PH^D

NUCLEAR FIELD THEORIES.

E. Marjorie Williamson, M. Sc.,

Bedford College.

Nuclear Field Theories. E. M. Williamson.

Abstract.

The thesis opens with an account of the five-dimensional relativity theory of Klein and Kaluza and its development by Flint as a five-dimensional matrix geometry. This leads to a fundamental equation which contains the Dirac equation for the electron as a special case.

The introduction of the nuclear field in a manner analogous to that of the electromagnetic field in four-dimensional relativity theory leads to an equation representing the behaviour of a fundamental particle in the nuclear field. Comparison of this equation with the Dirac equation applicable to such a particle shows the presence of extra terms giving rise to an additional amount of energy which is considered to be due to the interaction of the particle with the field.

By analogy with Maxwell's electromagnetic theory the nuclear field equations are obtained for the various types of field and compared with those of other authors. The interaction energy is calculated in each case and agrees with the accepted forms.

An interesting result emerges when the total energy of the system (that is, field and particle) is considered, suggesting a possible escape from the difficulty of the infinities arising in the calculation of the interaction energy.

Finally, the case of the electron is considered in the light of the nuclear field theory.

Index.

Chapter I

THE GEOMETRY OF THE FIVE DIMENSIONAL CONTINUUM

The general theory of relativity as developed by Einstein is concerned primarily with the gravitational field. The electromagnetic field equations may be expressed formally in terms of the theory, but the two kinds of field appear to be logically distinct structures.

There have been a number of attempts to remove this apparent dualism and form a unitary theory. Weyl (1918), by modifying the concept of parallel displacement, found a natural place in the geometry of the four-dimensional Riemannian space for the electromagnetic potential.

An alternative approach was suggested by Kaluza (1921) and developed by Klein (1927). The fundamental metric tensor, g_{mn}, of Einstein's theory, by reason of its symmetry, has ten components which are all associated with the gravitational potentials. In order to obtain disposable quantities of the same form which might represent the electromagnetic potentials, Kaluza proposed an extension to a five-dimensional continuum where the counterpart of the fundamental metric tensor would have fifteen components.

The four-dimensional space time continuum was considered to be a cross section (x^5 = constant) of the five-dimensional space and it was supposed that all derivatives with respect to the new dimension were zero. The significance of this fifth coordinate will be discussed later.

The Fundamental Tensor and the Line Element

Consider, following Klein, a five dimensional Riemannian space in which the line element $d\sigma$ is given by

$$d\sigma^2 = \gamma_{\mu\nu}\, dx^\mu dx^\nu \tag{1.1}$$

where x^1, x^2, x^3, x^4 are the ordinary four dimensional co-ordinates and the $\gamma_{\mu\nu}$ are all independent of the new co-ordinate x^5.

(Greek indices will throughout take the values 1-5 and Roman indices, except where indicated, the values 1-4.)

The general transformation is of the type

$$x'^\mu = f^\mu(x^1, x^2, x^3, x^4, x^5). \tag{1.2}$$

Then $$dx'^\mu = \frac{\partial f^\mu}{\partial x^\nu}\, dx^\nu.$$

But for $\gamma_{\mu\nu}$ to be independent of x^5 in all systems of co-ordinates, $\frac{\partial f^\mu}{\partial x^\nu}$ must be a constant for all values of μ.

Now let $\frac{\partial f^\mu}{\partial x^5} = 0$ so that the first four equations of (1.2) become

$$x'^\mu = f^\mu(x^1, x^2, x^3, x^4) \tag{1.3}$$

which is the usual transformation of the ordinary co-ordinates, and let

$$\frac{\partial f^5}{\partial x^5} = 1 \tag{1.4}$$

so that the fifth equation becomes

$$x'^5 = x^5 + f^5(x^1, x^2, x^3, x^4). \tag{1.5}$$

Extending the usual processes to five dimensions

$$\gamma'_{\mu\nu} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\tau}{\partial x'^\nu}\gamma_{\sigma\tau}. \tag{1.6}$$

Then

$$\gamma'_{55} = \frac{\partial x^\sigma}{\partial x'^5}\frac{\partial x^\tau}{\partial x'^5}\gamma_{\sigma\tau} = \gamma_{55}$$

so that γ_{55} is an invariant quantity. The reason for the special choice of the constant in (1.4) is now apparent.

The quantity $\gamma_{5\mu}\, dx^\mu$ is also invariant since

$$\gamma'_{5\mu}\, dx'^\mu = \frac{\partial x^\sigma}{\partial x'^5}\frac{\partial x^\tau}{\partial x'^\mu}\gamma_{\sigma\tau}\frac{\partial x'^\mu}{\partial x^\alpha}\, dx^\alpha$$

$$= \gamma_{5\alpha}\, dx^\alpha$$

and so $$d\theta = \frac{\gamma_{5\mu}\, dx^\mu}{\sqrt{\gamma_{55}}}$$ is invariant. (1.7)

The line element as defined by (1.1) is invariant and so the quantity

$$(d\sigma^2 - d\theta^2) = \left(\gamma_{\mu\nu} - \frac{\gamma_{5\mu}\gamma_{5\nu}}{\gamma_{55}}\right) dx^\mu\, dx^\nu$$

is another invariant.

Rearranging the terms it may be written as

$$(d\sigma^2 - d\theta^2) = \left(\gamma_{mn} - \frac{\gamma_{5m}\gamma_{5n}}{\gamma_{55}}\right) dx^m\, dx^n.$$

This is also invariant in the four-dimensional space, for if

$$\left(\gamma'_{\mu\nu} - \frac{\gamma'_{5\mu}\gamma'_{5\nu}}{\gamma_{55}}\right) dx'^\mu\, dx'^\nu = \left(\gamma_{\alpha\beta} - \frac{\gamma_{5\alpha}\gamma_{5\beta}}{\gamma_{55}}\right) dx^\alpha\, dx^\beta$$

then it is also true that

$$\left(\gamma'_{mn} - \frac{\gamma'_{5m}\gamma'_{5n}}{\gamma_{55}}\right) dx'^m\, dx'^n = \left(\gamma_{ab} - \frac{\gamma_{5a}\gamma_{5b}}{\gamma_{55}}\right) dx^a\, dx^b.$$

Hence $(d\sigma^2 - d\theta^2)$ may be identified with the invariant

$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ of the four-dimensional theory.

It follows that

$$\gamma_{\mu\nu} = g_{\mu\nu} + \frac{\gamma_{5\mu}\gamma_{5\nu}}{\gamma_{55}} \tag{1.8}$$

and

$$d\sigma^2 = ds^2 + d\theta^2 . \tag{1.9}$$

The form of this last equation suggests that $d\theta$ may be regarded as the projection of $d\sigma$ in a direction normal to the ordinary Riemannian space.

Vectors and Tensors in Five Dimensions

Let A^{μ} be a five-vector.

This will transform in the usual way, i.e.

$$A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha} .$$

For the first four components

$$A'^{m} = \frac{\partial x'^{m}}{\partial x^{\alpha}} A^{\alpha}$$

$$= \frac{\partial x'^{m}}{\partial x^{a}} A^{a} \qquad \text{by (1.3)}$$

i.e., they transform in the same way as the components of a vector $a^{m} = A^{m}$ in four dimensions.

The component A^{5} does not transform simply but since for the co-variant components

$$A'_{\mu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} A_{\alpha}$$

$$A'_{5} = \frac{\partial x^{\alpha}}{\partial x'^{5}} A_{\alpha} = A_{5}$$

i.e. the fifth (co-variant) component is an invariant which will be represented by a.

Hence, from the point of view of ordinary space, the five vector appears as a four vector together with a scalar.

The fundamental tensor $\gamma_{\mu\nu}$ will have in five dimensions the same properties of lowering suffixes that g_{mn} has in four.

Hence $$A_\mu = \gamma_{\mu\nu} A^\nu$$

i.e. $$A_5 = \gamma_{5n} A^n + \gamma_{55} A^5$$

or $$A^5 = \frac{1}{\gamma_{55}} \left(a_. - \gamma_{5n} a^n \right). \tag{1.10}$$

Also
$$\begin{aligned} A_m &= \gamma_{mn} A^n + \gamma_{m5} A^5 \\ &= \left(g_{mn} + \frac{\gamma_{m5}\gamma_{n5}}{\gamma_{55}} \right) a^n \qquad \text{from (1.8) and (1.10)} \\ &\qquad + \frac{\gamma_{m5}}{\gamma_{55}} \left(a_. - \gamma_{5n} a^n \right) \\ &= a_m + \frac{\gamma_{m5}}{\gamma_{55}} a_. \end{aligned} \tag{1.11}$$

For any tensor of rank two

$$T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x'^\nu}{\partial x^\tau} T^{\sigma\tau}$$

and $$T'_{\mu\nu} = \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\tau}{\partial x'^\nu} T_{\sigma\tau} .$$

Hence, as in the case of the fundamental tensor, T_{55} is invariant.

Since
$$\begin{aligned} T'^{mn} &= \frac{\partial x'^m}{\partial x^\sigma} \frac{\partial x'^n}{\partial x^\tau} T^{\sigma\tau} \\ &= \frac{\partial x'^m}{\partial x^s} \frac{\partial x'^n}{\partial x^t} T^{st} \end{aligned}$$

these components transform as a contravariant tensor in four dimensions and since

$$T'^{\mu}{}_{5} = \frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial x^{\tau}}{\partial x'^{5}} T^{\sigma}{}_{\tau}$$

$$= \frac{\partial x'^{\mu}}{\partial x^{s}} T^{s}{}_{5}$$

these components transform as a vector in four dimensions.

So, from the point of view of ordinary space, the tensor appears to consist of a four-dimensional tensor T^{mn} together with a four vector $T^{n}{}_{5}$ and a scalar quantity T_{55}.

Returning now to the fundamental tensor $\gamma_{\mu\nu}$, its contravariant components $\gamma^{\mu\nu}$ are defined in the usual way and this results in the special property

$$\begin{aligned} \gamma^{\mu}{}_{\nu} &= 0 \quad \text{if } \mu \neq \nu, \\ &= 1 \quad \text{if } \mu = \nu. \end{aligned} \tag{1.12}$$

The components γ^{mn} have the same invariance properties as the g^{mn} and will be identified with them, and γ_{55} is an invariant whose significance is not yet clear. There remain the components having one index equal to five which are to be associated with the four vector ϕ_{m} of the electromagnetic field.

The vector quantities would be the $\gamma^{m}{}_{5}$ and by (1.12) these do not exist. However, ϕ_{m} may be associated with γ_{m5} although this quantity is not a four dimensional vector.

For consider a five vector Φ of which the invariant component $\Phi_5 = \phi_{.} = 0$. The co-variant components

$$\Phi_m' = \phi_m + \frac{\gamma_{m5}}{\gamma_{55}}\phi_{.} = \phi_m \qquad \text{from (1.11)}$$

transform according to

$$\Phi_m' = \frac{\partial x^\alpha}{\partial x'^m}\Phi_\alpha .$$

On the other hand

$$\gamma'_{m5} = \frac{\partial x^\alpha}{\partial x'^m}\frac{\partial x^\beta}{\partial x'^5}\gamma_{\alpha\beta}$$

$$= \frac{\partial x^\alpha}{\partial x'^m}\gamma_{\alpha 5}$$

so that in five dimensions Φ_m and γ_{m5} have the same invariance.

Hence it is possible to put, as Kaluza does,

$$\frac{\gamma_{m5}}{\gamma_{55}} = \alpha\Phi_m' = \alpha\phi_m \qquad (1.13)$$

where α is independent of the co-ordinates.

It must be noted that the two five vectors are not completely identified for the fifth components are quite different since $\quad = 0$ and $\gamma_{55} \neq 0$.

Furthermore, while ϕ_m is the true four-dimensional counterpart of Φ_m there is no such counterpart for γ_{m5} since quantities such as $g_{m.}$ do not exist.

Now since $\quad \gamma_{5\mu}\gamma^{\mu n} = 0$

$$\gamma_{55}\gamma^{5n} + \gamma_{5m}\gamma^{mn} = 0$$

and $\therefore$
$$\gamma^{5n} = -\frac{\gamma_{5m}}{\gamma_{55}} g^{mn} = -\alpha\phi^{n}. \qquad (1.14)$$

<u>Summarising</u>:-

For the fundamental tensor

$$\gamma^{mn} = g^{mn},$$

$$\gamma_{mn} = g_{mn} + \gamma_{55}\alpha^2\phi_m\phi_n.$$

For a five vector

$$A^m = a^m,$$

$$A_5 = a_.\,,$$

$$A_m = a_m + \alpha\phi_m a_.\,,$$

$$A^5 = \frac{a_.}{\gamma_{55}} - \alpha\phi_m a^m.$$

The relations between all the various components of the general five tensor and its four dimensional counterparts may be obtained in similar ways, but are not required here.

The Geodesics of the Space

In the theory of relativity the path of a particle of mass m in a gravitational field only is a geodesic in four-dimensional space whose equations may be written

$$\frac{d}{d\tau}\left(g_{mn}\frac{dx^n}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{ab}}{\partial x^m}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = 0 \qquad (1.15)$$

where $d\tau$ is the element of proper time given by $ds = c\,d\tau$.

Klein, following Kaluza, suggested that the path of a charged particle in a gravitational and an electromagnetic field should be a geodesic in the five-dimensional continuum.

By analogy with (1.15) the equations of this path will be

$$\frac{d}{d\tau}\left(\gamma_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\frac{\partial \gamma_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \qquad (1.16)$$

The fifth of these will be

$$\frac{d}{d\tau}\left(\gamma_{5\nu}\frac{dx^\nu}{d\tau}\right) = 0$$

since $\gamma_{\alpha\beta}$ does not contain x^5, i.e.

$$\gamma_{5\nu}\frac{dx^\nu}{d\tau} = a_0 \qquad \text{which is some constant.} \qquad (1.17)$$

The first four equations are

$$\frac{d}{d\tau}\left(\gamma_{mn}\frac{dx^n}{d\tau}\right) + \frac{d}{d\tau}\left(\gamma_{m5}\frac{dx^5}{d\tau}\right) - \frac{1}{2}\frac{\partial \gamma_{ab}}{\partial x^m}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} - \frac{\partial \gamma_{a5}}{\partial x^m}\frac{dx^a}{d\tau}\frac{dx^5}{d\tau} = 0$$

and substituting for γ_{mn}, γ_{m5} (1.8 and 1.13) and making use of (1.17), these become

$$\frac{d}{d\tau}\left(g_{mn}\frac{dx^n}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{ab}}{\partial x^m}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = a_0\alpha\left(\frac{\partial\phi_n}{\partial x^m} - \frac{\partial\phi_m}{\partial x^n}\right)\frac{dx^n}{d\tau}$$

$$= a_0\alpha\, B_{mn}\frac{dx^n}{d\tau}$$

(1.18)

where B_{mn} represents the electromagnetic field in the usual way

i.e. $B_{12} = B_z$, $iB_{14} = E_x$, etc. (1.19)

and (1.18) is the path of the particle in four dimensional space.

Now let the rest mass of the particle be m_0 and the charge upon it be e. According to classical theory its equation of motion is

$$\frac{d}{dt}\left(m\frac{dx^K}{dt}\right) = e\left(E + \frac{v}{c}\times B\right)_K \qquad K = 1,2,3,$$

or

$$\frac{d}{d\tau}\left(\frac{dx^K}{d\tau}\right) = \frac{e}{m_0 c}B_{Kn}\frac{dx^n}{d\tau} \quad , \qquad n = 1,2,3,4.$$

for

$$\frac{d\tau}{dt} = (1 - v^2/c^2)^{1/2} \quad , \qquad m = m_0(1 - v^2/c^2)^{-1/2}$$

The generalised form of this equation in Riemannian space is

$$\frac{d}{d\tau}\left(g_{mn}\frac{dx^n}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{ab}}{\partial x^m}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = \frac{e}{m_0 c}B_{mn}\frac{dx^n}{d\tau} \tag{1.20}$$

and this will be identical with (1.18) if

$$a_0\alpha = e/m_0 c \tag{1.21}$$

However, α is still not expressed in terms of known fundamental quantities.

Kaluza's work was extended by Fisher (1929) who supposed that the path of the charged particle in the five-dimensional continuum was, in addition, a null geodesic. This suggestion was put forward as a possible way of avoiding the well known difficulties which arise in both classical and quantum theories in connection with the structure and energy of an electron. The electron could then be treated in five dimensions in the same way as the photon is treated in four.

For a null geodesic

$$d\sigma^2 = ds^2 + d\theta^2 = 0. \qquad (1.22)$$

But $$ds^2 = -c^2 d\tau^2$$

and by definition $$d\theta = \frac{\gamma_{5\mu}}{\sqrt{\gamma_{55}}} dx^\mu = \frac{a_0}{\sqrt{\gamma_{55}}} d\tau. \qquad \text{by (1.17)}$$

Hence the condition (1.22) becomes

$$a_0^2 = c^2 \gamma_{55},$$

i.e. $$a_0 = c\sqrt{\gamma_{55}}, \qquad (1.23)$$

the positive square root being taken, since a negative sign would only mean that the geodesic is traversed in the reverse direction.

Thus $$\alpha = \frac{1}{\sqrt{\gamma_{55}}} \frac{e}{m_0 c^2} \quad \text{for a charge e.} \qquad (1.24)$$

Now since $\dfrac{\gamma_{m5}}{\gamma_{55}} = \alpha\phi_m$ etc. it seems that the geometry of the continuum depends not only upon the fields but also upon the charge and mass of the test particle used to explore them. This differs from the usual theories of Kaluza and Klein, who both suppose α to be a world constant, and means that the test particle will influence the properties of the space which it explores.

The constant γ_{55} still remains arbitrary, for having put

$$d\sigma^2 = \gamma_{\mu\nu}\, dx^\mu\, dx^\nu = 0$$

it is possible to divide throughout by any one of the $\gamma_{\mu\nu}$.

Also it is the ratio γ_{m5}/γ_{55} which has been given physical significance by associating it with ϕ_m.

So, without any loss of generality it is possible to put $\gamma_{55} = 1$. It will be seen later that this choice is particularly convenient.

Then
$$\begin{aligned} \gamma^{mn} &= g^{mn}, \\ \gamma_{m5} &= \alpha\phi_m, \\ \gamma^{m5} &= -\alpha\phi^m \end{aligned} \qquad (1.25)$$

and
$$\gamma_{mn} = g_{mn} + \alpha^2\phi_m\phi_n .$$

The Nature of the Fifth Co-ordinate

If the co-ordinates of a charged particle in the five dimensional continuum are represented by the five-vector x^μ then, as above, the four contravariant components correspond to the ordinary co-ordinates x^m of the four dimensional space and the covariant quantity x_5 is an invariant length.

In quantum theory there are two invariant lengths which occur, namely the "Compton wave length" h/m_0c and the so-called radius of the electron e^2/m_0c^2 and this suggests that for a fundamental particle, such as an electron, the fifth (covariant) co-ordinate might be identified with one of these.

Flint has recently developed the quantum form of the co-ordinate velocity momentum and force operators and from the latter derived the equation of motion of an electron in an electromagnetic field. Comparing this with the classical form of the equation, in the case of small velocities, they are seen to be identical if

$$x_5 = \frac{2}{3}\, e^2/m_0c^2 .$$

There has always been some doubt about the exact value of the multiplying constant and so it will be convenient to put

$$x_5 = K\, e^2/m_0c^2 . \tag{1.26}$$

The quantity dx_5 is also an invariant and in fact, since the path of the particle is a null geodesic

$$dx_5 = \gamma_{5\mu}\, dx^\mu = d\theta = c\,d\tau \tag{1.27}$$

Since

$$x^5 = x_5 - \alpha\phi_m x^m$$

for a particle without charge

$$x^5 = x_5 = 0$$

and the problem becomes a four dimensional one.

That is, it is only charged particles that have components in the fifth dimension.

The Momentum Vector

In four-dimensional space this is

$$p^m = m_0 \frac{dx^m}{d\tau} .$$

The analogue, in five-dimensional space, is the vector

$$P^\mu = m_0 \frac{dx^\mu}{d\tau} \qquad (1.28)$$

and $P^m = p^m$ as usual so that the first four components form the usual momentum vector.

