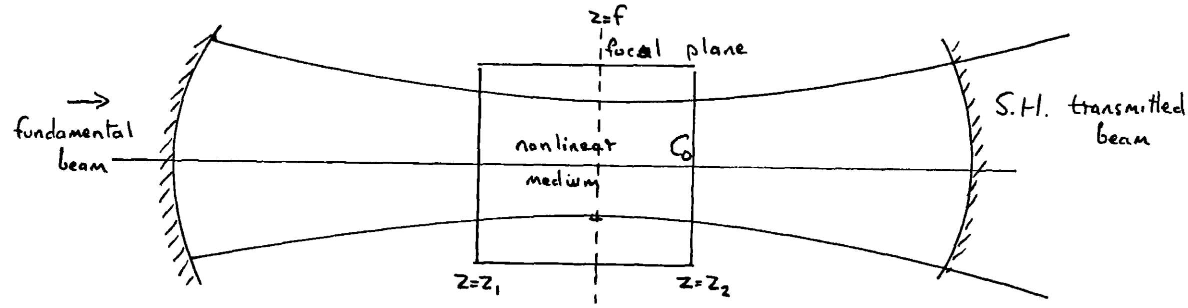
Absorption

The presence of absorption can be taken into the foregoing theory without difficulty. It of course modifies the theory reducing the output amplitudes, but gives no new effects except that an optimum crystal length can now be defined which will depend on the amount of absorption present at each frequency.

4.6 Calculation of the Output from the Resonant Cavity

To calculate the output of the resonant cavity on a given mode we refer to diagram 4.1 where C_0 is defined as the amplitude of the relevant mode just inside the crystal exit face and $r \exp(i\phi)$ is the power loss parameter of the cavity including diffraction losses, reflection losses and the loss due to the output. In the absence of any second harmonic generation after a single round trip the amplitude of the mode would be $C = C_0 r \exp(i\phi)$ since the second harmonic power is only generated as the wave travels from left to right the amplitude in the cavity is given by the self consistency equation.

fig 4.1



4.29 $$C = C_0 \, r \exp(i\phi) + C_1$$

Where C_1 is the S.H. generated in a single pass, the quantity we have been considering. Therefore from 4.29,

4.30 $$C_0 = \frac{C_1}{(1 - r\exp(i\phi))}$$

and in terms of the power present

4.31 $$P = \frac{P_1}{(1 - 2r\cos\phi + r^2)}$$

Therefore the power transmitted from the cavity is given by

4.32 $$P = \frac{t\,P_1}{(1 - 2r\cos\phi + r^2)}$$

where t is the transmission coefficient of the exit mirror.

at resonance

4.33 $$P = \frac{tP_1}{(1 - r)^2}$$

As pointed out by Ashkin, Boyd and Dziedzic, 23, this can be much greater than P_1 the power obtained in a single pass e.g. the case $r = 0.99$, $t = 0.01$, $P = 10^2 P$

This optimisation process will also discriminate against the higher modes since from equation 2.33 when the 0 – 0 mode is resonating $\phi = 0$ the higher modes will not be resonant. For example for a confocal cavity using the data given previously the ratio of the power output from the cavity in the 0 – 2 mode to that in the 0 – 0 mode is approximately

$$0.25 \times 10^{-4} \quad \frac{P_{02}}{P_{00}}$$

where P_{02} and P_{00} are the power generated in a single pass of the non linear in the 0 – 2 and 0 – 0 mode respectively. Hence the power output in the 0 – 2 mode is negligible.

In general the fundamental laser beam will consist of a number of longitudinal modes of the laser resonator and hence the fundamental electric field will be of the form

$$4.34 \quad E(\omega) = \frac{1}{2} \sum_n{}' \left(E^{\omega_n} \delta(\omega - \omega_n) + E^{*\,\omega_n} \delta(\omega + \omega_n) \right)$$

where from equation 2.33 ω_n is given by

$$\omega_n = \frac{c}{d} \left(n\pi \mp \tan^{-1}(\psi) \right)$$

To a good approximation the spot sizes of each of these modes will be equal and so the general form of each mode will be the same and so in considering the frequency term of the modes we need not consider the mode shape.

Using equation 1.18 the second order polarization will be set up at frequencies.

4.35 $\omega_q = \omega_m + \omega_n$, where ω_m, ω_n are given by equation 4.34. The resonances of the second harmonic resonator occur at frequencies.

4.36 $\omega_q = \frac{c}{d_1} \left(q\pi \pm \tan^{-1} (\psi_1) \right)$ where d_1 is the optical length of the resonator cavity and ψ_1 is calculated from equation 2.32 using the optical length d_1. Putting together 4.34, 35, 36 resonance occurs when

4.37 $d_1\left((m + n)\pi + 2 \tan^{-1} (\psi) \right) = d \left(q\pi \pm \tan^{-1} (\psi_1) \right)$

From this equation knowing the laser cavity parameters suitable parameters for the second harmonic cavity can be calculated.

Chapter 5

Parametric Amplification

5.1 The second problem to which the formalism developed in the previous chapters has been applied is that of parametric amplification and oscillation from a travelling wave pump optical field. Under the approximation that the pump field is much larger than either the signal or idler fields and therefore may be regarded as constant the problem splits into two parts as with second harmonic generation. First the calculation of the single pass amplification and secondly the resonator calculation which gives the threshold of a given resonator and its output for a given pump field above threshold. Guided by the experience of the last chapter only the interaction between the lowest order, 0-0 modes of each field has been examined, contributions from higher modes have been neglected. Under these approximations an analytical solution to the coupled mode equations is given for the weak focussing - near field limit and preliminary results of numerical computations are given for the general solution.

We consider a non linear dielectric enclosed in a resonator, orientated as before. Impinging on the dielectric will be optical fields at frequencies ω_3, ω_2, ω_1 which will be referred to as the pump, idler, and signal fields respectively. To achieve phase matching in Lithium Niobate we consider the pump field to be propagating as an extraordinary wave and the signal and idler fields as ordinary waves. The coupled mode equations governing the amplitudes of the signal and idler beams are from equations 3.51

$$5.1 \qquad \frac{dA_1}{dz} = C^{\omega_1} A_2^{*} B$$

$$\frac{dA_2}{dz} = C^{\omega_2} A_1^{*} B$$

where C^{ω_1}, C^{ω_2} are the coupling coefficients and B is the pump mode amplitude. The C^{ω} are given by equation 3.61 or its three dimensional generalization

5.2 <u>Weak Focussing - Near Field Limit</u>

Under the same approximations as have been used in section 4.2(a) equations 5.1 can be written

$$5.2 \quad (a) \qquad \frac{dA_1}{dz} = -i\alpha_1 A_2^{*} \exp(iyz)$$

$$(b) \qquad \frac{dA_2}{dz} = -i\alpha_2 A_1^{*} \exp(iyz)$$

$$\text{where } (c)\ \alpha_{1,2} = \frac{\omega^2_{1,2}\, d_{15}\, \pi^{\frac{1}{2}}\, 2^{\frac{7}{2}}\, W_1\, W_2\, B}{k_{1,2}\, c^2\, e^{\frac{1}{4}}\, W_{o3}\, (W_1^2 + W_2^2 + 1)^{\frac{1}{2}}\, (e^2 W_1^2 + e^2 W_2^2 + 1)^{\frac{1}{2}}}$$

$$\text{and } (d)\ y = \Delta k + (e^2+1)\left\{W_1^2 \left(\frac{1}{z_{o3}} - \frac{1}{z_{o2}}\right) + W_2^2 \left(\frac{1}{z_{o3}} - \frac{1}{z_{o1}}\right) \quad \left(\frac{1}{z_{o1}} + \frac{1}{z_{o2}}\right)\right\}$$

$$\text{and} \qquad W_1 = \frac{W_{o3}}{W_{o1}}, \quad W_2 = \frac{W_{o3}}{W_{o1}}$$

All other variables are as defined previously.

From 5.2(c) the variation of the interaction with the various spot sizes is given by the function

$$5.3 \qquad F(W) = \frac{W_1 W_2}{W_{o3} (W_1^2 + W_2^2 + 1)^{\frac{1}{2}} (e^2 W_1^2 + e^2 W_2^2 + 1)^{\frac{1}{2}}}$$

When $e = 1$ the maximum as a function of W_{o3} occurs when

$$\frac{1}{W_{o3}^2} = \frac{1}{W_{o2}^2} + \frac{1}{W_{o1}^2}$$

and it can be shown by the method of Lagranges undetermined multipliers that the absolute maximum will occur when $W_{o1} = W_{o2}$. Since the $W_1 \sim W_2$ symmetry is retained when $e \neq 1$ the maximum value would still be expected at $W_{o1} = W_{o2}$. Then F (W) reduces to the function studied in section 4.2(a).

One method of solution of equations 5.2 is to separate the real and imaginary parts and solve the resulting equations 58,16,

Thus putting

$$A_1 = \hat{A}_1 e^{i\theta_1} , \quad \alpha_1 = \hat{\alpha}_1 e^{i\psi_1} \text{ etc.}$$

and

$$\Phi = \frac{-\pi}{2} - (\theta_1 + \theta_2) + \psi + yz$$

(Note $\psi = \psi_1 = \psi_2$

from equation 5.2(c))

we have

$$5.5 \quad (a) \quad \frac{d\hat{A}_1}{dz} = \hat{\alpha}_1 \hat{A}_2 \cos \Phi$$

(b) $$\frac{d\hat{A}_2}{dz} = \hat{\alpha}_2 \hat{A}_1 \cos \bar{\Phi}$$

(c) $$\frac{d\bar{\Phi}}{dz} = y - \left(\frac{\hat{\alpha}_1 \hat{A}_2}{\hat{A}_1} + \frac{\hat{\alpha}_2 \hat{A}_1}{\hat{A}_2} \right) \sin \bar{\Phi}$$

These equations give immediately information about the maximum possible amplification. From 5.5(c) we see that as long as $y = 0$ all the derivatives of $\bar{\Phi}$ are identically zero when $\bar{\Phi} = 0$ and hence $\bar{\Phi} = 0$ is a solution of 5.5(c). This reduces the problem to solving the two simple coupled equations.

5.6 $$\frac{d\hat{A}_1}{dz} = \hat{\alpha}_1 \hat{A}_2$$

$$\frac{d\hat{A}_2}{dz} = \hat{\alpha}_2 \hat{A}_1$$

which will give the optimum amplification.

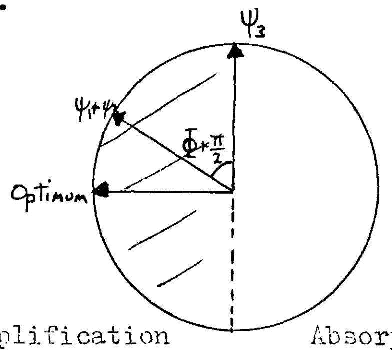
The second point which can be deduced from 5.5(c) is that if at the crystal entry face the idler mode amplitude A_2 is zero (i.e. non-resonant idler) then the initial value of $\bar{\Phi}$ is fixed at the optimum value $\bar{\Phi} = 0$.

Thirdly it can be seen from equation 5.5(a) that the signal is amplified when $-\frac{\pi}{2} < \bar{\Phi} < \frac{\pi}{2}$ and absorbed otherwise. This has a simple interpretation in terms of the phases of the three interacting waves. From equation 5.4, in the plane wave limit $y = \Delta k$, we have

$$\bar{\Phi} = \frac{-\pi}{2} + (\psi + k_3 z) - (\theta_1 + k_1 z) - (\theta_2 + k_2 z)$$

$$= \frac{-\pi}{2} + \bar{\Psi}_3 - (\bar{\Psi}_1 + \bar{\Psi}_2)$$

where ψ_1, ψ_2, ψ_3 are the phases of the three interacting waves at any point in the crystal.



Amplification Absorption

Thus if $\psi_1 + \psi_2$ lags ψ_3 then both signal and idler are amplified if $\psi_1 + \psi_2$ leads ψ_3 then both signal and idler are absorbed. The possibility of signal and idler behaving differently is ruled out since the phase of α_1 is equal to that of α_2. In the non plane wave case y can still be split into three parts, one associated with each frequency. Then ψ_3, ψ_2, ψ_1 may be interpreted as an average value of the phases of the constituent plane waves of the resonator mode at each frequency.

To obtain the general solution it is simpler to proceed with the complex equations. Multiplying equation 5.2(a) by exp (−iyz) and differentiating we can then substitute from equation 5.2(b) and obtain the equation

$$5.7 \qquad \frac{d^2A_1}{dz^2} - iy\frac{dA_1}{dz} - \alpha_1\alpha_2^* A_1 = 0$$

for A_1 and by a similar procedure a similar equation for A_2. The solution to this equation depends on the familiar (16, 58, 20) gain factor

$$5.8 \qquad \beta^2 = \alpha_1\alpha_2^* - y^2/4$$

The solution for A_1 increases exponentially with increasing distance from the entry face when $\beta > 0$ but oscillates sinusoidally when $\beta < 0$. In terms of the power present in each mode which is directly proportional to $|A_1|^2$, $|A_2|^2$, the solution to equation 5.7 can be written

5.9 (a) for $\beta > 0$.

$$P_1(z) = P_1 + (\alpha^2 P_1 + |\alpha_1|^2 P_2 - |\alpha_1| y \sqrt{P_1 P_2} \cos(\Phi_o + \tfrac{\pi}{2})) \frac{\sinh^2(2\beta(z-a))}{\beta^2}$$

$$+ \sin(\Phi_o + \tfrac{\pi}{2}) |\alpha_1| \sqrt{P_1 P_2} \frac{\sinh(2\beta(z-a))}{\beta}$$

(b) for $\beta < 0$.

$$P_1(z) = P_1 + (\alpha^2 P_1 + |\alpha_1|^2 P_2 - |\alpha_1| y \sqrt{P_1 P_2} \cos(\Phi_o + \tfrac{\pi}{2}) \frac{\sin^2(2\beta(z-a))}{\beta^2}$$

$$+ \sin(\Phi_o + \tfrac{\pi}{2}) |\alpha_1| \sqrt{P_1 P_2} \frac{\sin(2\beta(z-a))}{\beta}$$

where $\alpha^2 = \alpha_1 \alpha_2^*$; P_1, P_2 are the values of $P_1(z)$, $P_2(z)$ the power present in the signal and idler modes respectively and Φ_o is the value of Φ at the crystal entry face $z = a$.

It can be verified that the optimum value of this function with respect to y and Φ occurs at the expected values $\Phi_o = 0$ $y = 0$. It can be seen that for a general value of Φ_o the power output function is not symmetric with respect to the phase matching parameter y. This will only be important in the degenerate case when the signal and idler fields are one, $A_1 = A_2$. Since, in the non-degenerate case, the idler is generated as a side effect and if it is non-resonant, or if it is resonant in an optimised cavity then $\tilde{\Phi}_0 = 0$ and the function is symmetric. But in the degenerate case $\tilde{\Phi}_0$ is fixed by external parameters and need not be zero. Hence in degenerate parametric amplification it is likely to be difficult to obtain the optimum output experimentally. For the important case $\tilde{\Phi}_0 = 0$ we have

5.10 (a) $\beta > 0$

$$P_1(z) = P_1 + (\alpha^2 P_1 + |\alpha_1|^2 P_2) \frac{\sinh^2(\beta(z-a))}{\beta^2} + |\alpha_1|\sqrt{P_1 P_2} \frac{\sinh(2\beta(z-a))}{\beta}$$

(b) $\beta < 0$

$$P_1(z) = P_1 + (\alpha^2 P_1 + |\alpha_1|^2 P_2) \frac{\sin^2(\beta(z-a))}{\beta^2} + |\alpha_1|\sqrt{P_1 P_2} \frac{\sin(2\beta(z-a))}{\beta}$$

For the case $P_2 = 0$, non-resonant idler, there is gain for all values of z and y.

