AN ESSAY

ON THE DISTRIBUTION OF VALUES OF THE σ-FUNCTION

AND RELATED TOPICS

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PREFACE

The function of n, $\sigma(n)$ is defined to be the sum of all positive integer divisors of n.

This dissertation is a survey of major work done during the last 30 years on problems connected with numbers n satisfying

$$\frac{\sigma(n)}{n} \geqslant \lambda, \qquad \lambda > 1.$$

For real values of λ these numbers are called λ-abundant. When $\lambda = 2$, the numbers are simply called abundant. When λ is any integer, but the equality sign only holds, the numbers satisfying the equation are called multi-perfect, or, for the special case $\lambda = 2$, perfect. There is also some discussion on pairs of amicable numbers (m, n), for which

$$\sigma(m) = \sigma(n) = m + n.$$

The first chapter contains those standard results of number theory which are essential for the development of this dissertation. The reader is advised to proceed direct to Chapter 11 and to refer to these results as they arise in the text.

Chapter 11 is an introductory chapter, in which the historical back-ground of these numbers is presented. It includes theorems discussing the general behaviour of the function $\frac{\sigma(n)}{n}$, for example its average value, and true maximum order, and a deeper theorem finding the number of distinct numbers $\frac{\sigma(m)}{m}$, for all integers m not exceeding n.

The third chapter contains a very full discussion of abundant and λ-abundant numbers. Conditions for consecutive abundant numbers are found, the sequences of abundant numbers and λ-abundant numbers are shown to possess density, and bounds for the numbers of primitive abundant, and primitive λ-abundant numbers not exceeding n are determined.

Chapter 1V concerns perfect and multi-perfect numbers. The density of the sequences of both these types of numbers is shown to be zero, and upper bounds for the numbers of both perfect and multi-perfect numbers not exceeding n are found.

The final chapter contains a brief discussion of amicable numbers. Considerably less is known about these, and the chapter contains only one major theorem: namely that the density of the sequence of amitable numbers is zero.

The theorems incorporated into this dissertation are mainly those of P.Erdös, whose work in this particular field has been outstanding.

NOTATION.

$a, b, m, n, \ldots\ldots$	denote integers. The letters and will be used without exception to denote primes.
$x, y, \ldots\ldots$	denote real numbers.
$c, c_1, c_2, \ldots\ldots$	denote absolute constants.
$\sum_{m \le n} f(m)$, $\prod_p f(p)$,	(with various modifications and extensions which will be explained in the text) indicate sums and products respectively over all positive integers m, or all primes p, within the specified ranges. In the case of the product, where no range is indicated, it is understood that all primes are to be included.
$o, O, \sim$,	are used in the classical sense. If $f(n)$ and $\phi(n)$ are functions of the integral variable n, then (1) $f = o(\phi)$, means that $\lim_{n\to\infty} f/\phi = 0$. (2) $f = O(\phi)$, means that $\lvert f \rvert < c\phi$, for all values of n. (3) $f \sim \phi$, means that $\lim_{n\to\infty} f/\phi = 1$.
$\min(a,b)$, $\max(a,b)$,	denote respectively the lesser and greater of a and b.
$[x]$	denotes the largest integer which does not exceed x.
$b \mid a$, $b \nmid a$,	mean that b divides a, or b does not divide a, respectively.
$a \equiv b \pmod{m}$,	means that m divides $a-b$.
$(a,b) = d$,	means that d is the highest common factor of a and b. Thus $(a,b)=1$ means that a and b are co-prime.
${}^nC_r$,	denotes the binomial coefficient $\frac{n!}{(n-r)!\,r!}$
$\zeta(s)$,	the Riemann-zeta function, denotes $\sum_{n=1}^{\infty} \frac{1}{n^s}$, where $\Re(s) > 1$. (In our case s will only take real values).
s_m, r_m,	denote respectively the square-free and quadratic parts of an integer m. i.e. if the prime decomposition of m is $p_1 p_2 \cdots p_r q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$, where each $\alpha_i \ge 2$, then $s_m = p_1 p_2 \cdots p_r$ and $r_m = q_1^{\alpha_1} \cdots q_s^{\alpha_s}$.

CHAPTER 1.

PRELIMINARIES.

1.1. This chapter contains a number of auxiliary results which will be used in the sequel. These fall, broadly speaking, into three main groups: one covering standard results on the distribution of primes, another giving some simple relations between an integer m and its square-free and quadratic parts, and a third defining some number-theoretical functions and proving any necessary results concerning them.

We list these results as a number of lemmas, (though some, of course, are major theorems in their own field), and give the proof whenever it is short. In the case of non-elementary results we give a reference only. The ordinary analytic theory of natural logarithms and exponentials is taken for granted, but it is important to lay stress on one property of $\log x$. Since

$$e^x = 1 + x + \cdots\cdots + \frac{x^r}{r!} + \frac{x^{r+1}}{(r+1)!} + \cdots\cdots,$$

$$x^{-r}e^x > \frac{x}{(r+1)!} \to \infty, \quad \text{as } x \to \infty.$$

Hence, $e^x \to \infty$ more rapidly than any power of x. It follows that $\log x$, the inverse function, tends to infinity more slowly than any positive power of x ; $\log x \to \infty$, but

$$\frac{\log x}{x^{\delta}} \to 0,$$

for every positive δ . Similarly, $\log\log x \to \infty$ more slowly than any power of $\log x$.

At this stage, however, the reader is advised to proceed direct to Chapter II, and to refer to the results of this chapter as they arise in the text.

1.2. In this section we collect together some standard results on the distribution of primes.

LEMMA 1.

$$\sum_{m \leq x} \frac{1}{m} = \log x + \gamma + O\left(\frac{1}{x}\right),$$

where γ is a constant, known as Euler's constant.

Proof. We have

$$\begin{aligned}
\sum_{m \leq x} \frac{1}{m} &= 1 + \sum_{1 \leq m \leq [x]-1} m\left(\frac{1}{m} - \frac{1}{m+1}\right) \\
&= 1 + \sum_{1 \leq m \leq [x]-1} \int_m^{m+1} \frac{[t]}{t^2}\, dt \\
&= 1 + \int_1^{[x]} \frac{[t]}{t^2}\, dt \\
&= 1 + \int_1^{x} \frac{[t]}{t^2}\, dt - \int_{[x]}^{x} \frac{[t]}{t^2}\, dt \\
&= 1 + \log x + \int_1^{x} \frac{[t]-t}{t^2}\, dt - [x]\left(\frac{1}{[x]} - \frac{1}{x}\right)
\end{aligned}$$

$$= \log x + \frac{[x]}{x} - \int_1^x \frac{t-[t]}{t^2}\,dt$$

$$= \log x + \gamma + E_x ,$$

where

$$\gamma = 1 - \int_1^\infty \frac{t-[t]}{t^2}\,dt ,$$

is independent of x , and

$$E_x = \int_x^\infty \frac{t-[t]}{t^2} + \frac{[x]}{x} - 1 = \int_x^\infty \frac{O(1)}{t^2}\,dt + O(\tfrac{1}{x}) = O(\tfrac{1}{x})$$

which completes the proof of Lemma 1.

LEMMA 2.

$\prod (1-\frac{1}{p})$ diverges to zero.

Proof. Consider the finite product

$$\prod_{p\le N} (1-\tfrac{1}{p})^{-1} = \prod_{p\le N} (1+\tfrac{1}{p}+\tfrac{1}{p^2}+\cdots\cdots)$$

$$= \sum_{m=1}^{\infty} \frac{\theta(m)}{m} ,$$

where

$$\theta(m) = \begin{cases} 1, \text{ if each prime factor of } m \text{ does not exceed } N, \\ 0, \text{ otherwise}. \end{cases}$$

Therefore, by Lemma 1,

$$\prod_{p\le N} (1-\tfrac{1}{p})^{-1} \geq \sum_{m\le N} \frac{1}{m} = \log N + \gamma + O(\tfrac{1}{N}).$$

Hence

$$\prod_{p\le N} (1-\tfrac{1}{p})^{-1} \to \infty, \text{ as } N \to \infty,$$

and Lemma 2 follows immediately.

LEMMA 3.

$$\sum \frac{1}{p} \text{ diverges} \; ; \quad \sum \frac{1}{p^2} \text{ converges.}$$

Proof. The first part follows from Lemma 2, and the second part by comparison with the convergent series $\sum \frac{1}{n^2}$.

LEMMA 4.*

$$\sum_{p \leq n} \frac{\log p}{p} = \log n + O(1).$$

LEMMA 5.*

$$\sum_{p \leq n} \frac{1}{p} = \log\log n + B + O\left(\frac{1}{\log n}\right),$$

where B is constant.

LEMMA 6.

$$\prod_{p \leq n} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log n},$$

where γ is Eüler's constant.

LEMMA 7.

$$c'' \log n > \prod_{p \leq n} \left(1 + \frac{1}{p}\right) > c' \log n,$$

for suitable constants $c' > 0$ and $c'' > 0$.

Proof.

$$\prod_{p \leq n} \left(1 + \frac{1}{p}\right) = \prod_{p \leq n} \left(1 - \frac{1}{p^2}\right) \prod_{p \leq n} \left(1 - \frac{1}{p}\right)^{-1},$$

and the result follows from Lemmas 3 and 6.

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* E. A. INGHAM, "Distribution of Prime Numbers," Cambridge Tract. No. 30, Theorem 7.

LEMMA 8.

$$\sum_{\substack{p,\alpha \\ p^\alpha \le n}} \frac{1}{p^\alpha} < 2\log\log n.$$

Proof. The left hand sum may be written as

$$\sum_{\substack{p,\alpha \\ p^\alpha \le n}} \frac{1}{p^\alpha} = \sum_{p\le n} \frac{1}{p} + \sum_{\substack{p^\alpha \le n \\ \alpha \ge 2}} \frac{1}{p^\alpha} < \sum_{p\le n} \frac{1}{p} + \sum_{p \le n} \left(\frac{1}{p^2} + \frac{1}{p^3} + \cdots \right)$$

$$= \sum_{p\le n} \frac{1}{p} + \sum_{p\le n} \frac{1}{p(p-1)}$$

$$< \sum_{p\le n} \frac{1}{p} + \sum_{2\le m\le n} \frac{1}{m(m-1)}$$

$$< \sum_{p\le n} \frac{1}{p} + 1$$

$$< 2\log\log n,$$

by Lemma 5 for all sufficiently large n.

LEMMA 9.

$$\prod_{p\le n} p < 4^n.$$

Proof. Consider the product $\prod_{N+1<p\le 2N+1} p$. It is clearly a factor of ${}^{2N+1}C_{N+1}$, and it is easily proved by induction that ${}^{2N+1}C_{N+1} < 4^N$.

Hence,

$$\prod_{N+1<p\le 2N+1} p < 4^N.$$

We now prove Lemma 9 by induction on n. The inequality is clearly true for $n = 2, 3$. If n is even, then

$$\prod_{p \leq n} p = \prod_{p \leq n-1} p .$$

Hence, we need only prove the result for n odd. Put $n = 2N+1$. Then by the induction hypothesis

$$\prod_{p \leq 2N+1} p = \prod_{p \leq N+1} p \cdot \prod_{N+1 < p \leq 2N+1} p$$
$$< 4^{N+1} \cdot 4^{N}$$
$$= 4^{2N+1},$$

and this completes the proof of Lemma 9.

LEMMA 10.[*1] (Tchebycheff)

If $\pi(n)$ denotes the number of primes not exceeding n, then there exist two absolute constants $C_1 > 0$ and $C_2 > 0$ such that

$$\frac{C_1 n}{\log n} < \pi(n) < \frac{C_2 n}{\log n} .$$

LEMMA 11.[*2] (Prime Number Theorem).

$$\pi(n) = \int_2^n \frac{dt}{t} + O\left(\frac{n}{(\log n)^{\Delta}}\right),$$

for any fixed (positive) $\Delta > 1$.

LEMMA 12.[*3] (Alternative form of Prime Number Theorem)

$$\lim_{n \to \infty} \frac{1}{n} \sum_{p \leq n} \log p = 1 .$$

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*1 E. A. INGHAM. loc.cit. Theorem 4. (The right hand inequality is equivalent to our Lemma 9.)

*2 E. A. INGHAM. loc.cit. Theorem 23.

*3 E. A. INGHAM. loc.cit. Theorem 3. The result then follows from Lemma 11.

1.3. We now prove some simple relations between an integer m, its square-free part s_m, and its quadratic part r_m. We have already shown that if the prime decomposition of m is

$$m = p_1 p_2 \cdots p_r \, q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}, \qquad \alpha_i \geq 2,$$

then

$$s_m = p_1 p_2 \cdots p_r, \qquad r_m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots\cdots q_r^{\alpha_r}$$

Trivially, we have the following:

LEMMA 13.

$$(s_m, r_m) = 1.$$

We now prove two simple results about r_m.

LEMMA 14.

r_m may always be written as the product of a square and a cube.

Proof. Every integer greater than or equal to 2, (and hence every α_i), may be written in the form $2a + 3b$ for some integers a and b, where $a \geq 0$ and $b \geq 0$. The proof of Lemma 14 now follows immediately.

LEMMA 15.

r_m is always divisible by a square not less than $r_m^{2/3}$.

Proof. By Lemma 14 we may write

$$r_m = a^2 b,$$

where b is square-free, and where for every $p \mid b$ we have $p^2 \mid a$.

Hence

$$b \leq a,$$

$$r_m \leq a^3,$$

i.e. $$a^2 \geq r_m^{2/3}.$$

Finally, let $R(n, A)$ denote the number of integers $m \leq n$ for which $r_m > A$, where A is a positive constant. We prove

LEMMA 16.

$$R(n, A) < cnA^{-1/2},$$

where c is a positive absolute constant.

Proof. By Lemma 15,

$$R(n, A) \leq \sum_{r_m > A} \frac{n}{r_m} < \sum_{k^2 L^3 > A} \frac{n}{k^2 L^3}$$

$$< n \sum_{L=1}^{\infty} \frac{1}{L^3} \sum_{k^2 > \frac{A}{L^3}} \frac{1}{k^2}$$

$$< n \sum_{L=1}^{\infty} \frac{1}{L^3} \left(\frac{L^3}{A}\right)^{1/2}$$

$$= nA^{-\frac{1}{2}} \sum_{L=1}^{\infty} \frac{1}{L^{3/2}}$$

$$= cnA^{-\frac{1}{2}},$$

where $c > 0$ is an absolute constant.

1.4. In this last section we consider some properties of some of the better known arithmetic functions. An arithmetic function is a real function defined on the natural numbers, which describes some special property of these numbers.

We define first the divisor function $\tau(n)$ as the number of divisors of n. We shall show that $\tau(n)$ is multiplicative, i.e.

(1.1) $$\tau(mn) = \tau(m)\tau(n), \quad \text{if } (m,n) = 1.$$

Suppose that the prime decomposition of n is

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots\cdots p_r^{\alpha_r}.$$

Then a typical divisor of n has the form

$$p_1^{\beta_1} p_2^{\beta_2} \cdots\cdots p_r^{\beta_r},$$

where $0 \le \beta_i \le \alpha_i$. The total number of divisors of n is then the number of distinct sets of exponents $\beta_1, \beta_2, \cdots \beta_r$, with $0 \le \beta_i \le \alpha_i$.

Hence,

(1.2) $$\tau(n) = (\alpha_1+1)(\alpha_2+1)\cdots\cdots(\alpha_r+1),$$

and (1.1) follows from (1.2).

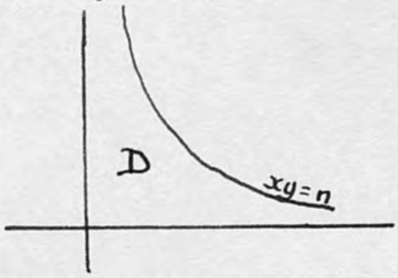
We prove also two simple results about the behaviour of $\tau(n)$ as a function of n. First:

LEMMA 17. (Dirichlet).

The average value of $\tau(n)$ is $\log n$.

Proof.

We denote by D the region in the upper right-hand quadrant contained between the axes and the rectangular hyperbola $xy = n$.



We count the lattice points in D, including those on the hyperbola, but not those on the axes. Every lattice point in D appears on a hyperbola

$$xy = s, \quad (1 \leq s \leq n),$$

and the number on such a hyperbola is $\tau(s)$. Hence, the number of lattice points in D is

$$\sum_{m \leq n} \tau(m) = \tau(1) + \tau(2) + \dots + \tau(n).$$

Of these points, $n = [n]$ have the x-co-ordinate 1, $[\frac{1}{2}n]$ have the x-co-ordinate 2, and so on. Hence, by Lemma 1, their number is

$$\sum_{m \leq n} \tau(m) = [n] + [\tfrac{n}{2}] + \dots + [\tfrac{n}{n}] = n\left(1 + \tfrac{1}{2} + \dots + \tfrac{1}{n}\right) + O(n)$$

$$= n \log n + O(n),$$

and this completes the proof of Lemma 17.

Secondly, we find an upper bound for the order of magnitude of $\tau(n)$.

LEMMA 18.

$$\tau(n) = o(n^{\delta}),$$

for all positive δ .

Proof. The assertions that $\tau(n) = o(n^{\delta})$, and $\tau(n) = O(n^{\delta})$, for all positive δ , are equivalent, since $n^{\delta'} = o(n^{\delta})$ when $0 < \delta' < \delta$. Therefore, let the prime decomposition of n be

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots\cdots p_r^{\alpha_r}.$$

By (1.2),

$$\frac{\tau(n)}{n^{\delta}} = \prod_{i=1}^{r} \left(\frac{\alpha_i + 1}{p_i^{\alpha_i \delta}} \right). \tag{1.3}$$

Since

$$\alpha \delta \log 2 \leq e^{\alpha \delta \log 2} = 2^{\alpha\delta} \leq p^{\alpha\delta},$$

We have

$$\frac{\alpha + 1}{p^{\alpha\delta}} \leq 1 + \frac{\alpha}{p^{\alpha\delta}} \leq 1 + \frac{1}{\delta \log 2} \leq \exp\left(\frac{1}{\delta \log 2}\right). \tag{1.4}$$

Now if $p \geq 2^{1/\delta}$, we have

$$\frac{\alpha + 1}{p^{\alpha\delta}} \leq \frac{\alpha + 1}{2^{\alpha}} \leq 1,$$

Therefore, by (1.3) and (1.4),

$$\frac{\tau(n)}{n^{\delta}} \leq \prod_{p \leq 2^{1/\delta}} \exp\left(\frac{1}{\delta \log 2}\right) < \exp\left(\frac{2^{1/\delta}}{\delta \log 2}\right) = O(1),$$

and this proves Lemma 18.

The second arithmetic function we consider is the

Moebius function $\mu(n)$, where

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \text{ is divisible by a square}, \\ (-1)^r, & \text{if } n \text{ is square-free and has } r \text{ distinct prime factors.} \end{cases}$$

It is clear from this definition that $\mu(n)$ is also multiplicative.

We shall prove three simple lemmas about this function.

LEMMA 19.

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. The result is obvious for $n = 1$. Suppose $n > 1$, and that the prime decomposition of n is

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}.$$

By the definition of $\mu(n)$ all we need consider is the sum

$$\sum_{d|p_1p_2\cdots p_r} \mu(d) = 1 + r(-1) + \frac{r(r-1)}{1.2.}(-1)^2 + \cdots\cdots + (-1)^r$$

$$= (1 + (-1))^r$$

$$= 0.$$

LEMMA 20.

$$\sum_{a^2|m} \mu(a) = |\mu(m)|.$$

Proof. Let b be the largest square dividing m. Then

$$\sum_{a^2|m} \mu(a) = \sum_{a|b} \mu(a).$$

But, by Lemma 19, the sum on the right-hand side is 1 if $b=1$, (i.e. m is square-free), and is 0 otherwise. Our lemma then follows by the definition of $|\mu(m)|$.

LEMMA 21.

Let

(1.5) $$F(n,r) = \sum_{\substack{m \le n \\ (m,r)=1}} |\mu(m)|.$$

Then

$$F(n,r) = \frac{6n}{\pi^2} \prod_{p|r} \left(1+\frac{1}{p}\right)^{-1} + O(n^{1/2}\tau(r)).$$

Proof. By Lemma 20,

$$F(n,r) = \sum_{\substack{m \le n \\ (m,r)=1}} \sum_{a^2 b = m} \mu(a) = \sum_{\substack{a \le \sqrt{n} \\ (a,r)=1}} \mu(a) \sum_{\substack{b \le \frac{n}{a^2} \\ (b,r)=1}} 1$$

$$= \sum_{\substack{a \le \sqrt{n} \\ (a,r)=1}} \mu(a)\, G\left(\frac{n}{a^2}, r\right),$$

where, by Lemma 19,

$$G(x,r) = \sum_{b \le x} \sum_{d|(b,r)} \mu(d) = \sum_{d|r} \mu(d) \sum_{\substack{b \le x \\ d|b}} 1$$

$$= \sum_{d|r} \mu(d) \left[\frac{x}{d}\right] = x \sum_{d|r} \frac{\mu(d)}{d} + O(\tau(r)).$$

Hence ,

$$F(n,r) = n\left(\sum_{d|r}\frac{\mu(d)}{d}\right)\sum_{\substack{a\le\sqrt{n}\\(a,r)=1}}\frac{\mu(a)}{a^2} + O(n^{1/2}\tau(r))$$

$$= n\left(\sum_{d|r}\frac{\mu(d)}{d}\right)\sum_{\substack{a=1\\(a,r)=1}}^{\infty}\frac{\mu(a)}{a^2} + O(n^{1/2}\tau(r))$$

$$= n\prod_{p|r}\left(1-\frac{1}{p}\right)\prod_{p\nmid r}\left(1-\frac{1}{p^2}\right) + O(n^{1/2}\tau(r)).$$

But, by Lemma 3, $\prod(1-\frac{1}{p^2})$ converges. Therefore ,

$$\prod\left(1-\frac{1}{p^2}\right) = \prod\left(\frac{1}{1-\frac{1}{p^2}}\right)^{-1} = \prod\left(1+\frac{1}{p^2}+\frac{1}{p^4}+\dots\right) = \left(\sum\frac{1}{n^2}\right)^{-1} = \frac{6}{\pi^2},$$

so that

$$F(n,r) = \frac{6n}{\pi^2}\prod_{p|r}\left(1+\frac{1}{p}\right)^{-1} + O(n^{1/2}\tau(r)).$$

The last arithmetic function that we consider here is $\nu(n)$, where $\nu(n)$ is the number of distinct prime divisors of n . i.e.

$$\nu(n) = \sum_{p|n} 1, \qquad \nu(1) = 0.$$

We note that $\nu(n)$ is an additive, not a multiplicative, function:

$$\nu(nm) = \nu(n)+\nu(m), \qquad \text{if } (n,m) = 1.$$

We now prove one last lemma:

LEMMA 22.

$$\nu(n) \le \frac{\log n}{\log 2}.$$

Proof. Let the prime decomposition of n be

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \ge 2^{\alpha_1+\alpha_2 \cdots +\alpha_r} \ge 2^r.$$

Then

$$\nu(n) = r \le \frac{\log n}{\log 2}.$$

This now completes the list of the auxiliary results which it will be necessary to use in the following chapters.