The covariant components will be

$$P_\mu = \gamma_{\mu\nu} P^\nu$$
$$= \gamma_{\mu\nu} m_0 \frac{dx^\nu}{d\tau}$$

and the fifth of these

$$P_5 = m_0 \gamma_{5\nu} \frac{dx^\nu}{d\tau}$$

$$= m_0 a_0 \qquad \text{by } (1.17)$$

$$= m_0 c \qquad \text{since } a_0 = c \qquad (1.29)$$

which is, as expected, an invariant quantity.

Also $\quad P_m = p_m + \alpha \phi_m P_5$

$$= p_m + \frac{e}{c}\phi_m \text{ , on substituting for } \alpha \text{ and } P_5.$$

Now, in a four-dimensional continuum $p_m = \frac{\partial L}{\partial x^m}$ is conjugate to the co-ordinate x^m in the absence of an electromagnetic field.

When a field is present, if the motion of a particle of charge e is considered, then it is the quantity $\left(p_m + \frac{e}{c}\phi_m\right)$ which is the conjugate of x^m, i.e. in the five-dimensional theory, whether or not there is a field present, P_μ is the quantity conjugate to x^μ.

Thus it is P_μ which is replaced by the operator $\frac{h}{2\pi i}\frac{\partial}{\partial x^\mu}$ in the usual processes of quantum mechanics.

i.e.

$$(P_m)_{op} = \frac{h}{2\pi i}\frac{\partial}{\partial x^m} = \left(p_m + \frac{e}{c}\phi_m\right)_{op}$$

or

$$(p_m)_{op} = \frac{h}{2\pi i}\frac{\partial}{\partial x^m} - \frac{e}{c}\phi_m \quad ,$$

which is the familiar form of the operator when there is a field present, and shows the physical importance of the identification of γ_{m5} with ϕ_m.

The fifth operator $P_5 = m_0 c = \frac{h}{2\pi i}\frac{\partial}{\partial x^5}$.

Hence for a quantity F containing x^5

$$\frac{\partial F}{\partial x^5} = \frac{2\pi i}{h} m_0 c F \quad , \qquad (1.30)$$

that is, F contains x^5 as a factor $e^{\frac{2\pi i}{h} m_0 c x^5}$.

Then if x^5 changes by an amount h/m_0c this factor remains unaltered and it has been suggested by Flint (1940) that x^5 is similar in nature to an angular co-ordinate θ which may have an apparently fixed value in spite of comparatively large changes of 2π.

It is not necessary however for the present purpose to make any assumptions about x^5; all that is needed is the relation (1.30) above.

In four dimensions, in the absence of an electromagnetic field

$$p_x^2 + p_y^2 + p_z^2 - \frac{W^2}{c^2} = -m_0^2 c^2$$

i.e.

$$p_\mu p^\mu + m_0^2 c^2 = 0.$$

In five dimensions

$$\begin{aligned} P_\mu P^\mu &= (p_\mu + \alpha\phi_\mu m_0 c)p^\mu + m_0 c(m_0 c - \alpha\phi_\mu p^\mu) \\ &= p_\mu p^\mu + m_0^2 c^2 \\ &= 0. \end{aligned} \tag{1.31}$$

Covariant Differentiation in the Five-dimensional Continuum

The concept of parallel displacement may be extended to five dimensions. For such a displacement of the vector A^{μ} the change in its components

$$dA^{\mu} = -\Delta^{\mu}_{\rho\nu} A^{\rho} dx^{\nu} \tag{1.32}$$

where $\Delta^{\mu}_{\rho\nu}$ is the five-dimensional counterpart of the Christoffel index symbol.

Likewise for the covariant components

$$dA_{\mu} = \Delta^{\rho}_{\mu\nu} A_{\rho} dx^{\nu}. \tag{1.33}$$

Generalising the four-dimensional expressions

$$\Delta^{\mu}_{\rho\nu} = \frac{1}{2}\gamma^{\mu\sigma}\left(\frac{\partial\gamma_{\nu\sigma}}{\partial x^{\rho}} + \frac{\partial\gamma_{\rho\sigma}}{\partial x^{\nu}} - \frac{\partial\gamma_{\rho\nu}}{\partial x^{\sigma}}\right). \tag{1.34}$$

Hence $\Delta^{m}_{rn} = \frac{1}{2}\gamma^{ms}\left(\frac{\partial\gamma_{ns}}{\partial x^{r}} + \frac{\partial\gamma_{rs}}{\partial x^{n}} - \frac{\partial\gamma_{rn}}{\partial x^{s}}\right) + \frac{1}{2}\gamma^{m5}\left(\frac{\partial\gamma_{n5}}{\partial x^{r}} + \frac{\partial\gamma_{r5}}{\partial x^{n}} - \frac{\partial\gamma_{rn}}{\partial x^{5}}\right)$

and so, by means of the relations (1.25)

$$\Delta^{m}_{rn} = \Gamma^{m}_{rn} + \frac{\alpha^2}{2} g^{ms}\left[\phi_n\left(\frac{\partial\phi_s}{\partial x^{r}} - \frac{\partial\phi_r}{\partial x^{s}}\right) + \phi_r\left(\frac{\partial\phi_s}{\partial x^{n}} - \frac{\partial\phi_n}{\partial x^{s}}\right)\right]$$

since the $\gamma_{\mu\nu}$ do not contain x^5

$$= \Gamma^{m}_{rn} + \frac{\alpha^2}{2} g^{ms}\left(\phi_n B_{rs} + \phi_r B_{ns}\right)$$

(1.35)

where $$B_{rs} = \frac{\partial \phi_s}{\partial x^r} - \frac{\partial \phi_r}{\partial x^s} \qquad (1.36)$$

represents the electromagnetic field.

$$\Delta^{m}_{5n} = \frac{1}{2}\gamma^{ms}\left(\frac{\partial \gamma_{5s}}{\partial x^n} + \frac{\partial \gamma_{ns}}{\partial x^5} - \frac{\partial \gamma_{5n}}{\partial x^s}\right) + \frac{1}{2}\gamma^{m5}\left(\frac{\partial \gamma_{55}}{\partial x^n} + \frac{\partial \gamma_{n5}}{\partial x^5} - \frac{\partial \gamma_{5n}}{\partial x^5}\right)$$

$$= \frac{1}{2}\alpha g^{ms} B_{ns}. \qquad (1.37)$$

$$\Delta^{5}_{rn} = \frac{1}{2}\gamma^{5s}\left(\frac{\partial \gamma_{rs}}{\partial x^n} + \frac{\partial \gamma_{ns}}{\partial x^r} - \frac{\partial \gamma_{rn}}{\partial x^s}\right) + \frac{1}{2}\gamma^{55}\left(\frac{\partial \gamma_{r5}}{\partial x^n} + \frac{\partial \gamma_{n5}}{\partial x^r} - \frac{\partial \gamma_{rn}}{\partial x^5}\right)$$

$$= -\alpha\phi^s\Gamma_{nr,s} + \frac{1}{2}\alpha\left(\frac{\partial \phi_r}{\partial x^n} + \frac{\partial \phi_n}{\partial x^r}\right) - \frac{1}{2}\alpha^3\phi^s\left(\phi_r B_{ns} + \phi_n B_{rs}\right). \qquad (1.38)$$

$$\Delta^{5}_{r5} = \frac{1}{2}\gamma^{5s}\left(\frac{\partial \gamma_{rs}}{\partial x^5} + \frac{\partial \gamma_{5s}}{\partial x^r} - \frac{\partial \gamma_{r5}}{\partial x^s}\right) + \frac{1}{2}\gamma^{55}\left(\frac{\partial \gamma_{r5}}{\partial x^5} + \frac{\partial \gamma_{55}}{\partial x^r} - \frac{\partial \gamma_{r5}}{\partial x^5}\right)$$

$$= \frac{1}{2}\alpha^2\phi^s B_{sr}. \qquad (1.39)$$

$$\Delta^{m}_{55} = \Delta^{5}_{55} = 0. \qquad (1.40)$$

Chapter II

MATRIX GEOMETRY AND ITS APPLICATION TO QUANTUM THEORY.

The ideas of the quantum theory must now be introduced into this unified five-dimensional theory of gravitation and electromagnetism. The wave functions of the quantum theory are matrices and therefore such quantities must appear as a fundamental part of the metric of the continuum. For this reason the element of length is expressed in matrix form in the following way (Flint 1935).

For each point (x^σ) five fundamental matrices γ^μ (of four rows and four columns) are defined, related to the fundamental tensor (i.e. to the geometry of the space) by the equations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\gamma^{\mu\nu}. \tag{2.1}$$

Associated with these are the covariant matrices

$$\gamma_\mu = \gamma_{\mu\nu}\gamma^\nu \quad \text{etc.} \tag{2.2}$$

Because of their tensor character, the four-dimensional counterparts of the matrices γ^μ will be the matrices

$\beta^m = \gamma^m$ and the matrix $\beta_\cdot = \gamma_5$.

For the other components

$$\begin{aligned} \gamma_m &= \beta_m + \alpha\phi_m\beta_\cdot \,, \\ \gamma^5 &= \beta_\cdot + \alpha\phi_m\beta^m . \end{aligned} \tag{2.3}$$

The matrices γ_μ play a part analogous to that of the unit vectors of ordinary analysis so that the length of an ordinary five-vector whose components are A^μ will be

$\gamma_\mu A^\mu$ or $\gamma^\mu A_\mu$.

This is a matrix and will be referred to as the matrix length. A similar operator, but in four dimensions, has been suggested by Mimura (1935).

In order to obtain the value of this length it must be multiplied by some scalar matrix, ψ. In accordance with the general practice of quantum mechanics, a second scalar matrix η is associated with ψ and the length of the vector is defined to be

$$L = \eta \gamma^{\mu} A_{\mu} \psi . \tag{2.4}$$

Introduced in this way ψ may be described as a gauging factor. The connection between η and ψ will become apparent later on.

For the present it is supposed that η and ψ have such properties that when the vector quantity (A_{μ}) undergoes a parallel displacement to a neighbouring point there is no change in the value of L

i.e. $$d(\eta \gamma^{\mu} A_{\mu} \psi) = 0 . \tag{2.5}$$

It follows that

$$\frac{\partial \eta}{\partial x^{\nu}} dx^{\nu} \gamma^{\mu} A_{\mu} \psi + \eta \frac{\partial \gamma^{\mu}}{\partial x^{\nu}} dx^{\nu} A_{\mu} \psi + \eta \gamma^{\mu} \Delta^{\lambda}_{\mu\nu} dx^{\nu} \psi + \eta \gamma^{\mu} A_{\mu} \frac{\partial \psi}{\partial x^{\nu}} dx^{\nu} \equiv 0,$$

and since this must be true for all vectors A_{μ} and all displacements dx^{ν},

$$\frac{\partial \eta}{\partial x^{\nu}} \gamma^{\mu} \psi + \eta \left(\frac{\partial \gamma^{\mu}}{\partial x^{\nu}} + \gamma^{\rho} \Delta^{\mu}_{\rho\nu} \right) \psi + \eta \gamma^{\mu} \frac{\partial \psi}{\partial x^{\nu}} = 0 . \tag{2.6}$$

Let

$$\frac{\partial \gamma^\mu}{\partial x^\nu} + \gamma^\rho \Delta^\mu_{\rho\nu} = K^\mu{}_\nu \tag{2.7}$$

and consider the form of this quantity.

The five dimensional covariant derivative of the will vanish

i.e. $$\frac{\partial \gamma^{\mu\nu}}{\partial x^\sigma} + \Delta^\mu_{\alpha\sigma}\gamma^{\alpha\nu} + \Delta^\nu_{\alpha\sigma}\gamma^{\mu\alpha} = 0. \tag{2.8}$$

Eliminating the $\gamma^{\mu\nu}$ by means of equation (2.1)

$$\frac{\partial \gamma^\mu}{\partial x^\sigma}\gamma^\nu + \gamma^\mu \frac{\partial \gamma^\nu}{\partial x^\sigma} + \frac{\partial \gamma^\nu}{\partial x^\sigma}\gamma^\mu + \gamma^\nu \frac{\partial \gamma^\mu}{\partial x^\sigma} + \Delta^\mu_{\alpha\sigma}\left(\gamma^\alpha\gamma^\nu + \gamma^\nu\gamma^\alpha\right)$$
$$+ \Delta^\nu_{\alpha\sigma}\left(\gamma^\mu\gamma^\alpha + \gamma^\alpha\gamma^\mu\right) = 0$$

i.e., rearranging,

$$K^\mu{}_\sigma\gamma^\nu + \gamma^\nu K^\mu{}_\sigma + K^\nu{}_\sigma\gamma^\mu + \gamma^\mu K^\nu{}_\sigma = 0. \tag{2.9}$$

This is satisfied by

$$K^\mu{}_\nu = \Delta_\nu\gamma^\mu - \gamma^\mu\Delta_\nu \tag{2.10}$$

and also by

$$K^\mu{}_\nu = a_{\nu\lambda}\left(\gamma^\lambda\gamma^\mu - \gamma^\mu\gamma^\lambda\right) \tag{2.11}$$

where $a_{\nu\lambda}$ is not a matrix.

It is convenient to consider these solutions separately, for equation (2.7) must be satisfied also.

i.e., for the form (2.10)

$$\Delta_\nu\gamma^\mu - \gamma^\mu\Delta_\nu = \frac{\partial \gamma^\mu}{\partial x^\nu} + \gamma^\rho\Delta^\mu_{\rho\nu}. \tag{2.12}$$

For $\mu = u$, $\nu = u$,

$$\Delta_u\gamma^u - \gamma^u\Delta_u = \frac{\partial \gamma^u}{\partial x^u} + \gamma^r\Delta^u_{ru} + \gamma^5\Delta^u_{5u}$$

so that in terms of the four dimensional quantities

$$\Delta_n \beta^m - \beta^m \Delta_n$$

$$= \frac{\partial \beta^m}{\partial x^n} + \beta^r \left[\Gamma^m_{rn} + \tfrac{1}{2}\alpha^2 g^{ms}(\phi_n B_{rs} + \phi_r B_{ns}) \right] + (\beta_. - \alpha \phi_r \beta^r)\tfrac{1}{2}\alpha g^{ms} B_{ns} . \qquad (2.13)$$

It is usual in the quantum mechanics of the electron and other fundamental particles to neglect the effect of the gravitational field, this being small in comparison with the electromagnetic field.

Then the four dimensional quantities Γ^m_{rn} are zero, the distinction between co-variance and contravariance disappears and the matrices β_m are independent of the co-ordinates.

Hence equation (2.13) becomes

$$\Delta_n \beta_m - \beta_m \Delta_n = \tfrac{1}{2}\alpha \beta_. B_{nm} + \tfrac{1}{2}\alpha^2 \phi_n \beta_r B_{rm} \qquad (2.14)$$

which is satisfied if

$$\Delta_n = \tfrac{1}{4}\alpha B_{ln} \beta_l \beta_. + \tfrac{1}{8}\alpha^2 \phi_n B_{lr} \beta_l \beta_r . \qquad (2.15)$$

Similarly from equation (2.12) for $\mu = m$, $\nu = 5$

$$\Delta_5 \beta_m - \beta_m \Delta_5 = \tfrac{1}{2}\alpha \beta_r B_{rm} \qquad (2.16)$$

and this is satisfied if

$$\Delta_5 = \tfrac{1}{2}\alpha B_{lr} \beta_l \beta_r . \qquad (2.17)$$

The other equations of (2.12), i.e. $\mu = 5$, $\nu = n$ and $\mu = 5$, $\nu = 5$ are satisfied respectively by the expressions (2.15) and (2.17), providing a check upon these values of the quantities Δ_ν.

Considering the solution (2.11) on the other hand it is not possible to find a form for the quantities $a_{\nu\lambda}$ that permits of the equations

$$a_{\nu\lambda}\left(\gamma^{\lambda}\gamma^{\mu} - \gamma^{\mu}\gamma^{\lambda}\right) = \frac{\partial \gamma^{\mu}}{\partial x^{\nu}} + \gamma^{\rho}\Delta^{\mu}_{\rho\nu} \tag{2.18}$$

being satisfied.

Equation (2.7) may now be written

$$\frac{\partial \gamma^{\mu}}{\partial x^{\nu}} = -\Delta^{\mu}_{\rho\nu}\gamma^{\rho} + \Delta_{\nu}\gamma^{\mu} - \gamma^{\mu}\Delta_{\nu} \tag{2.19}$$

and expresses the change in the components of the matrix upon parallel displacement. The first term on the right hand side arises from the vector nature of the quantity and the additional terms are due to its matrix character.

A similar relation, but in four dimensions only, was given by Schrödinger (1932) - and it can be shown that equation (2.19), like its four-dimensional counterpart, is invariant with respect to a similarity transformation

$$\gamma'_{\alpha} = S^{-1}\gamma_{\mu} S .$$

It may be noted that a term of the form $f_{\nu} I$ can be added to the matrix Δ_{ν} , where I is the unit matrix and f_{ν} an ordinary quantity (not a matrix).

The Fundamental Equation

Returning to equation (2.6), of p. 22,

i.e. $$\frac{\partial \eta}{\partial x^\nu}\gamma^\mu\psi + \eta K^\mu{}_\nu\psi + \eta\gamma^\mu\frac{\partial\psi}{\partial x^\nu} = 0,$$

a comparison with the theory of gravitation suggests that there may be some restriction imposed upon the tensor $K^\mu{}_\nu$. The formal statement of this restriction will be the quantum law.

If this should be $K^\mu{}_\nu = 0$ (2.20)

then $$\frac{\partial \eta}{\partial x^\nu}\gamma^\mu\psi + \eta\gamma^\mu\frac{\partial\psi}{\partial x^\nu} = 0$$

i.e. $$\frac{\partial}{\partial x^\nu}(\eta\gamma^\mu\psi) - \eta\frac{\partial\gamma^\mu}{\partial x^\nu}\psi = 0.$$

Consider now the equations

$$\frac{\partial}{\partial x^u}(\eta\gamma^m\psi) - \eta\frac{\partial\gamma^m}{\partial x^u}\psi = 0,$$

i.e. $$\frac{\partial}{\partial x^u}(\eta\beta^m\psi) - \eta\frac{\partial\beta^m}{\partial x^u}\psi = 0$$

which if there is no gravitational field become

$$\frac{\partial}{\partial x^u}(\eta\beta_m\psi) - \eta\frac{\partial\beta_m}{\partial x^u}\psi = 0$$

or, since the β_m are independent of the co-ordinates,

$$\frac{\partial}{\partial x^u}(\eta\beta_m\psi) = 0$$

i.e. the quantity $\eta\beta_m\psi$ is independent of the co-ordinates and the whole significance of ψ as a gauging factor is lost. Hence $K^{\mu}{}_{\nu} = 0$ is too restrictive a condition and an alternative must be looked for.

The situation is similar to that arising in Einstein's theory when a general law of gravitation in empty space was sought. If the Riemann-Christoffel tensor R^{t}_{mns} vanished then space-time became flat and there could be no gravitational field. Einstein therefore proposed instead to set the reduced form of the tensor equal to zero.

In the present case a similar procedure is adopted and the quantum law is assumed to be

$$K^{\mu}{}_{\mu} = 0. \tag{2.21}$$

Hence from equation (2.6)

$$\frac{\partial \eta}{\partial x^{\mu}} \gamma^{\mu} \psi + \eta \gamma^{\mu} \frac{\partial \psi}{\partial x^{\mu}} = 0 \tag{2.22}$$

or, in the absence of gravitational and electromagnetic fields,

$$\frac{\partial}{\partial x^{\mu}} (\eta \gamma^{\mu} \psi) = 0.$$

However, without this restriction (i.e. the absence of fields, equation (2.6) may be written

$$\frac{\partial}{\partial x^{\nu}} (\eta \gamma^{\mu} \psi) + \Delta^{\mu}_{\rho\nu} \eta \gamma^{\rho} \psi = 0$$

which means that $\eta \gamma^{\mu} \psi$ is unchanged by a parallel displacement and may be regarded as a vector. The reduced form of this equation is

$$\frac{\partial}{\partial x^{\mu}} (\eta \gamma^{\mu} \psi) + \Delta^{\mu}_{\rho\mu} \eta \gamma^{\rho} \psi = 0. \tag{2.23}$$

But, by analogy with the four-dimensional theory,

$$\Delta^{\mu}_{\rho\mu} = \frac{1}{\sqrt{-\gamma}} \frac{\partial}{\partial x^{\rho}} (\sqrt{-\gamma})$$ where γ is the determinant $|\gamma_{\mu\nu}|$,

so that equation (2.23) becomes

$$\frac{1}{\sqrt{-\gamma}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-\gamma}\, \eta \gamma^{\mu} \psi \right) = 0. \qquad (2.24)$$

But $|\gamma_{\mu\nu}| = \gamma_{55} |g_{mn}|$ and it has been assumed that $\gamma_{55} = 1$. Hence $\sqrt{-\gamma} = \sqrt{-g}$ and this may be put equal to unity as is frequently done in the theory of relativity.

