FUNDAMENTAL SEQUENCES IN THE SECOND AND THIRD ORDINAL NUMBER CLASSES

a thesis by

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ABSTRACT

The text is concerned with the definition and investigation of some classes of ordinal valued-functions; and with the determining of a particular assignment function Ω. This function assigns to each limit ordinal α smaller than a constant θ_0 and belonging to the second number class a fundamental sequence $\{\Omega_\alpha(n)\}_{n<\omega}$; that is, a strictly increasing sequence satisfying $\alpha = \lim_{n<\omega} \Omega_\alpha(n)$.

The operations used to determine Ω are derived from a class $\mathfrak{D}^{(2)}$ of functions $\mathfrak{D}_\alpha : \omega_1^{(2)} \to \omega_1$, $\alpha < \omega_1$. Since $\mathfrak{D}^{(2)}$ generalizes the standard ordinal arithmetic operations, some of the properties of the $\mathfrak{D}_\alpha$ are studied, together with the operations Γ_α which are generalizations of transfinite sum and product. Also investigated is a related class $\mathcal{D}_\Omega^{(1)}$ of number- theoretic functions. in this context Ω is an arbitrary assignment bounded by ω_1 in place of $\theta_0 < \omega_1$.

It is proved that the functions $\mathfrak{D}_\alpha$ are normal in the second argument, and these functions are compared with a hierarchy of normal functions obtained by Veblen's process of iteration. The $\mathfrak{D}_\alpha$ provide a natural means of extending the notion of epsilon number, and some of the properties of the generalized epsilon numbers are presented.

The function of α $\Phi_\alpha^2(\omega)$ is normal, and the notation $\{\theta_\mu\}_{\mu<\omega_1}$ is adopted for the sequence of countable fixed points of the function. The generalized epsilon numbers determine a hierarchical classification of limit numbers $<\theta_0$,and on this basis a normal form is determined for each, and thence the function Ω is defined by transfinite recursion.

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§1. INTRODUCTION.

ORDER TYPES AND ORDINAL NUMBERS

A binary relation $<$ on a set S is an ordering relation on S iff

(i) $(\forall x, y, z \in S)\ [x < y \ \&\ y < z \Rightarrow x < z]$

(transitivity)

(ii) $(\forall x, y \in S)\ [x \neq y \Rightarrow x < y \vee y < x]$

(Connectivity)

(iii) $(\forall x, y \in S)\ [x < y \Rightarrow \neg\, y < x]$

(asymmetry)

$(S, <)$ denotes the set S ordered by $<$.

The relation $<$ is a well-ordering on the set S iff the following condition is also satisfied:

(iv) $(\forall T \subseteq S)_{T \neq \emptyset} (\exists x \in T)(\forall y \in T)[y \neq x \Rightarrow x < y]$

Two ordered sets $(S, <)$, $(S', <')$ are said to be <u>similar</u> if there exists a bijective function f such that

i) $f : S \rightarrow S'$

ii) $(\forall x, y \in S)[x < y \Rightarrow f(x) <' f(y)]$

An order type is the class of ordered sets similar to a given set. An ordinal number is an order type of well-ordered sets.

Ordinal numbers are characterized in set theory by regarding them as certain canonical sets x satisfying:

(i) $(\forall y)[y \in x \Rightarrow y \subseteq x]$ or equivalently $(\forall y, z)[y \in x \;\&\; z \in y \Rightarrow z \in x]$. This property of x is written $\mathrm{Trans}(x)$.

(ii) $(\forall y)[y \in x \Rightarrow \mathrm{Trans}(y)]$

These sets are each well-ordered by the membership relation $\in$, and constitute a sequence well ordered by $\in$. (c.f. Schoenfield [17], p 246).

Zero is the smallest ordinal number, viz. the empty set $\emptyset$. Suppose S is a canonical set determining the ordinal α. Then the set $\{x \mid x = S \vee x \in S\}$ is the canonical set determining the least ordinal greater than α, and this number is written $\alpha + 1$, and is called a successor number, or the successor of α. Non-zero ordinals which are not successors all called limit numbers.

The finite numbers $0, 1, 2, \ldots$ start the sequence of ordinal numbers.

The least transfinite ordinal is denoted by the symbol 'ω', or by 'ω_0'.

Some of the notations adopted in this text for ordinals are as follows:

$$(\forall \alpha)\ \mathrm{Suc}(\alpha) \Leftrightarrow_{Df} 0 < \alpha \ \&\ (\exists \beta < \alpha)\ \alpha = \beta + 1$$

$$\mathrm{Lim}(\alpha) \Leftrightarrow 0 < \alpha \ \&\ (\forall \beta < \alpha)\ \beta + 1 < \alpha$$

The notation LIM (α) is also used as an alternative to Lim (α). Furthermore, Lim $(\alpha, \beta, \ldots)$, Suc $(\alpha, \beta, \ldots)$ are used to denote Lim (α) & Lim (β) & ..., and Suc (α) & Suc (β) & ... respectively.

For Lim (α) $(\forall m, n < \omega)$ $P^m(\alpha + n) =_{Df} \alpha + (n \dot{-} m)$,

where $n \dot{-} m =_{Df} \begin{cases} n - m & \text{if } m < n \\ 0 & \text{otherwise} \end{cases}$

Also, for each β, $P^1(\beta) =_{Df} P(\beta)$.

SEGMENTS AND NUMBER CLASSES

W denotes the class of ordinal numbers.

For each ordinal α, W_α denotes the set of ordinals smaller than α. This set is called a <u>segment</u> (of type α) of W.

N denotes the set consisting of zero and the natural numbers, and is called the <u>first number class</u>. $Z(\aleph_0)$ denotes the set of denumerable ordinals (that is, ordinals α for which there exist a bijection from W_α onto N). $Z(\aleph_0)$ is called the <u>second number class</u>.

For each ordinal $\mu > 0$, $Z(\aleph_\mu)$ denotes the set of ordinals of cardinality or power equal to $\aleph_\mu$, and ω_μ denotes the first ordinal of $Z(\aleph_\mu)$. $Z(\aleph_\mu)$ is called the $(2+\mu)^{-th}$ number class.

SEQUENCES AND LIMITS

Five classes of ordinal sequences are characterized here. Suppose $\{\alpha_\xi\}_{\xi<\lambda}$ is a sequence of ordinals

$\alpha_0, \alpha_1, \ldots, \alpha_\xi, \alpha_{\xi+1}, \cdots$ of limit type λ. Then

1. $\{\alpha_\xi\}_{\xi<\lambda}$ is <u>strictly increasing</u> if
   $$(\forall \xi', \xi < \lambda)\ \xi' < \xi \Rightarrow \alpha_{\xi'} < \alpha_\xi$$
2. $\{\alpha_\xi\}_{\xi<\lambda}$ is <u>internally unbounded</u> if
   $$(\forall \xi' < \lambda)(\exists \xi < \lambda)\ \xi' < \xi \ \&\ \alpha_{\xi'} < \alpha_\xi$$
3. $\{\alpha_\xi\}_{\xi<\lambda}$ is <u>internally bounded</u> if
   $$(\exists \xi' < \lambda)(\forall \xi < \lambda)\ \xi' < \xi \Rightarrow \alpha_\xi \leq \alpha_{\xi'}$$

4. $\{\alpha_\xi\}_{\xi<\lambda}$ is <u>non-decreasing</u> if

$$(\forall \xi', \xi < \lambda) \quad \xi' < \xi \Rightarrow \alpha_{\xi'} \leq \alpha_\xi$$

5. $\{\alpha_\xi\}_{\xi<\lambda}$ is <u>ultimately increasing</u> if it is non-decreasing and is internally unbounded.

Now suppose $\{\alpha_\xi\}_{\xi<\lambda}$ is an arbitrary sequence of limit type λ.

DEFINITION 1.1. (Rotman & Kneebone, [16], p. 87)

The sequence $\{\alpha_\xi\}_{\xi<\lambda}$ is said to have an ordinal α as a <u>limit</u> if

$$(\forall \beta < \alpha)(\exists \mu < \lambda)(\forall \xi < \lambda)_{\xi > \mu} \quad \beta < \alpha_\xi \leqslant \alpha,$$

and this property of α is expressed by

$$\alpha = \lim_{\xi<\lambda} \alpha_\xi .$$

□ 1.1

Symbols $\vec{\alpha}, \vec{\beta}, \ldots$ are used to denote finite sequences of ordinals which arise as function parameters.

LEAST NUMBER OPERATION μ, AND λ-NOTATION.

Suppose P is a unary predicate defined on W. Then

$$(\mu\beta)[P(\beta)] =_{Df} \begin{cases} \text{the least } \beta \text{ such that } P(\beta), \\ \text{if there is such a } \beta, \text{ or} \\ 0 \ \text{ otherwise.} \end{cases}$$

In particular, for all ordinals α, β such that $\alpha \geqslant \beta$, $\quad \alpha - \beta =_{Df} (\mu\gamma)[\alpha = \beta + \gamma]$.

The λ-notation adopted is as follows:

Suppose $f : A_1 \times A_2 \times \ldots \times A_n \to B$, for some n such that $0 < n < \omega$.

Suppose $(\forall i \leqslant n)_{i>0}\ \alpha_i$ is any element of A_i.

Then $(\forall m \leqslant n)_{>0} (\forall a_1, a_2, \ldots, a_m)_{\leqslant n}\ a_1 < a_2 < \ldots < a_m \Rightarrow$

$\lambda \alpha_{a_1}, \alpha_{a_2}, \ldots, \alpha_{a_m} \cdot f(\vec{\beta}_1, \alpha_{a_1}, \vec{\beta}_2, \alpha_{a_2}, \vec{\beta}_3, \ldots, \vec{\beta}_m, \alpha_{a_m}, \vec{\beta}_{m+1})$

denotes the function g of m arguments obtained from the function f of n arguments by fixing the values of arguments

$\alpha_1, \alpha_2, \ldots, \alpha_{a_1 - 1}$ of f with the values $\vec{\beta}_1$, when $1 < a_1$
$\alpha_{a_1+1}, \alpha_{a_1+2}, \ldots, \alpha_{a_2 - 1}$ of f with the values $\vec{\beta}_2$, when $a_1 + 1 < a_2$
$\vdots$
$\alpha_{a_m+1}, \alpha_{a_m+2}, \ldots, \alpha_n$ of f with the values $\vec{\beta}_{m+1}$, when $a_m < n$

The parameter strings $\vec{\beta}$ are empty in those cases where the inequalities are not satisfied.

Thus $g : A_{a_1} \times A_{a_2} \times \ldots \times A_{a_m} \to B$.

In particular, for $a_i = i$, for each i such that $1 \leq i \leq n$,

$$g = f = \lambda \alpha_1, \alpha_2, \ldots, \alpha_n . f(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Although the symbols 'μ', 'λ' are also used to denote ordinals, this is only in contexts where confusion with the least number operator or with the λ-notation symbol will not arise.

FUNCTIONS

A function $f : W \to W$ is *non-decreasing* if $(\forall \alpha'; \alpha)$ $\alpha' < \alpha \Rightarrow f(\alpha') \leq f(\alpha)$. f is *strictly increasing* if $(\forall \alpha'; \alpha)$ $\alpha' < \alpha \Rightarrow f(\alpha') < f(\alpha)$. f is *continuous* if for every strictly increasing sequence $\{\alpha_\xi\}_{\xi < \lambda}$ of limit type

$$f(\lim_{\xi < \lambda} \alpha_\xi) = \lim_{\xi < \lambda} f(\alpha_\xi).$$

f is *normal* if f is both strictly increasing and continuous. An ordinal α is a *fixed point* of f if $f(\alpha) = \alpha$.

LEMMA 1.2 If f is a normal function, then $(\forall \alpha)$ $\alpha \leq f(\alpha)$.

PROOF. Suppose otherwise, such that β is the least ordinal such that $f(\beta) < \beta$. Now $f(0) \geq 0$, therefore $\beta > 0$ and $(\forall \gamma < \beta)\ \gamma \leq f(\gamma)$.

If $Suc(\beta)$, then $f(P(\beta)) < f(\beta)$ as f is strictly increasing, therefore

$$f(\beta) \geq P(\beta) + 1 \text{ by hypothesis}$$
$$= \beta .$$

If $Lim(\beta)$, then

$$f(\beta) = \lim_{\beta' < \beta} f(\beta'), \text{ as } f \text{ is continuous}$$
$$> f(\beta') \text{ for each } \beta' < \beta, \text{ as } f \text{ is strictly increasing}$$
$$\geq \beta' \text{ by hypothesis .}$$

Therefore $f(\beta) \geq \beta$, which contradicts hypothesis $\square$ 1.2.

Transfinite iteration of any normal function f is defined as follows:

$$(\forall \alpha)\Big[f^1(\alpha) =_{Df} f(\alpha)\ \&\ (\forall \beta)_{>1}\big[[Suc(\beta) \Rightarrow f^{\beta}(\alpha) = f\{f^{P(\beta)}(\alpha)\}]\ \&\ [Lim(\beta) \Rightarrow f^{\beta}(\alpha) = \lim_{\beta' < \beta} f^{(\beta')}(\alpha)]\big]\Big]$$

THEOREM 1.3. If f is a normal function, then
a) the least fixed point of f is $f^{\omega}(0)$
b) $(\forall \beta)(\exists \alpha)[\beta < \alpha\ \&\ f(\alpha) = \alpha]$

PROOF a)

$$f(f^{\omega}(0)) = f\{\lim_{n<\omega} f^{n}(0)\}$$

$$= \lim_{n<\omega} ff^{n}(0)$$

$$= \lim_{n<\omega} f^{n+1}(0)$$

$= f^{(\omega)}(0)$, therefore $f^{\omega}(0)$ is a fixed point of f. Suppose $\beta < f^{\omega}(0)$ is a fixed point of f. Then if $f(0) = 0$, $f^{\omega}(0) = 0$, therefore $\beta \neq f^{\omega}(0)$. Thus suppose $f(0) > 0$. Therefore $f(0) \neq \beta$, since f is strictly increasing, and so $f(f(0)) = f^{2}(0) > f(0)$, and as β belongs to f[domain of f], $f(0) \leq \beta$, therefore $f(0) < \beta$. And if $f^{n}(0) < \beta$, $n > 1$, then $f(f^{n}(0)) = f^{n+1}(0) > f^{n}(0)$ therefore $f^{n+1}(0) = f(f^{n}(0)) < f(f^{n+1}(0)) = f^{n+2}(0)$, and therefore $f^{n+1}(0) \neq \beta$, and as $\beta \in f[\text{dom} f]$, by hypothesis $f(f^{n}(0)) = f^{n+1}(0) \leq \beta$, therefore $f^{n+1}(0) < \beta$, and so by induction $(\forall n < \omega) f^{n}(0) < \beta$, therefore $f^{\omega}(0) \leq \beta$, which contradicts hypothesis.

b) For some ordinal β, take $\alpha = f^{\omega}(\beta+1)$
$\geqslant \beta+1 > \beta$.

$$\text{Then } f(\alpha) = f(f^{\omega}(\beta+1))$$
$$= f\{\lim_{n<\omega} f^{n}(\beta+1)\}$$
$$= \lim_{n<\omega} f^{n+1}(\beta+1) = \alpha.$$

□THEOREM 1.3

Note that α is the least fixed point of f such that $\alpha > \beta$. For suppose not, and let there be a γ such that $\beta+1 \leqslant \gamma < \alpha$ and $f(\gamma) = \gamma$. Then $f(\beta+1) \leqslant \gamma$, and $(\forall n < \omega)$ if $f^{n}(\beta+1) \leqslant \gamma$, then $f^{n+1}(\beta+1) \leqslant \gamma$, so $(\forall n<\omega)$ $f^{n}(\beta+1) \leqslant \gamma$, therefore $\alpha = f^{\omega}(\beta+1) \leqslant \gamma$ which contradicts the hypothesis.

LEMMA 1.4. If f is normal, then the fixed points of f form a set closed under the operation of taking limits.

PROOF. For some limit number μ, let $\{\alpha_{\xi}\}_{\xi<\mu}$ be a strictly increasing sequence of fixed points of a normal function f. Then

$$f(\lim_{\xi<\mu} \alpha_{\xi}) = \lim_{\xi<\mu} f(\alpha_{\xi}) = \lim_{\xi<\mu} \alpha_{\xi} \quad \square\ 1.4$$

DEFINITION 1.5. The operation 'F' upon normal functions f is defined by $f^F(\alpha) = \alpha$-th fixed point of f. Thus by theorem 1.3:

$$f^F(0) = f^\omega(0)$$

$$f^F(\alpha+1) = f^\omega(f^F(\alpha)+1), \quad \text{and } \mathrm{Lim}(\alpha) \Rightarrow$$

$$f^F(\alpha) = \lim_{\alpha'<\alpha} f^F(\alpha').$$

Thus f^F is continuous, and if $\beta < \alpha$, then $f^F(\beta) < f^F(\alpha)$, So f^F is normal. (c.f. Veblen, [20], p 285).

THEOREM 1.5a. Suppose f, g are normal functions. Then $f \circ g$ is normal.

PROOF. Suppose $\alpha' < \alpha$. Then $g(\alpha') < g(\alpha)$, therefore $f(g(\alpha')) < f(g(\alpha))$. Now suppose $\mathrm{Lim}(\alpha)$. Then $fg(\alpha) = f\{\lim_{\alpha'<\alpha} g(\alpha')\}$

$$= \lim_{\alpha'<\alpha} fg(\alpha')$$

$\square$ 1.5a

ELEMENTARY FUNCTIONS.

The class $\mathcal{E}$ of elementary functions on the ordinals $< \omega_1$ is now defined. $\mathcal{E}$ contains many of the simpler operations found in ordinal arithmetic, and functions taken from this class are used to form an initial basis for the equations of nested course-of-values transfinite recursion called Schemata R.

DEFINITION 1.561

The class $\mathcal{E}$ is the smallest class satisfying:

a. Initial functions:

$\lambda\alpha.\ S'\alpha,\ P\alpha,\ I\alpha,\ Z\alpha \in \mathcal{E}$ (Successor, predecessor, identity and zero functions)

$\lambda\alpha_0,\alpha_1.\ U_0(\alpha_0,\alpha_1),\ U_1(\alpha_0,\alpha_1),\ \alpha_0\cdot\alpha_1,\ \alpha_0^{\alpha_1} \in \mathcal{E}$
(First and second projection functions, multiplication and exponentiation)

b. Closure:

I Superposition:

Suppose $\lambda\vec{\alpha}.\ \Psi(\alpha_0,\ldots,\alpha_i,\alpha_{i+1},\alpha_{i+2},\ldots,\alpha_n)$ and $\lambda\vec{\beta}.\ \Phi(\beta_0,\ldots,\beta_m) \in \mathcal{E}$

Then $\lambda\alpha_0,\ldots,\alpha_i,\vec{\beta},\alpha_{i+2},\ldots,\alpha_n.$

$$\Psi(\alpha_0,\ldots,\alpha_i,\Phi(\vec{\beta}),\alpha_{i+2},\ldots,\alpha_n) \in \mathcal{E}.$$

II Bounded sum and product:

Suppose $\lambda\beta,\vec{\alpha}.\ \Psi(\beta,\vec{\alpha}) \in \mathcal{E}$. Then each of the functions $\Psi^{(1)}(\beta,\vec{\alpha}) = \sum_{\beta'<\beta} \Psi(\beta',\vec{\alpha})$

$$\Psi^{(2)}(\beta,\vec{\alpha}) = \prod_{\beta'<\beta} \Psi(\beta',\vec{\alpha})$$

belongs to $\mathcal{E}$.

(The operations $\sum$ and $\prod$ are defined on page ,
and in Bachmann, [4], pp 53, 56).

SCHEMATA R FOR NESTED TRANSFINITE COURSE-OF-VALUES RECURSION

The schemata provide the definition for certain ordinal valued functions of finitely many (>2) arguments, and are used in §3 to obtain the higher arithmetic operations on the ordinal numbers $<\omega_1$. The functions of schemata R are defined inductively. The recursive scheme for functions Υ of three arguments is given, then for some integer $n \geq 3$, the recursive scheme for functions $\boxed{\vdash\!\dashv}$ of $n+1$ arguments is defined, upon the basis of a function $\boxed{-}$, which is itself defined from the recursive scheme for functions of n arguments.

DEFINITION 1.5c

Suppose the function Υ of three arguments is defined by: $(\forall \alpha_1, \alpha_2, \alpha_3)_{<\omega_1}$

$$\Upsilon(i, \alpha_1, \alpha_2) = \mu_i(\alpha_1, \alpha_2), \; i \leq 2, \; \mu \in \mathcal{E}$$

and for $\alpha_0 \geqslant 3$:

$$\Upsilon(\alpha_0, 0, \alpha_2) = \tau, \text{ for some } \tau < \omega_1$$

$$\Upsilon(\alpha_0, 1, \alpha_2) = \alpha_2$$

$$\Upsilon(\alpha_0, 2, \alpha_2) = ON^{(1)}(\{\Upsilon(\alpha', \alpha_2, \alpha_2)\}_{\alpha' < \alpha_0})$$

and for $\alpha_0 = 3$ and $\alpha_1 \geqslant 3$:

$$\Upsilon(3, \alpha_1, \alpha_2) =$$

$$ON^{(2)}(\{\Upsilon(n, \alpha_1, ON^{(3)}(\{\Upsilon(3, \alpha, \alpha_2)\}_{\alpha < \alpha_1}))\}_{n < 3})$$

and for $\alpha_0 \geqslant 4, \alpha_1 \geqslant 3$:

$$\Upsilon(\alpha_0, \alpha_1, \alpha_2) =$$

$$ON^{(5)}(\{\Upsilon(\alpha, ON^{(4)}(\{\Upsilon(\alpha_0, \alpha', \alpha_2)\}_{\alpha' < \alpha_1}), \alpha_2)\}_{\alpha < \alpha_0})$$

where $ON^{(j)}$, $0 < j < 6$, are given operations, defined on each sequence of countable ordinals $\{\alpha_\xi\}_{\xi < \mu}$, for $\mu < \omega_1$, and such that for each such sequence $ON^{(j)}(\{\alpha_\xi\}_{\xi < \mu}) < \omega_1$. (In many instances, these operations are derived from the operation of limit, and in some cases reduce to: $ON^{(j)}(\{\alpha_\xi\}_{\xi < \mu}) = \lim_{\xi < \mu} \alpha_\xi$.)

Then the function Υ is said to be defined by schemata R from the functions μ_i, $i \leq 2$, and from the constant τ, and from the operations $ON^{(j)}$, $0 < j < 6$.