In the other extreme $P_2 = P_1$, degenerate, we have from equation 5.10(b)

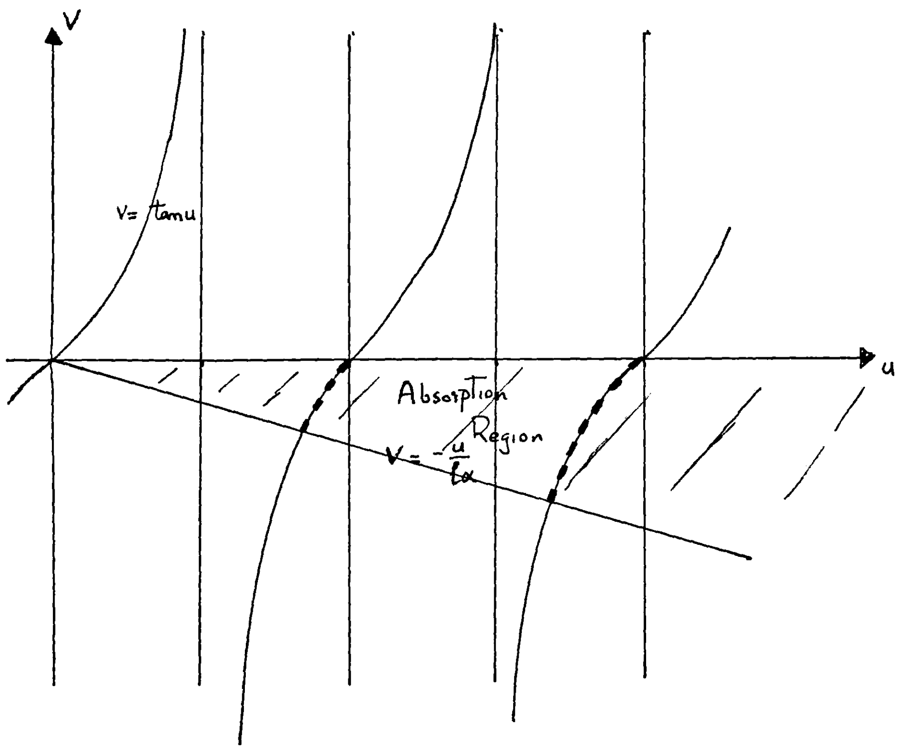
5.11 $$P_1(z) = P_1((1 + \frac{2\alpha^2}{2} \sin^2 (\beta(z-a)) + \frac{\alpha}{\beta} \sin (2\beta(z-a)))$$

and there is gain as long as

5.12 $$\frac{\beta}{\alpha} > - \tan (\beta(z-a))$$

Putting $z-a = l$, $u = \beta l$ this condition becomes

5.13 $$u/\alpha l > - \tan u$$



In the diagram the curves $v = \tan u$ and $v = \frac{-u}{l}$ are sketched and the regions of absorption are shown dotted.

When the initial value of the idler power lies between these extremes the behaviour of the output value of the signal would be expected to lie between these extremes.

5.2 General Case

The single pass amplification is governed by equations 5.1 in the general case

where 5.14

$$C^{\omega_1} = \frac{2^{5/2}\pi^{\frac{1}{2}}\omega_1^{\,2}\, d_{15}\, W_1\, W_2\, \exp\,(i\Delta kz)}{c^2\, k_1\, e^{\frac{1}{4}} W_{o3}}$$

$$\times\left(W_1^{\,2}\,(1+i\,\xi_3)\,(1-i\,\xi_2) + W_2^{\,2}\,(1+i\,\xi_3)\,(1-i\,\xi_1) + (1-i\,\xi_1)\,(1-i\,\xi_2)\right)^{-\frac{1}{2}}$$

$$\times\left(e^2\, W_1^{\,2}(1+i\,\xi_3)\,(1-i\,\xi_2) + e^2\, W_2^{\,2}\,(1+i\,\xi_3)\,(1-i\,\xi_1) + (1-i\,\xi_1)\,(1-i\,\xi_2)\right)^{-\frac{1}{2}}$$

in the three dimensional case. A similar equation defines C^{ω_2}. A computer program has been developed to solve these equations numerically using Milnes predictor-corrector method of integration,[57]. The program is described in Appendix III.

Graphs 13 - 17 are preliminary results obtained from this program. Graphs 13, 14, 15 show the typical variation in amplification of the signal beam

(defined as:-

$$\text{amplification} = \frac{A_1 - A_{10}}{A_{10}} \quad)$$

with the phase matching parameter $\Delta k \frac{1}{2}$ for non-degenerate case when the initial idler amplitude is zero. As has been pointed out this will correspond to the situation when the idler is non resonant. The pump beam is focussed at the centre of a 1 cm crystal. Graph 13 shows the behaviour in a relatively weakly focussed case $\frac{1}{Z_{03}} = 0.5$ and has just the appearance expected in this limit. As the focussing increases the main peak moves away from $\Delta k = 0$ and those peaks on the side $\Delta k < \Delta k$ (optimum) increase in height. This behaviour is somewhat different than that which occurs for second harmonic generation.

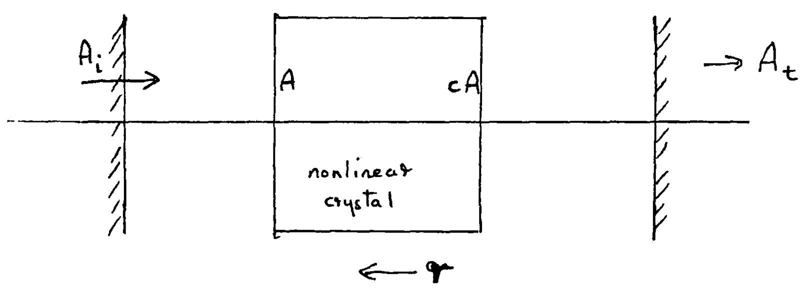
Graph 17 shows the variation of the gain with focussing when the phase matching is optimised both for degenerate parametric amplification (with optimised initial phase) and for non-resonant idler. Note that the scales differ for the two curves so that the degenerate amplification is over ten times greater than the other. The maxima in both cases appears to be about $\frac{l}{Z_{03}} = 3.5$ although the strongly focussed region is not yet well defined.

Graph 16 represents the variation of the gain in each of these cases, optimised with respect to phase matching, and at the focussing $\frac{l}{Z_{03}} = 3.33$ with the effective non-linear coefficient. The effective non-linear coefficient is defined as

$$5.15 \qquad d = 64\pi^2 \; d_{15} \sqrt{\frac{20P}{c}} \; 10^7$$

where P is the power present in the pump beam in watts. This graph can be used to calculate the threshold of a given cavity knowing its losses as is shown in the next section. Note that for this graph also the scales for the gain in the two cases differ by a factor of ten.

5.3 Calculation of Threshold and Resonator Output



Consider the schematic diagram of the signal resonator figure 5.2

A_i is the signal mode amplitude entering the resonator, A the amplitude of this mode just inside the crystal entry face. C is the gain coefficient which has been examined in the previous section and t the transmission loss of the output mirror. In the real case there will be some loss of the input beam as it enters the cavity but this has been neglected. Since there is gain only in one direction we have

$$A = A_i + A\, r\, c \tag{5.16}$$

also $A_t = c\, t\, A$

therefore

$$A_t = \frac{ct\, A_i}{1 - rc} \tag{5.17}$$

The condition for gain is given by

$$\left| \frac{A_t}{A_i} \right| \geqslant 1$$

Putting $rc = r\, c\, e^{i\phi}$ where ϕ is the change of phase of the amplitude in a round trip and substituting from equation 5.17 the gain condition becomes

$$|c|^2 \left(|r|^2 - |t|^2 \right) - 2|r|\,|c| \cos\phi + 1 \leq 0 \tag{5.18}$$

giving the gain condition

$$|c| \geq \frac{|r| \cos\phi - \sqrt{|t|^2 - |r|^2 \sin^2\phi}}{|r|^2 - |t|^2} \tag{5.19}$$

which in the case of resonance, $\phi = 0$, reduces to

5.20 $$|C| \geq \frac{1}{|r| + |t|}$$

For gain to take place at all, the cavity must be tuned so that the square root in equation 5.19 remains real

i.e. $$\sin^2 \phi \leq \frac{t^2}{r^2}$$

For $r = 0.9$ i.e. 10% loss in the resonator and the output mirror having a transmission coefficient of $t = 0.05$ i.e. 5% transmission, we have at resonance from equation 5.20 that $|c| \geq 1.05$. Under the conditions of graph 16 this occurs when the effective non-linear coefficient d is greater than 4.6×10^{-4} for degenerate parametric amplification and 6.6×10^{-3} for non-degenerate amplification with a non-resonant idler.

Taking the approximate values for d_{15}, 3×10^{-8} e.s.u. and for c, 3×10^{10} cms/sec, and substituting into equation 5.15 it can be seen that threshold occurs for this cavity at P = 10 milliwatts for the degenerate case and P = 1 watt in the non-degenerate case. This power of course must be in a single longtitudinal pump mode. This method of estimating the threshold of a given cavity should give better results than previous methods 15, 25 since there are fewer doubtful assumptions made.

Appendix I

Evaluation of the Integral $I = \int_C \exp(-z^2)\, dz$

Where the contour C is a straight line in the complex z plane given by $z = bx + c$, where b, c are complex constants, x a real variable. As x runs from $-\infty$ to $+\infty$, z runs on the line C.

$$\text{Putting} \quad z = u + iv$$
$$b = b_1 + ib_2$$
$$c = c_1 + ic_2$$

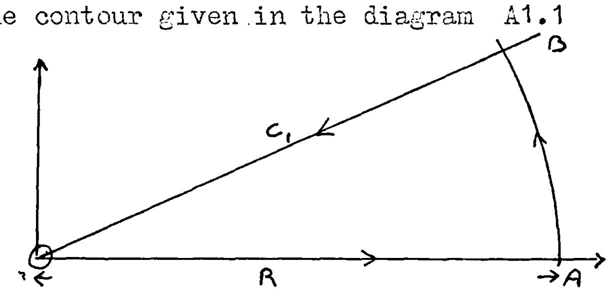
the equation for the line $z = bx + c$ is

A1.1 $$v = \left(\frac{b_2}{b_1}\right) u - c_1 b_2/b_1 + c_2$$

Consider $I_{c_1} = \int_{C_1} \exp(-z^2)\, dz$ where

G is the contour given in the diagram A1.1

Fig. A1·1



By Cauchy's theorum $I_{c_1} = 0$. as the integral has no poles inside the contour.

Therefore we have

$$\text{A2.2} \quad I_{c_1} = \int_0^R \exp(-u^2)\, du - \int_0^B \exp(-z^2)\, dz + \int_A^B \exp(-z^2)\, dz = 0$$

Now

$$|I_{AB}| = \left| \int_A^B \exp(-z^2)\, dz \right|$$

$$= \left| \int_0^{\alpha} \exp(-R^2 \cos 2\theta - iR^2 \sin 2\quad) \, iRe^{i\theta}\, d\theta \right|$$

$$\leq R \int_0^{|\alpha|} \exp(-R^2 \cos 2\theta)\, d\theta$$

$$\leq R \int_0^{\pi/4} \exp(-R^2 \cos 2\theta)\, d\theta \quad \text{if } |\alpha| < \frac{\pi}{4}$$

substituting $\theta = \pi/4 - \phi/2$

$$|I_{AB}| \leq \int_0^{\pi/2} \exp(-R^2 \sin\phi)\, d\phi$$

$$< \frac{\pi}{4R} \quad \text{by Jordans Inequality}$$

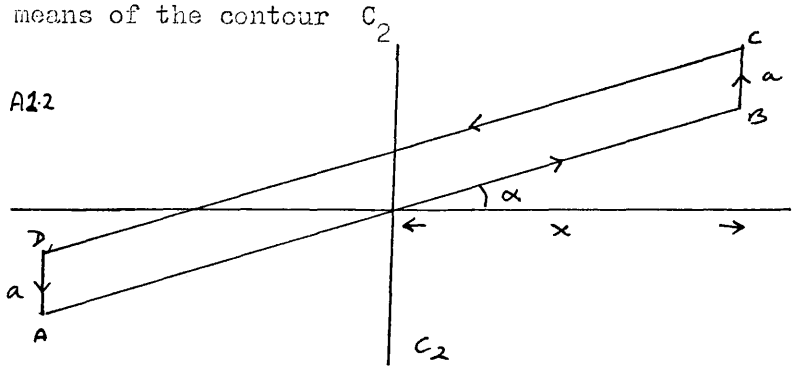
and as $R \to \infty$ $|I_{AB}| \to 0$.

Therefore

$$\text{A1.3} \quad \lim_{R\to\infty} \int_{OB} \exp(-z^2)\, dz = \frac{\sqrt{\pi}}{2}$$

This result can be extended to a line not passing through the origin by means of the contour C_2

Fig. A1.2



Again by Cauchy's theorum $\int_{C_2} \exp(-z^2)\, dz = 0$

$$|I_{BC}| = \left| \int_{BC} \exp(-X^2 + y^2 - 2iXy)\; i\,dy \right|$$

$$\leq \exp(-X^2) \int_{BC} \exp(y^2)\, dy$$

$$= \exp - X^2 (1-\tan^2 x) \int_0^a \exp(2X\tan u + u^2)\, du$$

and therefore $|I_{BC}| \to 0$ as $X \to \infty$ if $|\alpha| < \frac{\pi}{4}$

Similarly $|I_{DA}| \to 0$ as $X \to \infty$ and hence

A1.4 $$\lim_{X \to \infty} \int_{DC} \exp(-z^2)\, dz = \lim_{X \to \infty} \int_{AB} \exp(-z^2)\, dz$$

$$= \sqrt{\pi} \qquad \text{for } |\alpha| < \frac{\pi}{4}$$

from A1.1

$\alpha = \tan^{-1}(b_2/b_1) = \arg(b)$. Therefore the condition

$|\alpha| < \frac{\pi}{4}$ implies that

$$\arg(b^2) < \frac{\pi}{2}$$

ie that the real part of b^2 is greater than zero which is true for all the integrals which have been considered.

Appendix 2

Numerical Integration of the Coupling Coefficients

The integrals on which the second harmonic output in each mode of the resonator depend are all of the same form, increasing in complexity with increasing mode number. Thus the following method can be used to evaluate any particular one. The Integration was carried out on the London University Atlas Computer, the program being written is Fortran V. Use was made of the Fortran statement function procedure in order to facilitate change of the integrand.

The program evaluates the real and imaginary parts of the integral using Simpson's Rule and prints the modulus and argument. It is an extension of one given by Pennington[57] the basic Simpson's Rule integration procedure of which has an error control. The integral is calculated using successively decreasing step lengths until two successive calculations differ by less than a required amount. This procedure only gives good results if the integrand varies consistently over the range of integration because of the approximation introduced by Simpson's Rule. The coupling coefficients satisfy this condition.

The input data for the program was arranged in three nested loops. For each value of the parameters W, spot size, a and b, crystal and positions, the integral was calculated over a specified range of y the phase mismatch from y_1 to y_2 in steps Δy. Having completed this new values of a and b were taken and the procedure repeated. This can be seen from the flow diagram for the program, figure A2.1. The spot size parameter W was placed in the outermost loop since this varies slowly near the maximum. The speed of the program varied with the length of the interval a, b as would be expected but in general it calculated four to eight integrals per second to a specified accuracy of four decimal places. This accuracy was adequate except for determining the maximum values of the focus curve and spot size curve when six places were used.