Hence $$\mathrm{div}\,(\eta \gamma^{\mu} \psi) = 0 .$$

This is then suggested as the fundamental equation of quantum mechanics.

The Dirac Equation

Consider now the equation

$$\gamma^{\mu} \frac{\partial\psi}{\partial x^{\mu}} = 0 \qquad (2.26)$$

i.e.
$$\gamma^{m} \frac{\partial\psi}{\partial x^{m}} + \gamma^{5} \frac{\partial\psi}{\partial x^{5}} = 0 .$$

In terms of four-dimensional quantities

$$\beta^{m} \frac{\partial\psi}{\partial x^{m}} + (\beta_{.} - \alpha\phi_{m}\beta^{m})\frac{h}{2\pi i} m_{0}c\psi = 0$$

i.e.
$$\beta^{m}\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^{m}} - \frac{e}{c}\phi_{m}\right)\psi + \beta_{.} m_{0}c\psi = 0$$

on substituting for a.

In the absence of a gravitational field $\beta^{m} = \beta_{m}$ and the equation becomes

$$\beta_{m}\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^{m}} - \frac{e}{c}\phi_{m}\right)\psi + \beta_{.} m_{0}c\psi = 0 . \qquad (2.27)$$

This may be written

$$\beta_{K}\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^{K}} - \frac{e}{c}\phi_{K}\right)\psi + \beta_{4}\left(\frac{h}{2\pi i}\frac{1}{ic}\frac{\partial}{\partial t} - \frac{e}{c}i\phi\right)\psi + \beta_{.} m_{0}c\psi = 0$$

where $K = 1,2,3$ only

and so, multiplying by $i\beta_{4}$,

$$\left[\left(\frac{h}{2\pi i}\frac{1}{c}\frac{\partial}{\partial t} + \frac{e}{c}\phi\right) + i\beta_{4}\beta_{K}\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^{K}} - \frac{e}{c}\phi_{K}\right) + i\beta_{4}\beta_{.} m_{0}c\right]\psi = 0 .$$

Comparing this with the usual form of Dirac's equation

$$\left(\frac{W}{c} + \alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta m_0 c\right)\psi = 0,$$

i.e. $\left[-\left(\frac{h}{2\pi i}\frac{1}{c}\frac{\partial}{\partial t} + \frac{e}{c}\phi\right) + \alpha_k\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^k} - \frac{e}{c}\phi_k\right) + \beta m_0 c\right]\psi = 0,$

the equations are seen to be identical if

$$\begin{aligned} \alpha_k &= -i\beta_4\beta_k \quad, \\ \beta &= -i\beta_4\beta. \end{aligned} \qquad (2.28)$$

The equation conjugate to (2.27) is

$$\beta_k^{\times}\left(-\frac{h}{2\pi i}\frac{\partial}{\partial x^k} - \frac{e}{c}\phi_k\right)\psi^{\times} + \beta_4^{\times}\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^4} + \frac{e}{c}\phi_4\right)\psi^{\times} + \beta.^{\times} m_0 c\,\psi^{\times} = 0$$

or in terms of the adjoint

$$-\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^k} + \frac{e}{c}\phi_k\right)\psi^{+}\beta_k + \left(\frac{h}{2\pi i}\frac{\partial}{\partial x^4} + \frac{e}{c}\phi_4\right)\psi^{+}\beta_4 + m_0 c\,\psi^{+}\beta. = 0.$$

Multiplying by β_4 from the right and making use of the fact that the βs anticommute

$$\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^k} + \frac{e}{c}\phi_k\right)\psi^{+}\beta_4\beta_k + \left(\frac{h}{2\pi i}\frac{\partial}{\partial x^4} + \frac{e}{c}\phi_4\right)\psi^{+}\beta_4\beta_4 - m_0 c\,\psi^{+}\beta_4\beta. = 0. \qquad (2.29)$$

Now consider the other relation suggested by equation (2.22)

$$\frac{\partial\eta}{\partial x^\mu}\gamma^\mu = 0. \qquad (2.30)$$

η being a quantity conjugate to ψ will contain x^5 in an exponential factor $e^{-\frac{2\pi i}{h} m_0 c x^5}$, so that

$$\frac{\partial\eta}{\partial x^m}\beta^m - \frac{2\pi i}{h} m_0 c\,\eta\left(\beta. - \alpha\phi_m\beta^m\right) = 0$$

or, in the absence of a gravitational field

$$\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^{u}} + \frac{e}{c}\phi_{u}\right)\eta\beta_4 - m_0 c\,\eta\beta. = 0.$$

This is identical with equation (2.29) if

$$\eta = \psi^{+}\beta_4 \tag{2.31}$$

and so from equation (2.22), both Dirac's equation and the complex conjugate equation have been obtained.

Equation (2.22) has, however, been divided somewhat arbitrarily into two parts, and this division should properly be made at an earlier stage.

The fundamental equation is (equation 2.25)

$$\frac{\partial\eta}{\partial x^{\nu}}\gamma^{\mu}\psi + \eta K^{\mu}{}_{\nu}\psi + \eta\gamma^{\mu}\frac{\partial\psi}{\partial x^{\nu}} = 0$$

with the restriction that $K^{\mu}{}_{\mu} = 0$.
Written in its reduced form it becomes

$$\frac{\partial\eta}{\partial x^{\mu}}\gamma^{\mu}\psi + \eta\left(\Delta_{\mu}\gamma^{\mu} - \gamma^{\mu}\Delta_{\mu}\right)\psi + \eta\gamma^{\mu}\frac{\partial\psi}{\partial x^{\mu}} = 0$$

and instead of (2.26) and (2.30) the equation and its conjugate should be

$$\gamma^{\mu}\left(\frac{\partial\psi}{\partial x^{\mu}} - \Delta_{\mu}\psi\right) = 0 \tag{2.32}$$

and
$$\left(\frac{\partial\eta}{\partial x^{\mu}} + \eta\Delta_{\mu}\right)\gamma^{\mu} = 0 \tag{2.33}$$

for although

$$\Delta_{\mu}\gamma^{\mu} - \gamma^{\mu}\Delta_{\mu} = K^{\mu}{}_{\mu} = 0$$

$$\gamma^\mu \Delta_\mu = \gamma^m \Delta_m + \gamma^5 \Delta_5$$

$$= \beta_m \left(\tfrac{1}{4} \alpha B_{lm} \beta_l \beta_. + \tfrac{1}{8} \alpha^2 \phi_m B_{lp} \beta_l \beta_p \right.$$
$$\left. + (\beta_. - \alpha \phi_m \beta_m) \tfrac{1}{8} \alpha B_{lp} \beta_l \beta_p \right.$$

$$= -\tfrac{1}{8} \alpha B_{lp} \beta_. \beta_l \beta_p \qquad \text{in the absence of gravitation}$$

$$= -\beta_. \Delta_5 \neq 0 .$$

Equation (2.32) is in fact the same as that suggested by Schrödinger (1932).

This suggests the addition of a term to the Dirac equation (Flint 1943) and this will modify the usual expression for the Hamiltonian.

Neglecting the gravitational field, as is usual, equation (2.34) becomes

$$\left[\beta_m \left(\frac{h}{2\pi i} \frac{\partial}{\partial x^m} - \frac{e}{c} \phi_m \right) + \beta_. m_0 c + \frac{h}{2\pi i} \beta_. \Delta_5 \right] \psi = 0$$

i.e. $$\left[\beta_k \left(\frac{h}{2\pi i} \frac{\partial}{\partial x^k} - \frac{e}{c} \phi_k \right) + \beta_4 \left(-\frac{h}{2\pi c} \frac{\partial}{\partial t} - \frac{e}{c} i\phi \right) + \beta_. m_0 c + \frac{h}{2\pi i} \beta_. \Delta_5 \right] \psi = 0 .$$

Multiplying through by $i\beta_4$ and putting the equation in terms of the matrices α, β

$$\left[\left(\frac{h}{2\pi i} \frac{\partial}{\partial t} + e\phi \right) - c\alpha_k \left(\frac{h}{2\pi i} \frac{\partial}{\partial x^k} - \frac{e}{c} \phi_k \right) - \beta m_0 c^2 + \frac{hc}{2\pi} \beta_4 \beta_. \Delta_5 \right] \psi = 0$$

or

$$-\frac{h}{2\pi i} \frac{\partial \psi}{\partial t} = H\psi + \frac{hc}{2\pi} \beta_4 \beta_. \Delta_5 \psi$$

where H is the usual Hamiltonian operator.

There is thus an operator H' added to the usual Hamiltonian where

$$H'\psi = \frac{hc}{2\pi}\,\beta_4\beta_.\,\Delta_5\,\psi$$

$$H' = \frac{hc}{2\pi}\,\frac{\alpha}{8}\,B_{lp}\,\beta_4\beta_.\beta_l\beta_p$$

$$= \frac{hc\alpha}{16\pi}\left[2B_{jk}\,\beta_4\beta_.\beta_j\beta_k + 2B_{j4}\,\beta_4\beta_.\beta_j\beta_4\right], \qquad j, k = 1, 2, 3$$

in cyclic order,

$$= \frac{hc\alpha}{8\pi}\left[B_{jk}\,i\beta\alpha_j\alpha_k + B_{j4}\,\beta\alpha_j\right]$$

in terms of the Dirac matrices,

$$= \frac{he}{8\pi m_0 c}\left[B_x\,i\beta\alpha_y\alpha_z + \cdots \text{ etc.} - E_x\,i\beta\alpha_x - \cdots \text{ etc.}\right].$$

This represents a mean energy density given by

$$\psi^+H'\psi = \frac{he}{8\pi m_0 c}\left[\left(B_x\,\psi^+ i\beta\alpha_y\alpha_z\psi + \cdots\right) - \left(E_x\,\psi^+ i\beta\alpha_x\psi + \cdots\right)\right]$$

and the proper values of both terms are real since the matrices $i\beta\alpha_j\alpha_k$ and $i\beta\alpha_j$ are Hermitian.

This extra term is of exactly the same form as the additional term which Pauli, working on different lines (1941), introduces into the Dirac equation (although his term appears to contain a disposable constant). He regards it as being unnececessary in the case of the electron, since the original

Dirac equation has already provided for the intrinsic magnetic and electric moments, but suggests that it might prove of value in the theory of the neutron.

More recently Borsellino and Caldironi (1946) make use of this same extra term in developing a theory of the interaction between fundamental particles, but do not commit themselves about the value of the magnetic moments λ which they associate with the various particles.

Allard (1950) also introduces the same expression, but with a smaller numerical factor, and regards it as a perturbation which is, however, too large for the case of the electron in an atom.

The extra term will be discussed further in a later chapter. It would seem that Dirac's first-order equation for the electron, in its original form, is not the most fundamental equation. Solutions of the Dirac equation are also solutions of the more fundamental equation

$$\mathrm{div}\,(\eta \gamma^{r} \psi) = 0.$$

Chapter III

THE NUCLEAR FIELD

The nuclear field is now introduced (Flint, 1935, 1945) in a way which corresponds to Weyl's treatment (1921) of the electromagnetic field in the four dimensional continuum. This, as developed by Eddington (1923) is briefly as follows. With every point of space-time is associated a standard of interval length so that an interval at a point is measured in terms of the standard at that point.

The length l of a vector at P, the point x^n, is measured in terms of the standard at P but when the vector has been transferred, by parallel displacement, to a neighbouring point P', $(x^n + dx^n)$, its length must be measured in terms of the standard at P' and may be denoted by $(l + dl)$. The change in length, dl, may be assumed to depend upon the original length and the displacement PP', and the relation may be supposed linear, so that

$$dl = \kappa_n l \, dx^n . \qquad (3.1)$$

Evidently κ_n is a vector quantity characteristic of the neighbourhood of P and Weyl's theory identifies this with the four-vector electromagnetic potential ϕ_n at the point P so that

$$dl = \phi_n l \, dx^n . \qquad (3.2)$$

In the five-dimensional continuum the significant quantity is the matrix length $\gamma^\mu A_\mu$ of a vector (A^μ). It will be supposed that for a parallel displacement of (A^μ) to a neighbouring point the change in the matrix length is $\Phi_\nu \gamma^\mu A_\mu \psi$ where Φ_ν is descriptive of the nuclear field in the neighbourhood of the original point

Φ_ν is therefore a matrix operator having the character of a co-variant vector.

To obtain the length of a vector the quantities η, ψ must be introduced as on p. 22 and the relation corresponding to equation (3.2) is then

$$d(\eta \gamma^\mu A_\mu \psi) = \eta \Phi_\nu \gamma^\mu A_\mu \psi \, dx^\nu \tag{3.3}$$

This must be satisfied for all vectors (A^μ) and all small displacements (dx^ν), hence

$$\frac{\partial \eta}{\partial x^\nu} \gamma^\mu \psi + \eta K^\mu_{\ \nu} \psi + \eta \gamma^\mu \frac{\partial \psi}{\partial x^\nu} = \eta \Phi_\nu \gamma^\mu \psi . \tag{3.4}$$

As in the previous chapter, it is the reduced form of this which gives the fundamental equations.

These may be written in the form

$$\gamma^\mu \left(\frac{\partial \psi}{\partial x^\mu} - \Delta_\mu \psi \right) = \gamma^\mu H_\mu \psi \tag{3.5}$$

and
$$\left(\frac{\partial \eta}{\partial x^\mu} + \eta \Delta_\mu \right) \gamma^\mu = \eta H_\mu^{+} \gamma^\mu . \tag{3.6}$$

The nuclear forces are so great compared with those of gravitation and electromagnetism that it is usually not necessary to take these latter into account. The term containing Δ_μ, which depends upon the electromagnetic field will therefore disappear and the fundamental equation for a particle in a nuclear field becomes

$$\gamma^{\mu} \frac{\partial \psi}{\partial x^{\mu}} = \gamma^{\mu} \#_{\mu} \psi \qquad (3.7)$$

together with the conjugate equation.

The particle under consideration will be taken to be a nucleon since electrons do not exist within the nuclear structure, i.e. within range of the nuclear forces. Equation (3.7) then refers to a proton or a neutron in the field of other nucleons.

As before, the co-ordinate x^5 when it occurs explicitly does so in the form exp. $\left(\frac{2\pi i}{h} Mcx^5\right)$ where M is a mass to be associated with the nucleon.

In the absence of any kind of field equation (3.7) reduces to

$$\beta_\mu \frac{\partial \psi}{\partial x^\mu} + \beta . \frac{2\pi i}{h} Mc\psi = 0 \tag{3.8}$$

which is Dirac's equation for a free particle of mass M.

Multiplying throughout by $\frac{hc}{2\pi i} i\beta_4$ and so expressing the equation in terms of the (four-dimensional) Dirac matrices α_K and β, the energy operator is given by

$$W\psi = -\frac{h}{2\pi i}\frac{\partial \psi}{\partial t} = -\frac{h}{2\pi i} c\alpha_K \frac{\partial \psi}{\partial x^K} - Mc^2\beta\psi . \tag{3.9}$$

In the presence of the nuclear field additional terms $\gamma^\mu H_\mu \psi$ occur on the right hand side of (3.8) and hence terms of the form

$$-\frac{hc}{2\pi}\beta_4 \gamma^\mu H_\mu \psi \tag{3.10}$$

are added to the energy operator of equation (3.9).

These terms represent additional energy due to the interaction of the nucleon with the nuclear field and the value of this additional energy is given by

$$-\psi^+\theta \frac{hc}{2\pi}\beta_4 \gamma^\mu H_\mu \psi . \tag{3.11}$$

The factor θ is introduced, as will be shown presently, to preserve the correct invariance, for any term added to the right hand side of equation (3.9) must be, like the term $\psi^+\beta Mc^2\psi$, invariant in the four-dimensional continuum.

The addition of a complex conjugate expression, in order that real values may be obtained, will be assumed throughout.

The field operator H_μ may be expressed more generally in the form

$$H_\mu = g_1 T_\mu + g_2 \gamma^\nu T_{\mu\nu} + g_3 \gamma^\nu \gamma^\rho T_{\mu\nu\rho} + g_4 \gamma^\nu \gamma^\rho \gamma^\sigma T_{\mu\nu\rho\sigma} \tag{3.12}$$

where g_1 , g_2 .... are constants and the antisymmetric tensors T_μ , $T_{\mu\nu}$ .... are representative of different types of field. Such a superposition of fields was suggested by Møller and Rosenfeld (1940).

In the first case, if $\theta = 1$, the interaction terms of (3.11) are $\quad -g_1 \frac{hc}{2\pi} \psi^+ \beta_4 (\beta_m t_m + \beta . t.) \psi$,
where, as before, the small letters t_m, t. denote the four-dimensional quantities corresponding to T_μ .

In terms of the Dirac matrices α , β this expression becomes

$$-g_1 \frac{hc}{2\pi} i \left(\psi^+ \alpha_K \psi \, t_K - \psi^+ \psi \, i \, t_4 + \psi^+ \beta \psi \, t. \right)$$

or $\qquad (3.13)$

which, as required, is an invariant quantity in four dimensions.

In the second case it is necessary to put $\theta = \beta$. when the interaction terms are

$$-g_2 \frac{hc}{2\pi} i \left(\psi^+ \alpha_K \psi \, t_K . - i \psi^+ \psi \, t_4 . \right)$$
$$- g_2 \frac{hc}{\pi} \left[\psi^+ (-i \beta \alpha_K \alpha_\ell) \psi \, t_{K\ell} + \psi^+ (\beta \alpha_K) \psi \, t_{4K} \right]. \tag{3.14}$$

For the third type of field quantity $\theta = 1$ and the interaction terms are

$$-g_3 \frac{hc}{2\pi}\Big[\psi^+(\alpha_K\alpha_\ell)\psi\, t_{4K\ell} + \psi^+(i\alpha_1\alpha_2\alpha_3)\psi\, t_{123} - \psi^+(-i\beta\alpha_K\alpha_\ell)\psi\, t_{K\ell.} - \psi^+(\beta\alpha_K)\psi\, t_{4K.}\Big], \qquad (3.15)$$

and for the fourth type, $\theta = \beta$. and the interaction terms are

$$-g_4 \frac{hc}{2\pi}\Big[\psi^+(\alpha_K\alpha_\ell)\psi\, t_{4K\ell.} + \psi^+(i\alpha_1\alpha_2\alpha_3)\psi\, t_{123.} + \psi^+(\beta\alpha_1\alpha_2\alpha_3)\psi\, t_{4123}\Big]. \qquad (3.16)$$

(The changing numerical factor ~~arises~~ arising from the antisymmetry of the field quantities is absorbed into the constants g.)

It is noticeable that all the terms in expressions (3.13) to (3.16) are in the form of scalar products of field quantities (t) and quantities such as $\psi^+\alpha_K\psi$ which refer to the nucleon. The quantities occurring are:-

<u>Scalars</u> $\psi^+\beta\psi$ and $t.$, (3.17)

<u>Four-vectors</u> $\psi^+\alpha_K\psi$, $-i\psi^+\psi$ and t_K , t_4
or $t_{K.}$, $t_{4.}$,

<u>Six vectors</u> $\psi^+(-i\beta\alpha_K\alpha_\ell)\psi$, $\psi^+(\beta\alpha_K)\psi$ and $t_{K\ell}$, t_{4K}
or $t_{K\ell.}$, $t_{4K.}$

<u>Pseudo-vectors</u> $\psi^+(\alpha_K\alpha_\ell)\psi$, $\psi^+(i\alpha_1\alpha_2\alpha_3)\psi$ and $t_{4K\ell}$, t_{123}
or $t_{4K\ell.}$, $t_{123.}$

and finally

<u>Pseudoscalars</u> $\psi^+(\beta\alpha_1\alpha_2\alpha_3)\psi$ and t_{4123} .