Now let $\boxminus$ be a function of n ordinal arguments, where $3 \leq n < \omega$, which is defined by schemata R from certain functions. Suppose the function $\boxed{H}$ of $n+1$ arguments is defined by: $(\forall \alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \gamma)_{<\omega}$,

(i) $\boxed{H}(0, \alpha_1, \ldots, \alpha_{n-1}, \gamma) = \boxminus(\alpha_1, \ldots, \alpha_{n-1}, \gamma)$

(ii) for $\alpha_0 > 0$ $(\forall i < n-1)$

$$\boxed{H}(\alpha_0, \ldots, \alpha_i, 0, \alpha_{i+2}, \ldots, \alpha_{n-1}, \gamma) = 1$$

$$\boxed{H}(\alpha_0, \ldots, \alpha_i, 1, \alpha_{i+2}, \ldots, \alpha_{n-1}, \gamma) = \gamma$$

$$\boxed{H}(\alpha_0, \ldots, \alpha_i, 2, \alpha_{i+2}, \ldots, \alpha_{n-1}, \gamma) =$$

$$ON'_i(\{\boxed{H}(\alpha_0, \ldots, \alpha_{i-1}, \alpha, \alpha_{i+2}, \alpha_{i+2}, \alpha_{i+3}, \ldots \ldots, \alpha_{n-1}, \gamma)\}_{\alpha < \alpha_i})$$

where $\alpha_{i+2} =_{df} \gamma$, when $i = n-2$, and where ON'_i are given operations satisfying the conditions specified in the definition of the operations $ON^{(j)}$ above

(iii) $\boxed{H}(\alpha_0, \ldots, \alpha_{n-1}, 0) = \boxed{H}(\alpha_0, \ldots, \alpha_{n-1}, 1) = 1$

(iv) This final inductive clause is given in equations containing function symbols Λ_i, $i \leq n-1$, from which the parameters have been omitted:

for $\alpha_0 > 0$, $\alpha_i > 2$ for $0 < i \leq n-1$, $\gamma > 1$:

$$\boxed{H}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \gamma) = \Lambda_{n-1}, \text{ where}$$

$$\Lambda_0 = ON_0(\{\boxed{H}(\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha, \gamma)\}_{\alpha < \alpha_{n-1}})$$

and for all $i < n-1$

$$\Lambda_{i+1} = ON_{i+1}(\{\boxed{H}(\alpha_0, \alpha_1, \ldots, \alpha_{n-i-3}, \alpha, \Lambda_i, \alpha_{n-i}, \ldots, \alpha_{n-1}, \gamma)\}_{\alpha < \alpha_{n-i-2}})$$

and where ON_i, $i < n$, are operations satisfying the conditions specified in the definition of the operations $ON^{(j)}$ above, and where α_{n-i-3} = the empty sequence, when $i = n-2$.

Then $\boxed{H}$ is said to be defined by schemata R from the function $\boxminus$ and from the operations ON_i', $0 < i < n$; ON_i, $i < n$.

□ DEFINITION 1.5c.

Schemata R are based on a generalization of the Ackermann number theoretic function,(1), and this has determined the choice of initial and inductive clauses.

THEOREM 1.5d. Suppose the function $\boxed{H}$ is defined by schemata R. Then $\boxed{H}$ is uniquely defined.

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(1) See appendix I.

PROOF. This is by induction on the number of arguments of $\boxminus$.

Firstly, there cannot be more than one such function $\boxminus$ of three arguments. For suppose $\boxminus^{(1)}, \boxminus^{(2)}$ are such distinct functions, satisfying identical equations of schemata R. Then $(\forall \alpha_1, \alpha_2)$

$$\boxminus^{(1)}(i, \alpha_1, \alpha_2) = \boxminus^{(2)}(i, \alpha_1, \alpha_2), \; i \leqslant 2 .$$

Let α_0 be the least number > 2 for which $(\exists \alpha_1, \alpha_2)$

$\boxminus^{(1)}(\alpha_0, \alpha_1, \alpha_2) \neq \boxminus^{(2)}(\alpha_0, \alpha_1, \alpha_2)$. Thus $(\forall \alpha_0' < \alpha_0)$

$\boxminus^{(1)}(\alpha_0', \alpha_1, \alpha_2) = \boxminus^{(2)}(\alpha_0', \alpha_1, \alpha_2)$.

From the definition of schemata R, the two functions clearly coincide at $\alpha_1 = 0, 1$.

Suppose $\alpha_1 = 2$. Then

$$\begin{aligned} \boxminus^{(1)}(\alpha_0, 2, \alpha_2) &= ON^{(1)}(\{\boxminus^{(1)}(\alpha_0', 2, \alpha_2)\}_{\alpha_0' < \alpha_0}) \\ &= ON^{(1)}(\{\boxminus^{(2)}(\alpha_0', 2, \alpha_2)\}_{\alpha_0' < \alpha_0}) \\ &= \boxminus^{(2)}(\alpha_0, 2, \alpha_2). \end{aligned}$$

Thus, $\alpha_1 \neq 2$.

Suppose $\alpha_0 = 3$.

Then let α_1 be the least number greater than 2 and such that

$$\boxed{H}^{(1)}(3, \alpha_1, \alpha_2) \neq \boxed{H}^{(2)}(3, \alpha_1, \alpha_2).$$

$$\begin{aligned}\boxed{H}^{(1)}(3, \alpha_1, \alpha_2) &= ON^{(2)}(\{\boxed{H}^{(1)}(n, \alpha_1, ON^{(3)}(\{\\ &\qquad \boxed{H}^{(1)}(3, \alpha_1', \alpha_2)\}_{\alpha_1' < \alpha_1}))\}_{n<3})\\ &= ON^{(2)}(\{\boxed{H}^{(2)}(n, \alpha_1, ON^{(3)}(\{\\ &\qquad \boxed{H}^{(2)}(3, \alpha_1', \alpha_2)\}_{\alpha_1' < \alpha_1}))\}_{n<3})\\ &= \boxed{H}^{(2)}(3, \alpha_1, \alpha_2).\end{aligned}$$

Thus $\alpha_0 \neq 3$.

Then suppose $\alpha_0 > 3$. Then, as before, $\alpha_1 \neq 2$.

Let α_1 be the smallest number such that

$$\boxed{H}^{(1)}(\alpha_0, \alpha_1, \alpha_2) \neq \boxed{H}^{(2)}(\alpha_0, \alpha_1, \alpha_2).$$

Then $\boxed{H}^{(1)}(\alpha_0, \alpha_1, \alpha_2) =$

$$ON^{(5)}(\{\boxed{H}^{(1)}(\alpha_0', ON^{(4)}(\{\boxed{H}^{(1)}(\alpha_0, \alpha_1', \alpha_2)\}_{\alpha_1' < \alpha_1}),$$
$$\alpha_2)\}_{\alpha_0' < \alpha_0}) =$$

$$= ON^{(5)}(\{\boxed{H}^{(1)}(\alpha_0', ON^{(4)}(\{\boxed{H}^{(2)}(\alpha_0, \alpha_1', \alpha_2)\}_{\alpha_1' < \alpha_1}), \alpha_2)\}_{\alpha_0' < \alpha_0})$$

$$= ON^{(5)}(\{\boxed{H}^{(2)}(\alpha_0'; ON^{(4)}(\{\boxed{H}^{(2)}(\alpha_0, \alpha_1', \alpha_2)\}_{\alpha' < \alpha}), \alpha_2)\}_{\alpha < \alpha_0}$$

$$= \boxed{H}^{(2)}(\alpha_0, \alpha_1, \alpha_2).$$

Thus the assumption of two different functions $\boxed{H}^{(1)}$, $\boxed{H}^{(2)}$, each of three arguments and satisfying identical equations of schemata R, leads to a contradiction.

Now it is proved that for every constant τ, and pair $(\{\mu_i\}_{i<3}, \{ON^{(j)}\}_{0<j<6})$, there exists a function $\boxed{H}$ satisfying the equations of schemata R. Suppose there is no such function. Now $\boxed{H}$ satisfies R for each α_1, α_2, and each $i \leq 2$, so suppose α_0 is the least number for which $(\exists \alpha_1, \alpha_2)\ \boxed{H}(\alpha_0, \alpha_1, \alpha_2)$ is not defined. Clearly $\alpha_1 \neq 0, 1$.

Suppose $\alpha_1 = 2$. Then $(\forall \alpha_0' < \alpha_0)\ \boxed{H}(\alpha_0', 2, \alpha_2)$ is, by hypothesis, defined, and so by R, $\boxed{H}(\alpha_0, 2, \alpha_2)$ is defined, thus $\alpha_1 \neq 2$.

Suppose $\alpha_0 = 3$. Then let α_1 be the least number > 2 such that $(\exists \alpha_2)\ \boxed{H}(3, \alpha_1, \alpha_2)$ is not defined.

Then $(\forall \alpha_0' < \alpha_0)(\forall \alpha_1' < \alpha_1)(\forall \alpha_2)$ $\boxed{H}(\alpha_0', \alpha_1, \alpha_2)$, $\boxed{H}(\alpha_0, \alpha_1', \alpha_2)$ are defined, and thus by R $\boxed{H}(\alpha_0, \alpha_1, \alpha_2)$ too is defined.

Thus the assumption that there is no function $\boxed{H}$ satisfying, for given τ, $\{\mu_i\}_{i<3}$, $\{ON'^{(j+1)}\}_{j<5}$, schemata R, leads to a contradiction, so theorem 1.5D is now proved for functions of three arguments.

Suppose, for induction hypothesis, that it has been proved that for some $n \geq 3$, theorem 1.5D is satisfied for all functions $\boxminus$ of n ordinal arguments defined by application of schemata R.

Let $\boxminus$ be such a function, and let $\{ON_i'\}_{0<i<n}$, $\{ON_i\}_{i<n}$ be sequences of ordinal valued ($<\omega_1$) operations defined on countable sequences of countable ordinals. Further, let $\boxed{H}$ be a function of $n+1$ arguments defined from these functions by schemata R. Suppose that for certain sets of argument values, $\boxed{H}$ is not uniquely defined. Then

DEFINITION. Let the sequence of numbers $\{\bar{\alpha}_0, \bar{\alpha}_1, \ldots \bar{\alpha}_{n-1}, \bar{\gamma}\}$ be defined by induction as follows:

$\bar{\alpha}_0 =_{Df}$ the least ordinal number such that

$(\exists \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \gamma)$ $\boxed{H}(\bar{\alpha}_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \gamma)$ is not uniquely defined (u.d.).

Suppose for some i such that $0 < i < n-2$, numbers $\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_i$ have been defined. Then

$\bar{\alpha}_{i+1}$ = the least ordinal number such that

$(\exists \alpha_{i+2}, \alpha_{i+3}, \ldots, \alpha_{n-1}, \gamma) \boxed{\vdash\!\dashv} (\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_i, \bar{\alpha}_{i+1},$
$\alpha_{i+2}, \ldots, \alpha_{n-1}, \gamma)$ is not u.d.

and

$\bar{\alpha}_{n-1}$ = the least ordinal number such that

$(\exists \gamma) \boxed{\vdash\!\dashv} (\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{n-1}, \gamma)$ is not u.d.

and

$\bar{\gamma}$ = the least ordinal number such that

$\boxed{\vdash\!\dashv} (\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{n-1}, \bar{\gamma})$ is not u.d.

LEMMA. Let $\bar{\alpha}_0, \ldots, \bar{\gamma}$ be as defined above. Then $\bar{\alpha}_0 > 0$, $(\forall i)_{0 < i < n-1}\ \bar{\alpha}_i > 2$ and $\bar{\gamma} > 1$.

PROOF. By definition $(\forall \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \gamma) \boxed{\vdash\!\dashv} (0, \alpha_1, \ldots, \alpha_{n-1}, \gamma)$ is u.d. Thus $\bar{\alpha}_0 > 0$, and this proves the first clause of the lemma. Furthermore, $(\forall \alpha_0' < \bar{\alpha}_0)$ $(\forall \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \gamma) \boxed{\vdash\!\dashv} (\alpha_0', \alpha_2, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \gamma)$ is u.d., and thus $(\forall \alpha_2, \ldots, \alpha_{n-1}, \gamma) \boxed{\vdash\!\dashv} (\bar{\alpha}_0, 2, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \gamma)$ is u.d. too.

Therefore $\bar{\alpha}_1 > 2$.

Now suppose for induction hypothesis that it has been shown for some i such that $0 < i < n-1$ that

$(\forall k \leq i)_{>0}\ \bar{\alpha}_k > 2$. Then $(\forall \alpha'_k < \bar{\alpha}_k)\ \forall \alpha_{k+2}, \ldots, \alpha_{n-1}, \gamma)\ \boxed{\vdash\!\dashv}\ (\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{k-1}, \alpha'_k, \alpha_{k+2}, \alpha_{k+2}, \alpha_{k+3}, \ldots, \alpha_{n-1}, \gamma)$ is u.d., and thus $(\forall \alpha_{k+2}, \alpha_{k+3}, \ldots, \alpha_{n-1}, \gamma)$ is u.d., and thus $(\forall \alpha_{k+2}, \alpha_{k+3}, \ldots, \alpha_{n-1}, \gamma)$ $\boxed{\vdash\!\dashv}\ (\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_k, 2, \alpha_{k+2}, \alpha_{k+3}, \ldots, \alpha_{n-1}, \gamma)$ too is u.d. Thus the second clause of the lemma is proved.

Furthermore, by definition of schemata R, $\boxed{\vdash\!\dashv}$ is uniquely defined for $\gamma = 0, 1$.

Thus for $\boxed{\vdash\!\dashv}\ (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \gamma)$ not u.d., $\alpha_0 > 0$, $\alpha_1 > 2, \ldots, \alpha_{n-1} > 2$, $\gamma > 1$. ☐ LEMMA

By the hypothesis that $(\exists \alpha_0, \ldots, \alpha_{n-1}, \gamma)\ \boxed{\vdash\!\dashv}\ (\alpha_0, \ldots, \alpha_{n-1}, \gamma)$ is not u.d., and by the definition of numbers $\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{n-1}, \bar{\gamma}$, it follows that $(\forall \alpha'_{n-1} < \bar{\alpha}_{n-1})(\forall \gamma)$ $\boxed{\vdash\!\dashv}\ (\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{n-2}, \alpha'_{n-1}, \gamma)$ is u.d., and thus so is Λ_0.

Suppose, for some $i < n-1$, that Λ_i is u.d. Then $(\forall \alpha'_{n-i-2} < \bar{\alpha}_{n-i-2})(\forall \alpha_{n-i}, \ldots, \alpha_{n-1}, \gamma)\ \boxed{\vdash\!\dashv}\ (\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{n-i-3}, \alpha'_{n-i-2}, \Lambda_i, \alpha_{n-i}, \ldots, \alpha_{n-1}, \gamma)$ is u.d., and thus so is Λ_{i+1}.

Hence it follows by induction that $\Lambda_{n-1} =$ $\boxed{\vdash\!\dashv}\ (\bar{\alpha}_0, \ldots, \bar{\alpha}_{n-1}, \bar{\gamma})$ is uniquely defined, contradicting the hypothesis.

Thus it is proved by contradiction that if theorem 1.5D is satisfied for all functions of $n \geqslant 3$ ordinal arguments defined by application of schemata R, then the theorem is satisfied for all such functions of $n+1$ arguments, and so by induction it follows that theorem 1.5D is proved. $\square$

§2. FUNDAMENTAL SEQUENCES.

In this section, some standard definitions and results are outlined, and an unsolved problem arising in connection with a scheme of G. H. Hardy is stated.

If α is an ordinal such that $Lim(\alpha)$, and $\alpha \in Z(\aleph_0)$, then a fundamental sequence for α is an infinite sequence of countable ordinals $\{\alpha_n\}_{n<\omega}$ such that

i) $(\forall n < \omega)\ \alpha_n < \alpha_{n+1}$

ii) $\lim\limits_{n<\omega} \alpha_n = \alpha$

THEOREM 2.1. For every limit number belonging to $Z(\aleph_0)$, there exists a fundamental sequence.

PROOF (Sierpinski, [18], p 384). Suppose $\alpha < \omega_1$, and $Lim(\alpha)$. Now the set W_α is thus denumerable, and does not contain a greatest element. So let $S = \{\xi_n\}_{n<\omega}$ be an infinite sequence containing exactly the ordinals in the set W_α. Let $\alpha_0 = \xi_0$, and for every $n < \omega$, let $\alpha_{n+1} = (\mu\xi)[\xi \in S \;\&\; \alpha_n < \xi]$.

Thus the sequence $\{\alpha_n\}_{n<\omega}$ is strictly increasing. Now $(\forall p<\omega)(\exists n<\omega)\ \xi_p<\alpha_n$, thus no element of S is greater than every element of $\{\alpha_n\}_{n<\omega}$, so α is the least ordinal greater than each element of $\{\alpha_n\}_{n<\omega}$, therefore $\alpha=\lim\limits_{n<\omega}\alpha_n$.

□ THEOREM 2.1

Theorem 2.1 does not enable a function to be exhibited, such that the function assigns to each countable limit number one and only one fundamental sequence. On the assumption of the existance of such a function, Alonzo Church [8] theorem A_4, p 191 has proved:

$$\aleph_1 \leq 2^{\aleph_0}$$

without the aid of the axiom of choice.

DEFINITION 2.2 $\lambda\alpha, n. \Omega_\alpha(n)$ is called a *fundamental sequence assignment* (f.s.a.) if $(\forall\alpha<\omega_1)$ $(\forall n<\omega)$:

i) $\Omega:\omega_1\times\omega\rightarrow\omega_1$

ii) $(\alpha=0 \vee Suc(\alpha))\Rightarrow\Omega_\alpha(n)=0$

iii) $Lim(\alpha)\Rightarrow\Omega_\alpha(n)<\Omega_\alpha(n+1)$

iv) $Lim(\alpha)\Rightarrow\lim\limits_{m<\omega}\Omega_\alpha(m)=\alpha$

Thus Ω assigns to each countable limit ordinal α, a fundamental sequence $\{\Omega_\alpha(n)\}_{n<\omega}$.

EPSILON NUMBERS

Let $\beta_0 = 0$, and $(\forall n<\omega)\ \beta_{n+1} = \omega^{\beta_n}$.

Let $\varepsilon_0 = \lim_{n<\omega} \beta_n$.

Then $\omega^{\varepsilon_0} = \lim_{n<\omega} \omega^{\beta_n}$, as $\lambda\beta.\omega^\beta$ is continuous

$$= \lim_{n<\omega} \beta_{n+1} = \varepsilon_0$$

By theorem 1.3, and since $\lambda\beta.\omega^\beta$ is strictly increasing, it follows that ε_0 is the least number with this property.

The solutions ξ to the equation

$$\omega^\xi = \xi$$

are called by Cantor ε-numbers, $\{\varepsilon_\mu\}_{\mu<\omega_1}$ denoting the sequence of countable ε-numbers, ordered according to magnitude.

According to the normal form theorem of Cantor, every limit number $\alpha < \varepsilon_0$ can be expressed uniquely as

$$\alpha = \sum_{i=1}^{r} \omega^{\alpha_i} \cdot a_i$$

where $0 < \alpha_r < \alpha_{r-1} < \dots < \alpha_1 < \alpha$, and where r and $a_1, a_2, \ldots, a_r$ are natural numbers.

It is thus possible to define a fundamental sequence $\{\Omega_\alpha(n)\}_{n<\omega}$ for limit $\alpha < \varepsilon_0$ by transfinite induction: $(\forall n < \omega)$

(i) If $\alpha = \omega$, then $\Omega_\alpha(n) = n$

(ii) If Suc (α_r), then

$$\Omega_\alpha(n) = \sum_{i=1}^{r-1} \omega^{\alpha_i} \cdot a_i + \omega^{\alpha_r} \cdot (a_r - 1) + \omega^{\alpha_r - 1} \cdot n$$

(iii) If Lim (α_r), and $\lambda m.\, \Omega_{\alpha_r}(m)$ has already been defined, then

$$\Omega_\alpha(n) = \sum_{i=1}^{r-1} \omega^{\alpha_i} \cdot a_i + \omega^{\alpha_r} \cdot (a_r - 1) + \omega^{\Omega_{\alpha_r}(n)} .$$

A PROBLEM CONCERNING FUNDAMENTAL SEQUENCES.

There is an unsolved problem in the theory of ordinal numbers which arises in connection with G. H. Hardy's scheme [12] for determining a set of points belonging to the continuum whose cardinal number is $\aleph_1$.

Stated in it's general form, the problem is: find a function of two arguments $\lambda\alpha, \beta \cdot \Omega_{\alpha}(\beta)$ such that for every ordinal number μ, and for every limit ordinal λ such that

$$\lambda \in Z(\aleph_{\mu})$$

the function Ω determines a unique strictly increasing sequence $\{\Omega_{\lambda}(\eta)\}_{\eta < \omega_{\sigma}}$, for some $\sigma \leqq \mu$, such that

$$\lambda = \lim_{\eta < \omega_{\sigma}} \Omega_{\lambda}(\eta) \quad .$$

The sequence $\{\Omega_{\lambda}(\eta)\}_{\eta < \omega_{\sigma}}$ is called a <u>fundamental sequence</u> for the ordinal λ. (c.f. Bachmann [3], p. 141).

In connection with this problem, as it applies to the second number class, several classes of ordinal-valued functions are presented, and some of their properties investigated.

§3. THE CLASS $\Phi^{(2)}_{\Omega}$ OF FUNCTIONS $\Phi^{\beta}_{\alpha} : \omega_1 \to \omega_1$.

Certain arithmetic operations on the countable ordinals, including the familiar operations of addition, multiplication and exponentiation, generate on $\omega_1 \times \omega_1$ array of functions:

$$
\begin{array}{lcc|cccccr}
 & \vdots & \vdots & \vdots & \vdots & & \vdots & & \\
 & \Phi^{\beta}_{0} & \Phi^{\beta}_{1} & \Phi^{\beta}_{2} & \Phi^{\beta}_{3} & \cdots & \Phi^{\beta}_{\alpha} & \cdots & \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & & \\
[1] & \Phi^{3}_{0} & \Phi^{3}_{1} & \Phi^{3}_{2} & \Phi^{3}_{3} & \cdots & \Phi^{3}_{\alpha} & \cdots & [2] \\
 & \Phi^{2}_{0} & \Phi^{2}_{1} & \Phi^{2}_{2} & \Phi^{2}_{3} & \cdots & \Phi^{2}_{\alpha} & \cdots & \\
\cline{4-8}
 & \Phi^{1}_{0} & \multicolumn{1}{c}{\Phi^{1}_{1}} & \Phi^{1}_{2} & \Phi^{1}_{3} & \cdots & \Phi^{1}_{\alpha} & \cdots & \\
 & \Phi^{0}_{0} & \multicolumn{1}{c}{\Phi^{0}_{1}} & \Phi^{0}_{2} & \Phi^{0}_{3} & \cdots & \Phi^{0}_{\alpha} & \cdots & \\
\end{array}
$$

In the following definition, those functions in region [1] are defined explicitly (case (i) below) and those in region [2] are defined by transfinite recursion (cases (ii), (iii), (iv)).