CHAPTER II.

INTRODUCTION.

2.1. One of the oldest and best known of all the arithmetic functions is $\sigma(n)$, the sum of all the positive divisors of n. The earliest reference to it seems to have been by the Greeks, who were aware of its multiplicative property:

(2.1) $$\sigma(mn) = \sigma(m)\sigma(n), \quad \text{if } (m,n) = 1.$$

The proof of (2.1) is very simple. Suppose the prime decomposition of n is

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}.$$

A typical divisor of n then has the form

$$p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r},$$

where $0 \le \beta_i \le \alpha_i$. Therefore, the sum of all the divisors of n is given by

$$\sigma(n) = \sum_{\beta_1=0}^{\alpha_1} \sum_{\beta_2=0}^{\alpha_2} \cdots \sum_{\beta_r=0}^{\alpha_r} p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$$

$$= \Big(\sum_{\beta_1=0}^{\alpha_1} p_1^{\beta_1}\Big)\Big(\sum_{\beta_2=0}^{\alpha_2} p_2^{\beta_2}\Big) \cdots \Big(\sum_{\beta_r=0}^{\alpha_r} p_r^{\beta_r}\Big)$$

$$= \prod_{i=1}^{r} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1},$$

and (2.1) follows immediately.

Since n and 1 are divisors of n always ~~n divides itself~~, we clearly have

LEMMA 22.

$$\frac{\sigma(n)}{n} \geqslant 1, \qquad n \geqslant 1.$$

This raises the interesting question of when

$$(2.2) \qquad \frac{\sigma(n)}{n} \geqslant \lambda,$$

and $\lambda > 1$. Strangely enough, although this question seems simple, it has lead to problems of great difficulty, some of which remain unsolved to-day.

In this essay we attempt to answer this particular question, and some others closely related to it, by giving a collected account of all the work done in this field, mainly during the last thirty years. It is important to emphasise the inspiration which this work has derived from experiment, which takes the form of testing possible general theorems by numerical examples. Such experiment, though necessary in some form to progress in every part of mathematics, has played a greater part in ~~of~~ the development of the theory of numbers than elsewhere, for in other branches of mathematics the evidence found in this way is too often fragmentary and misleading.

2.2. We consider first the case when

$$(2.3) \qquad \frac{\sigma(n)}{n} = 2.$$

A number n for which (2.3) holds is said to be perfect. This name originated from the Greeks, who thought of a perfect number as one which equalled the sum of its proper divisors.

Since the days of antiquity perfect numbers have been essential elements in all numerological speculations. God created the world in 6 days, a perfect number. The moon circles the earth in 28 days, again a symbol of perfection.

Only one general class of perfect numbers is known. In the ninth book of Euclid's Elements we can find the following Theorem.

THEOREM 1. (Euclid).

If $2^{n+1}-1$ is prime, then $2^n(2^{n+1}-1)$ is perfect.

Proof. Write $2^{n+1}-1=p$, $m=2^n p$; then

$$\sigma(m) = (2^{n+1}-1)(p+1) = 2^{n+1}(2^{n+1}-1) = 2m,$$

so that m is perfect.

Theorem 1 shows that to every Mersenne prime M_p (i.e. one of the form 2^p-1, where p is prime), there corresponds a perfect number. It is not yet known whether there are infinitely many perfect numbers of this form, since it is not known whether there are infinitely many Mersenne primes, but in the eighteenth century Euler succeeded in proving that every even perfect number must be of Euclid's form.

THEOREM 2. (Euler).

Any even perfect number is of the form $2^n(2^{n+1}-1)$, where $2^{n+1}-1$ is prime.

Proof. We can write any even perfect number in the form $\overset{m = 2^n b}{\cancel{2^n(2^{n+1}-1)}}$, where $n > 0$ and b is odd. Hence,

$$\sigma(m) = \sigma(2^n)\sigma(b) = (2^{n+1}-1)\sigma(b).$$

Since m is perfect,

$$\sigma(m) = 2m = 2^{n+1} b,$$

so that

$$\frac{b}{\sigma(b)} = \frac{2^{n+1}-1}{2^{n+1}}.$$

The fraction on the right-hand side of this equation is in its lowest terms, and therefore

$$b = (2^{n+1}-1)a, \quad \sigma(b) = 2^{n+1}a,$$

where a is an integer.

Now if $a > 1$, b at least has the divisors $a, b, 1$, so that

$$\sigma(b) \geq a + b + 1 = 2^{n+1}a + 1 > \sigma(b),$$

an evident contradiction. Hence $a = 1$, and we have

$$m = 2^n(2^{n+1}-1),$$

and

$$\sigma(2^{n+1}-1) = 2^{n+1}.$$

But, if $2^{n+1}-1$ is not prime, it has divisors other than itself and 1, and

$$\sigma(2^{n+1}-1) > 2^{n+1}.$$

Hence, $2^{n+1}-1$ is prime, and the theorem is proved.

In Barlow's Number Theory (London, 1811) the author gives the perfect numbers up to the one corresponding to M_{31}, at the time the greatest prime known. This perfect number 'is the greatest that will ever be discovered, for, as they are merely curious without being useful, it is not likely that any person will attempt to find one beyond it'. The great efforts expended since that time in such ~~in such~~ computations show how easy it is to underestimate human curiosity! In 1876 Lucas[*1] found a method for testing whether or not a particular integer is a Mersenne prime, and used it to verify the primality of $2^{127}-1$. This remained the largest known prime until recently, when the electronic computers developed during the second world war became available for peaceful purposes. In 1956 the S.W.A.C. computer at Los Angeles determined all primes of the form 2^p-1 for $p<2304$, giving seventeen Mersenne primes and hence seventeen known perfect numbers. Since $2^{2304}-1$ is a number of 925 digits this gives some idea of the rarity of the even perfect numbers. The first five are

6, 28, 496, 8,128, 33,550,336.

The question of the existence odd perfect numbers is one of the celebrated unsolved problems in number theory. It seems probable that there are no odd perfect numbers, since extensive numerical calculations[*2] have failed to find one less than e^{52729} or with less than 2,800 different prime factors.

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*1 L. E. DICKSON. History of the Theory of Numbers. Vol. 1. Ch. XV11.

*2 MITSUI. Sci. Papers. Coll. Gen. Ed. Unif. Tokyo. 6 (1956) 1 - 11.

It has been possible to find various criteria which any odd perfect number must satisfy, and the two classic results in this direction came from Euler and Sylvester.* The latter showed that an odd perfect number must have at least five distinct prime factors.* We do not prove this result, since it is interesting rather than useful, but we shall prove Euler's result in our next theorem.

THEOREM 3. (Euler).

Every odd perfect number is of the form

$$m = q^{\alpha} k^2, \qquad q \equiv \alpha \equiv 1 \bmod 4.$$

Proof. Let

$$m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_r^{\alpha_r},$$

where $q_1, q_2, \ldots q_r$ are distinct (odd) primes. Since m is perfect

$$\sigma(m) = \sigma(q_1^{\alpha_1})\sigma(q_2^{\alpha_2}) \cdots \sigma(q_r^{\alpha_r}) = 2q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_r^{\alpha_r}.$$

Thus, one of the numbers $\sigma(q_1^{\alpha_1}), \ldots\ldots \sigma(q_r^{\alpha_r})$, say $\sigma(q_1^{\alpha_1})$, is the double of an odd number, and the remaining ones are odd. But

$$\sigma(q^{\alpha}) = 1 + q + \cdots + q^{\alpha}, \tag{2.4}$$

and each of the $\alpha + 1$ terms on the right-hand side of (2.4) is odd.

Hence, $\alpha_2, \alpha_3 \cdots \alpha_r$, are even, and α_1 is of the form $4r+1$ or $4r+3$. If the latter, there are $4(r+1)$ odd terms on

* SYLVESTER. Comptes Rendus Paris. 106 (1888) 448 - 450.

the right-hand side of (2.4) and $\sigma(q_1^{\alpha_1})$ is the double of an even number.

Similarly, if q_1 is of the form $4r+3$, then $4 \mid 1+q_1$.

But, since

$$\sigma(q_1^{\alpha_1}) = 1+q_1 + q_1^2(1+q_1) + \cdots\cdots + q_1^{\alpha-1}(1+q_1),$$

it follows that $4 \mid \sigma(q_1^{\alpha_1})$, and $\sigma(q_1^{\alpha_1})$ is the double of an even number.

These contradictions complete the proof of Theorem 3.

In recent times, the most notable work on the question of odd perfect numbers $m = q^\alpha k^2$ has been by Kanold, who in a series of papers *1, eliminates possible forms of k^2, by Touchard *2, who shows that m must take one of the forms

$$36r+1,\ 36r+9,\ 36r+13,\ 36r+25,$$

and by McCarthy *3, who proves that $m \not\equiv 2 \bmod 3$, and if $m \equiv 1 \bmod 3$, then $\alpha \not\equiv -1 \bmod 3$.

2.3. Returning again to the question (2.2) we shall more generally consider the case when

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*1 H. J. KANOLD. J. Reine. Agnew Math. 1941,1942,1944,1950,1953.

*2 J. TOUCHARD. Scripta Math. 19 (1953) 35 - 39.

*3 P. McCARTHY. Amer. Math. Monthly. 64 (1957) 257 - 258.

$$\frac{\sigma(n)}{n} = \lambda, \tag{2.5}$$

and λ is an integer. A number n for which (2.5) holds is said to be multi-perfect, and when the value of λ is particularly relevant is said to be of class λ. The reason for this nomenclature is apparent. Clearly the perfect numbers are a sub-class of all the multi-perfect numbers.

The problem of finding multi-perfect numbers appears to have been formulated first in 1631 by Mersenne in a letter to Descartes. The latter must have speculated considerably over the proposed problem, because seven years later he responded with a list of multi-perfect numbers together with various general methods for finding them. The first two of these were the multi-perfect numbers of Class 3, $120 = 2^3.3.5$ and $672 = 2^5.3.7$. Fermat and Frenicle also worked on the problem, and letters exchanged between them contain several other multi-perfect numbers.

More recently many more have been discovered, notably by Lucas, Lehmer, Cunningham, Carmichael and Mason. The standard list of all known multi-perfect numbers was published by Poulet *[1] in 1929, and revised in 1934. This contained 334 numbers, some of class as high as 7. New discoveries since that date were tabulated by Brown *[2] in 1954.

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*[1] P. POULET. La Chasse aux Nombres. 2 Vols - Brussels 1929 - 1934.

*[2] A. L. BROWN. Multi-perfect numbers. Scripta Math. 20 (1954) 103 - 106.

2.4. For numbers that are not perfect there are two possibilities :

$$(2.6) \qquad \frac{\sigma(n)}{n} > 2, \quad \text{or} \quad \frac{\sigma(n)}{n} < 2,$$

Numbers of the first kind are called abundant, and those of the second kind deficient. Generally speaking, we class the perfect numbers with the abundant numbers, so that an abundant number is one for which

$$(2.7) \qquad \frac{\sigma(n)}{n} \geq 2.$$

A number n satisfying our general equation (2.2) will be called λ- abundant.

The distinction between abundant and deficient numbers has always been considered important in numerology. For instance, Alcuin (735-804), the advisor and teacher of Charlemagne, observes that the entire human race descends from the 8 souls in Noah's Ark. Since 8 is a deficient number, he concludes that this second creation was imperfect in comparison with the first, which was based on the principle of the perfect number 6!

The first few abundant numbers, all found by experiment, are,

6, 12, 18, 20, 24, 28, 30, 36, ...;

there are only 23 of them up to 100, as the reader may easily verify, and all of them are even. The first odd abundant number is $945 = 3^3.5.7$.

The very earliest mention of these numbers appears to have been by Nicomachus in A.D. 100, who cited 12 and 24 as abundant, and 8 and 14 as deficient. In 1296 Jordanus Nemoranus proved:

THEOREM 4.

A prime or a power of a prime is deficient.

Proof. The proof is very simple.

$$\frac{\sigma(p^\alpha)}{p^\alpha} = \frac{p^{\alpha+1}-1}{p^\alpha(p-1)} = 1 + \frac{p^\alpha - 1}{p^\alpha(p-1)} < 1 + \frac{1}{p-1} \leq 2,$$

and our theorem is proved.

We can also prove (c.f. Theorem 1):

THEOREM 5.

$2^n(2^{n+1}-1)$ is abundant, for all $n \geq 1$.

Proof. Since, for all values of a ,

$$\sigma(a) \geq 1 + a,$$

it follows that

$$\sigma(2^{n+1}-1) \geq 2^{n+1}.$$

Hence,

$$\sigma(2^n(2^{n+1}-1)) = (2^{n+1}-1)\sigma(2^{n+1}-1) \geq 2^{n+1}(2^{n+1}-1).$$

Therefore,

$$\frac{\sigma(2^n(2^{n+1}-1))}{2^n(2^{n+1}-1)} \geq 2.$$

The following lemma has enabled us to prove some more simple Theorems about abundant numbers.

LEMMA 23.

If $d|n$, then

$$\frac{\sigma(d)}{d} \leq \frac{\sigma(n)}{n}.$$

Proof. Let the prime decomposition of n be

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}.$$

Since $\sigma(n)$ is multiplicative, we need only show that if $0 \leq \beta \leq \alpha$, then

$$\frac{\sigma(p^\beta)}{p^\beta} \leq \frac{\sigma(p^\alpha)}{p^\alpha},$$

i.e. $$1 + \frac{1}{p} + \cdots + \frac{1}{p^\beta} \leq 1 + \frac{1}{p} + \cdots + \frac{1}{p^\alpha}, \quad 0 \leq \beta \leq \alpha,$$

which is clearly true.

It follows immediately from Lemma 2.3. that

THEOREM 6.

(1) A multiple of an abundant number is also abundant.

(2) A divisor of a deficient number is also deficient.

Theorem 6 raises another interesting question. It is clearly possible to find a minimal sub-set S of the abundant numbers, so that the proper divisors of each element of S are deficient. The set of all the abundant numbers is then the set formed by all multiples of the elements of S.

L. E. Dickson* was the first to see the possibilities that this question evoked, and in 1913 he defined a primitive abundant number as one for which

$$\frac{\sigma(m)}{m} \geq 2,$$

but for every proper divisor d of m,

$$\frac{\sigma(d)}{d} < 2.$$

Dickson's work included a number of minor results. He proved that there is only a finite number of primitive abundant numbers having a given number of distinct odd prime factors, and a given number of factors 2. He showed also that there is no odd abundant number with fewer than 3 distinct prime factors, and gave a list of the numerous primitive odd abundant numbers with four distinct prime factors, as well as lists of even abundant numbers of certain kinds. In particular he determined all primitive abundant numbers less than 15,000.

Extending the idea of primitive abundant numbers, we shall discuss integers n for which (2.2) holds generally. We shall say that n is λ-abundant if

(2.8) $$\frac{\sigma(n)}{n} \geq \lambda > 1,$$

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* L. E. DICKSON. Amer. Jour, Math, 35 (1913) 413 - 426.

and n is primitive λ-abundant if (2.8) holds, but for every proper divisor d of n,

$$\frac{\sigma(d)}{d} < \lambda .$$

2.5. We have already introduced the notion of a perfect number, and explained that the name originated from the Greek terminology

$$\sigma_1(n) = n,$$

where $\sigma_1(n) = \sigma(n) - n$. It seems natural to wonder whether there exist pairs of numbers (a,b) such that

$$\sigma_1(a) = b, \quad \sigma_1(b) = a,$$

or

$$\sigma(a) = \sigma(b) = a+b. \tag{2.9}$$

Pairs of numbers which satisfy (2.8) are said to be amicable, and numbers of this type have been even more prominent in the lore of number mysticism than the perfect numbers, having symbolized mutual harmony, perfect friendship, and love. Their existence seems to have been discovered somewhat later than the perfect numbers, probably in the period of the flowering of the Neo-Platonic mystical school in Greek philosophy. One of the most influential of the Neo-Platonic philosophers, Iamblichus of Chalcis (about A.D. 320), ascribes the knowledge of amicable numbers to the earliest Pythagorean school, about 500 B.C.

In Arab mathematical writings, the amicable numbers occur repeatedly. They play a role in magic and astrology, in the casting of horoscopes, in sorcery, in the concoction of love potions, and in the making of talismans. As an illustration we quote from the Historical Prolegomenon of the Arab scholar Ibn Khaldun (1332 - 1406).

'Let us mention that the practice of the art of talismans has also made us recognise the marvellous virtues of amicable numbers. These numbers are 220 and 284...... One prepares a horoscope theme for each individual...... On each one of these themes one inscribes one of the numbers just indicated but giving the strongest number to the person whose friendship one wishes to gain, the beloved person. I don't know if, by the strongest number one wishes to designate the greatest one or the one which has the greatest number of aliquot parts. There results a bond so close between the two persons that they cannot be separated'.

Through the Arabs knowledge of amicable numbers spread to Western Europe. They are mentioned in the works of many prominent mathematical writers about A.D. 1500, for instance Chuquet, Stiefel, Cardanus and Tartaglia. However, there is no indication of any other pair of amicable numbers (besides the pair (220, 284) known to the Greeks and the Arabs) having been discovered before the work of Fermat. This is somewhat odd in that Fermat found his new pair through the rediscovery of a rule that actually had been formulated by the Arab mathematician Abu-l-Hasan Thabit ben Korrah as early as the ninth century:

if

$$p = 3.2^n - 1, \quad q = 3.2^{n-1} - 1, \quad \text{and} \quad r = 9.2^{n-1} - 1,$$

are all primes, then the numbers

$$a = 2^n pq, \quad b = 2^n r,$$

are an amicable pair. (It is easy to verify, with this formulation, that $\sigma(a) = \sigma(b) = a + b$). The case $n = 2$, gives

$$p = 11, \quad q = 5, \quad r = 17,$$

and we obtain the classical pair (220, 284). The next two pairs found by this rule come from the value $n = 4$, and $n = 7$, giving the amicable numbers

$$\begin{cases} 17,296 = 2^4.23.47. \\ 18,416 = 2^4.1151. \end{cases} \qquad \begin{cases} 9,363,584 = 2^7.191.383. \\ 9,437,056 = 2^7.7327. \end{cases}$$

Euler took up the search for amicable numbers in a systematic fashion, and developed several methods for finding them. In 1747 he gave a list of 30 pairs, which he later expanded to more than 60, all of them exceedingly large. However, it is interesting to see how the purely experimental side of number theory can defeat the ablest of mathematicians. More than a hundred years later, a sixteen year-old Italian boy, Nicolo Paganini, published the very small pair

$$1,184 = 2^5.37, \qquad 1210 = 2.5.11^2,$$

which had eluded all previous investigators. There is no evidence as to a method of discovery: they were probably found by trial and error.

A complete survey of the existing knowledge about amicable numbers was published in 1946 by E. B. Escott*. This paper contains a list of the 390 known pairs in factor form, together with a discussion on various methods of discovery.

2.6. Since, in this essay, we are to be concerned with numbers arising from the function $\frac{\sigma(n)}{n}$ (in the case of perfect and abundant numbers), and from the function $\sigma(n)$ (in the case of amicable numbers), it is interesting, at this stage, to consider the average value of these functions. We prove

THEOREM 7.

The average value of $\frac{\sigma(n)}{n}$ is $\frac{1}{6}\pi^2$.

Proof. It is easy to prove that ~~As an easy application of multiplication of Dirichlet series involving the Riemann-zeta function $\zeta(s)$ we have~~

$$\cancel{\sum_{m=1}^{\infty} \frac{\sigma(m)}{m^{s+1}} = \zeta(s)\zeta(s+1) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{m=1}^{\infty} \frac{1}{m^{s+1}}}$$

~~where $R(s) > 1$. Putting $s = 0$ into this equation, and restricting the summation to the integers $m \leq n$, gives~~

(2.10) $$\sum_{m \leq n} \frac{\sigma(m)}{m} = \sum_{m \leq n} \frac{1}{m}\left[\frac{n}{m}\right] = n \sum_{m \leq n} \frac{1}{m^2} + O\left(\sum_{m \leq n} \frac{1}{m}\right).$$

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* E. B. ESCOTT. Amicable numbers. Scripta Math 12 (1946) 61 - 72.

But

(2.11) $$\sum_{m\le n} \frac{1}{m^2} = \sum_{m=1}^{\infty} \frac{1}{m^2} - \sum_{m>n} \frac{1}{m^2} = \frac{\pi^2}{6} + O\left(\frac{1}{n}\right),$$

and by Lemma 1,

(2.12) $$\sum_{m\le n} \frac{1}{m} = O(\log n).$$

Therefore,

$$\sum_{m\le n} \frac{\sigma(m)}{m} = \frac{\pi^2}{6} n + O(\log n),$$

and our Theorem follows.

Since $\frac{1}{6}\pi^2 \doteq 1\frac{2}{3}$, this suggests that there are far more deficient numbers than abundant. This is certainly true of the integers less than 1000. Only 22 of these are abundant.

The equivalent Theorem for the function $\sigma(n)$ is very similar:

THEOREM 8.

$$\sum_{m\le n} \sigma(m) = \frac{1}{12}\pi^2 n^2 + O(n\log n).$$

Proof. $\sum_{m\le n} \sigma(m) = \sum x$ where the summation extends over all the lattice points in the region D of Lemma 17. Hence,

$$\sum_{m\le n} \sigma(m) = \sum_{y=1}^{n} \sum_{x\le \frac{n}{y}} x = \sum_{y=1}^{n} \frac{1}{2}\left[\frac{n}{y}\right]\left(\left[\frac{n}{y}\right]+1\right)$$

$$= \frac{1}{2}\sum_{y=1}^{n} \left(\frac{n}{y} + O(1)\right)\left(\frac{n}{y} + O(1)\right)$$

$$= \frac{1}{2}n^2 \sum_{y=1}^{n} \frac{1}{y^2} + O\left(n \sum_{y=1}^{n} \frac{1}{y}\right) + O(n).$$

Then, by (2.11) and (2.12),

$$\sum_{m\le n} \sigma(m) = \tfrac{1}{12}\pi^2 n^2 + O(n\log n).$$

In particular, since $\sum_{m=1}^{n} n \sim \frac{1}{2}n^2$, we have the average value of $\sigma(n)$ is $\frac{1}{6}\pi^2 n$.