These are exactly the quantities enumerated by Kemmer (1938) whose field quantities χ and ϕ, as will be shown presently, are the four-dimensional counterparts of the five-dimensional (T).

Thus the interaction terms which Kemmer introduces into the Lagrangian function as extra terms occur naturally in the present theory.

Another interesting point arises here with regard to these additional energy terms. The energy operator (3.9) calculated from the usual Dirac equation (3.8) for a fundamental particle contains the rest mass M in the term $Mc^{\nu}\beta\psi$, the value of the corresponding energy density being $Mc^{\nu}\psi^{+}\beta\psi$. Hence if among the additional energy terms there should occur any of the form $\psi^{+}\beta\psi$, they might be interpreted as a corresponding alteration of mass and might account for the mass of the meson (Flint 1947).

Only the scalar field, however, gives rise to such a term,

$$\text{viz.} \quad -g_1 \frac{hc}{2\pi} i\, t_{.}\, \psi^{+}\beta\psi$$

and the total (real) energy contribution from this term

$$= + \frac{hc}{4\pi} \int_v \left(i g_1^{\times} t_{.}^{\times} - i g_1 t_{.} \right) \psi^{+}\beta\psi \, dv \quad . \qquad (3.18)$$

This integral can be evaluated when the form of the field quantity t. has been determined, so for the present the idea will not be developed further. It seems unlikely to provide a theory of the meson mass, since the occurrence of mesons is certainly not confined to the scalar field.

It will be supposed then that a nuclear field exists in the neighbourhood of a nucleon just as an electromagnetic field exists in the region surrounding an electric charge, the new field being described by certain tensors which are of different types, so that it may be a scalar field, a vector field, a pseudo-vector field or a pseudo-scalar field. It may also be a combination of two or more of these kinds of field.

The analogy with the electromagnetic field suggests that the interaction between the nucleons and the nuclear fields may be treated in a way similar to the interaction between the polarization of a material medium and the electric and magnetic fields which give rise to this polarization.

The terms of the expression (3.11) will be considered from this point of view.

First, however, it is convenient to consider the pure nuclear field in free space, and the comparison with the electromagnetic theory leads naturally to a consideration of the vector field in the first place.

The theory of the vector nuclear field was developed independently by Kemmer (1938) and Yukawa, Sakata and Taketani (1938), who based their analysis on the work of Proca (1936, 1937), representing the field by an antisymmetric tensor (Kemmer, χ_{ab} , Yukawa $\overline{F}, \overline{G}$) of the second rank analogous to the electric and magnetic fields, together with a four-vector, (Kemmer, ϕ^a , Yukawa $\overline{U}$. U_0), these quantities being related by field equations which are generalisations of Maxwell's equations for the electromagnetic field.

The equations of Proca's paper had already been obtained by Fisher and Flint (1930) in an attempt to develop a field theory of the electron. A number of other workers in this field, Darwin (1928), Whittaker (1928) and Frenkel (1928), had the same object but they were all (including Proca), in fact, developing a field theory appropriate to a particle of integral spin and therefore not an electron. Then the work of Fermi (1934) on β-disintegration, Yukawa's earlier papers (1935, 1937) on the nuclear forces and the discovery of the meson in the cosmic radiation (Anderson and Neddermeyer, 1936, 1937) led to a renewed interest in these field theories.

The five-dimensional form of the nuclear field theory as developed here is due to Flint (1945, 1947).

The Vector Field in Free Space. The Field Equations.

The vector nuclear field is represented by the five-dimensional antisymmetric tensor $T^{\mu\nu}$.

It is supposed that, by analogy with Maxwell's equations for the electromagnetic field in free space, the field equations are

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{3.19}$$

and

$$\frac{\partial T_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial T_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial T_{\nu\lambda}}{\partial x^{\mu}} = 0 \tag{3.20}$$

together with a similar set of equations for the complex conjugate quantities $T^{\times\mu\nu}$, for the field quantities are to describe a particle which may be charged and so, unlike those of the electromagnetic theory, cannot be real.

Some assumption must now be made as to the dependence of the field tensors upon the co-ordinate x^5. As before, this co-ordinate occurs explicitly only in the form exp. $\frac{2\pi i}{h} \mu_0 c x^5$ where μ_0 is the mass of the particle associated with the nuclear field. This particle is the meson, in the same way as the photon is associated with the electromagnetic field, and therefore μ_0 is taken to be the mass of the meson. As will be seen, this is in accord with Yukawa's theory and leads to his field equations and hence to the correct range for the nuclear forces.

Hence for any of the field quantities $T_{\mu\nu}$ the differential operation $\partial/\partial x^5$ is to be regarded as equivalent to multiplication by $\frac{2\pi i}{h}\mu_0 c$. For the complex conjugate field quantities $T^{\times}_{\mu\nu}$ the sign of this factor is reversed.

Let $$\frac{2\pi}{h}\mu_0 c = \kappa . \qquad (3.21)$$

Then from equation (3.20), for the case $\lambda = 5$

$$i\kappa T_{mn} + \frac{\partial T_{5m}}{\partial x^n} + \frac{\partial T_{n5}}{\partial x^m} = 0$$

or $$\kappa t_{mn} = \frac{\partial}{\partial x^m}(it_{n.}) - \frac{\partial}{\partial x^n}(it_{m.})$$ in the absence of any other field, gravitational or electro-magnetic.

$$= \frac{\partial f_n}{\partial x^m} - \frac{\partial f_m}{\partial x^n}$$ if $f_m = it_m.$ (3.22)

while equation (3.19) may be written

$$\frac{\partial T^{mn}}{\partial x^n} + i\kappa T^{m5} = 0$$

or $$\frac{\partial t^{mn}}{\partial x^n} = \kappa f^m . \qquad (3.23)$$

Equations (3.22) and (3.23) are exactly Kemmer's form of the field equations.

Expressing equations (3.19) and (3.20) in detail in terms of the co-ordinates (x, y, z, t) the field equations obtained are identical with those of Yukawa's paper III (1938).

Again, from equation (3.22) it is clear that all the field quantities t_{mn} can be derived from the four quantities $t_{m.}$.

But these, by (3.19), are related by the equation

$$\frac{\partial}{\partial x^{\mu}}\left(t_{\mu\cdot}\right) = 0 \tag{3.24}$$

and hence there are only three independent quantities which describe the field and its associated particle. This corresponds to the three orientations of the spin of the particle.

The Klein-Gordon Equation

For a particle of mass μ_0, the energy W and the momentum p are related by the equation

$$\frac{W^2}{c^2} = p_x^2 + p_y^2 + p_z^2 + \mu_0^2 c^2 \tag{3.25}$$

and the usual procedure of replacing p_x by $\frac{h}{2\pi i}\frac{\partial}{\partial x}$ and W by $-\frac{h}{2\pi i}\frac{\partial}{\partial t}$ (3.26)

leads to the relativistic second order equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \left(\frac{2\pi}{h}\right)^2 \mu_0^2 c^2 \psi$$

$$= \kappa^2\psi \tag{3.27}$$

which is usually referred to as the Klein-Gordon equation (Gordon, 1926).

That this is satisfied by all the vector field quantities of the present theory is seen by differentiating equation (3.20) with respect to x^{λ} and making use of equation (3.19), for

$$\frac{\partial^2}{\partial x^{\lambda 2}} T_{\mu\nu} + \frac{\partial}{\partial x^{\nu}}\frac{\partial T_{\lambda\mu}}{\partial x^{\lambda}} + \frac{\partial}{\partial x^{\mu}}\frac{\partial T_{\nu\lambda}}{\partial x^{\lambda}} = 0$$

and hence $$\frac{\partial^2}{\partial x^{\lambda 2}} T_{\mu\nu} = 0$$

i.e. since $$\frac{\partial}{\partial x^5} T_{\mu\nu} = i\kappa T_{\mu\nu} ,$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa^2\right) T_{\mu\nu} = 0 \qquad (3.28)$$

as required.

The Energy Momentum Tensor

A tensor $R^\mu{}_\nu$ may be defined in a way analogous to the energy momentum tensor for the electromagnetic field.

$$R^\mu{}_\nu = \frac{1}{2}\left(T^{\times\mu\alpha} T_{\nu\alpha} - \frac{1}{4}\gamma^\mu{}_\nu T^{\times\alpha\beta} T_{\alpha\beta} + \text{c. conjugate}\right). \qquad (3.29)$$

Then $$R^4{}_4 = \frac{1}{2}\Big[T^{\times 4k} T_{4k} + T^{\times 45} T_{45} - \frac{1}{2}\left(T^{\times kl} T_{kl} + T^{\times k4} T_{k4} + T^{\times k5} T_{k5} + T^{\times 45} T_{45}\right) + \text{c. conjugate}\Big], \qquad (k<l),$$

$$= -\frac{1}{2} t^\times_{kl} t_{kl} + \frac{1}{2} t^\times_{4k} t_{4k} - \frac{1}{2} t^\times_{k.} t_{k.} + \frac{1}{2} t^\times_{4.} t_{4.}$$

since there are no other fields.

Hence the energy density

$$W = -R_{44}$$

$$= \frac{1}{2}\left(t^\times_{kl} t_{kl} - t^\times_{4k} t_{4k} + t^\times_{k.} t_{k.} - t^\times_{4.} t_{4.}\right). \qquad (3.30)$$

It is often more convenient to have the field quantities expressed in terms of the real co-ordinate $x^0 = ct$ instead of the imaginary $x^4 = ict$.

Then $$t^{\kappa 4} = it^{\kappa 0}$$

and $$t_{\kappa 4} = -it_{\kappa 0} \; .$$

The complex conjugate of $t^{\kappa l}$ is $t^{\times \kappa l}$, and of $t^{\kappa 0}$ is $t^{\times \kappa 0}$. For the index 4 the starred quantity $t^{\times \kappa 4}$ is not, in the ordinary sense, the complex conjugate of $t^{\kappa 4}$ but is defined by the relations $$t^{\times \kappa 4} = it^{\kappa 0}$$

and $$t^{\times}_{\kappa 4} = -it^{\times}_{\kappa 0} \, .$$

The energy density may now be written

$$W = \tfrac{1}{2}\left(t^{\times}_{\kappa l} t_{\kappa l} + t^{\times}_{\kappa 0} t_{\kappa 0} + t^{\times}_{\kappa \cdot} t_{\kappa \cdot} + t^{\times}_{0 \cdot} t_{0 \cdot} \right) \qquad (3.31)$$

when it becomes clear that it is, as it should be, a positive quantity. It is, in fact, apart from a numerical multiplying factor, identical with the expression obtained by Kemmer from his Lagrangian.

The five-dimensional energy-momentum tensor contains additional terms of the type

$$R^{\mu}{}_{5} = \tfrac{1}{2}\left(T^{\times \mu a} T_{5a} + c.c \right)$$ (since $a \neq 5$ owing to the antisymmetry of $T_{\mu\nu}$)

and the quantity represented by such terms is conserved, for

$$\frac{\partial}{\partial x^{\mu}} R^{\mu}{}_{5} = \frac{1}{2}\left(\frac{\partial T^{\times \mu a}}{\partial x^{\mu}} T_{5a} + T^{\times \mu a} \frac{\partial T_{5a}}{\partial x^{\mu}} + c.c. \right)$$

and from the field equations (3.19) and (3.20)

$$\frac{\partial T^{a\mu}}{\partial x^{\mu}} + i\kappa T^{a5} = 0$$

$$\frac{\partial T^{\times a\mu}}{\partial x^{\mu}} - i\kappa T^{\times a5} = 0 \, ,$$

$$\frac{\partial T_{5a}}{\partial x^u} + \frac{\partial T_{u5}}{\partial x^a} + ik\,T_{au} = 0$$

and
$$\frac{\partial T^x_{5a}}{\partial x^u} + \frac{\partial T^x_{u5}}{\partial x^a} - ik\,T^x_{au} = 0.$$

Hence

$$2\frac{\partial}{\partial x^u} R^u{}_5 = \frac{1}{ik}\left(\frac{\partial T^x_{5a}}{\partial x^u} - \frac{\partial T^x_{5u}}{\partial x^a}\right)\frac{\partial T_{5a}}{\partial x^u} + ik\,T^{x5a}\,T_{5a}$$
$$- \frac{1}{ik}\left(\frac{\partial T_{5a}}{\partial x^u} - \frac{\partial T_{5u}}{\partial x^a}\right)\frac{\partial T^x_{5a}}{\partial x^u} - ik\,T^{5a}\,T^x_{5a}$$
$$= 0.$$

This quantity $R^u{}_5$ with a multiplying constant of appropriate dimensions, represents a current density, and again is of the same form as Kemmer's expression.

Thus for the case of the vector meson field in free space, all Kemmer's results follow directly from the concept of a five-dimensional field tensor

The other three types of field, as Kemmer first suggested, may now be treated in exactly the same way as the vector field.

The Scalar Field

This is the type of field considered by Yukawa in his earlier papers (1935, 1937). From the present viewpoint such a field may be described by the five-dimensional field quantity T^{μ} for which the field equations are similar to the vector case,

i.e.
$$\frac{\partial T^{\mu}}{\partial x^{\mu}} = 0 \tag{3.33}$$

and
$$\frac{\partial T_{\mu}}{\partial x^{\nu}} - \frac{\partial T_{\nu}}{\partial x^{\mu}} = 0 \tag{3.34}$$

together with similar relations for $T^{\times\mu}$.

These may be written

$$\frac{\partial T^{\mu}}{\partial x^{\mu}} = -i\kappa T^{5}$$

i.e.
$$\frac{\partial t_{\mu}}{\partial x^{\mu}} = \kappa(-it.) \tag{3.35}$$

and
$$\frac{\partial T_{5}}{\partial x^{\mu}} = i\kappa T_{\mu}$$

or
$$\frac{\partial}{\partial x^{\mu}}(-it.) = \kappa t_{\mu} \tag{3.36}$$

and equations (3.35) and (3.36) are the field equations used by Kemmer.

From equations (3.34) and (3.33)

$$\frac{\partial^2}{\partial x^{\nu 2}} T_{\mu} = \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial T_{\nu}}{\partial x^{\nu}}\right) = 0 \tag{3.37}$$

and therefore the field quantities all satisfy the Klein-Gordon equation and so are appropriate to a particle of mass μ_0 Since the four-vector quantities t_μ are all derived from the one scalar t. in this case the equations describe a particle of zero spin.

The energy-momentum current tensor will be

$$R^\mu{}_\nu = \frac{1}{2}\left(T^{\times\mu}T_\nu - \frac{1}{2}\gamma^\mu{}_\nu T^{\times\alpha}T_\alpha + c.c.\right) \tag{3.38}$$

and

$$\frac{\partial}{\partial x^\mu} R^\mu{}_\nu = \frac{1}{2}\left(\frac{\partial T^{\times\mu}}{\partial x^\mu}T_\nu + T^{\times\mu}\frac{\partial T_\nu}{\partial x^\mu} + \frac{\partial T^\mu}{\partial x^\mu}T^\times_\nu + T^\mu\frac{\partial T^\times_\nu}{\partial x^\mu} - T^{\times\alpha}\frac{\partial T_\alpha}{\partial x^\nu} - \frac{\partial T^\times_\alpha}{\partial x^\nu}T^\alpha\right)$$

$$= 0$$

by means of the field equations (3.33) and (3.34). Hence the conservation principles are satisfied.

The current density is proportional to

$$R^\mu{}_5 = \frac{1}{2}\left(T^{\times\mu}T_5 + T^\mu T^\times_5\right)$$

$$= \frac{1}{2}\left(t^\times_\mu t_. + t_\mu t^\times_.\right) \tag{3.39}$$

and the energy density

$$W = -R_{44} = -\frac{1}{2}\left(T^\times_4 T_4 - \frac{1}{2}T^{\times\alpha}T_\alpha + c.c\right)$$

$$= \frac{1}{2}\left(t_k{}^\times t_k + t^\times_0 t_0 + t^\times_. t_.\right) \tag{3.40}$$

and is positive, as it should be.

The Pseudo-vector Field

In this case the field tensor is $T^{\mu\nu\lambda}$ and completely antisymmetric.

The field equations are

$$\frac{\partial T^{\mu\nu\lambda}}{\partial x^\lambda} = 0 \qquad (3.41)$$

and
$$\frac{\partial T_{\mu\nu\lambda}}{\partial x^\rho} - \frac{\partial T_{\rho\mu\nu}}{\partial x^\lambda} + \frac{\partial T_{\lambda\rho\mu}}{\partial x^\nu} - \frac{\partial T_{\nu\lambda\rho}}{\partial x^\mu} = 0 \qquad (3.42)$$

and it follows, in the same way as before, that all these field quantities satisfy the Klein-Gordon equation.

The corresponding four-dimensional quantities are t_{kl4}, $t_{kl\cdot}$, $t_{k4\cdot}$ and t_{123} .

From equation (3.42), for μ, ν, λ, ρ = 1, 2, 3, 5

$$ik\, t_{123} = \frac{\partial}{\partial x^1} t_{23\cdot} + \frac{\partial}{\partial x^2} t_{31\cdot} + \frac{\partial}{\partial x^3} t_{12\cdot} = \operatorname{div} t_{kl\cdot} \qquad (3.43)$$

and for 1, 2, 4, 5

$$ik\, t_{kl4} = \operatorname{curl} t_{j4\cdot} - \frac{1}{c}\frac{\partial}{\partial t} t_{kl\cdot} \qquad (j, k, l \text{ cyclic}) \qquad (3.44)$$

Thus the field quantities t_{kl4} , t_{123} may be expressed in terms of those field quantities containing a suffix 5, just as in the case of the vector meson field t_{kl} , t_{k4} could be expressed in terms of the vector $f_\mu = i t_\mu$.

In the present case let $f_{\mu\nu} = i t_{\mu\nu\cdot}$

$$\begin{aligned} &\text{i.e.} \quad f_{kl} = i t_{kl\cdot} \\ &\text{and} \quad f_{k4} = i t_{k4\cdot} \end{aligned} \qquad (3.45)$$

By reason of the antisymmetry of $T_{\mu\nu\lambda}$ no such terms as $f_{\mu\cdot}$ will have any meaning and therefore (f) is a six-vector consisting of two (three-dimensional) vectors. Equations (3.43), (3.44) relate these vectors to the four quantities t_{kl4}, t_{123} which make up the pseudo-vector (in four dimensions) from which this type of field gets its name. Hence this field also will describe a particle of spin one.

The various field equations, which will not be written out in detail at this stage, show a form somewhat similar to those for the vector field.

The energy momentum current tensor is defined as

$$R^{\mu}{}_{\nu} = \frac{1}{2}\left(\frac{1}{2} T^{\times\mu\alpha\beta} T_{\nu\alpha\beta} - \frac{1}{12}\gamma^{\mu}{}_{\nu} T^{\times\alpha\beta\gamma} T_{\alpha\beta\gamma} + c.c\right) \tag{3.46}$$

the numerical factors allowing for the multiplicity of the tensor components arising from the antisymmetry properties. Then the energy density

$$W = -R_{44} = \frac{1}{2}\left(t^{\times}_{123} t_{123} + t^{\times}_{kl0} t_{kl0} + t^{\times}_{kl\cdot} t_{kl\cdot} + t^{\times}_{k0\cdot} t_{k0\cdot}\right) \tag{3.47}$$

and is positive.

The Pseudo-scalar Field

In this case the field tensor is the antisymmetric tensor

$$T^{\mu\nu\lambda\rho}.$$

The field equations are

$$\frac{\partial T^{\mu\nu\lambda\rho}}{\partial x^{\rho}} = 0 \tag{3.48}$$

and

$$\frac{\partial T_{\mu\nu\lambda\rho}}{\partial x^{\sigma}} + \frac{\partial T_{\sigma\mu\nu\lambda}}{\partial x^{\rho}} + \frac{\partial T_{\rho\sigma\mu\nu}}{\partial x^{\lambda}} + \frac{\partial T_{\lambda\rho\sigma\mu}}{\partial x^{\nu}} + \frac{\partial T_{\nu\lambda\rho\sigma}}{\partial x^{\mu}} = 0 \tag{3.49}$$

and from these equations all the field quantities may be shown to satisfy the Klein-Gordon equation.