DEFINITION 3.1 . For each countable ordinal γ, and some fundamental sequence assignment Ω (such that for each $n < \omega$, $\Omega_{\omega}(n) = n$; c.f. definition 2.2) :

(i) $(\forall\alpha<\omega_1)\left[(\Phi^0_\alpha(\gamma)=1)_{2<\alpha} \;\&\; \Phi^\alpha_0(\gamma)=0\right.$

$\left.\& \; \Phi^\alpha_1(\gamma)=\gamma+\alpha\right] \;\&$

$(\forall\alpha<\omega_1)_{1<\alpha}\;[\Phi^1_\alpha(\gamma)=\gamma \;\&\; \Phi^0_2(\gamma)=0]$

(ii) $(\forall\alpha<\omega_1)_{1<\alpha}\;\left[Suc(\alpha)\Rightarrow\Phi^2_\alpha(\gamma)=\Phi^\gamma_{P\alpha}(\gamma)\;\&\right.$

$\left.Lim(\alpha)\Rightarrow\Phi^2_\alpha(\gamma)=\lim_{n<\omega}\Phi^2_{\Omega_\alpha(n)+1}(\gamma)\right]$

(iii) For $\alpha=x=1,2$, and $2<\beta$:

$$\Phi^\beta_{x+1}(\gamma) = \begin{cases} \Phi^\gamma_x\{\Phi^{P\beta}_{x+1}(\gamma)\} \text{ if } Suc(\beta) \\ \lim_{n<\omega}\Phi^{\Omega_\beta(n)}_{x+1}(\gamma) \text{ if } Lim(\beta) \end{cases}$$

(iv) $(\forall\alpha,\beta<\omega_1)_{2<\beta,\,3<\alpha}$

$$\Phi^\beta_\alpha(\gamma) = \begin{cases} \Phi^{\Phi^{\omega\beta'}_\alpha(\gamma)+1}_{P\alpha}(\gamma) \text{ if } Suc(\alpha,\beta)\;\&\;x=1 \\ \Phi^{\Phi^{\omega\beta'+x-1}_\alpha(\gamma)}_{P\alpha}(\gamma) \text{ if } Suc(\alpha,\beta)\;\&\;1<x \\ \lim_{n<\omega}\Phi^{\Omega_\beta(n)}_\alpha(\gamma) \text{ if } Lim(\beta) \\ \lim_{n<\omega}\Phi^{\Phi^{P\beta}_\alpha(\gamma)+1}_{\Omega_\alpha(n)}(\gamma) \text{ if } Lim(\alpha)\;\&\;Suc(\beta). \end{cases}$$

In each case β is expressed in the unique form $\beta=\omega\beta'+x$, $\beta'\leq\beta$, $x<\omega$.

□ DEFINITION 3.1

This class of functions is denoted by the symbol $\Phi^{(2)}_\Omega$.

γ is called here the operand, β the exponent and α the index of the expression $\Phi_\alpha^\beta(\gamma)$ (1). Cantor [6] shows that each countable limit ordinal λ can be expressed uniquely in the form $\lambda = \delta + \omega^\beta$, the number β being called the exponent of λ. Definition 3.1 with respect to this terminology thus conforms to Cantor in the case $\lambda = \Phi_3^\beta(\omega) = \omega^\beta$.

The following notation is sometimes useful: $(\forall \alpha, \beta, \gamma < \omega_1)$

$$\gamma \,\textcircled{\alpha}\, \beta =_{Df} \Phi_\alpha^\beta(\gamma).$$

DEFINITION 3.2. THE OPERATIONS Σ, Π and Γ_α, $\alpha < \omega_1$.

There follow standard definitions for sum and product of finite and transfinite sequences of ordinals (c.f. Bachmann [4], pp 53, 56):

for each $f : \omega_1^{(n)} \to \omega_1$, $n > 0$, and each string $\vec{\xi}$ of parameters of length $n-1$, and each $\beta < \omega_1$:

$$\sum_{\beta' < 0} f(\beta', \vec{\xi}) = 0 \quad , \quad \prod_{\beta' < 0} f(\beta', \vec{\xi}) = 1$$

---

(1) See appendix II, p. 183.

$$\sum_{\beta' < \beta+1} f(\beta', \vec{\xi}) = \sum_{\beta' < \beta} f(\beta', \vec{\xi}) + f(\beta, \vec{\xi})$$

$$\prod_{\beta' < \beta+1} f(\beta', \vec{\xi}) = \left\{ \prod_{\beta' < \beta} f(\beta', \vec{\xi}) \right\} \cdot f(\beta, \vec{\xi})$$

and for λ a countable limit number

$$\sum_{\beta' < \lambda} f(\beta', \vec{\xi}) = \lim_{\mu < \lambda} \sum_{\beta' < \mu} f(\beta', \vec{\xi})$$

$$\prod_{\beta' < \lambda} f(\beta', \vec{\xi}) = \lim_{\mu < \lambda} \prod_{\beta' < \mu} f(\beta', \vec{\xi})$$

The operations Γ_α, $\alpha < \omega_1$, map 1-1 from $\omega_1^{[\omega_1^{[n]}]}$ into $\omega_1^{[\omega_1^{[n]}]}$, for each $n < \omega$, and are defined as follows from elements of the class $\mathfrak{D}_{\Omega}^{(2)}$ (the square brackets denote set-theoretic exponentiation). For each n such that $0 < n < \omega$, and for each

$f : \omega_1^{[n]} \to \omega_1$,

$(\forall \vec{\xi})_{\vec{\xi} \in \omega_1^{[n-1]}} (\forall \alpha, \beta)_{\alpha, \beta < \omega_1, 0 < \beta}$

$\Big[$ (i) $\underset{\beta' < 0}{\Gamma_\alpha} f(\beta', \vec{\xi}) = 1$, for $\alpha \geq 3$.

(ii) for $\alpha = 0$, $\displaystyle\Gamma_{0\,\beta' < \beta} f(\beta', \vec{\xi}) = f(0, \vec{\xi}) + P_\beta$

$$\alpha = 1,\ \Gamma_{1\,\beta' < \beta} f(\beta', \vec{\xi}) = \sum_{\beta' < \beta} f(\beta', \vec{\xi})$$

$$\alpha = 2,\ \Gamma_{2\,\beta' < \beta} f(\beta', \vec{\xi}) = \prod_{\beta' < \beta} f(\beta', \vec{\xi})$$

(iii) $3 \leq \alpha$ and $Suc(\beta) \Rightarrow$

$$\Gamma_{\alpha\,\beta' < \beta} f(\beta', \vec{\xi}) = \begin{cases} \underline{\Phi}_{\alpha}^{u+1}\{f(P_\beta, \vec{\xi})\} \text{ if } \alpha < \omega,\ x = 1 \\ \underline{\Phi}_{\alpha}^{u}\{f(P_\beta, \vec{\xi})\} \text{ if } \alpha < \omega,\ 1 < x \\ \underline{\Phi}_{P\alpha}^{u+1}\{f(P_\beta, \vec{\xi})\} \text{ if } \omega < \alpha,\ 1 < x \\ \underline{\Phi}_{P\alpha}^{u}\{f(P_\beta, \vec{\xi})\} \text{ if } \omega < \alpha,\ 1 < x \\ \quad \text{where } u = \Gamma_{\alpha\,\beta' < P_\beta} f(\beta', \vec{\xi}). \text{ The} \\ \quad \text{parameters for } u \text{ have been omitted.} \\ \sup_{\alpha' < \alpha} \Gamma_{\alpha\,\beta' < \beta} f(\beta', \vec{\xi}) \text{ if } Lim(\alpha). \end{cases}$$

(iv) for $3 \leq \alpha$ and $Lim(\beta)$

$$\Gamma_{\alpha\,\beta' < \beta} f(\beta', \vec{\xi}) = \sup_{\beta^* < \beta} \Gamma_{\alpha\,\beta' < \beta^*} f(\beta', \vec{\xi})$$

$\square$ DEFINITION 3.2

The operation Σ, Π form special cases on account of the property of the arithmetic operations ⓐ on the countable ordinals, which is discussed on pages 121-123 of § 4.

EXAMPLES. a) $\Phi_1^\beta(\gamma) = \gamma+\beta$ by definition

b) $\Phi_2^\beta(\gamma) = \gamma.\beta$. For suppose $\beta = 0, 1, 2$. Then $\Phi_2^\beta(\gamma)$ equals $0 = \gamma\cdot 0$; $\gamma = \gamma.1$; $\Phi_1^\gamma(\gamma) = \gamma+\gamma = \gamma.2$ respectively.

Now suppose $2 < \beta$, and that the equality holds for all numbers smaller than β. Then $Suc(\beta) \Rightarrow$

$$\Phi_2^\beta(\gamma) = \Phi_1^\gamma\{\Phi_2^{P\beta}(\gamma)\} \text{ by definition 3.1(iii)}$$

$$= \Phi_1^\gamma(\gamma.P\beta) \text{ by hypothesis}$$

$$= \gamma.P\beta + \gamma = \gamma.\beta$$

Furthermore, $Lim(\beta) \Rightarrow$

$$\Phi_2^\beta(\gamma) = \lim_{n<\omega} \Phi_2^{\Omega\beta(n)}(\gamma) \text{ by definition 3.1(iii)}$$

Therefore $\Phi_2^{\beta}(\gamma) = \lim_{n<\omega} \gamma . \Omega_\beta(n)$ by hypothesis

$$= \gamma . \beta$$

Thus by transfinite induction the equality is proved to hold for all numbers β.

Similarly, $\Phi_3^{\beta}(\gamma) = \gamma^{\beta}$

$$\Phi_4^{\beta}(\gamma) = \overset{\beta\text{-terms}}{\gamma^{\gamma^{\cdot^{\cdot^{\cdot^{\gamma}}}}}}$$

In particular, $\Phi_1^{2}(\gamma) = \gamma + 2$, $\Phi_2^{2}(\gamma) = \gamma . 2$, $\Phi_3^{2}(\gamma) = \gamma^2$, $\Phi_4^{2}(\gamma) = \gamma^{\gamma}$. Also, since

$$\varepsilon_0 = \lim \{ \omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \dots \}$$

$$\varepsilon_0 = \lim_{n<\omega} \Phi_4^{\Omega_\omega(n)}(\omega) = \Phi_4^{\omega}(\omega) = \Phi_5^{2}(\omega) .$$

Certain functions on the natural numbers, of interest in recursion theory, can be obtained from elements of the class $\Phi_{\Omega}^{(2)}$. Thus, by writing

$(\forall n < \omega)\ \phi(n) =_{Df} \Phi_4^{n}(2)$,

$$\phi(0) = \Phi_4^{0}(2) = 1$$

$$\phi(n+1) = \Phi_3^{\phi(n)}(2) = 2^{\phi(n)},$$ so that

$$\phi(n) = \overbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}^{n \text{ terms}} .$$

ϕ is considered by Parsons [14] to be, according to a suitable set of criteria 'the simplest function which is not elementary'.

It can be shown that for each $m, n < \omega$ $\lambda x. \Phi_n^m(x) \upharpoonright \omega$, (1), and for each $n < \omega$, $\lambda x, y. \Phi_n^y(x) \upharpoonright \omega^{[2]}$ are primitive recursive number theoretic functions; but $\lambda x. \Phi_\omega^2(x) \upharpoonright \omega$, although it is (a restriction of) the first 'new' function to be obtained from the above functions in definition 3.1, it is not a majorizing function of the Ackermann type, since it is always the limit of a transfinite increasing sequence of ordinals. Such majorizing functions are obtained in the array $\{D_\alpha^n\}_{n<\omega, \alpha<\omega_1}$, defined on p. 54.

Two important examples of the operations Γ_α are already defined explicitly: $\Gamma_1 = \Sigma$, $\Gamma_2 = \Pi$. Hausdorff, [13] p. 119, has defined the transfinite product, for functions f such that $(\forall \beta')\ 0 < f(\beta')$, as follows:

$$\prod_{\beta'<0}^{(H)} f(\beta') = 1, \text{ and for each } \beta > 0,$$

---

(1) The restriction of a function f to A', written $f \upharpoonright A'$, is defined as follows: if $f: A \to B$, and $A' \subseteq A$, then $f \upharpoonright A'$ denotes the function $g: A' \to B$ such that $(\forall x \in A')\ g(x) = f(x)$.

$$\prod_{\beta'<\beta}^{(H)} f(\beta') = (\mu\alpha)\left[(\forall\theta<\beta)\prod_{\beta'<\theta}^{(H)} f(\beta')\cdot f(\theta) \leq \alpha\right]$$

$$= \sup_{\theta<\beta}\left\{\prod_{\beta'<\theta}^{(H)} f(\beta')\right\}.f(\theta).$$

LEMMA. If $(\forall\beta')f(\beta')>0$, then both $\lambda\beta.\prod_{\beta'<\beta} f(\beta')$ and $\lambda\beta.\prod_{\beta'<\beta}^{(H)} f(\beta')$ are monotone.

PROOF by transfinite induction. Immediate for $\prod$, from definition 3.2.

For $\prod^{(H)}$, if $Suc(\beta)$, then $\prod_{\beta'<\beta}^{(H)} f(\beta') \geqslant \prod_{\beta'<P\beta}^{(H)} f(\beta').f(P\beta)$

$$\geqslant \prod_{\beta'<P\beta}^{(H)} f(\beta'),$$ as

$f(P\beta)$ is positive.

If $Lim(\beta)$,

$$(\forall\beta'<\beta)\prod_{\beta'<\beta}^{(H)} f(\beta') \geqslant \prod_{\beta^*<\beta'}^{(H)} f(\beta^*).f(\beta')$$

$$\geqslant \prod_{\beta^*<\beta'}^{(H)} f(\beta^*),$$ as $f(\beta')$ is positive.

□

COROLLARY 3.3. $\prod^{(H)} = \prod$.

PROOF. By transfinite induction. From the lemma it follows that for $suc(\beta)$, $\prod_{\beta'<\beta}^{(H)} f(\beta') =$

$= \{\prod^{(H)}_{\beta' < \rho_\beta} f(\beta')\} \cdot f(\rho_\beta)$, and for $Lim(\beta)$,

$\prod^{(H)}_{\beta' < \beta} f(\beta') = \sup_{\beta' < \beta} \prod^{(H)}_{\beta^* < \beta'} f(\beta^*)$ □COROLLARY 3.3

The operations Γ_{n+1}, $2 \leq n < \omega$, when applied to number theoretic functions, are operations under which are closed the classes $\mathcal{E}^{n+1}$, but not the classes $\mathcal{E}^n$, where those classes are members of A. Grzegorczyk's hierarchy of primitive recursive functions.

Two classes of maps are now defined, using definitions 3.1, 3.2.

DEFINITION 3.4

Some of the defining equations of definition 3.1 contain limit operations in their right hand sides. Let the functions D^n_α be those obtained in place of the functions Φ^β_α by restriction to finite values x, n of ordinals γ and β respectively, and by making the following substitutions within two of these equations, and by leaving the remaining equations unchanged.

In definition 3.1(iii):

$$Lim(\alpha) \Rightarrow D^2_\alpha(x) = D^2_{\Omega_\alpha(x)}(x)$$

In definition 3.1(iv):

$$Lim(\alpha) \Rightarrow D^{n+1}_\alpha(x) = D^2_{\Omega_\alpha\{D^n_\alpha(x)\}}(x)$$

$\{D^n_\alpha\}_{n<\omega, \alpha<\omega_1} =_{Df} \mathcal{D}^{(1)}_\Omega$. □ 3.4

Thus, number-theoretic functions have been obtained from some of the Φ^{β}_{α} by replacing the taking of limits by a process of diagonalization, and restricting the domain of arguments β, γ to finite ordinal values. Not all these functions are general recursive.

DEFINITION 3.5. Some of the defining equations of 3.2 contain in their r. h. s's expressions built up from Φ, and also limit operations. Let the operations $\mathcal{U}_\alpha$ which map 1-1 from $\omega^{[\omega^{[n]}]}$ into $\omega^{[\omega[n]]}$, $n < \omega$, be those obtained by considering only finite values $t > 0$ of the index β of 3.2, and by making the following substitutions within two of these equations, and leaving the remaining equations unchanged:

In 3.2 (iii): $t > 0$, $3 \leq \alpha$ and $\mathrm{Suc}(\alpha) \Rightarrow$

$$\mathcal{U}_{\alpha \atop x < t} f(x, \vec{y}) = \begin{cases} D^{exp}_{\alpha}\{f(t-1, \vec{y})\} & \text{if } \alpha < \omega \\ D^{exp}_{P\alpha}\{f(t-1, \vec{y})\} & \text{if } \omega < \alpha \end{cases}$$

where $exp =_{Df} \mathcal{U}_{\alpha \atop x < t-1} f(x, \vec{y})$

In 3.2 (iv):

$$\underset{x<t}{U_\alpha} f(x,\vec{y}) = \underset{x<t}{U_{\Omega_\alpha(t)}} f(x,\vec{y})$$

$$\{U_\alpha\}_{\alpha<\omega_1} =_{\mathrm{Df}} \mathcal{U}^{(1)}_{\Omega}$$ $\square$ DEFINITION 3.5

It is clear that the classes obtained in definitions 3.4 and 3.5 depend on the choice of the fundamental sequence assignment Ω. (However it is shown in §4 that the class $\mathfrak{D}^{(2)} = \mathfrak{D}^{(2)}_{\Omega}$ is independent of the choice of Ω)

Note that given an f.s.a. Ω an f.s.a. Ω' $\neq \Omega$ can readily be found. For example, let Ω, Ω' satisfy $(\forall\alpha<\omega_1)\,\Omega_\alpha(n.2) = \Omega'_\alpha(n)$, for each $n<\omega$ and for Lim (α). Then Ω' is an f.s.a iff Ω is an f.s.a, but since both sequences $\{\Omega_\alpha(n)\}_{n<\omega}, \{\Omega'_\alpha(n)\}_{n<\omega}$ are strictly increasing, for each n the term $\Omega_\alpha(n.2+1)$ will not occur in $\{\Omega'_\alpha(m)\}_{m<\omega}$.

It is now shown that the function Φ is an instance of definition by application of schemata R, definition 1.5C, p.20.

THEOREM 3.6. The functions Φ_α^β, $\alpha, \beta < \omega_1$ of definition 3.1 are well defined.

PROOF. By theorem 1.5D, every function $\boxminus$ defined by schemata R of definition 1.5C is well-defined.

Let Φ be the function of three arguments of 3.1, and put

$$(\forall \alpha, \beta, \gamma)\ \Phi_\alpha^\beta(\gamma) = \Phi(\alpha, \beta, \gamma)$$

Then $(\forall \beta, \gamma)\ \Phi(0, \beta, \gamma) = 0 = \mathbb{Z}(\mathcal{U}_0(\beta, \gamma))$

$$\Phi(2, \beta, \gamma) = \gamma \cdot \beta = \sum_{\eta < \beta} \mathcal{U}_1(\eta, \gamma)$$

$$\Phi(1, \beta, \gamma) = \gamma + \beta = \mathcal{L} \sum_{\eta < \beta + 1} \mathcal{U}_1(\eta, S(\gamma . 0^\eta))$$

Thus, by definition 1.5D1, $(\forall i)_{i \leq 3}$ $\lambda \beta, \gamma . \Phi(i, \beta, \gamma) \in \mathcal{E}$.

Furthermore, for each $\alpha \geq 3$, and each γ

$\Phi(\alpha, 0, \gamma) = 1$, and

$$\Phi(\alpha, 1, \gamma) = \begin{cases} \gamma \text{ if } \alpha > 1 \\ \gamma+1 \text{ if } \alpha = 1 \\ 0 \text{ if } \alpha = 0 \end{cases}$$

In which cases Φ corresponds in definition to function Υ of definition 1.5C.

For $\beta = 2$, and each γ

$$\Phi(\alpha, 2, \gamma) = \begin{cases} \Phi(P\alpha, \gamma, \gamma) \\ \text{if } Suc(\alpha) \\ \lim\limits_{n<\omega} \Phi(\Omega_\alpha(n)+1, 2, \gamma) \\ = \lim\limits_{n<\omega} \Phi(\Omega_\alpha(n), \gamma, \gamma) \\ \text{if } Lim(\alpha). \end{cases}$$

Thus, by putting $ON^{(1)}(\{\Phi(\alpha', \gamma, \gamma)\}_{\alpha'<\alpha})$

$$= \begin{cases} \Phi(P\alpha, \gamma, \gamma) \text{ if } Suc(\alpha) \\ \lim\limits_{n<\omega} \Phi(\Omega_\alpha(n), \gamma, \gamma), \text{ if } Lim(\alpha) \end{cases}$$

Φ and Υ again correspond in definition.

For $\alpha = 3$, $\beta > 2$, and each γ

$$\Phi(3, \beta, \gamma) = \Phi(2, \gamma, \Phi(3, P\beta, \gamma)) \text{ if } Suc(\beta)$$

$$\lim_{n<\omega} \Phi(3, \Omega_\beta(n), \gamma) \text{ if } Lim(\beta).$$

Thus by putting $ON^{(2)}(\{\Phi(n, \beta, ON^{(3)}(\{\Phi(3, \beta', \gamma)\}_{\beta' < \beta}))\}_{n < 3})$

$$=_{Df} \begin{cases} \Phi(2, \gamma, ON^{(3)}(\{\Phi(3, \beta', \gamma)\}_{\beta' < \beta}) & \text{if } Suc(\beta) \\ ON^{(3)}(\{\Phi(3, \beta', \gamma)\}_{\beta' < \beta}) & \text{if } Lim(\beta) \end{cases}$$

and $ON^{(3)}(\{\Phi(3, \beta', \gamma)\}_{\beta' < \beta})$

$$=_{Df} \begin{cases} \Phi(3, P_\beta, \gamma) & \text{if } Suc(\beta) \\ \lim\limits_{n<\omega} \Phi(3, \Omega_\beta(n), \gamma) & \text{if } Lim(\beta) . \end{cases}$$

Again Φ and Υ correspond in definition.

Finally, for $\alpha > 3$, $\beta > 2$ and each γ

$$\Phi(\alpha, \beta, \gamma) = \begin{cases} \Phi(P_\alpha, \Phi(\alpha, \omega\beta', \gamma) + 1, \gamma) \\ \text{if } Suc(\alpha, \beta) \text{ and } x = 1 \\ \Phi(P_\alpha, \Phi(\alpha, \omega\beta' + x - 1, \gamma) \\ \text{if } Suc(\alpha, \beta) \text{ and } 1 < x \\ \lim\limits_{n<\omega} \Phi(\alpha, \Omega_\beta(n), \gamma) \text{ if } Lim(\beta) \\ \lim\limits_{n<\omega} \Phi(\Omega_\alpha(n), \Phi(\gamma, P_\beta, \gamma) + 1, \gamma) \\ \text{if } Lim(\alpha) \text{ and } Suc(\beta). \end{cases}$$

where β is expressed uniquely as $\beta = \omega \cdot \beta' + x$, $x < \omega$.

Thus, by putting $ON^{(4)}(\{\Phi(\alpha, \beta', \gamma)\}_{\beta' < \beta})$

$$=_{Df} \begin{cases} \lim\limits_{n < \omega} \Phi(\alpha, \Omega_\beta(n), \gamma) \text{ if } Lim(\beta) \\ \Phi(\alpha, P_\beta, \gamma) + 1 \text{ if } Lim(\alpha) \text{ and } Suc(\beta) \\ \Phi(\alpha, \omega\beta' + x - 1, \gamma) \text{ if } Suc(\alpha, \beta) \text{ and } 1 < x \\ \Phi(\alpha, \omega\beta', \gamma) \text{ if } Suc(\alpha, \beta) \text{ and } x = 1 \end{cases}$$

and by putting $ON^{(5)}(\{\Phi(\alpha', ON^{(4)}(\{\Phi(\alpha, \beta', \gamma)\}_{\beta' < \beta}, \gamma)\}_{\alpha' < \alpha})$

$$=_{Df} \begin{cases} ON^{(4)}(\{\Phi(\alpha, \beta', \gamma)\}_{\beta' < \beta}) \text{ if } Lim(\beta) \\ \lim\limits_{n < \omega} \Phi(\Omega_\alpha(n), ON^{(4)}(\{\Phi(\alpha, \beta', \gamma)\}_{\beta' < \beta}) \\ +1, \gamma) \text{ if } Lim(\alpha) \text{ and } Suc(\beta) \\ \Phi(P_\alpha, ON^{(4)}(\{\Phi(\alpha, \beta', \gamma)\}_{\beta' < \beta}), \gamma) \\ \text{if } Suc(\alpha, \beta). \end{cases}$$

Again, Φ and Υ correspond in definition.