For the lowest order mode the integral from equation 4.5 of the form

A2.1

$$I = \int_a^l \frac{\exp(iyx)\,dx}{(1-ix)\,(c+idx)^{\frac{1}{2}}}$$

where $c = e^2W^2+1 \qquad d = e^2W^2$

the real and imaginary parts being

A2.2

$$Ir = \int_a^l F_1(x)\,dx \qquad Ii = \int_a^l F_2(x)\,dx$$

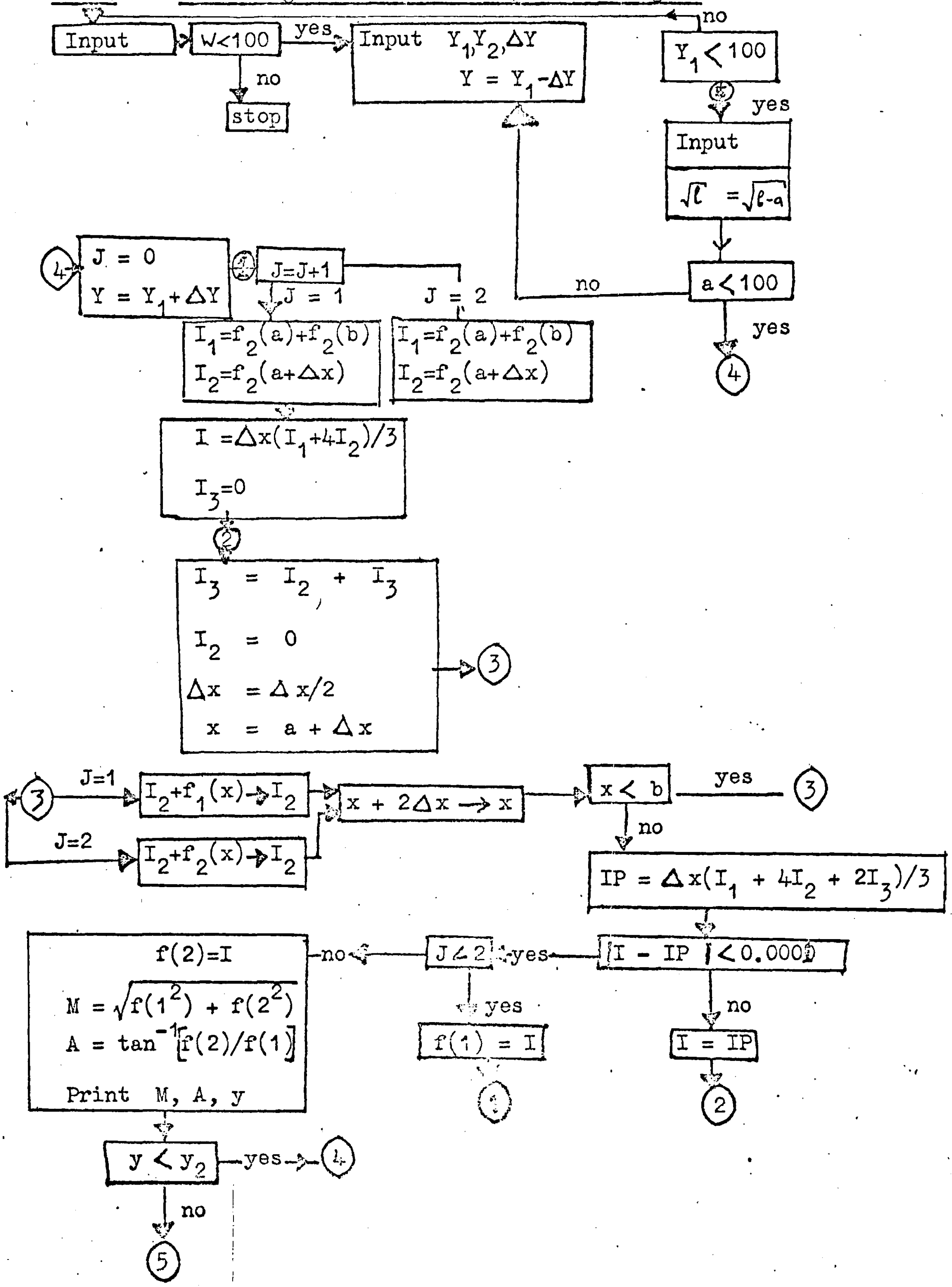
where

$$F_1(x) = \frac{\sin(yx)\left(c+dx^2+(c^2+d^2x^2)^{\frac{1}{2}}\right)+x\cos(yx)\left(c-d+(c^2+d^2x^2)^{\frac{1}{2}}\right)}{2^{\frac{1}{2}}\,(1+x^2)\,(c^2+d^2x^2)^{\frac{1}{2}}\,\left(c+(c^2+d^2x^2)^{\frac{1}{2}}\right)^{\frac{1}{2}}}$$

The general procedure is unaffected by the inclusion of absorption since Equation A2.1 just becomes

$$I = \int_a^l \frac{\exp(iyx-kx)\,dx}{(1-ix)\,(c+i\,dx)^{\frac{1}{2}}}$$

A 2.4 Flow Diagram for S.H. Integration Program



Integration Program

```
      DIMENSION F (2)
11    READ 101,W,E
      IF (W-100) 8,9,9
8     EP=1-1/(E*E)
      W=W*E
33    READ 103,Y1,Y2,DY
      IF (Y1-100) 29,11,11
29    READ 101,A,B
      PRINT 104,A,B
      IF (A-100) 31,33,33
31    Y=Y1-DY
      ACL=SQRT (B-A)
28    Y=Y+DY
      I=1
26    J=0
25    J=J+1
      FGR1 (X)=((W*W +1+SQRT((W*W+1)* *2+(W*W*X*EP)**2))*(COS(X*Y)-X*SIN(X
     1Y))-X*W*W*EP*(X*COS(X*Y)+SIN(X*Y)))/((1+X*X)*SQRT((W*W/(E*E)+1)*(W
     2 W+1+SQRT(W W+1)  2+(W W X EP)  2))  ((W W+1)  2+(W W EP X)  2) 2)
     3)
      FGR2(X)=((W*W+1+SQRT((W*W+1)**2+(W*W*X*EP)**2))*(COS(X*Y)*X+SIN(X
     1Y))+X*W*W*EP*(COS(X*Y)-X*SIN(X*Y)))/((1+X*X)*SQRT((W*W/(E*E)+1)*(W
     2*W+1+SQRT((W*W+1)**2+(W*W*X*EP)**2)*((W*W+1)**2+(W*W*EP*X)**2)*2)
     3)
1     DX=(B-A)/2
      GO TO (21,22),J
21    FI1=FGR1(B)+FGR1(A)
      FI2=FGR1(A+DX)
      GO TO 10
22    FI1=FGR2(B)+FGR2(A)
      FI2=FGR2(A+DX)
      GO TO 10
10    FI3=0
      FI=DX*(FI1+4*FI2)/3
2     FI3=FI2+FI3
      FI2=0
      TDX=DX
      DX=0.5*DX
      X=A+DX
3     GO TO(12,13),J
12    FI2=FI2+FGR1(X)
      GO TO 16
13    FI2=FI2+FGR2(X)
      GO TO 16
16    X=X+IDX
      IF(X-B)3,3,4
```

```
4     FIP=DX*(FI1+4,*FI2+2.*FI3)/3
      IF(ABS(FIP-FI)-0.000001)6,6,51
51    IF(DX-.0001)6,6,5
5     FI=FIP
      GO TO 2
6     F(J)=FIP
      IF(J-2)25,7,7
7     AMOD=SQRT(F(1) F(1)+F(2) F(2))
      ANG=ATAN(F(2)/F(1))
      BCL=AMOD/ACL
      PRINT 102,AMOD,ANG,Y,BCL
      IF(Y-Y2-,001)28,28,29
102   FORMAT(2F12,6,F8,3,F12.6)
101   FORMAT(2F10.3)
104   FORMAT(2F6,3)
103   FORMAT(3F10,3)
9     CALL EXIT
      END
```

Appendix 3

The parametric coupled differential equations have been integrated using Milnes predictor-corrector method of numerical integration with a variable step length for speed and accuracy. The program is a generalisation of one given in Pennington 57 written on Fortran V for the London University Atlas Computer, as before. The equations to be solved are of the form

A 3.1

$$\frac{dy_1}{dx} = \alpha_1 \left(f_1(x)\, y_4 - f_2(x)\, y_3\right)$$

$$\frac{dy_2}{dx} = \alpha_1 \left(f_1(x)\, y_3 + f_2(x)\, y_4\right)$$

$$\frac{dy_3}{dx} = \alpha_2 \left(f_1(x)\, y_2 - f_2(x)\, y_1\right)$$

$$\frac{dy_4}{dx} = \alpha_2 \left(f_1(x)\, y_1 + f_2(x)\, y_2\right)$$

where $f_1(x)$, $f_2(x)$ are defined in the program which follows this section. Since Milnes method is not self starting the seven starting values required are calculated by the method of Runga-Kutta. This subroutine is also based on the program given in Pennington 57.

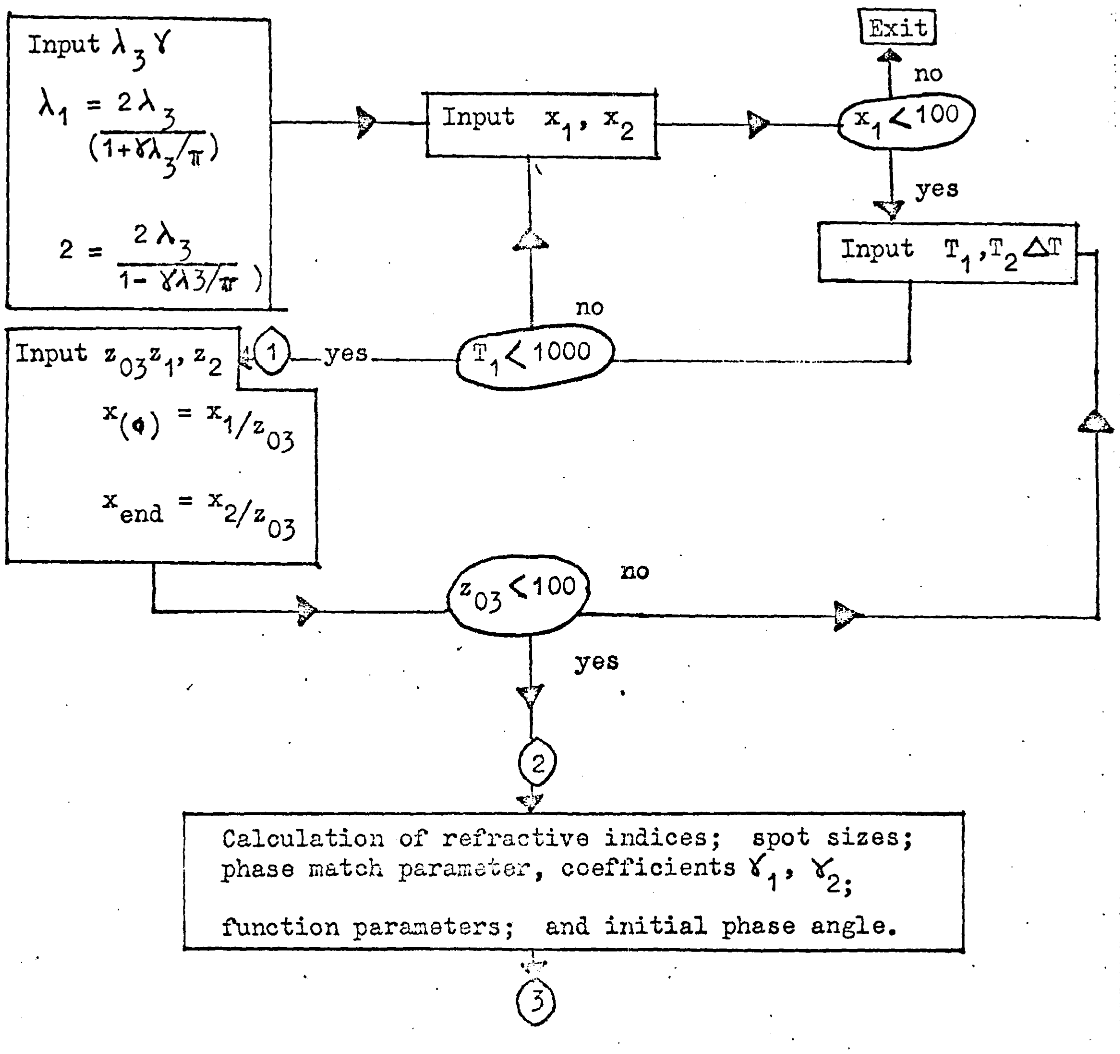
The first section of the program calculates all the input data. In order to simplify the input as much as possible the variables chosen as the basic set are those which can be varied directly in an experimental situation. They consist of the wavelength of the three beams, the length of the crystal, the temperature of the crystal, the confocal parameters of the three beams, and the effective non-linear coefficients, which includes the pump power. Section one of the flow diagram shows the way in which this is accomplished. Equation 1.66, 1.67 are used to define the refractive index of the crystal with wavelength and temperature. By using the important secondary parameters the phase matching parameter, the spot sizes, and the relative length $\frac{1}{Z_{C3}}$ when evaluating the results of the integration, the results can be regarded as quite general although the equations 1.66 1.67 represent a specific crystal.

The results so far have been restricted to (1) $\gamma = 0$ i.e. degenerate amplification, (2) $x_1 = -x_2$ focus at the centre of the crystal and (3) $z_1 = z_2 = 1$ the three confocal parameters equal, although the program is capable of dealing with these variations.

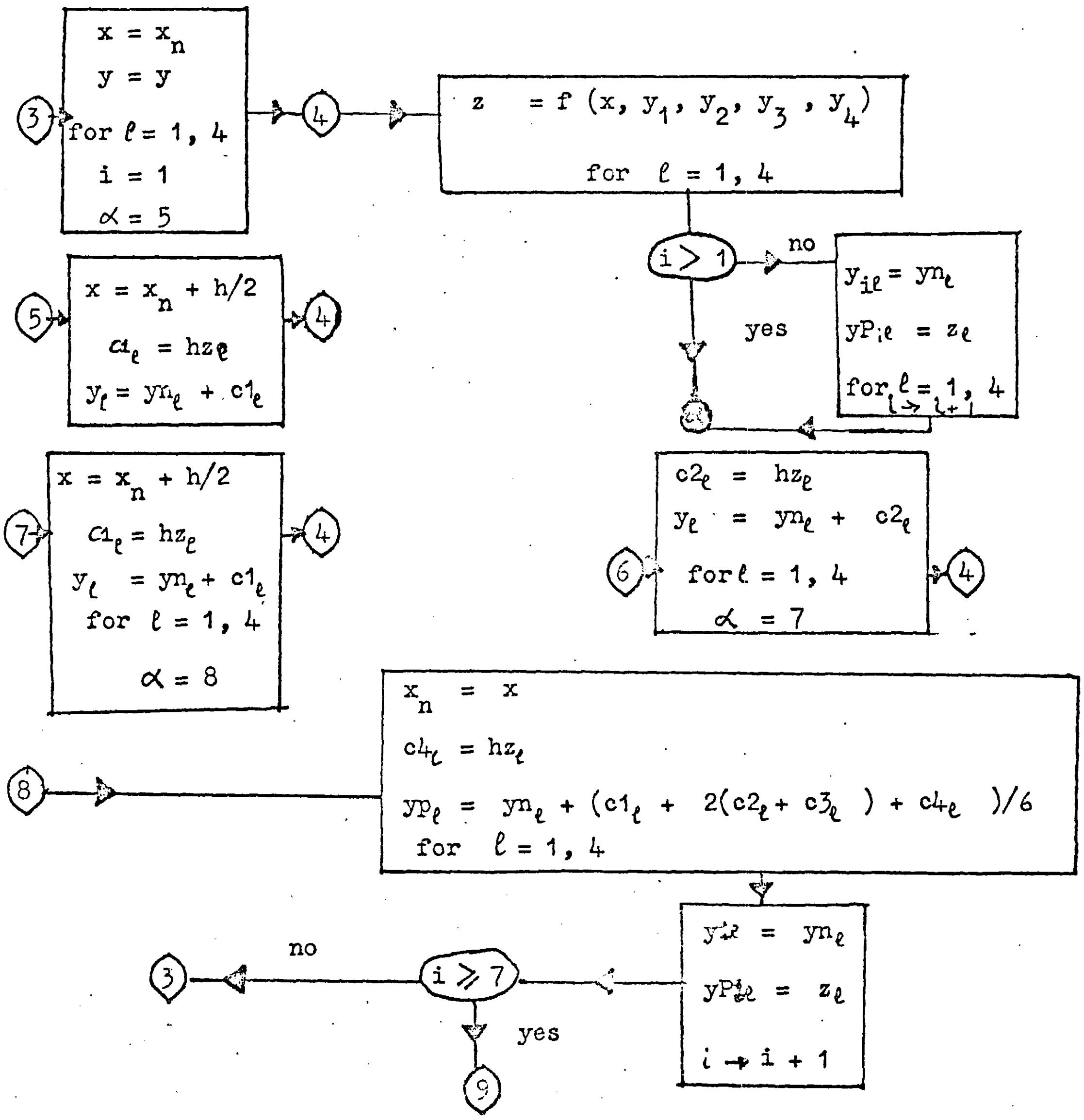
The first section of the following program also calculates the optimum phase angle of the initial value of the signal amplitude this being the most interesting value. By altering this specification any value can be taken into account.

Flow diagrams for the first two sections of the program are included but not one for the main part since this is only a minor modification of the diagram given in Pennington, 57.