From equation (3.48)

$$ik\, t_{124\cdot} = \frac{\partial}{\partial x^3} t_{1234} \quad \text{etc.} \tag{3.50}$$

and $\quad ik\, t_{123\cdot} = \dfrac{\partial}{\partial x^4} t_{1234}$.

Hence the field quantities $t_{123\cdot}$ and $t_{kl4\cdot}$ are all derived from the pseudo-scalar t_{1234} and hence as in the case of the scalar field, describe a particle of zero spin.

The energy momentum current tensor is

$$R^{\mu}{}_{\nu} = \frac{1}{2}\left(\frac{1}{6} T^{\times\mu\alpha\beta\gamma} T_{\nu\alpha\beta\gamma} - \frac{1}{48}\gamma^{\mu}{}_{\nu} T^{\times\alpha\beta\gamma\epsilon} T_{\alpha\beta\gamma\epsilon} + c.c\right) \tag{3.51}$$

from which the energy density

$$W = \frac{1}{2}\left(t^{\times}_{123\cdot} t_{123\cdot} + t^{\times}_{0123} t_{0123} + t^{\times}_{kl0\cdot} t_{kl0\cdot}\right) \tag{3.52}$$

which is again positive.

Chapter IV

THE INTERACTION OF THE NUCLEON WITH THE MESON FIELD

The analogy with the electromagnetic field is now carried a stage further (Flint, 1945). The interaction of the nucleon and the meson field is compared with that between an electric or magnetic field and the polarisation induced by the field in a material medium.

Maxwell's equations for the field in the presence of polarisable matter may be written

$$\frac{\partial F^{mn}}{\partial x^n} = 0 \tag{4.1}$$

and

$$\frac{\partial B_{mn}}{\partial x^l} + \frac{\partial B_{lm}}{\partial x^n} + \frac{\partial B_{nl}}{\partial x^m} = 0 \tag{4.2}$$

where the quantities B_{kl}, iB_{k4} $(k, l = 1, 2, 3)$ represent the magnetic induction B and the electric intensity E respectively, while F^{kl} , iF^{k4} represent the magnetic field H and the electric displacement D.

These are related by the equations

$$B = H + M \tag{4.3}$$

$$D = E + P \tag{4.4}$$

where M and P represent the magnetic and electric polarisations of the medium (in suitable units).

Neglecting the gravitational field, equations (4.3) and (4.4) may together be written

$$B_{mn} = F_{mn} + I_{mn} \tag{4.5}$$

It is supposed that in the presence of the nucleon the meson field equations (3.21) and (3.22) are modified in a similar way. T is still expressed as a curl, i.e.

$$\frac{\partial T_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial T_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial T_{\nu\lambda}}{\partial x^{\mu}} = 0 \tag{4.6}$$

the equation corresponding to (4.1) is

$$\frac{\partial V^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{4.7}$$

and, in the absence of other fields,

$$T_{\mu\nu} = V_{\mu\nu} + S_{\mu\nu} \tag{4.8}$$

where $S_{\mu\nu}$ is analogous to the polarisation J_{mn} of the electromagnetic theory and represents the effect of the field upon the medium, i.e. upon the nucleon.

The analogy has suggested the form of the equations for the case of the vector field; the other types of field may be treated in exactly the same way. It is, however, convenient to consider the vector field in more detail first.

L. de Broglie (1945) uses a somewhat similar method in that he also compares the propagation of the meson waves in the presence of the nucleons with that of electromagnetic waves in a polarisable medium, the sources of the meson field being represented by the various tensor quantities such as $\psi^{+}\alpha_{k}\psi$ etc. which are associated with the nucleon. The analogy leads him to adopt a certain Lagrange function for each type of field from which, with the aid of field equations which are assumed to be of a form suggested by the electromagnetic theory, he is able to find the Hamiltonian and so calculate the interaction energy.

The five-dimensional approach gives similar relations, and in addition leads to some interesting conclusions with regard to the energy.

The Vector Meson Field in the Presence of Nucleons

It will be assumed for the present that there is no gravitational or electromagnetic field.

The nuclear field equations are, as on p. 56,

$$\frac{\partial T_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial T_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial T_{\nu\lambda}}{\partial x^{\mu}} = 0 \tag{4.6}$$

$$\frac{\partial V^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{4.7}$$

while
$$T_{\mu\nu} = V_{\mu\nu} + S_{\mu\nu} \tag{4.8}$$
all these tensors being antisymmetric.

There will be a corresponding set of equations for the complex conjugate quantities.

Equation (4.6) gives,

for $\mu, \nu, \lambda = K, l, 5$
$$t_{Kl} = \frac{\partial}{\partial x^{K}}\left(\frac{i}{\kappa} t_{l\cdot}\right) - \frac{\partial}{\partial x^{l}}\left(\frac{i}{\kappa} t_{K\cdot}\right), \tag{4.9}$$

for $K, 4, 5$
$$i t_{K4} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{i}{\kappa} t_{K\cdot}\right) - \frac{\partial}{\partial x^{K}}\left(\frac{t_{4\cdot}}{\kappa}\right)$$

or
$$t_{K0} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{i}{\kappa} t_{K\cdot}\right) + \frac{\partial}{\partial x^{K}}\left(\frac{i}{\kappa} t_{0\cdot}\right), \tag{4.10}$$

for $K, l, 4$
$$\frac{1}{c}\frac{\partial}{\partial t} t_{Kl} = -\frac{\partial}{\partial x^{K}}(i t_{l4}) + \frac{\partial}{\partial x^{l}}(i t_{K4})$$

$$\text{or} = -\frac{\partial}{\partial x^{K}}(t_{l0}) + \frac{\partial}{\partial x^{l}}(t_{K0}) \tag{4.11}$$

and finally for $1, 2, 3$,
$$\operatorname{div} t_{Kl} = 0. \tag{4.12}$$

From equation (4.7)

for $\mu = k$, $$\left(\operatorname{curl} v_{kl}\right)_k = \frac{1}{c}\frac{\partial}{\partial t} v_{k0} - i\kappa v_{k.}\ , \tag{4.13}$$

for $\mu = 4$, $$\operatorname{div} v_{k0} = i\kappa v_{0.} \tag{4.14}$$

and for $\mu = 5$, $$\operatorname{div}(i\kappa v_{k.}) - \frac{1}{c}\frac{\partial}{\partial t}(i\kappa v_{0.}) = 0 . \tag{4.15}$$

A convenient notation is suggested by the similarity with the electromagnetic equations.

Let $t_{kl} = B_j\ ,\quad t_{k0} = E_k$

$v_{kl} = H_j\ ,\quad v_{k0} = D_k$

then $s_{kl} = M_j\ ,\quad s_{k0} = -P_k$ (j, k, l indicates a cyclic order of the indices 1,2,3) (4.16)

Also let $\frac{i}{\kappa} t_{k.} = U_k\ ,\quad \frac{i}{\kappa} t_{0.} = -U_0$

$i\kappa s_{k.} = J_k$ and $-i\kappa s_{0.} = J_0$.

Then the field equations may be written as follows:-

From (4.9) $$B = \operatorname{curl} U \tag{4.17}$$

and from (4.10) $$E = -\frac{1}{c}\frac{\partial U}{\partial t} - \operatorname{grad} U_0\ , \tag{4.18}$$

or alternatively $$H = \operatorname{curl} U - M \tag{4.19}$$

and $$D = -\frac{1}{c}\frac{\partial U}{\partial t} - \operatorname{grad} U_0 + P . \tag{4.20}$$

From (4.13) $$\operatorname{curl} H = \frac{1}{c}\frac{\partial D}{\partial t} - \kappa^2 U + J \tag{4.21}$$

and from (4.14) $$\operatorname{div} D = -\kappa^2 U_0 + J_0 . \tag{4.22}$$

Equations (4.19) to (4.22) are the field equations suggested by Yukawa, Sakata and Taketani (1938).

The Energy of the Field

Again by analogy with the electromagnetic theory, the energy-momentum-current tensor of p. 47, equation (3.39), becomes

$$R^{\mu}{}_{\nu} = \frac{1}{2}\left(V^{\times\mu\alpha}T_{\nu\alpha} - \frac{1}{4}\gamma^{\mu}{}_{\nu}V^{\times\alpha\beta}T_{\alpha\beta} + \text{c. conjugate.}\right) \qquad (4.23)$$

Then

$$R^{4}{}_{4} = \frac{1}{2}\left(\frac{1}{2}v^{\times}_{4k}t_{4k} + \frac{1}{2}v^{\times}_{4.}t_{4.} - \frac{1}{2}v^{\times}_{kl}t_{kl} - \frac{1}{2}v^{\times}_{k.}t_{k.} + c.c.\right). \qquad (4.24)$$

Hence the energy density

$$W = \frac{1}{4}\left(v^{\times}_{kl}t_{kl} + v^{\times}_{ok}v_{ok} + v^{\times}_{k.}t_{k.} + v^{\times}_{o.}t_{o.} + c.c.\right) \qquad (4.25)$$

Now by means of equation (4.8) this may be written

$$W = \frac{1}{4}\left[(t^{\times}_{kl}-s^{\times}_{kl})t_{kl} + (t^{\times}_{ok}-s^{\times}_{ok})t_{ok} + (t^{\times}_{k.}-s^{\times}_{k.})t_{k.} + (t^{\times}_{o.}-s^{\times}_{o.})t_{o.} + c.c.\right]$$

$$= \frac{1}{2}\left[|t_{kl}|^2 + |t_{ok}|^2 + |t_{k.}|^2 + |t_{o.}|^2\right] \qquad (4.26)$$

$$-\frac{1}{4}\left[s^{\times}_{kl}t_{kl} + s^{\times}_{ok}t_{ok} + s^{\times}_{k.}t_{k.} + s^{\times}_{o.}t_{o.} + c.c.\right]$$

The first bracket contains ohly the field quantities t_{kl} etc. and this term may be said to represent the energy density of the field, while the second bracket involves products of field quantities and something analogous to a polarisation, and hence this term represents the additional energy density due to the interaction of the field and the nucleon.

This second term therefore may reasonably be identified with the energy density associated with the nucleon on account of the modified form of the Dirac equation for such a particle.

For the vector field this latter is given by equation (3.14) (page 39), i.e.

$$-\frac{1}{2} g_2 \frac{hc}{\pi}\left[\psi^+(-i\beta\alpha_k\alpha_l)\psi\, t_{kl} + \psi^+(-i\beta\alpha_k)\psi\, t_{0k} + i\psi^+\alpha_k\psi\, t_{k.} - i\psi^+\psi\, t_{0.}\right] + c.c.$$

Comparing each term with the corresponding term from equation (4.26)

$$s^{\times}_{kl} = -g_2 \frac{hc}{\pi} \psi^+ i\beta\alpha_k\alpha_l\psi, \qquad s_{kl} = -g_2^{\times}\frac{hc}{\pi}\psi^+ i\beta\alpha_k\alpha_l\psi, \tag{4.27}$$

$$s^{\times}_{0k} = -g_2 \frac{hc}{\pi}\psi^+ i\beta\alpha_k\psi, \qquad s_{0k} = -g_2^{\times}\frac{hc}{\pi}\psi^+ i\beta\alpha_k\psi,$$

$$s^{\times}_{k.} = g_2\frac{hc}{\pi} i\,\psi^+\alpha_k\psi, \qquad s_{k.} = -g_2^{\times}\frac{hc}{\pi} i\,\psi^+\alpha_k\psi,$$

$$s^{\times}_{0.} = -g_2\frac{hc}{\pi} i\,\psi^+\psi \quad \text{and} \quad s_{0.} = g_2^{\times}\frac{hc}{\pi} i\,\psi^+\psi .$$

The quantities (S) are thus expressed in terms of the wave functions of the nucleon and the Dirac matrices.

Hence

$$J_k = g_2^{\times}\frac{hc\kappa}{\pi}\psi^+\alpha_k\psi, \tag{4.28}$$

$$J_0 = g_2^{\times}\frac{hc\kappa}{\pi}\psi^+\psi,$$

$$M = -g_2^{\times}\frac{hc}{\pi}\psi^+ i\beta\alpha_k\alpha_l\psi,$$

$$P = -g_2^{\times}\frac{hc}{\pi}\psi^+ i\beta\alpha_k\psi,$$

and these, apart from the multiplying constants, are the same expressions as those given by Yukawa (1938).

The Total Energy of the System

From equation (4.19) the total energy of the system, i.e. field plus particle,

$$\int W d\tau = \frac{1}{4}\int \left(v^{\times}_{\kappa l} t_{\kappa l} + v^{\times}_{\kappa o} t_{\kappa o} + v^{\times}_{\kappa .} t_{\kappa .} + v^{\times}_{o.} t_{o.} + c.c. \right) d\tau \qquad (4.29)$$

where the integral is taken throughout (three dimensional) space, $d\tau$ being the element of volume.

This integral may be evaluated with the help of the field equations (4.20) to (4.23) together with the corresponding complex conjugate equations.

The first term

$$= \frac{1}{4}\int H^{\times}\cdot B \, d\tau$$

$$= \frac{1}{4}\int H^{\times}\cdot \operatorname{curl} U \, d\tau, \qquad \text{by equation (4.17),}$$

$$= \frac{1}{4}\int \operatorname{curl} H^{\times}\cdot U \, d\tau, \qquad \text{integrating by parts, the field quantities vanishing at the boundaries,}$$

$$= \frac{1}{4}\int \left(\frac{1}{c}\frac{\partial D^{\times}}{\partial t} - \kappa^2 U^{\times} + J^{\times} \right)\cdot U \, d\tau, \qquad \text{by equation (4.21).}$$

The second term

$$= \frac{1}{4}\int D^{\times}\cdot E \, d\tau$$

$$= -\frac{1}{4}\int D^{\times}\cdot \left(\frac{1}{c}\frac{\partial U}{\partial t} + \operatorname{grad} U_0 \right) d\tau, \qquad \text{by equation (4.18),}$$

$$= -\frac{1}{4}\int \left(D^{\times}\frac{1}{c}\frac{\partial U}{\partial t} - \operatorname{div} D^{\times} U_0 \right) d\tau, \qquad \text{integrating by parts,}$$

$$= -\frac{1}{4}\int \left(D^{\times}\frac{1}{c}\frac{\partial U}{\partial t} + \kappa^2 U_0^{\times} U_0 - J_0^{\times} U_0 \right) d\tau \qquad \text{by (4.22).}$$

The third and fourth terms

$$= \frac{1}{4}\int\left[\left(t^{\times}_{K\cdot}-s^{\times}_{K\cdot}\right)t_{K\cdot}+\left(t^{\times}_{0\cdot}-s^{\times}_{0\cdot}\right)t_{0\cdot}\right]d\tau$$

$$= \frac{1}{4}\int\left(\kappa^2 U^{\times}\cdot U - J^{\times}\cdot U + \kappa^2 U_0^{\times}U_0 - J_0^{\times}U_0\right)d\tau$$

Hence the total integral

$$\int W\,d\tau = \frac{1}{4c}\int\left(\frac{\partial D^{\times}}{\partial t}\cdot U - D^{\times}\cdot\frac{\partial U}{\partial t} + c.c\right)d\tau . \qquad (4.30)$$

If, as is usual in this field of investigation at the present time, consideration is confined to the static case, then

$$\int W\,d\tau = 0.$$

This is an extremely interesting result, for it offers a possible way out of some of the well-known difficulties arising in nuclear field theories. In calculating the energy of the nuclear field and the energy of the nucleon, infinities have occurred and it may be that they are due to an injudicious division into two parts of a quantity of energy that is finite.

For the present theory suggests that the energy of the meson field and the interaction energy are to be taken together as a finite and constant whole and that if the nucleon gains energy it does so at the expense of the energy of the field.

The other possible types of field will now be considered and it will be shown that the total integrated energy is zero for these as well, in the so-called static case.

The Scalar Meson Field in the Presence of Nucleons

The nuclear field equations are

$$\frac{\partial T_\mu}{\partial x^\nu} - \frac{\partial T_\nu}{\partial x^\mu} = 0 \tag{4.31}$$

and

$$\frac{\partial V^\mu}{\partial x^\mu} = 0 \tag{4.32}$$

while

$$T_\mu = V_\mu + S_\mu \tag{4.33}$$

together with corresponding relations for the complex conjugate quantities.

From (4.31) in the case $\mu = m,\ \nu = 5$,

$$\frac{\partial t_m}{\partial x^5} - \frac{\partial t.}{\partial x^m} = 0$$

or

$$i\kappa\, t_m = \frac{\partial}{\partial x^m} t.$$

i.e.

$$t_K = -\frac{\partial}{\partial x^K}\left(\frac{i}{\kappa} t.\right) \tag{4.34}$$

and

$$t_0 = it_4 = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{i}{\kappa} t.\right). \tag{4.35}$$

In the case $\mu, \nu = K, l$,

$$\text{curl } t_K = 0$$

and for $\mu = K,\ \nu = 4$

$$\frac{1}{c}\frac{\partial}{\partial t} t_K = \frac{\partial}{\partial x^K} t.$$

From (4.32)

$$\frac{\partial v_K}{\partial x^K} - \frac{1}{c}\frac{\partial v_0}{\partial t} + i\kappa V. = 0. \tag{4.36}$$

The Scalar Meson Field in the Presence of Nucleons

The nuclear field equations are

$$\frac{\partial T_\mu}{\partial x^\nu} - \frac{\partial T_\nu}{\partial x^\mu} = 0 \tag{4.31}$$

and

$$\frac{\partial V^\mu}{\partial x^\mu} = 0 \tag{4.32}$$

while

$$T_\mu = V_\mu + S_\mu \tag{4.33}$$

together with corresponding relations for the complex conjugate quantities.

From (4.31) in the case $\mu = m, \nu = 5$,

$$\frac{\partial t_m}{\partial x^5} - \frac{\partial t.}{\partial x^m} = 0$$

or

$$i\kappa t_m = \frac{\partial}{\partial x^m} t.$$

i.e.

$$t_K = -\frac{\partial}{\partial x^K}\left(\frac{i}{\kappa} t.\right) \tag{4.34}$$

and

$$t_0 = it_4 = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{i}{\kappa} t.\right). \tag{4.35}$$

In the case $\mu, \nu = K, l$,

$$\text{curl } t_K = 0$$

and for $\mu = K, \nu = 4$

$$\frac{1}{c}\frac{\partial}{\partial t} t_K = \frac{\partial}{\partial x^K} t.$$

From (4.32)

$$\frac{\partial v_K}{\partial x^K} - \frac{1}{c}\frac{\partial v_0}{\partial t} + i\kappa v. = 0. \tag{4.36}$$

Let $t_K = G_K$, $t_0 = G_0$, $\frac{it.}{\kappa} = R$,

and let $v_K = F_K$, $v_0 = F_0$ and $\frac{iv.}{\kappa} = \mathcal{Q}$.

Then the field equations may be written

$$\begin{aligned} G &= -\operatorname{grad} R, \\ G_0 &= -\frac{1}{c}\frac{\partial R}{\partial t}, \end{aligned} \tag{4.37}$$

$$\operatorname{div} F = \frac{1}{c}\frac{\partial F_0}{\partial t} - \kappa^2 \mathcal{Q}. \tag{4.38}$$

The energy of the field

In this case the energy momentum current tensor

$$R^{\mu}{}_{\nu} = \frac{1}{2}\left(V^{\times\mu}T_{\nu} - \frac{1}{2}\gamma^{\mu}{}_{\nu}V^{\times\alpha}T_{\alpha} + c.c.\right) \tag{4.39}$$

so that

$$\begin{aligned} R_{44} &= \frac{1}{4}\left(v_4^{\times}t_4 - v_K^{\times}t_K - v_.^{\times}t_. + c.c\right) \\ &= \frac{1}{4}\left[(t_4^{\times}-s_4^{\times})t_4 + (t_K^{\times}-s_K^{\times})t_K - (t_.^{\times}-s_.^{\times})t_. + c.c.\right] \end{aligned}$$

by means of equation (4.33).

Hence the energy density

$$W = \frac{1}{2}\left(|t_K|^2 + |t_0|^2 + |t_.|^2\right) - \frac{1}{4}\left(s_K^{\times}t_K + s_0^{\times}t_0 + s_.^{\times}t_. + c.c.\right) \tag{4.40}$$

the first bracket, as previously, containing only the field quantities and the second containing only interaction terms.