The error terms in Theorems 7 and 8 are, of course, very far from being the best possible. We can re-arrange (2.10) and (2.11) so that

$$\sum_{m\le n} \frac{\sigma(m)}{m} = n\sum_{m\le n}\frac{1}{m^2} - \frac{1}{2}\sum_{m\le n}\frac{1}{m} - \rho(n)$$

$$= \tfrac{1}{6}\pi^2 n - \tfrac{1}{2}\log n - \rho(n) + O(1),$$

and

$$\sum_{m\le n} \sigma(m) = \tfrac{1}{12}\pi^2 n^2 - n\rho(n) + O(n),$$

where

$$\rho(n) = \sum_{m\le n} \frac{1}{m}\left(\frac{n}{m} - \left[\frac{n}{m}\right] - \frac{1}{2}\right).$$

Wigert *[1] showed as early as 1913, that

$$\rho(n) = O(\log n),$$ which follows, of course, from the above analysis.

Walfisz *[2] improved on this estimate of Wigert, by the use of Weyl's inequality for exponential sums, to

$$\rho(n) = O\left(\frac{\log n}{\log\log n}\right).$$

Further progress, however, became possible after Vinogradov's

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*[1] S. WIGERT. Sur quelques fonchons arithmetiques. ActaMathematica 37 (1913) 113 - 123.

*[2] A. WALFISZ. Tellerprobleme. Math Zeits. 26 (1927) 66 - 88.

remarkable improvements on Weyl's inequality, and Davenport *[1] and Walfisz *[2] proved independently that for any $\epsilon > 0$,

$$\rho(n) = O((\log n)^{4/5+\epsilon}).$$

By using an improved form of Vinogradov's inequalities for trigonometrical sums, a Chinese mathematician Pan Cheng Tung claims he can prove by Davenport's method, that $\frac{4}{5}+\epsilon$ may be replaced by $\frac{2}{3}+\epsilon$, and this is the best result available at present.

We have shown that the average value of the function $\frac{\sigma(n)}{n}$ is $\frac{1}{6}\pi^2$, but the maximum values of the function are considerably larger. We prove another result of Wigert.*[3]

THEOREM 9. (Wigert).

$$\overline{\lim_{n\to\infty}} \frac{\sigma(n)}{n \log\log n} = e^{\gamma},$$

where γ is Euler's constant.

Proof. Let the prime decomposition of n be

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$$

so that

$$(2.13) \qquad \frac{\sigma(n)}{n} = \prod_{i=1}^{r} \frac{p_i^{\alpha_i+1}-1}{p_i^{\alpha_i}(p_i-1)} < \prod_{i=1}^{r} \frac{1}{1-\frac{1}{p_i}}.$$

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*[1] H. DAVENPORT. A divisor problem. Quart.Journ. of Math. (Oxford) (1949) 37 - 44.

*[2] A. WALFISZ. Uber Gitterpunkte in mehrdimensionalen Ellipsoiden, achte Abhandlung. Travaux de l'Inst. Math de Tblissi. 5 (1938) 181 - 196.

*[3] S. WIGERT. loc. cit.

By Lemma 6, having chosen $\epsilon' > 0$ arbitrarily small, we have that

$$(2.14) \qquad \prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right) > \frac{(1-\epsilon')e^{-\gamma}}{\log\log n},$$

for all sufficiently large n. Therefore, given any $\epsilon > 0$, we choose

$$\epsilon' < \frac{\epsilon}{1+\epsilon},$$

and it then follows from (2.13) and (2.14) that, for all sufficiently large n,

$$\frac{\sigma(n)}{n} < (1+\epsilon)e^{\gamma}\log\log n.$$

We must now show that given any $\epsilon > 0$ the inequality

$$(2.15) \qquad \frac{\sigma(n)}{n} > (1-\epsilon)e^{\gamma}\log\log n$$

holds for infinitely many values of n.

Let N be a positive number, and $k > 1$ a positive integer to be fixed later. Put

$$(2.16) \qquad n = \prod_{p \le N} p^k,$$

so that

$$\frac{1}{k}\log n = \sum_{p \le N}\log p.$$

By Lemma 12, we have that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{p\le N}\log p = 1,$$

so that, if N is large enough,

$$(2.17) \qquad \frac{N}{2} < \frac{1}{k}\log n < 2N,$$

and

$$(2.18) \qquad \log N + \log\frac{k}{2} < \log\log n < \log N + \log 2k.$$

Now, by (2.16)

$$(2.19) \qquad \frac{\sigma(n)}{n} = \prod_{p\le N}\frac{1-\frac{1}{p^{k+1}}}{1-\frac{1}{p}}.$$

But, by Lemma 3, $\prod\left(1-\frac{1}{p^{k+1}}\right)$ converges. Therefore

$$\prod\left(1-\frac{1}{p^{k+1}}\right) = \prod\left(\frac{p^{k+1}}{1-p^{k+1}}\right)^{-1} = \left(\prod 1+\frac{1}{p^{k+1}}+\cdots\cdots\right)^{-1} = \frac{1}{\zeta(k+1)},$$

so that, given any $\epsilon > 0$, for all sufficiently large N we have

$$\prod_{p \le N} \left(1 - \frac{1}{p^{k+1}}\right) > \left(1 - \frac{\epsilon}{4}\right) \frac{1}{3(k+1)} .$$

Also, by Lemma 6, for all sufficiently large N,

$$\frac{1}{\log N} \prod_{p \le N} \left(1 - \frac{1}{p}\right)^{-1} > \left(1 - \frac{\epsilon}{4}\right) e^{\gamma}.$$

Therefore, by (2.19) and (2.18),

$$\frac{\sigma(n)}{n} > \left(1 - \frac{\epsilon}{4}\right)^2 \frac{e^{\gamma}}{3(k+1)} \log N \tag{2.20}$$

$$> \left(1 - \frac{\epsilon}{2}\right) \frac{e^{\gamma}}{3(k+1)} \left(\log\log n - \log 2k\right).$$

We now choose k large enough to ensure that

$$\frac{1 - \frac{\epsilon}{2}}{1 - \epsilon} > 3(k+1) > 1 .$$

Then, if N is sufficiently large, we have from (2.18)

$$\frac{\log 2k}{\log\log n} < \frac{\log 2k}{\log N + \log \frac{k}{2}} < 1 - \left(\frac{1-\epsilon}{1 - \frac{\epsilon}{2}}\right) 3(k+1),$$

which, after rearrangement, gives,

$$\left(1 - \frac{\epsilon}{2}\right)\left(\log\log n - \log 2k\right) > (1 - \epsilon)\, 3(k+1) \log\log n.$$

Hence, for all sufficiently large values of N, and hence for an infinite number of values of n, we have from (2.20),

$$\frac{\sigma(n)}{n} > (1 - \epsilon) e^{\gamma} \log\log n,$$

and our Theorem is proved.

2.7. Finally, since we are to be primarily concerned with numbers of the form $\frac{\sigma(n)}{n}$ it is interesting, at this stage, to consider the number of distinct values taken by the function $\frac{\sigma(m)}{m}$, with $1 \leq m \leq n$. We prove a result of Erdős *, whose work in ~~our particular~~ This field has been outstanding.

THEOREM 10. (Erdős).

The number of distinct numbers of the form

$$\frac{\sigma(m)}{m}, \qquad 1 \leq m \leq n,$$

equals $c_1 n + o(n)$, where $\frac{6}{\pi^2} < c_1 \leq 1$.

Proof. To prove this Theorem we shall require the results of two supplementary lemmas.

LEMMA 24.

$$\frac{\sigma(b_1)}{b_1} \neq \frac{\sigma(b_2)}{b_2},$$

if b_1, and b_2, are square-free integers.

Proof. The proof is very simple; we can clearly assume that $(b_1, b_2) = 1$. Let the prime decomposition of b_1 be

$$b_1 = p_1 p_2 \cdots p_r, \qquad p_1 < p_2 < \cdots < p_r,$$

so that

$$\sigma(b_1) = (1+p_1)(1+p_2)\cdots(1+p_r).$$

Then $p_r \nmid \sigma(b_1)$, since $p_r \nmid 1 + p_r$, and all the other factors

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* P. ERDŐS. Remarks on Number Theory 11. Acta Arithmetica V (1959) 171 - 177.

of $\sigma(b_1)$ are less than p_r. Hence the equation

$$b_2\,\sigma(b_1) = b_1\,\sigma(b_2), \qquad (b_1, b_2) = 1,$$

is impossible, since the right hand is a multiple of p_r and the left-hand side is not. This completes the proof of Lemma 24.

LEMMA 25.

Let a_1, and a_2, be two integers each of whose prime factors occurs with an exponent greater than 1, (i.e. whose square-free part is 1). Then there exists at most one pair of square-free integers b_1 and b_2 satisfying

$$(2.20)\qquad \begin{cases} \dfrac{\sigma(a_1 b_1)}{a_1 b_1} = \dfrac{\sigma(a_2 b_2)}{a_2 b_2}, \\[2ex] (a_1, b_1) = (a_2, b_2) = (b_1, b_2) = 1. \end{cases}$$

Proof. The proof is rather similar to the proof of Lemma 24. We suppose, on the contrary, that there is a second pair of square-free integers b_1' and b_2' satisfying (2.20). Then we should have, by (2.20), that

$$(2.21)\qquad \frac{\sigma(b_1)}{b_1}\cdot\frac{b_2}{\sigma(b_2)} = \frac{\sigma(b_1')}{b_1'}\cdot\frac{b_2'}{\sigma(b_2')},$$

and

$$(2.22)\qquad (b_1, b_2) = (b_1', b_2') = 1,$$

whereas we shall show that (2.21) and (2.22) have no solutions, (except if $b_1 = b_1'$, $b_2 = b_2'$, or $b_1 = b_2'$, $b_2 = b_1'$).

Assume that b_1, b_2, b_1', b_2', are solutions of (2.21) and (2.22) for which the product

(2.23) $$b_1 b_2 b_1' b_2'$$

is minimal. (This product is clearly greater than 1, since we are assuming that not all the b's equal 1). Let p_r be the greatest prime factor of the product (2.23), say

(2.24) $$p_r \mid b_1, \quad p_r \nmid b_2.$$

We can apply an argument similar to that used in Lemma 24 to show that $p_r \nmid \sigma(b_1)$. Similarly, $p_r \nmid \sigma(b_2')$, since the greatest prime factor of b_2' does not exceed p_r. Hence, by (2.21) and (2.22)

(2.25) $$p_r \mid b_1', \quad p_r \nmid b_2'.$$

But then, by (2.24) and (2.25)

$$\frac{b_1}{p}, \; b_2, \; \frac{b_1'}{p}, \; b_2'$$

also satisfy (2.21) and (2.22), which contradicts the minimality of the product (2.23). This completes the proof of Lemma 25.

We are now in a position to continue with the proof of Theorem 10.

Let $1 = a_1 < a_2 < \cdots\cdots$ be the infinite sequence of integers whose square-free part is 1, and denote by $m_1^{(i)} < m_2^{(i)} < \cdots\cdots$ the infinite sequence of integers whose quadratic part is a_i. Put

$$m_k^{(i)} = a_i b_k^{(i)},$$

where $b_k^{(i)}$ is square-free, and denote by S_i the set of all the numbers

(2.26) $$\frac{\sigma(m_k^{(i)})}{m_k^{(i)}}, \quad (k=1,2,\ldots\ldots)$$

Clearly, no two elements of the set S_i are the same, for if this were the case we should have

$$\frac{\sigma(b_k^{(i)})}{b_k^{(i)}} = \frac{\sigma(b_{k'}^{(i)})}{b_{k'}^{(i)}},$$

and by Lemma 24 this is impossible.

If an element of S_i is the same as an element of S_j, $j<i$, then ~~we should have~~

(2.27) $$\frac{\sigma(m_k^{(i)})}{m_k^{(i)}} = \frac{\sigma(m_{k'}^{(j)})}{m_{k'}^{(j)}},$$

for some $m_k^{(i)}$ and $m_{k'}^{(j)}$. Let

$$(b_k^{(i)},\ b_{k'}^{(j)}) = b,$$

so that

(2.28) $$m_k^{(i)} = a_i b d_k^{(i)}, \quad m_{k'}^{(j)} = a_j b d_{k'}^{(j)}, \quad (d_k^{(i)}, d_{k'}^{(j)}) = 1.$$

Then, if (2.27) holds, ~~we should have~~

(2.29) $$\frac{\sigma(a_i d_k^{(i)})}{a_i d_k^{(i)}} = \frac{\sigma(a_j d_{k'}^{(j)})}{a_j d_{k'}^{(j)}}, \qquad (d_k^{(i)}, d_{k'}^{(j)}) = 1.$$

and by Lemma 25 there is at most one pair of square-free integers $d_k^{(i)}$ and $d_{k'}^{(j)}$ such that (2.29) holds, with $j<i$. Therefore, there are at most $i-1$ of the integers $d_k^{(i)}$, for fixed i, and we denote these by

(2.30) $$d_{k_1}^{(i)}, d_{k_2}^{(i)}, \ldots\ldots d_{k_{i-1}}^{(i)},$$

and their product by

$$D_i = d_{k_1}^{(i)} \ldots\ldots d_{k_{i-1}}^{(i)}.$$

It follows that, if (2.27) holds for some $j<i$, then $m_k^{(i)}$ must be divisible by at least one of the integers (2.30) and

hence have a factor in common with D_i .

We now restrict attention to those integers $a_k^{(1)}$ which do not exceed n . We denote by N_i the number of elements in S_i which differ from all the elements of the classes S_j , $j < i$. Then N_i certainly does not exceed the total number of elements of S_i , and is at least as great as the number of integers $m_k^{(1)}$ not exceeding n which are co-prime with D_i . Therefore,

$$(2.31) \qquad \sum_{\substack{b \le \frac{n}{a_i} \\ (b,a_i)=1 \\ (b,D_i)=1}} |\mu(b)| \le N_i \le \sum_{\substack{b \le \frac{n}{a_i} \\ (b,a_i)=1}} |\mu(b)| .$$

By Lemma 21 and Lemma 18, the sum on the right-hand side of (2.31) equals

$$F\left(\frac{n}{a_i}, a_i\right) = \frac{6n}{\pi^2 a_i} \prod_{p|a_i} \left(1+\frac{1}{p}\right)^{-1} + O\left(\frac{n}{a_i}\right).$$

Similarly, the sum on the left-hand side equals

$$F\left(\frac{n}{a_i}, a_i D_i\right) = \frac{6n}{\pi^2 a_i} \prod_{p|a_i D_i} \left(1+\frac{1}{p}\right)^{-1} + O\left(\frac{n}{a_i}\right).$$

It follows that

$$(2.32) \qquad N_i = \alpha_i n + O\left(\frac{n}{a_i}\right),$$

where

$$(2.33) \qquad \frac{6n}{\pi^2 a_i} \prod_{p|a_i D_i} \left(1+\frac{1}{p}\right)^{-1} \le \alpha_i \le \frac{6n}{\pi^2 a_i} \prod_{p|a_i} \left(1+\frac{1}{p}\right)^{-1},$$

and to prove our Theorem we must evaluate the sums over all integers $i > 0$, of the expressions on the right and left-hand sides of (2.~~2~~3).

Now we have

$$(2.34) \qquad \sum_i \frac{1}{a_i} = \prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\right) = \prod_p \left(1 + \frac{1}{p(p-1)}\right) < \infty .$$

Also, by Lemma 21, the density of integers whose quadratic part is a_i equals

$$\lim_{n \to \infty} \frac{F\left(\frac{n}{a_i}, a_i\right)}{n} = \frac{6}{\pi^2 a_i} \prod_{p | a_i} \left(1 + \frac{1}{p}\right)^{-1}.$$

It follows from simple density considerations that

$$(2.35) \qquad \sum_{i=1}^{\infty} \frac{6}{\pi^2 a_i} \prod_{p | a_i} \left(1 + \frac{1}{p}\right)^{-1} = 1 .$$

In addition, since $a_1 = 1$, and $D_1 = 1$,

$$(2.36) \qquad \sum_{i=1}^{\infty} \frac{6}{\pi^2 a_i} \prod_{p | a_i D_i} \left(1 + \frac{1}{p}\right)^{-1} = \frac{6}{\pi^2} + \sum_{i=2}^{\infty} \frac{6}{\pi^2 a_i} \prod_{p | a_i D_i} \left(1 + \frac{1}{p}\right)^{-1}$$

$$> \frac{6}{\pi^2} .$$

It now follows from (2.32), (2.33), (2.34), (2.35) and (2.36), that

$$\sum_{i=1}^{\infty} N_i = c_1 n + o(n),$$

where $\frac{6}{\pi^2} < c_1 \leq 1$, and this completes the proof of our Theorem.

CHAPTER 111.

ABUNDANT NUMBERS.

3.1. We have already defined an abundant number n in 2.4 as one for which

$$(3.1) \qquad \frac{\sigma(n)}{n} \geq 2,$$

and a primitive abundant number n as one which satisfies (3.1) but is such that for every proper divisor d of n,

$$\frac{\sigma(d)}{d} < 2.$$

Over the last thirty years Erdős has succeeded by entirely elementary means in giving an astonishingly complete account of the distribution of these numbers. We shall prove below that there exist arbitrarily long runs of consecutive abundant numbers, we shall find the correct order of magnitude for the counting number of the sequence of primitive abundant number, and we shall show that the asymptotic density of the sequence of abundant numbers exists.

3.2. The most amazing part of Erdös's result that there exist arbitrarily long runs of consecutive abundant numbers *, is that at the time of publication of his paper there was no empirical evidence of any kind to suggest that such is the case.

* P. ERDÖS J. London Math. Soc. 10 (1935). 128 - 131.

Glaisher's number divisor tables for all the integers up to 10,000 were only published in 1940, but even these are not helpful in this respect, for there are only eleven odd abundant numbers amongst them, (the smallest being 945), and they do not lie close together. The only possible numerical hint that could provide any motivation for looking for such a result, comes from Salié *[1], who showed in 1955 that every integer greater than

$$33{,}426{,}748{,}355$$

is either an abundant number or the sum of two such numbers. We must remember, though, that Erdos proved his theorem some ~~thirty~~ twenty years earlier.

To prove this theorem we shall require the following two lemmas:

LEMMA 26

Let $m_1, m_2, \ldots m_r$ be r integers each of which is less than n. Then

$$v(m_1 m_2 \ldots m_r) < \pi(4r\log^2 n),$$

for all sufficiently large n.

Proof. We have already shown that $v(m)$ is an additive function, so that it follows *[2] ~~from Lemma 22~~ that

*[1] H. SALIÉ Math. Nachr. 14. (1955). 39 - 46.

*[2] The inequality $v(mn) \leq v(m) + v(n)$ is trivially true.

(3.2) $$\nu(m_1 m_2 \dots m_r) \le \frac{r \log n}{\log 2}.$$

But, by Tchebycheff's Theorem (Lemma 10),

(3.3) $$\pi(4r\log^2 n) > \frac{3r\log^2 n}{\log(4r\log^2 n)} > \frac{r\log n}{\log 2}, \quad (r<n),$$

providing n is sufficiently large. (3.2) and (3.3) complete the proof of Lemma 26.

LEMMA 27.

Denote by $m_1; m_2, \dots m_r$ the integers between $m-N+1$ and m which are not divisible by any prime less than or equal to a given prime q. Then

$$r > \frac{N}{2}\prod_{p<q}\left(1-\frac{1}{p}\right),$$

if N ~~is~~ and m are sufficiently large.

Proof. Let $\prod_{p<q} p = Q$. Then

$$r = \sum_{a=m-N+1}^{m} \sum_{\substack{d|a\\ d|Q}} \mu(d),$$

since, by Lemma 19, the inner sum equals 1 when $(a,Q)=1$, and is 0 otherwise. Reversing the order of summation we obtain

$$r = \sum_{d|Q}\mu(d)\left\{\left[\frac{m}{d}\right]-\left[\frac{m-N}{d}\right]\right\}$$

$$= N\sum_{d|Q}\frac{\mu(d)}{d} + O(\tau(Q))$$

$$> \frac{N}{2}\sum_{\substack{\cancel{p<q}\\ d|Q}}\frac{\mu(d)}{d},$$

$$= \frac{N}{2}\prod_{p<q}\left(1-\frac{1}{p}\right),$$

if N and m are sufficiently large compared with q .

We are now in a position to prove the following result.

THEOREM 11 (Erdős).

It is possible to find two positive constants c_1 and c_2 such that for all sufficiently large n there exist at least $c_1 \log\log\log n$, but not more than $c_2 \log\log\log n$, consecutive integers all abundant and less than n .

Proof. To prove the first part of this theorem it will ~~only be necessary~~ suffice to show that there is a sequence of N abundant numbers, $m_1, m_2, \ldots\ldots m_N$, say, which are relatively prime in pairs, and such that

(3.4) $$m_1 m_2 \ldots\ldots m_N < n .$$

For in this case the simultaneous congruences

$$m \equiv i - 1 \pmod{m_i}, \qquad (i = 1, 2, \ldots N),$$

have a solution which is unique $\bmod (m_1 m_2 \ldots m_N)$, i.e. a solution m with

$$0 < m < n .$$

Then, since any multiple of an abundant number is also abundant, it follows that

$$m, \; m-1, \ldots\ldots \; m - N + 1,$$

are N consecutive abundant numbers, each less than n .