Section I — Input and Calculation of Physical Parameters



Section II Calculation of the Starting Values of the Milne Program using the Method of Runge-Kutta



Parametric Integration Programme

```
      M = 4
      Dimension  Y(7,4), YP (7,4), Q(3,4), QP(3,4), YS(4), C(4),W1(4),
                 WP1(4), W2
      1(4),WP2(4)
      Dimension Z(4), V(4), YN(4), C1(4), C2(4), C3(4), C4(4), YEND (43)
      Dimension WAV (3), ROF(2)
      PI = 3.1415926536
      FGR1(X) = (((A1+A2*X*X+A3*X**4) + SQRT((A1+A2*X*X+A3*X**4)**2+(A4*x+A5
      1*X*X*X)**2))*COS(YM*X)+X*(A4+A5*X*X)*SIN(YM*X))/SQRT(((A1+A2*X*X+A
      23*X**4)+SQRT((A1+A2*X*X+A3*X**4)**2+(A4*X+A5*X*X*X)**2))*2*((A1+A2
      3*X*X+A3*X**4)**2+(A4*X+A5*X*X*X)XX2))
      FGR2(X)=(((A1+A2*X*X+A3*X**4)+SQRT((A1+A2*X*X+A3*X**4)**2+(A4*X+A5
      1*X*X*X)**2))*SIN(YM*X)-X*(A4+A5*X*X)*COS(YM*X))/SQRT(((A1+A2*X*X+A
      23*X**4)+SQRT((A1+A2*X*X+A3*X**4)**2+(A4*X+A5*X*X*X)**2))*2*((A1+A2
      3*X*X+A3*X**4)**2+(A4*X+A5*X*X*X)**2))
      YT=0.001
      ERR= 0.00001
      D = 0.001
      READ 110,WAV(3),GA
93    READ 110,X1,X2
      IF(X1-100.0)19,44,44
19    READ 112, T1,T2,DT
      IF(T1-1000.0)46,93,93
46    READ 112,ZOP,Z1,Z2
47    IF(ZOP-100.0)96,19,19
```

```
96  PRINT 112,ZOP, Z1,Z2
    XO=X1/7OP
    XEND=X2/ZOP
    PRINT 110,XO,XEND
    T=T1-DT
94  T=T+DT
    S=0,001*(T+273.0)
    WAV(1)=2*WAV(3)*PI/(PI+GA*WAV(3))
    WAV(2)=2*WAV(3)"PI/(PI-GA*WAV(3))
    DO 95,I=1,3
95  ROF(I)=SQRT(4.9130-0.0278*WAV(I)*WAV(I)+((11.73+1.65*S*S)/(100*WAV
    1(I)*WAV(I)-(2.12+0.27*S*S)**2)))
    REF=SQRT(4.5567+0.2605*S*S-0.0224*WAV(3)*WAV(3)+((9.7+2.7*S*S)/(10
    10*WAV(3)*WAV(3)=(2.01+0.54*S*S)**2)))
    WOP=0.01*SQRT(WAV(3)*ZOP/(PI*REF))
    WO2=WAV(3)*ROF(2)*Z2/(WAV(2)*REF)
    WO1=WAV(3)*ROF(1)*Z1/(WAV(1)*REF)
    YM=2*PI*ZOP*(REF/WAV(3)-ROF(1)/WAV(1)-ROF(2)/WAV(2))*10000.0
    AW=WO1+WO2+1
    BW=WO1*Z2+WO2*Z1-Z1*Z2
    CW=Z1+Z2+WO1*(Z2-1)+WO2*(Z1-1)
    E=(REF/ROF(3))**2
    EAW=E*WO1+E*WO2+1
    EBW=E*WO1*Z2+E*WO2*Z1-Z1*Z2
    ECW=Z1+Z2+E*WO1*(Z2-1)+E*WO2*(Z1-1)
```

```
      A1=AW*EAW
      A2+EAW*BW+AW*EBW-CW*ECW
      A3=BW*FBW
      A4=CW*EAW+ECW*AW
      A5=EBW*CW+ECW*BW
      YTI=SQRT(FGR1(XO)**2+FGR2(XO)**2)
      YN(1)=YT*(SQRT(1+FGR1(XO)/YTI)+SQRT(1-FGR1(X))/YTI))*0.5
      YN(2)=YT*(SQRT(1-FGR1(XO)/YTI)-SQRT(1+FGR1(XO)/YTI))*0.5
      YN(3)=YN(1)
      YN(4)=YN(2)
      AP1=-0.001*(WO1*ZOP/(WOP*SQRT(SQRT(E))))
      AP2=AP1/(WAV(2)*ROF(2))
      AP1=AP1/(WAV(1)*ROF(1))
      PRINT 110,AP1,AP2
45    XN=XO
      I = 1
      DD=.5*D
22    X=XN
      DO 28 I=1,M
28    V(L)=YN(L)
      J=1
23    Z(1)=AP1*(FGR1(X)*V(4)-FGR2(X)*V(3))
      Z(2)=AP1*(FGR1(X)*V(3)+FGR2(X)*V(4))
      Z(3)=AP2*(FGR1(X)*V(2)-FGR2(X)*V(1))
      Z(4)=AP2*(FGR1(X)*V(1)+FGR2(X)*V("))
      IF(I-1)99,99,16
```

```
99  DO 98 L=1,M
    Y(1,L)=YN(L)
98  YP(1,L)=7(L)
    I=I+1
    GO TO 16
16  GO TO (24, 25, 26, 27),J
24  X=XN+DD
    DO 29 L=1,M
    C1(L)=D*Z(L)
29  V(L)=YN(L)+C1(L)*0.5
    J=2
    GO TO 23
25  DO 32 L= 1,M
    C2(L)=D*Z(L)
32  V(L)=YN(L)+C2(L)*.5
    J=3
    GO TO 23
26  X=XN+D
    DO 33 L=1,M
    C3(L)=D*Z(L)
33  V(L)=YN(L)+C3(L)
    J =4
    GO TO 23
27  XN=X
    DO 34 L=1,M
    C4(L)=D*Z(L)
```

```
34  YN(L)=YN(L)+(C1(L)+2*(C2(L)+C3(L))+C4(L))/6
    DO 17 L=1,M
    Y(I,L)=YN(L)
17  YP(I,L)=7(L)
    I=I+1
    IF(I-7)22,91,91
91  X=X0+5*D
    ERP=29*ERR
    ERG=.01*ERP
10  A=1.3333333*D
    B=.3333333*D
    K=1
 1  DO 61 L=1,M
61  Y(7,L)=Y(3,L)+A*(YP(4,L)+YP(4,L)-YP(5,L)+2*YP(6,L))
    X=X+D
    J=1
 2  YP(1,1)=AP1*(FGR1(X)*Y(7,4)-FGR2(X)*Y(7,3))
    YP(7,2)=AP1*(FGR1(X)*Y(7,3)+FGR2(X)*Y(7,4))
    YP(7,3)=AP2*(FGR1(X)*Y(7,2)-FGR2(X)*Y(7,1))
    YP(7,4)=AP2*(FGR1(X)*Y(7,1)+FGR2(X)*Y(7,2))
    GO TO (3,4),J
 3  DO51 L=1,M
    YS(L)=Y(7,L)
51  Y(7,L)=Y(5,L)+B*(YP(5,L)+4*YP(7,L))
    J=2
    GO TO 2
```

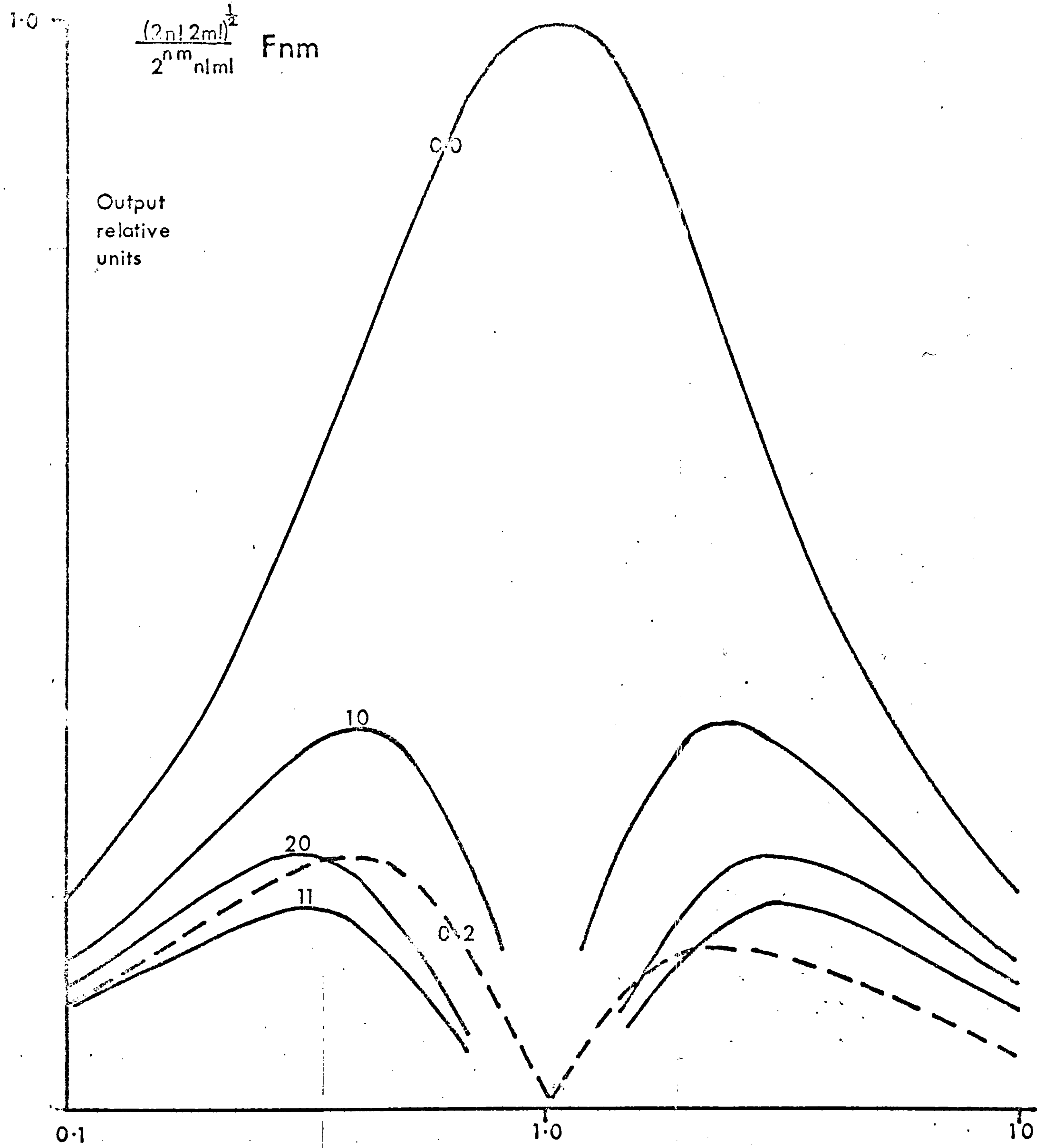
```
 4  L = 1
52  C(L)=ABS(Y(7,L)=YS(L))
    IF(C(L)-ABS(ERP*Y(7,L)))53,53,21
53  L=L+1
    IF(L-M-1)52,20,20
20  IF(X-XEND+.0001)5,43,43
43  IF(X-XEND-.0001)50,50,21
 5  L=1
72  IF(C(L)-ABS(EFG*Y(7,L)))54,54,30
54  L=L+1
    IF(L-M-1)72,7,7
30  K=1
 6  DO 55  =1,M
    DO 31 I=1,6
    Y(I,L)=y(I+1,L)
31  YP(I,L)=YP(I+1,L)
55  CONTINUE
    GO TO 1
 7  GO TO (11,12,13,14,15),K
11  DO 56 L=1,M
    W1(L)=Y(1,L)
56  WP1(L)=YP(1,L)
    K=2
    GO TO 6
12  K=3
    GO TO 6
```

```
13  DO 57 L=1,M
    W2(L)=Y(1,L)
57  WP2(L)=YP(1,L)
    K=4
    GO TO 6
14  K=5
    GO TO 6
15  DO 58 L=1,M
    Y(6,L)=y(7,L)
    Y(4,L)=Y(3,L)
    Y(3,L)=Y(1,L)
    Y(2,L)=W2(L)
    Y(1,L)=W1(L)
    YP(6,L)=YP(7,L)
    YP(4,L)=YP(3,L)
    YP(3,L)=YP(1,L)
    YP(2,L)=WP2(L)
58  YP(1,L)=WP1(L)
    D=D+D
    GO TO 10
21  IF(D-.0000001)40,40,41
40  PRINT 1001
    IF(T-T2)94,46,46
41  DO 59 L=1,M
    DO 42 I=1,3
```

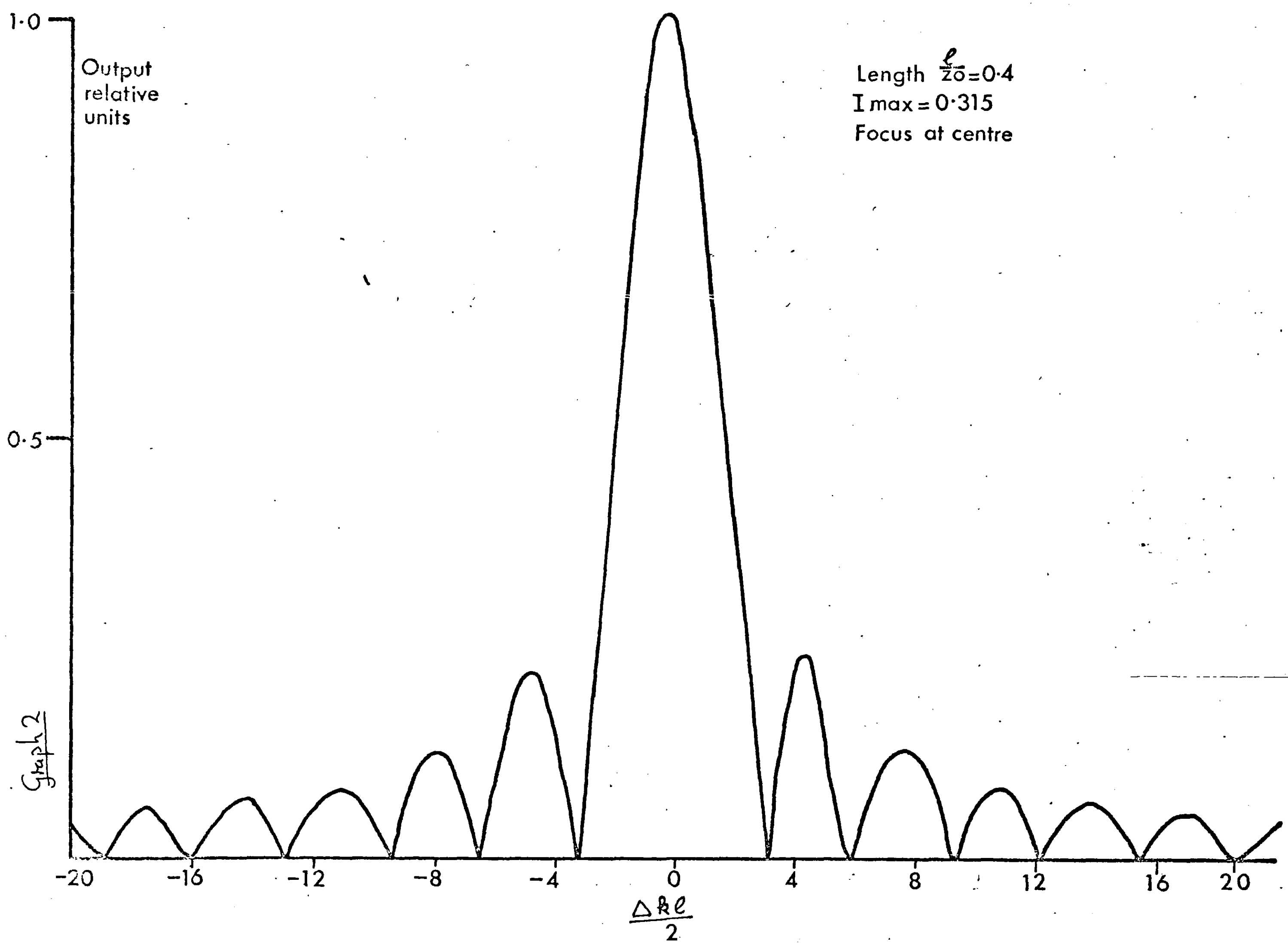
```
      Q(I,L)=0.5*(Y(5=I,L)+Y(6-I,L))-.0625*(Y(7-I,L)-U(6-I,L)-Y(5=I,L)+Y
     1(4-I,L))
42    QP(I,L)=0.5*(YP(5=I,L)+YP(6-I,L))-.0625*(YP(7-I,L)-YP(6=I,L)=YP(5=
     1I,L)+YP(4-I,L))
      Y(6,L)=Y(5,L)
      Y(2,L)=Y(3,L)
      Y(5,L)=Q(1,L)
      Y(3,L)=Q(2,L)
      Y(1,L)=Q(3,L)
      YP(6,L)=YP(5,L)
      YP(2,L)=YP(3,L)
      YP(5,L)=QP(1,L)
      YP(3,L)=QP(2,L)
59    YP(1,L)=QP(3,L)
      X=X-2*D
      D=.5*D
      GO TO 10
50    XEND=X
      DO 60 L =1,M
60    YEND(L)=Y(7,L)
      U1=(SQRT(YEND(1)**2+YEND(2)**2)/0.001)=1
      U2=SQRT(YEND(3)**2+YEND(4)**2)
      PRINT 101,T,YM,W01,W02,U1,U2
      IF(T=T2)94,46,46
44    CALL EXIT
```

```
110   FORMAT(2F10.4)
101   FORMAT(2F12.4,2F10.3,2E15.4)
112   FORMAT(3F10.3)
1001  FORMAT(32HROUNDOFF ERROR PREVENTS SOLUTION)
      END
```