Thus it is proved that the function Φ of definition

3.1 is an instance of a function defined from certain elementary functions μ_i , $i \leqslant 2$, a constant τ $(=1)$, and operations $ON^{(j)}$, $0 < j < 6$, by schemata R. Thus, by theorem 1.5D it follows that Φ is well-defined.

□ 3.6

THE FUNCTIONS $\{D^n_\alpha\}_{\substack{n<\omega \\ \alpha<\omega_1}} = \mathcal{D}^{(1)}_\Omega$

These functions were characterized in definition 3.4; they were obtained by a certain transformation from the operations $\mathbb{D}^\beta_\alpha$ of definition 3.1. The recursive definition of these functions is now given explicitly, with further constraints upon the f.s.a. Ω. And this is followed by a brief investigation of the class $\mathcal{D}^{(1)}_\Omega$ upon the basis of two unproved hypotheses (1) concerning a property of Ω and a consequent property of functions in the class. There follow some preliminary definitions

THE FUNCTION M_Ω

Suppose $\mathrm{Lim}(\alpha, \beta)$, $\alpha, \beta < \omega_1$, and $\beta < \alpha$. Then there is a least t such that $\beta \leq \Omega_\alpha(t)$. So for a given f.s.a. Ω it is possible to define a function $M_\Omega : \omega_1^{[2]} \rightarrow \omega_0$ such that

$$M_\Omega(\alpha, \beta) = 0 \text{ if } (\neg \mathrm{Lim}(\alpha, \beta) \vee \alpha \leq \beta), \text{ and}$$

$$M_\Omega(\alpha, \beta) = (\mu y)_{<\omega} [(\forall x < \omega)\, y < x \Rightarrow \Omega_\beta(x) < \Omega_\alpha(x)] \text{ o.w.}$$

---

(1) These hypotheses are discussed on p 70.

DEFINITION 3.7

The fundamental sequence $\{\alpha_n\}_{n<\omega}$ for the countable limit ordinal α is *perfect* if

$$(\forall x, y < \omega)_{y>2} \; D^2_{\alpha_x}(y) < D^2_{\alpha_{x+1}}(y)$$

DEFINITION 3.7a

Let Ω be an f.s.a. such that

(i) $(\forall \alpha < \omega_1)\{\Omega_\alpha(x)\}_{x<\omega}$ is a perfect fundamental sequence

(ii) it is the f.s.a. Ω upon which is based the definition of the D^n_α

(iii) $(\forall \alpha < \omega_1)\{\mathrm{Lim}(\alpha) \Rightarrow$

$$[\neg(\exists\beta<\alpha)[\alpha=\beta+\omega] \Rightarrow (\forall n<\omega)_{n>0} \; \mathrm{Lim}(\Omega_\alpha(n)) \;\&$$

$$(\exists\beta<\alpha)[\alpha=\beta+\omega] \Rightarrow (\forall n<\omega) \; \Omega_\alpha(n)=\beta+n]\}$$

Such an f.s.a. Ω is called a *perfect fundamental sequence assignment* (p.f.s.a.)

□ 3.7a.

There follows an explicit definition of the functions D^n_α, and an inductive proof of certain of their properties, that of theorem 3.8.

This contains parts a)(i), (ii), (iii), b) (i), (ii), (iii), and c). The transfinite induction is on the index α, and thus is in two major steps, for Suc (α) and Lim (α) respectively. The expression I. H. b (ii), for example, denotes the transfinite induction hypothesis, as it applies to part b (ii) of theorem 3.8.

As each part of the theorem is proved, within each of the two steps, there is, in the cases where necessary, an inner inductive hypothesis upon the index n.

HYPOTHESIS 3.7b. There exists a p.f.s.a.

DEFINITION 3.7c.

Let Ω be a p.f.s.a.

$$(\forall x, y < \omega) \quad D_0^y(x) = 0 \; ; \; D_1^y(x) = x + y \; ;$$

$$D_2^y(x) = x \cdot y$$

and for every countable ordinal $\alpha > 2$:

$$D_\alpha^0(x) = 1$$

$$D_\alpha^1(x) = x$$

$$\left.\begin{aligned} D_\alpha^2(x) &= D^2_{\Omega_\alpha(x)}(x) \\ D_\alpha^{y+1}(x) &= D^2_{\Omega_\alpha\{D_\alpha^y(x)\}}(x) \end{aligned}\right\} \text{Lim}(\alpha),\ y > 1$$

$$\left.\begin{aligned} D_\alpha^2(x) &= D^x_{P\alpha}(x) \\ D_\alpha^{y+1}(x) &= D^{D_\alpha^y(x)}_{P\alpha}(x) \end{aligned}\right\} \text{Suc}(\alpha),\ y > 1$$

THEOREM 3.8

a) $(\forall\alpha < \omega_1)_{>2}\ (\forall n, m, x, y < \omega)_{>2}$

(i) $x < y \Rightarrow D_\alpha^n(x) < D_\alpha^n(y)$

(ii) $m < n \Rightarrow D_\alpha^m(x) < D_\alpha^n(x)$

(iii) $D_\alpha^n(x) < D_{\alpha+1}^n(x)$

b) $(\forall\alpha < \omega_1)_{>2}\ (\forall n, x, y < \omega)_{>2}$

(i) —

(ii) $\text{Suc}(\alpha) \Rightarrow D_{P\alpha}^{n+1}(x) < D_\alpha^n(x)$

(iii) $\text{Lim}(\alpha) \Rightarrow D^{n+1}_{\Omega_\alpha(y)}(x) < D^n_{\Omega_\alpha(y+1)}(x)$

3.8 c) $(\forall \alpha < \omega_1)_{>2} (\forall n, x < \omega)_{>2}$

$$\max(n, x) + 2 < D^n_\alpha(x),$$

where in cases a) - c) for each n, α, $D^n_\alpha \in \mathcal{D}^{(1)}_\Omega$, such that Ω is a p.f.s.a., and such that $(\forall \alpha < \omega_1)$ $Lim(\alpha) \Rightarrow \Omega_\alpha(0) > 0$.

PROOF by transfinite induction on α.

Let $\alpha = 3$. Then $D^n_3(x) = x^n$. For $n, x > 2$, $x < y \Rightarrow x^n < y^n$, and $n < m \Rightarrow x^n < x^m$; and since $x^3 < x^x$, and for each $n \geq 3$,

$$x^n < \overbrace{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}^{n} \Rightarrow x^{n+1} = x^n \cdot x < \overbrace{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}^{n} \cdot x$$

$$< x^{\overbrace{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}^{n}},$$

it follows $(\forall n, x < \omega)_{>2}$ $x^n < \overbrace{x^{\cdot^{\cdot^{\cdot^{x}}}}}^{n}$.

With regard to part b(ii): to prove that $x^{n+1} < \overbrace{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}^{n}$, it is sufficient to prove $n+1 < D^{n-1}_4(x)$. Suppose $x > 2$ and $n > 5$. Let $n = 6$. Then

$$D^{n-1}_4(x) \geq \overbrace{2^{\cdot^{\cdot^{\cdot^{2}}}}}^{5} > 7$$

Suppose $D_4^{n-1}(x) > n+1$, for some $n > 5$.

Then $D_4^n(x) = x^{D_4^{n-1}(x)}$

$$\geq 2^{D_4^{n-1}(x)} > D_4^{n-1}(x).2$$

$$> (n-1).2$$

$$> n+2,$$ therefore for all $n > 5$

3.8 a), b) hold for $\alpha = 3$. It is easy to show that 3.8 a), b) hold for $n = 3, 4, 5$.

3.8 c) follows in this case as $x^n > x+2$; $x^n > n+2$, for $x, n > 2$.

Now suppose that for some α such that $3 < \alpha < \omega$, theorem 3.8 has been proved for all $\alpha' < \alpha$.

Then if $Suc(\alpha)$, suppose $x < y$, $x, y > 2$.

For 3.8a:

a)(i)

$D_\alpha^2(y+1) = D_{P\alpha}^{y+1}(y+1)$ by 3.7c

$> D_{P\alpha}^{y}(y+1)$ by I.H. a)(ii)

$> D_{P\alpha}^{y}(y)$ by I.H. a)(i)

$= D_\alpha^2(y)$.

Suppose for inner inductive hypothesis (i.i.h.) that $(\forall n')_{2<n'<n}\ D_\alpha^{n'}(y+1) > D_\alpha^{n'}(y)$.

Then $D_\alpha^n(y+1) = D_{P\alpha}^{D_\alpha^{n-1}(y+1)}(y+1)$

$> D_{P\alpha}^{D_\alpha^{n-1}(y)}(y+1)$ by i. i. h.

$> D_{P\alpha}^{D_\alpha^{n-1}(y)}(y)$ by I.H. a) (i).

Therefore $(\forall n)_{n>2}\ D_\alpha^n(y+1) > D_\alpha^n(y)$.

a) (ii).

For every y, $D_\alpha^{y+1}(x) = D_{P\alpha}^{D_\alpha^y(x)}(x)$

$> D_\alpha^y(x)$ by I.H. c)

a) (iii).

$D_{\alpha+1}^2(y) = D_\alpha^y(y) = D_{P\alpha}^{D_\alpha^{y-1}(y)}(y)$

Now $D_\alpha^{y-1}(y) > D_{P\alpha}^{y-1}(y)$ by I.H. a) (iii)

$> y$ by I.H. c)

Therefore

$D_{P\alpha}^{D_\alpha^{y-1}(y)}(y) > D^y_{P\alpha}(y)$ by I.H. a) (ii)

$= D_\alpha^2(y)$.

Suppose for inner inductive hypothesis that for some $n > 2$ it has been demonstrated, for every $n' < n$, that

$D_{\alpha+1}^{n'}(y) > D_\alpha^{n'}(y)$. Then

$D_{\alpha+1}^n(y) = D_\alpha^{D_{\alpha+1}^{n-1}(y)}(y)$.

But $D^{n-1}_{\alpha+1}(y) > D^{n-1}_{\alpha}(y)$ by i.i.h.

$> D^{n-1}_{P\alpha}(y)$ by I.H. a)(iii)

$> n$ by I.H. c)

Thus it follows that $D^{n-1}_{\alpha+1}(y) > n$.
From what has been shown for case a)(ii), it thus follows that $D^{n}_{\alpha+1}(y) > D^{n}_{\alpha}(y)$.

Now for 3.8 b):

b (ii)

$D^{2}_{\alpha}(x) = D^{x}_{P\alpha}(x) > max(x, x) + 2$ by I.H. c)

Now suppose for some $n > 3$ that $(\forall k < n-1)$ $D^k_\alpha(x) > \max(k, x) + 2$.

Then $D^{n-1}_\alpha(x) = D_{E\alpha}^{D^{n-2}_\alpha(x)}(x)$ by 3.7c

$> D^n_{E\alpha}(x)$ by i.i.h. and by I.H. a)(ii).

$> \max(n, x) + 2$ by I.H. c)

Then since

$D^n_\alpha(x) = D_{E\alpha}^{D^{n-1}_\alpha(x)}(x)$, it follows

$> D^{n+2}_{E\alpha}(x)$ by I.H. a)(ii)

$> D^{n+1}_{E\alpha}(x)$.

Therefore $(\forall n < \omega)$ $D^n_\alpha(x) > D^{n+1}_{E\alpha}(x)$

For 3.8 c), it is required to show that $(\forall n, x < \omega)_{>2}$ $\max(n, x) + 2 < D^n_\alpha(x)$. This is proved by induction on n. $D^2_\alpha(x) = D^x_{E\alpha}(x)$ by 3.7c

$> \max(x, x) + 2$ by I.H. c)

$\geq \max(2, x) + 2$.

Now suppose for i.i.h. that the inequality has been proved $(\forall k < n)$, for some $n > 2$. Then

$$D^n_\alpha(x) = D^{D^{n-1}_\alpha(x)}_{P\alpha}(x) \quad \text{by } 3.7c$$

$$> D^{\max(n-1,x)+2}_{P\alpha}(x) \quad \text{by i.i.h.}$$

$$> \max\{\max(n-1,x)+x\}+2 \quad \text{by I.H. c)}$$

$$> \max(n,x)+2.$$

Thus the inequality holds for every $n<\omega$, $n>2$, and this concludes the cases a) — c) of 3.8 for α a successor.

Now suppose α is a limit ordinal. Then for 3.8 a):

a) (i)

$$D^2_\alpha(x+1) = D^2_{\Omega_\alpha(x+1)}(x+1) \quad \text{by } 3.7c$$

$$> D^2_{\Omega_\alpha(x)}(x+1) \quad \text{since } \Omega \text{ is a perfect f.s.a.}$$

$$> D^2_{\Omega_\alpha(x)}(x) \quad \text{by I.H. a) (i)}$$

$$= D^2_\alpha(x)$$

Now suppose for inner inductive hypothesis that there is an $n>2$ such that $(\forall k<n)_{k\geq 2}$ $(\forall x)\ D^k_\alpha(x) < D^k_\alpha(x+1)$.

Then $D^n_\alpha(x+1) = D^2_{\Omega_\alpha\{D^{n-1}_\alpha(x+1)\}}(x+1)$ by 3.7 c)

$> D^2_{\Omega_\alpha\{D^{n-1}_\alpha(x+1)\}}(x)$, by

I.H. a) (i), since $\Omega_\alpha\{D^{n-1}_\alpha(x+1)\} < \alpha$

$> D^2_{\Omega_\alpha\{D^{n-1}_\alpha(x)\}}(x)$, since

$D^{n-1}_\alpha(x) < D^{n-1}_\alpha(x+1)$ by i.i.h,

and since Ω is perfect

$= D^n_\alpha(x)$, therefore $(\forall n < \omega)_{n \geq 2}$

$D^n_\alpha(x) < D^n_\alpha(x+1)$.

a) (ii)

By definition of Ω, $\Omega_\alpha(0) > 0$, therefore

$D^2_{\Omega_\alpha(0)}(x) > 0$, by I.H. c). Furthermore, since Ω is perfect, it follows that $(\forall n < \omega)$

$D^2_{\Omega_\alpha(n)}(x) < D^2_{\Omega_\alpha(n+1)}(x)$. Therefore $(\forall n < \omega)$

$D^2_{\Omega_\alpha(n)}(x) > n$.

Now $D^{n+1}_\alpha(x) = D^2_{\Omega_\alpha\{D^n_\alpha(x)\}}(x)$

$> D^n_\alpha(x)$ by above argument.

a)(iii)

$$D^2_{\alpha+1}(y) = D^y_\alpha(y)$$

$$= D^2_{\Omega_\alpha}\{D^{y-1}_\alpha(y)\}(y)$$

$$> D^{y-1}_\alpha(y)$$

Repetition of this step proves $D^2_{\alpha+1}(y) > D^{y-2}_\alpha(y)$; and finite iteration of this step proves $D^2_{\alpha+1}(y) > D^2_\alpha(y)$.

Suppose for i.i.h that for some $n > 2$ it has been demonstrated, for every $n' < n$, that $D^{n'}_{\alpha+1}(y) > D^{n'}_\alpha(y)$. Then $D^n_{\alpha+1}(y) = D_\alpha^{D^{n-1}_{\alpha+1}(y)}(y)$ by 3.7c. Now $D^{n-1}_{\alpha+1}(y) > D^{n-1}_\alpha(y)$ by i.i.h., and so by the proof already given that $(\forall x, x')\ D^{x'+1}_\alpha(x) > D^{x'}_\alpha(x)$, it follows $D^n_{\alpha+1}(y) > D_\alpha^{D^{n-1}_\alpha(y)}(y)$, therefore $D^{n-1}_\alpha(y) > n+1 > n$, since $D^2_\alpha(y) = D^2_{\Omega_\alpha(y)}(y) > \max(2,y)+2$ by I.H. c).

Thus $D^n_{\alpha+1}(y) > D^n_\alpha(y)$, and so it is proved by induction that $(\forall n, y < \omega)_{>2}\ D^n_{\alpha+1}(y) > D^n_\alpha(y)$.

For 3.8 b):

b) (iii). This case constitutes the 2-nd hypothesis of this section,

HYPOTHESIS 3.8 b) (iii).

$$(\forall n, x, y < \omega)_{\substack{x \geq 3 \\ n > 1}} \; (\forall \alpha < \omega_1)_{\mathrm{Lim}(\alpha)} \; D^{n+1}_{\Omega_\alpha(y)}(x) < D^{n}_{\Omega_\alpha(y+1)}(x)$$

This seems likely for Ω perfect, but difficult to prove without knowing more about Ω.

EXAMPLE. Let $\alpha = \omega.2, y = 0, n = 2$, $\{\Omega_\alpha(y)\}_{y<\omega} = \{\omega+n\}_{n<\omega}$. Then $D^3_{\Omega_\alpha(y)}(x) = D^3_\omega(x)$, the 3-rd power of the Ackermann function applied to x. Since it can readily be shown by induction that for $\{\Omega_\omega(y)\}_{y<\omega} = \{y\}_{y<\omega}$, $(\forall x)_{x>3}\ \lambda n.\ D^n_\omega(x)$ is strictly increasing, and since $x > 3$, it follows

$$D^3_{\Omega_\alpha(y)}(x) < D^2_{\Omega_\alpha(y+1)}(x).$$

b (ii). In this case there is nothing to prove, as $\neg\,\mathrm{Suc}(\alpha)$.

For 3.8 c): this also is proved by induction.

$$D^2_\alpha(x) = D^2_{\Omega_\alpha(x)}(x)$$

$$> \max(2, x) + 2 \quad \text{by I.H. c)}$$

Suppose the inequality of theorem 3.8 c) has been proved to hold for each number n' smaller than some $n > 2$, so that $n' \geq 2$. Then

$$D^n_\alpha(x) = D^2_{\Omega_\alpha\{D^{n-1}_\alpha(x)\}}(x)$$

$$\geq D^2_{\Omega_\alpha\{\max(n-1, x)+2\}}(x), \quad \text{by i.i.h.}$$

and by Ω being a p.f.s.α. Also since Ω is perfect and

$$D^2_{\Omega_\alpha(0)}(x) > 1, \quad (\forall k)\ D^2_{\Omega_\alpha(k)}(x) > k+1$$

therefore $D^n_\alpha(x) > \{\max(n-1, x) + 2\} + 1$

$$\geq \max(n, x) + 2$$

Thus it follows by transfinite induction that $(\forall \alpha < \omega_1)_{\alpha > 2}$, the inequalities of theorem 3.8 are proved to hold. □

MAJORIZATION THEOREMS. From theorem 3.8, several majorization properties of the D^n_α can be deduced

DEFINITION $(\forall f, g : N \to N)$ f *majorizes* g iff $(\exists y)(\forall x)\,[y < x \Rightarrow g(x) < f(x)]$. The relation is written $g <_m f$.

COROLLARY 3.9 $(\forall n < \omega)_{n > 2}\, (\forall \alpha < \omega_1)_{2 \leq \alpha}$

$$\lambda x.\, D^n_\alpha(x) <_m \lambda x.\, D^{n+1}_\alpha(x)$$
$$<_m \lambda x.\, D^n_{\alpha+1}(x)$$

PROOF follows from theorem 3.8 a)(ii) and b)(ii).

THEOREM 3.10 $(\forall n < \omega)_{n \geq 2}\, (\forall \alpha < \omega_1)_{\alpha > 2}\, (\forall \beta < \alpha)_{\beta > 0}$

$\lambda x.\, D^n_\beta(x) <_m \lambda x.\, D^n_\alpha(x)$.

PROOF by transfinite induction on α.

Let α be any finite ordinal > 2. Then 3.10 is a standard result for primitive recursive functions. Now suppose for induction hypothesis that 3.10 has been proved true for all ordinals $\alpha' < \alpha$, for some countable $\alpha \geq \omega$. Then if $Suc(\alpha)$, $(\forall n)_{n \geqslant 2} (\forall x)_{x > 2}$

$$D_\alpha^{n+1}(x) = D_{P\alpha}^{D_\alpha^n(x)}(x) \quad \text{by } 3.7\ c$$

$$> D_{P\alpha}^{\max(n,x)+2}(x) \quad \text{by } 3.8\ c)$$

$$> D_{P\alpha}^{n+1}(x) \quad \text{by } 3.8\ a)\ (ii)$$

Now suppose $Lim(\alpha)$. This case is proved by induction on the power n. For $n = 2$,

$D_\alpha^2(x) = D_{\Omega_\alpha(x)}^2(x)$ by 3.7c. Now suppose $\beta < \alpha$, $\beta > 0$. Then let y_β be the smallest integer such that $\beta < \Omega_\alpha(y_\beta)$. Then by I.H.

$$\lambda x . D_\beta^2(x) <_m \lambda x . D_{\Omega_\alpha(y_\beta)}^2(x).$$

But as Ω is a p.f.s.a., $(\forall x < \omega)\ x > y_\beta \Rightarrow$

$$D_{\Omega_\alpha(y_\beta)}^2(x) < D_{\Omega_\alpha(x)}^2(x) = D_\alpha^2(x)$$

Therefore $\lambda x . D_{\Omega_\alpha(y_\beta)}^2(x) <_m \lambda x . D_\alpha^2(x)$, and so $\lambda x . D^2_\beta(x) <_m \lambda x . D^2_\alpha(x)$, for each $\beta < \alpha$. Let $n \geqslant 2$, and suppose for i.i.h. that the case has been proved $(\forall m \leq n)$, and let β be any ordinal $< \alpha$.

