

"DICHROMATIC POLYNOMIALS OF LINEAR GRAPHS"

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ABSTRACT

The dichromatic polynomial of a graph [16] (or the Tutte polynomial) is a polynomial function of two variables from which a large amount of important information about the graph may be obtained, including the chromatic polynomial and the complexity of the graph. Some properties of the Tutte polynomial and an algorithm for its computation are given.

A recursive family of graphs is defined to be a family of graphs whose Tutte polynomials satisfy a homogeneous linear recurrence relation. The smallest possible order of such a recurrence relation is called the recursiveness of the family. The existence of such a recurrence relation enables us to consider the Tutte polynomials of large graphs.

Some elementary properties of recursive families are found and two large classes of recursive families of graphs are defined. The proof that the families in these classes are recursive is constructive and the methods used are applied to some families from the two classes with small recursiveness. The problem of the location of the chromatic roots of a graph is considered in the light of the information thus gained and several conjectures are made. The most important of these is a generalisation of Brooks' theorem [6] and states that for a graph whose greatest valency is k the chromatic roots all have modulus not greater than $k + 1$.

Much of the work may be generalised immediately

to matroid theory and where this is so the appropriate results are stated.

Acknowledgements

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1. Introduction

A graph G is an ordered triple (V, E, i) where V and E are finite sets and i is an incidence function

$$i : E \rightarrow V \cup V^{[2]}$$

($V^{[2]}$ denotes the set of unordered pairs of elements of V .) The elements of V and E are called the vertices and edges of G .

If $e \in E$ and $i(e) = v \in V$ then e is called a loop, and may sometimes be denoted by $\{v, v\}$

If $i(e) = \{u, v\} \in V^{[2]}$ we usually do not distinguish between e and $\{u, v\}$ unless there are two edges e_1, e_2 such that $i(e_1) = i(e_2) = \{u, v\}$ in which case G is said to have multiple edges.

Other graph theoretic terms, where not explicitly defined are used according to the definitions in [12] (where the structure defined above is called a pseudo-graph).

For a graph $G = (V, E, i)$ define the cycle rank (rank), $m(G)$ and the co-boundary rank (co-rank) $m^*(G)$ by

$$m(G) = |E| - |V| + p_0(G)$$

$$m^*(G) = |V| - p_0(G)$$

where $p_0(G)$ denotes the number of connected components of G .

Define the Whitney Rank Polynomial of G ,
 $R(G; x, y)$ by

$$R(G; x, y) = \sum_H x^{m^*(H)} y^{m(H)}$$

where the summation is taken over all spanning
subgraphs H of G . (See [21], [22] and [17].)
This definition is equivalent to that given in [17].)

For a graph $G = (V, E, i)$ and an edge $e \in E$
define the deletion of e to be the graph

$$G_e' = (V, E - \{e\}, i')$$

where

$$i' : E - \{e\} \longrightarrow V \cup V^{121}$$

is the restriction of i to $E - \{e\}$

If in addition e is not a loop, so that

$$i(e) = \{v_1, v_2\} \quad \text{define the contraction of}$$

to be the graph

$$G_e'' = ((V - \{v_1, v_2\}) \cup \{*\}, E - \{e\}, i'')$$

where

$$\begin{array}{lll} i''(f) = u_1 & \text{if} & i(f) = u_1 \\ i''(f) = \{u_1, u_2\} & \text{if} & i(f) = \{u_1, u_2\} \\ i''(f) = \{u_1, *\} & \text{if} & i(f) = \{u_1, v_1\} \text{ or } \{u_1, v_2\} \\ i''(f) = * & \text{if} & i(f) = \{v_1, v_2\} \end{array}$$

for all adges $f \in E - \{e\}$ with u_1, u_2
denoting any two distinct vertices in $V - \{v_1, v_2\}$.

Lemma 1.1:

$$R(G; x, y) = R(G'_e; x, y) + x R(G''_e; x, y) \quad (1)$$

for any edge e of G which is not a loop.

Proof: We set up a bijection between the set of spanning subgraphs of G and the union of the sets of spanning subgraphs of G'_e and G''_e as follows.

Let G_1 be a spanning subgraph of G with edge set E_1 . If $e \notin E_1$, then G_1 is a spanning subgraph of G'_e . If $e \in E_1$, then $G_{1,e}$ is a spanning subgraph of G''_e . Let G_2 be a spanning subgraph of G'_e . Then G_2 is a spanning subgraph of G . Let G_3 be a spanning subgraph of G''_e with $G_3 = (V_3, E_3, i_3)$. Then let $H = ((V - \{*\}) \cup \{v_1, v_2\}, E_3 \cup \{e\}, j)$ where the notation is that of the definition of G''_e above and j is the restriction of i to $E_3 \cup \{e\}$. H is clearly a spanning subgraph of G .

We now observe that

$$\begin{aligned} m(G_{1,e}) &= m(G_1), \\ m^*(G_{1,e}) &= m^*(G_1) - 1, \\ m(H) &= m(G_3), \\ \text{and } m^*(H) &= m^*(G_3) + 1 \end{aligned}$$

and the result follows from the definition of $R(G; x, y)$.



A colouring of a graph G is an assignment of colours to the vertices. More precisely it is a mapping

$$c: V \rightarrow C$$

of the vertices into any finite set C .

A colouring is said to be proper if G has no loops and for every edge $\{v_1, v_2\}$ of G

$$c(v_1) \neq c(v_2).$$

For a graph G , the chromatic polynomial of G , $P(G; n)$ is a function whose value for non-negative integer values of n is the number of proper colourings of G with a set of n distinct colours. We shall assume many of the elementary properties of $P(G; n)$ which are described in

[13] including the fact that $P(G; n)$ is a polynomial of degree $|V|$ in n . The $|V|$ zeros of $P(G; n)$ are called the chromatic roots of G .

Lemma 1.2 :

$$P(G; n) = P(G'_e; n) - P(G''_e; n) \quad 2)$$

for any edge e of G that is not a loop.

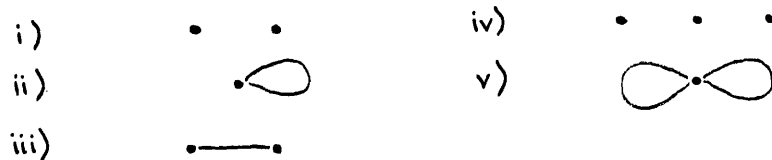
Proof : If $i(e) = \{v_1, v_2\}$, then the proper colourings of G'_e for which $c(v_1) \neq c(v_2)$ correspond to the proper colourings of G . On the other hand those for which $c(v_1) = c(v_2)$ correspond to the proper colourings of G''_e .

□

The following result is due to Birkhoff [5] and was also proved by Whitney [21] as an application of his formalisation of the principle of inclusion and exclusion.

Theorem 1.1 $P(G; n) = n^{|V|} R(G; -\frac{1}{n}, -1). \quad 3)$

Proof: If $|V| + |E| = 2$ or 3 the only possibilities are



and for these graphs both sides in the above equation take the value $n^2, 0, n(n-1), n^3, 0$ respectively.

If $|V| + |E| > 3$ and $|E| > 0$ let $e \in E$.

Then proceeding by induction

$$\begin{aligned}
 P(G; n) &= P(G'_e; n) - P(G''_e; n) \quad \text{by lemma 1.2} \\
 &= n^{|V|} R(G'_e; -\frac{1}{n}, -1) - n^{|V|-1} R(G''_e; -\frac{1}{n}, -1) \\
 &= n^{|V|} \left(R(G'_e; -\frac{1}{n}, -1) + (-\frac{1}{n}) R(G''_e; -\frac{1}{n}, -1) \right) \\
 &= n^{|V|} R(G; -\frac{1}{n}, -1) \quad \text{by lemma 1.1.}
 \end{aligned}$$

Finally if $|E| = 0$ then $P(G; n) = n^{|V|}$

and $R(G; x, y) = 1$. □

The dichromate of a graph was introduced by Tutte in

[16].

It is now often known as the Tutte polynomial and is defined as follows. (In his most recent papers Professor Tutte calls ^{this} the dichromatic polynomial and leaves the polynomial $Q(G; x, y)$,

defined in [17] and called the dichromatic polynomial, nameless.)

Consider a spanning tree T of a connected graph G . For any edge e of T , T_e' has two components T_1 and T_2 with vertex sets V_1 and V_2 ($V_1 \cup V_2 = V$). The cut set of e with respect to T is the set of edges $\{v_1, v_2\} \in E$ such that $v_1 \in V_1$ and $v_2 \in V_2$. For any edge $e = \{u_1, u_2\} \in E$ (not a loop) not in T there is a unique path $(e_1, e_2, \dots, e_p) = (\{u_1, v_1\}, \{v_1, v_2\}, \dots, \{v_{p-1}, u_2\})$ in T such that $v_{i-1} \neq v_{i+1}$

for $i = 2, 3, \dots, p-2$ and $v_2 \neq u_1, v_{p-2} \neq u_2$.

The set of edges $\{e_1, e_2, \dots, e_p\}$ in this path is the circuit of e with respect to T .

Suppose the edges E of G are ordered by a total ordering $<$. An edge e of T is defined to be internally active with respect to T if for every edge e' in the cut set of e

$$e > e'.$$

An edge e of G not in T is externally active with respect to T if for every edge e' in the circuit of e

$$e > e'.$$

An isthmus of G is internally active with respect to all spanning trees of G . A loop of G is externally active with respect to all spanning trees of G .

For a spanning tree T of the graph G the internal and external activities of T , $i(T)$ and $e(T)$ are the number of edges of G which are internally and externally active with respect to T .

For a connected graph G the Tutte polynomial $\chi(G; x, y)$ of G is defined by

$$\chi(G; x, y) = 1 \quad \text{if} \quad |E| = 0$$

$$\chi(G; x, y) = \sum_T x^{i(T)} y^{e(T)} \quad \text{otherwise,}$$

where the summation is taken over all spanning trees T of G .

Tutte proves in [16] that this polynomial is independent of the ordering $<$ placed on the set E .

The definition is extended to graphs which are not connected by defining the Tutte polynomial of a graph G to be the product of the Tutte polynomials of the connected components of G .

We state with out proof some elementary properties of $\chi(G; x, y)$ given in [16].

- i) $\chi(G; x, y)$ is a polynomial of degree $m^*(G)$ in x and $m(G)$ in y .
- ii) For any edge $e \in E$ which is neither a loop nor an isthmus

$$\chi(G; x, y) = \chi(G'_e; x, y) + \chi(G''_e; x, y).$$

iii) If G is a graph with $a(G)$ loops and $b(G)$ isthmuses and no other edges then

$$\chi(G; x, y) = x^{b(G)} y^{a(G)}. \quad 5)$$

iv) If G is composed of two connected graphs G_1, G_2 having just one vertex in common then

$$\chi(G; x, y) = \chi(G_1; x, y) \cdot \chi(G_2; x, y). \quad 6)$$

The number of spanning trees of G is called the complexity of G and denoted by $C(G)$.

$$v) \quad \chi(G; 1, 1) = C(G)$$

Theorem 1.2 : $\chi(G; x, y) = (x-1)^{|V|-P_0(G)} R(G; \frac{1}{x-1}, y^{-1}). \quad 7)$

Proof : If G has $a(G)$ loops and $b(G)$ isthmuses and no other edges then

$$\chi(G; x, y) = x^{b(G)} y^{a(G)}$$

A spanning subgraph S of G with i isthmuses and j loops has

$$m(S) = j$$

$$m^*(S) = i$$

$$\text{Hence } R(G; x, y) = \sum_{i,j} \binom{b(G)}{i} \binom{a(G)}{j} x^i y^j$$

$$= (x+1)^{b(G)} (y+1)^{a(G)}$$

$$\therefore (x-1)^{|V|-P_0(G)} R(G; \frac{1}{x-1}, y^{-1}) = (x-1)^{b(G)} \left(\frac{x}{x-1}\right)^{b(G)} y^{a(G)}$$

$$= \chi(G; x, y)$$

For other graphs G we proceed by induction on $|V| + |E|$. If $|V| + |E| = 2$ or 3 there are 5 possible G (see Theorem 1.1) and the theorem is true for these by

the argument above.

If $|V| + |E| > 3$ let e be an edge of G that is neither a loop nor an isthmus. Then

$$\begin{aligned} \chi(G; x, y) &= \chi(G_e'; x, y) + \chi(G_e''; x, y), \text{ lemma 1.2} \\ &= (x-1)^{|V|-p_0(G)} R(G_e'; \frac{1}{x-1}, y-1) \\ &\quad + (x-1)^{|V|-p_0(G)-1} R(G_e''; \frac{1}{x-1}, y-1) \\ &= (x-1)^{|V|-p_0(G)} \left(R(G_e'; \frac{1}{x-1}, y-1) + \frac{1}{x-1} R(G_e''; \frac{1}{x-1}, y-1) \right) \\ &= (x-1)^{|V|-p_0(G)} R(G; \frac{1}{x-1}, y-1), \text{ lemma 1.1.} \quad \square \end{aligned}$$

Corollary : $P(G; n) = (-1)^{|V|-p_0(G)} n^{p_0(G)} \chi(G; 1-n, 0). \quad 8)$

Proof: This is an immediate consequence of theorems 1.1 and 1.2. □

in other words

$$P(G; n) = n^{p_0(G)} \sum_{i=1}^{n^*(G)} \alpha_i (1-n)^i \quad 9)$$

where α_i is the number of spanning trees of G whose internal activity is i and whose external activity is 0 .

Because of theorem 1.2 and its corollary statements 1), 2) and 4) are different interpretations of the same relation. This relation is fundamental in what follows and will be referred to as relation *

* * * * *

we are primarily concerned with properties of colourings of graphs. However a graph may be considered as a particular kind of matroid and some of our results will be expressed as results about these more general structures.

A matroid M over a finite set E is defined by a class of ^{non-empty} subsets of E called the circuits of M and satisfying

- 1) No circuit is a proper subset of another ;
- 2) If X and Y are two circuits of M and $e_1, e_2 \in E$, are such that $e_1 \in X \cap Y$ and $e_2 \in X - Y$ then there is a circuit Z of M such that $e_2 \in Z \subseteq X \cup Y - \{e_1\}$.

The elements of E are the cells or edges of M . If E is the set of edges of a graph G then the circuits of G form the circuits of a matroid, called the circuit matroid of G , and the cut sets of G form the circuits of the bond matroid of G . A matroid which is the bond matroid or circuit matroid of a graph is called cographic or graphic respectively (See [20]).

If S is a subset of E , S is called independent if it contains no circuit of M . The rank of any set $S \subseteq E$ is the number of cells in a maximal independent subset of S . It is a consequence of axiom 2) that rank is well defined. The bases of M are the maximal independent subsets of E . If $S \subseteq E$ is maximal among subsets of E with the same rank, then S is closed.

Consider an order $<$ placed on the cells of a matroid M and extended lexicographically to an order on subsets of E having the same number of elements. Let T be a basis of M . Define T^- to be the smallest subset of T (least number of elements) such that the $<$ maximum basis of M containing T^- is T , and define T^+ to be the largest set containing T such that the $<$ minimum basis of M contained in T^+ is T . Put

$$i(T) = |T - T^-| ,$$

$$e(T) = |T^+ - T| .$$

Define the Tutte polynomial of M by

$$\chi(M; x, y) = \sum_T x^{i(T)} y^{e(T)}$$

with the summation taken over all bases T of M .

This is a direct generalisation of the definition

for graphs and is due to Crapo [8] who showed that the definition is independent of the ordering $<$. Also if M^* denotes the dual matroid of M (see [20]) then

$$\chi(M^*; x, y) = \chi(M; y, x).$$

C.A.B. Smith [15] proposes an alternative mode of definition for the Tutte polynomial which is as follows.

For a matroid M over E with $e \in E$ let the reduction of M by e , $R_e M$ be the matroid over $E - \{e\}$ whose circuits are those of M which do not contain e . Let the contraction of M by e , $C_e M$ be the matroid over $E - \{e\}$ whose circuits are ^{minimal sets from, the circuits} ~~those~~ of $R_e M$ together with those subsets $S \subseteq E$ such that $S \cup \{e\}$ is a circuit of M .

A cell e of M is an isthmus if it is contained in no circuit of M and is a loop if $\{e\}$ is a circuit of M . A sequence of matroids is a condensation of M if it is obtained from M by successive replacement of matroids M' by $R_e M'$ and $C_e M'$ for any cell e of M' which is neither a loop nor an isthmus. A matroid is degenerate if all its circuits are loops (or it has no circuits). A complete condensation of M is a condensation, all of whose members are degenerate. Smith shows that there is only one complete condensation of M .

The Tutte polynomial of a degenerate matroid M with $|E|$ cells and $a(M)$ loops is defined to be

$$\chi(M; x, y) = x^{|E| - a(M)} y^{a(M)} .$$

The Tutte polynomial of any matroid M is the sum of the Tutte polynomials of the members of the complete condensation of M .

It is not difficult to show the equivalence of these two definitions.

* * * * *

It is often convenient to denote any of the polynomials defined above for a graph G (or matroid M) just by $G = G(x, y)$ (or $M = M(x, y)$) instead of $\chi(G; x, y)$ etc. and where the context makes it clear which polynomial is being referred to, this will be done.

2. An algorithm.

In this chapter we describe an algorithm for the computation of the Tutte polynomial of a graph.

Let $G = (V, E, i)$ be a graph with no loops. (A loop just introduces a factor y into the Tutte polynomial so this restriction involves no loss of generality.) Let $V = \{v_1, v_2, \dots, v_n\}$, $n = |V|$, and $E = \{e_1, e_2, \dots, e_m\}$, $m = |E|$. For each $e \in E$ define an ordering on the pair $i(e) \in V^{\{2\}}$ so that e may be written $(v_i, v_k) \in V \times V$.

The incidence matrix $F = (f_{ij})$ of G is an $n \times m$ matrix whose entries are given by

$$f_{ij} = 1 \quad \text{if } e_j = (v_i, v_k) \text{ for any } k,$$

$$f_{ij} = -1 \quad \text{if } e_j = (v_k, v_i) \text{ for any } k,$$

$$f_{ij} = 0 \quad \text{otherwise.}$$

Denote by F_0 the $(n-1) \times m$ matrix whose rows are the first $n-1$ rows of F . For any set E' of $n-1$ edges partition F_0 into $[F_1 | F_2]$ where F_1 is an $(n-1) \times (n-1)$ square matrix whose columns correspond to the edges of E' . F_2 is accordingly an $(n-1) \times (m-n+1)$ matrix.

Lemma 2.1 F_1 is non-singular if and only if the edges E' form a spanning tree of G .

Proof : If E' is a tree it is a spanning tree since it has $n - 1$ edges (see [12]).

If E' is not a tree then some subset of E' must form a circuit. In this case the columns of F_1 corresponding to the edges of the circuit are linearly dependent and so $\det F_1 = 0$.

Conversely suppose E' is a tree. Consider the vertices of G to be ordered according to their distance in the tree from v_n , with the furthest vertices first in the ordering, and the ordering of vertices the same distance from v_n being arbitrary. Further consider the edges of E' to be ordered according to the position of their end points remote from v_n in the ordering of the vertices. Permuting the rows and columns of F_1 according to these orderings yields a matrix F_1' having

$$|\det F_1'| = |\det F_1|$$

Also F_1' is a lower triangular matrix whose diagonal elements are all ± 1 .