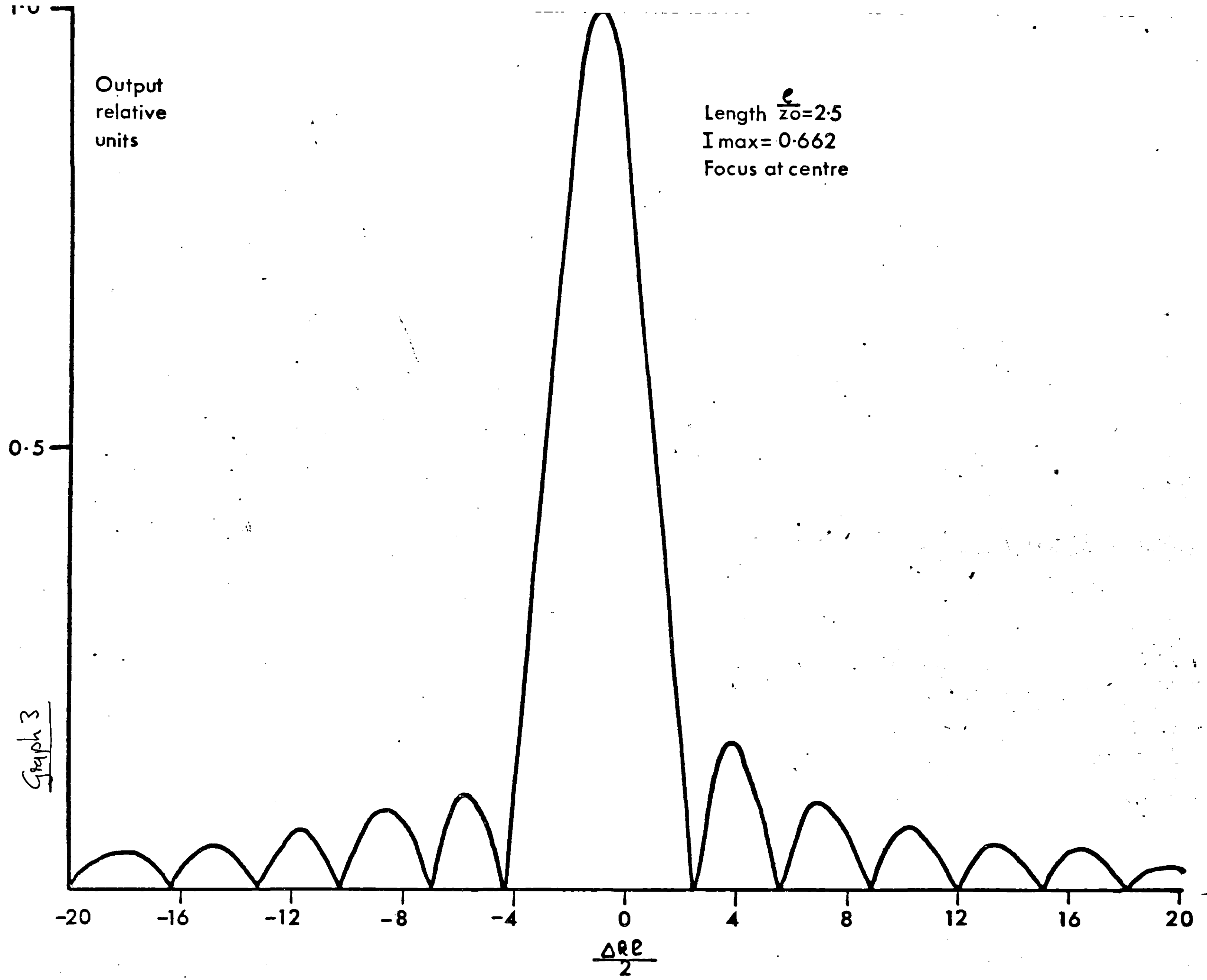
Graph 1



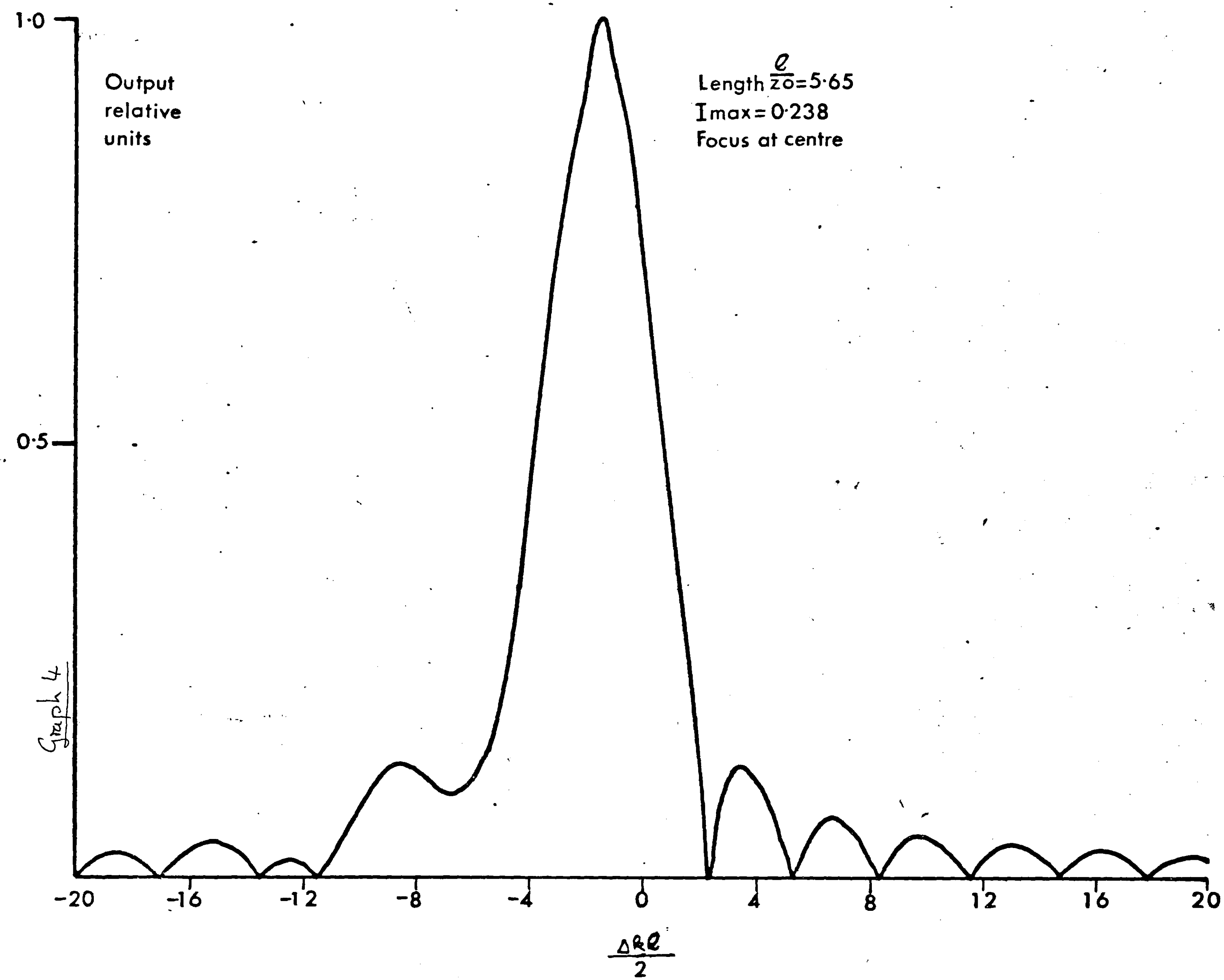
1·0
$\frac{(2n!\,2m!)^{\frac{1}{2}}}{2^{nm}\,n!\,m!}$ Fnm
Output relative units
00
10
20
11
02
0·1
1·0
10
W


1·0
0·5
Output
relative
units
Length $\frac{l}{zo}$=0·4
I max = 0·315
Focus at centre
-20
-16
-12
-8
-4
0
4
8
12
16
20
$\frac{\Delta k l}{2}$

Graph 2

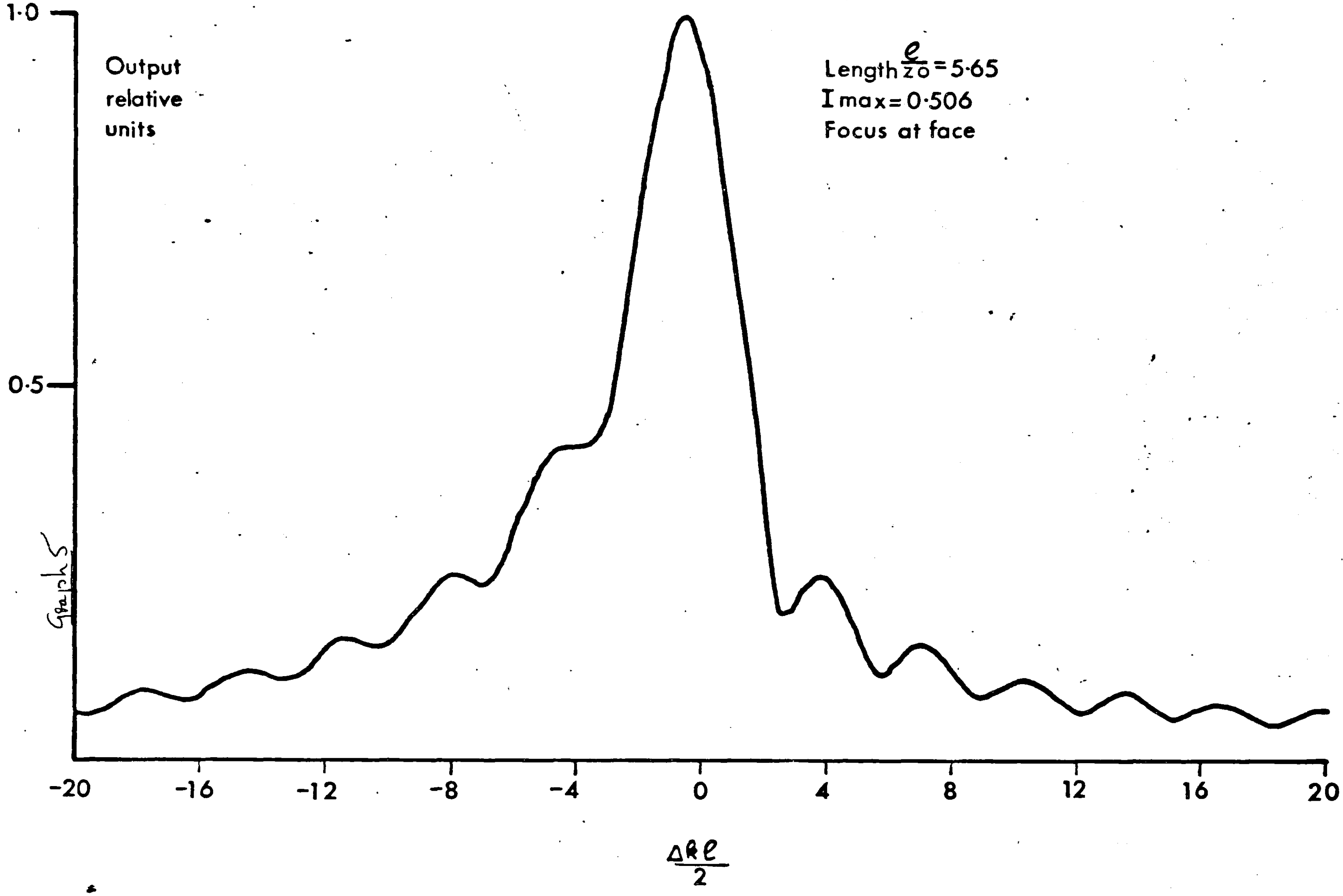

1·0
0·5
Output
relative
units
Length $\frac{\ell}{zo}$=2·5
I max= 0·662
Focus at centre
-20
-16
-12
-8
-4
0
4
8
12
16
20
$\frac{\Delta R \ell}{2}$

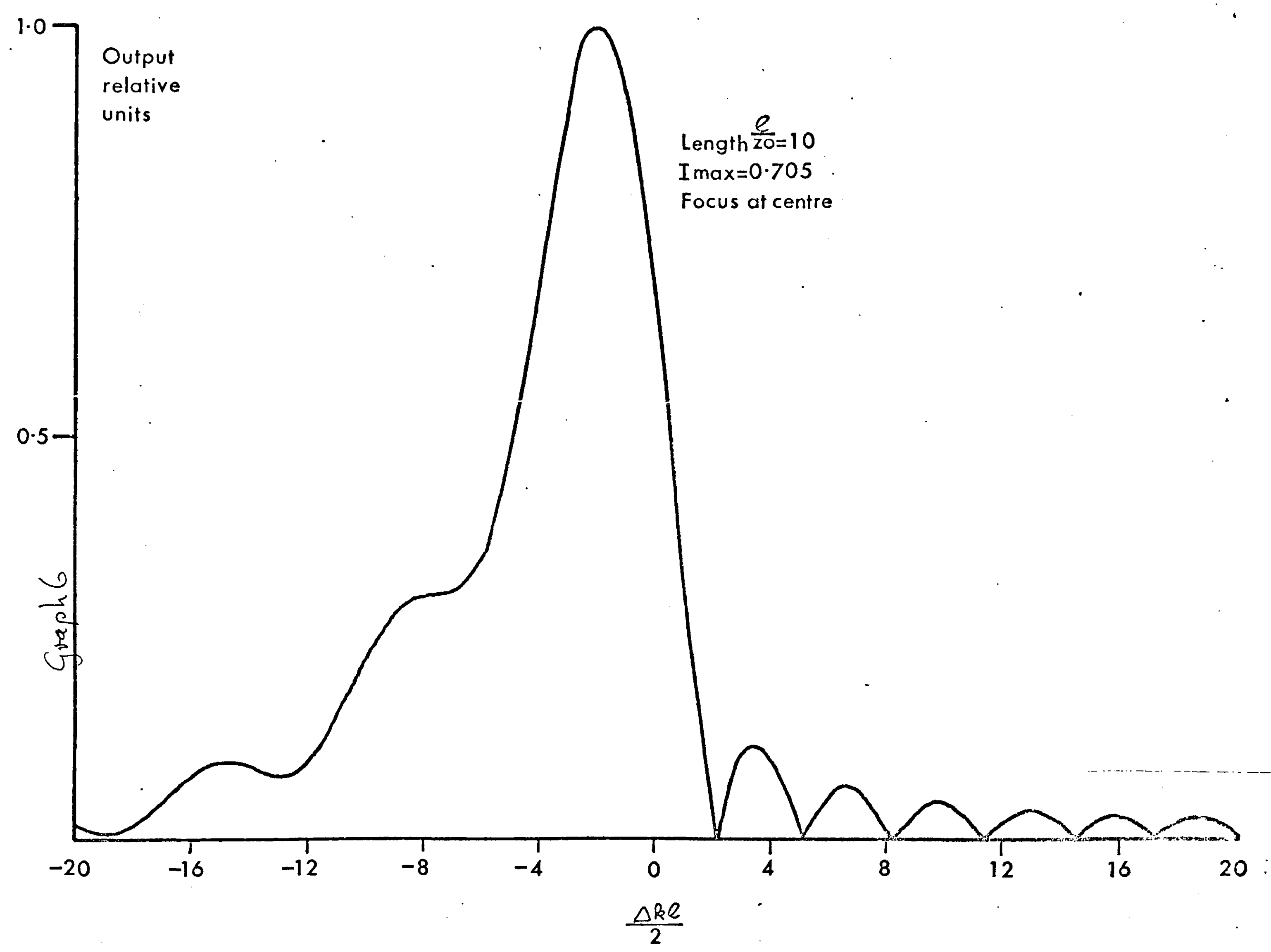
Graph 3


1·0
0·5
Output
relative
units
Length $\frac{\ell}{z_0}$=5·65
Imax=0·238
Focus at centre
-20
-16
-12
-8
-4
0
4
8
12
16
20
$\frac{\Delta k \ell}{2}$