The second term will again be identified with the additional energy density of p. 39 , equation (3.13), i.e.

$$-\frac{1}{2} g_1 \frac{hc}{\pi} i \left(\psi^+\alpha_K \psi \, t_K - \psi^+\psi \, t_0 + \psi^+\beta\psi \, t_.\right) + c.c.$$

Comparing each term with the corresponding term from equation (4.40)

$$s_K^\times = g_1 \frac{hc}{\pi} i \, \psi^+\alpha_K\psi, \qquad s_K = -g_1^\times \frac{hc}{\pi} i \, \psi^+\alpha_K\psi, \tag{4.41}$$

$$s_0^\times = -g_1 \frac{hc}{\pi} i \, \psi^+\psi, \qquad s_0 = g_1^\times \frac{hc}{\pi} i \, \psi^+\psi,$$

$$s_.^\times = g_1 \frac{hc}{\pi} i \, \psi^+\beta\psi, \qquad s_. = -g_1^\times \frac{hc}{\pi} i \, \psi^+\beta\psi.$$

Thus the interaction quantities (S) are identified and it appears that s_K , s_0 , apart from the multiplying constant, are the same as the quantities $s_{K.}$, s_0. of the vector case whereas S. is a new invariant quantity associated with the nucleon.

The total energy of the system is now found by a volume integration throughout space and is

$$\int W \, d\tau = \frac{1}{4}\int \left(F_0^\times G_0 + F^\times \cdot G + \kappa^2 Q^\times R + c.c. \right) d\tau$$

$$= \frac{1}{4}\int \left(-\frac{1}{c} F_0^\times \frac{\partial R}{\partial t} - F^\times \cdot \operatorname{grad} R + \kappa^2 Q^\times R + c.c. \right) d\tau$$

from equations (4.37)

The second term, integrated by parts, becomes

$$\frac{1}{4}\int \operatorname{div} F^\times R \, d\tau$$

$$= \frac{1}{4}\int \left(\frac{1}{c}\frac{\partial F_0^\times}{\partial t} - \kappa^2 Q^\times \right) R \, d\tau$$

from the equation conjugate to (4.38)

Hence

$$\int W \, d\tau = \frac{1}{4c} \int \left(\frac{\partial F_0^{\times}}{\partial t} R - F_0^{\times} \frac{\partial R}{\partial t} \right) d\tau + c.c. \qquad (4.42)$$

and again this is zero in the static case.

The Pseudo-Vector Field in the Presence of Nucleons

The field equations are

$$\frac{\partial T_{\mu\nu\lambda}}{\partial x^{\rho}} - \frac{\partial T_{\rho\mu\nu}}{\partial x^{\lambda}} + \frac{\partial T_{\lambda\rho\mu}}{\partial x^{\nu}} - \frac{\partial T_{\nu\lambda\rho}}{\partial x^{\mu}} = 0 \qquad (4.43)$$

and
$$\frac{\partial V^{\mu\nu\lambda}}{\partial x^{\lambda}} = 0 \qquad (4.44)$$

where
$$T_{\mu\nu\lambda} = V_{\mu\nu\lambda} + S_{\mu\nu\lambda}. \qquad (4.45)$$

Equation (4.44) gives

for $\mu, \nu = 1, 2$, $\quad V_{12\cdot} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{i}{\kappa} V_{120}\right) + \frac{\partial}{\partial x^3}\left(\frac{i}{\kappa} V_{123}\right)$,

for $\mu, \nu = 1, 4$, $\quad V_{10\cdot} = -\frac{\partial}{\partial x^2}\left(\frac{i}{\kappa} V_{120}\right) + \frac{\partial}{\partial x^3}\left(\frac{i}{\kappa} V_{310}\right)$,

for $\mu, \nu = 1, 5$, $\quad \frac{1}{c}\frac{\partial}{\partial t} V_{10\cdot} = \frac{\partial}{\partial x^2} V_{12\cdot} - \frac{\partial}{\partial x^3} V_{31\cdot}$ and

for $\mu, \nu = 4, 5$ $\quad \operatorname{div} V_{k0\cdot} = 0$.

The resemblance of these equations to equations (4.9) to (4.12) for the vector meson field suggests that a convenient notation for the quantities (V) would be

$$V_{kl\cdot} = E'_j, \qquad V_{k0\cdot} = -B'_k, \qquad (4.46)$$
$$\frac{i}{\kappa} V_{kl0} = U'_j \quad \text{and} \quad \frac{i}{\kappa} V_{123} = -U'_0.$$

Equation (4.43) gives

for 1, 2, 4, 5

$$\frac{\partial}{\partial x^1} t_{20\cdot} - \frac{\partial}{\partial x^2} t_{10\cdot} = -\frac{1}{c}\frac{\partial}{\partial t} t_{12\cdot} + i\kappa t_{120},$$

for 1, 2, 3, 5 $\quad \operatorname{div} t_{Kl.} = i\kappa t_{123}$ and

for 1, 2, 3, 4, $\quad \operatorname{div} t_{Klo} + \frac{1}{c}\frac{\partial}{\partial t}(-t_{123}) = 0.$

$$\left.\begin{array}{llll} \text{Now let} & t_{Kl.} = D'_j, & & t_{Ko.} = -H'_K \\ \text{and so let} & s_{Kl.} = P'_j & \text{and} & s_{Ko.} = M'_K \\ \text{and finally} & -i\kappa s_{Klo} = J'_j & \text{and} & i\kappa s_{123} = J_o'. \end{array}\right\} \quad (4.47)$$

Then making use of (4.45) the field equations may be written,

$$H' = \operatorname{curl} U' - M', \qquad (4.48)$$

$$D' = -\frac{1}{c}\frac{\partial U'}{\partial t} - \operatorname{grad} U_o' + P', \qquad (4.49)$$

$$\operatorname{curl} H' = \frac{1}{c}\frac{\partial D'}{\partial t} - \kappa^2 U' + J' \qquad (4.50)$$

and $\quad \operatorname{div} D' = -\kappa^2 U_o' + J_o'. \qquad (4.51)$

and are in the same form as the vector field equations. This was to be expected since the field quantities are the duals of the vector field quantities.

There will be a corresponding set of equations for the complex conjugate quantities.

The energy of the field

The energy-momentum current tensor of equation (3.46) now becomes

$$R^\mu{}_\nu = \frac{1}{2}\left(\frac{1}{2} V^{\times\mu\alpha\beta} T_{\nu\alpha\beta} - \frac{1}{12}\gamma^\mu{}_\nu V^{\times\alpha\beta\gamma} T_{\alpha\beta\gamma} + c.c.\right). \qquad (4.52)$$

Then

$$R_{44} = \tfrac{1}{2}\left(\tfrac{1}{2} v^{\times}_{kl4} t_{kl4} + \tfrac{1}{2} v^{\times}_{k4.} t_{k4.} - \tfrac{1}{2} v^{\times}_{kl.} t_{kl.} - \tfrac{1}{2} v^{\times}_{123} t_{123} + c.c.\right).$$

Hence the energy density

$$\begin{aligned} W &= \tfrac{1}{4}\left(v^{\times}_{kl.} t_{kl.} + v^{\times}_{123} t_{123} + v^{\times}_{kl0} t_{kl0} + v^{\times}_{k0.} t_{k0.} + c.c.\right) \\ &= \tfrac{1}{2}\left(|t_{kl.}|^2 + |t_{123}|^2 + |t_{kl0}|^2 + |t_{k0.}|^2\right) \\ &\quad - \tfrac{1}{4}\left(s^{\times}_{kl.} t_{kl.} + s^{\times}_{123} t_{123} + s^{\times}_{kl0} t_{kl0} + s^{\times}_{k0.} t_{k0.}\right), \end{aligned} \tag{4.53}$$

dividing the expression as before.

Identifying the second term with the energy density of the expression (3.15), p. 40,

i.e. $$-\tfrac{1}{2} g_3 \tfrac{hc}{2\pi}\left[\psi^{+}(-i\alpha_k\alpha_l)\psi\, t_{kl0} + \psi^{+}(i\alpha_1\alpha_2\alpha_3)\psi\, t_{123} - \psi^{+}(-i\beta\alpha_k\alpha_l)\psi\, t_{kl.} + \psi^{+}(-i\beta\alpha_k)\psi\, t_{k0.}\right] + c.c.$$

and comparing corresponding terms,

$$\begin{aligned} s^{\times}_{kl.} &= g_3 \tfrac{hc}{\pi}\, \psi^{+} i\beta\alpha_k\alpha_l \psi, & s_{kl.} &= g_3^{\times} \tfrac{hc}{\pi}\, \psi^{+} i\beta\alpha_k\alpha_l \psi, \\ s^{\times}_{k0.} &= -g_3 \tfrac{hc}{\pi}\, \psi^{+} i\beta\alpha_k \psi, & s_{k0.} &= -g_3^{\times} \tfrac{hc}{\pi}\, \psi^{+} i\beta\alpha_k \psi, \\ s^{\times}_{kl0} &= -g_3 \tfrac{hc}{\pi}\, \psi^{+} i\alpha_k\alpha_l \psi, & s_{kl0} &= -g_3^{\times} \tfrac{hc}{\pi}\, \psi^{+} i\alpha_k\alpha_l \psi, \\ s^{\times}_{123} &= g_3 \tfrac{hc}{\pi}\, \psi^{+} i\alpha_1\alpha_2\alpha_3 \psi, & s_{123} &= g_3^{\times} \tfrac{hc}{\pi}\, \psi^{+} i\alpha_1\alpha_2\alpha_3 \psi. \end{aligned} \tag{4.54}$$

The total energy of the system

In this case

$$\int W\,d\tau = \frac{1}{4}\int\left(V^{\times}_{\kappa l\cdot}F_{\kappa l\cdot} + V^{\times}_{\kappa o\cdot}F_{\kappa o\cdot} + V^{\times}_{\kappa l o}F_{\kappa l o} + V^{\times}_{123}F_{123} + c.c\right)d\tau$$

$$= \frac{1}{4}\int\left(E'^{\times}\cdot D' + B'^{\times}\cdot H' + \kappa^2 U'^{\times}U' + \kappa^2 U_0'^{\times}U_0' - U'^{\times}J' - U_0'^{\times}J_0' + c.c.\right)d\tau. \qquad (4.55)$$

The complete integral is thus in exactly the same form as that for the vector field, and since the field equations (4.48) to (4.51) for the primed field quantities here are the same as the vector field equations (4.19) to (4.22),

$$\int W\,d\tau = \frac{1}{4c}\int\left(\frac{\partial D'^{\times}}{\partial t}U' - D'^{\times}\frac{\partial U'}{\partial t}\right)d\tau + c.c.$$

Hence again, for the static case,

$$\int W\,d\tau = 0.$$

The Pseudo-Scalar Field in the Presence of Nucleons

The field equations are

$$\frac{\partial T_{\mu\nu\lambda\rho}}{\partial x^{\sigma}} + \frac{\partial T_{\sigma\mu\nu\lambda}}{\partial x^{\rho}} + \frac{\partial T_{\rho\sigma\mu\nu}}{\partial x^{\lambda}} + \frac{\partial T_{\lambda\rho\sigma\mu}}{\partial x^{\nu}} + \frac{\partial T_{\nu\lambda\rho\sigma}}{\partial x^{\mu}} = 0 \tag{4.56}$$

and

$$\frac{\partial V^{\mu\nu\lambda\rho}}{\partial x^{\rho}} = 0 \tag{4.57}$$

where

$$T_{\mu\nu\lambda\rho} = V_{\mu\nu\lambda\rho} + S_{\mu\nu\lambda\rho}. \tag{4.58}$$

They are similar in form to those of the scalar field, since from equation (4.57),

for $\mu, \nu, \lambda = 2, 3, 4$
$$V_{230.} = -\frac{\partial}{\partial x^1}\left(\frac{i}{\kappa} V_{1230}\right), \tag{4.59}$$

for 1, 2, 3
$$V_{123.} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{i}{\kappa} V_{1230}\right), \tag{4.60}$$

for 1, 4, 5 etc. $\quad \operatorname{curl} V_{kl0.} = 0$

and for 2, 3, 5 etc., $\frac{1}{c}\frac{\partial}{\partial t} V_{kl0.} = \operatorname{grad} V_{123.}$

while equation (4.56) gives

$$\operatorname{div} t_{kl0.} - \frac{1}{c}\frac{\partial}{\partial t} t_{123.} + i\kappa\, t_{1230} = 0. \tag{4.61}$$

If now $V_{230.} = G_1'$ etc., $\quad V_{123.} = G_0'$, $\quad \frac{i}{\kappa} V_{1230} = R'$

$t_{230.} = F_1'$ etc., $\quad t_{123.} = F_0'$ and $\frac{i}{\kappa} t_{1230} = \varphi'$, then the field equations (4.59), (4.60) and (4.61) become

$$\left.\begin{aligned} G' &= -\operatorname{grad} R', \\ G_0' &= -\frac{1}{c}\frac{\partial R'}{\partial t}. \end{aligned}\right\} \quad (4.62)$$

and $$\operatorname{div} F' = \frac{1}{c}\frac{\partial F_0'}{\partial t} - \kappa^2 \mathcal{G}'. \quad (4.63)$$

i.e., they are identical in form with those of the scalar field, although the primed symbols do not represent the same field quantities as the unprimed.

The energy of the field

The energy-momentum-current tensor

$$R^{\mu}{}_{\nu} = \frac{1}{2}\left(\frac{1}{6} V^{\times\mu\alpha\beta\gamma} T_{\nu\alpha\beta\gamma} - \frac{1}{48}\gamma^{\mu}{}_{\nu} V^{\times\alpha\beta\gamma\epsilon} T_{\alpha\beta\gamma\epsilon} + c.c.\right). \quad (4.64)$$

Then $R_{44} = \frac{1}{2}\left(\frac{1}{2} V^{\times}_{kl4.} t_{kl4.} + \frac{1}{2} V^{\times}_{1234} t_{1234} - \frac{1}{2} V^{\times}_{123.} t_{123.} + c.c\right)$.

Hence the energy density

$$\begin{aligned} W &= \frac{1}{4}\left(V^{\times}_{kl0.} t_{kl0.} + V^{\times}_{1230} t_{1230} + V^{\times}_{123.} t_{123.} + c.c.\right) \\ &= \frac{1}{2}\left(|t_{kl0.}|^2 + |t_{1230}|^2 + |t_{123.}|^2\right) \quad (4.65) \\ &\quad - \frac{1}{4}\left(S^{\times}_{kl0.} t_{kl0.} + S^{\times}_{1230} t_{1230} + S^{\times}_{123.} t_{123.} + c.c.\right). \end{aligned}$$

This second term, as before, is identified with the energy density of the expression (3.16) p. 40.

i.e.

$$-\frac{1}{2} g_4 \frac{hc}{2\pi}\Big[\psi^+(i\alpha_K\alpha_l)\psi\, t_{Kl0.} - \psi^+(i\alpha_1\alpha_2\alpha_3)\psi\, t_{123.}$$
$$- \psi^+(i\beta\alpha_1\alpha_2\alpha_3)\psi\, t_{1230}\Big] + c.c.$$

and comparing corresponding terms,

$$s^\times_{Kl0.} = g_4 \frac{hc}{\pi}\psi^+ i\alpha_K\alpha_l\psi, \qquad s_{Kl0.} = g_4^\times \frac{hc}{\pi}\psi^+ i\alpha_K\alpha_l\psi, \tag{4.66}$$

$$s^\times_{123.} = -g_4 \frac{hc}{\pi}\psi^+ i\alpha_1\alpha_2\alpha_3\psi, \qquad s_{123.} = -g_4^\times \frac{hc}{\pi}\psi^+ i\alpha_1\alpha_2\alpha_3\psi,$$

$$s^\times_{1230} = -g_4 \frac{hc}{\pi} i\,\psi^+\beta\alpha_1\alpha_2\alpha_3\psi, \qquad s_{1230} = g_4^\times \frac{hc}{\pi} i\,\psi^+\beta\alpha_1\alpha_2\alpha_3\psi.$$

The pseudo-scalar field equations, and these expressions apart from a multiplying factor, are in agreement with those of Tarikawa and Yukawa (1941).

The total energy of the system

In this case

$$\int W d\tau = \frac{1}{4}\int\left(v^\times_{Kl0.}t_{Kl0.} + v^\times_{123.}t_{123.} + v^\times_{1230}t_{1230} + c.c\right)d\tau$$
$$= \frac{1}{4}\int\left(G'^\times F' + G_0'F_0' + \kappa^2 R'^\times Q' + c.c.\right)d\tau. \tag{4.67}$$

The complete integral is in the same form as that for the scalar field, therefore since the field equations are also in the same form

$$\int W\, d\tau = \frac{1}{4c}\int\left(\frac{\partial F_0'^\times}{\partial t}R' - F_0'^\times\frac{\partial R'}{\partial t}\right)d\tau + c.c. \tag{4.68}$$

Then, as before, for the static case $\int W d\tau = 0$.

Chapter V

THE CALCULATION OF THE INTERACTION ENERGY

It has been shown in the last chapter that the various tensor quantities such as $\psi^+\alpha_K\psi$ associated with the nucleon are analogous to the polarisation of a material medium. Continuing the analogy with the electromagnetic theory, these polarisations may be regarded as sources of the meson field. Then for any given distribution of nucleons the field quantities may be calculated and hence the interaction energy may be found. This is clearly a complicated procedure involving a knowledge of the wave function ψ for each nucleon, or at least of the quantities such as $\psi^+\alpha_K\psi$ as functions of the co-ordinates.

The problem is simplified if the calculation is limited to the non-relativistic case, that is, if it is supposed that the nucleons are practically at rest. This is the so-called static case treated by most authors. The field quantities are then regarded as being constant in time; this limitation has already been introduced in the calculation of the total energy. Furthermore, some of the interaction terms will disappear and others will be modified if certain results, well known in the Dirac theory of the electron (de Broglie, 1934), are applied in the case of the nucleon. It can be shown that for any particle obeying Dirac's equation the components ψ_1, ψ_2 of the wave function are zero when the particle is at rest while the other components ψ_3 and ψ_4 remain finite. It follows that the quantities

$$\psi^+\alpha_K\psi\ , \quad \psi^+ i\beta\alpha_K\psi\ , \quad \psi^+ i\alpha_1\alpha_2\alpha_3\psi \quad \text{and} \quad \psi^+\beta\alpha_1\alpha_2\alpha_3\psi$$

are all zero since the non-zero terms occurring in these matrices contain either the suffix 1 or the suffix 2. Also $\psi^+\beta\psi$ has the same value as $-\psi^+\psi$ and $\psi^+ i\beta\alpha_K\alpha_\ell\psi$ the same value as $-\psi^+ i\alpha_K\alpha_\ell\psi$.

With these simplifications the various types of field will be dealt with in turn.

The Scalar Interaction

From equations (4.41) of the last chapter

$$s_k = -g_1^{\times} \frac{hc}{\pi} i \psi^{+}\alpha_k \psi$$

and may be neglected in the static case.

Therefore $t_k = v_k$ or, in the notation of Chapter IV, $G = F$.

The field equations (4.37) and (4.38) now reduce to

$$G = -\operatorname{grad} R \qquad (5.1)$$

$$G_0 = 0 \qquad \text{and} \qquad (5.2)$$

$$\operatorname{div} G = -\kappa^2 \varphi$$

$$= -\kappa^2 R + \kappa^2 I \qquad (5.3)$$

where $I = \frac{i}{\kappa} s.$

$$= -g_1^{\times} \frac{hc}{\kappa\pi} \psi^{+}\psi \qquad \text{from equation (4.41)} \qquad (5.4)$$

Hence from equations (5.1) and (5.3)

$$\nabla^2 R - \kappa^2 R = -\kappa^2 I . \qquad (5.5)$$

The solution of this equation is

$$R(\bar{r}) = \int \kappa^2 I_a \frac{e^{-\kappa|\bar{r}-\bar{r}_a|}}{4\pi(\bar{r}-\bar{r}_a)} d\tau_a$$

where $d\tau_a$ is an element of volume situated at the point $\bar{r} = \bar{r}_a$, I_a is the value of I at this point and the integration extends throughout space.