Hence $\det F_1 \neq 0$ □

For a spanning tree T of the directed graph G define the circuit matrix $C_T = (c_{ij}^T)$ and the cut set matrix $K_T = (k_{ij}^T)$ of T with respect to G as follows.

C_T is an $(m - n + 1) \times m$ matrix whose rows correspond to the edges of G not in T and whose columns correspond to the edges of G . c_{ij}^T will be used to denote the entry of C_T corresponding to edges e_i, e_j rather than the (i, j) entry. If e_i is an edge not in T then the direction of e_i induces a direction on the edges of the circuit of e_i with respect to T in the obvious way.

Put $c_{ij}^T = +1$ if e_j is in the circuit of e_i and has direction the same as that induced by e_i ,

$c_{ij}^T = -1$ if e_j is in the circuit of e_i and has direction different to that induced by e_i ,

$c_{ij}^T = 0$ otherwise.

K_T is an $(n - 1) \times m$ matrix whose rows correspond to the edges of T . It is defined similarly to C_T .

Namely $k_{ij}^T = +1$ if e_j is in the cut set of e_i and has direction the same as that induced

by e_i ,

$k_{ij}^T = -1$ if e_j is in the cut set of e_i and has direction different to that induced

by e_i ,

$k_{ij}^T = 0$ otherwise.

For the purposes of this algorithm we assume that an

edge is a member of its own cut set or circuit so by suitably permuting the columns of C_T and K_T we can partition them into

$[I_{m-n+1} | C_{T\pi}]$ and $[K_{T\pi} | I_{n-1}]$ respectively.

Lemma 2.2 : i) $C_{T\pi} = - (F_1^{-1} F_2)^t$
 ii) $K_{T\pi} = F_1^{-1} F_2$

(index t denotes the transpose of a matrix).

Proof: We show that

i') $F_0 C_T^t = 0$
 ii') $C_T K_T^t = 0$

which may be written in terms of the partitions as

and $F_2 + F_1 C_{\pi}^t = 0$
 $K_{T\pi} + C_{\pi} = 0$

respectively, which yield the results.

Row i of F has a non zero entry for each edge incident at vertex v_i . Row j of C_T has a non zero entry for each edge in the circuit of e_j with respect to T . There are either 0 or 2 edges in this circuit incident with vertex v_i . In the former case $(F_0 C_T^t)_{ij} = 0$ clearly. In the latter case let e_{k_1}, e_{k_2} be the two edges in the circuit incident with v_i . There are two possibilities. Either the directions of e_{k_1} and e_{k_2} are both the same as or both different to the direction induced by e_j or one has direction the same as and the other has direction different to that induced by e_j . In the first case one of e_{k_1}, e_{k_2} is directed

towards v_i and the other away from v_i ,
 in the second both are directed to or both away
 from v_i .

In the first case $c_{jk_1}^T = c_{jk_2}^T$ and $f_{ik_1} = -f_{ik_2}$
 and in the second case $c_{jk_1}^T = -c_{jk_2}^T$ and $f_{ik_1} = f_{ik_2}$

Hence $f_{ik_1} c_{jk_1}^T + f_{ik_2} c_{jk_2}^T = 0$ in any case
 and i') follows.

A similar argument yields ii'). □

Now an edge e_i not in a spanning tree T
 of G is externally active with respect to T if
 and only if $c_{ij}^T = 0$ for all $j > i$
 and an edge e_i in T is internally active with
 respect to T if and only if $k_{ij}^T = 0$
 for all $j > i$.

Hence lemmas 2.1 and 2.2 enable us to find
 the Tutte polynomial of a graph G from its
 incidence matrix F simply by checking the
 determinants of all possible F_i matrices and where
 appropriate looking at $F_1^{-1} F_2$. A computer
 program based on this algorithm is simple to write
 using standard subroutines for the matrix operations
 and the only data input required is the incidence
 matrix.

It should be noted that this is not the best
 possible algorithm from the point of view of computer
 time, the inversion of a matrix being a fairly
 lengthy operation. However the alternative is to

base the algorithm on the recursion * , and the choice of the edge to delete and contract at each stage and the recognition of graphs that require no further processing both increase the complexity of the program and the amount of data input required by a large factor.

As an appendix to this chapter we list the chromatic polynomials and chromatic roots of some small graphs, including all trivalent graphs with no more than 10 vertices as listed in [1] , which were computed by the matrix method on the university of London CDC 6600 machine.

Appendix.

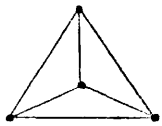
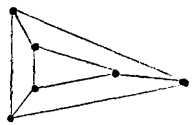
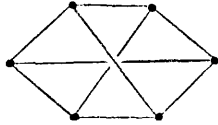
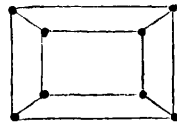
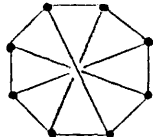
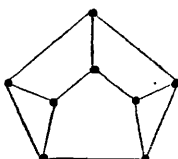
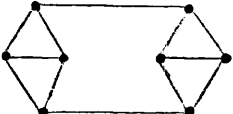
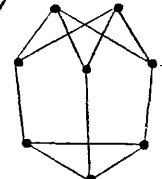
In the following list the graph diagrams are followed by a sequence of coefficients $\alpha_1, \alpha_2, \dots, \alpha_{M-1}$ where

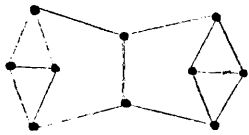
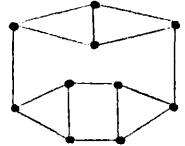
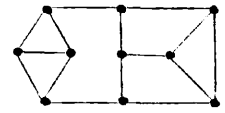
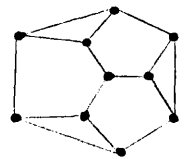
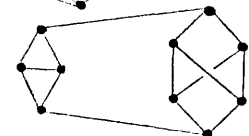
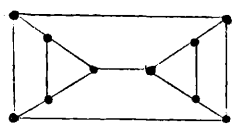
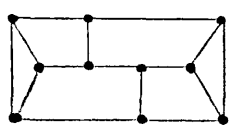
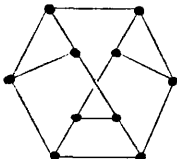
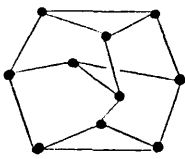
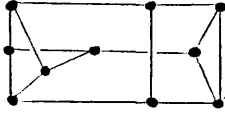
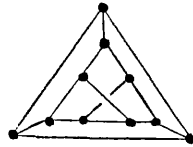
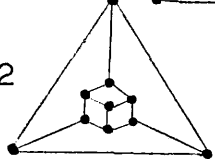
$$P(G; \lambda) = \lambda \sum_{i=1}^{M-1} \alpha_i (1-\lambda)^i$$

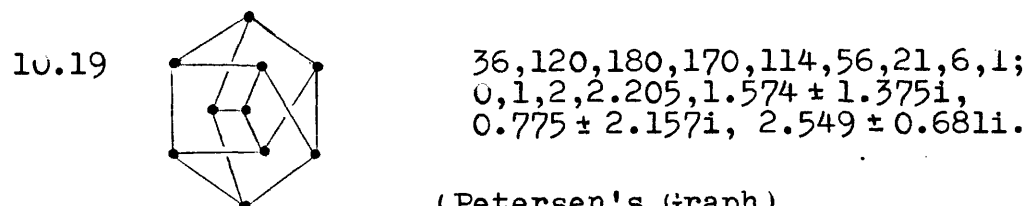
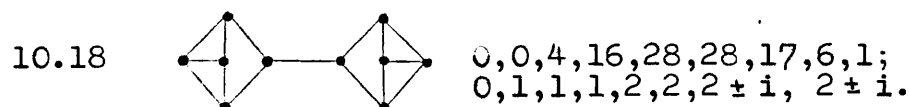
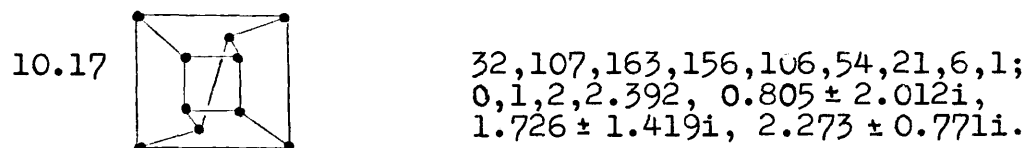
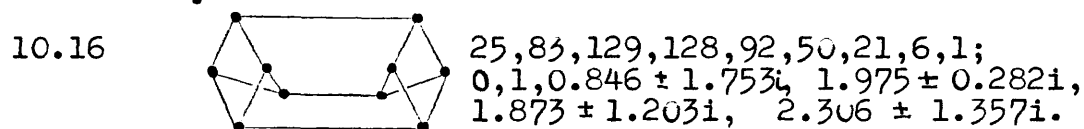
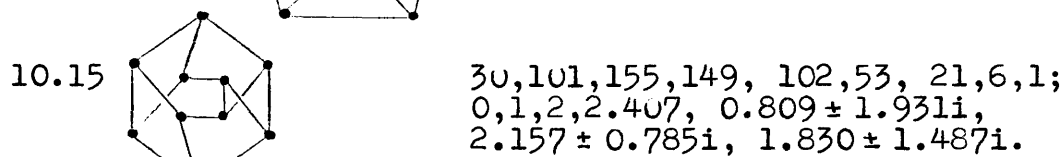
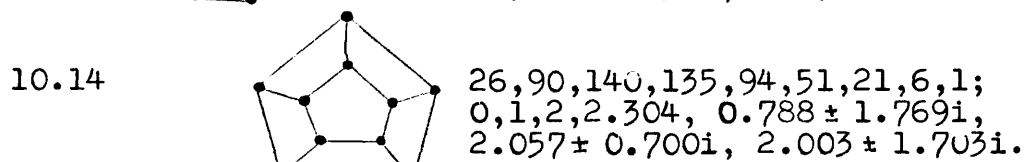
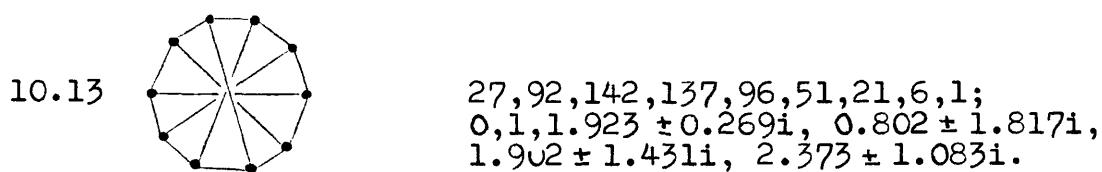
and then by the roots of the equation

$$P(G; \lambda) = 0$$

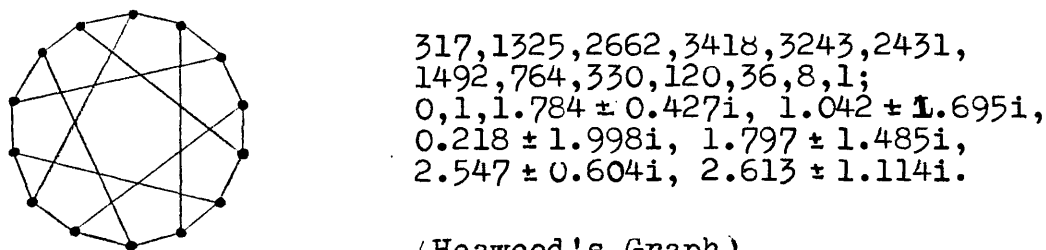
(these roots being correct to 3 decimal places).

4.1		2,3,1 ; 0,1,2,3.
6.1		4,9,8,4,1; 0,1,2,2.453, 1,773 ± 1,468i
6.2		5,11,10,4,1; 0,1,1.859 ± 0.492i, 2.141 ± 1.949i
8.1		11,32,40,29,15,5,1; 0,1,1.238 ± 1.856i, 1.863 ± 0.339i, 2.399 ± 1.304i.
8.2		12,34,42,31,15,5,1; 0,1,2,2.433, 2 ± i, 1.284 ± 2.027i.
8.3		8,23,29,33,13,5,1; 0,1,2,2.329, 2.226 ± 0.930i , 1,110 ± 1,591i.
8.4		4,12,16,15,11,5,1; 0,1,2,2, 2.545 ± 0.716i, 0.955 ± 1.174i.
8.5		10,28,35,27,14,5,1; 0,1,2,2.526, 2 ± i, 1.237 ± 1.795i.

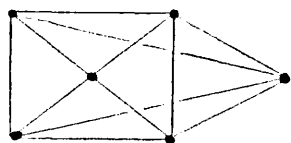
- 10.1  4.16, 32, 44, 44, 32, 17, 6, 1;
0, 1, 2, 2, 2.544, 2.544,
1.228 ± 1.115i, 1.228 ± 1.115i.
- 10.2  8, 28, 44, 47, 42, 31, 17, 6, 1;
0, 1, 2, 2, 0.654 ± 1.147i,
1.851 ± 1.273i, 2.495 ± 0.380i.
- 10.3  8, 30, 52, 60, 53, 36, 18, 6, 1;
0, 1, 2, 2, 0.853 ± 1.255i,
1.748 ± 1.318i, 2.399 ± 0.475i;
- 10.4  16, 54, 83, 83, 63, 38, 18, 6, 1;
0, 1, 2, 2.379, 0.667 ± 1.504i,
1.708 ± 1.296i, 2.435 ± 0.425i.
- 10.5  10, 36, 60, 66, 57, 39, 19, 6, 1;
0, 1, 2, 2.450, 0.797 ± 1.230i,
1.941 ± 0.517i, 2.038 ± 1.666i.
- 10.6  16, 55, 86, 87, 67, 41, 19, 6, 1;
0, 1, 2, 2.265, 0.735 ± 1.466i,
2.232 ± 0.645i, 1.901 ± 1.461i.
- 10.7  16, 56, 90, 93, 71, 42, 19, 6, 1;
0, 1, 2, 2.271, 0.786 ± 1.569i,
1.777 ± 1.328i, 2.302 ± 0.653i.
- 10.8  20, 66, 101, 101, 75, 43, 19, 6, 1;
0, 1, 2, 2.478, 0.761 ± 1.628i,
1.656 ± 1.358i, 2.344 ± 0.625i.
- 10.9  20, 68, 106, 107, 79, 44, 19, 6, 1;
0, 1, 2, 2.193, 0.781 ± 1.736i,
1.565 ± 1.298i, 2.558 ± 0.552i.
- 10.10  20, 68, 107, 108, 80, 46, 20, 6, 1;
0, 1, 2, 2.347, 0.814 ± 1.618i,
2.087 ± 0.797i, 1.926 ± 1.492i.
- 10.11  24, 81, 125, 123, 88, 48, 20, 6, 1;
0, 1, 2, 2.366, 0.804 ± 1.780i,
1.714 ± 1.423i, 2.299 ± 0.689i.
- 10.12  22, 76, 119, 117, 84, 47, 20, 6, 1;
0, 1, 2, 2.468, 0.772 ± 1.700i,
2.106 ± 0.633i, 1.888 ± 1.534i.



(Petersen's Graph)

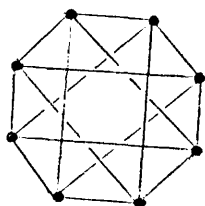


(Heawood's Graph)



11,25,20,7,1;
0,1,2,2.547, 3.227 ± 1.468i.

$(K_{2,2,2})$, (Octahedron)

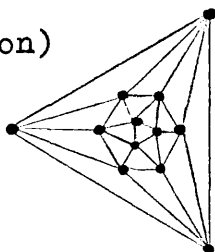


73,119,219,129,45,9,1;
0,1,2.554 ± 1.361i, 2.802 ± 3.097i,
2.143 ± 0.157i.

$(K_{4,4})$

The final two polynomials, which follow, were not obtained using the algorithm of chapter 2 but are included for completeness. Because of the size of the coefficients they have been left in the form in which they were found.

(Icosahedron)



20170, -40240, 36408,
-19698,6999,-1670,260,
-24,1;
0,1,2,3,2.618,3,223,
3.617 ± 1.776i, 3.755 ± 0.404i,
1.707 ± 2.721i.

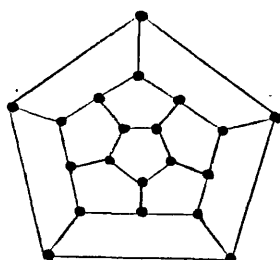
The coefficients given are b_0, b_1, \dots, b_8

where $P(G;n) = n(n-1)(n-2)(n-3) \sum_{i=0}^8 b_i n^i$.

(Dodecahedron)

The coefficients given are c_0, c_1, \dots, c_{17}

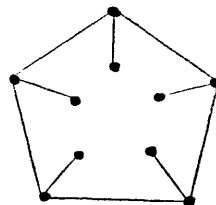
where $P(G;n) = n(n-1)(2-n) \sum_{i=0}^{17} c_i (1-n)^i$.



4412,21302,50648, 80332,
96556, 94118,77702, 56048,
36052, 20890, 10912, 5104,
2112,759, 230,56,10,1;
0,1,2,2.660, 2.366 ± 0.916i,
2.089 ± 0.363i, 1.853 ± 1.401i,
2.539 ± 0.235i, 1.311 ± 1.521i,
0.636 ± 1.999i, -0.009 ± 1.469i,
1.385 ± 1.862i.

Notes:

- 1) The chromatic polynomial of Heawood's graph was found in co-operation with N.L. Biggs and R.M. Damarell.
- 2) The chromatic polynomial of the Icosahedron is due to Whitney [22]
- 3) The chromatic polynomial of the Dodecahedron was found using a reduction for graphs containing the configuration.



with other edges of the graph incident only at the univalent vertices of this configuration.

The dodecahedron contains the configuration twice and the chromatic polynomial is readily found.

The calculations have been carefully checked but if further verification is required this polynomial gives 7,200 3-colourings of the dodecahedron which agrees with the 144 given in [2] after taking account of permutations not considered there. Also, at the suggestion of Professor Tutte, the value at $1 + \tau$ ($= 2.618\dots$, see chapter 6) has been compared with that of the dual of the truncated icosahedron [10] and shown to satisfy 2.2 of [18].

3. Properties of the Tutte Polynomial.

The first results in this chapter are reductions of the Tutte polynomial of composite graphs; which are related to the basic relation *

$$\chi(G; x, y) = \chi(G'_e; x, y) + \chi(G''_e; x, y)$$

for any edge e of G that is neither a loop nor an isthmus. The rest of the chapter is made up of relations between the coefficients of the Tutte polynomial and such invariants of the graph as the girth and connectivity.

Lemma 3.1: Let H be the graph constructed from the graph $G = (V, E, \dot{c})$ by introducing a new vertex v and joining it by an edge to each member of a subset $W \subseteq V$

$$\text{So } H = (V \cup \{v\}, E \cup W', j) \quad , \quad |W'| = |W|$$

where

$$j|E = i$$

$$j|W' = \{w, v\} \quad \text{for } w' \in W'$$

Then the Tutte polynomial of H is given by

$$H = (x-1)G + \sum_{S \subseteq W} G/S \quad 1)$$

where S ranges over all non-empty sub-sets of W and G/S denotes the graph obtained from G by identifying all the vertices in S .

$$G/S = (V \cup \{v\} - S, E, i)$$

where if $i(e) = \{u_1, u_2\}$

$$\begin{aligned}
 i_1(e) &= \{u_1, u_2\} & \text{if } u_1, u_2 \notin S \\
 &= \{u_1, v\} & \text{if } u_2 \in S, u_1 \notin S \\
 &= v & \text{if } u_1, u_2 \in S
 \end{aligned}$$

if $W = V$, H is sometimes called the cone on G .