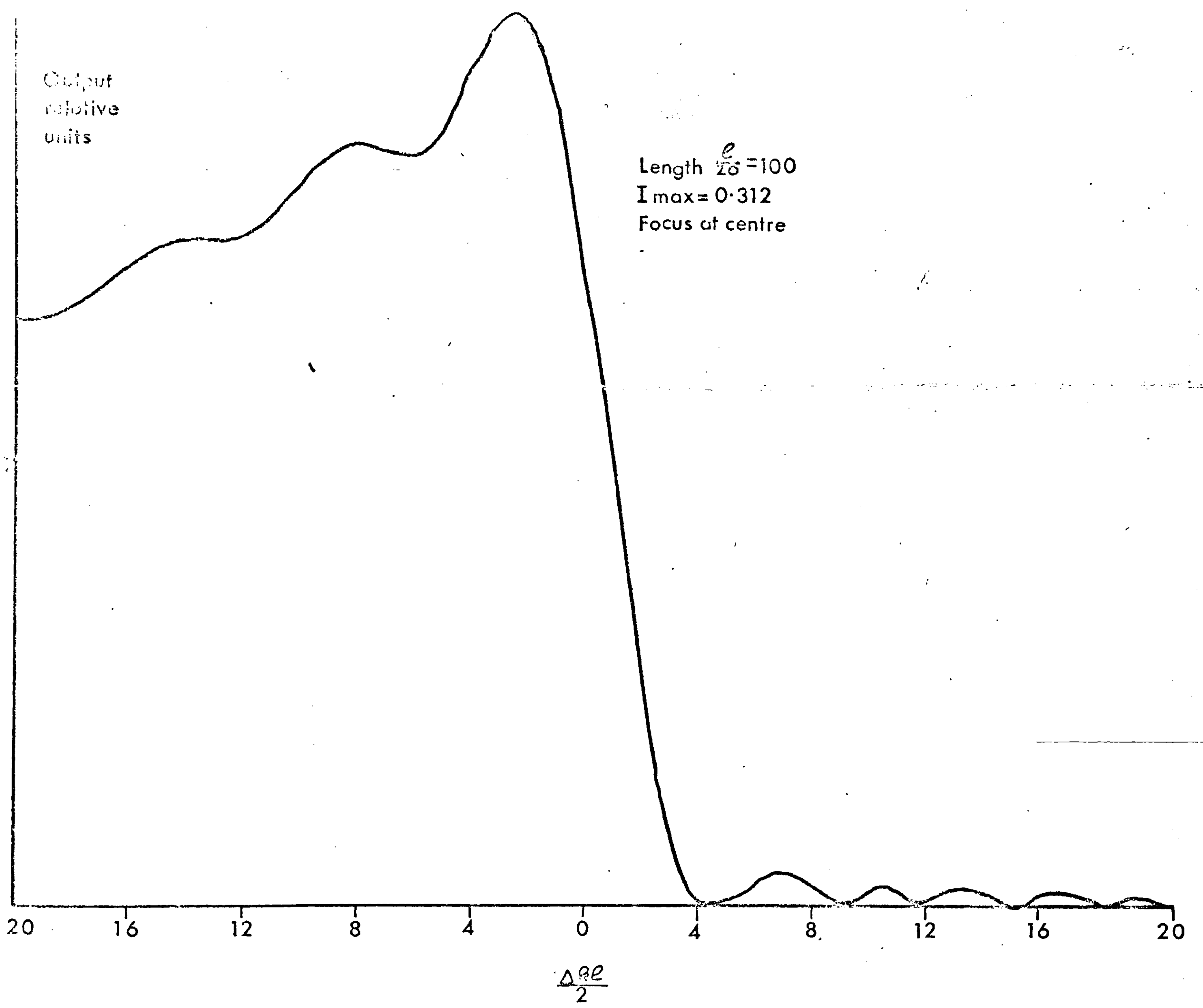
Graph 4

Graph 5

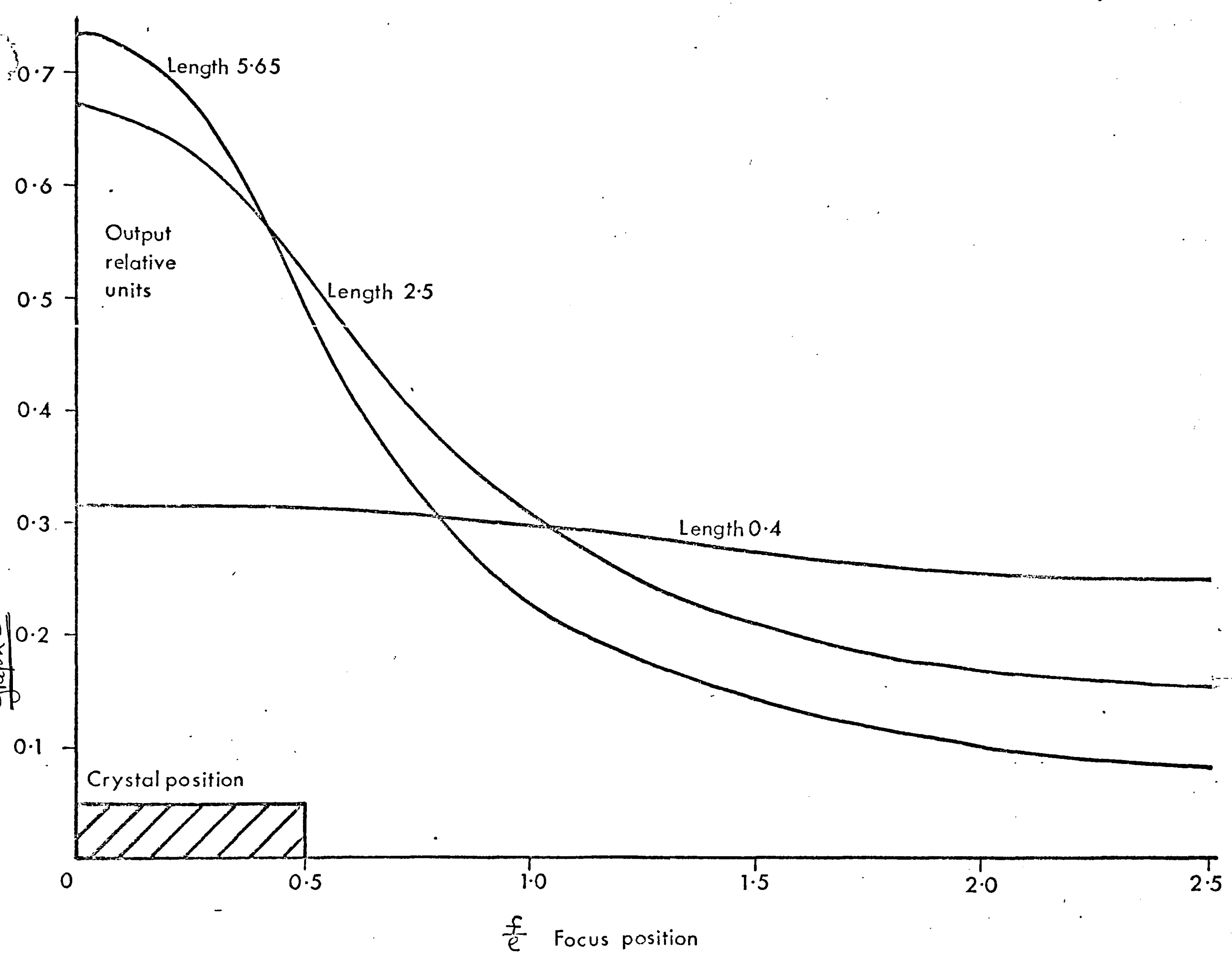

1·0
0·5
Output
relative
units
Length $\frac{\ell}{z_0}$ = 5·65
I max = 0·506
Focus at face
-20
-16
-12
-8
-4
0
4
8
12
16
20
$\frac{\Delta k \ell}{2}$


Graph 6
1·0
0·5
Output
relative
units
Length $\frac{\ell}{zo}$=10
Imax=0·705
Focus at centre
-20
-16
-12
-8
-4
0
4
8
12
16
20
$\frac{\Delta k \ell}{2}$

Graph 7

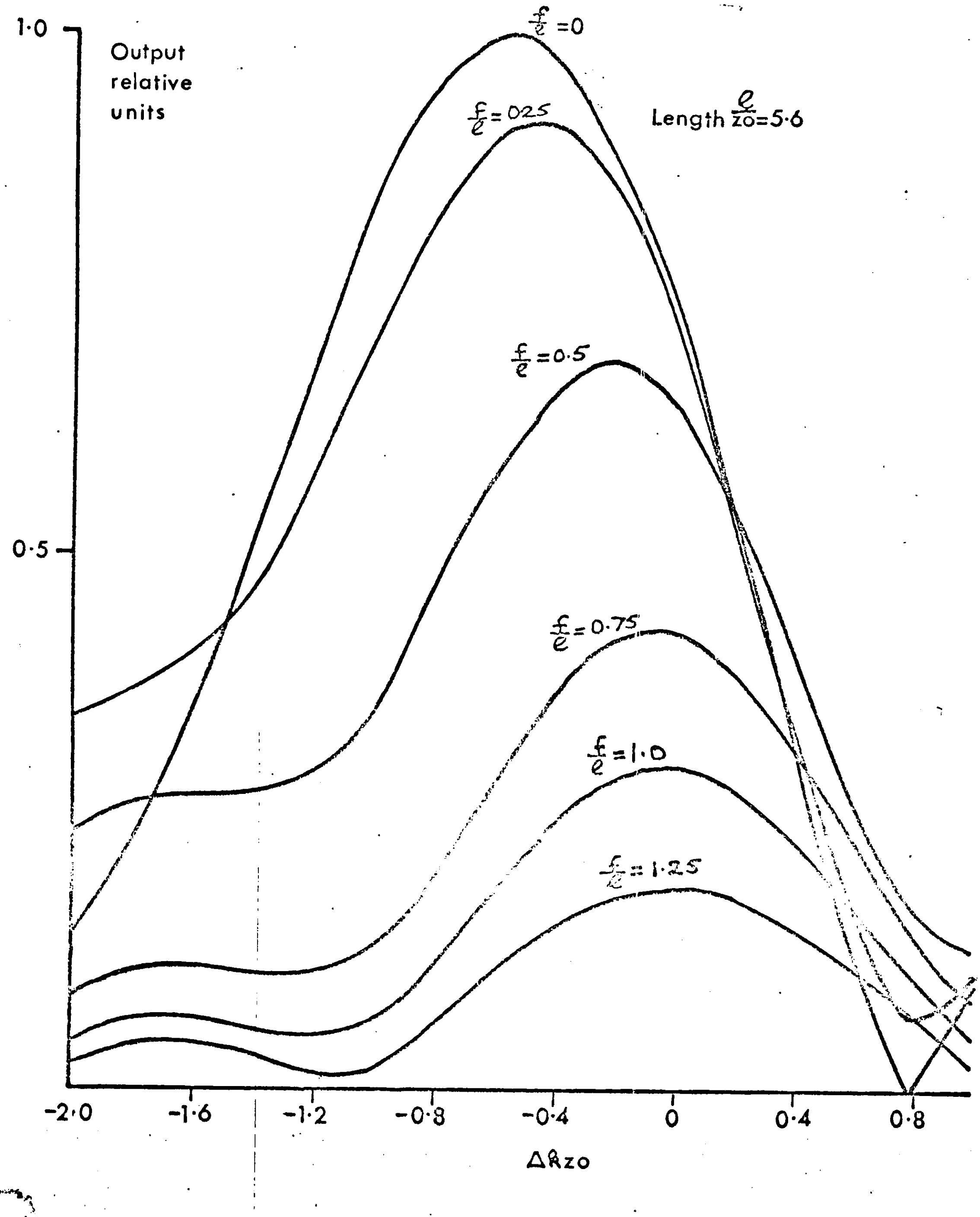

1.0
0.5
Output relative units
Length $\frac{\ell}{2\sigma}$ =100
I max = 0·312
Focus at centre
20
16
12
8
4
0
4
8
12
16
20
$\frac{\Delta \beta \ell}{2}$

Graph 8

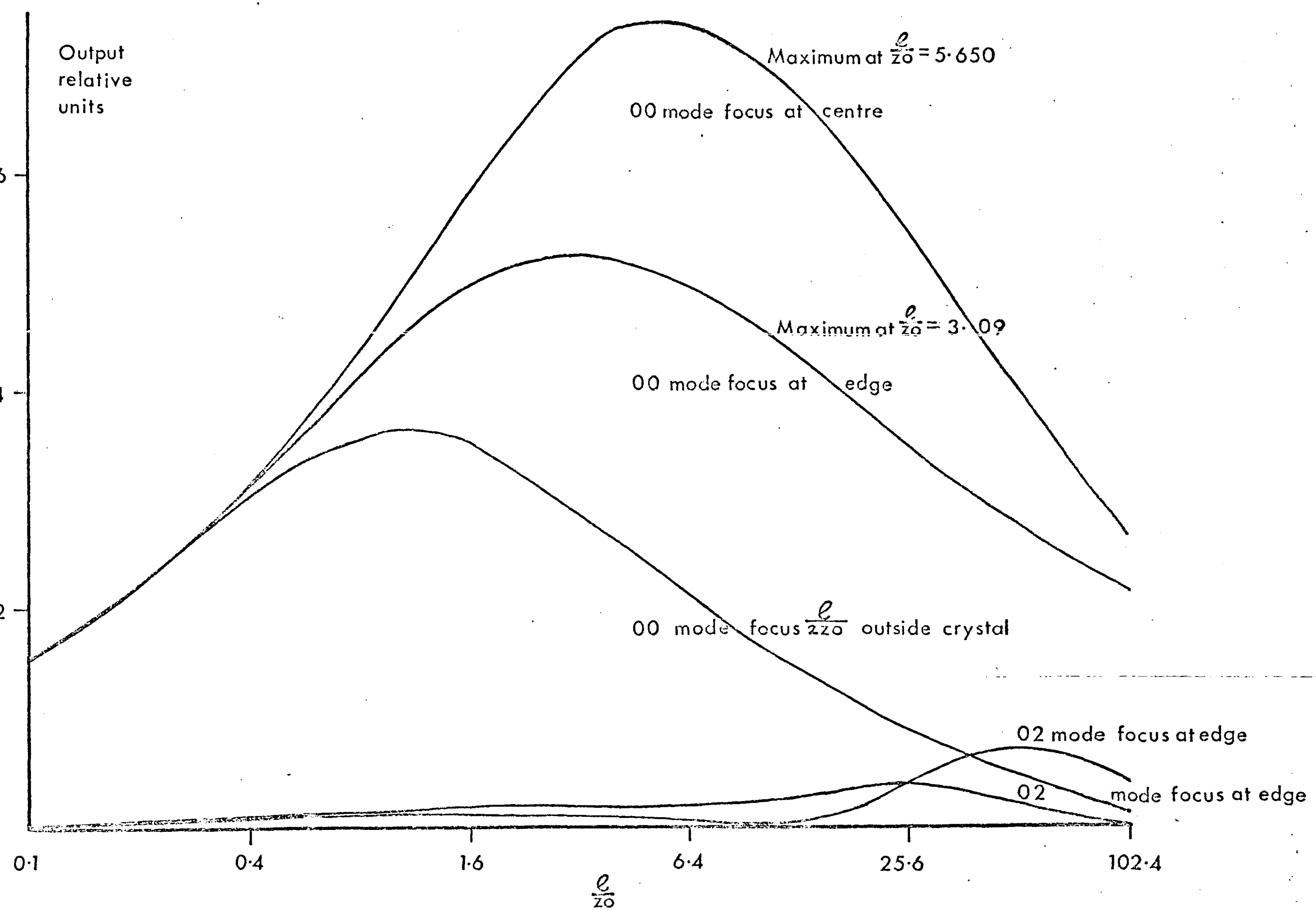


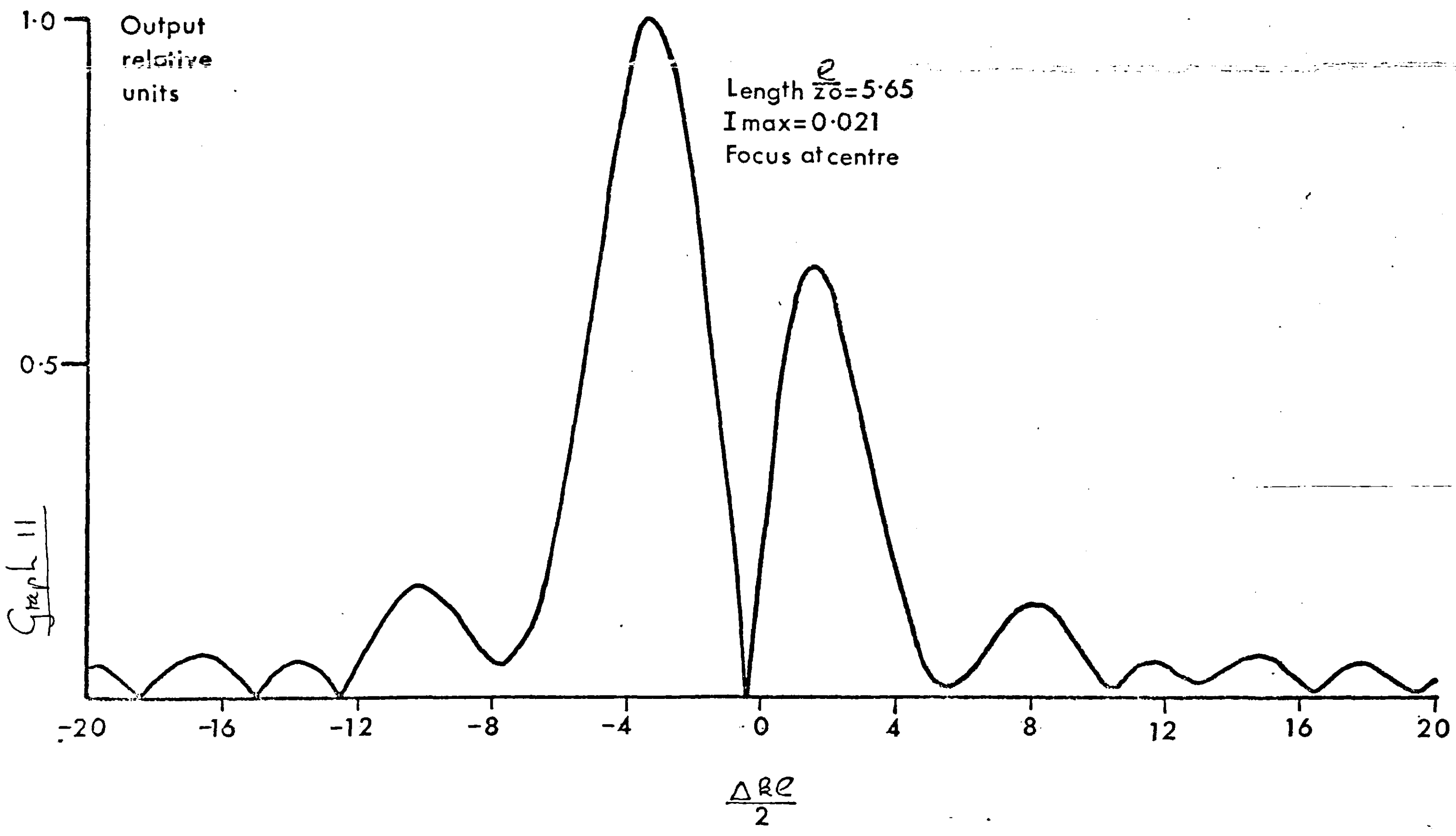
0·7
0·6
0·5
0·4
0·3
0·2
0·1
Output relative units
Length 5·65
Length 2·5
Length 0·4
Crystal position
0
0·5
1·0
1·5
2·0
2·5
$\frac{f}{e}$ Focus position