We therefore construct our sequence m_i as follows. Let

$$m_1 = 2.3, \quad m_2 = 5.7.\ldots p_4, \quad m_3 = p_5 \ldots\ldots p_6, \ldots\ldots,$$

where m_1 is an abundant number, p_4 denotes the smallest prime such that m_2, the product of primes from 5 to p_4, is an abundant number, p_5 is the prime following p_4, and p_6 is the smallest prime so that m_3, the product of primes between p_5 and p_6 is an abundant number, and so on. It will follow that p_{2r} and p_{2r-1} are respectively the largest and smallest prime factors of m_r. (Clearly this construction is possible since by Lemma 3 $\prod_p (1+\frac{1}{p})$ diverges to $+\infty$, and therefore given any p_{2r-1}, it is possible to find a p_{2r} so that

$$\frac{\sigma(p_{2r-1}\cdots p_{2r})}{p_{2r-1}\cdots p_{2r}} = \prod_{p_{2r-1}\le p\le p_{2r}} (1+\tfrac{1}{p}) \ge 2).$$

We now define

$$A = \prod_{\substack{\cancel{p_{2r-1}\le p\le p_{2r}} \\ p<\frac{1}{2}\log n}} \cancel{\varepsilon}\, p,$$

and observe firstly that, by Lemma 9,

$$(3.5) \qquad A < 4^{\frac{1}{2}\log n} = 2^{\log n} < n,$$

and secondly that, by Lemma 7,

$$(3.6) \qquad \frac{\sigma(A)}{A} = \prod_{p<\frac{1}{2}\log n} (1+\tfrac{1}{p}) > c_3 \log\log n,$$

where c_3 is a suitable absolute constant. Now we need only define N by the pair of inequalities

$$m_1 m_2 \cdots m_N \le A < m_1 m_2 \cdots m_{N+1},$$

and then, by (3.5),

$$m_1 m_2 \cdots m_N \le A < n.$$

Hence, (3.4) is satisfied, and by the definition of m_i, the numbers $m_1, m_2, \ldots m_N$ are relatively prime in pairs, so that there exists a sequence of N consecutive integers, all abundant and less than n .

It remains to show that

$$N > c_1 \log\log\log n,$$

for some absolute constant c_1 .

It is clear that A is a proper divisor of $m_1 m_2 \dots m_{N+1}$, since both are products of consecutive primes starting with 2, and so we can write

$$m_1 m_2 \dots m_{N+1} = Aa,$$

where $a > 1$, and $(A, a) = 1$. Then, since all the m's are relatively prime in pairs we have, by (3.6), that

$$(3.7) \qquad \frac{\sigma(m_1)}{m_1} \cdots \frac{\sigma(m_{N+1})}{m_{N+1}} = \frac{\sigma(A)}{A} \cdot \frac{\sigma(a)}{a} > \frac{\sigma(A)}{A} > c_3 \log\log n.$$

Also, since $\left(\frac{m_i}{p_{2i}}, p_{2i}\right) = 1$,

$$\frac{\sigma(m_i)}{m_i} = \frac{\sigma(m_i/p_{2i})}{m_i/p_{2i}} \cdot \frac{\sigma(p_{2i})}{p_{2i}}.$$

But as $\frac{m_i}{p_{2i}}$ is deficient, by definition of p_{2i} , the first factor on the right-hand side is less than 2. Also,

$$\frac{\sigma(p_{2i})}{p_{2i}} = 1 + \frac{1}{p_{2i}} < \frac{3}{2}, \qquad \text{for all } p_{2i},$$

and it follows that

$$(3.8) \qquad \frac{\sigma(m_i)}{m_i} < 3, \qquad (i = 1, 2, \cdots N).$$

Hence, by (3.7) and (3.8),

$$3^{N+1} > c_3 \log\log n,$$

and we can therefore find an absolute constant c_1 such that

$$(3.9) \qquad N > c_1 \log\log\log n.$$

We have now proved that for sufficiently large n, there exist at least $c_1 \log\log\log n$ consecutive integers all abundant and less than n. We let N be the length of the longest possible run of consecutive abundant numbers less than ~~#~~ n (i.e. let N be maximal), and we shall show that there exists an absolute constant $c_2 > c_1$, such that

(3.10) $$N < c_2 \log\log\log n.$$

We proved our inequality (3.9) by showing, on one hand, that

$$\frac{\sigma(m_i)}{m_i} < 3, \qquad (i = 1, 2, \dots N),$$

and on the other, that

$$\prod_{i=1}^{N} \frac{\sigma(m_i)}{m_i} > c_3 \log\log n.$$

We shall prove (3.10) in a similar manner, i.e. by computing upper and lower bounds for the product

(3.11) $$\prod_{i=1}^{r} \frac{\sigma(m_i)}{m_i}$$

where, in this case $m_1, m_2, \dots m_r$ are a special sub-set of the maximal set of N consecutive abundant numbers $m, m-1, \dots m-N+1$, not exceeding n. (We note that this means that $r < N$).

Firstly, since the m's are abundant,

(3.12) $$\frac{\sigma(m_i)}{m_i} \geq 2.$$

Secondly,

$$\begin{aligned} \frac{\sigma(m_i)}{m_i} &= \prod_{p^\alpha \| m_i} \left(1 + \frac{1}{p} + \dots\dots + \frac{1}{p^\alpha}\right) \\ &\leq \prod_{p | m_i} \left(1 + \frac{1}{p} + \dots\dots\right) \\ &\leq \prod_{p | m_i} \left(1 + \frac{1}{p-1}\right). \end{aligned}$$

We now let the numbers m_i, $(i=1,2,\ldots r)$, be the integers between m and $m-N+1$ which are not divisible by any prime less than a certain fixed prime q, to be determined later. We denote by M the product

$$m_1 m_2 \cdots\cdots m_r .$$

Then,

$$(3.13) \qquad \prod_{i=1}^{r} \frac{\sigma(m_i)}{m_i} < \prod_{\substack{p>q \\ p \mid M}} \left(1+\frac{1}{p-1}\right)^{\left[\frac{N}{p}\right]+1},$$

since at most $\left[\frac{N}{p}\right]+1$ of the integers m_i are divisible by the prime p. To compute an upper bound for the product on the right-hand side of (3.13) we divide the primes $p>q$ into two classes.

In the first we place those primes p satisfying

$$q < p \leq N.$$

For these, we have

$$\left[\frac{N}{p}\right] + 1 \leq \frac{2N}{p},$$

and it follows that

$$\prod_{\substack{q<p\leq N \\ p \mid M}} \left(1+\frac{1}{p-1}\right)^{\left[\frac{N}{p}\right]+1} < \prod_{q<p\leq N} \left(1+\frac{1}{p-1}\right)^{\frac{2N}{p}}$$

$$< \prod_{q<p\leq N} \left(1+\frac{1}{p(p-1)}\right)^{2N}$$

$$< \prod_{q<p\leq N} \exp\left\{\frac{2N}{p(p-1)}\right\}.$$

But

$$\sum_{q<p\leq N} \frac{1}{p(p-1)} < \sum_{q<n\leq N} \frac{1}{n(n-1)} < \frac{1}{q},$$

and therefore,

$$(3.14) \qquad \prod_{q<p\leq N} \left(1+\frac{1}{p-1}\right)^{\left[\frac{N}{p}\right]+1} < e^{\frac{2N}{q}}.$$

In the second class we place all the primes $p > N$, so that

$$\left[\frac{N}{p}\right] = 0.$$

For these primes we have, from Lemma 26 and Lemma 7,

(3.15)
$$\prod_{\substack{p>N \\ p\mid M}} \left(1+\frac{1}{p-1}\right)^{\left[\frac{N}{p}\right]+1} = \prod_{\substack{p>N \\ p\mid M}} \left(1+\frac{1}{p-1}\right)$$

$$< \prod_{p<4r\log^2 n} \left(1+\frac{1}{p-1}\right)$$

$$< c_4(2\log\log n + \log r)$$

$$< c_4(2\log\log n + \log N).$$

Hence, by (3.13), (3.14) and (3.15),

$$\prod_{i=1}^{r} \frac{\sigma(m_i)}{m_i} < c_4 e^{\frac{2N}{q}}(2\log\log n + \log N),$$

so that, from (3.12),

(3.16)
$$2^r < c_4 e^{\frac{2N}{q}}(2\log\log n + \log N).$$

It is at this point that the reason for this particular method of proof becomes clear. Originally it may have seemed artificial to consider, instead of the entire run $m, m-1, \ldots m-N+1$ of consecutive abundant numbers, a particular sub-sequence $m_1, m_2, \ldots m_r$ of integers not divisible (in a precise sense) by small primes, but in (3.16) we see the justification of this device. If the entire run were taken we should have N in place of r on the left of (3.16) and 1 in place of q on the right. In these circumstances the above line of argument breaks down, for (3.16) is then trivially true and yields no new information.

On the other hand, by taking q to be the least prime such that

$$\prod_{p<q}\left(1-\frac{1}{p}\right) > \frac{8}{q},$$

we have, by Lemma 27, that

$$(3.17) \qquad r > \frac{4N}{q},$$

providing N is sufficiently large (and since $N > c_1 \log \log \log \log n$ we can choose n so large as to make this possible). By (3.17), we then have

$$2^r > e^{\frac{3N}{q}},$$

so that, from (3.16),

$$e^{\frac{N}{q}} < c_4(2\log\log n + \log N)$$

$$< 2c_4^2 \log\log n \,.\, \log N,$$

since $a+b < ab$, if $a > b > 2$. But for all sufficiently large n,

$$2c_4^2 \log N < e^{\frac{N}{2q}}.$$

It then follows that

$$e^{\frac{N}{2q}} < \log\log n.$$

Hence, finally, we can find a constant $c_2 > 0$ such that for all sufficiently large n,

$$N < c_2 \log\log\log n,$$

and this completes the proof of Theorem 11.

3.3. In this section we consider the behaviour of the counting numbers of the sequences of abundant and primitive abundant numbers as functions of n. We shall denote by $\left\{\begin{matrix} D(n) \\ A(n) \end{matrix}\right.$ the number of $\left\{\begin{matrix} \text{deficient} \\ \text{abundant} \end{matrix}\right.$ numbers not exceeding n, and by $N(n)$ the number of primitive abundant numbers not exceeding n. More generally, we shall write $A(n,\lambda)$ for the number of λ- abundant numbers not exceeding n.

One of the most interesting questions to ask at this point is whether there is a fairly constant proportion of abundant numbers amongst the first n integers where n is large, i.e. does $\lim_{n\to\infty} \frac{A(n)}{n}$ exist? If it does, what proportion of these integers can we expect to be abundant numbers?

We can prove very simply the following theorem:

THEOREM 12.

$$\frac{A(n)}{n} < \frac{2}{3},$$

if n is sufficiently large.

Proof. Since, for an abundant number m we have $\frac{\sigma(m)}{m} \geq 2$, and for a deficient number m we have $\frac{\sigma(m)}{m} > 1$, it follows that

$$\sum_{m\leq n} \frac{\sigma(m)}{m} \geq 2A(n) + D(n) = A(n) + n.$$

Therefore,

$$\frac{A(n)}{n} \leq \frac{1}{n} \sum_{m\leq n} \frac{\sigma(m)}{m} - 1.$$

But in Theorem 7 we showed that

$$\sum_{m\leq n} \frac{\sigma(m)}{m} = \frac{\pi^2}{6} n + O(\log n),$$

and therefore, for sufficiently large n,

$$\frac{A(n)}{n} \le \frac{\pi^2}{6} - 1 + \epsilon,$$

i.e. $$\frac{A(n)}{n} < \frac{2}{3}.$$

Since there are only 23 abundant numbers amongst the first 100 integers, it is very likely that ~~much more than~~ Theorem 12 is far from best possible. ~~true.~~ Certainly, if we consider solely the odd abundant numbers less than n, we can prove a result much better than would be expected from Theorem 12.

THEOREM 13. $$\frac{A'(n)}{n} < \frac{1}{8},$$

where $A'(n)$ denotes the number of odd abundant numbers not exceeding n.

Proof. By an argument similar to that used in Theorem 12, we can show that

(3.18) $$\frac{1}{n}\sum_{m\le n}\frac{\sigma(2m-1)}{2m-1} \ge \frac{2A'(2n)+D'(2n)}{n} = 1 + \frac{A'(2n)}{n}.$$

where $D'(n)$ denotes the number of odd deficient numbers not exceeding n.

But

$$\sum_{m\le n}\frac{\sigma(2m-1)}{2m-1} = \sum_{m\le n}\sum_{d\mid 2m-1}\frac{1}{d}$$

$$\le \sum_{(d,2)=1}\frac{1}{d}\left[\frac{n}{d}\right] \qquad \cancel{= \sum_{(d,2)=1}\frac{1}{d}\left(\frac{2n+d-1}{2d}\right)}$$

$$\le \sum_{(d,2)=1}\frac{n}{d^2} + O\Big(\sum_{d\le 2n-1}\frac{1}{d}\Big), \qquad \cancel{< n\sum_{\substack{d=1\\(d,2)=1}}^{2n-1}\frac{1}{d^2} + \frac{1}{2}\sum_{d=1}^{2n-1}\frac{1}{d}},$$

Since $\sum_{\substack{d=1\\(d,2)=1}}^{\infty} \frac{1}{d^2} = \frac{\pi^2}{8}$, and $\sum_{d=1}^{2n-1} \frac{1}{d} \sim \log 2n$, we therefore have

$$(3.19) \qquad \frac{1}{n} \sum_{m \le n} \frac{\sigma(2m-1)}{2m-1} < \frac{\pi^2}{8} + \epsilon < \frac{5}{4}.$$

Theorem 13 now follows from (3.18) and (3.19), if n is sufficiently large.

Theorems 12 and 13 together suggest that there are fewer odd than even abundant numbers. This is certainly borne out by the limited numerical evidence at our disposal: there is only one odd abundant number in the first 1,000 integers, and eleven in the first 10,000, compared with 253 and 2601 even abundant numbers. However, this evidence suggests that the $\frac{A(n)}{n}$ quotient lies approximately between 0.23 and 0.3.

In 1933 Behrend *1 proved, by an exceedingly difficult method, that in fact it lies between 0.241 and 0.314. A year later Davenport *2, Behrend and Chowla, each proved independently by the same method the existence of the limit

$$\lim_{n\to\infty} \frac{A(n,\lambda)}{n} = f(\lambda), \qquad \lambda > 1,$$

and further, that $f(\lambda)$ is a continuous function of λ. Salié *3, in a paper published much later, sharpened inequalities due to Behrend and proved that $f(\frac{3}{2}) > 0.569$, $f(2) > 0.246$, and $f(3) > 0.018$.

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*1 F. BEHREND. Berlin. Akad. Sitzungberichte (1933) 280-293.

*2 H. DAVENPORT. Ibid. (1934) 830 - 837.

*3 H. SALIÉ. Math. Nachr. 14 (1955) 39 - 46.

However, a year after Davenport had obtained his result, Erdös* proved that $\frac{A(n)}{n}$ tends to a limit as $n \to \infty$, by a method which differs entirely from that of Davenport, and which requires only elementary considerations. He makes an ingenious use of the following lemma:

LEMMA 28.

Let $m_1 < m_2 < \cdots$ be an infinite sequence of positive integers, and let $A_1(n)$ denote the number of integers not exceeding n which are divisible by at least one m_k. Then $\frac{A_1(n)}{n}$ tends to a limit as $n \to \infty$ providing only that $\sum \frac{1}{m_k}$ converges.

Proof. Let $A_1^{(k)}(n)$ denote the number of integers not exceeding n which are divisible by m_k but not by any of the integers $m_1, m_2, \ldots m_{k-1}$. Then

$$A_1^{(k)}(n) = \left[\frac{n}{m_k}\right] - \sum_{i<k}\left[\frac{n}{[m_i, m_k]}\right] + \sum_{i,j<k}\left[\frac{n}{[m_i, m_j, m_k]}\right] - \cdots$$

Since

$$[x] = x + O(1),$$

it follows that

$$\lim_{n\to\infty} \frac{A_1^{(k)}(n)}{n} = \frac{1}{m_k} - \sum_{i<k}\frac{1}{[m_i, m_k]} + \sum_{i,j<k}\frac{1}{[m_i, m_j, m_k]} - \cdots$$
$$= A_k \text{, say.}$$

Now, trivially we have that

(3.20) $$0 \leq \frac{A_1^{(k)}(n)}{n} \leq \frac{1}{n}\left[\frac{n}{m_k}\right] \leq \frac{1}{m_k},$$

Hence

(3.21) $$A_k < \frac{1}{m_k}.$$

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* P. ERDÖS. ~~Oxford Quart. Journ.~~ J. Lond. Math. Soc. (1934) 10. 278 - 282.

It follows from (3.20) that if $\sum \frac{1}{m_k}$ converges, then $\sum_{k=1}^{\infty} \frac{A_1^{(k)}(n)}{n}$ converges uniformly in n, and by (3.21), $\sum A_k$ converges. Hence,

$$\lim_{n\to\infty} \frac{A_1(n)}{n} = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{A_1^{(k)}(n)}{n} = \sum_{k=1}^{\infty} \lim_{n\to\infty} \frac{A_1^{(k)}(n)}{n} = \sum_{k=1}^{\infty} A_k < \infty,$$

i.e. $\lim_{n\to\infty} \frac{A_1(n)}{n}$ exists, and our proof is completed.

To show that $\frac{A(n)}{n}$ tends to a limit as $n \to \infty$ it will now only be necessary to show that the sum of the reciprocals of the primitive abundant numbers converges, for since we obtain all the abundant numbers by taking all multiples of the primitive abundant numbers, it will follow from Lemma 28 that $\lim_{n\to\infty} \frac{A(n)}{n}$ exists.

THEOREM 14. (Erdös).

$\frac{A(n)}{n}$ tends to a limit as $n \to \infty$.

Proof. Let $m_1 < m_2 < \cdots\cdots$ be the infinite sequence of primitive abundant numbers. (In future we shall write p.a.n. for primitive abundant number). We have already shown that to prove Theorem 14 we need only show that $\sum \frac{1}{m_k}$ converges.

Now if $N(n)$ denotes the number of p.a.n's not exceeding n we have

$$\sum_{M \le m_k \le N} \frac{1}{m_k} = \sum_{m=M}^{N} \frac{N(m) - N(m-1)}{m}$$

$$= \sum_{m=M}^{N} N(m)\left(\frac{1}{m} - \frac{1}{m+1}\right) + \frac{N(N)}{N+1} - \frac{N(M-1)}{M}$$

$$= \sum_{m=M}^{N} \frac{N(m)}{m(m+1)} + \frac{N(N)}{N+1} - \frac{N(M-1)}{M}.$$

Therefore to show that $\sum \frac{1}{m_k}$ converges we must show that

$$N(n) = \theta\left(\frac{n}{g(n)}\right),$$

where $g(n) \to \infty$ rapidly enough to ensure that $\sum \frac{1}{mg(m)} < \infty$. ~~as slowly as we please~~. It would then follow that

$$\sum_{M \leq m_k \leq N} \frac{1}{m_k} = \theta\left(\sum_{m=M}^{N} \frac{1}{mg(m)} + \frac{1}{g(N)} + \frac{1}{g(M)}\right),$$

and the proof of Theorem 14 would be complete.

We therefore prove the following theorem:

THEOREM 15 (Erdős).

$$N(n) = o\left(\frac{n}{(\log n)^5}\right).$$

Proof. Before counting the number of p.a.n's not exceeding n we shall eliminate the integers $m \leq n$ which do not satisfy any of the following conditions:

(1) If $p^\alpha | m$, and $\alpha > 1$, then $p^\alpha < (\log n)^{10}$,

(2) $\nu(m) < \rho$, where $\rho = 10 \log\log n$,

(3) The greatest prime factor of m is greater than $2n^{1/2\rho}$,

by showing that the number of these integers is $o\left(\frac{n}{(\log n)^5}\right)$.

We consider first those integers $m \leq n$ which do not satisfy (1). Clearly, for these integers the quadratic part r_m

is at least $(\log n)^{10}$ and by Lemma 16, the number of these integers is .

(3.22) $$O\left(\frac{n}{(\log n)^5}\right).$$

Next, if m is an integer not satisfying (2), then we have
$$\tau(m) \geq 2^\rho = (\log n)^{10\log 2} > (\log n)^{\frac{13}{2}}.$$
But, by Lemma 17,
$$\sum_{m\leq n} \tau(m) = O(n\log n),$$
and it follows immediately that the number of integers which do not satisfy (2) is

(3.23) $$o\left(\frac{n}{(\log n)^5}\right).$$

As regards the integers $m \leq n$ not satisfying (3) we may suppose also, from the above, that they satisfy (1) and (2). But, for sufficiently large n,
$$(\log n)^{10} < n^{1/2\rho}.$$
Hence,

(3.24) $$m < (2n^{1/2\rho})^\rho = 2^\rho . n^{1/2} = o\left(\frac{n}{(\log n)^\Delta}\right),$$

for any $\Delta > 0$.

It follows from (3.22), (3.23), and (3.24), that the number of integers $m \leq n$ which do not satisfy any of the conditions (1), (2), and (3), is $o\left(\frac{n}{(\log n)^5}\right)$.

To estimate $N(n)$ it is now sufficient to consider only those p.a.n's not exceeding n, which satisfy our three conditions. We denote these numbers by $m_1, m_2, \ldots m_N$.

Our aim now is to show that each m_i has an unrepeated prime factor p_i which lies between $(\log n)^{10}$ and $n^{1/4\rho}$, so that firstly

$$\frac{m_i}{p_i} < \frac{n}{(\log n)^{10}}, \qquad (i = 1, 2, \ldots N),$$

and secondly, the integers $\frac{m_i}{p_i}$ are distinct. This clearly will suffice to prove our Theorem.

By (1) any prime divisor of m_i, which is greater than $(\log n)^{10}$ must necessarily be an unrepeated prime-factor, and so we need only show that m_i has some prime divisor between these limits. We assume, on the contrary, that

$$m_i = u_i v_i, \qquad (i = 1, 2, \ldots N),$$

where all the prime factors of u_i are less than $(\log n)^{10}$, and all the prime factors of v_i are greater than $n^{1/4\rho}$; by (3), $v_i > 1$. For simplicity of notation, we now drop the suffixes.