Proof: Let the vertices of W be ordered by a total ordering $<$, and denote them by $\{1, 2, \dots, p\}$.

Denote by H_k the graph obtained from H by deleting those edges joining v to vertices $w \in W$ satisfying

$$w > k. \quad \text{Thus } H_p = H.$$

For $S \subseteq W$, $|S| > 1$, $S = \{v_1, v_2, \dots, v_r\}$ with $v_1 < v_2 < \dots < v_r$, denote by G^S the graph obtained from G/S by joining by an edge not in G the vertex v to those vertices $w \in W$ satisfying $v_{r-1} < w < v_r$.

Finally denote by K_q the graph constructed from G by joining vertex $q+1$ by an edge to each vertex $w \in W$ satisfying $w \leq q$ ($q < p$).

Now applying * we have (all for Tutte polynomials)

$$\begin{aligned}
 H_1 &= xG, \\
 H_k &= H_{k-1} + K_{k-1}, \quad 1 < k \leq p, \\
 H &= H_p
 \end{aligned}$$

and so

$$H = xG + K_1 + K_2 + \dots + K_{p-1} \quad 2)$$

Also
$$K_k = G/\{k\} + G^{\{1, k\}} + G^{\{2, k\}} + \dots + G^{\{k, k\}}, \quad k \leq p-1,$$

$$G^{\{k_1, k_2\}} = G/\{k_1, k_2\} + G^{\{k_1, k_1+1, k_2\}} + G^{\{k_1, k_1+2, k_2\}} + \dots + G^{\{k_1, k_2-1, k_2\}},$$

and generally

$$\begin{aligned}
 G^{\{k_1, k_2, \dots, k_r\}} &= G/\{k_1, k_2, \dots, k_r\} + G^{\{k_1, k_2, \dots, k_{r-1}, k_{r-1}+1, k_r\}} \\
 &\quad + \dots + G^{\{k_1, \dots, k_{r-1}, k_{r-1}, k_r\}}
 \end{aligned} \quad 3)$$

$$\text{for } k_1 < k_2 < \dots < k_r \leq p$$

Note that $G/\{k\} = G$ for all $k \in W$ and if

$$S = \{k_1, k_2, \dots, k_{r-1}, k_{r-1}+1\}, \quad G^S = G/S.$$

Hence using 2) and 3) we obtain an expression for H in the form

$$\begin{aligned} H &= \alpha G + \sum_{\substack{S \in W \\ S \neq \{p\}}} c_s G/S \\ &= (\alpha - 1)G + \sum_{S \in W} c_s G/S \end{aligned} \quad 4)$$

for some coefficients c_s .

To complete the proof we show by induction that for all r satisfying $1 \leq r \leq p$ in the partial expansion of $H - \alpha G$ involving only terms G/S and G^T where $|S| \leq r-1$ and $|T| \leq r$ all coefficients are +1 (except $G/\{p\}$ having coefficient 0).

Equation 2) yields the result for $r = 2$ on substituting once for the k_1, k_2, \dots, k_{p-1} .

If $|S| = r+1$ let $S = \{k_1, \dots, k_{r+1}\}, k_1 < \dots < k_{r+1}$.

Then applying 3) to the r^{th} partial expansion of

$H - \alpha G$ a term G^S only occurs in the expansion of $G^{S - \{k_i\}}$ and in that expansion it has coefficient 1.

Similarly for terms G/S

Thus by induction the result is true when $r = p$ so in 4) $c_s = 1$ for all $S \in W$. \square

Lemma 3.2 : Let $G = (V_1, E_1, i_1)$, $H = (V_2, E_2, i_2)$ be graphs and let $e_1 \in E_1$, $i_1(e_1) = \{u_1, v_1\}$ and $e_2 \in E_2$, $i_2(e_2) = \{u_2, v_2\}$. Denote by G' , G'' , H' , H'' the graphs G_{e_1}' , G_{e_1}'' , H_{e_2}' , H_{e_2}'' respectively. Let G_1 be the graph formed by identifying in G' and H' the vertices u_1 and u_2 and the vertices v_1 and v_2 and let G_2 be the

graph formed by identifying v_1 and v_2 the edges e_1 and e_2 (so that G_2 is just G_1 with an additional edge).

Then the Tutte polynomials of G_1 and G_2 are given by

$$i) \quad G_1 = \frac{1}{xy-x-y} [(y-1)G'H' - G'H'' - G''H' + (x-1)G''H'']$$

$$ii) \quad G_2 = \frac{1}{xy-x-y} [(y-1)G'H' - G'H'' - G''H' + (xy-y-1)G''H''].$$

(This second result reduces to the well known result for chromatic polynomials given in [13] when the value $y = 0$ is substituted.)

Proof : i) We proceed via the Whitney Rank polynomial and theorem 1.2.

Recall that $R(G; x, y)$ is defined by

$$R(G; x, y) = \sum_H x^{m^*(H)} y^{m(H)} \quad 5)$$

the summation being over all spanning subgraphs H of G .

Now there is a bijection between the spanning subgraphs of G_1 and pairs of spanning subgraphs of G' and H' . Represent this by

$$G_i \longleftrightarrow (G'_i, H'_i)$$

Then the expression 5) for $R(G; x, y)$ may be split into four parts corresponding to the four possibilities of whether or not u_1 and v_1 are in the same component of G'_i and u_2 and v_2

are in the same component of H'_i .

$$\text{Also } R(G'_i; x, y) = \sum_{G'_i} x^{n^*(G'_i)} y^{m(G'_i)}$$

and this in turn may be split into two parts according as u_1, v_1 are/are not in the same component of G'_i . Let the two parts be $R_1(x, y)$ and $R_2(x, y)$ respectively. Similarly split up $R(H'_i; x, y)$ so that

$$\begin{aligned} R(G'_i; x, y) &= R_1(x, y) + R_2(x, y) \\ R(H'_i; x, y) &= R_3(x, y) + R_4(x, y). \end{aligned} \quad (6)$$

Considering subgraphs of G and H in a manner similar to that used in lemma 1.1 we readily obtain

$$\begin{aligned} R(G''; x, y) &= \left(\frac{y}{x}\right) R_1(x, y) + R_2(x, y) \\ R(H''; x, y) &= \left(\frac{y}{x}\right) R_3(x, y) + R_4(x, y). \end{aligned} \quad (7)$$

Now in the case when u_1 and v_1 are in the same component of G'_i and u_2 and v_2 are in the same component of H'_i then the number of components in G_i is one less than the sum of the number of components in G'_i and H'_i . In the other three cases the number of components in G_i is two less than that sum, and in all cases the number of vertices in G_i is two less than the sum of the number of vertices in G'_i and H'_i . Hence

$$\begin{aligned} R(G_i; x, y) &= \left(\frac{y}{x}\right) R_1(x, y) R_3(x, y) + R_1(x, y) R_4(x, y) \\ &\quad + R_2(x, y) R_3(x, y) + R_2(x, y) R_4(x, y). \end{aligned} \quad (8)$$

Equations 6), 7) and 8) together yield a relation between the Whitney Rank polynomials of $G_i, G', G'', H',$ and H'' :

$$G_1 = \left(\frac{x}{x-y}\right)^2 \left[\frac{y}{x} (G' - G'')(H' - H'') + (G' - G'')(H'' - \frac{y}{x} H') \right. \\ \left. + (G'' - \frac{y}{x} G')(H' - H'') + (G'' - \frac{y}{x} G')(H'' - \frac{y}{x} H') \right]$$

i.e.

$$G_1 = \frac{x}{y-x} \left[\frac{y}{x} G'H' - G'H'' - G''H' + G''H'' \right] \quad 9)$$

Now $\chi(G; x, y) = (x-1)^{|V|-p_0(G)} R(G; \frac{1}{x-1}, y-1)$

and so by transforming 9) by $x \rightarrow \frac{1}{x-1}$ and $y \rightarrow y-1$ and multiplying by $(x-1)^{|V|-p_0(G)}$

gives

$$\chi(G; x, y) = \frac{(x-1)^{|V|-p_0(G)}}{xy-x-y} \left[(x-1)(y-1)G'H' - G'H'' - G''H' + G''H'' \right]$$

where on the right hand side

$$G' = R(G'; \frac{1}{x-1}, y-1), \text{ etc.}$$

Since G'' and H'' have one less vertex than G' and H' respectively, the result follows.

ii) This follows from i) by a simple application of * . □

It should be noted that the results of lemma 3.2 may be considered as the interpretation to cographic matroids of theorem 6.15 of Brylawski [7] .

* * * * *

Let $t(i, j)$ be the number of spanning trees of the graph G with internal activity i and external activity j (so that $t(i, j)$ is the coefficient of $x^i y^j$ in $\chi(G; x, y)$), $i, j \geq 0$.

Let $|V| = n$, $|E| = m$, $\text{girth } G = \gamma$,

and edge connectivity of $G = \lambda$

Assume G has no isthmuses or loops (for every isthmus or loop just multiplies $\chi(G; x, y)$ by a factor x or y respectively).

- Lemma 3.3:
- i) $t(n-1, 0) = 1$;
 - ii) $t(n-2, 0) = m-n+1$ if G has no multiple edges,
 - iii) $t(n-3, 0) \leq \binom{m-n+2}{2}$ with equality if $\lambda > 3$,
 - iv) $t(0, m-n+1) = 1$;
 - v) $t(i, 1) > 0 \iff i \leq n-\lambda$;
 - vi) $t(1, i) > 0 \iff i \leq m-n+2-\lambda$;
 - vii) $t(1, 0) = t(0, 1)$.

Proof: The first three results may be deduced from the corresponding properties of the chromatic polynomial of G , but we shall prove then as results concerning the internal and external activities of spanning trees so that the analagous results for matroids can be seen to be true.

All the proofs depend on the crucial theorem of Tutte [16] that the Tutte polynomial is independent of

the ordering of the edges. For simplicity we represent the ordering as a labelling of the edges by the integers $\{1, 2, \dots, m\}$.

i) Let the edges of G be labelled so that the largest labels are assigned to the edges of a spanning tree T . Then it is clear that $e(T) = 0$ and $i(T) = n - 1$.

Now if T' is any other spanning tree of G let a be an edge (and the label assigned to the edge) of T' not in T . We show that a is not internally active with respect to T' , and ^{thus} ~~that~~ that T is the only spanning tree with internal/external activities $n-1/0$.

Let C be the circuit of a with respect to T so that C is a path in G between the two components of $T' - \{a\}$. Then at least one edge of C is in the cut set of a with respect to T' and so a is not internally active in T' .

ii) Let the edges be labelled as in i) and let T and a have the same meaning. Let C be the circuit of a with respect to T and let b be the smallest edge (edge having the smallest label) of C . Denote by T_a the spanning tree $(T - \{b\}) \cup \{a\}$ (making no distinction between the tree and its edge set as the meaning is clear). As above a is not internally active with respect to T_a , but every other edge of T_a is, since the only edges having b in their cut sets are

those of $(C - \{b\}) \cup \{a\}$ and by definition these are all greater than b (except a). If any edge of C other than b is chosen for interchange with a then in the resulting tree both a and b are not internally active.

Since there are no multiple edges the only edge which could be externally active with respect to T_a is b , but the circuit of b with respect to T_a is $(C - \{b\}) \cup \{a\}$ and by definition this contains an edge greater than b .

Thus we have constructed $m - n + 1$ spanning trees of G contributing to $t(n-2, 0)$. But in i) we showed that any edge of T' not in T is not internally active so these are the only spanning trees with internal/external activities $n-2/0$.

iii) Again from i) we see that we need only consider those spanning trees of G which have no more than two edges not in the spanning tree T (as defined in i)).

Further the methods of ii) show that there are just $m - n + 1$ spanning trees of G with internal/external activities $n-3/0$ and containing just one edge not in T . These are $(T - \{d\}) \cup \{a\}$ where d is the second smallest edge of C .

All spanning trees contributing to $t(n-3, 0)$ and containing two edges not in T may be constructed

by a single insertion and deletion from the $m-n+1$ trees found in ii) (for the construction can always be reversed), and provided there is no circuit of length 3 this yields $\frac{(m-n+1)(m-n)}{2}$ trees.

$$\begin{aligned} \text{Thus } t(n-3, 0) &\leq \frac{(m-n+1)(m-n)}{2} + (m-n+1) \\ &= \binom{m-n+2}{2} \end{aligned}$$

with equality if $\gamma > 3$

iv) Assigning the $n-1$ smallest labels to a spanning tree together with an argument similar to that used in i), yields the result.

Alternatively the graph may be considered as a matroid and duality invoked.

The results dual to ii) and iii) are omitted.

v) Let the edges of G be labelled so that the largest label is assigned to an edge of a circuit C of length γ , and the next $n-1$ labels are assigned to the edges of a spanning tree T of G that contains every other edge of C .

Then clearly $i(T) = n - \gamma$, $e(T) = 1$.

Now let T_1 be a spanning tree of G with $i(T_1) = j > 0$, $e(T_1) = 1$.

Let e_1 be the smallest edge of T_1 which is internally active and let e_2 be the largest edge in the cut set of e_1 with respect to T_1 . Let T_2 be the tree $(T_1 - \{e_1\}) \cup \{e_2\}$

Then $e(T_2) = 1$ and $i(T_2) = j - 1$.

The result follows by induction, and the observation that if e is externally active with respect to T_3 then every edge in the circuit of e is not internally active so $i(T_3) \leq n - \delta$

vi) This is proved similarly to v) or by the duality.

vii) Let T be a spanning tree with internal activity 1 and external activity 0. Then the edge m is in T for otherwise it is externally active, and thus the edge $m-1$ is not in T , for otherwise both m and $m-1$ are internally active. Also $m-1$ is not externally active so m is in the circuit of $m-1$.

Let T' be the spanning tree $(T - \{m\}) \cup \{m-1\}$

Clearly $i(T') = 0$ and $e(T') = 1$

This construction sets up a bijection between the two sets of spanning trees so $t(0,1) = t(1,0)$.

□

The nature of the proofs given above is such that the validity of the results when applied to matroids is clear (using Crapo's generalisation of internal and external activity).

Finally we note that observation suggests that the conjecture stated by Read [13] concerning the coefficients of the chromatic polynomial may be extended to the Tutte polynomial as follows.

With the notation of lemma 3.3 not both of

$$t(i-1, j) > t(i, j) < t(i+1, j),$$

and not both of $t(i, j-1) > t(i, j) < t(i, j+1)$

are true for any i, j .

4. Recursive Families of Graphs.

A recursive family of graphs is defined to be an infinite sequence $\{G_n\}$ of graphs whose Tutte polynomials satisfy a homogeneous linear recurrence relation of the form

$$G_{n+p} + \alpha_1(x,y)G_{n+p-1} + \dots + \alpha_p(x,y)G_n = 0 \quad 1)$$

where $\alpha_i(x,y)$, $1 \leq i \leq p$, is a fixed polynomial in x and y .

In order to eliminate uninteresting cases we further stipulate that no subsequence of a recursive family may be a recursive family. For instance if $\{G_n\}$ and $\{H_n\}$ are both families satisfying 1) above then the sequence $\{K_n\}$ defined by

$$K_{2r} = G_r, \quad K_{2r-1} = H_r, \quad r \geq 1$$

satisfies the relation

$$K_{n+2p} + \alpha_1(x,y)K_{n+2p-2} + \dots + \alpha_p(x,y)K_n = 0$$

but we do not call the sequence $\{K_n\}$ a recursive family.

The smallest integer p such that $\{G_n\}$ satisfies a relation of the form of 1) is called the recursiveness of the family, $\{G_n\}$.

The next chapter is given over to the consideration of some particular families, but as a simple example let C_n be the graph whose vertices and edges are those of an n -gon.

Using the recursion *,

$$G = G'_e + G''_e$$

we readily obtain the recurrence relation

$$C_{n+2} - (x+1)C_{n+1} + xC_n = 0 \quad 2)$$

For the Tutte polynomial of G_n . Thus the recursiveness of the family of n -gons is 2. The recurrence relation 2) may be solved in the usual way (see, for example [11]) to give

$$\begin{aligned} C_n &= \left(\frac{x}{x-1}\right)(x^{n-1}-1) + y \\ &= x^{n-1} + x^{n-2} + \dots + x + y \end{aligned}$$

which may be verified by induction.

Lemma 4.1 : If $\{G_n\}$ is a recursive family of graphs, with $G_n = (V_n, E_n, \alpha_n)$ then $\{|V_n|\}$ and $\{|E_n|\}$ are both monotonically increasing sequences.

Proof: The highest powers of x and y appearing in $\chi(G_n; x, y)$ are $|V_n|-1$ and $|E_n|-|V_n|+1$ respectively. Since the coefficients $\alpha_i(x, y)$, $1 \leq i \leq p$, are polynomials in x and y containing only non-negative powers $|V_n|-1 \geq |V_{n-1}|-1$, $n \geq 2$,

and $|E_n|-|V_n|+1 \geq |E_{n-1}|-|V_{n-1}|+1$, $n \geq 2$.

Thus $|V_n| - |V_{n-1}| \geq 0$

and $|E_n| - |E_{n-1}| \geq |V_n| - |V_{n-1}| \geq 0$.

□

Lemma 4.2 : If $\{G_n\}$ is a recursive family of graphs and $\phi(G_n)$ denotes the average valency of G_n ($\phi(G_n) = 2|E_n|/|V_n|$), then either $\{\phi(G_n)\}$ is a bounded sequence or $|V_n|$ is constant for all but a finite number of G_n .

Proof: If $\{|V_n|\}$ is an unbounded sequence at least one of the coefficients $a_i(x, y)$ contains a non zero power of x .

Thus if $A_1(n)$ is the highest power of x in $\chi(G_n; x, y)$

$$A_1(n) \geq \left[\frac{n}{p} \right] + B_1$$

for some constant B_1 , depending on the first p graphs in the family. So $A_1(n) \geq \frac{n}{2p}$

Also if $A_2(n)$ is the highest power of y in $\chi(G_n; x, y)$

$$A_2(n) \leq nB_2 + B_3$$

where B_2 is the highest power of y appearing in any of the coefficients $a_i(x, y)$, and B_3 is a constant depending on the first p graphs in the family.

Now

$$A_1(n) = |V_n| - 1,$$

$$A_2(n) = |E_n| - |V_n| + 1$$

so

$$\phi(G_n) = \frac{2|E_n|}{|V_n|} = \frac{2(A_1(n) + A_2(n))}{A_1(n) + 1}$$

$$\leq \frac{2A_2(n)}{A_1(n)} + 2 \leq \frac{2(nB_2 + B_3)}{\binom{n}{2p}} + 2$$

$$= 4pB_2 + \frac{2pB_3}{n} + 2$$

for all $n > p$. □

(If $|V_n|$ is constant for all but a finite number of G_n it is constant for all $n > p$. Further since the term of $\chi(G_n; x, y)$ including the power $x^{|V|-1}$ has coefficient 1 (lemma 3.3 i) it follows that only one of the $a_i(x, y)$, $i = 1, 2, \dots, p$ is non-zero and that that non-zero $a_i(x, y)$ is of the form y^τ for some $\tau \geq 0$. In this case G_n is obtained from G_{n-1} by adding τ loops.)

Lemma 4.3: If $\{G_n\}$ is a recursive family of graphs and $C(G_n)$ denotes the complexity of G_n then the sequence $\{\sqrt[p]{C(G_n)}\}$ is bounded.