Let $$\frac{e^{-\kappa|\bar{r}-\bar{r}_a|}}{4\pi(\bar{r}-\bar{r}_a)} = \phi(\bar{r}-\bar{r}_a) \qquad (5.6)$$

then $$R = \int \kappa^2 I_a \, \phi(\bar{r}-\bar{r}_a) \, d\tau_a \, . \tag{5.7}$$

In order to evaluate the integral and so determine R for any particular case the form of I as a function of the co-ordinates must be known. First, let it be supposed that it is one nucleon, situated at the point $\bar{r} = \bar{r}_i$ which gives rise to the meson field. The distribution of the polarisation density I of the nucleon is that of the quantity $\psi^+\psi$ which is associated with it and this will be vanishingly small except at the point occupied by the nucleon. This may be represented by the inclusion of Dirac's δ-function so that

$$I(r) = -g_1^\times \frac{hc}{\kappa\pi} \, \delta(\bar{r}-\bar{r}_i) \, \psi^{+i}(r) \, \psi^i(r) \, . \tag{5.8}$$

Then $$R(r) = -g_1^\times \frac{hc\kappa}{\pi} \, (\psi^+\psi)^i \, \phi(\bar{r}-\bar{r}_i)$$

by reason of the properties of the δ-function.

The potential function of the scalar meson field of the nucleon is thus similar to that of the Coulomb field but falls off more rapidly with distance on account of the exponential factor. The constant $g_1^\times \frac{hc\kappa}{\pi}$ is evidently analogous to the electric charge.

Now let there be a second nucleon at the point $\bar{r} = \bar{r}_k$ in this field.

The interaction energy, i.e. that of this second particle with the field, is, from equation (4.40),

$$\frac{1}{4}\int\left(s_k^{\times} t_k + s_0^{\times} t_0 + s_{\cdot}^{\times} t_{\cdot} + c.c.\right)_b d\tau_b$$

$$= \frac{1}{4}\int\left(s_{\cdot}^{\times} t_{\cdot} + s_{\cdot} t_{\cdot}^{\times}\right)_b d\tau_b \qquad \text{in the static case}$$

$$= \frac{\kappa^2}{4}\int\left(I^{\times} R + I R^{\times}\right)_b d\tau_b .$$

But at $\bar{r} = \bar{r}_b$

$$I^{\times} = -g_1 \frac{hc}{\kappa\pi} \delta(\bar{r}_b - \bar{r}_k)\, \psi^{+k}(r_b)\, \psi^{k}(r_b) ,$$

$$R = -g_1^{\times} \frac{hc\kappa}{\pi} (\psi^{+}\psi)^{i} \phi(\bar{r}_b - \bar{r}_i). \qquad \text{from (5.9)}$$

Thus the first term of the integral

$$= \frac{1}{4} g_1 g_1^{\times} \left(\frac{hc\kappa}{\pi}\right)^2 (\psi^{+}\psi)^{i} \int \psi^{+k}(r_b)\psi^{k}(r_b)\, \delta(\bar{r}_b - \bar{r}_k)\, \phi(\bar{r}_b - \bar{r}_k)\, d\tau_b$$

$$= \frac{1}{4} g_1 g_1^{\times} \left(\frac{hc\kappa}{\pi}\right)^2 (\psi^{+}\psi)^{i} (\psi^{+}\psi)^{k}\, \phi(\bar{r}_k - \bar{r}_i)$$

and, adding the complex conjugate quantity, the total interaction energy =

$$\frac{1}{2} |g_1|^2 \left(\frac{hc\kappa}{\pi}\right)^2 (\psi^{+}\psi)^{i} (\psi^{+}\psi)^{k} \frac{e^{-\kappa|\bar{r}_k - \bar{r}_i|}}{4\pi|\bar{r}_k - \bar{r}_i|} . \qquad (5.10)$$

This is of the same form as the expression obtained by Kemmer (1938), de Broglie (1945) and others.

The usual infinities arise here if the interaction of a nucleon with its own field is considered for this is the case $\bar{r}_i = \bar{r}_k$. This appears to be the same difficulty as that which arises in the problem of the self energy of the electron. But, in the present theory, the infinity occurring in the interaction energy is balanced by a corresponding infinite term, $\int R^{*} R \, d\tau$, in what has been called (in Chapter IV) the field energy and the total energy remains constant and finite.

The Vector Interaction

In this case $s_{k.} = 0$, $s_{k0} = 0$,

i.e. $J_k = 0$, $P_k = 0$, (5.11)

and since all the time derivatives are zero, the field equations (4.17), (4.18), (4.21), (4.22) reduce to

$$B = \operatorname{curl} U, \tag{5.12}$$

$$E = -\operatorname{grad} U_0, \tag{5.13}$$

$$\operatorname{curl} B = -\kappa^2 U + \operatorname{curl} M, \quad \text{since } B = H + M, \tag{5.14}$$

$$\operatorname{div} E = -\kappa^2 U_0 + J_0, \quad D = E + P. \tag{5.15}$$

Also, from equation (4.15),

and therefore $\operatorname{div} t_{k.} = 0$ since $s_{k.} = 0$

i.e. $\operatorname{div} U = 0.$ (5.16)

From equations (5.13), (5.15)

$$\nabla^2 U_0 - \kappa^2 U_0 = -J_0 \tag{5.17}$$

and from equations (5.12), (5.14) and (5.16)

$$\nabla^2 U - \kappa^2 U = -\operatorname{curl} M. \tag{5.18}$$

Hence $U_0(r) = \int (J_0)_a \, \phi(\bar{r} - \bar{r}_a) \, d\tau_a$

and $U(r) = \int (\operatorname{curl} M)_a \, \phi(\bar{r} - \bar{r}_a) \, d\tau_a$

From (4.28) $J_0 = g_2^{\times} \frac{hc\kappa}{\pi} \psi^+ \psi$

and $M_k = g_2^{\times} \frac{hc}{\pi} \psi^+ \sigma_k \psi$, $(\sigma_1 = i\alpha_2\alpha_3 \text{ etc.})$

in the present (static) case. The δ-function may be introduced as before, so that for a single nucleon at $\bar{r} = \bar{r}_i$.

$$U_0(r) = g_2^{\times} \frac{hc\kappa}{\pi} (\psi^+\psi)^i \phi(\bar{r}-\bar{r}_i) \tag{5.19}$$

and

$$U(r) = g_2^{\times} \frac{hc}{\pi} \int \mathrm{curl}_a \left[\psi^{+i}\sigma^i\psi^i \, \delta(\bar{r}_a-\bar{r}_i) \right] \phi(r-\bar{r}_a)\, d\tau_a \, .$$

Integrating by parts

$$U(r) = g_2^{\times} \frac{hc}{\pi} \int \psi^{+i}\sigma^i\psi^i \, \delta(\bar{r}_a-\bar{r}_i) \times \mathrm{grad}_a \, \phi(\bar{r}-\bar{r}_a)\, d\tau_a$$

$$= g_2^{\times} \frac{hc}{\pi} (\psi^+\sigma\psi)^i \times \mathrm{grad}_i \, \phi(\bar{r}-\bar{r}_i) . \tag{5.20}$$

The interaction energy of equation (4.26), in the static case, reduces to

$$= \frac{1}{4} \int \left(S^{\times}_{kl} t_{kl} + S^{\times}_{0.} t_{0.} + c.c \right)_b d\tau_b$$

$$= \frac{1}{4} \int \left(M^{\times} \mathrm{curl}\, U + J_0^{\times} U_0 + c.c \right)_b d\tau_b \tag{5.21}$$

and may now be calculated, as before, for a second nucleon at $\bar{r} = \bar{r}_k$ in the field of the first.

The first term of the integral (5.21)

$$= \frac{1}{4} g_2 g_2^{\times} \left(\frac{hc}{\pi}\right)^2 \int \psi^{+k}\sigma^k\psi^k \, \delta(\bar{r}_b-\bar{r}_k)\, \mathrm{curl}_b \left[(\psi^+\sigma\psi)^i \times \mathrm{grad}_i \, \phi(\bar{r}_b-\bar{r}_i) \right] d\tau_b . \tag{5.22}$$

But $\mathrm{curl}_b \left[(\psi^+\sigma\psi)^i \times \mathrm{grad}_i \phi \right] = (\psi^+\sigma\psi)^i \, \mathrm{div}_b(\mathrm{grad}_i \phi)$

$$- \mathrm{grad}_i\phi \, \mathrm{div}_b (\psi^+\sigma\psi)^i + (\mathrm{grad}_i \phi \, \mathrm{grad}_b)(\psi^+\sigma\psi)^i - \left[(\psi^+\sigma\psi)^i \cdot \mathrm{grad}_b \right] \mathrm{grad}_i \phi$$

and the sedond and third terms vanish since $(\psi^+\sigma\psi)^i$ does not depend upon r_b .

Also $\qquad \text{grad}_i\, \phi(\bar r_b - \bar r_i) = -\text{grad}_b\, \phi(\bar r_b - \bar r_i)$

and $\qquad \text{div}\left[\text{grad}_i\, \phi(\bar r_b - \bar r_i)\right] = -\text{div}_i\, \text{grad}_i\, \phi(\bar r_b - \bar r_i)$

$$= -\nabla_i^2\, \phi(r_b - r_i)$$

With the aid of the δ-function

$$\phi(\bar r_b - \bar r_i) = \int \delta(\bar r_b - \bar r_i)\, \phi(\bar r - \bar r_i)\, d\tau$$

and is thus a solution of the equation

$$\nabla^2 \phi(\bar r_b - \bar r_i) - \kappa^2 \phi(\bar r_b - \bar r_i) = -\delta(\bar r_b - \bar r_i).$$

Thus the expression (5.22) becomes

$$\frac{1}{4}\, g_2 g_2^{\times} \left(\frac{hc}{\pi}\right)^2 \int \psi^{+k} \sigma^k \psi^k\, \delta(\bar r_b - \bar r_k) \Big\{ -(\psi^+\sigma\psi)^i\, \nabla_i^2\, \phi(\bar r_b - \bar r_i)$$

$$- \left[(\psi^+\sigma\psi)^i \cdot \text{grad}_i\right] \text{grad}_b\, \phi(\bar r_b - \bar r_i) \Big\}\, d\tau_b$$

$$= \frac{1}{4}\, g_2 g_2^{\times} \left(\frac{hc}{\pi}\right)^2 \int \psi^{+k} \sigma^k \psi^k\, \delta(\bar r_b - \bar r_k) \Big\{ -(\psi^+\sigma\psi)^i\, \kappa^2 \phi(\bar r_b - \bar r_i) + (\psi^+\sigma\psi)^i\, \delta(\bar r_b - \bar r_i)$$

$$- \left[(\psi^+\sigma\psi)^i \cdot \text{grad}_i\right] \text{grad}_b\, \phi(\bar r_b - \bar r_i) \Big\}\, d\tau_b$$

$$= \frac{1}{4}\, g_2 g_2^{\times} \left(\frac{hc}{\pi}\right)^2 \Big\{ -\kappa^2 (\psi^+\sigma\psi)^k (\psi^+\sigma\psi)^i\, \phi(\bar r_k - \bar r_i)$$

$$- \left[(\psi^+\sigma\psi)^k \cdot \text{grad}_k\right]\left[(\psi^+\sigma\psi)^i \cdot \text{grad}_i\right] \phi(\bar r_k - \bar r_i) \Big\} \qquad (5.23)$$

the second term vanishing if the possibility $\bar r_i = \bar r_k$ is excluded.

The second term of the integral (5.21)

$$= \frac{1}{4} g_2 g_2^{*} \left(\frac{hc\kappa}{\pi}\right)^2 \int \psi^{+k} \psi^{k} \, \delta(\bar{r}_b - \bar{r}_k) \, (\psi^{+}\psi)^{i} \, \phi(r_b - r_i) \, d\tau_b$$

$$= \frac{1}{4} g_2 g_2^{*} \left(\frac{hc\kappa}{\pi}\right)^2 (\psi^{+}\psi)^{k} \, (\psi^{+}\psi)^{i} \, \phi(\bar{r}_k - \bar{r}_i) . \tag{5.24}$$

Hence, adding (5.23), (5.24) and the complex conjugate expressions, the total interaction energy

$$= \frac{1}{2} |g_2|^2 \left(\frac{hc\kappa}{\pi}\right)^2 \Big\{ (\psi^{+}\psi)^{k} (\psi^{+}\psi)^{i} - (\psi^{+}\sigma\psi)^{k} (\psi^{+}\sigma\psi)^{i} - \frac{1}{\kappa^2} \left[(\psi^{+}\sigma\psi)^{k} \cdot \operatorname{grad}_k \right] \left[(\psi^{+}\sigma\psi)^{i} \cdot \operatorname{grad}_i \right] \Big\} \, \phi(\bar{r}_k - \bar{r}_i). \tag{5.25}$$

The Pseudo-vector Interaction.

Although the field equations and the expression for the energy associated with this field are of the same form as those of the vector field, the interaction energy is quite different in the static case.

For now, from equations (4.54),

$$s_{k0\cdot} = 0\,, \qquad s_{123} = 0,$$

i.e.
$$M'_k = 0\,, \qquad J'_0 = 0. \tag{5.26}$$

Thus equations (4.48) to (4.51) become

$$H' = \operatorname{curl} U'\,, \tag{5.27}$$

$$D' = -\operatorname{grad} U'_0 + P'\,, \tag{5.28}$$

$$\operatorname{curl} H' = -\kappa^2 U' + J' \tag{5.29}$$

and
$$\operatorname{div} D' = -\kappa^2 U'_0\,. \tag{5.30}$$

Also, since

$$\operatorname{div} t_{kl0} = \frac{1}{c}\frac{\partial}{\partial t} t_{123} = 0\,,$$

$$\operatorname{div} v_{kl0} + \operatorname{div} s_{kl0} = 0$$

giving
$$\operatorname{div} U' = \frac{1}{\kappa^2} \operatorname{div} J'. \tag{5.31}$$

From equations (5.28) and (5.30)

$$\nabla^2 U'_0 - \kappa^2 U'_0 = \operatorname{div} P' \tag{5.32}$$

and from equations (5.27), (5.29) and (5.31)

$$\nabla^2 U' - \kappa^2 U' = -J' + \frac{1}{\kappa^2}\operatorname{grad}\operatorname{div} J' \tag{5.33}$$

Hence
$$U'_0 = -\int \operatorname{div}_a P'\,\phi(\bar{r}-\bar{r}_a)\,d\tau_a$$

$$= +\int P'_a \operatorname{grad}_a \phi(\bar{r}-\bar{r}_a)\,d\tau_a\,.$$

From equations (4.47) and (4.54)

$$P' = - g_3^{\times} \frac{hc}{\pi} \psi^{+} i\alpha_K\alpha_L \psi .$$

Thus $$P'_a = - g_3^{\times} \frac{hc}{\pi} \psi^{+i} \sigma^i \psi^i \, \delta(\bar{r}_a - \bar{r}_i) \tag{5.34}$$

in the present case and

$$U_0' = - g_3^{\times} \frac{hc}{\pi} (\psi^{+}\sigma\psi)^i \operatorname{grad}_i \phi(\bar{r} - \bar{r}_i) . \tag{5.35}$$

From equation (5.33)

$$U' = \int \left(J' - \frac{1}{K^2} \operatorname{grad} \operatorname{div} J' \right)_a \phi(\bar{r} - \bar{r}_a) \, d\tau_a \tag{5.36}$$

where, from equations (4.47) and (4.54)

$$\begin{aligned} J'_a &= iK s_{Kl0} \\ &= g_3^{\times} \frac{hcK}{\pi} i \psi^{+i} \sigma^i \psi^i \, \delta(\bar{r}_a - \bar{r}_i) \end{aligned} \tag{5.37}$$

The interaction energy, from equation (4.53),

$$\begin{aligned} &= \frac{1}{4} \int \left(s^{\times}_{Kl.} f_{kl.} + s^{\times}_{Kl0} f_{kl0} + c.c. \right)_b d\tau_b \\ &= \frac{1}{4} \int \left(P'^{\times} D' - J'^{\times} U' + \frac{1}{K^2} J'^{\times} J' + c.c. \right)_b d\tau_b \\ &= \frac{1}{4} \int \left(- P'^{\times} \operatorname{grad} U_0' + P'^{\times} P' - J'^{\times} U' + \frac{1}{K^2} J'^{\times} J' + c.c \right) d\tau_b . \end{aligned} \tag{5.38}$$

For the two-nucleon problem, the first term

$$\begin{aligned} &= -\frac{1}{4} g_3 g_3^{\times} \left(\frac{hc}{\pi}\right)^2 \int \psi^{+K} \sigma \psi^K \delta(\bar{r}_b - \bar{r}_K) \operatorname{grad}_k \left[(\psi^{+}\sigma\psi)^i \cdot \operatorname{grad}_i \right] \phi(\bar{r}_b - \bar{r}_i) \, d\tau_b \\ &= -\frac{1}{4} |g_3|^2 \left(\frac{hc}{\pi}\right)^2 \left[(\psi^{+}\sigma\psi)^K \cdot \operatorname{grad}_k \right] \left[(\psi^{+}\sigma\psi)^i \cdot \operatorname{grad}_i \right] \phi(\bar{r}_K - \bar{r}_i) . \end{aligned} \tag{5.39}$$

The second term

$$= \frac{1}{4}\,|g_3|^2\left(\frac{hc}{\pi}\right)^2 \int (\psi^{+k}\sigma^k\psi^k)\,\delta(\bar{r}_b-\bar{r}_k)(\psi^{+i}\sigma^i\psi^i)\,\delta(\bar{r}_b-\bar{r}_i)\,d\tau_b$$

$$= 0 \tag{5.40}$$

if the possibility $\bar{r}_k = \bar{r}_i$ is excluded.

The fourth term is similar to the second and is also zero. The third term

$$= -\frac{1}{4}\iint J_b'^{\,*}\left(J_a' - \frac{1}{\kappa^2}\,\mathrm{grad}_a\,\mathrm{div}_a\,J_a'\right)\phi(\bar{r}_b-\bar{r}_a)\,d\tau_a\,d\tau_b$$

of which the first part of the integral

$$= \frac{1}{4}\,|g_3|^2\left(\frac{hc\kappa}{\pi}\right)^2 \iint \psi^{+k}\sigma^k\psi^k\,\delta(\bar{r}_b-\bar{r}_k)\,\psi^{+i}\sigma^i\psi^i\,\delta(\bar{r}_a-\bar{r}_i)\,\phi(r_b-r_a)\,d\tau_a\,d\tau_b$$

$$= \frac{1}{4}\,|g_3|^2\left(\frac{hc\kappa}{\pi}\right)^2 (\psi^+\sigma\psi)^k\,(\psi^+\sigma\psi)^i\,\phi(\bar{r}_k-\bar{r}_i) \tag{5.41}$$

and the second part

$$= -\frac{1}{4}\,|g_3|^2\left(\frac{hc}{\pi}\right)^2 \iint \psi^{+k}\sigma^k\psi^k\,\delta(\bar{r}_b-\bar{r}_k)\,\mathrm{grad}_a\,\mathrm{div}_a\left[\psi^{+i}\sigma^i\psi^i\,\delta(\bar{r}_a-\bar{r}_i)\right]\phi(\bar{r}_b-\bar{r}_a)\,d\tau_a\,d\tau_b.$$

Integrating by parts and making use of the fact that

$$\mathrm{grad}_a\,\phi(\bar{r}_b-\bar{r}_a) = -\,\mathrm{grad}_b\,\phi(\bar{r}_b-\bar{r}_a)$$

this is found to be equal to

$$+\frac{1}{4}\,|g_3|^2\left(\frac{hc}{\pi}\right)^2\left[(\psi^+\sigma\psi)^k\cdot\mathrm{grad}_k\right]\left[(\psi^+\sigma\psi)^i\cdot\mathrm{grad}_i\right]\phi(\bar{r}_k-\bar{r}_i). \tag{5.42}$$

This contribution is exactly equal and opposite to that from the first term of the interaction energy.