Let β^* be the smallest limit ordinal such that $\beta \leq \beta^* < \alpha$, if there is such an ordinal. Then by I.H. $(\exists y)(\forall x)_{x > y} \; D^{n+1}_\beta(x) \leq D^{n+1}_{\beta^*}(x)$

Now let $x > \max(y, z^{(1)}, z^{(2)}, z^{(3)})$, where

$$z^{(1)} =_{Df} (\mu t)_{<\omega}\left[D^n_\beta(t) < D^n_\alpha(t)\right]$$ (there is such a t by i.i.h.)

$$z^{(2)} =_{Df} (\mu t)_{<\omega}\left[\beta^* < \Omega_\alpha\{D^n_\alpha(t)\}\right]$$

$$z^{(3)} =_{Df} (\mu t)_{<\omega}\left[(\forall r)\, t < r \Rightarrow D^2_{\Omega_{\beta^*}\{D^n_\alpha(z^{(2)})\}}(r) < D^2_{\Omega_\alpha\{D^n_\alpha(z^{(2)})\}}(r)\right]$$

$$D^{n+1}_{\beta^*}(x) = D^2_{\Omega_{\beta^*}\{D^n_{\beta^*}(x)\}}(x) \quad \text{by } 3.7c$$

$$< D^2_{\Omega_{\beta^*}\{D^n_\alpha(x)\}}(x), \text{ since } D^n_{\beta^*}(x) < D^n_\alpha(x)$$

by lower bound on x, and by i.i.h., and by Ω being a f.f.-s.a. Therefore

$D^{n+1}_{\beta^*}(x) < D^2_{\Omega_\alpha\{D^n_\alpha(x)\}}(x)$, since x is such that

$M_{\Omega}(\alpha, \beta^*) < D^n_\alpha(x)$, and by l. b. on x, and by I. H. Thus $D^{n+1}_\beta(x) < D^{n+1}_\alpha(x)$ by 3.7c. Alternatively, if each β^* such that $\beta \leq \beta^* < \alpha$ is a successor ordinal, then α is of the form $\alpha = \alpha' + \omega$, and β of the form $\beta = \alpha' + k$, some $k < \omega$, $k > 0$. Thus

$$D^{n+1}_\alpha(x) = D^{n+1}_{\alpha'+\omega}(x)$$

$= D^2_{\alpha' + D^n_{\alpha'+\omega}(x)}(x)$, by 3.7c, and where it is assumed that $(\forall t < \omega)$ $\Omega_\alpha(t) = \alpha' + t$

$> D^2_{\alpha'+k-1+D^n_{\alpha'+k}(x)}(x)$ by lower bound on x, and 3.8 a)(iii), where x

satisfies $x \geq \max(z^{(1)}, z^{(2)})$, for

$$z^{(1)} =_{Df} (\mu t)_{<\omega}\left[(\forall r)\, r > t \Rightarrow (k-1) + D^n_{\alpha'+k}(r) < D^n_{\alpha'+\omega}(r)\right]$$

(such a t can be found by iterating i. i. h. k times for ordinals $\beta = \alpha'+k, \alpha'+k+1, \ldots, \alpha'+k+(k-1)$)

$$z^{(2)} =_{Df} (\mu t)_{<\omega}\left[(\forall r)\, r > t \Rightarrow D^2_{\alpha'+k-1+D^n_{\alpha'+k}(z^{(1)})}(r) < D^2_{\alpha'+D^n_{\alpha'+\omega}(z^{(1)})}(r)\right]$$

(by I. H., there exists such a t)

Thus

$D^{n+1}_\alpha(x) > D^{2+D^n_{\alpha'+k}(x)}_{\alpha'+k-1}(x)$ by 3.8 b) (iii)

$$> D_{\alpha'+k-1}^{D^n_{\alpha'+k}(x)}(x) \text{ by } 3.8 \text{ a) (ii)}$$

$$= D^{n+1}_{\alpha'+k}(x) \text{ by } 3.7 \text{ c}$$

$$= D^{n+1}_{\beta}(x)$$

Hence it follows by transfinite induction that the theorem is proved. □ 3.10

INVERSES. Integer-valued inverses can readily be defined for the operations D_α:

DEFINITION 3.11 $(\forall \alpha < \omega_1)_{>1} (\forall n, x < \omega)_{x,n>0}$

$$rt_{\alpha,n}(x) = (\max y)[D^n_\alpha(y) \leq x]$$

$$\log_{\alpha,n}(x) = (\max y)[D^y_\alpha(n) \leq x].$$

Thus, for example, $rt_{3,n}(x) = [\sqrt[n]{x}]$; $\log_{3,n}(x) = [\log_n(x)]$. Also, $rt_{2,n}(x)$

$$= (\max y)\ y \cdot n \leq x$$

$$= \left[\frac{x}{n}\right]$$

$$= (\max y)\ n.y \leq x$$

$$= \log_{2,n}(x).$$

In these examples, square bracket expressions denote the integral part of the value denoted by their contents.

EVIDENCE IN SUPPORT OF THE HYPOTHESES

In the course of his construction of a set of points belonging to the continuum of cardinal number $\aleph_1$, G. H. Hardy remarks:

'In the case of the comparatively early members of the second (number) class, it is generally evident that a γ which has no predecessor is most naturally regarded as the limit of one particular sequence $\beta_1, \beta_2, \ldots$. Thus it is natural to regard ω^ω as the limit of

$$1, \omega, \omega^2, \omega^3, \ldots$$

ω^{ω^ω} as the limit of

$$\omega, \omega^\omega, \omega^{\omega^2}, \omega^{\omega^3}, \ldots$$

and ε_0 (1), the first of Cantor's ε-numbers, as the limit of

$$\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots,$$

S. S. Wainer [21] has presented a class of functions $F_\alpha^n : \omega \to \omega$, $n < \omega$, $\alpha < \omega_1$, defined by iteration and diagonalization at successor and limit ordinals respectively. For a choice of fundamental sequence assignment $\lambda\alpha, n . \{\alpha\}(n)$ of the kind envisaged by Hardy, restricted to limit ordinals $\alpha < \varepsilon_0$, Wainer has shown that $(\forall x < \omega)\ F^0_{\{\alpha\}(x)}(x+1) < F^0_{\{\alpha\}(x+1)}(x+1)$, which gives support to hypothesis 3.7b.

With regard to hypothesis 3.8b(iii), let $\alpha = \omega$. Then $\lambda y, n, x .\ D^n_{\Omega_\omega(y)}(x) = \lambda y, n, x .\ D^n_y(x)$, and this is the Ackermann function, which has the property of hypothesis 3.8b(iii).

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(1) In his paper, Hardy uses 'ϵ_1' to denote this number.

§4. SOME PROPERTIES OF FUNCTIONS IN THE CLASS $\mathfrak{D}^{(2)}_{\Omega}$.

Some normal functions in the class $\mathfrak{D}^{(2)}_{\Omega}$ are singled out and examined, and then the notion of leading term is developed.

Not every function in the class $\mathfrak{D}^{(2)}_{\Omega}$ is strictly increasing (s.i.); it can readily be shown that the statement '$(\forall \beta, \gamma < \omega_1)_{\beta > 1, \gamma \geq \omega} \lambda\alpha . \mathfrak{D}^{\beta}_{\alpha}(\gamma)$ is s.i' is false, by considering the following counter example: For each positive $n < \omega$ let λ_n be the principal number of multiplication expressed by $\overbrace{\omega^{\omega^{\cdots\omega}}}^{n \text{ terms}}$. Then

$$\mathfrak{D}^{\varepsilon_0}_{3}(\omega) = \lim_{n<\omega} \mathfrak{D}^{\lambda_n}_{3}(\omega) = \lim_{n<\omega} \omega^{\lambda_n}$$

$$= \lim_{n<\omega} \lambda_{n+1} = \varepsilon_0 \text{, and}$$

$$\mathfrak{D}^{\varepsilon_0}_{2}(\omega) = \lim_{n<\omega} \mathfrak{D}^{\lambda_n}_{2}(\omega) = \lim_{n<\omega} \omega \cdot \lambda_n$$

$$= \lim_{n<\omega} \lambda_n = \varepsilon_0$$

Thus $f = \lambda\alpha . \mathfrak{D}^{\varepsilon_0}_{\alpha}(\omega)$ is not s.i., as $f(2) = f(3)$. This property does not depend on any arbitrary features of definition 3.1, since only the standard operations of multiplication and exponentiation are used. And it is clear that similar equal functional values can be obtained by choosing an exponent suitably larger than ε_0. On the other hand, it is shown below that $(\forall \gamma, \beta < \omega_1)_{\beta > 1, \gamma \geq \omega}\ \lambda\alpha.\ \Phi_\alpha^\beta(\gamma)$ is ultimately increasing.

THEOREM 4.1 $(\forall \alpha, \gamma < \omega_1)_{\alpha > 0, \gamma \geq \omega}\ \lambda\beta.\ \gamma \textcircled{\alpha} \beta$ $= \lambda\beta.\ \Phi_\alpha^\beta(\gamma)$ is strictly increasing.

PROOF. This is by nested transfinite induction on the indices α, β. By definition 3.1, for $\alpha = 1$

$$\lambda\beta.\ \Phi_1^\beta(\gamma) = \lambda\beta.\ \gamma + \beta$$, which is strictly increasing.

For $\alpha = 2, 3$:

$$\lambda\beta.\ \Phi_2^\beta(\gamma) = \lambda\beta.\ \gamma\beta$$

$$\lambda\beta.\ \Phi_3^\beta(\gamma) = \lambda\beta.\ \gamma^\beta$$, both of which are strictly increasing.

For induction hypothesis (I.H.) suppose $(\forall \gamma < \omega_1)_{\geq \omega}$ $(\forall \alpha' < \alpha)_{> 0}$ $\lambda\beta.\ \Phi_{\alpha'}^\beta(\gamma)$ has been proved to be strictly increasings for some index $\alpha > 3$.

The proof is now in two steps. The initial step deals with the cases where β is finite in value, or is equal to ω, for α a successor or limit number. The second step deals with the remaining cases.

Suppose Suc(α). Then

$$\begin{aligned} \Phi_\alpha^0(\gamma) = 1 < \gamma &= \Phi_\alpha^1(\gamma) \\ &= \Phi_{P\alpha}^1(\gamma) \\ &< \Phi_{P\alpha}^\gamma(\gamma) \\ &= \Phi_\alpha^2(\gamma), \text{ by definition} \end{aligned}$$

3.1, and by inductive hypothesis. Also

$$\begin{aligned} \Phi_\alpha^3(\gamma) &= \Phi_{P\alpha}^{\Phi_\alpha^2(\gamma)}(\gamma) \\ &> \Phi_{P\alpha}^2(\gamma) \text{ by I.H., and by theorem} \end{aligned}$$

4.1a3a below. And if

$$\Phi_\alpha^{x+1}(\gamma) > \Phi_\alpha^x(\gamma), \text{ then by definition 3.1 and}$$

by I.H.

$$\begin{aligned} \Phi_\alpha^{x+2}(\gamma) = \Phi_{P\alpha}^{\Phi_\alpha^{x+1}(\gamma)}(\gamma) &> \Phi_{P\alpha}^{\Phi_\alpha^x(\gamma)}(\gamma) \\ &= \Phi_\alpha^{x+1}(\gamma), \text{ for} \end{aligned}$$

$x \geqslant 2$, so $(\forall x < \omega)\ \Phi_\alpha^{x+1}(\gamma) > \Phi_\alpha^x(\gamma)$.

Suppose Lim(α). Then for β = 0, 1:

$$\Phi^0_\alpha(\gamma) = 1 < \gamma = \Phi^1_\alpha(\gamma)$$

$$= \lim_{n<\omega} \Phi^1_{\Omega_\alpha(n)}(\gamma)$$

$$< \lim_{n<\omega} \Phi^2_{\Omega_\alpha(n)}(\gamma) \text{ by I.H.}$$

$$= \Phi^2_\alpha(\gamma) \text{, by definition 3.1.}$$

Now suppose $\beta = x$ such that $2 \leq x < \omega$. Then

$$\Phi^{x+1}_\alpha(\gamma) = \lim_{n<\omega} \Phi_{\Omega_\alpha(n)}^{\Phi^x_\alpha(\gamma)+1}(\gamma) \text{ by definition 3.1}$$

$$\geq \Phi^x_\alpha(\gamma) + 1 \text{ by I.H.}$$

$$> \Phi^x_\alpha(\gamma).$$

Thus it is proved that for α a successor or a limit number, $(\forall \beta < \omega)$

$$\Phi^{\beta+1}_\alpha(\gamma) > \Phi^\beta_\alpha(\gamma).$$

And since $\Phi^\omega_\alpha(\gamma) = \lim_{n<\omega} \Phi^n_\alpha(\gamma)$, it thus follows that $(\forall \beta < \omega)$

$$\Phi^\beta_\alpha(\gamma) < \Phi^\omega_\alpha(\gamma).$$

This completes the initial step. Proceeding now to the second step, for inner inductive hypothesis (i.i.h.)

suppose that for some β such that $\omega < \beta < \omega_1$ $(\forall \gamma < \omega_1)_{\geq \omega}\ \lambda \xi . \underline{\overline{\Phi}}_\alpha^\xi(\gamma) \restriction \beta$ is s.i.

In the first instance, suppose $Suc(\alpha, \beta)$ and $x = 1$, where β is expressed in the unique form $\beta = \omega\beta' + x$, $0 < \beta'$ $\beta' \leq \beta$, $x < \omega$. Then by definition 3.1,

$$\begin{aligned} \underline{\overline{\Phi}}_\alpha^\beta(\gamma) &= \underline{\overline{\Phi}}_{P\alpha}^{\underline{\overline{\Phi}}_\alpha^{\omega\beta'}(\gamma)+1}(\gamma) \\ &= \underline{\overline{\Phi}}_{P\alpha}^{\underline{\overline{\Phi}}_\alpha^{P\beta}(\gamma)+1}(\gamma) \text{ by definition of } \beta' \\ &\geqq \underline{\overline{\Phi}}_\alpha^{P\beta}(\gamma) + 1 \text{ by I.H.} \\ &> \underline{\overline{\Phi}}_\alpha^{P\beta}(\gamma) \end{aligned}$$

Thus $\lambda \xi . \underline{\overline{\Phi}}_\alpha^\xi(\gamma)$ is s.i. at $\xi = \beta$ for $Suc(\alpha, \beta)$ & $x = 1$.

In the second instance, suppose $Suc(\alpha, \beta)$ & $1 < x$, and in this instance, cases A, B are distinguished.

In case A, suppose $x = 2$. Then by definition 3.1;

$$\underline{\overline{\Phi}}_\alpha^\beta(\gamma) = \underline{\overline{\Phi}}_{P\alpha}^{\underline{\overline{\Phi}}_\alpha^{\omega\beta'+x-1}(\gamma)}(\gamma)$$

Now $\underline{\overline{\Phi}}_\alpha^{\omega\beta'+x-1}(\gamma) = \underline{\overline{\Phi}}_{P\alpha}^{\underline{\overline{\Phi}}_\alpha^{\omega\beta'}(\gamma)+1}(\gamma)$

$$\geqslant \underline{\overline{\Phi}}_\alpha^{\omega\beta'}(\gamma) + 1 \text{ by I.H.}$$

But Lim(l.h.s.) and Suc(r.h.s.) by corollary 4.1a4 below, therefore $\Phi_\alpha^{\omega\beta'+x-1}(\gamma) > \Phi_\alpha^{\omega\beta'}(\gamma)+1$

therefore $\Phi_\alpha{}^\beta(\gamma) > \Phi_{P\alpha}{}^{\Phi_\alpha^{\omega\beta'}(\gamma)+1}(\gamma)$ since $\lambda\xi.\Phi_{P\alpha}^{\xi}(\gamma)$ is s.i. by I.H.

$$= \Phi_\alpha^{P\beta}(\gamma).$$

In case B, suppose $x > 2$. Then

$$\Phi_\alpha{}^\beta(\gamma) = \Phi_{P\alpha}{}^{\Phi_\alpha^{\omega\beta'+x-1}(\gamma)}(\gamma)$$

Now $\Phi_\alpha^{\omega\beta'+x-1}(\gamma) > \Phi_\alpha^{\omega\beta'+x-2}(\gamma)$ by i.i.h.

Also $\lambda\xi.\Phi_{P\alpha}^{\xi}(\gamma)$ is s.i. by I.H., therefore

$$\Phi_{P\alpha}{}^{\Phi_\alpha^{\omega\beta'+x-1}(\gamma)}(\gamma) > \Phi_{P\alpha}{}^{\Phi_\alpha^{\omega\beta'+x-2}(\gamma)}(\gamma)$$

$$= \Phi_\alpha^{\omega\beta'+x-1}(\gamma)$$

$$= \Phi_\alpha^{P\beta}(\gamma)$$

Thus $\lambda\xi.\Phi_\alpha^{\xi}(\gamma)$ is strictly increasing at $\xi=\beta$ for Suc (α,β) and $x > 1$.

In the third instance which arises in the proof of theorem 4.1, suppose Lim (β). Then by definition 3.1:

$$\Phi_\alpha{}^\beta(\gamma) = \lim_{n<\omega} \Phi_\alpha^{\Omega_\beta(n)}(\gamma).$$ since Ω is an f.s.a.,

$(\forall\beta'<\beta)(\exists n<\omega)\ \beta' < \Omega_\beta(n)$. Therefore

$(\forall\beta'<\beta)(\exists n<\omega)\ \Phi_\alpha{}^{\beta'}(\gamma) < \Phi_\alpha{}^{\Omega_\beta(n)}(\gamma)$ by i.i.h.

But $(\forall n<\omega)\ \Phi_\alpha{}^{\Omega_\beta(n)}(\gamma) < \Phi_\alpha{}^{\Omega_\beta(n+1)}(\gamma)$ since Ω is an f.s.a., and by i.i.h.

therefore $(\forall n<\omega)\ \Phi_\alpha{}^{\Omega_\beta(n)}(\gamma) < \Phi_\alpha^\beta(\gamma)$

therefore $(\forall\beta'<\beta)\ \Phi_\alpha{}^{\beta'}(\gamma) < \Phi_\alpha{}^{\beta}(\gamma)$; that is, the function denoted by $\lambda\xi\,.\,\Phi_\alpha^\xi(\gamma)$ is strictly increasing at β.

In the fourth and final instance, suppose Lim (α) and Suc (β). Then by definition 3.1,

$$\Phi_\alpha{}^\beta(\gamma) = \lim_{n<\omega} \Phi_{\Omega_\alpha(n)}{}^{\Phi_\alpha{}^{P\beta}(\gamma)+1}(\gamma)$$

$$\geq \Phi_{\Omega_\alpha(n)}{}^{\Phi_\alpha{}^{P\beta}(\gamma)+1}(\gamma) \text{ for every } n<\omega$$

$$\geq \Phi_\alpha{}^{P\beta}(\gamma)+1 \text{ for } n>0, \text{ by I.H.}$$

$$> \Phi_\alpha{}^{P\beta}(\gamma),$$

therefore $\lambda\xi\,.\,\Phi_\alpha^\xi(\gamma)$ is s.i. at β.

Thus it follows by transfinite induction that $(\forall\gamma<\omega_1)_{\geq\omega}$ $(\forall\beta<\omega_1)_{\beta>1}$ $\lambda\xi\,.\,\Phi_\alpha{}^\xi(\gamma)\restriction\beta$ is s.i., therefore for every γ belonging to the second number class, $\lambda\beta . \Phi_\alpha{}^\beta(\gamma)$ is s.i.

Thus it follows by nested transfinite induction that

$(\forall\alpha,\gamma < \omega_1)_{\substack{\alpha \geq 0 \\ \gamma \geq \omega}} \lambda\beta . \Phi_\alpha^\beta(\gamma)$ is s.i. $\square$ 4.1

There follow results proving that in 'most' cases, the expression $\Phi_\alpha{}^\beta(\gamma)$ evaluates to a limit ordinal.

THEOREM 4.1a2 $(\forall\alpha,\beta,\gamma < \omega_1)_{\substack{\alpha \geq 1 \\ \beta > 0}}$ $\mathrm{Lim}(\gamma) \Rightarrow \mathrm{Lim}\{\Phi_\alpha{}^\beta(\gamma)\}$.

PROOF by nested transfinite induction.

For $\alpha = 2, \beta > 0$

$$\Phi_2{}^\beta(\gamma) = \gamma \cdot \beta = \begin{cases} \gamma \cdot P\beta + \gamma \text{ if } \mathrm{Suc}(\beta) \\ \lim\limits_{n<\omega} \gamma \cdot \beta_n \text{ if } \mathrm{Lim}(\beta), \end{cases}$$

therefore $\mathrm{Lim}\{\Phi_2{}^\beta(\gamma)\}$.

Suppose for inductive hypothesis that the theorem has been proved $(\forall\alpha' < \alpha)_{\alpha' \geq 1} (\forall\beta,\gamma < \omega_1)_{\beta > 0}$ for some $\alpha > 2$. Then for $\beta = 2$, $\Phi_\alpha^2(\gamma)$ $= \Phi_{P\alpha}^\gamma(\gamma)$ if $\mathrm{Suc}(\alpha)$ or $\lim\limits_{n<\omega} \Phi^2_{\Omega\alpha(n)}(\gamma)$ if $\mathrm{Lim}(\alpha)$, therefore in either case by I.H., $\mathrm{Lim}\{\Phi_\alpha^2(\gamma)\}$.

Now suppose for i.i.h. that for some $\beta > 2$, it has been proved that $(\forall \gamma < \omega_1)(\forall \beta' < \beta)_{\beta' > 0}$ $\mathrm{Lim}(\gamma) \Rightarrow \mathrm{Lim}\{\Phi_\alpha{}^{\beta'}(\gamma)\}$. Then if $\mathrm{Suc}(\alpha, \beta)$, by definition 3.1

$$\Phi_\alpha{}^\beta(\gamma) = \Phi_{P\alpha}{}^{\Phi_\alpha{}^{P\beta}(\gamma)}(\gamma) \text{ or } \Phi_{P\alpha}{}^{\Phi_\alpha^{P\beta}(\gamma)+1}(\gamma)$$

$$\text{or } \Phi_{P\alpha}{}^{\gamma}(\gamma) \text{ or } \gamma$$

thus in every case the I.H. gives $\mathrm{Lim}\{\Phi_\alpha{}^\beta(\gamma)\}$.

If $\mathrm{Lim}(\alpha)$ and $\mathrm{Suc}(\beta)$:

$$\Phi_\alpha^{\beta}(\gamma) = \lim_{n<\omega} \Phi_{\Omega_\alpha(n)}{}^{\Phi_\alpha{}^{P\beta}(\gamma)+1}(\gamma)\text{, and thus}$$

the I.H. gives $\mathrm{Lim}\{\Phi_\alpha{}^\beta(\gamma)\}$.

If $\mathrm{Lim}(\beta)$:

$$\Phi_\alpha{}^\beta(\gamma) = \lim_{n<\omega} \Phi_\alpha{}^{\Omega_\beta(n)}(\gamma)\text{, and thus the}$$

i.i.h. gives $\mathrm{Lim}\{\Phi_\alpha{}^\beta(\gamma)\}$. Thus it follows by transfinite induction that $(\forall \beta, \gamma < \omega_1)_{\beta > 0}$ $\mathrm{Lim}(\gamma) \Rightarrow \mathrm{Lim}\{\Phi_\alpha{}^\beta(\gamma)\}$, therefore by nested transfinite induction theorem 4.1a2 is proved. □

THEOREM 4.1a3. $(\forall \alpha, \beta, \gamma < \omega_1)_{\alpha > 2, \beta, \gamma \geq \omega}$ $\mathrm{Suc}(\gamma) \Rightarrow \mathrm{Lim}\{\Phi_\alpha^\beta(\gamma)\}$.

PROOF. Theorem 4.1a3 is a special case of theorem 4.1a3a which follows: □

THEOREM 4.1a3a

$(\forall \alpha, \beta, \gamma < \omega_1)_{\alpha \geq 3, \beta \geq \omega, \gamma \geq 2}$ $\mathrm{LIM}(\Phi_\alpha^\beta(\gamma))$.

PROOF, by nested transfinite induction.

In addition to considering successor and limit cases for α, β, there are separate cases for each of: $\gamma = 2$, $2 < \gamma < \omega$, $\omega \leq \gamma < \omega_1$. Furthermore, the cases $3 \leq \alpha < \omega$, $\alpha = \omega$ each have to be distinguished from the case $\omega \leq \alpha < \omega_1$. Thus the proof of this theorem is expatiated by a multiplicity of cases.

The proof is given in three steps: the primary proof deals with finite values of indices α, β, γ. The <u>initial proof</u>, which includes the primary proof, deals with all cases such that α lies in the range $3 \leq \alpha \leq \omega$.