Proof: $C(G_n) = \chi(G_n; 1, 1)$ and so we have
 $C(G_{n+p}) + a_1(1, 1)C(G_{n+p-1}) + \dots + a_p(1, 1)C(G_n) = 0.$

Hence

$$C(G_n) = A_1 t_1^n + A_2 t_2^n + \dots + A_p t_p^n$$

where A_i , $1 \leq i \leq p$, is constant and t_1, t_2, \dots, t_p are the roots of the equation

$$t^p + a_1(1, 1)t^{p-1} + \dots + a_p(1, 1) = 0$$

Thus $C(G_n) \leq |A_1| |t_1|^n + |A_2| |t_2|^n + \dots + |A_p| |t_p|^n$

$$\leq p \left(\max_{1 \leq i \leq p} |A_i| \right) \left(\max_{1 \leq i \leq p} |t_i| \right)^n.$$

□

As a consequence of these lemmas we see that such families as $\{K_n\}$, the complete graphs, $\{K_{n,n}\}$, the complete bipartite graphs, and $\{Q_n\}$,

the n -dimensional cubes are not recursive families.

note that if G_n is planar for all n then the dual family $\{G_n^*\}$ is defined in the natural way, and if $\{G_n\}$ is recursive with recurrence relation 1), then $\{G_n^*\}$ is recursive with recurrence relation

$$G_{n+p}^* + \alpha_1(y,x)G_{n+p-1}^* + \dots + \alpha_p(y,x)G_n^* = 0.$$

The following theorem describes a large class of recursive families of graphs and demonstrates the construction of the recurrence relation satisfied by their Tutte polynomials. We require the following notation, and lemma.

Let X and Y be graphs with vertex sets V, W . Let V' be an independent subset of V (i.e. no pair in V' are joined by an edge in X) with $|V'| = n$. Let Y be connected, and if X is not connected let V' contain at least one vertex from each component of X . Let Z be a subgraph of Y with vertex set W' satisfying $|W'| = n$. Let α, β be total orderings of the sets V' and W' and denote the ordered sets by V'_α, W'_β .

Let $G = X \circ Y(V'_\alpha, W'_\beta)$ be the graph obtained from X and Y by identifying the sets V' and W' , preserving the ordering.

Let \mathcal{P} be a partition of $W' = W'_1 \cup W'_2 \cup \dots \cup W'_r$. Denote by Y/\mathcal{P} the graph constructed from Y by identifying all the vertices in each subset $W'_i, 1 \leq i \leq r$ (so that Y/\mathcal{P} has $|W| = n + r$ vertices).

Lemma 4.4 : The Tutte polynomial of $G = X \circ Y(V'_\alpha, W'_\beta)$ may be expressed in the form

$$G = \sum_P b(X, V', P) Y/P$$

where the summation is taken over all partitions P of V' (which induce partitions P of W') and where $b(X, V', P)$ is a polynomial dependent only on X, V' and P .

proof: Once again we use the fundamental relation *

$$\chi(G; x, y) = \chi(G'_e; x, y) + \chi(G''_e; x, y),$$

on this occasion applying it repeatedly to every edge of X that is neither a loop nor an isthmus of G . In this way the Tutte polynomial of G is expressed as the sum of Tutte polynomials of graphs which are Y/P together with some loops and isthmuses joined at the vertices of W'/P . This is an expression of the required form.

It remains to show that this expression is independent of the order in which $*$ is applied to the edges of X . This requires only a slight modification of Smith's proof that his definition of the Tutte polynomial is valid [15], and is omitted.

□

Let X, Y, Z be three (possibly isomorphic) distinct graphs and let X_1, X_2 be two isomorphic subgraphs of X (not necessarily disjoint), Y_1, Y_2 two isomorphic

subgraphs of X , Z_1 , Z_2 two isomorphic subgraphs of \mathcal{L} with

$$X_1 \approx X_2 \approx Y_1 \approx Y_2 \approx Z_1 \approx Z_2 .$$

Let the vertex sets of $X, Y, Z, X_1, X_2, \dots, Z_2$ be $U, V, W, U_1, U_2, \dots, W_2$.

Now any ordering α on U_2 induces an ordering β on V_1 by the isomorphism and so if Y' is the graph obtained by deleting those edges of Y having both vertices in V_1 , then the graph $X \circ Y' (U_{2\alpha}, V_{1\beta})$

is the graph obtained by identifying the subgraphs X_2 and Y_1 . Denote this graph by $X \circ Y (X_2, Y_1)$

Let $H_1 = X \circ Y (X_2, Y_1)$ and inductively define

$$H_n = H_{n-1} \circ Y (Y_2, Y_1) \quad 3)$$

where the subgraphs X_1 and Y_2 of H_n are defined inductively to be the subgraphs X_1 of H_{n-1} and Y_2 of Y in 3).

Define two classes of families of graphs as follows:

Class i) is those families of graphs whose typical member is

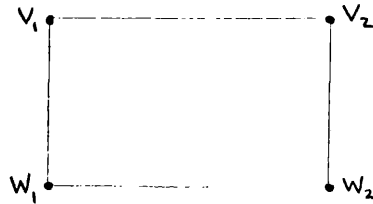
$$G_n = H_n \circ Z (Y_2, Z_1)$$

Class ii) is those families of graphs whose typical member is

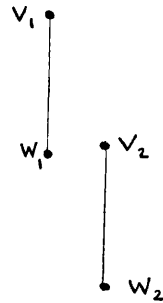
$$G_n = H_n \circ H_n (Y_2, X_1)$$

this slight abuse of notation meaning that the identification of subgraphs takes place in one graph H_n (i.e. we are not joining two isomorphic graphs.)

As an example if X, Y, Z are all isomorphic to the graph

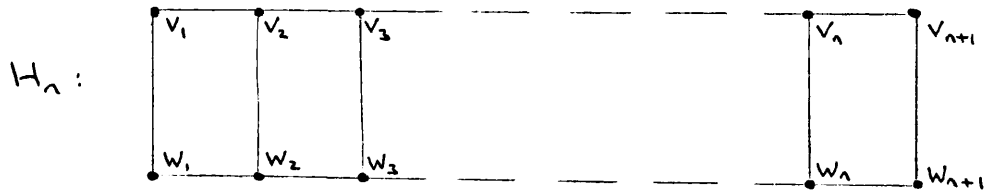


and X_1, Y_1, Z_1 correspond to

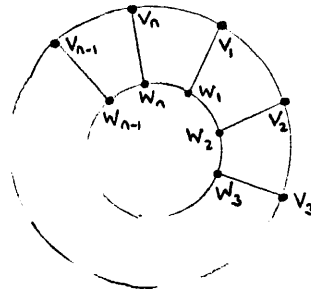


while X_2, Y_2, Z_2 correspond to

then the family of class i) as defined above has typical member



and the corresponding family of class ii) has typical member Π_n the skeleton of the prism on $2n$ vertices :



Theorem 4.1: The families of classes i) and ii) are all recursive families of graphs.

Proof: i) Let $G_n = H_n \circ Z(Y_2, Z_1)$ be a typical member of a family of class i). Then by lemma 4.4 (and the definition of the construction $H_n \circ Z(Y_2, Z_1)$ the Tutte polynomial of G_n may be

expressed in the form

$$G_n = \sum_P b(Z, W_1, P) H_{n/P} \quad 4)$$

the summation being over all partitions P of w_1 ,

Now for any P

$$H_{n/P} = H_{n-1/Y/P} (Y_2, Y_1) \quad \text{provided that}$$

Y_1 and Y_2 are disjoint, and so the Tutte polynomial of $H_{n/P}$ is given by

$$H_{n/P} = \sum_Q b(Y/P, V_1, Q) H_{n-1/Q} \quad 5)$$

the summation being taken over all partitions Q of V_1 ($\approx W_1$)

5) may be rewritten in matrix form as below, where

$\{P_1, P_2, \dots, P_q\}$ represents the set of partitions of V_1 ,

$$\begin{bmatrix} H_{n/P_1} \\ H_{n/P_2} \\ \vdots \\ H_{n/P_q} \end{bmatrix} = \underset{\sim}{B} \begin{bmatrix} H_{n-1/P_1} \\ H_{n-1/P_2} \\ \vdots \\ H_{n-1/P_q} \end{bmatrix} \quad 6)$$

where \underline{B} is a q -square matrix whose entries, b_{ij} , are the coefficients $b(Y/P_i, V_1, P_j)$.

Now if $f(t)$ is the minimum polynomial of \underline{B} then

each of the sets of Tutte polynomials $\{H_{n/P_i}\}_{n=1}^{\infty}$

(fixed i) satisfies a recurrence relation whose auxiliary equation is

$$f(t) = 0.$$

Thus from 4), $\{G_n\}_{n=1}^{\infty}$ also satisfies this

recurrence relation which is both linear and finite,

and so $\{G_n\}$ is a recursive family.

If Y_1 and Y_2 are not disjoint

$$H_n/P = H_{n-1}/P \circ Y/P (Y_2/P, Y_1/P)$$

and so equation 5) is still valid in this case.

ii) If $G_n = H_n \circ H_n (Y_2, X_1)$ is a typical member of a family of class ii) define K_n inductively

by
$$K_1 = 1$$

$$K_n = K_{n-1} \circ Y (Y_2, Y_1)$$

and then

$$G_n = K_n \circ X ((Y_1 \cup Y_2), (X_2 \cup X_1))$$

and so by lemma 4.4 the Tutte polynomial is given by

$$G_n = \sum_P b(X, U_2 \cup U_1, P) K_n/P, \quad 7)$$

the summation being taken over all partitions P of $U_2 \cup U_1$.

The proof now proceeds similarly to i) with H_n replaced by K_n .

□

Lemma 4.5: The auxiliary polynomial (of the recurrence relation for the Tutte polynomials) of a family of class ii) contains as a factor the auxiliary equation of the corresponding family of class i).

Proof: The result may be deduced from the elementary matrix theory result that the minimum polynomial of a matrix A divides any polynomial satisfying

$$f(A) = Q.$$

We give the following proof in order to demonstrate the construction of the other factor.

Equation 7) is

$$G_n = \sum_P b(X, U_2 \cup U_1, P) K_{n/P}$$

P ranging over all partitions of $U_2 \cup U_1$ and we have

$$\begin{bmatrix} K_{n/P_1} \\ K_{n/P_2} \\ \vdots \\ K_{n/P_q} \end{bmatrix} = \underset{\sim}{B} \begin{bmatrix} K_{n-1/P_1} \\ K_{n-1/P_2} \\ \vdots \\ K_{n-1/P_q} \end{bmatrix} \quad 8)$$

Partitioning $\underset{\sim}{B}$ according to those P_i which have no part containing vertices from both U_1 and U_2

(P_1, P_2, \dots, P_t) and those that do (P_{t+1}, \dots, P_q)

we have

$$\begin{bmatrix} K_{n/P_{t+1}} \\ K_{n/P_{t+2}} \\ \vdots \\ K_{n/P_q} \end{bmatrix} = \underset{\sim}{B}_1 \begin{bmatrix} K_{n-1/P_1} \\ K_{n-1/P_2} \\ \vdots \\ K_{n-1/P_q} \end{bmatrix} \quad 9)$$

Now each of the families $\{K_{n/P_i}\}$, $1 \leq i \leq t$ is of class i) and satisfies the same recurrence relation, since they only differ in the graphs X and Z (in the notation of theorem 4.1 i)) and the matrix $\underset{\sim}{B}$ there only depended on Y. Represent this recurrence relation by

$$\Delta_i (K_{n/P_i}) = 0 \quad 1 \leq i \leq t$$

Then operating on 9) by Δ_i ,

$$\begin{bmatrix} \Delta_1 (K_n/P_{t+1}) \\ \Delta_1 (K_n/P_{t+2}) \\ \vdots \\ \Delta_1 (K_n/P_q) \end{bmatrix} = \underline{B}_1 \begin{bmatrix} \vdots \\ \vdots \\ \vdots \\ \Delta_1 (K_{n-1}/P_{t+1}) \\ \Delta_1 (K_{n-1}/P_{t+2}) \\ \vdots \\ \Delta_1 (K_{n-1}/P_q) \end{bmatrix} \quad (10)$$

and so if $\Delta_2 (S_n) = 0$ is the recurrence relation whose auxiliary equation is $f_2(t) = 0$ where $f_2(t)$ is the minimum polynomial of \underline{B}'_1 (which is the last $(q-t)$ columns of \underline{B}_1)

$$\Delta_1 \Delta_2 (K_n/P_i) = 0, \quad 1 \leq i \leq q.$$

Thus from 7)

$$\Delta_1 \Delta_2 (G_n) = 0$$

and this recurrence relation has auxiliary equation

$$f_1(t) f_2(t) = 0$$

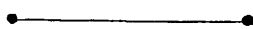
where $f_1(t) = 0$ is the auxiliary equation of $\Delta_1 (S_n) = 0$.

□

As an example we prove the following theorem.

Theorem 4.2: With the notation of theorem 4.1, if

$X_1 (\approx X_2 \approx Y_1, \text{ etc. })$ is the graph



and X_1 and X_2 are disjoint and Y_1 and Y_2

are disjoint, then the class $i)$ family has recursiveness at most 13.

Proof: $i)$ It is convenient to have a diagrammatic representation for the graphs. Since the only significant edges are those of X_1, X_2 etc. these are the only edges shown. The diagram for G_n is



and it is hoped that the meaning is clear.

To reach equation 4), $*$ is applied to all edges of Z except $\{v_n, w_n\}$ that are neither loops nor isthmuses.

$w_1 = \{v_n, w_n\}$ so there are just two partitions

$P_1 = \{v_n, w_n\}$

$P_2 = \{v_n\}, \{w_n\}$

Thus

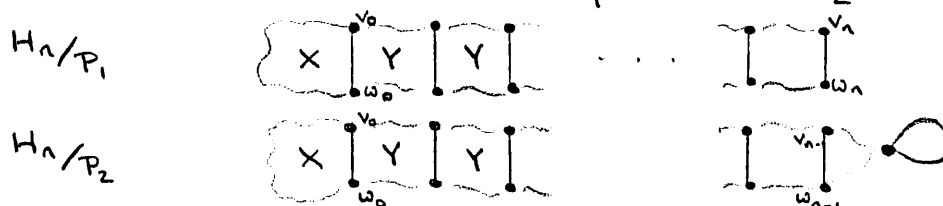
$$G_n = b(Z, w_1, P_1) H_n/P_1 + b(Z, w_n, P_2) H_n/P_2$$

and 6) becomes

$$\begin{bmatrix} H_n/P_1 \\ H_n/P_2 \end{bmatrix} = \begin{bmatrix} b(Y/P_1, v_1, (w_1), P_1) & b(Y/P_1, v_1, P_2) \\ b(Y/P_2, v_1, P_1) & b(Y/P_2, v_1, P_2) \end{bmatrix} \begin{bmatrix} H_{n-1}/P_1 \\ H_{n-1}/P_2 \end{bmatrix}$$

and so G_n has recursiveness at most 2.

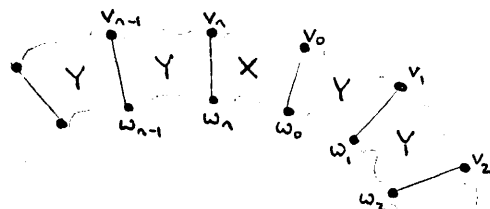
The diagrams for H_n/P_1 and H_n/P_2 are



and so the Tutte polynomial of H_n/P_2 contains a factor y . Thus the chromatic polynomials of G_n ($y=0$) satisfy a recurrence relation of order 1 as expected.

$$\left(P(G_n; x) = \frac{P(G_{n-1}; x) \cdot P(Y; x)}{x(x+1)} \right)$$

ii) The diagram for G_n in this case is



and \mathcal{U}_2 is the set $\{v_0, \omega_0\}$, \mathcal{U}_1 is the set $\{v_n, \omega_n\}$

By lemma 4.5 we need only consider those partitions of $\mathcal{U}_2 \cup \mathcal{U}_1$ for which there is a part containing vertices from both \mathcal{U}_2 and \mathcal{U}_1 , the others yield only a factor of degree 2, obtained above, in the auxiliary polynomial.

The only possible such partitions of $\mathcal{U}_2 \cup \mathcal{U}_1$ are

$$\begin{aligned} P_1 &= \{v_n\}, \{\omega_n, \omega_0\}, \{v_0\}, & P_2 &= \{v_n, v_0\}, \{\omega_n\}, \{\omega_0\}, \\ P_3 &= \{v_n, \omega_0\}, \{\omega_n\}, \{v_0\}, & P_4 &= \{v_n\}, \{\omega_n, v_0\}, \{\omega_0\}, \\ P_5 &= \{v_n, v_0, \omega_n\}, \{\omega_0\}, & P_6 &= \{v_n, v_0, \omega_0\}, \{\omega_n\}, \\ P_7 &= \{v_n, \omega_0, \omega_n\}, \{v_0\}, & P_8 &= \{v_0, \omega_0, \omega_n\}, \{v_n\}, \\ P_9 &= \{v_n, v_0, \omega_0, \omega_n\}, & P_{10} &= \{v_n, v_0\}, \{\omega_n, \omega_0\}, \\ P_{11} &= \{v_n, \omega_0\}, \{v_0, \omega_n\}. \end{aligned}$$

The theorem follows from theorem 4.1.



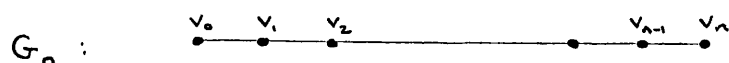
* * * * *

The notion of a recursive family may be extended in the natural way to matroids. The basic construction in the definitions of class i) and class ii) families of graphs is that of combining two graphs by identifying isomorphic subgraphs. The corresponding construction for matroids is that of identifying isomorphic closed subsets. Lemmas 4.4 and 4.5 and Theorem 4.1 all have their generalisations valid for matroids and if the notion of identifying vertices is replaced in the proofs given by that of contracting edges, the proofs of the more general results should be clear. (There is no necessity for the identified subsets to be closed but the assumption involves no loss of generality.)

5. Examples of Families

A major benefit gained from the methods described in the previous chapter is that we can consider the Tutte and chromatic polynomials of very large graphs by regarding them as members of families of graphs. In this chapter we shall derive the Tutte polynomials of some large graphs and in the next chapter we shall discuss the location of their chromatic roots.

5.1 The simplest example of a family of class i) is the one whose typical member G_n is just n edges joined in ^{series} parallel, thus

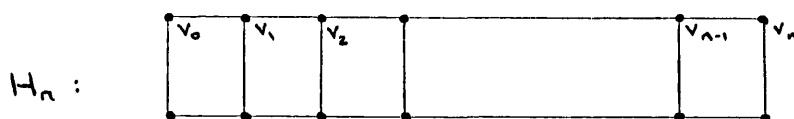


In this trivial case the Tutte polynomials satisfy the relation

$$G_{n+1} - xG_n = 0$$

and so $G_n = x^n$ and the family has recursiveness 1.

5.2 As a second example of a family of class i) we consider the family $\{H_n\}$ whose typical member is the "ladder" on $2n+2$ vertices which was mentioned briefly in the discussion preceding theorem 4.1.



An investigation of this family is a necessary preliminary to consideration of the next example, the family of prisms (see lemma 4.5). It will also be useful to show this investigation in complete detail as it is fairly straightforward, and then omit some of the voluminous and tedious computations involved in the later examples.