Adding the complex conjugate quantity the total interaction energy in the pseudo-vector case is therefore

$$\frac{1}{2}\,|g_3|^2\left(\frac{hc}{\pi}\right)^2 (\psi^+\sigma\psi)^k\,(\psi^+\sigma\psi)^i\,\phi(\bar{r}_k-\bar{r}_i) \tag{5.43}$$

The Pseudo-scalar Interaction

In this case $S_{123\cdot} = 0, \quad S_{1230} = 0$

therefore
$$\mathcal{Q}' = R', \tag{5.44}$$
$$F_0' = G_0' \tag{5.45}$$

and the field equations (4.62) and (4.63) become
$$G' = -\operatorname{grad} R' \tag{5.46}$$
$$G_0' = 0 \tag{5.47}$$
$$\operatorname{div} F' = -\kappa^2 \mathcal{Q}' = -\kappa^2 R'. \tag{5.48}$$

Let $\quad S_{kl0\cdot} = N' = g_4^{\times} \frac{hc}{\pi} \psi^+ \sigma \psi \qquad$ from equation (4.66)

then
$$F' = G' + N' \tag{5.49}$$

and equation (5.48) may be written
$$\operatorname{div} G' = -\kappa^2 R' - \operatorname{div} N'. \tag{5.50}$$

Hence from equation (5.46)
$$\nabla^2 R' - \kappa^2 R' = \operatorname{div} N' \tag{5.51}$$

and thus
$$R'(r) = -\int (\operatorname{div} N')_a \, \phi(\bar{r}-\bar{r}_a)\, d\tau_a$$
$$= +\int N_a' \cdot \operatorname{grad}_a (\bar{r}-\bar{r}_a)\, d\tau_a$$

on integrating by parts.

In this particular case
$$N_a' = g_4^{\times} \frac{hc}{\pi} \psi^{+i} \sigma^i \psi^i \, \delta(\bar{r}_a - \bar{r}_i) \tag{5.52}$$

and so
$$R'(r) = g_4^{\times} \frac{hc}{\pi} (\psi^+ \sigma \psi)^i \operatorname{grad}_i \phi(\bar{r}-\bar{r}_i). \tag{5.53}$$

The interaction energy, in the static case, is, from equation (4.65)

$$\frac{1}{4}\int\left(s^{\times}_{k\ell o.}\,t_{k\ell o.}+c.c\right)_b d\tau_b$$

$$=\frac{1}{4}\int\left(N'^{\times}\cdot F'+c.c\right)_b d\tau_b$$

$$=\frac{1}{4}\int\left(-N'^{\times}\,\mathrm{grad}\,R'+N'^{\times}N'+c.c\right)_b d\tau_b\,. \tag{5.54}$$

For the two-nucleon problem, the first term

$$=-\frac{1}{4}g_4^{\times}g_4\left(\frac{hc}{\pi}\right)^2\int\psi^{+k}\sigma^k\psi^k\,\delta(\bar{r}_b-\bar{r}_k)\,\mathrm{grad}_b\left[(\psi^+\sigma\psi)^i\cdot\mathrm{grad}_i\,\phi(\bar{r}_b-\bar{r}_i)\right]d\tau_b$$

$$=-\frac{1}{4}|g_4|^2\left(\frac{hc}{\pi}\right)^2\left[(\psi^+\sigma\psi)^k\cdot\mathrm{grad}_k\right]\left[(\psi^+\sigma\psi)^i\cdot\mathrm{grad}_i\right]\phi(\bar{r}_k-\bar{r}_i).$$

The second term

$$=\frac{1}{4}|g_4|^2\left(\frac{hc}{\pi}\right)^2\int\psi^{+k}\sigma^k\psi^k\,\delta(\bar{r}_b-\bar{r}_k)\,\psi^{+i}\sigma^i\psi^i\,\delta(\bar{r}_b-\bar{r}_i)\,d\tau_b$$

$$=0$$

since the possibility $\bar{r}_i=\bar{r}_k$ is excluded.

Thus, adding the complex conjugate, the total interaction energy

$$=-\frac{1}{2}|g_4|^2\left(\frac{hc}{\pi}\right)^2\left[(\psi^+\sigma\psi)^k\cdot\mathrm{grad}_k\right]\left[(\psi^+\sigma\psi)^i\cdot\mathrm{grad}_i\right]\phi(\bar{r}_k-\bar{r}_i)\,. \tag{5.55}$$

The expressions (5.10), (5.25), (5.43) and (5.55) for the interactions due to the various types of field are identical in form with those obtained by Kemmer, de Broglis and others, the pseudo-vector interaction (5.43) appearing slightly different only on account of the vanishing of the coefficient of the term containing

$$\left[(\psi^{+}\tau\psi)^{k}\cdot \mathrm{grad}_{k}\right]\left[(\psi^{+}\tau\psi)^{i}\cdot \mathrm{grad}_{i}\right]\phi(\bar{r}_{k}-\bar{r}_{i})$$

In Kemmer's paper (1938) this term is multiplied by a factor $(g_c^2 - f_c^2)$ where f_c and g_c are two complex constants introduced into the interaction terms of the Lagrangian and associated with field quantities $\chi^{\alpha\beta\gamma}$ and $\phi^{\alpha\beta}$ respectively. Kemmer's theory is completely symmetrical with respect to the χ and the ϕ ; expressing it in the five dimensional notation χ and $i\phi$ are seen to be corresponding quantities and f is equal to either $+ig$ or $-ig$ according to the type of field. Thus, in all cases, since f and g are complex,

$$g^2 = f^2$$

Hence in the particular case of the pseudo-vector field the above mentioned term vanishes when Kemmer's expressions are written in the five dimensional form. Since it is usually found necessary to assume a combination of the different types of interaction, in order to account for the stability of the deuteron for instance, this difference is not significant.

In the calculation of the interaction energy the usual difficulty arises, that of the non-central terms which become infinitely great as $(\bar{r}_i - \bar{r}_k)$ becomes very small. But it has been shown in Chapter IV that for all types of field the total energy is finite, at least in the Static case and when the electromegnetic field is neglected. This does suggest that when the method of "cutting off" (Bethe 1940) is used to avoid the infinities, nothing essential is being removed for the total energy may be divided at will into any two convenient parts. These may or may not contain infinities, but the total energy remains finite.

It is interesting to note that a somewhat similar conclusion has been reached recently in the field of cosmology by Professor Jordan of Hamburg who has given a short report of his work in Nature (1949).

It is well-known that a "red shift" occurs in the spectrum of a spiral nebula which is proportional to the distance of the nebula, and this may be interpreted as a Doppler effect meaning a general expansion of space. In order to account for the fact that the mean density of mass throughout the universe remains sensibly constant a number of cosmologists, Bondi, Gold and Hoyle, have postulated a continual creation of matter in the form of hydrogen atoms. Jordan, however, appears to be the only one to reconcile this creation of matter with the principle of conservation of energy.

It may be shown empirically that

$$\frac{8\pi}{c^2} GM \simeq R$$

where G is the Newtonian constant of gravitation, M the total mass and R the "radius" of the (observable) universe.

But this may be written in the form

$$\frac{GM^2}{R} \simeq Mc^2$$

which suggests that "the negative potential energy of gravitation for the whole universe is equal to the sum of the rest energies of the masses of the stars".

Hence it seems possible that the total energy of the universe is, and remains, exactly zero and that the continual creation of mass takes place at the expense of the energy of the gravitational field.

Jordan supposes further that this holds, not only for the universe as a whole, but also for a finite region in which creation of matter takes place.

Chapter VI

THE INTERACTION OF THE ELECTRON WITH THE ELECTROMAGNETIC FIELD.

In conclusion, the development of this theory of the nuclear field now suggests a new approach to the problem of the electron. As was shown in Chapter II, this particle is properly represented not by the Dirac equation

$$\gamma^{\mu}\frac{\partial\psi}{\partial x^{\mu}} = 0 \tag{6.1}$$

but by the equation

$$\gamma^{\mu}\left(\frac{\partial}{\partial x^{\mu}} - \Delta_{\mu}\right)\psi = 0 \tag{6.2}$$

where $\gamma^{\mu}\Delta_{\mu} = -\beta.\Delta_5$ (p.32) (6.3)

and $\Delta_5 = \frac{\alpha}{8} B_{mn}\beta_m\beta_n.$ (p.24) (6.4)

The extra term has been shown already (p.33) to give rise to an additional energy density of magnitude

$$\frac{he}{8\pi mc}\left[B_{kl}\,\psi^{+}i\beta\alpha_k\alpha_l\psi - E_k\,\psi^{+}i\beta\alpha_k\psi\right]. \tag{6.5}$$

As in the case of the nucleon this may be regarded as the energy of interaction between the electron and the external electromagnetic field for each term is the product of a field quantity and a quantity characteristic of the particle.

The electron, from the point of view of quantum mechanics, is spread out through space and as a result there exists everywhere something which may be regarded as polarised matter. If the external field is represented bythe vectors B and E and there exists a magnetic polarisation M and an electric polarisation P the interaction energy density will be

$$\frac{1}{2}\,(B.M + E.P). \tag{6.6}$$

On comparing this with (6.5)

$$M = \frac{he}{4\pi m_0 c}\,\psi^+ i\beta\alpha_K\alpha_L\psi \quad , \tag{6.7}$$

$$P = -\frac{he}{4\pi m_0 c}\,\psi^+ i\beta\alpha_K\psi\,. \tag{6.8}$$

These are the polarisation densities of the electron itself. Integrating throughout space the usual expressions for the magnetic and electric moments of the electron are obtained.

This therefore appears to be a reasonable interpretation of the extra term of equation (6.2) although it is open to the objection (Pauli 1941) that the original Dirac equation (6.1), when suitably manipulated, does give these magnetic and electric moments without the introduction of extra terms.

It is possible to add still other terms to the energy density since, as was pointed out in Chapter II (p.25) a quantity of the form $F_\mu I$ (where I is the unit matrix) may be added to Δ_μ.

The corresponding energy density is proportional to

$$\begin{aligned} &\psi^+ i\beta_4\gamma^\mu F_\mu\psi \\ &= -\psi^+\alpha_K\psi\, f_K + i\,\psi^+\psi f_4 - \psi^+\beta\psi f. \end{aligned} \tag{6.9}$$

in terms of four dimensional quantities, (f_K, f_4) being some vector quantity associated with the field and $f.$ a corresponding scalar. There will be a suitable multiplying constant.

The vector which comes to mind is the electromagnetic potential ϕ in which case $f. = 0$ (Chapter I, p.8) and the energy density (6.9) is then of the form

$$-\psi^{+}\alpha_{K}\psi \ \phi_{K} - \psi^{+}\psi\,\phi_{o}\,. \tag{6.10}$$

This is of the form of an interaction between field and particle but these terms are already present in the energy as derived from the Dirac equation and it may be concluded that it is not necessary to introduce them again.

Consideration of the last (scalar) addition however leads to something new. Returning to the expression (6.5) the progressive form of the two terms suggests the addition of a term

$$\frac{he}{8\pi m_{o}c}\ \psi^{+} i\beta\psi\ S \tag{6.11}$$

where S is some scalar (invariant) quantity to be associated with the electron and such that the whole expression has the dimensions of an energy density.

The five dimensional form of the interaction terms suggests a value for S. In the case of the vector meson field for instance the field quantities have all been expressed as the curl of a "potential" $\frac{i}{K}\,t_{m}.$ (p.45). If this is regarded as being part of a five-vector the fifth (invariant) component would be $\frac{i}{K} t_{..}$. This does not exist, on account of the antisymmetry of the field tensor, and some other suitable quantity must be found.

In developing the theory of the five dimensional space (Chapter I) the elctromagnetic potential is associated with the coefficients γ_{m5} by the relations

$$\gamma_{m5} = \gamma_{55} \alpha \phi_m .$$

It is generally supposed that $\phi_5 = 0$, but formally a fifth component might be defined by

$$\gamma_{55} = \gamma_{55} \alpha \phi_5$$

whence $$\phi_5 = 1/\alpha$$

The quantity then taking the place of $t_{..}$ is then

$$-i\kappa \phi_5 = -i\kappa/\alpha$$

In identifying this with S both κ and α will be supposed to refer to the electron and thus

$$S = -i \frac{2\pi m_0 c}{h} \Big/ \frac{e}{m_0 c^2}$$

and the additional energy density (6.11) becomes

$$\frac{1}{4} m_0 c^2 \psi^+ \beta \psi . \tag{6.12}$$

In the expression for the energy density as obtained from the usual Dirac equation the term

$$- m_0 c^2 \psi^+ \beta \psi \tag{6.13}$$

occurs (p.32). The new term (6.12) suggests an addition to this and an interesting feature is the appearance of the factor $\frac{1}{4}$. For, according to the classical theory, the electromagnetic energy of the electron is not $m_0c^2(1-v^2/c^2)^{-1/2}$

but only $m_0c^2(1-v^2/c^2)^{-1/2} - \frac{1}{4}\, m_0c^2(1-v^2/c^2)^{-1/2}$ (6.14)

and the expressions (6.12) and (6.13) integrated throughout space and taken together give just this.

Thus the term (6.11) might formally be used to account for that part of the energy usually described as non-electromagnetic, but this is only a tentative suggestion and offers no physical interpretation of the problem.

It may be mentioned here that de Broglie (1950), in developing the theory of the "subtractive field", has attempted to account for this energy discrepancy in a way which is, formally, very much the same. Attributing to the photon a finite, though extremely small, mass he assumes equations for the electromagnetic field which are of the same form as those for the nuclear fields with which are associated mesons of integral spin. In other words, the "photon field" is treated as a special type of meson field. The field potentials are then formally all of the same type and may be added together. It is assumed that all particles may possess "mesonic charges" as well as electric charge and thus an electron moving in a superposition of electromagnetic and meson fields will interact with the electromagnetic field because of its electric charge and with the meson fields by reason of its "mesonic charge".

de Broglie suggests that the total interaction is the sum of these separate interactions and that this may be the origin of the non-electromagnetic energy. Such an addition of interaction terms to the energy density is exactly equivalent to the method already described.

It is still difficult to understand the real significance of these extra terms. They are permissible on the grounds of relativistic invariance but the usual Dirac equation has given satisfactory results without them and so it may be wondered whether they are really necessary.

It might be suggested that, so far as this last tentative addition, that of the mass term, is concerned the modified equation is really the same as the usual one but that m_0 is different in each case. The question then arises as to the exact nature of the quantity m_0 occurring in the fundamental equations. Here lies one of the chief difficulties of a field theory of the electron.

Appendix.

The matrices of the Dirac theory.

$$\beta = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

$$\alpha_1 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \qquad \alpha_2 = \begin{bmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{bmatrix} \qquad \alpha_3 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ 1 & & & \\ & -1 & & \end{bmatrix}$$

$$i\beta\alpha_2\alpha_3 = \begin{bmatrix} & -1 & & \\ -1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \qquad i\beta\alpha_3\alpha_1 = \begin{bmatrix} & i & & \\ -i & & & \\ & & & -i \\ & & i & \end{bmatrix} \qquad i\beta\alpha_1\alpha_2 = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

$$i\beta\alpha_1 = \begin{bmatrix} & & & i \\ & & i & \\ & -i & & \\ -i & & & \end{bmatrix} \qquad i\beta\alpha_2 = \begin{bmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{bmatrix} \qquad i\beta\alpha_3 = \begin{bmatrix} & & i & \\ & & & -i \\ -i & & & \\ & i & & \end{bmatrix}$$

$$i\alpha_2\alpha_3 = \begin{bmatrix} & -1 & & \\ -1 & & & \\ & & & -1 \\ & & -1 & \end{bmatrix} \qquad i\alpha_3\alpha_1 = \begin{bmatrix} & i & & \\ -i & & & \\ & & & i \\ & & -i & \end{bmatrix} \qquad i\alpha_1\alpha_2 = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \qquad i\alpha_1\alpha_2\alpha_3 = \begin{bmatrix} & & -1 & \\ & & & -1 \\ -1 & & & \\ & -1 & & \end{bmatrix}$$

$$\beta\alpha_1\alpha_2\alpha_3 = \begin{bmatrix} & & i & \\ & & & i \\ -i & & & \\ & -i & & \end{bmatrix}$$

Transformation properties of these quantities.

Invariant. $\psi^+\beta\psi$

Four-vector. $\psi^+\alpha_K\psi$, $-\psi^+\psi$,

Six-vector. $\psi^+ i\beta\alpha_K\alpha_l\psi$, $\psi^+ i\beta\alpha_K\psi$,

Ps. vector. $\psi^+ i\alpha_K\alpha_K\psi$, $-\psi^+ i\alpha_1\alpha_2\alpha_3\psi$

Ps. invariant. $\psi^+\beta\alpha_1\alpha_2\alpha_3\psi$.

(in terms of co-ordinates $x_1\ x_2\ x_3\ x_0$)

References.

Chapter I.

Fisher, J.W.	1929.	Proc.Roy.Soc.A, 123, p.489.
Flint, H.T.	1940.	PhilMag.(7), 29, p.330.
Kaluza, T.	1921.	S.B.preuss.Akad.Wiss., p.966.
Klein, O.	1927.	Z.Phys., 46, p.188.
Weyl, H.	1918.	S.B.Berlin Akad., p.465.

Chapter II.

Allard, G.	1950.	J.Phys. et le Radium, 11, p.646.
Borsellino, A. and Caldironi, P.	1946.	Rend. dell' Accad.Nat.dei Lincei (8) 1. p.969.
Flint, H.T.	1935.	Proc.Roy.Soc.A, 150, p.421.
	1943.	Phil.Mag.(7), 34, p.496.
Mimura, Y.	1935.	J.Sci. Hiroshima Univ. A 5, p.99.
Pauli, W.	1941.	Rev.Mod.Phys. 13, p.223.
Schrodinger, E.	1932.	S.B.preuss.Akad.Wiss.(9), p.105.

Chapter III.

Anderson, C.D.and Neddermeyer,	1936.	Phys.Rev. 50, p.273.
	1937.	Phys.Rev. 51, p.884.
Darwin, C.G.	1928.	Proc.Roy.Soc.A, 118, p.654.
Eddington, A.S.	1923.	Math.Theory of Rel. (1st ed.), Ch. VII, p.200.

Fermi, E.	1934.	Z.Phys., 88, p.161.
Fisher, J.W. and Flint, H.T.	1930.	Proc.Roy.Soc., 126, p.644.
Flint, H.T.	1935.	Proc.Roy.Soc.A, 150, p.421.
	1945.	Proc.Roy.Soc.A, 185, p.14.
	1945.	Phil.Mag. (7), 36, p.635.
	1947.	Phil.Mag. (7), 38, p.22.
Frenkel, J.	1928.	Z.Phys., 52, p.356.
Gordon, W.	1926.	Z.Phys., 40, p.117.
Kemmer, N.	1938.	Proc.Roy.Soc.A, 166, p.137.
Møller, C. and Rosenfeld, L.	1940.	Proc.Cop. 17, no. 8.
Proca, A.	1936.	J.Phys. et le Radium, 7, p.346.
	1937.	J.Phys. et le Radium, 8, p.23.
Weyl, H.	1921.	Raum, Zeit. u. Materie (4th ed.), p.263.
Whittaker, J.M.	1928.	Proc.Roy.Soc.A, 121, p.543.
Yukawa, H.	1935.	Proc.Phys.Math.Soc.Japan, 17, p.48.
Yukawa, H., and Sakata, S.	1937.	Proc.Phys.Math.Soc.Japan, 19, p.1084.
Yukawa, H., Sakata, S. and Taketani, M.	1938.	Proc.Phys.Math.Soc.Japan, 20, p.319.

<u>Chapter IV.</u>

de Broglie, L.	1945.	La Théorie du Noyan, Vol.II, Ch.XV.

Flint, H.T. 1945. Proc.Roy.Soc.A, 185, p.14.

1945. Phil.Mag. (7), 36, p.635.

Tanikawa, Y. and
Yukawa,H. 1941. Proc.Phys.Math.Soc.Japan, 23, p.445.

Yukawa, H. et al.
1938. Proc.Phys.Math.Soc.Japan, 20, p.319.

Chapter V.

de Broglie, L. 1934. L'électron Magnétique, Ch.XII, p.164.

1945. La Théorie du Noyan, Vol.II, Ch. XV.

Jordan, P. 1949. Nature, 164, p.637.

Kemmer, N. 1938. Proc.Roy.Soc.A, 166, p.127.

Chapter VI.

de Broglie, L. 1950. J.Phys.et le Radium, 11, p.481.

Pauli, W. 1941. Rev.Mod.Phys., 13, p.223.