In the initial proof, suppose that α is such that $3 \leq \alpha \leq \omega$.

Suppose firstly that α is such that $3 \leq \alpha < \omega$.

Suppose γ is finite, with $2 \leq \gamma$.

Suppose $\beta = \omega$.

Then $\Phi_\alpha^\omega(\gamma) = \lim\limits_{n<\omega} \Phi_\alpha^n(\gamma)$

Now $(\forall \alpha, n, \gamma < \omega)_{\alpha \geq 3, n \geq 3, \gamma \geq 2}$

$\lambda\alpha,n,\gamma\,.\,\Phi^{n}_{\alpha}(\gamma) = \lambda\alpha,n,\gamma\,.\,g_{\alpha}(n,\gamma)$, the Ackermann function.

In the stated ranges of values of each of it's arguments, this number - theoretic function is strictly increasing. Therefore $(\forall\alpha,\gamma<\omega)_{\alpha\geq 3,\,\gamma\geq 2}$

$$\Phi^{\omega}_{\alpha}(\gamma) = \omega, \text{ i.e. } \mathrm{LIM}(\Phi^{\omega}_{\alpha}(\gamma)).$$

This is called the primary proof.

Suppose now β is such that $\omega < \beta < \omega_1$.

Then $\mathrm{LIM}(\gamma^{\beta})$ or $\mathrm{LIM}(\Phi^{\beta}_{3}(\gamma))$, because suppose $\gamma = x_{\gamma} < \omega$, $x_{\gamma} \geq 2$. Then if β is such that $\omega < \beta < \omega_1$, and $\beta = \omega\beta' + x_{\beta}$, then

$\gamma^{\beta} = x_{\gamma}^{\omega\beta'+x_{\beta}} = x_{\gamma}^{\omega.\beta'}.\,x_{\gamma}^{x_{\beta}}$, which is a limit number.

Suppose for O.I.H. that for some a such that $3 \leq a < \omega$ $(\forall\beta<\omega_1)_{>\omega}(\forall\gamma<\omega)_{\geq 2}\ \mathrm{LIM}(\Phi^{\beta}_{a}(\gamma))$.

Suppose for i.i.h. that for some $\beta < \omega_1$ such that $\beta > \omega$ $(\forall\beta'<\beta)\ \mathrm{LIM}(\Phi^{\beta'}_{a+1}(\gamma))$.

Then if $\mathrm{Suc}(\beta)$ and $\mathrm{Lim}(P\beta)$:

$$\Phi^{\beta}_{a+1}(\gamma) = \Phi_{a}^{\Phi^{P\beta}_{a+1}(\gamma)+1}(\gamma) \text{ by definition 3.1}$$

$$= \Phi_{a}^{\lambda+1}(\gamma) \text{ such that } \mathrm{Lim}(\lambda)$$

by i.i.h., therefore $\mathrm{LIM}(\Phi^{\beta}_{a+1}(\gamma))$ by O.I.H.

If $Suc(\beta, P_\beta)$:

$$\Phi_{a+1}^{\beta}(\gamma) = \Phi_a^{\Phi_{a+1}^{P_\beta}(\gamma)}(\gamma)$$ by definition 3.1

$$= \Phi_a^{\lambda}(\gamma)$$ such that $Lim(\lambda)$ by i.i.h.

therefore $LIM(\Phi_{a+1}^{\beta}(\gamma))$ by O.I.H.

If $Lim(\beta)$,

$$\Phi_{a+1}^{\beta}(\gamma) = \lim_{n<\omega} \Phi_{a+1}^{\Omega\beta(n)}(\gamma)$$ by definition 3.1

= the limit of an increasing sequence of finite numbers, or a limit of finite numbers and limit numbers by i.i.h., therefore $LIM(\Phi_{a+1}^{\beta}(\gamma))$.

Therefore $(\forall\beta<\omega_1)_{\geq\omega}\ LIM(\Phi_{a+1}^{\beta}(\gamma))$ by transfinite induction and by primary proof.

Therefore $(\forall a<\omega)_{a\geq 3}(\forall\beta<\omega_1)_{\geq\omega}\ LIM(\Phi_a^{\beta}(\gamma))$ by transfinite induction and by primary proof.

Now suppose γ is transfinite.

Then $(\forall\beta<\omega_1)_{\beta\geq\omega}\ LIM(\gamma^{\beta})$ or $LIM(\Phi_3^{\beta}(\gamma))$, because suppose the ordinals γ, β are expressed in the unique form

$$\gamma = \omega\gamma' + x_\gamma$$

$$\beta = \omega\beta' + x_\beta, \quad x_\gamma, x_\beta < \omega$$

Then $\gamma^\beta = (\omega\gamma' + x_\gamma)^{\omega\beta' + x_\beta}$

$= (\omega\gamma' + x_\gamma)^{\omega\beta'} . (\omega\gamma' + x_\gamma)^{x_\beta}$

Now either LIM$\{(\omega\gamma'+x_\gamma)^{x_\beta}\}$ or SUC$\{(\omega\gamma'+x_\gamma)^{x_\beta}\}$.

The former implies LIM(γ^β).

The latter implies $\gamma^\beta = (\omega\gamma'+x_\gamma)^{\omega\beta'} . [P\{(\omega\gamma'+x_\gamma)^{x_\beta}\}+1]$

$$= \lim_{\mu < (\omega\gamma'+x_\gamma)^{\omega\beta'}} [(\omega\gamma'+x_\gamma)^{\omega\beta'} . P\{(\omega\gamma'+x_\gamma)^{x_\beta}\}+\mu],$$

since $0<\beta'$, and thus LIM$(\omega\gamma'+x_\gamma)^{\omega\beta'}$. Therefore LIM$(\gamma^\beta)$.

Thus, suppose for O.I.H. that for some a such that $3 < a < \omega$ it has been proved that $(\forall\gamma<\omega_1)_{\geq\omega}(\forall\beta<\omega_1)_{\beta\geq\omega}$ LIM$(\Phi_a^\beta(\gamma))$. Then

$$\Phi_{a+1}^{\omega}(\gamma) = \lim_{n<\omega} \Phi_{a+1}^{n}(\gamma)$$

$$= \lim\{\Phi_a^\gamma(\gamma), \Phi_a^{\Phi_a^\gamma(\gamma)}(\gamma), \ldots\} \quad \ldots \quad (0)$$

Now $\omega \leq \gamma$, therefore LIM$(\Phi_a^\gamma(\gamma))$ by O.I.H, therefore $\Phi_{a+1}^\omega(\gamma)$ is the limit of a sequence of limit numbers by O.I.H., therefore LIM$(\Phi_{a+1}^\omega(\gamma))$.

Now suppose for i.i.h. that for some β such that $\omega < \beta < \omega_1$, $(\forall\beta'<\beta)_{\geq\omega}(\forall\gamma<\omega_1)_{\geq}$ LIM$(\Phi_{a+1}^{\beta'}(\gamma))$.

Then if Suc(β) and Lim($P\beta$), $(\forall\gamma<\omega_1)_{\geq\omega}$

$$\Phi^{\beta}_{a+1}(\gamma) = \Phi_a^{\Phi^{P\beta}_{a+1}(\gamma)+1}(\gamma) \text{ by definition 3.1}$$

$$= \Phi_a^{\lambda+1}(\gamma) \text{ such that Lim}(\lambda) \text{ by}$$

i.i.h., therefore LIM$(\Phi^{\beta}_{a+1}(\gamma))$ by O.I.H.

If Suc(β, $P\beta$), $(\forall\gamma<\omega_1)_{\geq\omega}$

$$\Phi^{\beta}_{a+1}(\gamma) = \Phi_a^{\Phi^{P\beta}_{a+1}(\gamma)}(\gamma) \text{ by definition 3.1}$$

$$= \Phi_a^{\lambda}(\gamma) \text{ such that Lim}(\lambda) \text{ by i.i.h.}$$

therefore LIM$(\Phi^{\beta}_{a+1}(\gamma))$ by O.I.H.

If Lim(β)

$$\Phi^{\beta}_{a+1}(\gamma) = \lim_{n<\omega} \Phi_{a+1}^{\Omega\beta(n)}(\gamma),$$ which is a limit of limit numbers by property of sequence (0), and by i.i.h., therefore LIM$(\Phi^{\beta}_{a+1}(\gamma))$.

Thus it follows by transfinite induction that $(\forall\gamma,\beta<\omega_1)_{\omega\leq\beta,\gamma}$ LIM$(\Phi^{\beta}_{a+1}(\gamma))$.

Thus it follows by induction that $(\forall a<\omega)_{\geq 3}$ $(\forall\gamma,\beta<\omega_1)_{\omega\leq\beta,\gamma}$ LIM$(\Phi^{\beta}_{a}(\gamma))$.

Thus by primary proof, and proof for the case $2\leq\gamma<\omega$, $\omega<\beta<\omega_1$, $3\leq a<\omega$, it follows:

$(\forall\alpha<\omega)_{\alpha\geq 3}(\forall\beta,\gamma<\omega_1)_{\beta\geq\omega,\gamma\geq 2}$ LIM$(\Phi^{\beta}_{\alpha}(\gamma))$.

To complete the initial proof, suppose now $\alpha = \omega$.
Suppose $\beta = \omega$.
Suppose $\gamma = 2$. Then

$$\Phi_{\omega}^{\omega}(2) = \lim_{n<\omega} \Phi_{\omega}^{n}(2)$$

$$= \lim\{\Phi_{\omega}^{2}(2), \lim_{n<\omega} \Phi_{n}^{\Phi_{\omega}^{2}(2)+1}(2), \ldots\} \quad \ldots (2)$$

Now $\Phi_{\omega}^{2}(2) = \lim_{n<\omega} \Phi_{n}^{2}(2) = 4$ by theorem 4.15(i) (This theorem relies only upon definition 3.1)

$$\lim_{n<\omega} \Phi_{n}^{5}(2) = \lim\{\Phi_{3}^{5}(2), \Phi_{4}^{5}(2), \Phi_{5}^{5}(2) \ldots\}$$

$= \omega$ by the strictly increasing property of the Ackemann function, and by the identity of the finite restriction of Φ with this function.

Therefore (2) is the limit of limit numbers by the proofs for the cases previously considered where α is such that $3 \leq \alpha < \omega$, therefore $\mathrm{LIM}(\Phi_{\omega}^{\omega}(2))$.

Now suppose $\gamma > 2$ but still finite. Then

$$\Phi_{\omega}^{\omega}(\gamma) = \lim_{n<\omega} \Phi_{\omega}^{n}(\gamma)$$

$$= \lim\{\Phi_{\omega}^{2}(\gamma), \lim_{n<\omega} \Phi_{n}^{\Phi_{\omega}^{2}(\gamma)+1}(\gamma), \ldots\} \quad \ldots (3)$$

Now $\Phi_\omega^2(\gamma) = \lim_{n<\omega} \Phi_n^2(\gamma) = \lim_{n<\omega} \Phi_{n+1}^2(\gamma)$

$= \lim_{n<\omega} \Phi_n^\gamma(\gamma)$, which is a limit of limit numbers by proof for cases already considered where α is such that $3 \leq \alpha < \omega$.

Therefore $\mathrm{LIM}(\Phi_\omega^2(\gamma))$, therefore (3) is the limit of limit numbers by the same proof, therefore $\mathrm{LIM}(\Phi_\omega^2(\gamma))$.

Now suppose γ is transfinite. Then

$$\Phi_\omega^\omega(\gamma) = \lim_{n<\omega} \Phi_\omega^n(\gamma)$$

$$= \lim\{\Phi_\omega^2(\gamma), \lim_{n<\omega} \Phi_n^{\Phi_\omega^2(\gamma)+1}(\gamma), \ldots\} \quad \ldots (3')$$

the first term, and therefore the whole sequence is a limit of limit numbers by the proof for finite $\alpha \geq 3$, therefore $\mathrm{LIM}(\Phi_\omega^\omega(\gamma))$.

Suppose for i.h. that for some $\beta > \omega$ it has been proved that $(\forall \beta' < \beta)_{\geq\omega} (\forall \gamma < \omega_1)_{\geq 2}\, \mathrm{LIM}(\Phi_\omega^{\beta'}(2))$.

Suppose γ is such that $2 \leq \gamma < \omega_1$.

Then if $\mathrm{Suc}(\beta)$:

$$\Phi_\omega^\beta(\gamma) = \lim_{n<\omega} \Phi_n^{\Phi_\omega^{P\beta}(\gamma)+1}(\gamma)$$

$$= \lim_{n<\omega} \Phi_n^{\lambda+1}(\gamma)$$, which is a limit number by proof for cases where α is such that $3 \leq \alpha < \omega$, as λ is a limit number by i. h.

If Lim(β), $\Phi_\omega^\beta(\gamma) = \lim_{n<\omega} \Phi_\omega^{\Omega_\beta(n)}(\gamma)$, which is a limit of limit numbers by properties of sequences (2), (3), (3′), and by i. h., therefore LIM$(\Phi_\omega^\beta(\gamma))$.

Therefore by transfinite induction it follows that $(\forall \beta, \gamma < \omega_1)_{\beta \geq \omega, \gamma \geq 2}$ LIM$(\Phi_\omega^\beta(\gamma))$.

Thus so far it has been shown that $(\forall \alpha \leq \omega)_{3 \leq \alpha}$ $(\forall \beta, \gamma < \omega_1)_{\beta \geq \omega, \gamma \geq 2}$ LIM$(\Phi_\alpha^\beta(\gamma))$.

This part of the proof of theorem 4.1a 3a will be referred to as the initial proof.

The remaining part of the proof concerns the cases for which α is such that $\omega < \alpha < \omega_1$. The argument proceeds by nested transfinite induction, upon indices α, β, where β lies in the range $\omega \leq \beta < \omega_1$.

Suppose for O.I.H. that for some α such that $\omega < \alpha < \omega_1$ it has been shown $(\forall \alpha' < \alpha)_{\geq \omega}$ $(\forall \beta, \gamma < \omega_1)_{\beta \geq \omega, \gamma \geq 2}$ $\mathrm{LIM}(\Phi_{\alpha'}^{\beta}(\gamma))$.

Suppose $\mathrm{Suc}(\alpha)$ such that $\alpha = \omega\alpha' + k$, $k < \omega$.

Suppose $\beta = \omega$.

Suppose $\gamma = 2$.

Then $\Phi_\alpha^\omega(2) = \lim\limits_{n<\omega} \Phi_\alpha^n(2)$

$$= \lim_{n<\omega} \left\{ \overbrace{\Phi_{P_\alpha}^{\Phi_{P_\alpha}^{\cdot^{\cdot^{\cdot^{\Phi_{P_\alpha}^{\Phi_\alpha^2(2)}(2)}}}\cdots(2)}}}^{n-2}(2) \right\} \quad \ldots (4)$$

and $\Phi_{P_\alpha}^{\Phi_\alpha^2(2)}(2) = \Phi_{P_\alpha}^4(2)$ by theorem 4.15(i)

$$= \Phi_{P_\alpha^2}^{\Phi_{P_\alpha^2}^4(2)}(2) ;$$

repeating this decomposition $(k-1)$ times to the topmost term leads to $\Phi_{P^k{}_\alpha}^4(2)$

$$= \lim_{\alpha' < P^k{}_\alpha} \Phi_{\alpha'}^{\Phi_{P_\alpha^k}^3(2)+1}(2) ;$$

$$\Phi_{P_\alpha^k}^3(2) = \lim_{n<\omega} \Phi_{\Omega_{P_\alpha^k}(n)}^{\Phi_{P_\alpha^k}^2(2)+1}(2)$$

$$= \lim_{n<\omega} \Phi_{\Omega_{P_\alpha^k}(n)}^5(2) \text{ by theorem } 4.15(i) .$$

Let $t_1 = (\mu n)\ \omega \leq \Omega_{P_\alpha^k}(n)$. Then

$$\overline{\Phi}^3_{P^k_\alpha}(2) \geq \overline{\Phi}^5_{\Omega_{P^k_\alpha}(t_1)}(2).$$

Let $\Omega_{P^k_\alpha}(t_1) = \omega\xi + a$, $a < \omega$. Then for $a > 0$:

$$\overline{\Phi}^5_{\Omega_{P^k_\alpha}(t_1)}(2) = {}^{3}\overline{\Phi}_{\omega\xi+a-1}^{\cdot^{\cdot^{\cdot\,\overline{\Phi}_{\omega\xi+a-1}^{\overline{\Phi}^2_{\omega\xi+a}(2)}(2)}}\cdots}(2)$$

Now

$$\overline{\Phi}_{\omega\xi+a-1}^{\overline{\Phi}^2_{\omega\xi+a}(2)}(2) = \overline{\Phi}^4_{\omega\xi+a-1}(2) \text{ by theorem 4.15(i)}$$

$$= \overline{\Phi}_{\omega\xi+a-2}^{\overline{\Phi}_{\omega\xi+a-2}^{\overline{\Phi}^2_{\omega\xi+a-1}(2)}(2)}(2), \text{ and}$$

$$\overline{\Phi}_{\omega\xi+a-2}^{\overline{\Phi}^2_{\omega\xi+a-1}(2)}(2) = \overline{\Phi}^4_{\omega\xi+a-2}(2) \text{ by this theorem.}$$

Thus, by repeating this decomposition a times, the number $\overline{\Phi}^4_{\omega\xi}(2)$ is obtained from $\overline{\Phi}^5_{\Omega_{P^k_\alpha}(t_1)}(2)$.

Alternatively, if $a = 0$, then $\Omega_{P^k_\alpha}(t_1) = \omega\xi$.

Now $\overline{\Phi}^4_{\omega\xi}(2) = \lim\limits_{n<\omega} \overline{\Phi}_{\Omega_{\omega\xi}(n)}^{\lim\limits_{m<\omega} \overline{\Phi}^5_{\Omega_{\omega\xi}(m)}(2)+1}(2)$

by definition 3.1 and by theorem 4.15(i).

Now let $t_2 = (\mu n)\,\omega \leq \Omega_{\omega\xi}(n)$.

Then $\overline{\Phi}^3_{\omega\xi}(2) \geq \overline{\Phi}^5_{\Omega_{\omega\xi}(t_2)}(2) + 1$.

Call <u>process P</u> the process by which number $\Omega_{\omega\xi}(t_2)$ was obtained from number $\Omega_{P^k_\alpha}(t_1)$.

Then $(\exists r, q < \omega)$ after r repetitions of process P, a strictly decreasing sequence $\{\Omega_{\alpha_{r'}}(t_{r'})\}_{1 \leq r' < r}$ is obtained, for certain ordinals $\{\alpha_{r'}\}_{1 \leq r' < r}$, such that $\Omega_{\alpha_r}(t_r) = \omega + q$.

Now

$$\underline{\Phi}^{5}_{\omega+q}(2) = \overset{3}{\curvearrowright}\ \underline{\Phi}_{\omega+q-1}^{\cdot^{\cdot^{\cdot^{\underline{\Phi}_{\omega+q-1}^{\underline{\Phi}^{2}_{\omega+q}(2)}(2)}}}\cdots(2)},$$

and by repeating the decomposition used above q times, the number $\underline{\Phi}^{4}_{\omega}(2)$ is obtained.

$$\underline{\Phi}^{4}_{\omega}(2) = \lim_{m < \omega} \underline{\Phi}_m^{\lim\limits_{n<\omega} \underline{\Phi}^{5}_{n}(2)+1}(2) \text{ by theorem 4.15(i)}$$

$$= \lim_{m < \omega} \underline{\Phi}_m^{\omega+1}(2) \text{ by initial proof}$$

Therefore $\mathrm{LIM}(\underline{\Phi}^{4}_{\omega}(2))$ by initial proof.

Therefore $\mathrm{LIM}(\underline{\Phi}^{5}_{\omega+q}(2))$ by O.I.H.

Therefore $(\forall r' < r)_{r' > 0}$ $\mathrm{LIM}(\underline{\Phi}^{5}_{\Omega_{\alpha_{r'}}(t_{r'})}(2))$ by O.I.H.

In particular, for $\alpha_1 = \underline{P}^{k}_{\alpha}$

$$\mathrm{LIM}(\underline{\Phi}^{5}_{\Omega_{\underline{P}^{k}_{\alpha}}(t_1)}(2)),$$

therefore $\underline{\Phi}^{3}_{\underline{P}^{k}_{\alpha}}(2)$ is transfinite.

Therefore $\Phi^4_{P^k\alpha}(2) = \lim_{n<\omega} \Phi_{\Omega_{P^k_\alpha(n)}}^{\Phi^3_{P^k\alpha}(2)+1}(2)$,
as a limit of limit numbers by O.I.H., is itself a limit number.

DEFINITION. The route taken from the number $\Phi^3_{P^k\alpha}(2)$ to the number $\Phi^4_{\omega}(2)$ is called <u>route RR</u>, for subsequent reference.

Therefore, by O.I.H., at each step in the (k−1) decompositions (that is, the decompositions starting with the ordinal $\Phi^4_{P\alpha}(2)$), a limit number is obtained, therefore $\mathrm{LIM}(\Phi_{P\alpha}^{\Phi_\alpha^2(2)}(2))$, and therefore by O.I.H., $\Phi^\omega_\alpha(2)$ is the limit of limit numbers, therefore $\mathrm{LIM}(\Phi^\omega_\alpha(2))$.

Suppose now γ is finite and > 2. Then

$$\Phi^\omega_\alpha(\gamma) = \lim_{n<\omega} \Phi^n_\alpha(\gamma)$$

$$= \lim_{n<\omega}\left\{ \overbrace{\Phi_{P\alpha}^{\Phi_{P\alpha}^{\cdot^{\cdot^{\cdot^{\Phi_{P\alpha}^{\Phi_\alpha^2(\gamma)}(\gamma)}}}\cdots(\gamma)}}}^{n-2}(\gamma) \right\}$$

$$\ldots (5)$$

Now $\Phi_{\alpha}^{2}(\gamma) = \Phi_{P\alpha}^{\gamma}(\gamma)$

$$= \overbrace{\Phi_{P^2\alpha}^{\cdot^{\cdot^{\cdot^{\Phi_{P^2\alpha}^{\Phi_{P\alpha}^{2}(\gamma)}}}}}}^{(\gamma-2)}{}^{(\gamma)}\cdots(\gamma).$$

Repeating this decomposition k times to the topmost term leads to $\Phi_{P^k\alpha}^{2}(\gamma)$, and Lim$(P^k\alpha)$.

Therefore $\Phi_{P^k\alpha}^{2}(\gamma) = \lim_{n<\omega} \Phi^{2}_{\Omega_{P^k\alpha}(n)}(\gamma)$

$\geqslant \Phi_{\omega}^{\gamma}(\gamma)$ by the argument of route RR, taken in the case $\gamma = 2$ (the case for argument γ prior to the current case), but by replacing the substitution

$$\Phi_{\eta}^{\Phi_{\eta+1}^{2}(2)}(2) = \Phi_{\eta}^{4}(2), \text{ for some ordinal}$$

η (by theorem 4.15(i)), by the substitution:

$$\Phi_{\eta+1}^{2}(\gamma) = \Phi_{\eta}^{\gamma}(\gamma), \text{ for some ordinal } \eta$$

(by definition 3.1).