So, using identity * for Tutte polynomials,

$$G = G_e' + G_e''$$

we have

$$\begin{aligned} H_n &= \text{Diagram 1} + \text{Diagram 2} \\ &= (x^2 + x + 1) H_{n-1} + \text{Diagram 3} \\ &= (x^2 + x + 1) H_{n-1} + H_{n-1}/p \quad \text{say.} \end{aligned}$$

Also

$$\begin{aligned} H_{n-1}/p &= \text{Diagram 4} + \text{Diagram 5} \\ &= (xy + y) H_{n-1} + y H_{n-1}/p. \end{aligned}$$

Hence

$$\begin{bmatrix} H_n \\ H_{n-1}/p \end{bmatrix} = \begin{bmatrix} x^2 + x + 1 & 1 \\ xy + y & y \end{bmatrix} \begin{bmatrix} H_{n-1} \\ H_{n-1}/p \end{bmatrix}$$

thus H_n (and H_n/p) satisfies a recurrence relation whose auxiliary equation is

$$(x^2+x+1-t)(y-t) - (xy+y) = 0$$

i.e. $t^2 - (x^2+x+y+1)t + x^2y = 0$

and so we have

$$H_{n+2} - (x^2+x+y+1)H_{n+1} + x^2y H_n = 0$$

and the recursiveness of the family $\{H_n\}$ is 2.

Solving the recurrence in the usual way gives

$$H_n = A t_1^n + B t_2^n$$

where A and B are functions of x and y independent of n and

$$t_1 = \frac{1}{2} (x^2+x+y+1 + \alpha(x,y))$$

$$t_2 = \frac{1}{2} (x^2+x+y+1 - \alpha(x,y))$$

where $\alpha(x,y) = \sqrt{(x^2+x+y+1)^2 - 4x^2y}$.

The constants A and B are found from the Tutte polynomials of graphs H_1 and H_2 which are

$$x^3 + x^2 + x + y$$

and

$$x^5 + 2x^4 + 3x^3 + 2x^2 + x + 2x^2y + 2xy + y + y^2$$

respectively.

We find $A+B = x$,

$$A-B = (x^3+x^2+x+2y-xy) / \alpha(x,y)$$

and so $A = \frac{1}{2} (x + (x^3+x^2+x+2y-xy) / \alpha(x,y))$,

$$B = \frac{1}{2} (x - (x^3+x^2+x+2y-xy) / \alpha(x,y)).$$

In particular the chromatic polynomial of H_n (putting $y=0$ and $x=1-z$) is

$$\begin{aligned} P(H_n; z) &= -z(1-z) [(1-z)^2 + (1-z) + 1]^n \\ &= z(z-1)(z^2-3z+3)^n \end{aligned}$$

as we should expect using elementary methods.

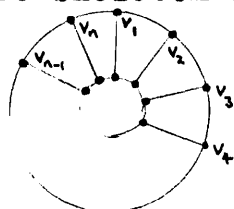
Finally, putting $x = y = 1$ we find that the complexity is

$$\begin{aligned} C(H_n) &= \frac{1}{2} \left(\left(1 + \frac{2}{\sqrt{3}}\right) (2 + \sqrt{3})^n + \left(1 - \frac{2}{\sqrt{3}}\right) (2 - \sqrt{3})^n \right) \\ &= \frac{1}{2\sqrt{3}} \left((2 + \sqrt{3})^{n+1} - (2 - \sqrt{3})^{n+1} \right) . \end{aligned}$$

5.3 We define two related families of graphs, the prisms and the (even) Mobius Ladders. A prism,

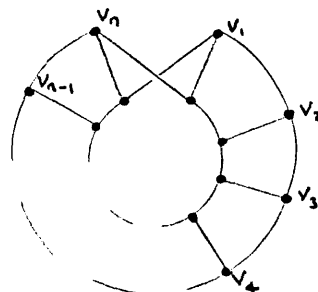
Π_n , is the skeleton of an n-gonal prism.

Π_n :



the graphs known as the Mobius ladders are defined in [9]. We denote by M_n the even Mobius ladder on 2_n vertices.

M_n :



M_n is not planar.

The odd Mobius ladders defined in [9] are 4-valent and will be considered later.

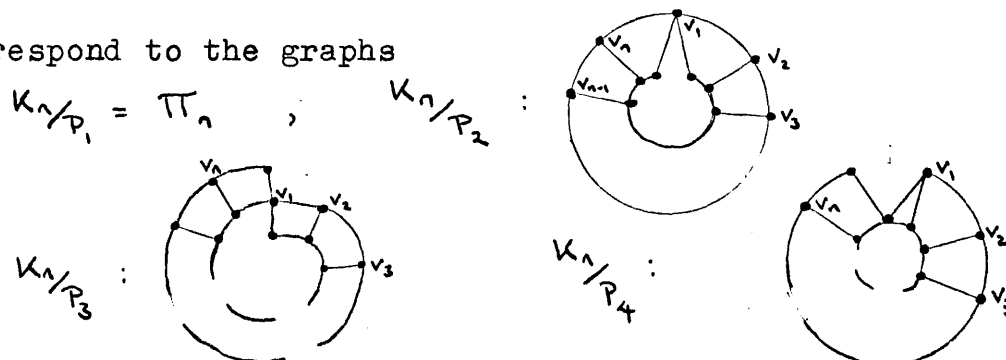
both $\{\pi_n\}$ and $\{M_n\}$ are families of class ii) and we first note that the recurrence relations satisfied by the Tutte polynomials of both families are the same, for following theorem 4.1 the matrix B is the same for both families although the graphs called there K_n/P_i differ between the two cases in the same way as M_n differs from π_n .

The following is the submatrix of B corresponding to those K_n/P_i which have connectivity > 2 (i.e. those of class ii)).

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & x & 0 & 1 \\ 0 & 0 & x & 1 \\ 0 & x & x & y+2 \end{bmatrix}$$

Because of lemma 4.5 the auxiliary polynomial of the recurrence relation satisfied by the Tutte polynomial of π_n (and M_n) is the product of the minimum polynomial of this matrix and the polynomial of degree 2 derived in 5.2.

The rows and columns of the submatrix above correspond to the graphs



respectively.

(The other partitions given in theorem 4.2 ii) are either equivalent to these or redundant.)

Hence the recurrence relation has auxiliary equation

$$(1-t)(x-t)(xy - (x+y+2)t + t^2)(x^2y - (x^2+x+y+1)t + t^2) = 0$$

and so the Tutte polynomial of \mathbb{T}_n satisfies

$$\mathbb{T}_{n+6} + a_1 \mathbb{T}_{n+5} + \dots + a_6 \mathbb{T}_n = 0$$

where

$$a_1 = -(x^2 + 3x + 2y + 4),$$

$$a_2 = 2x^3 + 6x^2 + 9x + 2x^2y + 5xy + y^2 + 5y + 5,$$

$$a_3 = -(x^4 + 5x^3 + 8x^2 + 8x + 4x^2y + 8x^2y + 9xy + 3y + x^2y^2 + 2xy^2 + y^2 + 2),$$

$$a_4 = x^4 + 3x^3 + 3x^2 + 2x + 2x^4y + 7x^3y + 7x^2y + 4xy + 2x^3y^2 + 2x^2y^2 + 2xy^2,$$

$$a_5 = -(2x^4y + 3x^3y + x^2y + x^4y^2 + 2x^3y^2 + x^2y^2),$$

$$a_6 = x^4y^2.$$

We could now solve the recurrence relation to obtain an explicit expression for the Tutte polynomial of \mathbb{T}_n and \mathbb{M}_n . However this will clearly be rather messy and is unnecessary since most information that can be found from the Tutte polynomial can be found directly and more simply from the recurrence relation.

For example, the chromatic polynomial of the prisms is found by putting $y = 0$. This reduces the auxiliary equation to

$$(1-t)(x-t)(x+2-t)(x^2+x+1-t) = 0$$

and so substituting $x = 1-z$ we obtain

$$P(\Pi_n; z) = z(A + B(1-z)^n + C(3-z)^n + D(z^2-3z+3)^n)$$

where A, B, C and D are polynomials in z , independent of n .

The chromatic polynomials of M_n is given by a similar expression differing only in the constants. Using the chromatic polynomials for

$$\Pi_n \quad \text{and} \quad M_n \quad \text{when} \quad n = 2, 3, 4 \text{ and } 5,$$

which are given in the appendix to chapter 2 we calculate the constants and obtain

$$P(\Pi_n; z) = z^2 - 3z + 1 + (z-1)(1-z)^n + (z-1)(3-z)^n + (z^2-3z+3)^n$$

and

$$P(M_n; z) = -1 - (z-1)(1-z)^n + (z-1)(3-z)^n + (z^2-3z+3)^n.$$

In a similar way we find the complexity of Π_n and M_n by putting $x = y = 1$ in the recurrence relation, reducing its auxiliary equation to

$$(1-t)^2(1-4t+t^2)^2 = 0$$

so

$$C(\Pi_n) = A_n + B + (C_n + D)(2 + \sqrt{3})^n + (E_n + F)(2 - \sqrt{3})^n,$$

for some real constants A, B, C, D, E and F and similarly for $C(M_n)$

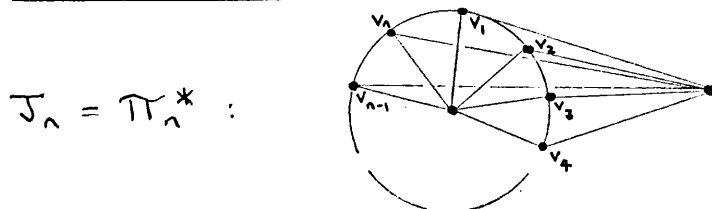
Using direct methods to calculate $C(\pi_n)$ and $C(M_n)$ for small values of n yields

$$C(\pi_n) = \frac{n}{2} \left((2+\sqrt{3})^n + (2-\sqrt{3})^n \right) - n$$

$$C(M_n) = \frac{n}{2} \left((2+\sqrt{3})^n + (2-\sqrt{3})^n \right) + n$$

the second of which is given in [14]

Now π_n is planar and so has a dual graph which is easily seen to be the skeleton of the double pyramid on an n -gon, which we shall call also a double pyramid, and denote by J_n .



The chromatic polynomial of J_n is found by putting $x = 0$ in the Tutte polynomial of π_n . This substitution reduces the auxiliary equation of the recurrence relation to

$$(1-t)(y+2-t)(y+1-t) = 0$$

and so

$$P(J_n; z) = z(A + B(3-z)^n + C(2-z)^n)$$

yielding

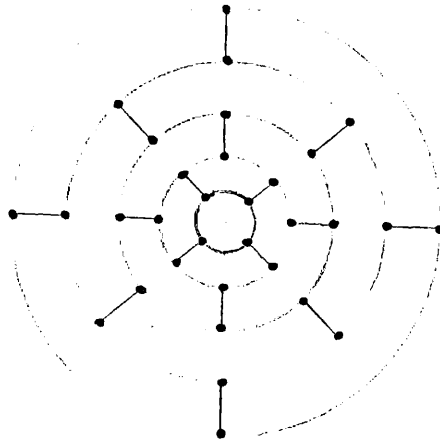
$$P(J_n; z) = (-1)^n z(z^2 - 3z + 1) + z(z-1)(z-3)^n + z(z-2)^n$$

as expected.

The Mobius ladders, being non-planar, have no dual graphs.

5.4 In the next chapter we shall be particularly concerned with the chromatic roots of trivalent graphs and so to complement the information gained from the prisms and Mobius ladders above we now consider the graphs $\Gamma_{i,j}$ consisting of i polygons P_1, P_2, \dots, P_i , with each consecutive pair P_n, P_{n+1} joined by j edges.

For example, the diagram shows $\Gamma_{5,4}$



These may be grouped into families by fixing either i or j . $\{\Gamma_{i,j}\}_{j=1}^{\infty}$ is a recursive family of class ii) for any $i \geq 2$ and

$\{\Gamma_{i,j}\}_{i=2}^{\infty}$ is a recursive family of class i) for any $j \geq 1$. In particular

$\{\Gamma_{2,j}\}$ is the family of prisms and $\Gamma_{3,5}$ is the dodecahedron.

The family $\{\Gamma_{3,j}\}$ is found to have recursiveness greater than 30, demonstrating one difficulty involved in dealing with families of class ii).

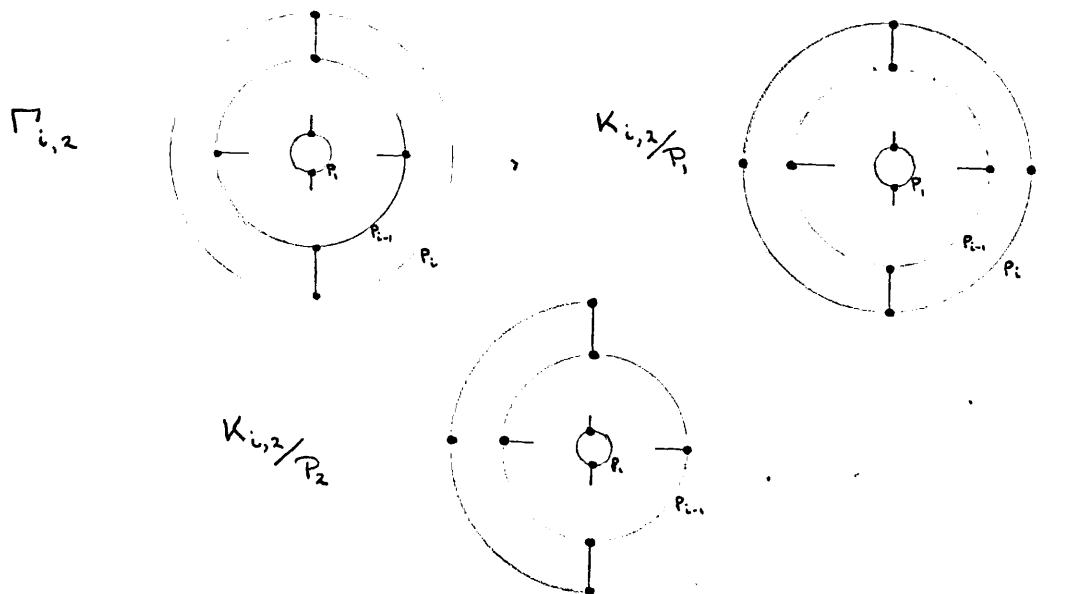
On the other hand the families $\{\Gamma_{i,j}\}_{i=2}^{\infty}$ have comparatively small recursiveness and for small values of j the recurrence relation may be found.

The family $\{\Gamma_{i,1}\}$ is trivial.

The family $\{\Gamma_{i,2}\}$ is of recursiveness 2 and the matrix is

$$\mathbb{B} = \begin{bmatrix} (x-1)(x^3+x^2+2x+y+1) & 2x+y+1 \\ x^3+x^2+x+1+y^2+2y+2x^2y+xy^2+2xy & (y+1)^2 \end{bmatrix}$$

with the rows and columns corresponding to $K_{i,2}/P_2$ and $K_{i,2}/P_1$ respectively.



The relevant recurrence relation for the chromatic polynomial is readily solved and $\Gamma_{2,2}$ and $\Gamma_{3,2}$ are in the list in the appendix to chapter 2 and we obtain

$$\chi(\Gamma_{i,2}; x, 0) = A \left(\frac{1}{2}(x^4+2x^3+3x^2+3x+2 + \beta(x)) \right)^{i-1} + B \left(\frac{1}{2}(x^4+2x^3+3x^2+3x+2 - \beta(x)) \right)^{i-1},$$

where

$$A = \frac{x}{2(x+1)} \left(1 - \frac{(x^4 - x^2 - x)}{\beta(x)} \right)$$

and

$$B = \frac{x}{2(x+1)} \left(1 + \frac{(x^4 - x^2 - x)}{\beta(x)} \right)$$

where

$$\beta(x) = \sqrt{(x^4 + 2x^3 + 3x^2 + 3x + 2)^2 + 4x^3(x+1)}$$

which expression may be transformed to the chromatic polynomial using the corollary to theorem 1.2.

when the dual family $\{\Gamma_{i,2}^*\}$ is considered the expression for the chromatic polynomial simplifies even further (the recurrence relation is of order one) and we find

$$P(\Gamma_{i,2}^*; z) = z(z-1)(z-2)^2((z-2)(z-3))^{i-1}$$

as we would expect from theorem 3 of [13].

The family $\{\Gamma_{i,3}\}$ has recursiveness 3 and the matrix \underline{B} is (b_{ij}) where

$$b_{11} = x^6 + 3x^5 + 6x^4 + 10x^3 + 12x^2 + 11x + 2y + 4 + xy,$$

$$b_{12} = 3x^3 + 6x^2 + 12x + 3y + 6, \quad b_{13} = 3x + y + 2,$$

$$b_{21} = x^5 + 3x^4 + 6x^3 + 9x^2 + 8x + 2y^2 + 6y + 4 + 7x^2y + 3x^2y + x^4y + 9xy + xy^2,$$

$$b_{22} = 4x^2 + 7x + 3y^2 + 9y + 6 + 2x^2y + 7xy,$$

$$b_{23} = (x+y+2)(y+1),$$

$$b_{31} = x^4 + 2x^3 + 3x^2 + 4x + 2y^3 + 6y^2 + 6y + 2 + 3x^2y + 3x^2y^2 + y^3x + 6xy^2 + 6xy^2 + 9xy,$$

$$b_{32} = 3(x+y+1)(y+1)^2, \quad b_{33} = (y+1)^3$$

Once again in the case of the chromatic polynomials

of the dual family $\{\Gamma_{i,3}^*\}$ everything is very easy and we obtain

$$\mathcal{P}(\Gamma_{i,3}^*; z) = z(z-1)(z-2)(z-3)^2(z^3-9z^2+29z-32)^{i-1}$$

as expected ($z(z-1)(z-2)(z^3-9z^2+29z-32)$ being the chromatic polynomial of the octahedron).

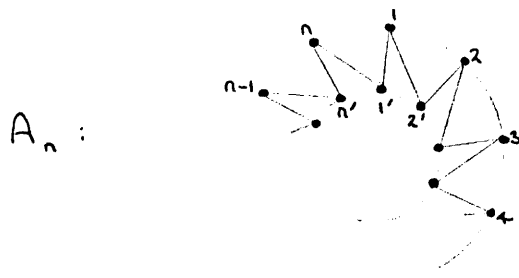
On the other hand we find that to obtain the chromatic polynomial of $\Gamma_{i,3}$ itself it is necessary to solve the recurrence relation whose auxiliary equation is

$$t^3 - (x^6 + 3x^5 + 6x^4 + 10x^3 + 16x^2 + 18x + 11)t^2 + x(x^7 + 4x^6 + 4x^5 + 5x^4 - 16x^3 - 19x^2 - 14x - 7)t - x^5(x^2 - 2)(x + 1) = 0$$

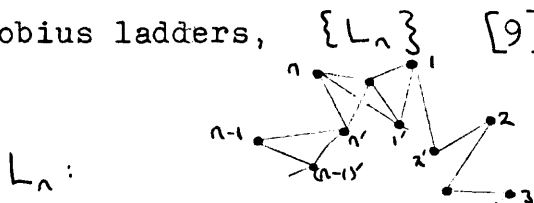
which has no "reasonable" linear factor. As we already have a complete analysis of two trivalent families there seems little point in performing the long winded calculations required to find $\mathcal{P}(\Gamma_{i,3}; z)$ (but see chapter 6).