Graph 9

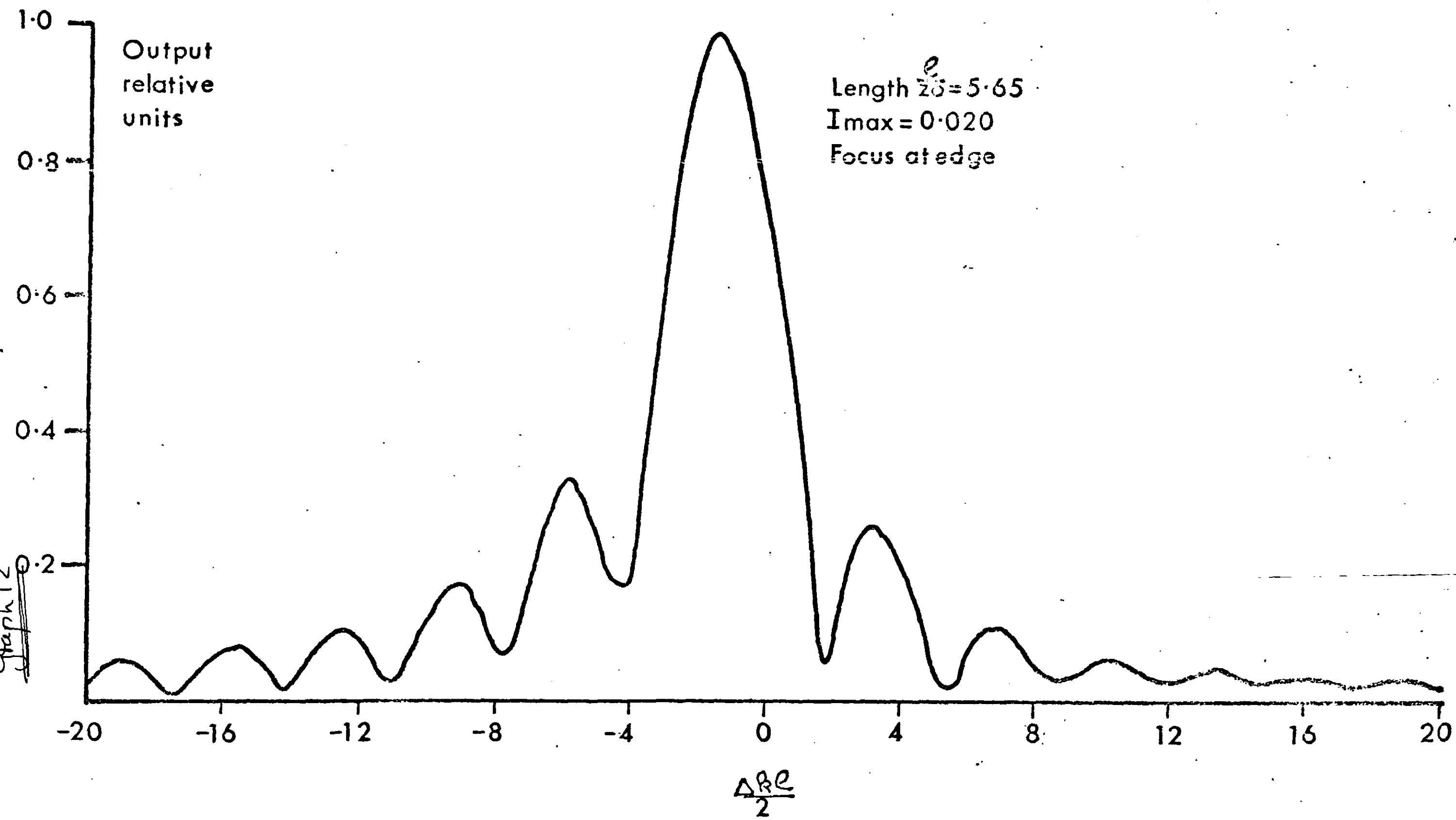

1·0
0·5
Output
relative
units
f/ℓ =0
f/ℓ = 0·25
f/ℓ =0·5
f/ℓ = 0·75
f/ℓ =1·0
f/ℓ = 1·25
Length ℓ/zo=5·6
-2·0
-1·6
-1·2
-0·8
-0·4
0
0·4
0·8
Δkzo

Graph 10

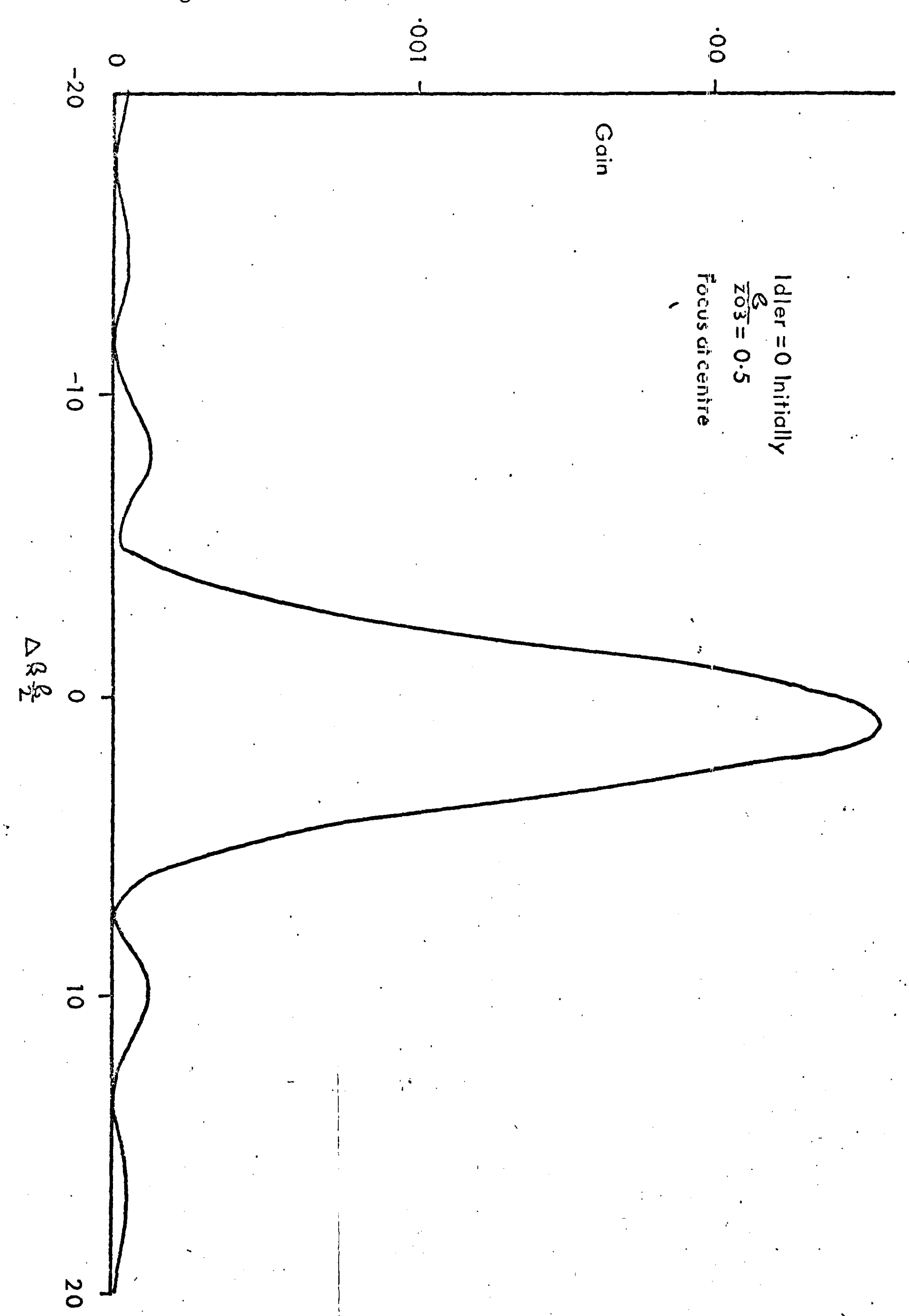

Output
relative
units
0·6
0·4
0·2
0
Maximum at $\frac{\ell}{zo}$ = 5·650
00 mode focus at centre
Maximum at $\frac{\ell}{zo}$ = 3·09
00 mode focus at edge
00 mode focus $\frac{\ell}{2zo}$ outside crystal
02 mode focus at edge
02 mode focus at edge
0·1
0·4
1·6
6·4
25·6
102·4
$\frac{\ell}{zo}$

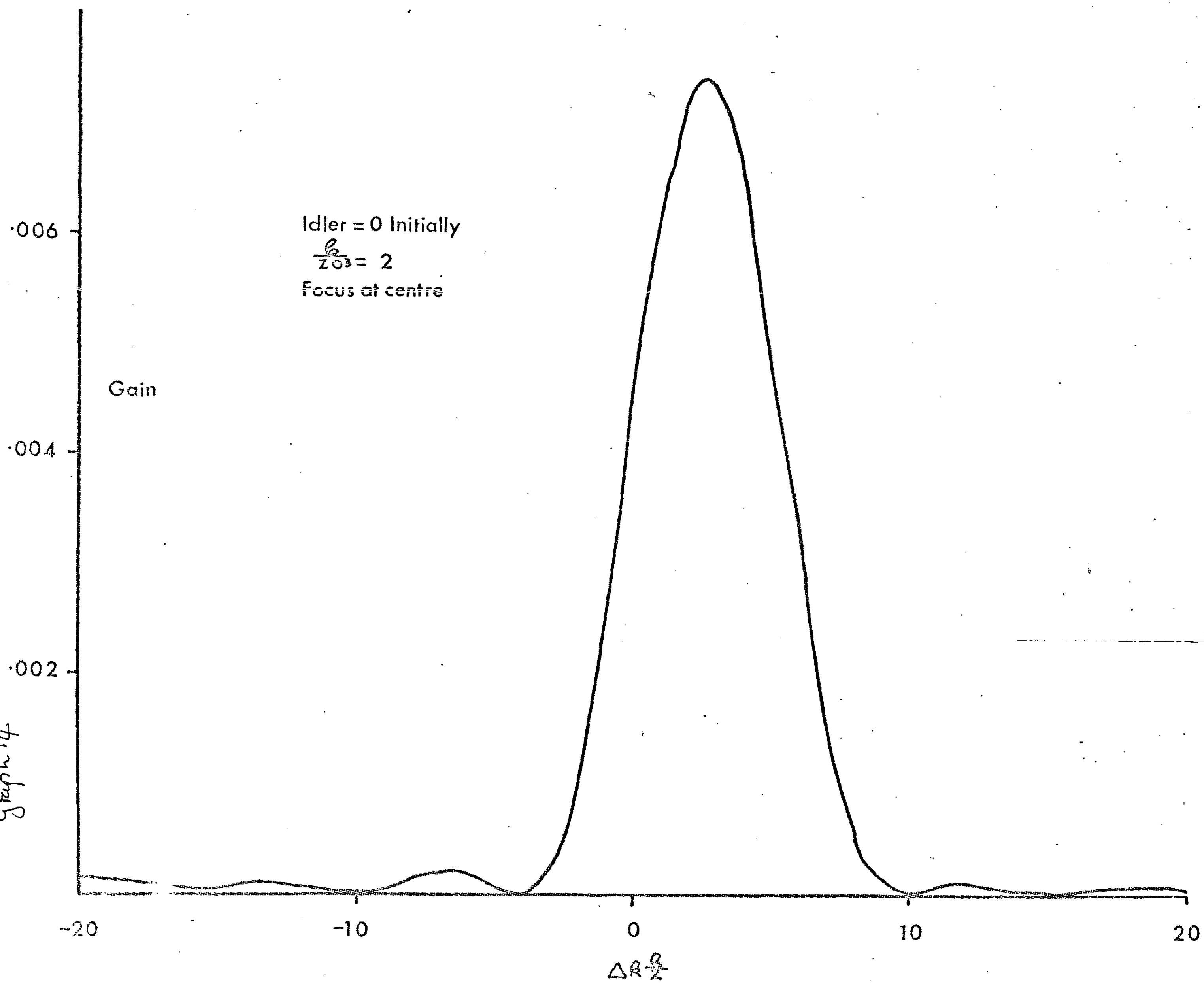

Graph 11
1·0
0·5
Output relative units
Length $\frac{\ell}{z_0}$=5·65
Imax=0·021
Focus at centre
-20
-16
-12
-8
-4
0
4
8
12
16
20
$\frac{\Delta k \ell}{2}$

Graph 12


1·0
0·8
0·6
0·4
0·2
Output
relative
units
Length $\frac{\ell}{2\delta}$ = 5·65
Imax = 0·020
Focus at edge
-20
-16
-12
-8
-4
0
4
8
12
16
20
$\frac{\Delta\beta\ell}{2}$

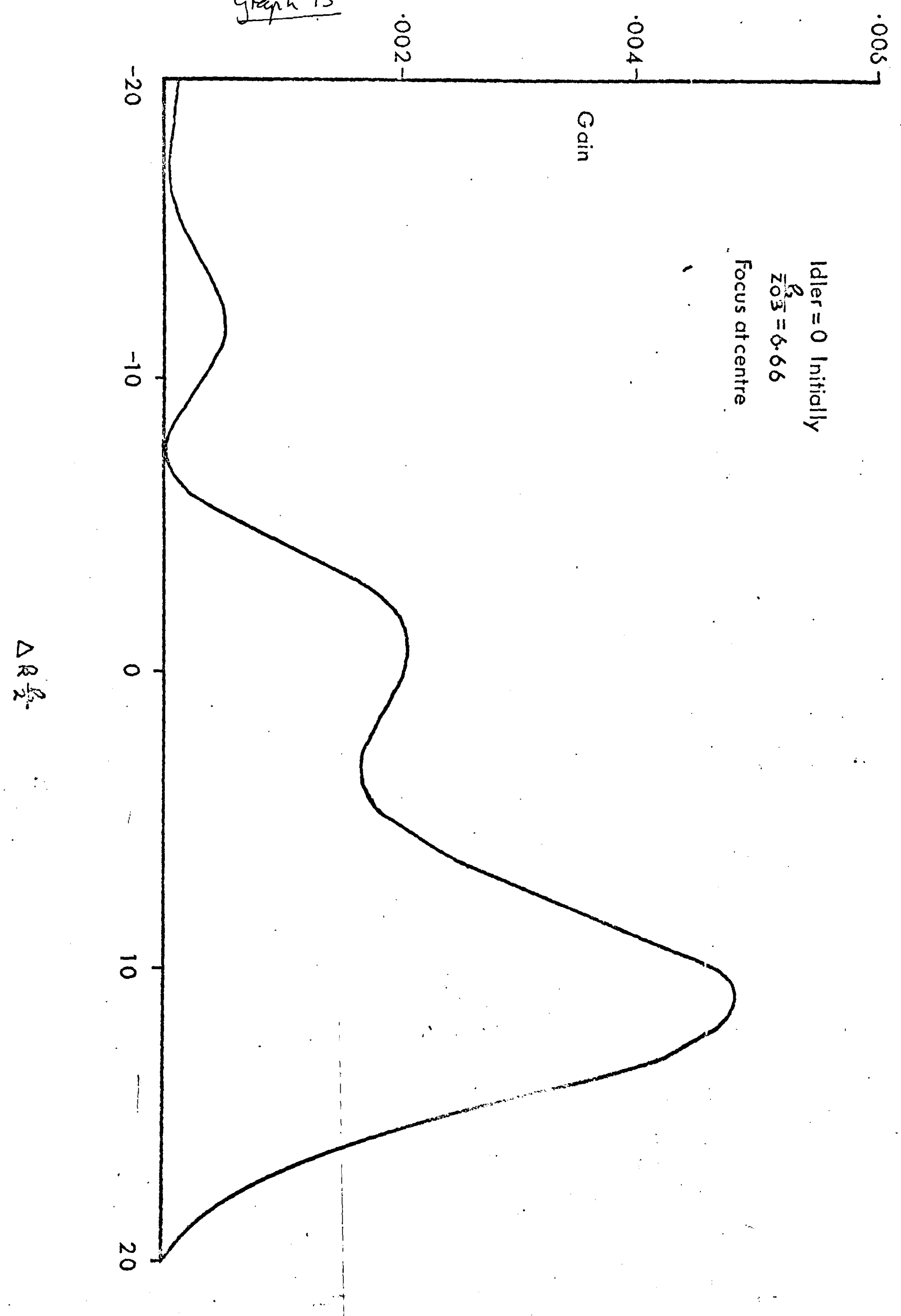
Graph 13




·006
·004
·002
Gain
Idler = 0 Initially
$\frac{b}{z_{03}} = 2$
Focus at centre
-20
-10
0
10
20
$\Delta k \frac{b}{2}$