Since m is a p.a.n. and $v > 1$, u is deficient. Therefore, since $\sigma(u)$ and $2u$ are integers.

$$\sigma(u) \leq 2u - 1$$

Hence, by (1) and (2),

$$(3.25) \qquad \frac{\sigma(u)}{u} \leq 2 - \frac{1}{u} < 2 - \frac{1}{(\log n)^{10\rho}}.$$

Also, by (2) and (3), for sufficiently large n *,

(3.26) $$\frac{\sigma(v)}{v} = \prod_{p \mid v}\left(1+\frac{1}{p}\right) < \left(1+\frac{1}{n^{1/4\rho}}\right)^{\rho} < 1+\frac{2\rho}{n^{1/4\rho}}.$$

But $(u,v)=1$, and it therefore follows from (3.25) and (3.26) that

$$\frac{\sigma(m)}{m} = \frac{\sigma(u)}{u}\cdot\frac{\sigma(v)}{v} < \left(2-\frac{1}{(\log n)^{10\rho}}\right)\left(1+\frac{2\rho}{n^{1/4\rho}}\right) < 2,$$

for sufficiently large n, which contradicts the hypothesis that m is abundant. Hence, each m_i has an unrepeated prime divisor p_i where

(3.26) $$(\log n)^{10} < p_i < n^{1/4\rho},$$

so that

(3.27) $$\frac{m_i}{p_i} < \frac{n}{(\log n)^{10}}.$$

To prove our theorem it remains to show that the integers $\frac{m_i}{p_i}$ are distinct. We suppose, on the contrary, that

$$\frac{m_\nu}{p_\nu} = \frac{m_\mu}{p_\mu},$$

for some $\mu \neq \nu$. We observe that, since $\left(\frac{m_\nu}{p_\nu}, p_\nu\right)=1$,

$$\frac{\sigma(m_\nu)}{m_\nu} = \frac{\sigma(m_\nu/p_\nu)}{m_\nu/p_\nu}\cdot\frac{1+p_\nu}{p_\nu},$$

and similarly with μ for ν. Therefore, on division,

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* We use here the inequality $(1+a)^b < 1+2ab$, if $0 < ab < \frac{1}{2}$, $\begin{cases} a>0. \\ b>1. \end{cases}$
For $(1+a)^b < 1+ba+\frac{b(b-1)}{1\cdot 2}a^2\ldots < 1+ba+b^2a^2+\ldots\ldots < 1+\frac{ba}{1-ab} < 1+2ab.$

(3.28) $$\frac{\sigma(m_\nu)}{m_\nu}\cdot\frac{m_\mu}{\sigma(m_\mu)} = \frac{p_\mu(1+p_\nu)}{p_\nu(1+p_\mu)}.$$

But since $m_\nu \neq m_\mu$, $p_\nu \neq p_\mu$, and we may therefore suppose, without loss of generality, that the right-hand side of (3.28) is greater than 1. Then by (3.26),

(3.29) $$\frac{\sigma(m_\nu)}{m_\nu}\cdot\frac{m_\mu}{\sigma(m_\mu)} \geqslant 1+\frac{1}{p_\nu(1+p_\mu)} \geqslant 1+\frac{1}{2n^{1/2\rho}}.$$

Now let p_ν' be the greatest prime divisor of m_ν. By (1) and (3), $p_\nu'^2 \nmid m_\nu$, and $p_\nu' > 2n^{1/2\rho}$. Hence, by (3), since $\frac{m_\nu}{p_\nu'}$ is deficient,

$$\frac{\sigma(m_\nu)}{m_\nu} = \frac{\sigma(m_\nu/p_\nu')}{m_\nu/p_\nu'}\left(1+\frac{1}{p_\nu'}\right) < 2\left(1+\frac{1}{2n^{1/2\rho}}\right).$$

But $\frac{\sigma(m_\mu)}{m_\mu} \geqslant 2$, and it follows that

$$\frac{\sigma(m_\nu)}{m_\nu}\cdot\frac{m_\mu}{\sigma(m_\mu)} < 1+\frac{1}{2n^{1/2\rho}},$$

in contradiction of (3.29).

We have now shown that the number of p.a.n's is ~~less than~~ $O\left(\frac{n}{(\log n)^{10}}\right)$, and this finally establishes our theorem.

The proof of Theorem 15 now completes the proof of Theorem 14.

3.4. Theorem 15 is in fact very far from being the best possible. The weakness of the method ~~result~~ lies in the stringency of the conditions (1), (2) and (3). Because the number of integers which do not satisfy them is comparatively large, the accuracy of our result is limited from the outset.

However, having suitably weakened conditions (1), (2) and (3), and eliminated the integers not satisfying them, it will be correspondingly harder to show that the number of p.a.n's among the remaining integers (which do satisfy the three conditions) is sufficiently small.

We recall that in estimating the number of the latter, we showed that each m_i has an unrepeated prime divisor p_i which is large enough to ensure that each $\frac{m_i}{p_i} < \frac{n}{(\log n)^{10}}$, and small enough to ensure that the integers $\frac{m_i}{p_i}$ are distinct. To improve on our result we would clearly have to show that each m_i has an unrepeated prime divisor lying in a corresponding interval for which the lower bound, at least, is larger.

In showing the integers $\frac{m_i}{p_i}$ to be distinct we used firstly that $(\frac{m_i}{p_i}, p_i) = 1$, and secondly that

$$\frac{\sigma(p_\nu)}{p_\nu} \neq \frac{\sigma(p_\mu)}{p_\mu}, \qquad p_\mu \neq p_\nu,$$

But in Lemma 24 we saw that

$$\frac{\sigma(a)}{a} \neq \frac{\sigma(b)}{b},$$

whenever a and b are square-free. Therefore the method of Theorem 15 can remain substantially unchanged if we can show that the square-free part of each m_i has some divisor d_i (nor necessarily prime) which is neither too large nor too small. Clearly, this will be more probable if the implied lower limit on d_i is larger than th~~ose~~at for the prime divisor p_i. (For example, the square-free part of each m_i

may have two prime divisors $p_1^{(1)}$, $p_2^{(1)}$, say, both too small to lie in our new interval, whereas the product $p_1^{(1)} p_2^{(1)}$ is large enough).

Working along these lines Erdős * found an improved upper estimate for $N(n)$, and then went on to find a lower estimate. His combined work gives us an amazing result: the correct order of magnitude for the function $N(n)$. He proves

$$\frac{n}{\exp(25(\log n \log\log n)^{1/2})} < N(n) < \frac{n}{\exp(\frac{1}{8}(\log n \log\log n)^{1/2})},$$

By better choice of the constants involved, and improved calculations, we apply Erdős's method to prove our next Theorem.

THEOREM 16.

$$\frac{n}{\exp(\frac{13}{2}(\log n \log\log n)^{1/2}} < N(n) < \frac{n}{\exp(\frac{1}{5}(\log n \log\log n)^{1/2}},$$

for all sufficiently large n .

Proof. For brevity we shall write

(3.30) $$x = (\log n \log\log n)^{1/2}, \qquad y = \left(\frac{\log n}{\log\log n}\right)^{1/2},$$

so that we have to prove that

(3.31) $$n e^{-\frac{13}{2}x} < N(n) < n e^{-\frac{1}{5}x}.$$

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* P. ERDOS. J. Lond. Math. Soc. 9. (1935) 49 - 58.

We shall note the following direct consequences of definitions (3.31) which will be in constant use below:

$$xy = \log n, \qquad \frac{x}{y} = \log\log n, \tag{3.32}$$

and

$$\left(\tfrac{1}{2} - c_3\right)x < y\log y < \tfrac{1}{2}x, \tag{3.33}$$

for all sufficiently large n, and any fixed positive constant $c_3 < \frac{1}{2}$.

We began the proof of Theorem 15 by eliminating from further consideration all integers $m \leq n$ having quadratic part r_m at least $(\log n)^{10}$, (condition (1)), and possessing many small prime factors, (conditions (2) and (3).) In the same way we now let S denote the set of all integers $m \leq n$ which do not satisfy

(1) $r_m < e^{c_1 x}$, where $c_1 = \frac{20}{49}$,

(2) The greatest prime factor of m is greater than $2e^{c_2 x}$, where $c_2 = \frac{1}{c_1} - c_3$.

We shall prove that the number N_1, of elements of S satisfies

$$N_1 < \tfrac{1}{2} n e^{-\frac{1}{5}x}. \tag{3.34}$$

By Lemma 16, the number of integers $m \leq n$ which do not satisfy (1) is less than

$$cne^{-\frac{1}{2}c_1 x}. \tag{3.35}$$

The integers $m \leq n$ not satisfying (2) may be divided into two classes.

In the first class we place those integers $m \leq n$ for which $\nu(m) \leq z$, where

$$z = [c_1 y] + 1.$$

We may suppose also, from the above, that their quadratic part is less than $e^{c_1 x}$. Hence, since

$$(3.36) \qquad c_1 y < z \leq c_1 y + 1 < c_1 x + 1,$$

the number N' of such integers is, by (3.32), at most

$$(2e^{c_2 x})^{c_1 y + 1} e^{c_1 x} = 2^{c_1 y + 1} e^{(1 - c_3 c_1) xy + (c_1 + c_2) x}$$
$$< e^{(1 - c_3 c_1) xy + (c_1 + c_2) x + c_1 y + 1}.$$

But $y < x$, and it therefore follows that

$$(3.37) \qquad N' < e^{xy - \frac{1}{2} c_1 x} = n e^{-\frac{1}{2} c_1 x},$$

if n is sufficiently large.

For the integers $m \leq n$ of the second class, $\nu(m) > z$. We shall denote the number of these integers by N'' . Clearly every integer counted in N'' is divisible by an integer a such that $\nu(a) = z$. Let $a_1, a_2, \dots a_r$ be all the integers $m \leq n$ which have exactly z different prime factors. Then since the integers of the second class are multiples of the a's we have

$$N'' \leq \frac{n}{a_1} + \frac{n}{a_2} + \dots + \frac{n}{a_r}$$

$$\leq \frac{n}{z!} \Big(\sum_{\substack{p, \alpha \\ p^\alpha \leq n}} \frac{1}{p^\alpha} \Big)^z .$$

But by Lemma 8,

$$\sum_{\substack{p,\alpha \\ p^\alpha \le n}} \frac{1}{p^\alpha} \le 2\log\log n.$$

Therefore, since $e^z > \frac{z^z}{z!}$,

$$N'' \le \frac{n(2\log\log n)^z}{z!} < n\left(\frac{2e\log\log n}{z}\right)^z,$$

Now it is easily verified that for fixed b the function $\left(\frac{b}{z}\right)^z$ is uniformly decreasing, provided that $z > \frac{b}{e}$. Hence, for sufficiently large n, we have, from (3.36), that

$$N'' \le n\left(\frac{2e\log\log n}{c_1y}\right)^{c_1y} < n\left(\frac{e^3\log\log n}{y}\right)^{c_1y}$$

$$= n\exp\{3c_1y + c_1y\log\log\log\log n - c_1y\log y\},$$

But by (3.33), $y\log y > (\frac{1}{2} - c_3)x$, so that from (3.30) we have

(3.38) $$N'' < ne^{-(\frac{1}{2}c_1 - c_3)x}, \qquad (c_1 < 1).$$

Now $c_1 = \frac{20}{49}$, and we may choose c_3 small enough to ensure that

(3.39) $$\tfrac{1}{2}c_1 - c_3 > \tfrac{1}{5}.$$

It will then follow from (3.35), (3.37), and (3.38) that for all sufficiently large n the number N_1, of integers $m \le n$ which do not satisfy (1) and (2), is, at most

$$\tfrac{1}{2}ne^{-\frac{1}{5}x}.$$

To prove the right-hand inequality in (3.31), it now suffices to show that the number of p.a.n's not exceeding n which satisfy (1) and (2) is at most $\frac{1}{2}ne^{-\frac{1}{5}x}$. We denote these p.a.n's by $m_1, m_2, \ldots m_N$.

We have already discussed the method we shall now use: we shall show that the square-free part of each p.a.n. m_i satisfying (1) and (2) has a divisor d_i satisfying

(3.40) $$e^{(\frac{1}{4}c_2 - c_1)x} < d_i < e^{\frac{1}{2}c_2 x},$$

so that for sufficiently large n ,

(3.41) $$\frac{m_i}{d_i} < ne^{-(\frac{1}{4}c_2-c_1)x}.$$

We shall then conclude by showing that the integers $\frac{m_i}{d_i}$ are distinct.

But since

$$\tfrac{1}{4}c_2 - c_1 = \frac{1}{4c_1} - c_1 - \tfrac{1}{4}c_3 > \tfrac{1}{2}c_1 - \tfrac{1}{4}c_3,$$

it will follow from (3.39), and (3.41) that for all sufficiently large n ,

(3.42) $$N < \tfrac{1}{2}ne^{-\frac{1}{5}x},$$

and the first part of our Theorem will be complete.

We now prove (3.40). If any one of the prime factors of m_i lies between $e^{\frac{1}{4}c_2 x}$ and $e^{\frac{1}{2}c_2 x}$ it occurs to the first power only by virtue of (1), and there is nothing to prove. Hence, let

(3.43) $$m_i = u_i v_i,$$

where u_i has only prime factors less than $e^{\frac{1}{4}c_2 x}$, and

v_i has only prime factors greater than $e^{\frac{1}{2}c_2x}$.

Since, from (1), the quadratic part of m_i is less than e^{c_1x}, to prove (3.40) it suffices to show that each m_i has a divisor between $e^{\frac{1}{2}c_2x}$ and $e^{\frac{1}{4}c_2x}$.

To simplify the notation we temporarily drop the suffixes in (3.43), and let the prime decomposition of u be

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\omega^{\alpha_\omega}.$$

If $\alpha_r = 1$, $p_r < e^{\frac{1}{4}c_2x}$, since each prime factor of u is less than $e^{\frac{1}{4}c_2x}$, and if $\alpha_r \geq 2$, we have $p_r^{\alpha_r} < e^{\frac{1}{4}c_2x}$, since the quadratic part of m is less than e^{c_1x}. Therefore

$$p_r^{\alpha_r} < e^{\frac{1}{4}c_2x}, \qquad (r = 1, 2, \ldots \omega). \tag{3.44}$$

We now consider the numbers

$$p_1^{\alpha_1},\ p_1^{\alpha_1}p_2^{\alpha_2},\ \ldots\ldots,\ p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\omega^{\alpha_\omega}.$$

if we can show that

$$u = p_1^{\alpha_1} \cdots p_\omega^{\alpha_\omega} > e^{\frac{1}{4}c_2x},$$

then there must exist a λ so that

$$p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\lambda^{\alpha_\lambda} \leq e^{\frac{1}{4}c_2x} < p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{\lambda+1}^{\alpha_{\lambda+1}};$$

since, by (3.44)

$$p_{\lambda+1}^{\alpha_{\lambda+1}} < e^{\frac{1}{4}c_2x},$$

It follows that

$$e^{\frac{1}{4}c_2x} < p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{\lambda+1}^{\alpha_{\lambda+1}} < e^{\frac{1}{2}c_2x},$$

and m has a divisor in the required interval.

We shall now prove that

(3.45) $$u > e^{\frac{1}{4}c_2 x}.$$

We assume, on the contrary, that

$$u \le e^{\frac{1}{4}c_2 x}.$$

Then, since u is deficient, (c.f. (3.25)).

(3.46) $$\frac{\sigma(u)}{u} \le 2 - \frac{1}{u} \le 2 - \frac{1}{e^{\frac{1}{4}c_2 x}}.$$

By (1), v is square-free, so that

$$\frac{\sigma(v)}{v} = \prod_{p|v}\left(1+\frac{1}{p}\right).$$

But, by Lemma 22, every number less than n has at most $\frac{\log n}{\log 2} < 2\log n$ prime factors. Hence, (c.f. footnote for (3.26))

(3.47) $$\frac{\sigma(v)}{v} < \left(1+\frac{1}{e^{\frac{1}{2}c_2 x}}\right)^{2\log n} < 1 + \frac{4\log n}{e^{\frac{1}{2}c_2 x}}.$$

But $(u,v) = 1$. Therefore, by (3.46) and (3.47),

$$\frac{\sigma(m)}{m} = \frac{\sigma(u)}{u}\cdot\frac{\sigma(v)}{v} < \left(2-\frac{1}{e^{\frac{1}{4}c_2 x}}\right)\left(1+\frac{4\log n}{e^{\frac{1}{2}c_2 x}}\right) < 2,$$

for sufficiently large n , and this contradicts the hypothesis that m is abundant. Hence, (3.45) holds, and this completes the proof of (3.40), and, in particular, the proof of (3.41).

We show finally that the integers $\frac{m_i}{d_i}$ are distinct, so that the number of p.a.n's m_i is less than $\frac{1}{2}ne^{-\frac{1}{5}x}$.

We suppose, on the contrary, that

$$\frac{m_\nu}{d_\nu} = \frac{m_\mu}{d_\mu},$$

for some $\mu \neq \nu$. Since $(\frac{m_i}{d_i}, d_i) = 1$,

$$\frac{\sigma(m_\nu)}{m_\nu} = \frac{\sigma(m_\nu/d_\nu)}{m_\nu/d_\nu} \cdot \frac{\sigma(d_\nu)}{d_\nu},$$

and similarly with μ for ν . Therefore, on division,

$$(3.48) \qquad \frac{\sigma(m_\nu)}{m_\nu} \cdot \frac{m_\mu}{\sigma(m_\mu)} = \frac{\sigma(d_\nu)}{d_\nu} \cdot \frac{d_\mu}{\sigma(d_\mu)} .$$

But $m_\nu \neq m_\mu$, so that $d_\nu \neq d_\mu$. Therefore, by Lemma 24, since d_μ and d_ν are square-free, the right-hand side of (3.48) cannot equal 1. We may therefore suppose, without loss of generality, that it is greater than 1.

i.e. $\qquad \sigma(d_\nu) d_\mu > \sigma(d_\mu) . d_\nu$.

But both sides of this inequality are integers, so we must have

$$\sigma(d_\nu) d_\mu \geq \sigma(d_\mu) d_\nu + 1 .$$

But since d_μ is deficient, it follows from (3.48) and (3.40) that

$$(3.49) \qquad \frac{\sigma(m_\nu)}{m_\nu} \cdot \frac{m_\mu}{\sigma(m_\mu)} \geq 1 + \frac{1}{d_\nu \sigma(d_\mu)} > 1 + \frac{1}{2 d_\nu d_\mu} \geq 1 + \frac{1}{2e^{c_2 x}} .$$

Now let p_ν' be the greatest prime factor of m_ν . By (1) and (2), $p_\nu' > 2e^{c_2 x}$ and $p_\nu'^2 \nmid m_\nu$. Since $\frac{m_\nu}{p_\nu'}$ is deficient it follows that

$$\frac{\sigma(m_\nu)}{m_\nu} = \frac{\sigma(m_\nu/p_\nu')}{m_\nu/p_\nu'} \left(1 + \frac{1}{p_\nu'}\right) < 2\left(1 + \frac{1}{2e^{c_2 x}}\right).$$

Hence, since m_μ is abundant,

$$\frac{\sigma(m_\nu)}{m_\nu} \cdot \frac{m_\mu}{\sigma(m_\mu)} < 1 + \frac{1}{2e^{c_2 x}},$$

in contradiction of (3.49).

Thus, we have shown that the integers $\frac{m_i}{d_i} < \frac{1}{2}ne^{-\frac{1}{5}x}$, $(i=1,2,\dots N)$, are all different, and hence, for sufficiently large n,

(3.50) $$N < \tfrac{1}{2}ne^{-\frac{1}{5}x}.$$

Relations (3.34) and (3.50) together prove that, for all sufficiently large n,

$$N(n) < ne^{-\frac{1}{5}x},$$

and this completes the proof of the first part of Theorem 16.*

To find a lower bound for $N(n)$ we construct a particular sequence of p.a.n's, each less than n, containing at least $ne^{-\frac{13}{2}x}$ members.

Let $k>1$ and ℓ be natural numbers to be fixed later, and let $p_1, p_2, \dots p_k$ be any k primes satisfying

(3.51) $$(k-1)2^{\ell+1} < p_k < p_{k-1} < \dots < p_1 < k2^{\ell+1}.$$

We shall prove that the number

(3.52) $$2^{\ell} p_1 p_2 \cdots p_k$$

is primitive abundant.

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* It is interesting to consider why the constant c_1 was chosen originally as $\frac{20}{49}$. By this method (see (3.35), (3.37), (3.38), (3.41)) the best result we can obtain is $N(n) < ne^{-c_4 x}$, where $c_4 = \min\{\frac{1}{2}c_1, \frac{1}{2}c_1 - c_3, \frac{1}{4}c_2 - c_1\}$, and $c_2 = \frac{1}{c_1} - c_3$. We therefore choose $c_3 > 0$, as small as possible, and $\frac{1}{2}c_1 = \frac{1}{4c_1} - c_1$, i.e. $c_1 = \frac{1}{\sqrt{6}} \doteq \frac{20}{49}$. It follows that $c_4 > \frac{1}{5}$.

First we show that a number of type (3.52) is abundant.

We observe that

$$p_r \leq k\,2^{e+1} - (2r-1), \quad (r = 1, 2, \ldots k),$$

and therefore

$$p_1 + p_2 + \ldots + p_k \leq k^2 . 2^{e+1} - \sum_{r=1}^{k} (2r-1) = k^2(2^{e+1}-1).$$

By a double application of the A.P. - G.P. inequality,

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} \geq k^2 (p_1 + p_2 + \cdots p_k)^{-1} \geq \frac{1}{2^{e+1}-1},$$

and it follows that

$$\begin{aligned} \frac{\sigma(2^e p_1 p_2 \cdots p_k)}{2^e p_1 p_2 \cdots p_k} &= \left(\frac{2^{e+1}-1}{2^e}\right)\left(1 + \frac{1}{p_1}\right) \cdots \left(1 + \frac{1}{p_k}\right) \\ &\geq \left(\frac{2^{e+1}-1}{2^e}\right)\left(1 + \frac{1}{p_1} + \cdots + \frac{1}{p_k}\right) \\ &\geq \left(\frac{2^{e+1}-1}{2^e}\right)\left(1 + \frac{1}{2^{e+1}-1}\right) \\ &= 2, \end{aligned}$$

proving the abundance of $2^e p_1 p_2 \cdots p_k$.