The family $\{\Gamma_{i,4}\}$ has recursiveness 6. The matrix B has been found but its nature is such that it is not practical to deal with it, as might have been expected following $\{\Gamma_{i,3}\}$.

5.5 We now consider a 4-valent family, the family of antiprisms. Let Π_n , the n-gonal prism have vertices $\{1, 2, \dots, n, 1', 2', \dots, n'\}$ with vertex i joined by edges to $i', i-1$, and $i+1$ (modulo n). Then the n-gonal antiprism A_n has the same vertices and edges as Π_n together with edges joining i and $i'+1$ for $i = 1, 2, \dots, n$.



The antiprisms are clearly closely related to the odd Mobius ladders, $\{L_n\}$ [9].



As with the prisms and Mobius ladders both these families are of class ii) and both satisfy the same recurrence relation.

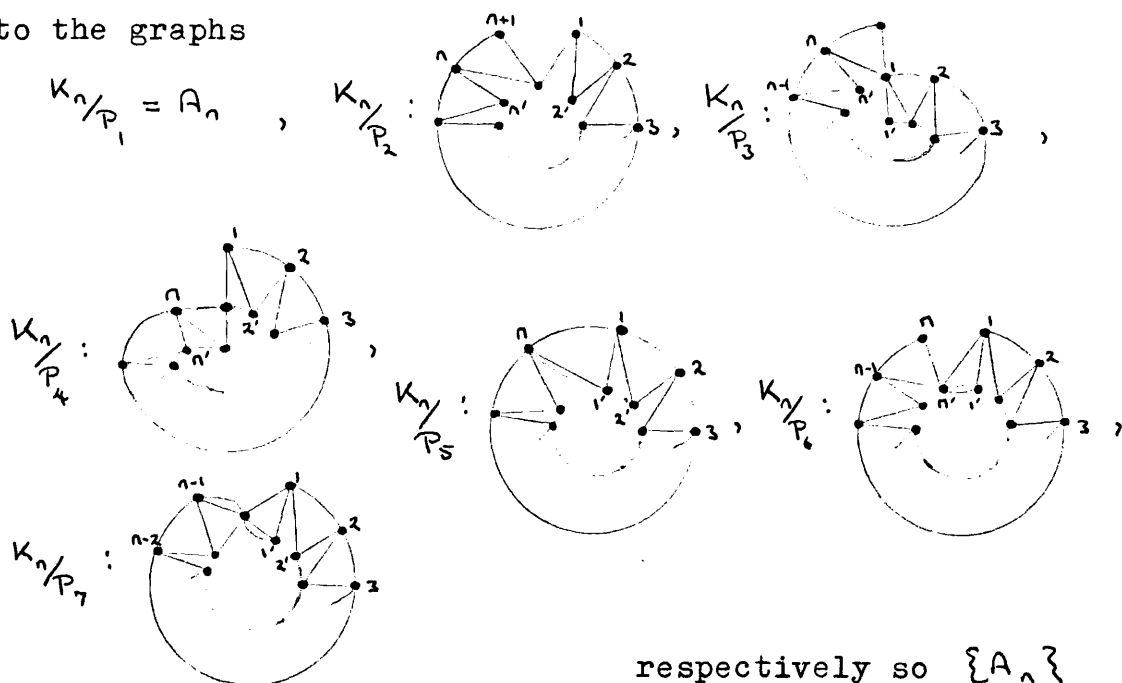
The recurrence relation for the corresponding families of class i), (see lemma 4.5), has auxiliary equation

$$t^2 - (x^2 + 2x + y^2 + 2y + 1)t + y^2x^2 = 0$$

and the residual submatrix of \underline{B} is found to be

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 2y & 0 & y(y+1) \\ 0 & x+1 & 0 & x+y+1 & y(x+y+1) & 0 & 0 \\ 0 & x+y+1 & x+1 & 0 & 0 & y(x+y+1) & 0 \\ 0 & 1 & 0 & x+y+2 & y(y+2) & 0 & 0 \\ 0 & 1 & 0 & y+1 & y(y+1) & 0 & 0 \\ 0 & y+1 & 1 & 0 & 0 & y(y+1) & 0 \\ 0 & 0 & 0 & 0 & y+1 & 1 & y(y+1) \end{pmatrix}$$

where the rows and columns of the matrix correspond to the graphs



and $\{L_n\}$ are of recursiveness 9. (K_n/P_5 , K_n/P_6 and K_n/P_7 are not precisely as defined in theorem 4.1 - loops have been omitted - but this does not affect the recurrence relation.)

In this case the characteristic polynomial of the matrix factorises. It is

$$(t^2 - (x^2 + 2x + y^2 + 2y + 1)t + y^2x^2)(t-1)(t-y(y+1))(t^2 - (y^2 + y + x + 1)t + y^2x)$$

$$(t^3 - (2x + 2y + y^2 + 3)t^2 + (x^2 + 2x + 2xy + 2xz + 1)t - x^2y^2).$$

thus $\chi(A_n; x, 0)$ and $\chi(L_n; x, 0)$ may be expressed in the form

$$A + B(x-1)^n + C(x+1)^{2n} + D\left(\frac{2x+3+\sqrt{4x+5}}{2}\right)^n + E\left(\frac{2x+3-\sqrt{4x+5}}{2}\right)^n$$

for suitable polynomials in x A, B, C, D, E independent of n .

The chromatic polynomials of the first five A_n have been calculated (starting

with $\mathcal{P}(A_n; z) = 0$,

A_n : 

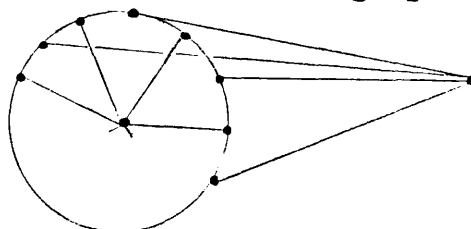
and we obtain the following values of the constants
(for A_n)

$$A = \frac{x^2 + x - 1}{x - 1} , \quad B = 0 , \quad C = \frac{1}{x - 1} ,$$

$$D = E = -\frac{x}{x - 1}$$

Since A_n is planar there is a dual graph A_n^*

and



$\chi(A_n^*; x, 0)$, = $\chi(A_n; 0, x)$, is of the form

$$A + B(x+1)^{2n} + Cx^n(x+1)^n + D(x^2+x-1)^n + Et_1^n + Ft_2^n$$

where $t_1 = \frac{1}{2}(x^2 + 2x + 3 + \delta(x))$,

$$t_2 = \frac{1}{2}\{(x^2 + 2x + 3) - \delta(x)\} ,$$

$$\delta(x) = \sqrt{x^4 + 4x^3 + 10x^2 + 12x + 5} ,$$

with A, B, C, D, E and F independent of n .

The polynomials A, B, C, D, E and F have not been found in this case but $\chi(A_n^*; x, 0)$ has been calculated for $n = 1, 2, 3$ and 4 and these are discussed in chapter 6.

5.6 Finally we consider a family of graphs which does not satisfy the definition of a recursive family given in chapter 4. These are the complete graphs and we use lemma 3.1 to obtain a non-linear recursive expression for their Tutte polynomials.

(For this section only) we define the following notation.

K_n denotes the complete graph on n vertices.

K_i^j denotes the graph constructed from K_{i-1} by joining a new vertex v by j edges to every vertex of K_{i-1} (that is inserting $(i-1)(j-1)$ edges into K_i).

cG is the cone of G as defined in lemma 3.1 and inductively $c^k G = c(c^{k-1} G)$.

All equations are relations between Tutte polynomials.

$$K_i = 1 = K_i^i \quad \text{for all } i.$$

$$q_j = y^{j-1} + y^{j-2} + \dots + y = \frac{y^{j-1}}{y-1}.$$

$$x_j = q_j + x.$$

$$A_{i,j}^1 = (i-2)(q_j+1).$$

$$A_{i,j}^k = (-1)^{k-1} (q_j+1) q_{j+1} q_{j+2} \dots q_{j+k-1} \sum_{p=k}^{i-2} \binom{p-1}{k-1},$$

for $k > 1$.

By lemma 3.1

$$K_n = (x+n-2)K_{n-1} + \binom{n-1}{2} y K_{n-2}^2 + \binom{n-1}{3} y^3 K_{n-3}^3 + \dots$$

$$+ \binom{n-1}{r} y \binom{r}{2} K_{n-r}^r + \dots + y \frac{(n-1)(n-2)}{2} K_1^{n-1}, \quad n > 1$$

which we abbreviate to

$$K_n = \mathcal{B}_1^\wedge K_{n-1} + \mathcal{B}_2^\wedge K_{n-2} + \dots + \mathcal{B}_{n-1}^\wedge K_1. \quad 1)$$

Now applying the reduction * to successively delete all but one of the edges of K_i^j incident at v gives

$$K_i^j = \gamma_j K_{i-1} + (q_j + 1) \sum_{k=1}^{i-2} c^{k-1} K_{i-k}^{j+1}. \quad 2)$$

Again applying *, this time to insert $(j-1)$ edges in $c K_i^j$ to give K_{i+1}^j yields

$$c K_i^j = K_{i+1}^j - q_j K_i^{j+1} \quad 3)$$

and so

$$\begin{aligned} c^{k-1} K_{i-k}^{j+1} &= K_{i-k}^{j+1} - (k-1) q_{j+1} K_{i-k-1}^{j+2} + \binom{k-1}{2} q_{j+1} q_{j+2} K_{i-k-2}^{j+3} \\ &\quad - \dots + (-1)^k q_{j+1} q_{j+2} \dots q_{j+k-1} K_{i-k}^{j+k}, \quad 4) \end{aligned}$$

and so from 2)

$$K_i^j = \gamma_j K_{i-1} + (q_j + 1) \sum_{k=1}^{i-2} \sum_{p=1}^k (-1)^{p-1} \binom{k-1}{p-1} q_{j+1} q_{j+2} \dots q_{j+p-1} K_{i-p}^{j+p}, \quad 5)$$

where the product $q_{j+1} q_{j+2} \dots q_{j+p-1}$ is taken to be 1 when $p = 1$.

Reversing the order of the summation signs in 5) gives

$$K_i^j = \gamma_j K_{i-1} + \sum_{k=1}^{i-2} A_{i,j}^k K_{i-k}^{j+k} \quad 6)$$

and now substituting this repeatedly in 1), gives

$$K_n = B_1^n K_{n-1} + \gamma_2 \delta_3^n K_{n-3} + \dots + \gamma_{n-3} \delta_{n-2}^n K_2 + \gamma_{n-2} \delta_{n-1}^n K_1 + B_{n-1}^n, \quad 7)$$

where δ_q^n is defined recursively by

$$\delta_q^n = \delta_{q-1}^n A_{n-q+2, q-2}^1 + \delta_{q-2}^n A_{n-q+3, q-3}^2 + \dots + \delta_3^n A_{n-2, 2}^{q-3} + B_{q-1}^n$$

for $q > 3$ and

$$\delta_3^n = B_2^n.$$

The final result probably looks better in matrix form.

6) may be written

$$\begin{bmatrix} \gamma_2 K_{n-3} \\ \gamma_3 K_{n-4} \\ \vdots \\ \gamma_{n-2} K_1 \end{bmatrix} = P_n \begin{bmatrix} K_{n-2}^2 \\ K_{n-3}^3 \\ \vdots \\ K_2^{n-2} \end{bmatrix}$$

where $P_n =$

$$\begin{bmatrix} -1 & A_{n-2,2}^1 & A_{n-2,2}^2 & \dots & A_{n-2,2}^{n-4} \\ & -1 & A_{n-3,3}^1 & \dots & A_{n-3,3}^{n-3} \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 & A_{3,n-3}^1 \\ & & & & & -1 \end{bmatrix}$$

and then 7) is simply

$$K_n - \mathcal{B}_1^n K_{n-1} = (\mathcal{B}_2^n \mathcal{B}_3^n \dots \mathcal{B}_{n-2}^n) P_n^{-1} \begin{bmatrix} \gamma_2 K_{n-3} \\ \gamma_3 K_{n-4} \\ \vdots \\ \gamma_{n-2} K_1 \end{bmatrix} + \mathcal{B}_{n-1}^n$$

This may be more amenable than the expression given by Tutte [17].

The first few A_{ij}^k are

$$A'_{32} = y+1, \quad A'_{33} = y^2+y+1, \quad A'_{34} = y^3+y^2+y+1,$$

$$A'_{42} = 2(y+1), \quad A'_{43} = 2(y^2+y+1),$$

$$A'_{52} = 3(y+1),$$

$$A^2_{42} = -(y+1)(y^2+y), \quad A^2_{43} = -y(y^2+y+1)^2,$$

$$A^3_{52} = y^2(y+1)^2(y^2+y+1),$$

$$A^4_{62} = -y^3(y+1)^2(y^2+y+1)(y^3+y^2+y+1),$$

and for example we have

$$B_1^5 = x+3, \quad B_2^5 = 6y, \quad B_3^5 = 4y^2, \quad B_4^5 = y^6,$$

and so

$$K_5 = (x+3)K_4 + (x+y)\delta_3^5 K_2 + (y^2+y+x)\delta_4^5 K_1 + y^6$$

now
$$\delta_3^5 = B_2^5 = 6y$$

and
$$\delta_4^5 = \delta_3^5 A'_{32} + B_3^5 = 6y(y+1) + 4y^2$$

and thus assuming $K_1 = 1$, $K_2 = x$, and

$$K_4 = x^3 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3$$

we find

$$K_5 = (1 \ y \ y^2 \ y^3 \ y^4 \ y^5 \ y^6) \begin{bmatrix} 0 & 6 & 11 & 6 & 1 \\ 6 & 20 & 20 & & & & \\ 15 & 15 & & & & & \\ 15 & 5 & & & & & \\ 10 & & & & & & \\ 4 & & & & & & \\ 1 & & & & & & \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

This raises the question of whether it is possible to extend the notion of recursiveness usefully to include such families as the complete graphs.

The recursion given above for K_n certainly suggests not, as it is not linear, not homogeneous, and most relevant it cannot be 'solved' to give a general expression for the Tutte polynomial of K . It

seems likely that the definition in chapter 4 is the best possible in this context.

6. The Location of the Chromatic Roots.

In this chapter we report on the location of the chromatic roots of the graphs discussed in chapter 5 and attempt to generalise these results to statements about the chromatic roots of any graph. This is in fact being rather ambitious; nevertheless some light is shed on the problem.

The information we have for a particular family of graphs is of two sorts. First, using the University of London CDC 6600 machine the chromatic roots of those members of families which have less than about 38 vertices have been computed. In most cases these results show a very clear pattern which we may confidently claim is followed by the larger members of the family. (The pattern is usually that the roots lie on one or more closed curves in the complex plane.) In other cases, while there is a pattern to the location of the chromatic roots it is not clear at some points how the pattern will extend to the larger graphs (e.g. when the roots lie on a curve in the complex plane that is not closed).

It is in this second case that information of the other kind is most useful. This is deduced from the explicit form of the chromatic polynomial and

can either be in the form of a Rouché's Theorem type of result, putting a bound on the modulus of the chromatic roots or, less usefully, in a similar form to Tutte's result on the existence of the "golden root" [3], [18], [19]. This is a theoretical explanation of the presence of a chromatic root of certain graphs at $2.618\dots = 1 + \tau$

$$= 1 + \frac{1 + \sqrt{5}}{2} \quad \text{and states that}$$

$$P(G; 1 + \tau) \leq \tau^{s-n}$$

if G has n vertices.

Such results are not very useful for our purposes for two reasons. They are not really conclusive; the graphs $\Gamma_{i,3}^*$ (see 5.4) satisfy the conditions of Tutte's theorem but have no chromatic root nearer $1 + \tau$ than $2.547\dots$ for all i . Also we are unlikely to be able to account for every chromatic root of a graph by such methods.

Note also that as a consequence of theorem 11 of [13] (c.f. lemma 3.3 ii) the centroid of the chromatic roots of a graph G , in the complex plane lies at the point $(\phi(G)/2, 0)$ where $\phi(G)$ is the average valency of G .

The first non trivial case to be considered is 5.3, the prisms and even Mobius ladders. These have chromatic polynomials

$$z^2 - 3z + 1 + (z-1)(1-z)^n + (z-1)(3-z)^n + (z^2 - 3z + 3)^n$$

and

$$-1 - (z-1)(1-z)^n + (z-1)(3-z)^n + (z^2 - 3z + 3)^n$$

respectively.

The chromatic roots of both these tend to lie near the cardioid-like curve C , shown in figure 1 or near that portion of the line $\operatorname{Re} z = 2$ which lies inside C . (A reasonable form of the equation of C has not been found but the possibility that it is a cardioid in the strict sense has been eliminated.)

This pattern has been confirmed by the following lemma due to R.M. Damarell [4].

Lemma 6.1 The chromatic roots of the prisms and even Mobius ladders have modulus not greater than 3.

Proof: The proofs for the two families being similar we give only that for the prisms.

When $|z| = 3$ we have by elementary calculus

$$(1.4) \quad |3-z| < |z^2-3z+3| ,$$

$$(1.4) \quad |1-z| < |z^2-3z+3| ,$$

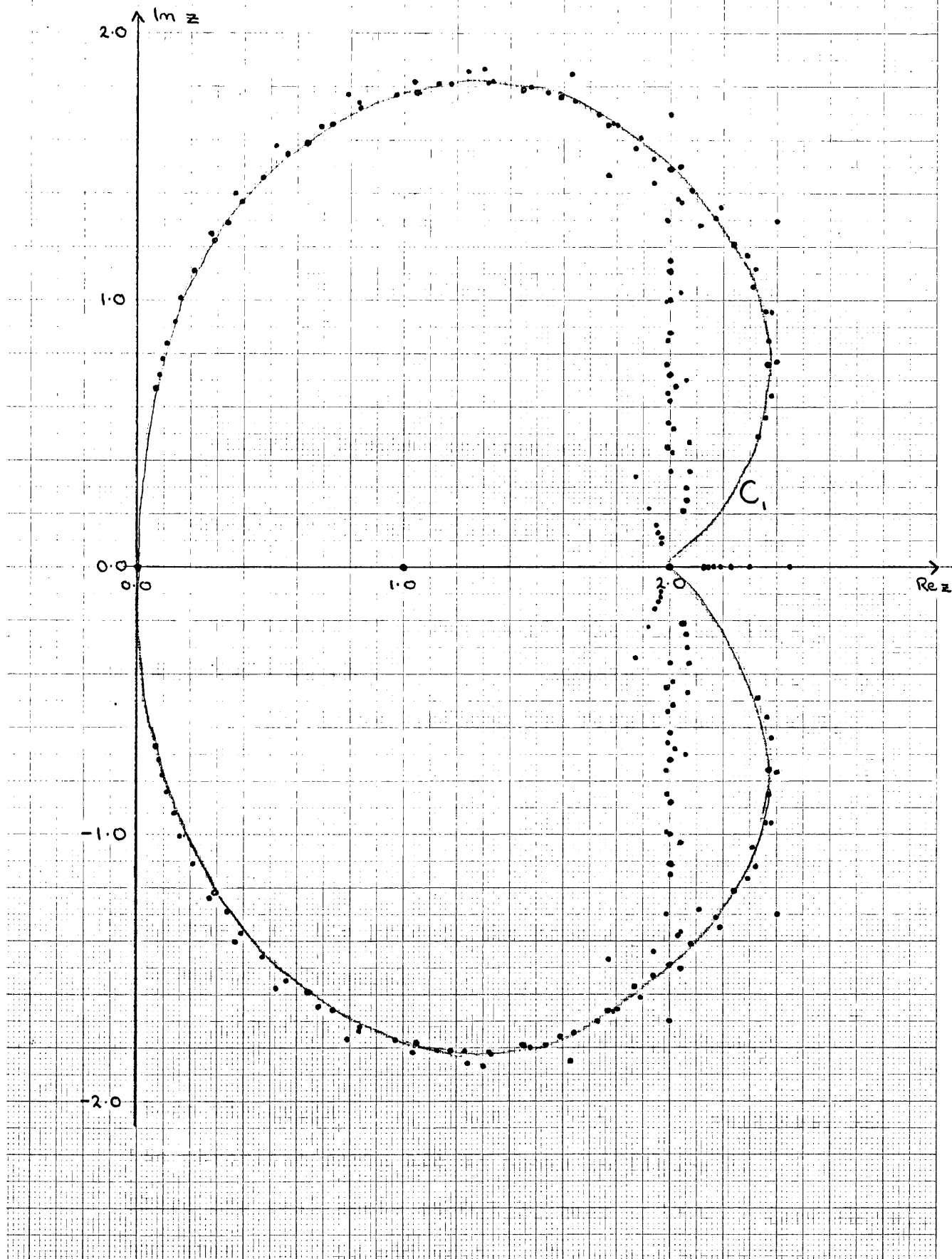
$$3 \leq |z^2-3z+3| ,$$

$$|1-z| \leq 4 .$$

Thus

$$\begin{aligned} |z^2-3z+1| , &\leq \frac{5}{3} |z^2-3z+3| , \\ &\leq \frac{5}{3^2} |z^2-3z+3|^2 \end{aligned}$$

Figure 1: The chromatic roots of the prisms, π_n
 for $3 \leq n \leq 15$
 (The chromatic roots of π_{15} are
 shown in red.)