Thus

$$\Phi_{P^k\alpha}^{2}(\gamma) \geqslant \lim_{n_1<\omega} \Phi_{n_1}^{\cdot^{\cdot^{\cdot^{\lim_{n_{\gamma-2}<\omega} \Phi_{n_{\gamma-2}}^{\Phi_{\omega}^{2}(\gamma)+1}(\gamma)+1}}}}\cdot^{\cdot^{\cdot}}(\gamma)$$

But LIM$(\Phi_{\omega}^{2}(\gamma))$ by initial proof, therefore LIM(r. h. s.)

by initial proof, therefore $\underline{\Phi}^{2}_{P^k\alpha}(\gamma)$ is transfinite. Therefore, by O.I.H., at each step in the k decompositions, a limit number is obtained, therefore $\mathrm{LIM}(\underline{\Phi}^{2}_{\alpha}(\gamma))$, therefore by O.I.H. $\underline{\Phi}^{\omega}_{\alpha}(\gamma)$ is the limit of limit numbers, therefore $\mathrm{LIM}(\underline{\Phi}^{\omega}_{\alpha}(\gamma))$.

Suppose now γ is transfinite. Then

$$\underline{\Phi}^{\omega}_{\alpha}(\gamma) = \lim_{n<\omega} \underline{\Phi}^{n}_{\alpha}(\gamma)$$

$$= \lim\left\{\underline{\Phi}^{2}_{\alpha}(\gamma), \underline{\Phi}_{P\alpha}^{\underline{\Phi}^{2}_{\alpha}(\gamma)}(\gamma), \underline{\Phi}_{P\alpha}^{\underline{\Phi}_{P\alpha}^{\underline{\Phi}^{2}_{\alpha}(\gamma)}(\gamma)}(\gamma)\ldots\right\} \qquad \ldots (6)$$

Now $\underline{\Phi}^{2}_{\alpha}(\gamma) = \underline{\Phi}^{\gamma}_{P\alpha}(\gamma)$, which is a limit number by O.I.H., therefore $\mathrm{LIM}(\underline{\Phi}^{\omega}_{\alpha}(\gamma))$.

Suppose for i.i.h. that for some $\beta < \omega_1$ such that $\omega < \beta$ it has been shown $(\forall\gamma<\omega_1)_{\geq 2}$ $(\forall\beta'<\beta)_{\geq\omega}$ $\mathrm{LIM}(\underline{\Phi}^{\beta'}_{\alpha}(\gamma))$.

Then if $\mathrm{Suc}(\beta)$ and $\mathrm{Lim}(P\beta)$, $(\forall\gamma<\omega_1)_{\geq 2}$

$$\underline{\Phi}^{\beta}_{\alpha}(\gamma) = \underline{\Phi}_{P\alpha}^{\underline{\Phi}^{P\beta}_{\alpha}(\gamma)+1}(\gamma)$$

$$= \underline{\Phi}^{\lambda+1}_{P\alpha}(\gamma), \text{ where } \mathrm{Lim}(\lambda) \text{ by i.i.h.}$$

Therefore $\mathrm{LIM}(\underline{\Phi}^{\beta}_{\alpha}(\gamma))$ by O.I.H.

If Suc$(\beta, P\beta)$, $(\forall \gamma < \omega_1)_{\geq 2}$

$$\Phi_\alpha^\beta(\gamma) = \Phi_{P\alpha}^{\Phi_\alpha^{P\beta}(\gamma)}(\gamma)$$

$$= \Phi_{P\alpha}^{\lambda}(\gamma), \text{ where Lim}(\lambda) \text{ by i.i.h.,}$$

therefore LIM$(\Phi_\alpha^\beta(\gamma))$ by OIH.

If Lim(β), $(\forall \gamma < \omega_1)_{\geq 2}$

$\Phi_\alpha^\beta(\gamma) = \lim_{n<\omega} \Phi_\alpha^{\Omega_\beta(n)}(\gamma)$, which is a limit of limit numbers by properties of sequences (4), (5), (6) and by i.i.h., therefore LIM$(\Phi_\alpha^\beta(\gamma))$.

Suppose now Lim(α).

Suppose $\beta = \omega$

Suppose first that $\gamma = 2$. Then

$$\Phi_\alpha^\omega(2) = \lim_{n<\omega} \Phi_\alpha^n(2)$$

$$= \lim \left\{ \Phi_\alpha^2(2), \lim_{n<\omega} \Phi_{\Omega_\alpha(n)}^{\Phi_\alpha^2(2)+1}(2), \dots \right\} \quad \dots(7)$$

Now $\Phi_\alpha^2(2) = 4$ by theorem 4.15(i), and

$\lim_{n<\omega} \Phi_{\Omega_\alpha(n)}^5(2)$ is related by route RR to number $\Phi_\omega^4(2)$ which is a limit, therefore

$$\text{LIM}\left(\lim_{n<\omega} \Phi_{\Omega_\alpha(n)}^5(2)\right),$$

So by O.I.H. LIM$(\Phi_\alpha^\omega(2))$.

Now suppose γ is finite and greater than 2. Then

$$\Phi_\alpha^\omega(\gamma) = \lim_{n<\omega} \Phi_\alpha^n(\gamma)$$

$$= \lim\{\Phi_\alpha^2(\gamma), \lim_{n<\omega} \Phi_{\Omega_\alpha(n)}^{\Phi_\alpha^2(\gamma)+1}(\gamma), \dots\}$$

$$\dots (8)$$

Now $\Phi_\alpha^2(\gamma) = \lim_{n<\omega} \Phi_{\Omega_\alpha(n)}^2(\gamma)$, and this number is related to $\Phi_\omega^\gamma(\gamma)$ by route RR, in which is made, for some ordinal η, the substitution

$$\Phi_\eta^2(\gamma) = \Phi_\eta^\gamma(\gamma) \text{ (by definition 3.1)}$$

in place of the substitution

$$\Phi_\eta^{\Phi_{\eta+1}^2(2)}(2) = \Phi_\eta^4(2) \text{ (by theorem 4.15(i))}.$$

Therefore LIM$(\Phi_\alpha^2(\gamma))$, therefore by O.I.H. $\Phi_\alpha^\omega(\gamma)$ is the limit of limit numbers, and therefore itself a limit.

Now suppose γ is transfinite. Then

$$\Phi_\alpha^\omega(\gamma) = \lim_{n<\omega} \Phi_\alpha^n(\gamma)$$

$$= \lim\{\Phi_\alpha^2(\gamma), \lim_{n<\omega} \Phi_{\Omega_\alpha(n)}^{\Phi_\alpha^2(\gamma)+1}(\gamma), \dots\}$$

$$\dots (9)$$

Now $\Phi_\alpha^2(\gamma) = \lim_{n<\omega} \Phi^2_{\Omega_\alpha(n)}(\gamma)$ . . . . (10)

Then $(\forall n<\omega)$, if Suc$(\Omega_\alpha(n))$, then

$$\Phi^2_{\Omega_\alpha(n)}(\gamma) = \Phi^\gamma_{P\Omega_\alpha(n)}(\gamma) \quad . \ . \ . \ . \ (11)$$

which is a limit number by O.I.H., as the exponent γ is greater than or equal to ω.

If LIM$(\Omega_\alpha(n))$, then

$\Phi^2_{\Omega_\alpha(n)}(\gamma) = \lim_{m<\omega} \Phi^2_{\Omega_{\Omega_\alpha(n)}(m)}(\gamma)$, and the decomposition following expression (10) is repeated.

Always after a finite number of such repetitions a successor fundamental sequence term is reached, which gives a limit number of the form of expression (11).

Thus, by O.I.H., (10) is the limit of limit numbers, therefore LIM$(\Phi_\alpha^2(\gamma))$, and thus by O.I.H. (9) is the limit of limit numbers, therefore

$$\text{LIM}\,(\Phi_\alpha^\omega(\gamma)).$$

Suppose for i.i.h. that for some $\beta<\omega_1$ such that $\omega<\beta$ it has been shown $(\forall\gamma<\omega_1)_{\geq 2}$

$(\forall \beta' < \beta)_{\geq \omega} \quad \mathrm{LIM}(\Phi_\alpha^{\beta'}(\gamma))$.

Then if $\mathrm{Suc}(\beta)$, $(\forall \gamma < \omega_1)_{\geq 2}$:

$$\Phi_\alpha^{\beta}(\gamma) = \lim_{n < \omega} \Phi_{\Omega_\alpha(n)}^{\Phi_\alpha^{p_\beta}(\gamma)+1}(\gamma)$$

$$= \lim_{n < \omega} \Phi_{\Omega_\alpha(n)}^{\lambda+1}(\gamma),$$ where $\mathrm{Lim}(\lambda)$ by i.i.h.

Therefore $\mathrm{LIM}(\Phi_\alpha^{\beta}(\gamma))$ by O.I.H.

If $\mathrm{Lim}(\beta)$, $(\forall \gamma < \omega_1)_{\geq 2}$

$\Phi_\alpha^{\beta}(\gamma) = \lim_{n < \omega} \Phi_\alpha^{\Omega_\beta(n)}(\gamma)$, which is a limit of limit numbers, by properties of sequences (7), (8), (9) and by i. i. h., therefore $\mathrm{LIM}(\Phi_\alpha^{\beta}(\gamma))$.

Thus it follows by transfinite induction on β that $(\forall \beta < \omega_1)_{\beta \geq \omega} (\forall \gamma < \omega_1)_{\geq 2} \ \mathrm{LIM}(\Phi_\alpha^{\beta}(\gamma))$, thus it follows by transfinite induction on α, and by the initial proof, that $(\forall \alpha < \omega_1)_{\alpha \geq 3} (\forall \beta < \omega_1)_{\beta \geq \omega}$ $(\forall \gamma < \omega_1)_{\gamma \geq 2} \quad \mathrm{LIM}(\Phi_\alpha^{\beta}(\gamma))$

☐ THEOREM 4.1a3a.

Note that for $\alpha = 2$, $Suc(\beta,\gamma) \Rightarrow Suc(\Phi_2^{\beta}(\gamma))$,
since $\Phi_2^{\beta}(\gamma) = \gamma.(\omega\beta' + x_\beta)$

$= \gamma.\omega.\beta' + \gamma x_\beta$ by the left distributive property of ordinals, and $Suc(\gamma.x_\beta)$.

Combining theorems 4.1a2 and 3 gives:

COROLLARY 4.1a4 $(\forall\alpha,\beta,\gamma < \omega_1)_{\alpha > 2, \beta,\gamma \geq \omega}$
$Lim\{\Phi_\alpha^{\beta}(\gamma)\}$ □

Of course, since $(\forall n, x, y < \omega)_{>0}\ \Phi_n^{x}(y) < \omega$, it follows that $Suc\{\Phi_n^{x}(y)\}$.

It is now a simple matter to show that the exponent functions are normal:

COROLLARY 4.1a1 $(\forall\alpha,\gamma < \omega_1)_{\substack{\alpha \geq 0 \\ \gamma \geq \omega}}$
$\lambda\beta.\ \Phi_\alpha^{\beta}(\gamma)$ is normal.

PROOF. By theorem 4.1 the functions in question are strictly increasing. It remains to be shown that they are continuous.

$(\forall\alpha,\gamma < \omega_1)_{\substack{\alpha \geq 0 \\ \gamma \geq \omega}}\ (\forall\beta < \omega_1)\ Lim(\beta) \Rightarrow$

$\Phi_\alpha^{\beta}(\gamma) = \lim_{n<\omega} \Phi_\alpha^{\Omega_\beta(n)}(\gamma)$ by definition 3.1

$= \lim_{\beta' < \beta} \Phi_\alpha^{\beta'}(\gamma)$, since $\Phi_\alpha(\gamma)$ is s.i.

by theorem 4.1, $\{\Omega_\beta(n)\}_{n<\omega}$ is s.i. as Ω is an f.s.a. and thus $(\forall\beta' < \beta)(\exists n < \omega)\ \Phi_\alpha^{\beta'}(\gamma) < \Phi_\alpha^{\Omega_\beta(n)}(\gamma)$

□ 4.1a1

THEOREM 4.1b. $\lambda\alpha . \Phi^2_\alpha(\omega)$ is normal.

PROOF by transfinite induction. The function is strictly increasing, since

$$\Phi^2_0(\omega) = 0 < \Phi^2_1(\omega) = \omega + 2 \;;$$

and suppose for some $\alpha > 4$ that the function is strictly increasing at non-zero ordinals smaller than α. Then $Suc(\alpha) \Rightarrow$

$$\Phi^2_\alpha(\omega) = \Phi^\omega_{P\alpha}(\omega)$$
$$> \Phi^2_{P\alpha}(\omega), \text{ by theorem } 4.1$$

$Lim(\alpha) \Rightarrow$

$$\Phi^2_\alpha(\omega) = \lim_{n < \omega} \Phi^2_{\Omega_\alpha(n)}(\omega)$$
$$> \Phi^2_{\Omega_\alpha(n)}(\omega), \; (\forall n < \omega)_{>0},$$

since $\{\Omega_\alpha(n)\}_{n<\omega}$ is strictly increasing as Ω is an f.s.a., and $(\forall n)$

$$\Phi^2_{\Omega_\alpha(n)}(\omega) < \Phi^2_{\Omega_\alpha(n+1)}(\omega) \text{ by I.H.}$$

Since $(\forall \alpha' < \alpha)(\exists n < \omega)\; \Phi^2_{\alpha'}(\omega) < \Phi^2_{\Omega_\alpha(n)}(\omega)$, it follows $(\forall \alpha' < \alpha)\; \Phi^2_\alpha(\omega) > \Phi^2_{\alpha'}(\omega)$.

Therefore the function is strictly increasing at α, and so by transfinite induction $\lambda\alpha.\ \Phi^2_\alpha(\omega)$ is strictly increasing.

Furthermore, this property implies

$$(\forall\alpha)\ \mathrm{Lim}(\alpha) \Rightarrow \Phi^2_\alpha(\omega) = \lim_{n<\omega} \Phi^2_{\Omega\alpha(n)}(\omega) \text{ by } 3.1$$

$$= \lim_{\alpha'<\alpha} \Phi^2_{\alpha'}(\omega)$$

Thus $\lambda\alpha.\ \Phi^2_\alpha(\omega)$ is normal. $\square$ 4.1b

THEOREM 4.1c $(\forall\alpha,\beta,\gamma<\omega_1)_{\alpha>0,\beta>1,\gamma\geq\omega}$

$\max(\alpha,\beta,\gamma) \leq \Phi^\beta_\alpha(\gamma)$

PROOF by transfinite induction. 4.1c holds for given values of the index β, by theorem 4.1. The proof for index α follows.

Let $\gamma = \omega$, $\beta = 2$. Then $\Phi^2_1(\omega) = \omega + 2 > \alpha = 1$.

Suppose for i.i.h. for some $\alpha < \omega_1$ such that $\alpha > 1$ that $(\forall\alpha'<\alpha)_{>0}\ \alpha' \leq \Phi^2_{\alpha'}(\omega)$

Then if $\mathrm{suc}(\alpha)$, $\Phi^2_\alpha(\omega) = \Phi^\omega_{P\alpha}(\omega)$

$> \Phi^2_{P\alpha}(\omega)$ by theorem 4.1

$\geq P\alpha$ by i.i.h.

Therefore $\alpha \leq \Phi^3_\alpha(\omega)$.

If Lim(α), $\underline{\overline{\Phi}}_{\alpha}^{2}(\omega) = \lim_{n<\omega} \underline{\overline{\Phi}}^{2}_{\Omega_\alpha(n)}(\omega)$

$$\geqslant \lim_{n<\omega} \Omega_\alpha(n) \text{ by t.i.h.}$$

$$= \alpha$$

Thus the case for index α is proved by transfinite induction, for $\beta = 2$.

Suppose for I.H. $(\forall\alpha<\omega_1)_{>0}(\forall\beta'<\beta)_{\beta'>1}$ for some $\beta<\omega_1$ such that $\beta>2$, that $\alpha \leqslant \underline{\overline{\Phi}}_{\alpha}^{\beta'}(\omega)$.

Then consider the expression $\underline{\overline{\Phi}}_{\alpha}^{\beta}(\omega)$. By definition 3.1, (iii), (iv), there are six cases:

For $\alpha = 3$: $\underline{\overline{\Phi}}_3^{\beta}(\omega) = \begin{cases} \underline{\overline{\Phi}}_2^{\omega}\{\underline{\overline{\Phi}}_3^{P\beta}(\omega)\} & \text{if Suc}(\beta) \\ \lim_{n<\omega} \underline{\overline{\Phi}}_3^{\Omega_\beta(n)}(\omega) & \text{if Lim}(\beta) \end{cases}$

$$\underline{\overline{\Phi}}_2^{\omega}\{\underline{\overline{\Phi}}_3^{P\beta}(\omega)\} = \underline{\overline{\Phi}}_3^{P\beta}(\omega).\omega$$

$$> \underline{\overline{\Phi}}_3^{P\beta}(\omega)$$

$$\geqslant \alpha = 3 \text{ by I.H.}$$

and $(\forall n<\omega)\ \underline{\overline{\Phi}}_3^{\Omega_\beta(n)}(\omega) \geqslant \alpha = 3$ by I.H.

therefore $\lim_{n<\omega} \underline{\overline{\Phi}}_3^{\Omega_\beta(n)}(\omega) \geqslant \alpha = 3$.

For $\alpha > 3$:

$$\underline{\Phi}_\alpha^\beta(\omega) = \begin{cases} \underline{\Phi}_{P\alpha}^{\underline{\Phi}_\alpha^{\omega\beta'}(\omega)+1}(\omega) & \text{if } Suc(\alpha,\beta) \ \& \ x = 1 \\ \underline{\Phi}_{P\alpha}^{\underline{\Phi}_\alpha^{\omega\beta'+x-1}(\omega)}(\omega) & \text{if } Suc(\alpha,\beta \ \& \ x < 1 \\ \lim\limits_{n<\omega} \underline{\Phi}_\alpha^{\Omega\beta(n)}(\omega) & \text{if } Lim(\beta) \\ \lim\limits_{n<\omega} \underline{\Phi}_{\Omega\alpha(n)}^{\underline{\Phi}_\alpha^{P\beta}(\omega)+1}(\omega) & \text{if } Lim(\alpha) \text{ and } Suc(\beta) \end{cases}$$

$(\text{for } \beta = \omega\beta' + x,\ x < \omega)$

by definition 3.1. By theorem 4.1, the first two of these expressions are each strictly greater than $\underline{\Phi}_\alpha^{\omega\beta'}(\omega) > \alpha$ by I.H. Furthermore

$(\forall n < \omega)\ \underline{\Phi}_\alpha^{\Omega\beta(n)}(\omega) \geq \alpha$ by I.H., therefore

$\lim\limits_{n<\omega} \underline{\Phi}_\alpha^{\Omega\beta(n)}(\omega) \geq \alpha$ by I.H.

And finally, $(\forall n < \omega)_{n>0}\ \underline{\Phi}_{\Omega\alpha(n)}^{\underline{\Phi}_\alpha^{P\beta}(\omega)+1}(\omega)$

$> \underline{\Phi}_\alpha^{P\beta}(\omega)$ by theorem 4.1

$\geq \alpha$ by I.H.

therefore $\lim\limits_{n<\omega} \underline{\Phi}_{\Omega\alpha(n)}^{\underline{\Phi}_\alpha^{P\beta}(\omega)+1}(\omega) > \alpha$.

Thus it is shown that $(\forall\alpha < \omega_1)_{>0}\ \Phi_\alpha^\beta(\omega) \geqslant \alpha$, and so by transfinite induction and the result for the values of the index β, it follows:

$$(\forall\alpha, \beta < \omega_1)_{\alpha>0, \beta>1}\ \max(\alpha, \beta) \leqslant \Phi_\alpha^\beta(\omega).$$

The case for values $> \omega$ of the index γ remains.

LEMMA 4.1d $(\forall\beta,\gamma)_{\beta>1, \gamma\geqslant\omega}\ \lambda\xi \,.\, \Phi_\xi^\beta(\gamma)$ is non-decreasing (n. d.).

PROOF. $\Phi_0^\beta(\gamma) = 0 \leqslant \Phi_1^\beta(\gamma) = \gamma + \beta$

$$< \Phi_2^\beta(\gamma) = \gamma.\beta < \Phi_3^\beta(\gamma) = \gamma^\beta$$

Suppose for I.H. that for some $\alpha > 3$, the functions of Lemma 4.1d have been shown to be n.d. for all $\xi < \alpha$. Thus there are the four cases of definition 3.1 (iv) to be considered. For $\beta = \omega\beta' + x,\ x < \omega$:

if $Suc(\alpha, \beta)$ & $x = 1$

$$\Phi_{P\alpha}^{\Phi_\alpha^{\omega\beta'}(\gamma)+1}(\gamma) \geqslant \Phi_{P\alpha}^\beta(\gamma), \text{ by}$$

$\Phi_\alpha^{\omega\beta'}(\gamma) + 1 \geqslant \beta$, and by theorem 4.1.

If $Suc(\alpha, \beta)$ & $x > 1$,

$\mathrm{Lim}\{\underline{\Phi}_{\alpha}^{\omega\beta'+x-1}(\gamma)\}$ by corollary 4.1a4

and $\underline{\Phi}_{\alpha}^{\omega\beta'+x-1}(\gamma) \geqslant \omega\beta'+x-1$ by theorem 4.1

$= P\beta$, which is a successor number,

therefore $\underline{\Phi}_{\alpha}^{\omega\beta'+x-1}(\gamma) > P\beta$ and $\underline{\Phi}_{\alpha}^{\omega\beta'+x-1}(\gamma) \geq \beta$

so by theorem 4.1

$$\underline{\Phi}_{R\alpha}^{\underline{\Phi}_{\alpha}^{\omega\beta'+x-1}(\gamma)}(\gamma) \geqslant \underline{\Phi}_{R\alpha}^{\beta}(\gamma) .$$

If $\mathrm{Lim}(\alpha)$ and $\mathrm{Suc}(\beta)$, $(\forall n<\omega)_{>0}$

$$\underline{\Phi}_{\Omega_\alpha(n)}^{\underline{\Phi}_{\alpha}^{P\beta}(\gamma)+1}(\gamma) \geqslant \underline{\Phi}_{\Omega_\alpha(n)}^{\beta}(\gamma)$$ as $\underline{\Phi}_{\alpha}^{P\beta}(\gamma)+1$

$\geqslant \beta$ by theorem 4.1, therefore by I.H.

$$\lim_{n<\omega} \underline{\Phi}_{\Omega_\alpha(n)}^{\underline{\Phi}_{\alpha}^{P\beta}(\gamma)+1}(\gamma) \geqslant \underline{\Phi}_{\alpha'}^{\beta}(\gamma),$$ for

all $\alpha'<\alpha$, since $(\forall\alpha'<\alpha)(\exists n<\omega)\ \alpha'<\Omega_\alpha(n)$.

Finally, if $\mathrm{Lim}(\beta)$, then $\underline{\Phi}_{\alpha}^{\beta}(\gamma) =$

$$\lim_{n<\omega} \underline{\Phi}_{\alpha}^{\Omega_\beta(n)}(\gamma),$$ and $(\forall\alpha'<\alpha)$

$$\underline{\Phi}_{\alpha'}^{\beta}(\gamma) = \lim_{n<\omega} \underline{\Phi}_{\alpha'}^{\Omega_\beta(n)}(\gamma) .$$

Thus, after a finite number of substitutions in this case of definition 3.1 (iv), these two l.h.s's will be expressed in terms of the three remaining cases of definition 3.1(ii), all of which have already been dealt with.