To prove that a number of type (3.52) is primitive abundant we have to show if d is a proper divisor of $2^e p_1 p_2 \cdots p_k$, then $\frac{\sigma(d)}{d} < 2$. A proper divisor d may take either the form

$$(3.53) \qquad 2^e d',$$

where d' is a proper divisor of $p_1 p_2 \cdots p_k$, or the form

$$(3.54) \qquad 2^r d'',$$

where $0 \leq r \leq \ell-1$, and $d'' | p_1 p_2 \cdots p_k$. But we have shown, in Lemma 23, that if $a | b$, then

$$\frac{\sigma(a)}{a} \leq \frac{\sigma(b)}{b}.$$

Therefore the largest quotient $\frac{\sigma(d)}{d}$ which needs to be considered is

$$(3.55) \qquad \frac{\sigma(2^{\ell} p_2 p_3 \cdots p_k)}{2^{\ell} p_2 p_3 \cdots p_k},$$

in the case of a proper divisor of type (3.53), and

$$(3.56) \qquad \frac{\sigma(2^{\ell-1} p_1 p_2 \cdots p_k)}{2^{\ell-1} p_1 p_2 \cdots p_k},$$

in the case of a proper divisor of type (3.54). But,

$$\frac{\sigma(2^{\ell-1} p_1)}{2^{\ell-1} p_1} = \left(\frac{2^{\ell}-1}{2^{\ell-1}}\right)\left(1+\frac{1}{p_1}\right) \leq \frac{2^{\ell}-1}{2^{\ell-1}}\left(1+\frac{1}{2^{\ell+1}}\right)$$

$$= \frac{2^{\ell+1}-1}{2^{\ell}} - \frac{1}{2^{2\ell}} < \frac{\sigma(2^{\ell})}{2^{\ell}},$$

whence (3.55) is the larger of the two quotients (3.55) and (3.56). Thus it suffices to prove that

$$(3.57) \qquad \frac{\sigma(2^{\ell} p_2 p_3 \cdots p_k)}{2^{\ell} p_2 p_3 \cdots p_k} < 2.$$

Now

$$\frac{\sigma(2^{\ell} p_2 p_3 \cdots p_k)}{2^{\ell} p_2 p_3 \cdots p_k} < \left(\frac{2^{\ell+1}-1}{2^{\ell}}\right)\left(1+\frac{1}{(k-1)2^{\ell+1}}\right)^{k-1},$$

and writing λ for $2^{-(\ell+1)}$, the right-hand side of this inequality * is

$$2(1-\lambda)(1+\frac{\lambda}{k-1})^{k-1} < 2(1-\lambda)\{1+\lambda+\frac{\lambda^2}{2}(1+\lambda+\cdots)\}$$

$$= 2(1-\lambda)(1+\lambda+\frac{\lambda^2}{2(1-\lambda)})$$

$$= 2-\lambda^2$$

$$< 2;$$

and this completes the proof of (3.57).

In order to estimate the number of p.a.n's of type (3.52) we show that, for k and ℓ suitably chosen, the number of primes between $(k-1)2^{\ell+1}$ and $k\,2^{\ell+1}$ is at least $\frac{2^\ell}{\ell+1}$.

By the Prime Number Theorem (Lemma 11), with $\Delta = 3$, we have

$$\pi(u) = \int_2^u \frac{dt}{\log t} + O\left(\frac{u}{(\log u)^3}\right).$$

It follows that

$$\pi(k2^{\ell+1}) - \pi((k-1)2^{\ell+1}) = \int_{(k-1)2^{\ell+1}}^{k2^{\ell+1}} \frac{dt}{\log t} + O\left(\frac{k2^\ell}{\ell^3}\right)$$

$$> \frac{2^{\ell+1}}{\log(k\,2^{\ell+1})} + O\left(\frac{k2^\ell}{\ell^3}\right).$$

Next, we impose the condition

$$k < \ell,$$

---

* $${}^mC_r\, m^{-r} = \frac{m(m-1)\cdots(m-r+1)}{r!}\cdot\frac{1}{m^r} \leq \frac{1}{r!} \leq \frac{1}{2}, \quad r \geq 2.$$

and observe that

$$\log(k.2^{\ell+1}) < \log \ell + (\ell+1)\log 2 < \ell+1,$$

if ℓ is chosen sufficiently large. Hence,

$$\pi(k.2^{\ell+1}) - \pi((k-1)2^{\ell+1}) > \frac{2^{\ell+1}}{\ell+1} + O\left(\frac{2^\ell}{\ell^2}\right) > \frac{2^\ell}{\ell+1}.$$

We shall now prove that k and ℓ, $1<k<\ell$, may be chosen so that each p.a.n. of type (3.52) is less than n, and so that the number of them is at least $ne^{-\frac{13}{2}x}$. We choose ℓ so that

(3.58) $$2^{\ell-1} \leq e^{x+1.3} < 2^\ell,$$

and

$$k = [y - 4.5].$$

Then, by (3.58),

(3.59) $$e^{x+1.3} < 2^\ell < e^{x+2},$$

and for sufficiently large n,

$$1 < k < y < x < \ell.$$

By (3.51) and (3.33), each p.a.n. of type (3.52) is less than

$$\begin{aligned} 2^\ell(k.2^{\ell+1})^k &< e^{x+2}\cdot y^y e^{(x+3)(y-4.5)} \\ &< e^{x+2+0.5x+xy+3x-4.5x-13.5} \\ &< e^{xy} \\ &= n. \end{aligned}$$

Finally, since $N(n)$ is greater than the number of p.a.n's of type (3.52), $N(n)$ is greater than the number of ways of choosing k primes from at least $\frac{2^\ell}{\ell+1}$ primes lying between $(k-1)2^{\ell+1}$ and $k.2^{\ell+1}$.

i.e.

$$N(n) > {}^{[\frac{2^\ell}{\ell+1}]}C_k .$$

But

$$^rC_k = \frac{r}{k}\cdot\frac{r-1}{k-1}\cdot\ldots\cdot\frac{r-k+1}{1} > \left(\frac{r}{k}\right)^k,$$

and, by (3.59),

$$\left[\frac{2^\ell}{\ell+1}\right] > \frac{e^{x+1\cdot3}}{2x} - 1 > \frac{e^x}{x},$$

for all sufficiently large n. Therefore, by (3.32),

$$N(n) > \frac{e^{x(y-5\cdot5)}}{x^{y-4\cdot5}(y-4\cdot5)^{y-4\cdot5}} > \frac{e^{xy}}{(xy)^y e^{5\cdot5x}}$$

$$= ne^{-\frac{13}{2}x},$$

for all sufficiently large n.

This completes the proof of the left-hand inequality in (3.31), and hence the proof of Theorem 16.

3.5. In the last section of this chapter we shall discuss the behaviour of our generalised functions $A(n,\lambda)$ and $N(n,\lambda)$, where $A(n,\lambda)$ and $N(n,\lambda)$ denote respectively the number of λ- abundant and primitive λ-abundant numbers not exceeding n. Clearly, by a proof identical to that used in Theorem 12 we could show that

$$\frac{A(n,\lambda)}{n} < \frac{2}{3(\lambda-1)}, \qquad \lambda > 1.$$

We proved, however, the existence of the density of the abundant numbers by proving that the sum of the reciprocals of the primitive abundant numbers converges, (Theorem 14), but a similar method cannot be used with the λ-abundant numbers, since the sum of the reciprocals of the primitive λ-abundant numbers diverges for some values of λ. We can show this very simply.

THEOREM 17.

There exists a number $\lambda > 1$ for which

$$\sum \frac{1}{m_i} \to \infty,$$

where $m_1 < m_2 < \dots$ is the infinite sequence of primitive λ-abundant numbers.

Proof. Let $p_1, p_2, \dots p \dots$ be an infinite sequence of primes satisfying

$$p_{i+1} > e^{e^{p_i^2}}. \tag{3.59}$$

It is clear that

$$\sum \frac{1}{p_i} < \sum \frac{1}{p_{i-1}^2} < \infty,$$

so that $\prod \left(1 + \frac{1}{p_i}\right)$ converges. Put

$$\lambda = \prod_{i=1}^{\infty} \left(1 + \frac{1}{p_i}\right) = \lim_{k \to \infty} \frac{\sigma(p_1 p_2 \cdots p_k)}{p_1 p_2 \cdots p_k};$$

we shall show that for every k the integers

$$p_1 p_2 \cdots p_k \cdot p, \qquad p_k < p < p_{k+1}, \tag{3.60}$$

are primitive λ-abundant.

First we show that a number of type (3.60) is abundant. To prove this we must show that

$$\frac{\sigma(p_1 p_2 \cdots p_k p)}{p_1 p_2 \cdots p_k p} \geq \lambda,$$

or

$$(3.61) \qquad \frac{1+\frac{1}{p}}{1+\frac{1}{p_{k+1}}} \geq \prod_{i=k+2}^{M} \left(1+\frac{1}{p_i}\right).$$

But, by (3.59),

$$\prod_{i=k+2}^{M}\left(1+\frac{1}{p_i}\right) \leq \left(1+\frac{1}{p_{k+2}}\right)\prod_{i=k+2}^{M}\left(1+\frac{1}{p_i^2}\right)$$

$$\leq \left(1+\frac{1}{p_{k+2}}\right)\left(1+\sum_{n \geq p_{k+2}} \frac{1}{n^2}\right)$$

$$\leq \left(1+\frac{1}{p_{k+2}}\right)\left(1+\frac{1}{p_{k+2}-1}\right)$$

$$\leq \left(1+\frac{1}{p_{k+2}-1}\right)^2$$

$$< 1+\frac{3}{p_{k+2}-1}.$$

Also, since $p \leq p_{k+1}-2$,

$$\frac{1+\frac{1}{p}}{1+\frac{1}{p_{k+1}}} \geq \frac{1+\frac{1}{p_{k+1}-2}}{1+\frac{1}{p_{k+1}}}$$

$$= 1 + \frac{2}{(p_{k+1}+1)(p_{k+1}-2)}$$

$$\geq 1 + \frac{2}{(p_{k+1}-\frac{1}{2})^2}.$$

Therefore, to prove (3.61) it suffices to prove that

$$\frac{2}{(p_{k+1}-\frac{1}{2})^2} \geq \frac{3}{p_{k+2}-1}.$$

i.e.

$$p_{k+2} \geq 1 + \tfrac{3}{2}(p_{k+1}-1)^2.$$

But this is clearly true, by virtue of (3.59), and it follows that a number of type (3.60) is λ- abundant.

To prove that $p_1p_2\cdots p_kp$ is primitive λ-abundant we have to show that if d is a proper divisor of $p_1p_2\cdots p_kp$ then $\frac{\sigma(d)}{d} < \lambda$. If d is of the form pd', where d' is a proper divisor of $p_1p_2\cdots p_k$, then

$$\frac{\sigma(d)}{d} \leq \frac{\sigma(p_1p_2\cdots p_k)}{p_1p_2\cdots p_k} < \lambda,$$

since $\frac{\sigma(p)}{p} < \frac{\sigma(p_i)}{p_i}$, if $p_i < p$. On the other hand, if d is of the form d'', where $d'' \mid p_1p_2\cdots p_k$, then clearly

$$\frac{\sigma(d'')}{d''} < \frac{\sigma(p_1p_2\cdots p_k)}{p_1p_2\cdots p_k} < \lambda.$$

Hence, the numbers $p_1p_2\cdots p_kp$ are primitive λ-abundant for any k.

The proof of our theorem now follows easily. By Lemma 5,

$$\sum_{p\leq n} \frac{1}{p} = \log\log n + B + O\left(\frac{1}{\log n}\right),$$

where B is constant. It follows from (3.59) that

$$\sum_{p_k<p<p_{k+1}} \frac{1}{p} > \frac{1}{2}p_k^2.$$

But by our definition of the prime p_k, we have

$$p_k > p_1p_2\cdots p_{k-1}.$$

Thus

$$\sum_{p_k<p<p_{k+1}} \frac{1}{p_1p_2\cdots p_kp} > \frac{1}{2},$$

and

$$\sum \frac{1}{m_k} > \sum_k \sum_{p_k<p<p_{k+1}} \frac{1}{p_1p_2\cdots p_kp} \to \infty,$$

Theorem 17 demonstrates that it is not possible to use the same type of argument for the λ- abundant numbers as for the abundant numbers to show the sequence of these numbers possesses density. This really raises an interesting question of a more general nature: what are the most general conditions on a sequence m_i under which the sequence of multiples of m_i possesses density? Such a necessary and sufficient condition was discovered by Erdös *1 in 1948. Although this condition is rather complicated the proof of it is elementary in nature. The condition implies that if the number of numbers m_i not exceeding n is $O(\frac{n}{\log n})$, then the sequence of multiples of m_i possesses density.

In our case it follows that the sequence of λ-abundant numbers possesses density provided that

$$N(n,\lambda) = O\left(\frac{n}{\log n}\right).$$

In fact we shall prove the following theorem:

THEOREM 18.*2 (Erdös).

$$N(n,\lambda) = o\left(\frac{n}{\log n}\right).$$

Proof. Our method for the first part of this Theorem will now be familiar to the reader. We begin by eliminating the integers $m \leq n$ with large quadratic part, yet possessing many

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*1 P. ERDÖS. Bull. Amer. Math. Soc. 54.8 (1948) 685 - 692.

*2 P. ERDÖS. Acta. Arithmetica. V. (1958). 25-33.

small prime factors. We show that the numbers of integers $m \leq n$ which do not satisfy any of the following conditions (c.f. Theorem 15) is $O\left(\frac{n}{(\log n)^2}\right)$:

(1) $\frac{n}{(\log n)^2} < m$,

(2) if $p^\alpha \mid m$, and $\alpha > 1$, then $p^\alpha < (\log n)^{10}$.

(3) $\nu(m) < \rho$, where $\rho = 10 \log\log n$.

(4) The greatest prime-factor of m is greater than n^{1/ρ^2}.

It is trivial that the numbers of integers $m \leq n$ not satisfying (1) is $o\left(\frac{n}{(\log n)^2}\right)$, and we have already shown in (3.22) and (3.23) of Theorem 15 that the number of integers $m \leq n$ which do not satisfy (2) and (3) is $o\left(\frac{n}{(\log n)^5}\right)$. For the integers $m \leq n$ not satisfying (4), we may assume without loss of generality, that they satisfy (2) and (3). Let the prime decomposition of m be

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}.$$

By (2), if $\alpha \geq 2$, then $p^\alpha < (\log n)^{10} < n^{1/\rho^2}$, for sufficiently large n, and if $\alpha = 1$, then $p < n^{1/\rho^2}$, by (4). Hence, by (3),

$$m < (n^{1/\rho^2})^\rho = n^{1/\rho} = o\left(\frac{n}{(\log n)^\Delta}\right),$$

for any $\Delta \geq 0$.

We therefore need consider only those primitive λ - abundant numbers not exceeding n, which satisfy conditions (1) to (4). We denote these by $m_1, m_2, m_3, \dots\dots m_N$.

It is here, however, that our method differs from that of Theorem 15. Whereas we were there able to show that each m_i had an unrepeated prime divisor p_i which was large enough to ensure that $\frac{m_i}{p_i}$ was small, and small enough to ensure that the integers $\frac{m_i}{p_i}$ were distinct, it is not true that each λ-abundant number has a similar property. We can, of course, eliminate those λ-abundant numbers which do have such a divisor by a method similar in every respect to that used in Theorem 15.

We let each

$$m_i = u_i v_i ,$$

where all the prime factors of u_i do not exceed $(\log n)^{10}$, and all the prime factors of v_i are greater than $(\log n)^{10}$. We split the numbers m_i into two classes.

In the first class we place those m_i for which v_i is not a single prime. (By (2), (3), and (4), $v_i > 1$ and v_i is square-free). Let the prime decomposition of v_i be $v_i = p_1^{(i)} \cdots p_{r_i}^{(i)}, \quad p_1^{(i)} < \cdots < p_{r_i}^{(i)}$. Then by (4), we have

$$(3.62) \qquad (\log n)^{10} < p_1^{(i)} < \cdots < p_r^{(i)}, \quad p_r^{(i)} > n^{1/\rho^2}.$$

Now we split the numbers m_i of the first class into two sub-classes. In the first sub-class we place the numbers m_i with

$$(3.63) \qquad p_1^{(i)} < n^{1/4\rho^2}.$$

($p_1^{(i)}$ is, in fact, the unrepeated prime divisor we have already discussed). We shall show that if (3.63) is satisfied then the integers

$$(3.64) \qquad \frac{m_i}{p_1^{(i)}} < \frac{n}{(\log n)^{10}},$$

are all different, and it will then follow from (3.62) that the number of integers of the first sub-class is less than

$$\frac{n}{(\log n)^{10}} = o\left(\frac{n}{\log n}\right).$$

We assume, on the contrary, that

$$\frac{m_\nu}{p_1^{(\nu)}} = \frac{m_\mu}{p_1^{(\mu)}},$$

for some $\mu \neq \nu$. By a method identical to that used in Theorem 15 we can show by (3.63), that (c.f. (3.29)).

$$(3.65) \qquad \frac{\sigma(m_\nu)}{m_\nu} \cdot \frac{m_\mu}{\sigma(m_\mu)} \geq 1 + \frac{1}{p_1^{(\nu)}(p_1^{(\mu)}+1)} \geq 1 + \frac{1}{2n^{4/\rho^2}}.$$

But, on the other hand, if $p_r^{(\nu)}$ is the greatest prime factor of m_ν, then $\frac{m_\nu}{p_r^{(\nu)}}$ is λ-deficient, and, by (4),

$$\frac{\sigma(m_\nu)}{m_\nu} \leq \lambda\left(1 + \frac{1}{p_r^{(\nu)}}\right) \leq \lambda\left(1 + \frac{1}{n^{1/\rho^2}}\right).$$

But m_μ is λ-abundant, so it follows that

$$\frac{\sigma(m_\nu)}{m_\nu} \cdot \frac{m_\mu}{\sigma(m_\mu)} \leq 1 + \frac{1}{n^{1/\rho^2}},$$

in contradiction of (3.65).

This completes the proof that the integers $\frac{m_i}{p_1^{(i)}} < \frac{n}{(\log n)^{10}}$ are distinct, and thus disposes of the first sub-class.

In the second sub-class, we have the numbers for which v_i is not a single prime, where all the prime-factors of v_i are greater than $n^{1/4p^2}$. In Theorem 15 we were able to show that the corresponding sub-class was empty, by proving that if m_i did belong to this sub-class, m_i would be deficient. In the course of the argument we used the fact that if

$$\sigma(u_i) < 2u_i,$$

then

$$\sigma(u_i) \leq 2u_i - 1 .$$

On replacing 2 by λ here it becomes clear that this step is no longer valid, unless λ is an integer.

The most we can show now is that for the numbers m_i of the second sub-class $\frac{\sigma(u_i)}{u_i}$ has a constant value (clearly less than λ) i.e. there exists a number K_n such that

$$\frac{\sigma(u_i)}{u_i} = K_n < \lambda . \tag{3.66}$$

We assume, on the contrary, that for some m_1 and m_2 of the second sub-class we have

$$\frac{\sigma(u_1)}{u_1} < \frac{\sigma(u_2)}{u_2},$$

so that

$$\frac{\sigma(u_2)}{u_2} - \frac{\sigma(u_1)}{u_1} \geq \frac{1}{u_1 u_2}. \tag{3.67}$$

Now, by (2) and (3), we have

$$u_i < (\log n)^{10p}, \tag{3.68}$$

It follows from (3.67) and (3.68) that

$$\frac{\sigma(u_2)}{u_2} - \frac{\sigma(u_1)}{u_1} > (\log n)^{-20\rho},$$

or, since u_2 is necessarily deficient,

(3.69) $$\frac{\sigma(u_1)}{u_1} < \lambda - (\log n)^{-20\rho}.$$

But since v_i is square-free, and for every $p \mid v_i$ we have $p > n^{1/4\rho^2}$, it follows from (3) (c.f. footnote (3.26)) that

$$\frac{\sigma(v_i)}{v_i} = \prod_{p \mid v_i} \left(1 + \frac{1}{p}\right) < \left(1 + \frac{1}{n^{1/4\rho^2}}\right)^{\rho} < 1 + \frac{2\rho}{n^{1/4\rho^2}}.$$

But $(u_1, v_1) = 1$, and therefore

$$\frac{\sigma(m_1)}{m_1} = \frac{\sigma(u_1)}{u_1} \cdot \frac{\sigma(v_1)}{v_1} < \left(\lambda - \frac{1}{(\log n)^{20\rho}}\right)\left(1 + \frac{2\rho}{n^{1/4\rho^2}}\right) < \lambda,$$

if n is sufficiently large, and this contradicts the hypothesis that m_1 is primitive λ-abundant. This completes the proof of (3.66).

We now let

$$p_1 = \min \{ p_1^{(i)} \},$$

where the minimum is taken over the members of the second sub-class. We may immediately dismiss those m_i's for which the smallest prime factor $p_i^{(1)}$ of v_i satisfies,

(3.70) $$n^{1/4\rho^2} < p_1 \leq p_1^{(1)} < p_1\left(1 + \frac{1}{\log n}\right).$$

By Lemma 5, the number of these m_i's is less than

(3.71) $$n \sum_{p_1 \le p < p_1(1+\frac{1}{\log n})} \frac{1}{p} < \frac{c_1 n (\log\log n)^2}{(\log n)^2} = o\left(\frac{n}{\log n}\right).$$

Therefore, we consider the primitive λ- abundant numbers of the second sub-class for which

(3.72) $$p_1^{(1)} \geq p_1\left(1+\frac{1}{\log n}\right).$$

We shall show that for each of these numbers m_i we have

(3.73) $$p_1\left(1+\frac{1}{\log n}\right) \le p_1^{(1)} < p_2^{(1)} < p_1(\log n)^2,$$

i.e. each m_i has two unrepeated prime factors lying in the interval $(p_1, (\log n)^2 p_1)$, so that the number of these numbers is at most

(3.74) $$n \sum_{p_1 < q < r < p_1(\log n)^2} \frac{1}{qr} < \left(\sum_{p_1 < s < p_1(\log n)^2} \frac{1}{s}\right)^2 < c_2 n \left(\frac{\log\log n}{\log p_1}\right)^2.$$

By (3.70) the right-hand side of (3.74) is less than

(3.75) $$\frac{16 c_2 n \rho^4 (\log\log n)^2}{(\log n)^2} = o\left(\frac{n}{\log n}\right),$$

and (3.71) and (3.75) will dispose of the second sub-class.

It remains to prove (3.73). First we show that for every v_i

(3.76) $$\frac{\sigma(v_i)}{v_i} > 1 + \frac{1}{p_1}.$$

To see this we have only to remark that p_1 is a prime factor of some $m_j = u_j v_j$. Thus, since v_j is not a prime we have, by (3.66), that

$$\frac{\sigma(u_j p_1)}{u_j p_1} = K_n\left(1+\frac{1}{p_1}\right) < \lambda \le \frac{\sigma(u_i)}{u_i}\cdot\frac{\sigma(v_i)}{v_i} = K_n\frac{\sigma(v_i)}{v_i},$$

which proves (3.76). It follows that (c.f. footnote, (3.26))

$$1+\frac{1}{p_1} < \frac{\sigma(v_i)}{v_i} < \left(1+\frac{1}{p_1^{(i)}}\right)\cancel{\left(1+\frac{1}{p_2^{(i)}}\right)^{\rho}},$$

or

$$p_1^{(i)} < 2\rho p_1 = p_1(\log n)^2,$$

and by (3.72) $p_1^{(i)}$ lies in the desired interval.