Hence when $|z| = 3$

$$\begin{aligned}
 & |z^2 - 3z + 1 + (z-1)(1-z)^n + (z-1)(3-z)^n| \\
 & \leq |z^2 - 3z + 1| + |z-1|(|1-z|^n + |3-z|^n) \\
 & \leq \left(\frac{8}{1.4^n} + \frac{5}{3^n} \right) |z^2 - 3z + 3|^n \\
 & \leq |z^2 - 3z + 3|^n \quad \text{for } n \geq 7.
 \end{aligned}$$

applying Rouché's theorem we have that

$$(z^2 - 3z + 3)^n + (z-1)(1-z)^n + (z-1)(3-z)^n + z^2 - 3z + 1 \quad \text{has}$$

the same number of zeros inside the circle $|z| = 3$
 as $(z^2 - 3z + 3)^n$, namely $2n$, for $n \geq 7$.
 Direct computation confirms the result for $n < 7$.



The duals of the prisms, the double pyramids have
 chromatic polynomial

$$z(z-1)(z-3)^n + z(z-2)^n + (-1)^n z(z^2 - 3z + 1)$$

where the graph J_n has $n+2$ vertices, and
 the chromatic roots are as shown in figure 2.

There are real roots at 0, 1, 2, near $1 + \tau$ (see above),
 and when n is odd at 3. There are roots (including
 real roots) on the curve made up of

Hence when $|z| = 3$

$$\begin{aligned}
 & |z^2 - 3z + 1 + (z-1)(1-z)^n + (z-1)(3-z)^n| \\
 & \leq |z^2 - 3z + 1| + |z-1|(|1-z|^n + |3-z|^n) \\
 & \leq \left(\frac{8}{1.4^n} + \frac{5}{3^n} \right) |z^2 - 3z + 3|^n \\
 & \leq |z^2 - 3z + 3|^n \quad \text{for } n \geq 7.
 \end{aligned}$$

applying Rouché's theorem we have that

$$(z^2 - 3z + 3)^n + (z-1)(1-z)^n + (z-1)(3-z)^n + z^2 - 3z + 1 \quad \text{has}$$

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 Direct computation confirms the result for $n < 7$.



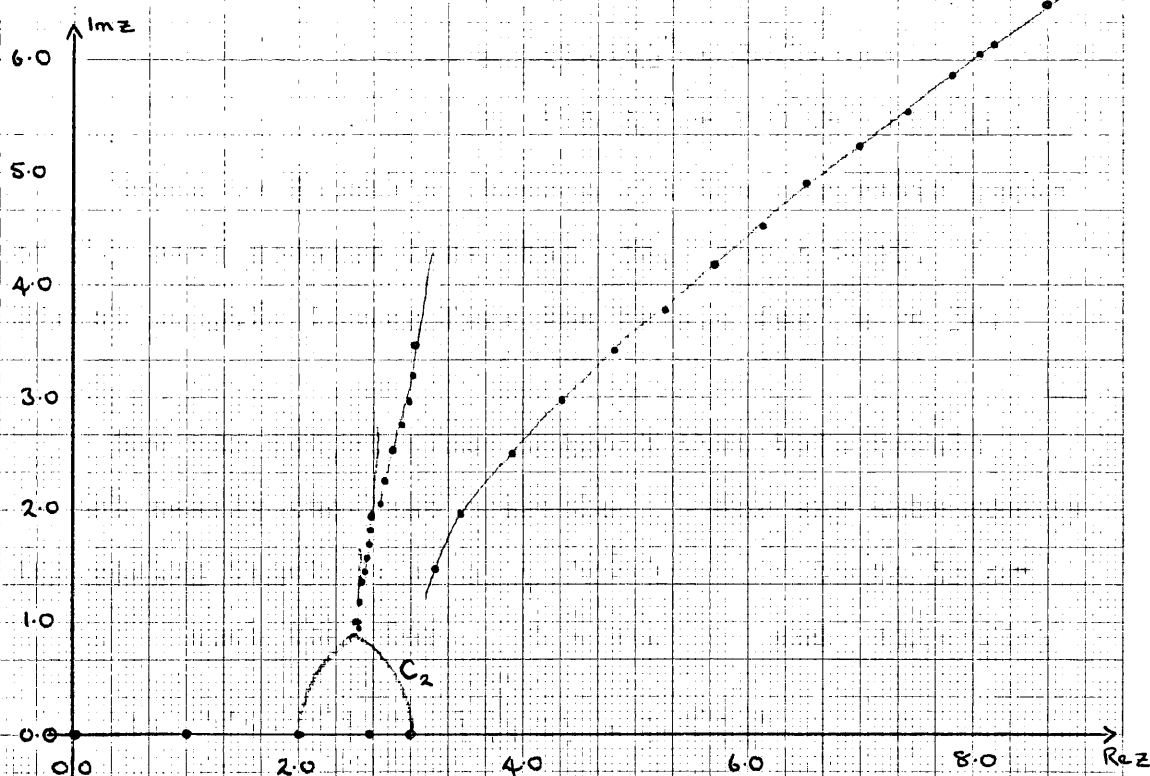
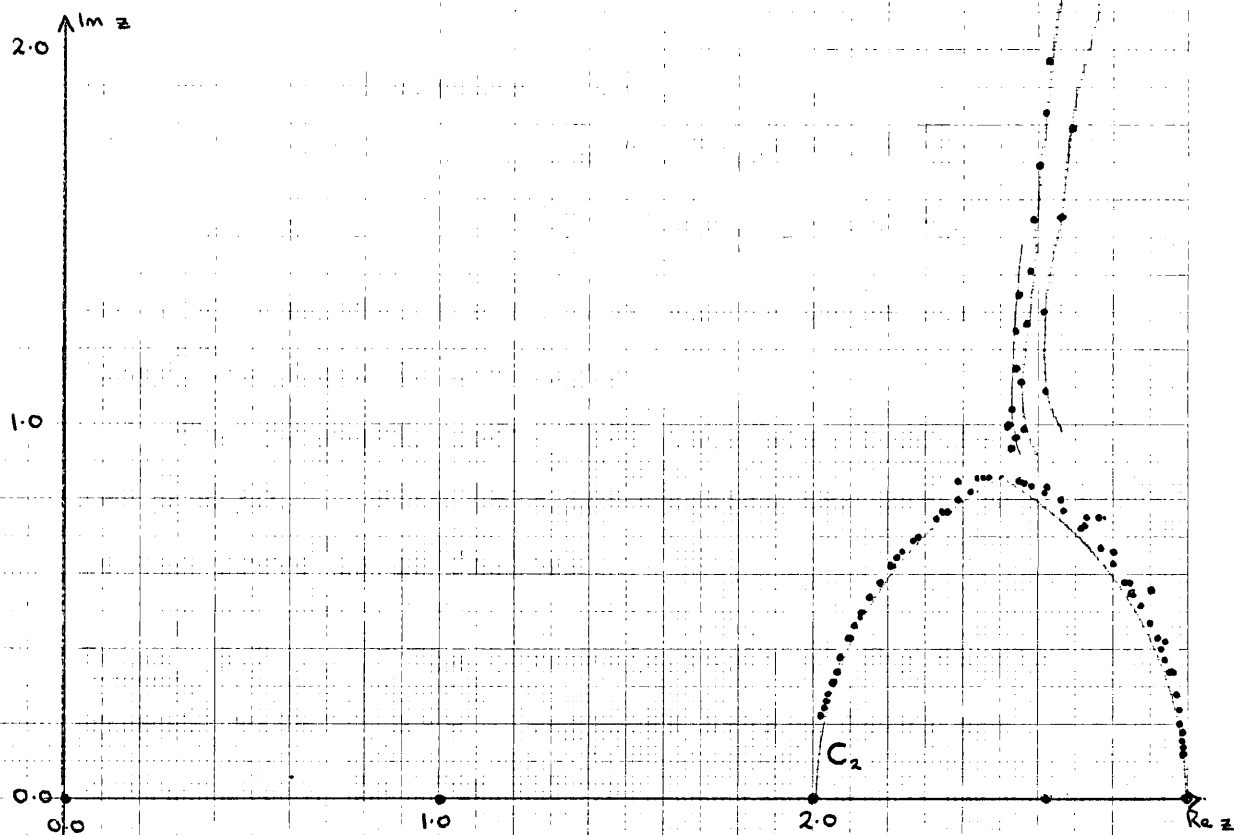
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 the chromatic roots are as shown in figure 2.

There are real roots at 0, 1, 2, near $1 + \tau$ (see above),
 and when n is odd at 3. There are roots (including
 real roots) on the curve made up of

Figure 2: The chromatic roots (with $\text{Im } z \geq 0$) of the double pyramids J_n , n even, $4 \leq n \leq 30$. The behaviour for n odd is identical except for one additional root at $z = 3$. (The chromatic roots of J_{30} are shown in red.)



the two arcs of circles (C_2)

$$|z-2|=1 \quad -\frac{\pi}{3} \leq \arg(z-2) \leq \frac{\pi}{3}$$

$$|z-3|=1 \quad \frac{2\pi}{3} \leq \arg(z-3) \leq \frac{4\pi}{3}$$

and the remainder lie on what appear to be straight lines, as shown in the figure.

The chromatic roots have only been computed for $n \leq 30$. With this information the best result we can prove is as follows.

Lemma 6.2 : The chromatic roots of the double pyramid with $n+2$ vertices have modulus not greater than $3n$ for all n .

Proof: For all $n \geq 30$ and $|z| = 3n$

$$\begin{aligned} |(z-1)(z-3)^n| &\geq (3n-1)(3n-3)^n \\ &\geq (3n+2)^n + 3n^2 + 9n + 1 \\ &\geq |(z-2)^n + (-1)^n(z^2 - 3z + 1)|. \end{aligned}$$

Hence by Rouché's theorem all the chromatic roots of J_n have modulus less than $3n$, for $n \geq 30$.

Direct computations confirm the result for $n < 30$.

□

Now our computations suggest that this result is not nearly the best possible. For any unboundedly increasing function of n , $\psi(n)$ there is an n_0 such that for $n \geq n_0$ and $|z| = \psi(n)$

$$|(z-1)(z-3)^n| \geq |(z-2)^n + (-1)^n(z^2 - 3z + 1)|.$$

Thus if we can show that the moduli of the chromatic roots of T_n are bounded by $\psi(n)$ for $n < n_0$, then $\psi(n)$ is a bound for all n . It appears that $\psi(n) = 2.2 + n f(n)$, where $f(n)$ is a decreasing function lying between 0.3 and $\frac{1}{n}$ (for $n \geq 4$) is the form of the best bound. Certainly $\psi(n) = 2.2 + 0.3n$ will do, assuming that the regular behaviour shown in figure 2 is maintained for larger n .

The chromatic roots of some of the members of the families $\{\Gamma_{i,2}\}$ and $\{\Gamma_{i,3}\}$ are given in figures 3 and 4 respectively.

$$\Gamma_{i,2} \quad \text{has} \quad \chi(\Gamma_{i,2}; x, 0) = A t_1^{i-1} + B t_2^{i-1}$$

where

$$t_1, t_2 = \frac{1}{2} (x^4 + 2x^3 + 3x^2 + 3x + 2 \pm \beta(x)),$$

$$A, B = \frac{x}{2(x+1)} \left(1 \mp \frac{(x^4 - x^2 - x)}{\beta(x)} \right),$$

$$\beta(x) = \sqrt{(x^4 + 2x^3 + 3x^2 + 3x + 2)^2 + 4(x^3(x+1))}.$$

In this case there is no obvious way to apply Rouché's theorem and the pattern of the chromatic roots of the small members of the family give only an incomplete indication of the likely location of the chromatic roots of the larger graphs. The

Figure 3 : The chromatic roots of the graphs $\Gamma_{i,2}$, $i \leq 9$

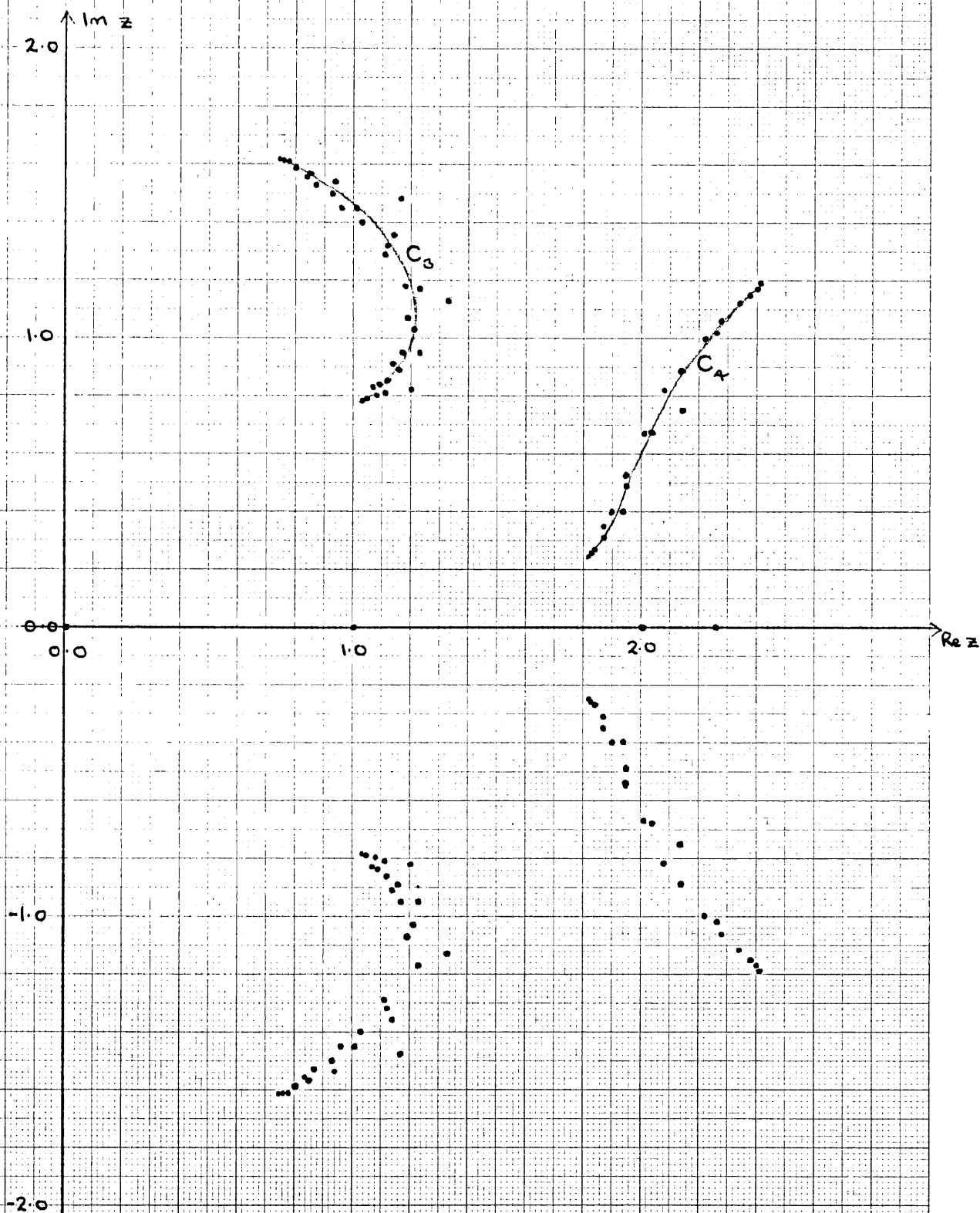
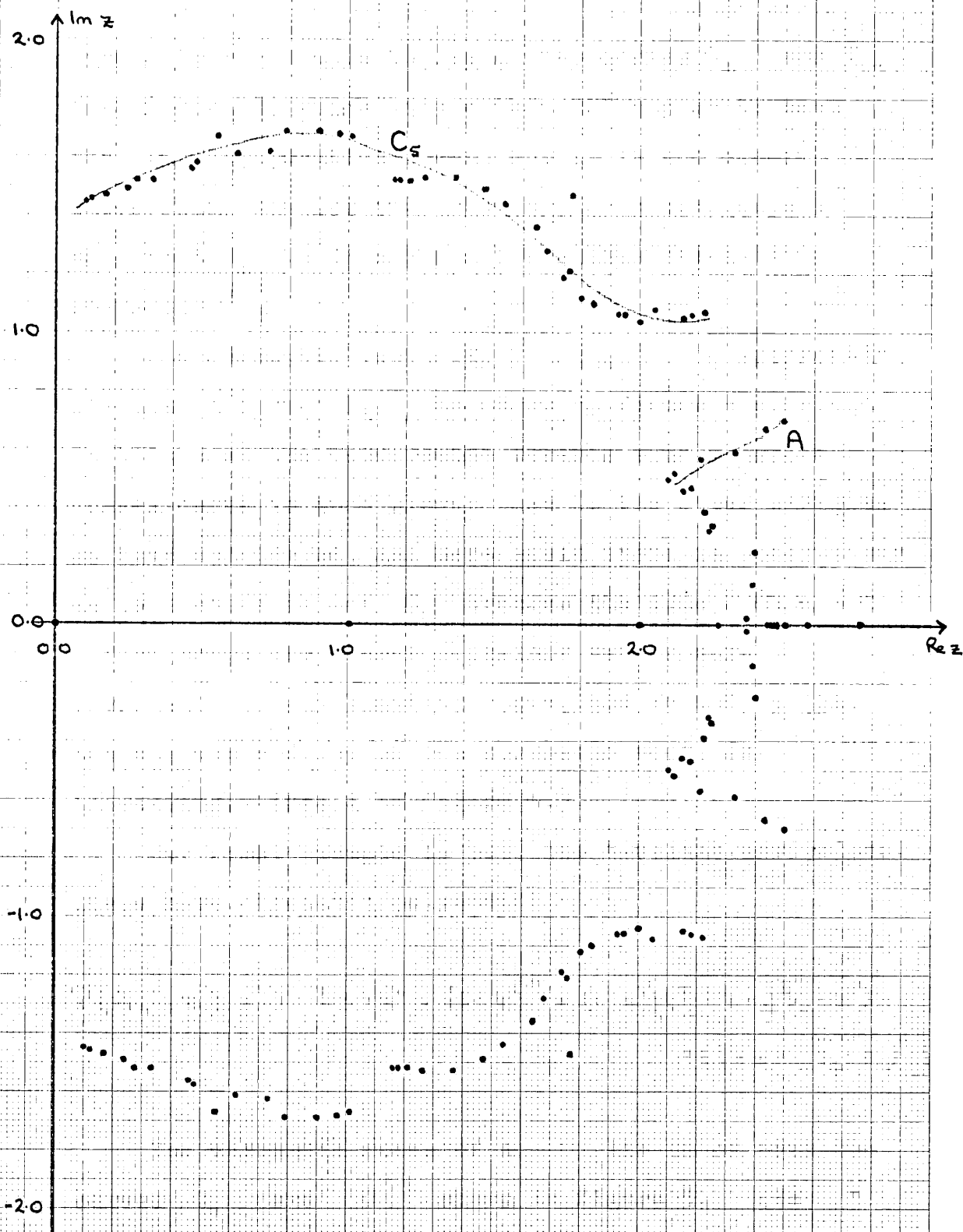