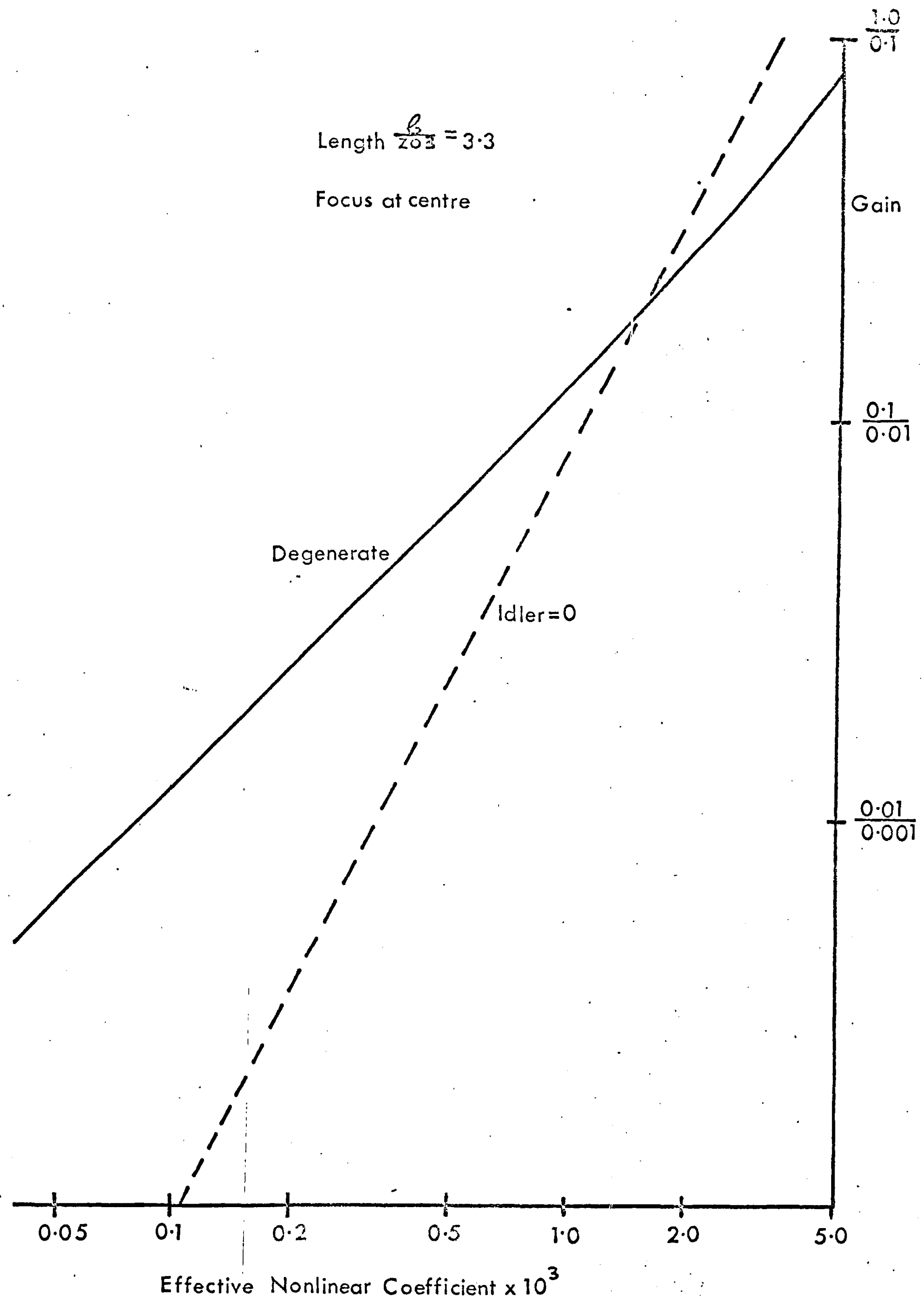
Graph 14

Graph 15



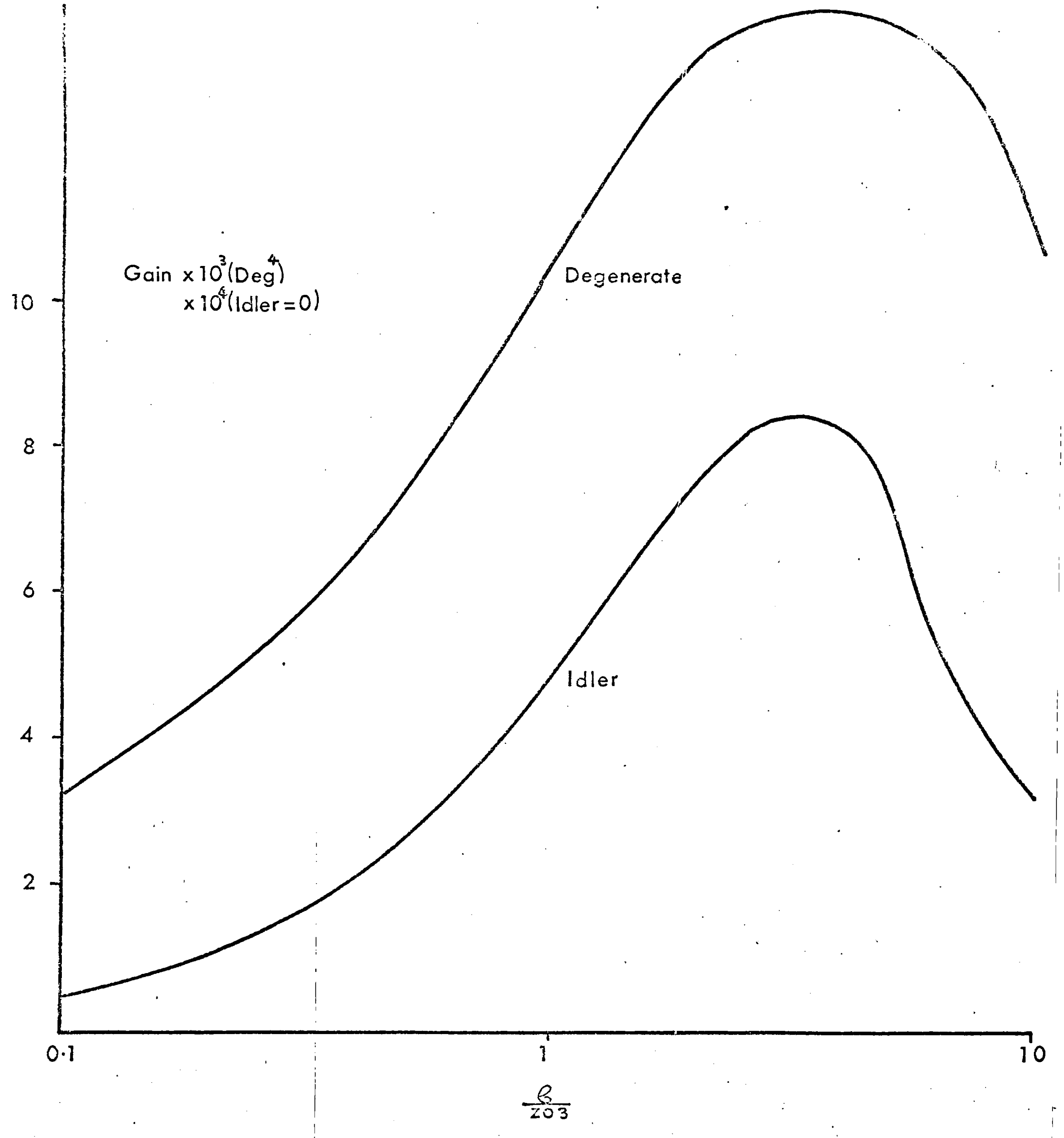
·006
·004
·002
Gain
Idler = 0 Initially
zo3 = 6·66
Focus at centre
-20
-10
0
10
20

Graph 16



Length $\frac{\ell}{z_{03}}$ = 3·3
Focus at centre
Gain
1·0 / 0·1
0·1 / 0·01
0·01 / 0·001
Degenerate
Idler=0
0·05
0·1
0·2
0·5
1·0
2·0
5·0
Effective Nonlinear Coefficient x 10^3

Graph 17



Gain x 10³ (Deg⁴)
x 10⁴ (Idler = 0)
Degenerate
Idler
10
8
6
4
2
0·1
1
10
β / z03

Conclusions

A method has been developed with which non-linear optical problems involving the integration of the modes of the optical resonator can be studied directly. This enables meaningful approximations to be made about the number of modes present in any given situation. The method developed here is for travelling waves but the same principles should apply to standing waves.

Two examples of the use of method have been given. In the first the problem of resonant second harmonic generation in the small conversion approximation has been studied: it has been shown that the results are consistent with previous theories and in the general case the results given by Boyd and Kleinman[15] for non-resonant second harmonic generation.

In applying this method to the second problem, parametric amplification, it has been shown that it gives results which agree with previous theories in the weak focussing limit. The preliminary numerical results indicate the behaviour in the general case. These results extend the previously established theory into the finite focussed region and indicate that further research will prove profitable.

Acknowledgements

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REFERENCES

1. P.A. Franken, C.W. Peters, G. Weinreich.
   Phys. Rev. Letters 7, 118 (1961)
2. F. Brown, R.E. Parks, A.M. Sleeper.
   Phys. Rev. Letters 14, 1029 (1965)
3. J.E. Midwinter, J. Warner.
   Journal of Applied Physics 38 519 (1967)
4. P.D. Maker, R.W. Terhune, C.M. Savage.
5. C.K.N. Patel.
   Applied Physics letters 9, 332 (1966)
6. L.D. Landau, E.M. Lifshitz.
   Electrodynamics of Continuous Media. Pergamon Press N.Y. 1960
7. J.A. Stratton.
   Electromagetic Theory McGaw-Hill N.Y. 1941
8. T.P. Maclean.
   Lectures on Quantum Electronics.
9. N. Bloembergen.
   Non-linear Optics. Benjamin N.Y. 1965
10. C.G.B. Garret, F.N.H. Robinson.
    J. Quantum Electronics QE-2 328 (1966)
11. F.N.H. Robinson.
    Bell System Tech. Journal XLVI No. 5 913
12. D.A. Kleinman.
    Phys. Rev. 126, 1977 (1962)
13. N.F.M. Henry, Kathleen Lonsdale (Editors)
    International Tables for X-ray Crystallography Vol. 1
    "Symmetry Groups" Kynoch Press Birmingham 1952.
14. J.F. Nye.
    The Physical Properties of Crystals. Clarendon Press Oxford 1957

15. G.D. Boyd, D.A. Kleinman
J. Applied Physics (Scheduled July 1968)

16. J.A. Armstrong, M. Bloembergen, J. Dunning, P. Pershan
Phys. Rev. 127 1918 (1962)

17. P.N. Butcher, T.P. Maclean
Proc. Phys. Soc. 81 219 1963
Proc. Phys. Soc. 83 579 1964

18. P.L. Kelley
J. Phys. Chem of Solids 24 607 (1963)

19. J.F. Ward, P.A. Franken
Phys. Rev. 133 A.183 1964

20. A. Yariv
Quantum Electronics John Wiley 1967

21. D.A. Kleinman
Phys. Rev. 128 1761 (1962)

22. P.N. Butcher
Nonlinear Optical Phenomena. Ohio State University Engineering Publication Bulletin 200

23. M.V. Hobden, J. Warner
Physics letters 22 243 (1966)

24. R.C. Miller, G.D. Boyd, A. Savage
Appl. Physics letters 6, 77.

25. G.D. Boyd, A. Ashkin
Phys. Rev. 146, 187

26. G.D. Boyd, R.E. Miller, K. Nassau, W.L. Bond, A. Savage
App. Physics letters 15, 234

27. D.A. Kleinman, G.D. Boyd
Phys. Rev. 145, 338

28. D.A. Kleinman, R.C. Miller
Phys. Rev. 148 302

29. G.D. Boyd, A. Ashkin, J.M. Dziedzic, D.A. Kleinman
Phys. Rev. 137, 1305 (1962)

30. J.E. Bjorkholm
Phys. Rev. 142, 126 (1966)

31. J.E. Bjorkholm, A.E. Seigman
Phys. Rev. 154, 851 (1967)

32. A. Ashkin, G.D. Boyd, J.M. Dziedzic
J. Quantum Electronics QE-2 109 (1966)

33. Cady. Piezoelectricity Mcgraw Hill
International in Pure and Applied Physics

34. R.H. Dicke
U.S. Patent Z, 851,652 September 9, 1958

35. A.L. Shawlow, C.H. Townes
Phys. Rev. 112 1940 (1958)

36. A.M. Prokhurov
J.E.T.P. 34 1658 (1958)

37. A.G. Fox, T. li
Bell Syst Tech. J. 40 453

38. G.D. Boyd, J.P. Gordon
Bell Syst Tech. J. 40 489

39. G.D. Boyd, H. Kogelnik
Bell Syst Tech. J. 41 1347

40. G. Toraldo di Francia
Optica Acta 13, 323 (1966)

41. C. Flammer, Spheroidal Wave Functions
Stanford Univ. Press. Palo Alto Calif. 1957

42. D.D. Bhawalker Ph.D. Thesis. Univ. of Southampton 1966

43. H. Kogelnik, T. li
Applied Optics 5, 1550 (1966)

44. H. Kogelnik
Proc. of Symposium on Quasi-optics. Pol. Inst. Brooklyn (New York) pp. 333

45. D.D. Bhawalker, A.M. Goncharenko, R.C. Smith.
Brit. J. Applied Physics 18, 1431 (1967)

46. D.D. Bhawalker, A.M. Goncharenko, R.C. Smith.
Private Communication

47. L. Bergstein, T. Zachos
J. Opt. Soc. of America 56 931

48. L. Bergstein, T. Zachos
IEEE J. of Quantum Electronics QE-2 677 (1966)

49. L.A. Vainshtein
J.E.T.P. 17 709 (1963)

50. M. Born, E. Wolf. Principles of Optics
Pergamon Press 1959

51. J.A. Giordmaine
Phys. Rev. letters 8 19 (1962)

52. P.D. Maker, R.W. Terhune, M. Nisenoff, C.M. Savage
Phys. Rev. letters 8, 21 (1962)

53. B. Ya. Zel'Dovich
J.E.T.P. 23 451 (1966)

54. S.A. Akhmanov, A.P. Sukhorukov, R.V. Khokhlov
J.E.T.P. 23 316 (1966)

55. R.H. Kingston, A.L. McWhorter
IEEE J. Quantum Electronics QE-2 4, 1965

56. W.H. Louisell. Coupled Mode Theory and Parametric Electronics
John Wiley NY. 1960

57. R.H. Pennington Introduction to Computer Methods and
Numerical Analysis Macmillan NY. 1965

58. S.A. Akhmanov, V.G. Dmitriyev, V.P. Modenov, V.V. Fadeyev,
Radiotekhnika i Elektronika, 196