Next, we find an estimate for $p_2^{(i)}$. It follows from (3.76) that

$$1+\frac{1}{p_1} < \frac{\sigma(v_i)}{v_i} < \left(1+\frac{1}{p_1^{(i)}}\right)\left(1+\frac{1}{p_2^{(i)}}\right)^{\rho},$$

and substituting from (3.72) gives

$$\left(1+\frac{1}{p_1}\right)\left(1+\frac{\log n}{p_1(1+\log n)}\right)^{-1} \le \left(1+\frac{1}{p_1}\right)\left(1+\frac{1}{p_1^{(i)}}\right)$$

(3.77)
$$< \left(1+\frac{1}{p_2^{(i)}}\right)^{\rho}.$$

But for all real a we have

(3.78) $$(1+a)^{-1} > 1-a,$$

and applying the inequality (3.78) to the left-hand side of (3.77) we obtain (after simple re-arrangement)

$$\rho_2^{(i)} < 2\rho p_1 A_n,$$

where, by (3.70),

$$A_n = (1+\log n)\left(1-\frac{\log n}{p_1}\right)^{-1} < 2\log n,$$

for sufficiently large n , so that $p_2^{(1)}$ also lies in the required interval. This completes the proof of (3.73).

We have now shown that the total number of primitive λ- abundant numbers in the first class is $o(\frac{n}{\log n})$. Finally, we consider the numbers m_i of the second-class, i.e. numbers m_i for which v_i is a single prime.

Let

(3.79) $$m_i = u_i p_i .$$

By (3.68) and (1), it suffices to consider the numbers satisfying

(3.80) $$p_i > n^{1/2} ;$$

But then we may again assume that (3.66) holds (i.e. $\frac{\sigma(u_i)}{u_i}$ is of constant value), the proof being exactly the same as previously. It will follow that the number of numbers in the second class equals

$$\sum{}' \pi(\tfrac{n}{u_i}) + o(\tfrac{n}{\log n}) = \frac{n}{\log n} \sum{}' \frac{1}{u_i} + o(\tfrac{n}{\log n}),$$

where the dash indicates that the summation extends over the u_i satisfying

$$\frac{\sigma(u_i)}{u_i} = K_n .$$

Hence, to prove our theorem it will suffice to show that

(3.81) $$\sum{}' \frac{1}{u_i} = o(1).$$

Now denote by $a_1 < a_2 < \cdots$ the infinite sequence of all integers satisfying

(3.82) $$\frac{\sigma(a_i)}{a_i} = K_n = \frac{a_n}{b_n} < \lambda , \qquad (a_n, b_n) = 1;$$

Clearly, we have

$$\sum{}' \frac{1}{u_i} \leq \sum \frac{1}{a_i}.$$

First we show that as $n \to \infty$, $b_n \to \infty$. By (3.79) and (3.80), we have

(3.83) $$\frac{\sigma(a_i)}{a_i} = \frac{a_n}{b_n} < \lambda, \qquad \sigma(m_i) = \frac{a_n}{b_n}\left(1 + \frac{1}{p_i}\right) \geq \lambda,$$

or

(3.84) $$\frac{a_n}{b_n} \geq \lambda\left(1 + \frac{1}{p_i}\right)^{-1}.$$

But $p_i > n^{1/2} \to \infty$, and therefore $\frac{a_n}{b_n} \to \lambda$, and this is impossible if b_n (and hence a_n) assume only a finite number of values. Therefore $b_n \to \infty$, $n \to \infty$.

Thus, to complete our proof it will suffice to show that

(3.85) $$\sum \frac{1}{a_i} < \frac{c}{b_n^{1/2}},$$

where c is an absolute constant. From (3.82),

$$b_n \mid a_i.$$

Therefore, by Lemma 7,

(3.86) $$\sum{}'' \frac{1}{a_i} \leq \frac{1}{b_n} \sum{}' \frac{1}{t} = \frac{1}{b} \prod_{p \leq b_n} \left(1 + \frac{1}{p-1}\right) < c_1 \frac{\log b_n}{b_n},$$

where the double-dash denotes that all the prime factors of a_i are less than b_n.

Now, consider the sum

(3.87) $$\sum{}''' \frac{1}{a_i},$$

where the triple-dash ~~denotes~~ signifies that at least one prime-factor of a_i is greater than b_n. Let $p^{(1)}$ be the greatest prime factor of a_i. Then $p^{(1)}$ is clearly greater than b_n. But since

$$\frac{\sigma(a_i)}{a_i} = \frac{a_n}{b_n},$$

we must have

$$(3.88) \qquad p^{(1)} \mid \sigma(a_i),$$

or, for some $q_i^{\alpha} \mid a_i$, we must have $p^{(1)} \mid \sigma(q_i^{\alpha})$. Since $p^{(1)}$ is the greatest prime factor of a_i,

$$p^{(1)} \nmid 1 + q_i = \sigma(q_i),$$

for any $q_i \leq p^{(1)}$, and we must therefore have $\alpha \geq 2$. Hence

$$p^{(1)} \leq \frac{q_i^{\alpha+1} - 1}{q_i - 1} < 2q_i^{\alpha}.$$

It then follows that the sun (3.87) is less than

$$\sum_{p^{(1)} > b_n} \frac{1}{p^{(1)}} \sum_1 \frac{1}{q^{\alpha}} \sum_2 \frac{1}{t},$$

where in Σ_1, $q^{\alpha} > \frac{1}{2} p^{(1)}$, and $\alpha > 1$, and in Σ_2 all prime factors of t are not greater than $p^{(1)}$. Thus, finally, by Lemma 7,

$$(3.89) \qquad \sum{}''' \frac{1}{a_i} < \sum_{p^{(1)} > b_n} \frac{1}{p^{(1)}} \sum_1 \frac{1}{q^{\alpha}} \prod_{p \leq p^{(1)}} \left(1 + \frac{1}{p-1}\right)$$

$$< \sum_{p^{(1)} > b_n} \frac{c_3 \log p^{(1)}}{p^{(1)}} \sum_1 \frac{1}{q^{\alpha}}$$

$$< c_4 \sum_{p^{(1)} > b_n} \frac{\log p^{(1)}}{p^{(1)3/2}}$$

$$< \frac{c_5}{b_n^{1/2}}.$$

Relations (3.86) and (3.89) together prove (3.85), and this completes the proof of Theorem 18.

Erdos states that this Theorem is the best possible in the following sense: it can be proved that if $g(n) \to \infty$ as slowly as we like, there always exists a λ so that for infinitely many n,

$$N(n,\lambda) > \frac{n}{g(n)\log n}.$$

CHAPTER IV.

PERFECT AND MULTI-PERFECT NUMBERS.

4.1. The perfect and multi-perfect numbers are perhaps even more fascinating than the abundant numbers. The reader will recall that a perfect number n is one for which

$$\frac{\sigma(n)}{n} = 2,$$

and more generally, a multi-perfect number n of class λ is one for which

$$\frac{\sigma(n)}{n} = \lambda,$$

where λ is an integer.

There is, however, a striking difference between these numbers and the abundant numbers. Very few ~~of them~~ perfect numbers are known, and yet the results that we have about them are so poor. We shall, in this chapter, find upper estimates for the counting numbers of the sequences of both the perfect and multi-perfect numbers. These results are probably very far from being the best possible, but even so they are not easy to prove.

4.2. Since the known perfect and multi-perfect numbers are so rare it seems natural to expect the following result:

THEOREM 19.

The density of the sequence of perfect numbers is zero. More generally, the density of the sequence of multi-perfect numbers of class λ is zero for any λ .

Proof. The proof of this Theorem is an immediate consequence of the following Lemma.

LEMMA 29.

Denote by $N(\lambda, \eta)$ the number of positive integers $m \leq n$ for which

(4.1) $$\lambda \leq \frac{\sigma(m)}{m} \leq \lambda + \eta.$$

Then for every $\delta > 0$, there exists a positive integer η such that

(4.2) $$N(\lambda, \eta) < \delta n,$$

for all sufficiently large n .

Proof. By Lemma 3, $\prod_p (1 - \frac{1}{p})$ diverges to zero and $\sum \frac{1}{p^2}$ converges. We may therefore choose a sequence of consecutive primes $p_1 < p_2 < \dots < p_t$ such that, for any $\delta > 0$,

(4.3) $$p_1 > \frac{3}{\delta},$$

(4.4) $$\prod_{i=1}^{t} (1 - \frac{1}{p_i}) < \frac{\delta}{4},$$

(4.5) $$\sum_{i=1}^{t} \frac{1}{p_i^2} < \frac{\delta}{3}.$$

We choose η to satisfy *

(4.6) $$1 + \frac{\eta}{\lambda} < \min_{1 \leq i < j \leq t} \left\{ \frac{\sigma(p_i)}{p_i} \cdot \frac{p_j}{\sigma(p_j)} \right\}.$$

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* This step is justified since $\frac{\sigma(p_i)}{p_i} \cdot \frac{p_j}{\sigma(p_j)} > 1$, if $p_i < p_j$.

and we shall show that, with this choice of η , (4.2) is satisfied.

We show first that the number of integers $m \leq n$ which do not satisfy either of the two conditions

(1) m is divisible by one of the primes p_i ,

(2) m is not divisible by any one of the p_i^2 ,

is less than $\frac{2}{3}\delta n$ for sufficiently large n .

The number N_1 , of integers $m \leq n$ not satisfying (1) is given by

(4.7) $$N_1 = n - \left[\frac{n}{p_1}\right] - \left[\frac{n}{p_2}\right] - \cdots + \left[\frac{n}{p_1 p_2}\right] + \cdots - \left[\frac{n}{p_1 p_2 p_3}\right] - \cdots$$

$$= n \prod_{i=1}^{t} \left(1 - \frac{1}{p_i}\right) + R,$$

where

$$|R| \leq 1 + {}^tC_1 + \cdots + {}^tC_t = 2^t.$$

But t is independent of n , so that for sufficiently large n,

(4.8) $$|R| < \frac{1}{12}\delta n.$$

Hence, by (4.4), (4.7) and (4.8)

(4.9) $$N_1 < \frac{1}{3}\delta n.$$

The number ~~of~~ N_2 of integers $m \leq n$ which do not satisfy (2) is at most

$$\sum_{i=1}^{t} \left[\frac{n}{p_i^2}\right] \leq n \sum_{i=1}^{t} \frac{1}{p_i^2}.$$

Therefore, by (4.5),

(4.10) $$N_2 < \tfrac{1}{3}\delta n,$$

and the total number of integers $m \leq n$ not satisfying (1) and (2) is, by (4.9) and (4.10), less than $\frac{2}{3}\delta n$, for sufficiently large n .

It now only remains to prove that the number of integers $m \leq n$ satisfying (1), (2) and (4.1) is, with the definition of η given in (4.6), less than $\frac{1}{3}\delta n$. We denote these (distinct) integers by $m_1, m_2, \ldots m_N$.

From (1) and (4.3) if m_i is divisible by p_{r_i} then

$$\frac{m_i}{p_{r_i}} < \tfrac{1}{3}\delta n,$$

and we shall show that the integers $\frac{m_i}{p_{r_i}}$ are distinct. This will clearly be sufficient to prove our lemma.

We suppose, on the contrary, that

$$\frac{m_\nu}{p_{r_\nu}} = \frac{m_\mu}{p_{r_\mu}},$$

for some $\mu \neq \nu$. It is clear that $p_{r_\nu} \neq p_{r_\mu}$, since $m_\nu \neq m_\mu$, and we may therefore suppose, without loss of generality, that $p_{r_\nu} < p_{r_\mu}$. By (2), $\left(\frac{m_i}{p_{r_i}}, p_{r_i}\right) = 1$; Therefore,

$$\frac{\sigma(m_\nu)}{m_\nu} = \frac{\sigma(m_\nu/p_{r_\nu})}{m_\nu/p_{r_\nu}} \cdot \frac{\sigma(p_{r_\nu})}{p_{r_\nu}},$$

and similarly with μ for ν . Hence, on division, (4.11) $$\frac{\sigma(m_\nu)}{m_\nu}\cdot\frac{m_\mu}{\sigma(m_\mu)} = \frac{\sigma(p r_\nu)}{p r_\nu}\cdot\frac{p r_\mu}{\sigma(p r_\mu)}.$$

But, by (4.1),

(4.12) $$\frac{\sigma(m_\nu)}{m_\nu}\cdot\frac{m_\mu}{\sigma(m_\mu)} \leq 1+\frac{\eta}{\lambda},$$

and by the definition of η given in (4.6),

(4.13) $$\frac{\sigma(p r_\nu)}{p r_\nu}\cdot\frac{p r_\mu}{\sigma(p r_\mu)} > 1+\frac{\eta}{\lambda}.$$

(4.12) and (4.13) together clearly contradict (4.11). Hence the integers $\frac{m_i}{p r_i}$ are distinct and

(4.14) $$N < \tfrac{1}{3}\delta n.$$

Thus, by (4.9), (4.10) and (4.14),

$$N(\lambda,\eta) < \delta n.$$

The proof of Theorem 19 follows immediately. If $P(n)$ and $M(n)$ denote respectively the number of perfect and multi-perfect integers m of class λ not exceeding n, then

$$P(n) \leq N(2,\eta),$$

$$M(n) \leq N(\lambda,\eta),$$

for any $\eta > 0$. Therefore, by Theorem 19,

$$\frac{P(n)}{n} < \delta,$$

$$\frac{M(n)}{n} < \delta,$$

for every $\delta > 0$. Since δ may be chosen arbitrarily small, it follows that the density of the sequence of perfect numbers is zero, and more generally, the density of the sequence of multi-perfect numbers of class λ is zero for any λ.

4.3. However, much more than Theorem 19 can be proved. In 1955 Kanold [*1] showed $M(n) = o(n)$, and in the same year Hornfeck [*2] proved that $P(n)$ is less than $n^{1/2}$. A year later, though, Erdös [*3] improved on both these estimates to $n^{3/4+\epsilon}$ for every $\epsilon > 0$, in the case of the multi-perfect numbers, and to $n^{1/2-c}$, for some constant $0 < c < \frac{1}{2}$, in the case of the perfect numbers.

THEOREM 20. (Erdös).

For every $\epsilon > 0$,

$$M(n) < n^{3/4+\epsilon},$$

if n is sufficiently large.

Proof. In estimating an upper bound for $M(n)$ it suffices to show that

$$M(n) - M(\tfrac{n}{2}) < n^{3/4+\epsilon},$$

for then, if N is a positive integer such that $2^N > n$, we have

$$M(n) = \sum_{r=1}^{N}\left\{M(\tfrac{n}{2^{r-1}}) - M(\tfrac{n}{2^r})\right\} + M(\tfrac{n}{2^N}) < \sum_{r=1}^{N}\left(\tfrac{n}{2^{r-1}}\right)^{3/4+\epsilon} < n^{3/4+2\epsilon}.$$

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*1 H. J. KANOLD. J. Reine Agnew Math. 194 (1955) 218 - 220.

*2 B. HORNFECK. Archiv. dev. Math. 6 (1955) 442 - 443.

*3 P. ERDÖS. Annali di Matematica. 42 (1956) 253 - 258.

Therefore, to prove our theorem we need only count the number of multi-perfect integers m satisfying

(4.18) $$\frac{n}{2} < m \leq n.$$

Of these, by Lemma 16, the number whose quadratic part r_m exceeds $n^{1/2}$ is at most

(4.19) $$c_1 n^{3/4},$$

and therefore we consider only those multi-perfect integers m satisfying (4.18) and

(4.20) $$r_m \leq n^{1/2}.$$

We denote these integers by $m_1, m_2, \ldots m_N$.

To estimate N we aim to show that each m_i has a divisor $d_i > A_n$, say, (where A_n is a number dependent on n), such that

(4.21) $$d\sigma(d) \mid \sigma(m)$$

Now since we are dealing with multiply perfect numbers, $\frac{\sigma(m)}{m}$ is an integer. Also, by Theorem 9,

$$\frac{\sigma(m)}{m} < 2\log\log n,$$

and it is therefore certain that

$$\frac{\sigma(m)}{m} \mid B_n$$

where $B_n = [2\log\log n]!$ It will then follow that for each m_i, $(i=1,2,\ldots N)$, $m_i B_n$ is divisible by an integer of the form $d\sigma(d)$ with $d > A_n$. Hence,

(4.22) $$N < nB_n \sum_{d>A_n} \frac{1}{d\sigma(d)} < nB_n \sum_{d>A_n} \frac{1}{d^2} < n\frac{B_n}{A_n},$$

and our Theorem will be proved provided only that

(4.23) $$\frac{B_n}{A_n} < n^{-\frac{1}{4}+\frac{1}{2}\epsilon}.$$

The proof of our Theorem is now reduced to finding a sufficiently large divisor d of each m which satisfies (4.21).

We can simplify this by one stage more. If this divisor divided the square-free part of m, then we should have

$$\sigma(m) = \sigma(\tfrac{m}{d})\sigma(d),$$

or, since $\sigma(m) = \lambda m$, where λ is an integer,

$$d \mid \sigma(m), \qquad \sigma(d) \mid \sigma(m).$$

If, in addition,

(4.24) $$(d, \sigma(d)) = 1,$$

then it would follow immediately that

$$d\,\sigma(d) \mid \sigma(m),$$

and (4.21) would be satisfied.

We therefore complete the proof of Theorem 20 by showing that the square-free part s_m of each integer m satisfying (4.18) and (4.20), has a divisor $d > A_n$ so that (4.23) and (4.24) hold.

Let the prime decomposition of s_m be

$$s_m = p_1 p_2 \cdots p_r, \qquad (p_1 < p_2 < \cdots < p_r).$$

We choose a subsequence of the primes p_i,

(4.25) $$p_{r_1} > p_{r_2} > \cdots > p_{r_\omega},$$

selected according to the following rule:

(4.26) $$\begin{cases} p_{r_1} = p_r, \\ p_{r_j} = \max\limits_{p_i < p_{r_{j-1}}} p_i, \quad \text{such that } p_{r_j} \nmid \sigma(p_{r_1} p_{r_2} \cdots p_{r_{j-1}}), \\ \qquad\qquad\qquad\qquad (1 < j \leq \omega). \end{cases}$$

and we define our divisor, $d = p_{r_1} p_{r_2} \cdots \cdots p_{r_\omega}$.

The reason for this particular construction is now clear. By (4.25), it follows that

$$p_{r_j} \nmid (1+p_{r_j})(1+p_{r_{j+1}}) \cdots \cdots (1+p_{r_\omega}),$$

i.e.

$$p_{r_j} \nmid \sigma(p_{r_j} p_{r_{j+1}} \cdots \cdots p_{r_\omega}),$$

and so, by (4.26),

$$p_{r_j} \nmid \sigma(p_{r_1} p_{r_2} \cdots \cdots p_{r_\omega}), \qquad (j = 1, 2, \ldots \omega).$$

Therefore,

$$(d, \sigma(d)) = 1 .$$

It now only remains to show that d is sufficiently large.

By our construction of d, it follows that

$$\frac{s_m}{d} \mid \sigma(d),$$

so that

$$s_m \leq d\sigma(d).$$

But since $r_m \leq n^{1/2}$, and $\frac{n}{2} < m \leq n$, we have $s_m \geq \frac{1}{2} n^{1/2}$.
Hence, by Theorem 9,

$$\tfrac{1}{2} n^{1/2} < s_m \leq d\sigma(d) < 2d^2 \log\log n,$$

i.e.

$$d > A_n,$$

where $A_n = \frac{1}{2} n^{1/4} (\log\log n)^{-\frac{1}{2}}$. It is now clear that d is sufficiently large, since with this value of A_n,

$$\frac{B_n}{A_n} < n^{-\frac{1}{4} + \frac{1}{2}\epsilon},$$

if n is sufficiently large.

Therefore, by (4.22),

$$N < n^{3/4 + \frac{1}{2}\epsilon}.$$

It now follows from (4.19) that the total number of multi-perfect integers m not exceeding n is at most

$$n^{3/4 + \epsilon},$$

and this completes the proof of Theorem 20.

The weakness of this result really lies in the initial elimination of the integers $m \leq n$ for which $r_m > n^{1/2}$. This leaves for consideration integers $m \leq n$ for which $s_m > \frac{1}{2}n^{1/2}$. Since our divisor $d > \frac{1}{2}(s_m)^{1/2}(\log\log n)^{-\frac{1}{2}}$, the best possible result by this method is

$$\frac{B_n}{A_n} < n^{-\frac{1}{4} + \frac{1}{2}\epsilon},$$

which limits our upper estimate for $M(n)$ to $n^{3/4+\epsilon}$.

4.4. However, we now turn our attention to finding an upper estimate for $P(n)$. Before proving the second theorem of Erdos, we shall prove a supplementary Lemma.

LEMMA 30.

Let d be a given square-free number, and $p_1'' > p_2'' > \dots > p_s''$ a sequence of distinct primes defined in some way so that to each p_i'' there corresponds a prime $p_i \mid d$ such that

$$p_i \mid \sigma(p_i''^2), \qquad p_i > p_i'', \quad (i = 1, 2, \dots s)$$

Let $d'' = p_1'' p_2'' \dots p_s''$. Then

$$d'' < d^2.$$

Proof. It is easy to see that in the product $p_1 p_2 \cdots p_s$ no three primes p_i, p_j, p_k can be equal. For if we assume *

(4.15) $$p_i \mid p_i''^2 + p_i + 1, \qquad p_i > p_i'',$$

(4.16) $$p_j \mid p_j''^2 + p_j + 1, \qquad p_j > p_j'',$$

(4.17) $$p_k \mid p_k''^2 + p_k + 1, \qquad p_k > p_k'',$$

and $p_i = p_j = p_k$, $(i < j < k)$, then, from (4.15) and (4.16),

$$p_i \mid p_i'' + p_j'' + 1,$$

Similarly,

$$p_i \mid p_k'' + p_j'' + 1,$$

Therefore,

$$p_i \mid p_i'' - p_k'',$$

an evident contradiction.

Hence, since d is square-free

$$d \geq (\sigma(d''^2), d) \geq (p_1 p_2 \cdots p_s)^{1/2} > d''^{1/2},$$

and our Lemma is proved.

We now prove the second of our two Theorems.

THEOREM 21. (Erdős).

There exists a positive constant $c < \frac{1}{2}$ such that

$$P(n) < n^{1/2 - c},$$

for sufficiently large n.

Proof. As in Theorem 20, we need only consider the perfect numbers satisfying

(4.27) $$\frac{n}{2} < m \leq n.$$

---

* $\sigma(p^2) = p^2 + p + 1.$

We can estimate the number N' of even perfect numbers $m \le n$ very simply. We apply the Euler-Euclid Theorems 1 and 2 : an even integer m is perfect if and only if

$$m = 2^{p-1}(2^p - 1) \le n,$$

where p and $2^p - 1$ are primes. Hence

$$2^{2p-2} < n,$$

and the number of possible choices for p is clearly less than $\log n$. Hence, for any positive constant $c_1 < \frac{1}{2}$,

(4.28) $$N' < \log n < n^{1/2 - c_1},$$

for all sufficiently large n.