Figure 4: The chromatic roots of the graphs $\Gamma_{i,3}$, $i \leq 7$.



chromatic roots seem to lie on two curves C_3 and C_4 (see figure 3) which vary with n and appear to have the limiting position shown, but it is not clear, inter alia, how (or even whether) we should expect to extend C_4 to describe all possible chromatic roots of $\Gamma_{i,2}$

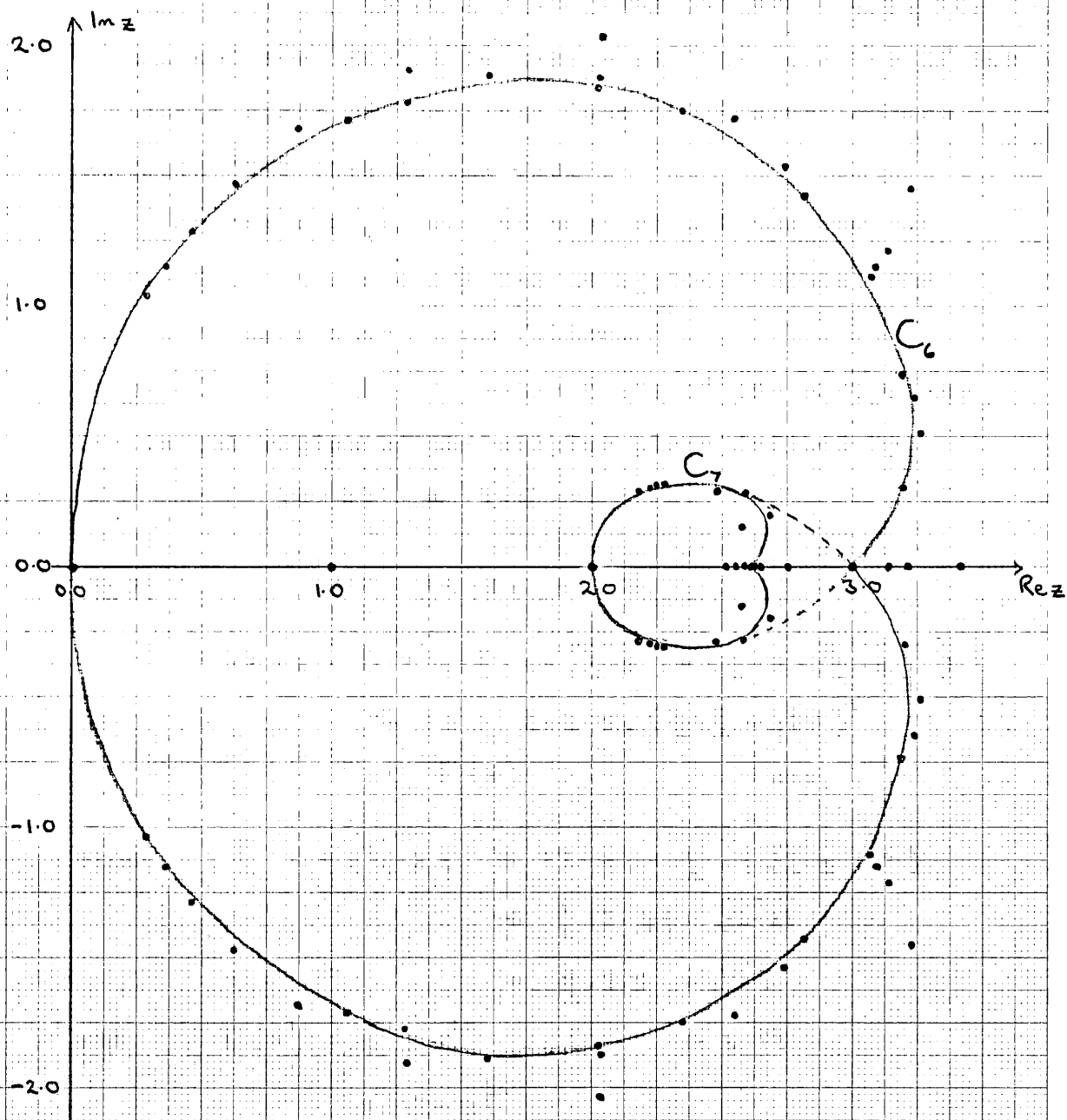
The chromatic roots of $\Gamma_{i,3}$ apparently lie near a curve C_5 , as shown in figure 4, although in this case the convergence towards such a smooth convex curve is much slower than in the other examples we have considered, and here again there is uncertainty as to whether or not C_5 crosses the line $\operatorname{Re} z = 0$ indeed there are almost certainly two or more components to any limiting curve, and we notice that the maximum modulus of the chromatic roots is increasing. (See A on figure 4.)

Our final example is 5.5, the 4-valent antiprisms. These have chromatic polynomial

$$P(A_n; z) = 1 - 3z + z^2 + (2-z)^{2n} + (1-z) \left[\left(\frac{5}{2} - z + \sqrt{\frac{9}{4} - z} \right)^n + \left(\frac{5}{2} - z - \sqrt{\frac{9}{4} - z} \right)^n \right]$$

The chromatic roots of A_n lie on or near the curves C_6 and C_7 shown in figure 5. (We

Figure 5: The chromatic roots of the antiprisms, A_n , $2 \leq n \leq 10$ (The chromatic roots of A_{10} are shown in red.)



expect the chromatic roots of the odd Mobius ladders, $\{L_n\}$ to follow the same pattern - c.f. the prisms and even Mobius ladders - and the location of the chromatic roots of the first four odd Mobius ladders suggests that this is the case.)

For the antiprisms the best result that we can obtain from Rouché's theorem using the computations that have been done is as follows.

Lemma 6.3 . The chromatic roots of the antiprisms lie in the region of the complex plane defined by

$$|z-2| \leq 2.5$$

Proof : When $|2-z| = 2.5$

$$\begin{aligned} & \left| 1 - 3z + z^2 - (1-z) \left(\left(\frac{5}{2} - z + \sqrt{\frac{9}{4} - z} \right)^n + \left(\frac{5}{2} - z - \sqrt{\frac{9}{4} - z} \right)^n \right) \right| \\ & \leq 13 + 2 \times 3 \cdot 5 \times \left(\frac{3}{2} \times \frac{5}{2} + \sqrt{2.5} \right)^n \\ & = F(n) \qquad \text{say,} \end{aligned}$$

$$\begin{aligned} \text{since } \left| \frac{5}{2} - z + \sqrt{\frac{9}{4} - z} \right| & \leq |2-z| + \frac{1}{2} + \left| \sqrt{2-z} \right| + \frac{1}{2} \\ & \leq \frac{3}{2} |2-z| + \sqrt{|2-z|} \\ & = \frac{3}{2} \times 2.5 + \sqrt{2.5} \end{aligned}$$

$$\begin{aligned} \text{and } F(n) & \leq 2.5^{2n} \quad \text{for } n \geq 18 \\ & = |2-z|^{2n} \end{aligned}$$

Thus by Rouché's theorem and direct computation up to $n = 18$ the result follows. □

The chromatic roots of the duals of the antiprisms, $\{A_n^*\}$ for $n = 2, 3, 4$, are shown in figure 6.

From these results and considering other information such as that given in the appendix to chapter 2, and the chromatic roots of the dual of the truncated icosahedron given in [10] (we are concerned with vertex colourings, [10] deals with face colourings) we make the following conjectures about the location of the chromatic roots.

These conjectures are based on complete information on 70 or more graphs with up to 38 vertices, and bounds and suggestive trends for several infinite families. It must be remembered however that the large members of the families considered possess considerable symmetry, and although their chromatic roots follow a pattern the first few members of the family usually exhibit considerable variation from the pattern. Most large graphs have no symmetry and are not 'large' members of a recursive family (although of course any graph may be considered as the first member of many families).

Figure 6: The chromatic roots of the duals of the antiprisms, A_n^* for $n = 2, 3, 4$.

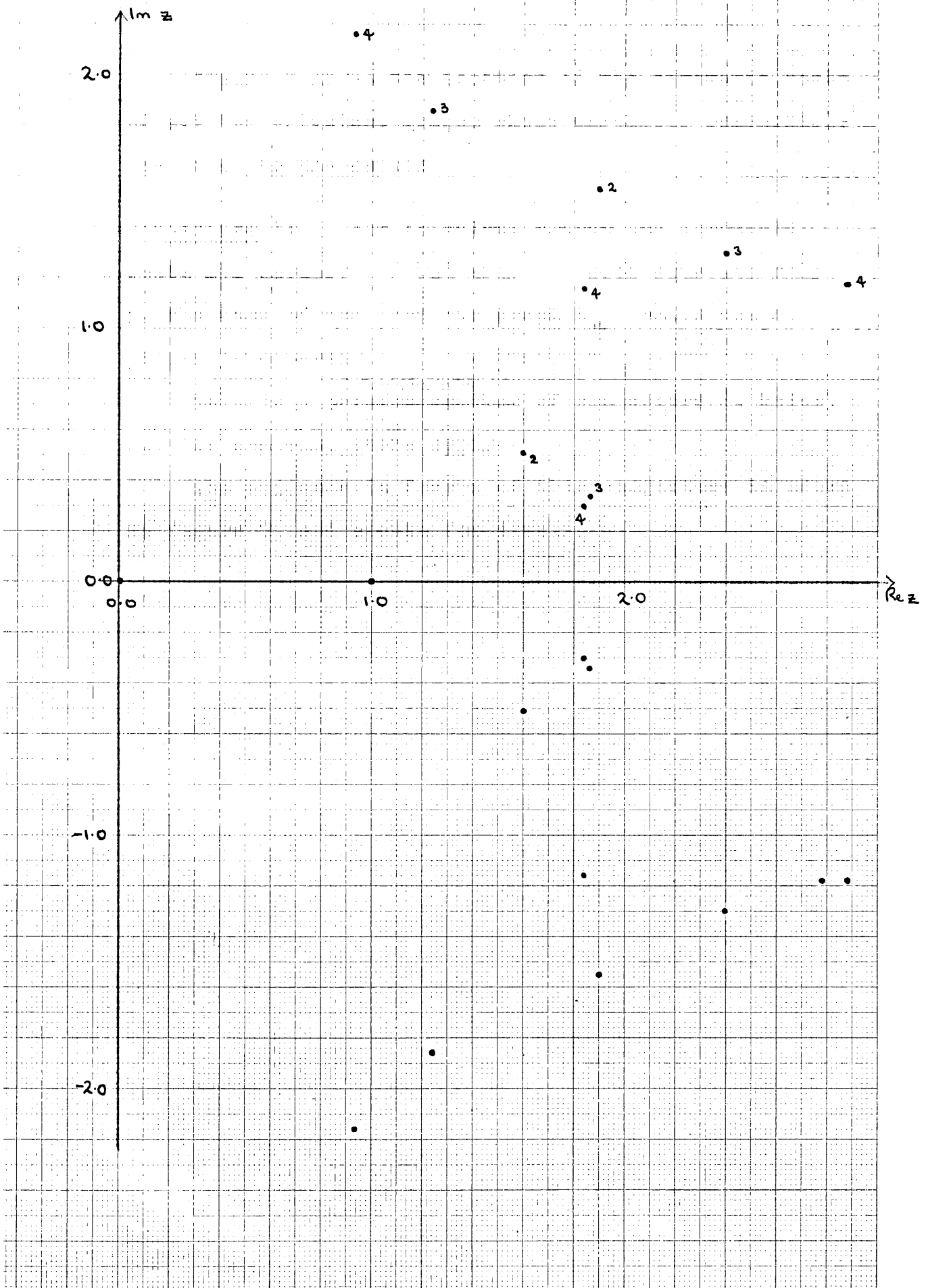


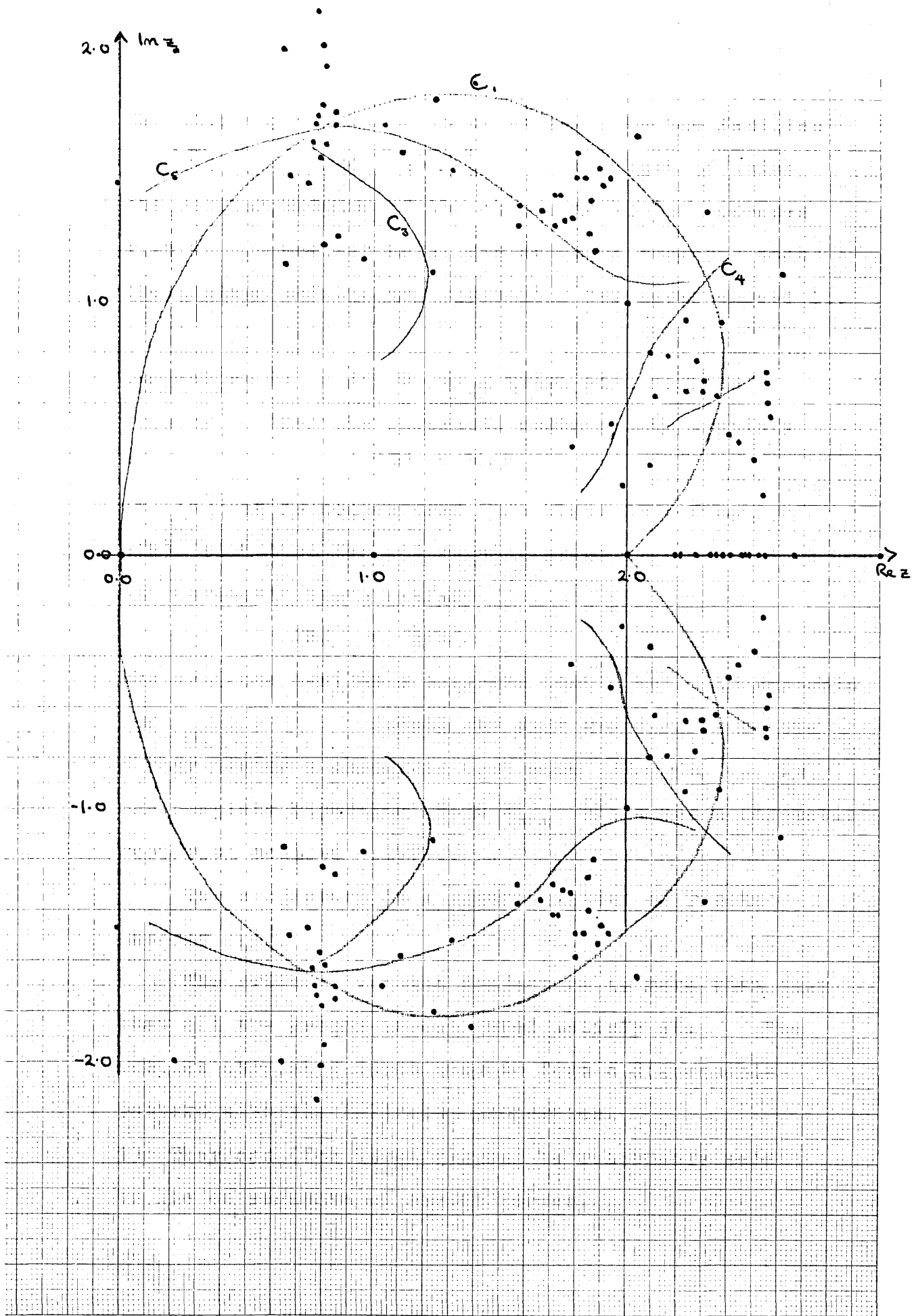
Figure 7 shows the location of the chromatic roots of all the trivalent graphs whose chromatic polynomials are known to the author.

Conjecture 1: For trivalent graphs the chromatic roots all satisfy $|z| \leq 3$ with equality only in the case of the complete graph K_4 which has no proper 3-colouring.

This is a ^{strengthening} ~~particular case~~ of conjecture 2, below, and the comments there apply.

The evidence in favour of this conjecture is shown in figure 7, and speaks for itself.

Figure 7: The chromatic roots of some trivalent graphs.



The only suggestion of doubt arises from the families $\{\Gamma_{i,2}\}$ and $\{\Gamma_{i,3}\}$ for both of which the maximum modulus of the chromatic roots increases with i . However here, the evidence suggests that the maximum modulus may approach 3 but never exceed it.

Conjecture 2: If G is a graph with maximum valency k then the chromatic roots of G satisfy

$$|z| < k+1$$

In [4] it is conjectured that there is a function $B(k)$ such that for all regular graphs of valency k the chromatic roots satisfy

$$|z| < B(k)$$

The evidence available suggests that $k+1$ is suitable for $B(k)$. Indeed the only graph known to the author having a chromatic root with $|z| > k$ is the complete bipartite graph $K_{4,4}$ which has a chromatic root with modulus 4.177... .

The conjecture is a generalisation of Brooks' theorem [6].

The evidence for the conjecture is more than just that all the graphs investigated satisfy it. The complex chromatic roots seem to form a discernable pattern with the following properties.

i) They can (for large graphs) be reasonably said to lie on segments of smooth curves.

- ii) The point on these curves extreme from the origin is much closer to the real axis than the imaginary axis.
- iii) The larger real roots follow the pattern set by the complex roots.
- iv) (The remark on page 78), The centroid of the chromatic roots lies at $(\phi(G)/2, 0)$
- v) (Brooks' theorem). The largest integer root is not greater than $k - 1$ (except for complete graphs, which have no complex chromatic roots).

Taking these five observations together, the behaviour of the chromatic roots of any counter-example to the conjecture must clearly be very unusual.

We conclude with a less significant observation.

Conjecture 3: For any graph G none of the complex chromatic roots of G lies inside the rhombus whose vertices are at $(0,0)$, $(2,0)$ and $(1, \pm \frac{1}{2})$ in the complex plane.

Combining these conjectures gives a very restricted region of the complex plane for the location of the chromatic roots. In addition ~~the~~ only one of our examples has a chromatic root with negative real part (the dodecahedron, which has a root at $-0.009 \pm 1.469i$) so it is almost certain there is a bound of the form $\text{Re } z \geq -\alpha$, for some real $\alpha > 0$ probably with $\alpha \leq 1$.

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