We must now estimate the number of odd perfect numbers satisfying (4.27). By Theorem 3, we know that each odd perfect number m is of the form

$$m = q^{\alpha} k^2, \qquad q \equiv \alpha \equiv 1 \bmod 4,$$

where q is prime. A simple argument shows that to every k there is at most one q^{α} so that $q^{\alpha} k^2$ is perfect, for if this were not the case we should have

$$\frac{\sigma(q^{\alpha})}{q^{\alpha}} = \frac{\sigma(p^{\beta})}{p^{\beta}} = \frac{2k^2}{\sigma(k^2)}, \qquad (p, q) = 1.$$

For the first equality sign to hold we must have $q^{\alpha} \mid \sigma(q^{\alpha})$ and $p^{\beta} \mid \sigma(p^{\beta})$, i.e. q^{α} and p^{β} must be multi-perfect and of the same class λ, where λ is an integer not less than 2. Hence,

$$\frac{2k^2}{\sigma(k^2)} = \lambda \ge 2,$$

which is clearly impossible. Therefore, in counting the number of odd perfect numbers m it is only necessary to count the possible number of integers k, where $m = q^\alpha k^2$. (We note That Hornfeck's result $P(n) \ll n^{1/2}$ (c.f. P. 98) follows immediately)

As in Theorem 20 we apply Lemma 16 to eliminate those numbers m for which r_k, the quadratic part of k, is greater than n^{4c_2}, for some constant $c_2 > 0$, to be fixed later. Since $m \leq n$, we have $k^2 \leq n$, and it follows that the number of these odd perfect numbers is at most

(4.29) $$R(n^{1/2}, n^{4c_2}) < c_1 n^{1/2 - 2c_2} < n^{1/2 - c_2},$$

if n is sufficiently large.

We may also eliminate those numbers m for which

$$q^\alpha > n^{2c_2},$$

for, by (4.27), it follows that

(4.30) $$k < n^{1/2 - c_2}.$$

Finally, we have to estimate the number of odd perfect numbers m satisfying (4.27), and for which

(4.31) $$q^\alpha < n^{2c_2}, \quad r_k \leq n^{4c_2}.$$

We denote these numbers by $m_1, m_2, \ldots\ldots m_N$, and the corresponding integers k by $k_1, k_2, \ldots\ldots k_N$.

We find our upper bound for N by a method similar to that used in Theorem 20. We start by showing that the square-free part of k has a sufficiently large divisor d. Unfortunately, in this case we cannot obtain the simple result

$$d\sigma(d) \mid \sigma(k),$$

since in dealing with k (which is not perfect) rather than with m, we forego the condition $d \mid \sigma(k)$. Instead, we show that

(4.32) $$db \mid k,$$

where $b = (k, \sigma(d^2))$, and b is also sufficiently large, say $b > B_n$, where B_n is a function of n. Then the number of integers $k \le n^{1/2}$, such that $m = q^2k^2$ satisfies (4.27) and (4.31) is (for fixed d) clearly at most

$$\frac{n^{1/2}}{d} \sum_{\substack{b \mid \sigma(d^2) \\ b > B_n}} \frac{1}{b} < \frac{n^{1/2}}{d} \frac{\tau(\sigma(d^2))}{B_n} < \frac{n^{1/2 + 1/2\epsilon}}{B_n d},$$

for every $\epsilon > 0$, and sufficiently large n. Hence

$$N < \frac{n^{1/2 + 1/2\epsilon}}{B_n} \cdot \sum_{d < n} \frac{1}{d} < \frac{n^{1/2 + \epsilon}}{B_n} < n^{1/2 - c_2},$$

provided only that

(4.33) $$b > B_n > n^{c_2 + \epsilon}.$$

Our problem is now reduced to finding a sufficiently large divisor d of the square-free part of k, so that (4.32) holds, and also (4.33).

We simplify this problem by one further step. Since it is trivially true that

$$d \mid k,$$
and $$b \mid k,$$

we need only ensure that $(d, b) = 1$, or even $(d, \sigma(d^2)) = 1$, to prove (4.32), and we construct our divisor d of the square-free part of k with this end in view.

Let the prime decomposition of S_k be

$$S_k = p_1 p_2 \cdots p_r, \qquad (p_1 < p_2 < \cdots < p_r).$$

We choose a sub-sequence of the primes p_i, (4.34) $$p_{r_1} > p_{r_2} > \cdots\cdots > p_{r_\omega},$$

selected according to the following rule:

$$p_{r_1} = p_r,$$

$$p_{r_j} = \max_{p_i < p_{r_{j-1}}} p_i \quad \text{such that}$$

(4.35) $$p_{r_j} \nmid \sigma(p_{r_1}^2 \cdots\cdots p_{r_{j-1}}^2),$$

and

(4.36) $$p_{r_i} \nmid \sigma(p_{r_j}^2), \qquad 1 \le i < j < \omega.$$

We define $d = p_{r_1} p_{r_2} \cdots\cdots p_{r_\omega}$.

By our particular construction of d it follows that

$$p_{r_j} \nmid \sigma(p_{r_1}^2 \cdots\cdots p_{r_\omega}^2), \qquad 1 \le j \le \omega.$$

so that $(d, \sigma(d^2)) = 1$, and condition (4.32) is satisfied.

It now only remains to show that d and b are sufficiently large.

It follows from (4.35) and (4.36) that if $p \mid s_k$, but $p \nmid d$, then either (if (4.35) does not hold),

(4.37) $$p \mid \sigma(d^2),$$

or, (if (4.36) does not hold),

(4.38) $$p_{r_i} \mid \sigma(p^2),$$

for some $p_{r_i} > p$.

Now let

(4.39) $$s_k = dd'd'',$$

where d' is the product of all primes $p \mid s_k$, $p \nmid d$, satisfying (4.37). Clearly, by (4.37), and since d' is square-free,

(4.40) $$d' \mid \sigma(d^2).$$

Therefore, by Theorem 9 and (4.40),

(4.41) $$d' \leq \sigma(d^2) < 2d^2 \log\log n.$$

Also, by (4.38) and Lemma 30,

(4.42) $$d'' < d^2.$$

Now since $r_k \leq n^{4c_2}$, $m > \frac{1}{2}n$, $q^\alpha \leq n^{2c_2}$, it follows from (4.39), (4.41) and (4.42) that

$$\tfrac{1}{2}n^{1/2 - 5c_2} < s_k < 2d^5 \log\log n.$$

Hence,

(4.43) $$d > A_n,$$

where $A_n = \frac{1}{4}n^{1/10 - c_2}(\log\log n)^{-1}$.

We complete our proof by showing that $B_n > n^{c_2+\epsilon}$.

Since $q^\alpha k^2$ is perfect, and $(k,q) = (\frac{k}{d}, d) = 1$, we have

$$\sigma(q^\alpha k^2) = 2q^\alpha k^2 = 2q^\alpha \left(\tfrac{k}{d}\right)^2 d^2 = \sigma(q^\alpha)\,\sigma\left(\left(\tfrac{k}{d}\right)^2\right)\sigma(d^2).$$

But since d is square-free, and all the prime factors of d are odd, $\sigma(d^2)$ is also odd, and therefore

$$\sigma(d^2) \mid q^\alpha k^2.$$

Hence,

$$\frac{\sigma(d^2)}{q^\alpha} \leq (k^2, \sigma(d^2)),$$

and it follows from (4.31), (4.33), and our definition of b, that

$$b \geq (k^2, \sigma(d^2))^{1/2} \geq \left(\frac{\sigma(d^2)}{q^\alpha}\right)^{1/2} \geq \left(\frac{d^2}{q^\alpha}\right)^{1/2} > A_n n^{-c_2} > n^{c_2+\epsilon},$$

providing that c_2 is chosen to be sufficiently small. Therefore our final condition (4.33) will be satisfied with $B_n = A_n n^{-c_2}$, and the total number of odd perfect numbers $m \leq n$ is less than

$$3n^{1/2 - c_2}.$$

Therefore we can find a positive constant $c < \frac{1}{2}$, so that

$$P(n) < n^{1/2 - c},$$

if n is sufficiently large.

As with Theorem 20, the factor which limits our result here is the size of S_K. Clearly both Theorems 19 and 20, are far from being the best possible. Since it is probable that there are no odd perfect numbers, it is also probable that

$$P(n) < n^{\epsilon},$$

for every $\epsilon > 0$, since the number $P'(n)$ of even perfect numbers not exceeding n is at most

$$\log n < n^{\epsilon},$$

for every $\epsilon > 0$, providing n is sufficiently large. However, a result as good as this seems inaccessible at present.

CHAPTER V.

AMICABLE NUMBERS.

5.1. We have already defined an amicable pair of numbers $\{a,b\}$, $a<b$, as integers a and b for which

$$\sigma(a) = \sigma(b) = a+b. \tag{5.1}$$

In Chapter 11 we discussed the history of these number pairs, and showed what fascination they have held for both the mathematician and the non-mathmatician since the earliest days of the ancient Greek civilisation. It is therefore surprising to find that in comparison with the abundant numbers, and even with the perfect numbers, very little is known about the amicable pairs. There seems little hope, at the moment, of even showing that there are infinitely many of these pairs, though it seems likely that this is true. It has been conjectured, in fact, that the number of amicable numbers less than n is greater than $n^{1-\epsilon}$ for every $\epsilon>0$, and sufficiently large n .

The only certain result which is available at present concerns the density of the sequence of amicable numbers, that is to say, the density of the sequence of integers a for which there exists an integer b satisfying (5.1). In 1954 Kanold * showed that this density must be less than

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* H.J. KANOLD. Proceedings of the International Congress of Mathematicians Amsterdam. 1954. p. 30.

0.204, and a year later Erdös [*1] proved, by elementary means, that in fact it equals zero. This result forms the major theorem of this chapter.

5.2.

THEOREM 22. (Erdös).

The density of the amicable numbers is zero.

Proof. To prove this theorem we shall require two supplementary lemmas.

LEMMA 31. (Turan). [*2]

Let Q be a sequence of primes $q_1 < q_2 < \cdots$ satisfying $\sum_{i=1}^{\infty} \frac{1}{q_i} = \infty$, and denote by $\nu_q(m)$ the number of distinct primes q that divide an integer m. Then, for every $A > 0$, and for every $\delta > 0$, the number of integers $m \leq n$ with $\nu_q(m) < A$ is less than $\frac{1}{4}\delta n$, for all sufficiently large n.

Proof. We apply a special case of a theorem of Turan.

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*1 P. ERDÖS. On amicable numbers. Publicationes Mathematicae. 4. (1955) 108 - 111.

*2 P. TURAN. Uber Einige Verallgemeinerungen Eines Satzes Von Hardy and Ramanujan. Satz 1. J. Lond. Math. Soc. 11 (1936).

This asserts that if

(1) $0 \le \psi(p) \le K$, for all primes p ,

(2) $\sum_{p \le n} \frac{\psi(p)}{p} \to \infty$, as $n \to \infty$,

(3) $\psi(m) = \sum_{p \mid m} \psi(p)$,

Then, for all but $o(n)$ integers $m \le n$, we have

$$(5.2) \qquad \Big| \psi(m) - \sum_{p \le n} \frac{\psi(p)}{p} \Big| < \phi(n) \Big(\sum_{p \le n} \frac{\psi(p)}{p} \Big)^{1/2},$$

providing $\phi(n) \to \infty$, as $n \to \infty$.

In our case we take

$$\psi(p) = \begin{cases} 1 \ , & p \in Q \\ 0 \ , & \text{otherwise} . \end{cases}$$

Then $\psi(m) = \sum_{p \mid m} \psi(p) = \nu_q(m)$, and $\sum_{p \le n} \frac{\psi(p)}{p} = \sum_{q_i \le n} \frac{1}{q_i} \to \infty$.
Thus, by (5.2), with $\phi(n) = \big(\sum_{q_i \le n} \frac{1}{q_i} \big)^{1/4} \to \infty$, we have

$$\nu_q(m) > \sum_{q_i \le n} \frac{1}{q_i} - \Big(\sum_{q_i \le n} \frac{1}{q_i} \Big)^{3/4} > A ,$$

providing n is sufficiently large. Hence for every $\delta > 0$, the number of integers $m \le n$ with $\nu_q(m) < A$ is less than $\frac{1}{4}\delta n$, for all sufficiently large n . This proves Lemma 31.

LEMMA 32.

Define

$$\sigma_A(m) = \sum_{\substack{d \mid m \\ d \le A}} \frac{m}{d} .$$

Then for any given pair of positive numbers ϵ and η we have, except for at most $\frac{1}{4}\epsilon n$ of the integers $m \le n$, that

$$\sigma(m) - \sigma_A(m) < \eta m,$$

providing A is sufficiently large.

Proof. Clearly we have that

$$(5.3) \qquad \sum_{m\le n}\{\sigma(m)-\sigma_A(m)\} = \sum_{m\le n}\sum_{\substack{d|m\\ d>A}} \frac{m}{d} = \sum_{d_1>A}\sum_{d_2\le \frac{n}{d_1}} d_2 < \sum_{d>A}\left(\frac{n}{d}\right)^2 < \frac{n^2}{A}.$$

We now assume, on the contrary, that

$$\sigma(m) - \sigma_A(m) \geq \eta m,$$

for at least $\frac{1}{4}\epsilon n$ of the integers $m \le n$. Then

$$(5.4) \qquad \sum_{m\le n}\{\sigma(m)-\sigma_A(m)\} \geq \eta \sum_{m\le \frac{1}{4}\epsilon n} m > \eta\frac{\epsilon^2 n^2}{64}.$$

Hence, for $A > \frac{64}{\eta\epsilon^2}$, (5.4) contradicts (5.3) proving Lemma 4.3.

We now continue with the proof of Theorem 22.

We denote by $\{a_i, b_i\}$ the sequence of ordered pairs of amicable numbers $a_i < b_i$, and we shall show that the density of integers a_i for which there exists an integer b_i satisfying

$$\sigma(a_i) = \sigma(b_i) = a_i + b_i,$$

is 0.

Let A and B be positive integers which we shall fix later, $p_1 < p_2 < \dots < p_r$ be all the distinct primes $p \le A$, and Q_i, $(i=1,2,\dots r)$ the infinite sequences of primes $q_{i_1}, q_{i_2}, \dots$ satisfying

$$(5.5) \qquad q_i \equiv -1 \bmod p_i,$$

$$(5.6) \qquad q_i > Br.$$

First we show that, given any $\epsilon > 0$, the number of integers $m \le n$ which do not satisfy either of the conditions

(1) m is divisible by at least A of the primes in the sequence Q_i, for every $1 \le i \le r$,

(2) m is divisible by no q_i^2, for every $1 \le i \le r$,

is less than $\frac{1}{2}\epsilon n$, if n is sufficiently large. Then we show that number of amicable numbers not exceeding n which do satisfy the two given conditions, is also less than $\frac{1}{2}\epsilon n$. Since we may choose ϵ arbitrarily small it will follow that the density of the amicable numbers is 0.

We observe that, for fixed i, the primes q_{ij}, are in arithmetical progression.* Hence,

$$\sum_{j=1}^{\infty} \frac{1}{q_{ij}} = \infty, \qquad (i = 1, 2, \dots r).$$

Therefore the condition of Lemma 31 is satisfied by each sequence Q_i, $(i=1,2,\dots r)$. Hence, by Lemma 31 with $\delta = \frac{\epsilon}{r}$, the number N_1, of integers $m \le n$ divisible by fewer than A of the primes q_i, for some $1 \le i \le r$, is (for fixed i) at most

$$\frac{1}{4} \cdot \frac{\epsilon}{r} \cdot n,$$

* H.N. SHAPIRO. On primes in arithmetic progression. 11. Ann. of Math. 2. 52. (1950) 231 - 243.

and it follows that

(5.7) $$N_1 < \tfrac{1}{4}\epsilon n.$$

Also, the number N_2 of the integers $m \le n$ divisible by q_{ij}^2, for any j and $1 \le i \le r$, is, by (5.6), clearly less than

$$\sum_{i=1}^{r} \sum_{j=1}^{M} \frac{n}{q_{ij}^2} < \frac{n}{B}.$$

But we may choose $B > \frac{4}{\epsilon}$. Hence

(5.8) $$N_2 < \tfrac{1}{4}\epsilon n.$$

Therefore, by (5.7) and (5.8), the total number of integers $m \le n$ not satisfying (1) and (2) is less than $\frac{1}{2}\epsilon n$.

We estimate the number of integers a_i not exceeding n satisfying conditions (1) and (2), by showing that, except for at most $\frac{1}{4}\epsilon n$ exceptions, for these numbers we have

(5.9) $$2 < \frac{\sigma(a_i)}{a_i} < 2 + \eta.$$

Then, with the notation of Lemma 29, the number of amicable numbers $a \le n$ does not exceed

$$N(2,\eta) + \tfrac{1}{4}\epsilon n.$$

Since, by Lemma 2.1 with $\delta = \frac{1}{4}\epsilon$, we have $N(2,\eta) < \frac{1}{4}\epsilon n$, our result will then follow.

Since $\sigma(q_{ij}) = 1 + q_{ij} \equiv 0 \bmod p_i$, it follows that for any amicable number a satisfying (1) we have, by (2),

$$\sigma(a) \equiv 0 \bmod p^A,$$

for every prime $p \le A$.

We observe that if $p^{\alpha} < A$, then $p < A$ and $\alpha < A$. Hence, if $d|a$, and $d < A$,

$$\sigma(a) \equiv 0 \bmod d,$$

and

(5.10) $$b \equiv \sigma(a) - a \equiv 0 \bmod d,$$

i.e. every divisor $d_k \leq A$ of a also divides b, $(k = 1, 2, \cdots z)$. Denote by d' the L.C.M. of $d_1, d_2, \ldots\ldots d_z$. Then trivially

$$\sigma_A(a) = \sigma(d'),$$

and

$$\frac{\sigma_A(a)}{a} \leq \frac{\sigma(d')}{d'}.$$

But since $d'|b$, by Lemma 23, we have

(5.11) $$\frac{\sigma(b)}{b} \geq \frac{\sigma(d')}{d'} \geq \frac{\sigma_A(a)}{a}.$$

It follows from Lemma 32 that we may choose A sufficiently large so that for any given η we have that

(5.12) $$\frac{\sigma_A(a)}{a} \geq \frac{\sigma(a)}{a} - \eta,$$

except for at most $\frac{1}{4}\epsilon n$ of the amicable numbers a not exceeding n. Hence, by (5.11) and (5.12),

$$\frac{\sigma(b)}{b} \geq \frac{\sigma(a)}{a} - \eta.$$

But $\sigma(a) = \sigma(b) = a + b$. Therefore

$$\eta \geq \frac{\sigma(a)}{a} - \frac{\sigma(b)}{b} = \frac{b}{a} - \frac{a}{b}, \qquad (a < b),$$

i.e. $$1 < \frac{b}{a} < 1 + \eta,$$

and

$$2 < \frac{\sigma(a)}{a} < 2 + \eta.$$

Then, by Lemma 29, with $\delta = \frac{1}{4}\epsilon$, we may choose η sufficiently small so that $N(2,\eta) < \frac{1}{4}\epsilon n$, and the number of amicable numbers a less than or equal to n is equal to ϵn. Since ϵ may be chosen arbitrarily small, it follows that the density of the amicable numbers is zero.

This completes the proof of Theorem 22.

It is clear from Theorem 22 that if $B(n)$ denotes the number of integers a_i not exceeding n for which there exists an integer b_i satisfying (5.1), then

$$B(n) = o(n).$$

However, a better estimate for the order of magnitude for the counting number of the sequence a_i seems inaccessible at the moment, and perhaps it is along these lines that future research into the numbers related to the σ – function will proceed.

BIBLIOGRAPHY

The following list of papers contain work directly incorporated in this dissertation:-

L. E. DICKSON. History of the Theory of Numbers.
Vol.1. Chapters 1 and XV11.

E. A. INGHAM. Distribution of Prime Numbers.
Cambridge Tract. No.30. Theorems 3,4,7,23.

E. B. ESCOTT. Amicable Numbers.
Scripta Math. 12 (1946) 61 - 72.

S. WIGERT. Sur quelques fonctions arithmetiques.
Acta Math. 37 (1913) 113 - 123

H. DAVENPORT. A divisor problem.
Quart. Journ. of Math (Oxford) 1949. 37 - 44.

P. ERDÖS. Remarks on Number Theory 11.
Acta Arithmetica.V.(1959) 171 - 177

Note on Consecutive abundant numbers.
J. Lond. Math.Soc.10. (1935) 128 - 131.

On The density of the abundant numbers.
J. Lond. Math.Soc. 9.(1934) 278 - 282.

On primitive abundant numbers.
J. Lond. Math.Soc. 9.(1935) 49 - 58.

Remarks on Number Theory 1.
Acta Arithmetica V. 1958. 25 - 33.

On perfect and multi-perfect numbers.
Annali di Matematica 42 (1956) 253 - 258.

On amicable numbers.
Publicationes Mathematica 4.(1955) 108-111.

P. TURAN. Uber Einige Verallgemeingrungen Eines Satzes
Von Hardy and Ramanujan. Satz.1.
J. Lond.Math.Soc.11. (1936) 125 - 132.

Reference is also made to the following:-

S. MITSUI. SCI.Papers. Unif.Tokyo. 6 (1956) 1 - 11.

J. J. SYLVESTER. Comptes Rendus Paris 106 (1888) 448-450.

H. J. KANOLD. J. Reine.Agnew Math.1941,1942,1944,1950,1953,1955.

J. TOUCHARD. Scripta Math. 19 (1953) 35 - 39.

P. McCARTHY. Amer.Math.Monthly 64 (1957) 257-258.

P. POULET. La Chasse Aux Nombres. 2 Vols.Brussels.

A. L. BROWN. Scripta Math. 20 (1954) 103-106.

L. E. DICKSON. Amer. J. Math. 35 (1913) 413-426.

A. WALFISZ. Math. Zeits. 26 (1927) 66 - 88.
Travaux.Inst.Math.Tblissi 5.(1938) 181-196.

H. SALIÉ. Math.Nachr. 14 (1955) 39 - 46.

F. BEHREND. Berlin Akad.Sitzung (1933) 280 - 293.

P. ERDÖS. Bull. Amer.Math.Soc.(1948) 685 - 692.

B. HORNFECK. Archiv. dev. Math.6 (1955) 442 - 443.

H.N.SHAPIRO~~HN~~. Ann. of Math.2.52 (1950) 231 - 243.