Thus it is proved that $(\forall\alpha'<\alpha)(\forall\beta,\gamma<\omega_1)_{\substack{\beta>1\\ \gamma\geq\omega}}$ $\Phi^{\beta}_{\alpha'}(\gamma)\leq\Phi^{\beta}_{\alpha}(\gamma)$, and so by transfinite induction lemma 4.1d is proved. □

To return to the proof of theorem 4.1c:

$(\forall\gamma<\omega_1)\ \Phi^2_i(\gamma)\geq\gamma$, $i=1,2,3$. By lemma 4.1d $\lambda\xi.\,\Phi^2_\xi(\gamma)$ is non-decreasing, therefore $(\forall\alpha<\omega_1)_{>0}\,(\forall\gamma<\omega_1)\ \Phi^2_\alpha(\gamma)\geq\gamma$. And by theorem 4.1 the exponent function of Φ is strictly increasing and hence non-decreasing, therefore $(\forall\alpha,\beta,\gamma<\omega_1)_{\alpha>0,\beta>1,\gamma\geq\omega}\,\gamma\leq\Phi^\beta_\alpha(\gamma)$, and so theorem 4.1c is proved.

COROLLARY 4.2 $(\forall\beta,\gamma<\omega_1)_{\beta>1,\gamma\geq\omega}$ $\lambda\xi.\,\Phi^\beta_\xi(\gamma)$ is ultimately increasing (u.i.).

PROOF. By lemma 4.1d, the functions of this corollary are non-decreasing, and by theorem 4.1c, $(\forall\beta,\gamma<\omega_1)_{\beta>1,\gamma>\omega}\ \alpha\leq\Phi^\beta_\alpha(\gamma)$, $0<\alpha$, so $\{\Phi^\beta_\alpha(\gamma)\}_{\alpha<\omega_1}$ is internally unbounded. □ 4.2

COROLLARY 4.3 $(\forall \alpha, \beta < \omega_1)_{\alpha, \beta > 0}\ \lambda\gamma \,.\, \underline{\Phi}_\alpha^\beta(\gamma)$ is n-d.

PROOF Definition 3.1 gives the explicit definitions:

$$(\forall \beta < \omega_1) \quad \underline{\Phi}_1{}^\beta(\gamma) = \gamma + \beta$$

$$\underline{\Phi}_1^1(\gamma) = \gamma + 1$$

$$(\forall \alpha < \omega_1)_{>1} \quad \underline{\Phi}_\alpha^1(\gamma) = \gamma$$

These functions of γ are n-d., and all the functions $\underline{\Phi}_\alpha{}^\beta$, $\alpha, \beta > 1$ are obtained from these functions by the application of superposition of non-decreasing functions, or by taking limits. Since neither of these operations leads out of the class of non-decreasing functions, $(\forall \alpha, \beta < \omega_1)_{\alpha, \beta > 0}$ $\lambda\gamma \,.\, \underline{\Phi}_\alpha^\beta(\gamma)$ is n-d. ☐ 4.3

COROLLARY 4.3a $(\forall \alpha, \beta < \omega_1)_{\alpha > 0, \beta > 1}$ $\lambda\gamma \,.\, \underline{\Phi}_\alpha{}^\beta(\gamma)$ is n-i. for $\omega \leq \gamma$.

PROOF By theorem 4.1c, $\gamma \geq \omega \Rightarrow$

$(\forall \alpha, \beta < \omega_1)_{\alpha > 0, \beta > 1}\ \gamma \leq \underline{\Phi}_\alpha{}^\beta(\gamma)$ ☐ 4.3a

Now suppose for $f : \omega_1^{[2]} \to \omega_1$:

(i) $(\forall\alpha<\omega_1)\ \lambda\beta . f(\alpha,\beta)$ is n.i

(ii) $(\forall\beta<\omega_1)\ \lambda\alpha . f(\alpha,\beta)$ is n.i

Thus for each $\alpha<\omega_1$, if $\alpha'<\alpha$, then

$$\begin{aligned} g(\alpha) =_{Df} f(\alpha,\alpha) &\geq f(\alpha',\alpha) \text{ by (ii)} \\ &\geq f(\alpha',\alpha') \text{ by (i)} \\ &= g(\alpha'), \text{ therefore} \end{aligned}$$

$g(\alpha')\leq g(\alpha)$, and so g is non-decreasing.

Now, for each α, let $\alpha^{(1)}, \alpha^{(2)}$ denote the least ordinals such that $f(\alpha,\alpha)<f(\alpha^{(1)},\alpha)$; $f(\alpha,\alpha)<f(\alpha,\alpha^{(2)})$. Now let $\alpha^* =_{Df} \max(\alpha^{(1)},\alpha^{(2)})$. Then $\alpha<\alpha^*$, and

$$\begin{aligned} g(\alpha) = f(\alpha,\alpha) &< f(\alpha^{(1)},\alpha) \\ &\leq f(\alpha^*,\alpha) \\ &\leq f(\alpha^*,\alpha^*) \\ &= g(\alpha^*). \end{aligned}$$

Thus $g(\alpha)<g(\alpha^*)$, and therefore g is n.i.

In general, there is the following theorem, which can readily be proved by induction:

THEOREM 4.4. Let $f:\omega_1^{[n]}\rightarrow\omega_1$ be such that for any $\vec{\alpha}\in\omega_1^{[r]}$, $\vec{\beta}\in\omega_1^{[n-(r+1)]}$ for natural numbers

n, r such that $r < n$ and $\lambda\xi . f(\vec{\alpha}, \xi, \vec{\beta})$ is u.i. Then if the function $g : \omega_1 \to \omega_1$ is obtained from f by choosing fixed values for a subsequence of the parameters of f, and identifying the remaining parameters of f, then g is u.i. □ 4.4

Thus

COROLLARY 4.4a The following functions are u.i: $(\forall \alpha, \beta, \gamma < \omega_1)_{\alpha > 0, \beta > 1, \gamma \geq \omega}\ \lambda\xi . \underline{\Phi}_{\xi}^{\xi}(\gamma)$, $\lambda\xi . \underline{\Phi}_{\alpha}^{\xi}(\xi)$, $\lambda\xi . \underline{\Phi}_{\xi}^{\beta}(\xi)$, $\lambda\xi . \underline{\Phi}_{\xi}^{\xi}(\xi)$. □

COROLLARY 4.5 Let $\alpha_1, \alpha_2; \alpha_3$ be the arguments of the function $\underline{\Phi}$ of definition 3.1. If $\underline{\Phi}$ is written as $\lambda\alpha_1, \alpha_2, \alpha_3 . \boxminus(\alpha_1, \alpha_2, \alpha_3)$, then whatever substitution is made for the arguments $\alpha_1, \alpha_2, \alpha_3$, $(\forall \alpha_1, \alpha, \alpha' < \omega_1)_{\geq \omega}\ \alpha' < \alpha \Rightarrow$

$$\lambda\alpha_2 . \boxminus(\alpha_1, \alpha_2, \alpha') \leq_m \lambda\alpha_2 . \boxminus(\alpha_1, \alpha_2, \alpha)$$

(In corollary 4.5, '$\leq_m$' denotes partial majorization.)

PROOF by theorems 4.1 and corollories 4.2, 4.3 □

Definitions 3.1, 3.2 were chosen in order that the definitions of functions D_α^n and operations U_α^n may be derived directly from them. It is now shown, on the basis of theorem 4.1, that definition 3.1 can be simplified. The only change to this definition is that limits of sequences whose indices form fundamental sequences are replaced by limits of sequences whose indices form segments. Thus, in order to avoid re-writing much of the definition, the function symbols Λ, Λ' are used to denote certain of it's functions.

DEFINITION 4.6. Let Λ, Λ' be such that

$$\Phi_\alpha^\beta(\gamma) = \lim_{n<\omega} \Lambda(\alpha, \beta, \Omega_\alpha(n), \gamma) \quad \ldots\ldots (1a)$$

$$\Phi_{\alpha'}^{\beta'}(\gamma') = \lim_{n<\omega} \Lambda'(\alpha', \beta', \Omega_{\beta'}(n), \gamma') \quad \ldots (1b)$$

are equations within the system of transfinite recursion of definition 3.1 whose right hand sides are defined by limiting operations in the cases Lim(α), Lim(β) respectively. Then the class $\Phi^{(2)}$ of functions $\overset{*}{\Phi}_\alpha^\beta$ is defined exactly as for the class $\Phi_\Omega^{(2)}$ of 3.1, excepting that every occurrence of an equation of form (1a) is replaced by:

$$\overset{*}{\Phi}_\alpha^\beta(\gamma) = \lim_{\mu<\alpha} \Lambda(\alpha, \beta, \mu, \gamma)$$

and of form (1b) replaced by:

$$\overset{*}{\Phi}_{\alpha'}^{\beta'}(\gamma') = \lim_{\xi<\beta'} \Lambda'(\alpha', \beta', \xi, \gamma')$$

Thus, definition 4.6 makes no reference to fundamental sequences for limit ordinals which occur as parameters in the definitions.

THEOREM 4.8 For each f.s.a. Ω,

$$\mathfrak{D}_\Omega^{(2)} = \mathfrak{D}^{(2)}$$

PROOF. Suppose that element $\overset{*}{\Phi}_\alpha{}^\beta$ of $\mathfrak{D}^{(2)}$ is defined by the clause

$$\overset{*}{\Phi}_\alpha{}^\beta(\gamma) = \lim_{\mu<\alpha} \Lambda(\alpha,\beta,\mu,\gamma)$$

By definition 4.6 and lemma 4.1d, the function Λ is non decreasing in the 3-rd argument. Thus

$$\{\Lambda(\alpha,\beta,\mu,\gamma)\}_{\mu<\alpha} \text{ and } \{\Lambda(\alpha,\beta,\Omega_\alpha(n),\gamma\}_{n<\omega} \quad (1)$$

are non-decreasing, and so

$$(\forall\mu<\alpha)(\exists n<\omega)\ \Lambda(\alpha,\beta,\mu,\gamma) \leq \Lambda(\alpha,\beta,\Omega_\alpha(n),\gamma)$$

and

$$(\forall n<\omega)(\exists\mu<\alpha)\ \Lambda(\alpha,\beta,\Omega_\alpha(n),\gamma) \leq \Lambda(\alpha,\beta,\mu,\gamma).$$

Thus the sequences (1) are congruent.

Similarly, if $\overset{*}{\Phi}_\alpha{}^\beta$ is defined in $\mathfrak{D}^{(2)}$ by

$$\overset{*}{\Phi}_\alpha{}^\beta(\gamma) = \lim_{\mu<\beta} \Lambda(\alpha,\beta,\mu,\gamma),$$ the sequences corresponding to (1) to limit β are congruent.

The remaining clauses of definitions 3.1, 4.6 coincide, so they are identical. ☐ 4.8

It was shown on p. 46 that distinct f.s.a's can be exhibited; however the following is now clear:

CORCLLARY 4.10 For all f.s.a's Ω, Ω', $\mathbb{D}_{\Omega}^{(2)} = \mathbb{D}_{\Omega'}^{(2)}$ ☐

COROLLARY 4.11 $(\forall \alpha, \beta, \gamma < \omega_1)_{\alpha > 0}$

$$\Gamma_{\substack{\alpha \\ \beta' < \beta}} \mathfrak{U}_2(\beta', \gamma) = \begin{cases} \mathbb{D}_{\alpha}^{\beta}(\gamma), & \omega \leq \alpha \\ \mathbb{D}_{\alpha+1}^{\beta}(\gamma), & \alpha < \omega, \end{cases}$$

where $\mathfrak{U}_2$ is the projection function, satisfying $(\forall \alpha, \beta < \omega_1)\ \mathfrak{U}_2(\beta, \alpha) = \alpha$.

PROOF. Since $\mathfrak{U}_2(\beta', \gamma) = \gamma$, for $\alpha = 1$ it follows

$$(\forall \beta, \gamma < \omega_1)\ \Gamma_{\substack{1 \\ \beta' < \beta}} \mathfrak{U}_2(\beta', \gamma) = \sum_{\beta' < \beta} \mathfrak{U}_2(\beta', \gamma) \text{ by definition 3.2}$$

$$= \underbrace{\gamma + \gamma + \cdots}_{\beta} = \gamma \cdot \beta$$

$$= \mathbb{D}_{2}^{\beta}(\gamma)$$

checking through the remaining clauses of definitions 3.2, 4.6 proves corollary 4.11 for $\mathbb{D}_{\alpha}^{\beta}(\gamma)$ as defined in 4.6, and so the corollary is proved by theorem 4.8 ☐ 4.11

In particular, from this corollary

$$\alpha \textcircled{4} (\beta+1) = \alpha \textcircled{3} \Big\{ \Gamma_{3\atop \beta'<\beta} \mathfrak{U}_2(\beta',\alpha) \Big\}$$

$$\alpha \textcircled{3} (\beta+1) = \Big\{ \prod_{\beta'<\beta} \mathfrak{U}_2(\beta',\alpha) \Big\} \textcircled{2} \alpha .$$

THE LEADING TERM

In the operational notation, for each countable γ, $\beta \geqslant 2$, $\alpha \geqslant 3$

$$\mathfrak{D}^{\beta}_{\alpha+1}(\gamma) = \gamma \textcircled{\alpha+1} \beta$$

$$= \underbrace{\dots \textcircled{\alpha} \gamma \textcircled{\alpha} \{ (\underbrace{\dots \textcircled{\alpha} \gamma \textcircled{\alpha} \gamma}_{\omega \text{ terms}}) + 1 \}}_{\beta \text{ terms}}$$

Now, the operations Γ_α of definition 3.2 give

$$\Gamma_{\alpha\atop \beta'<\beta} \gamma_{\beta'} = \begin{cases} \underbrace{\{ (\underbrace{\gamma_0 \textcircled{\alpha} \gamma_1 \textcircled{\alpha} \dots}_{\omega \text{ terms}}) + 1 \} \textcircled{\alpha} \gamma_\omega \dots \textcircled{\alpha} \gamma_{\beta'} \dots}_{\beta \text{ terms}} & \text{for } \alpha \leqslant 2 \\ \underbrace{\dots \textcircled{\alpha} \gamma_{\beta'} \dots \textcircled{\alpha} \gamma_\omega \textcircled{\alpha} \{ (\underbrace{\dots \textcircled{\alpha} \gamma_1 \textcircled{\alpha} \gamma_0}_{\omega \text{ terms}}) + 1 \}}_{\beta \text{ terms}} & \text{for } 3 \leqslant \alpha \end{cases}$$

In the case for $3 \leq \alpha$, γ_0 is called the leading term. There is a special case, which can arise quite often, in which each term in the expression takes the same value γ, except the leading term, γ'.

Thus, for $0 < \beta' \leq \beta'' < \beta$, $\gamma_{\beta'} =_{Df} \gamma_{\beta''} = \gamma$ for each such countable index, and for $\gamma_0 = \gamma'$, the following notation is adopted:

$$\Gamma_{\substack{\alpha \\ \beta' < \beta}} \gamma_{\beta'} =_{Df} \begin{cases} \Phi^{\beta}_{\alpha+1}(\gamma;\gamma') \text{ for } \alpha < \omega \\ \Phi^{\beta}_{\alpha}(\gamma;\gamma') \text{ for } \omega \leq \alpha \end{cases}$$

SOME PROPERTIES OF THE LEADING TERM

In the first instance, attention is restricted to the finite ordinals.

$(\forall x < \omega)_{x > 1}$ let $h(0) \underset{4}{=} 0$; $g(0) = 1$; $f(0) = 2$

$h(1) = 2$; $g(1) = 4$; $f(1) = 2$

$h(x+1) = 3^{h(x)}$; $g(x+1) = 2^{g(x)}$; $f(x+1) = (x+2)^{f(x)}$

Thus, for example

$h(5) = 3^{3^{3^{3^2}}}$, $g(5) = 2^{2^{2^{2^4}}}$, $f(5) = 6^{5^{4^{3^2}}}$. then

THEOREM 4.12 $h <_m g <_m f$.

PROOF by induction.

$$h(0) = 0 < 1 = g(0)$$
$$h(1) = 2 < 4 = g(1)$$
$$h(2) = 3^2 = 9 < 16 = 2^4 = g(2)$$
$$h(3) = 3^9 = (27)^3 < (32)^3 . 2 = 2^{16} = g(3)$$

Now in fact $h(3) = 19,683$; $g(3) = 32,768 \times 2$

Therefore $$\frac{g(3)}{h(3)} > 1.5 \times 2 = 3$$

Also, $h(4) = 3^{h(3)}$; $g(4) = 2^{g(3)}$

$$= \left(2^{\log_2(3)}\right)^{h(3)} \qquad > 2^{3 \cdot h(3)}$$
$$= 2^{\log_2(3) \cdot h(3)}$$

Therefore

$$\frac{g(4)}{h(4)} > \frac{2^{3 \cdot h(3)}}{2^{\log_2(3) \cdot h(3)}} = 2^{h(3)(3-\log_2(3))}$$
$$> 2^{h(3)}$$
$$> 4 \text{ since } 2 < h(3).$$

Now, for induction hypothesis, suppose for some $n > 4$ that $$\frac{g(n)}{h(n)} > 4.$$ Then

$$h(n+1) = 3^{h(n)} = \left(2^{\log_2(3)}\right)^{h(n)}$$
$$g(n+1) = 2^{g(n)} > 2^{4 \cdot h(n)}$$ by I.H., therefore

$$\frac{g(n+1)}{h(n+1)} > \frac{2^{4.h(n)}}{2^{\log_2(3).h(n)}} = 2^{h(n)(4-\log_2(3))}$$

$$> 2^{h(n).2}$$

$$> 4, \text{ since } 2 < h(n).$$

Thus it is proved by induction that for each $n>3$ $4.h(n) < g(n)$.

On the other hand, it can readily be seen that

$g(0) = 1$; $f(0) = 2$
$g(1) = 4$; $f(1) = 2$
$g(2) = 2^4 = 16$; $f(2) = 3^2 = 9$
$g(3) = 2^{16}$; $f(3) = 4^9 = (2^2)^9 = 2^{18}$

Thus $g(3) < f(3)$. Since for $n>3$, $f(n)$ is defined by the exponentiation of a larger base than that for $g(n)$, it follows $g(n) < f(n)$, and

$g <_m f$ ☐ 4.12

The leading term in the expression of $g(n)$, $n>0$, is 4. A less sharp but more general result follows. It shows that despite the r.h. relation of theorem 4.12, if the leading term is large enough, then in many cases it is decisive.

THEOREM 4.13. Suppose $\lambda x . m_x$ is majorized by $\lambda x . 2^x$. Then there is an integer M such that

$$\lambda x . \underset{i<x+1}{\Gamma_3} m_i <_m \lambda x . \Phi_4^x(2;M)$$

PROOF Make the definitions $g(1)=2$, $f(1)=8$, and for $n>0$, $g(n+1)=(2^{n+1})^{g(n)}$

$$f(n+1) = 2^{f(n)} .$$

Then $g(2) = 4^2 < 2^5 = \frac{f(2)}{2^3}$. For induction hypothesis, Suppose for some $n>2$ $g(n) . 2^{n+1} < f(n)$. Then $f(n+1) = 2^{f(n)} > 2^{2^{(n+1)} . g(n)}$ by I.H., also $g(n+1) = 2^{(n+1) . g(n)}$ therefore

$$\frac{f(n+1)}{g(n+1)} > \frac{2^{2^{(n+1)} . g(n)}}{2^{(n+1) . g(n)}} = 2^{g(n) . (2^{n+1}-(n+1))}$$

$$> 2^{2 . (2^{n+1}-n+1)}$$

$$> 2^{2^{n+1}}$$

$$> 2^{n+1},$$

thus by induction $(\forall n<\omega)\, g(n) . 2^n < f(n)$.

Now suppose $\{M_x\}_{x<\omega}$ is majorized by $\{2^x\}_{x<\omega}$ at $x = \bar{x}$. Then put $M = \underset{i<\bar{x}+1}{\Gamma_3} m_i$. Then since $\lambda m_1, m_2, \dots, m_x . \underset{i<x+1}{\Gamma_3} m_i$ is strictly increasing in each argument, $(\forall x<\omega) \underset{i<x+1}{\Gamma_3} m_i < \Phi_4^x(2;M)$ □ 4.13

It seems likely that 4.13 can be improved by replacing the majorizing function $\lambda x . 2^x$ for $\{m_x\}$ by the function of x $\overbrace{s^{s^{\cdot^{\cdot^{\cdot^{s^x}}}}}}^{t}$, for fixed integers s, t.

LEADING TERMS IN EXPRESSIONS CONTAINING TRANSFINITE ORDINALS

Since, according to definition 3·2, transfinite sum and product expressions are built up by additive and multiplicative right hand concatenation respectively, these expressions do not have right hand leading terms when the bounds of the operations are limit ordinals. Thus a definition of the left hand Hausdorff transfinite product for countable ordinals, for each $f : \omega_1^{(n)} \to \omega_1$, is now given

$$(\forall \vec{\xi} \in \omega_1^{[n-1]})(\forall \beta < \omega_1) \quad \beta \leq \omega \Rightarrow$$

$$f(\beta', \vec{\xi}) \prod_{\beta' < \beta} =_{Df} \prod_{\beta' < \beta} f(\beta', \vec{\xi}), \text{ and } \beta > \omega \Rightarrow$$

$$f(\beta', \vec{\xi}) \prod_{\beta' < \beta} =_{Df} (\mu \alpha < \omega_1)\left[(\forall \beta^* < \beta)\left[f(\beta^*, \vec{\xi}) \cdot f(\beta', \vec{\xi}) \prod_{\beta' < \beta^*} \leq \alpha\right]\right]$$

for some ~~positive~~ $n < \omega$.

Clearly this product differs from that of definition 3·2, as ordinal multiplication is not commutative. The leading term of $f(\beta', \vec{\xi}) \prod_{\beta' < \beta}$ is $f(0, \vec{\xi})$.

THEOREM 4.14

$$(\forall \beta < \omega_1)_{>\omega} \quad \omega^\omega = \mathfrak{U}_2(\beta', \omega) \prod_{\beta' \leq \beta} < \prod_{\beta' < \beta} \mathfrak{U}_2(\beta', \omega) = \omega^\beta$$

PROOF by t.i. First note that

$$\mathcal{U}_2(\beta';\omega)\prod_{\beta'<\omega} = \prod_{\beta'<\omega}\mathcal{U}_2(\beta';\omega)$$

$$= \lim_{n<\omega}\prod_{\beta'<n}\mathcal{U}_2(\beta';\omega) \text{ by definition 3.2}$$

$$= \omega^{\omega}.$$

It is readily shown by induction that $(\forall\beta<\omega_1)_{>\omega}$

$\prod_{\beta'<\beta}\mathcal{U}_2(\beta';\omega) = \omega^{\beta}$. But

$$\mathcal{U}_2(\beta';\omega)\prod_{\beta'<\omega+1} = (\mu\alpha<\omega_1)[(\forall\beta^*<\omega+1)[\mathcal{U}_2(\beta^*;\omega).\mathcal{U}_2(\beta';\omega)\prod_{\beta'<\beta^*}\leq\alpha]]$$

$$= \omega.\,\mathcal{U}_2(\beta';\omega)\prod_{\beta'<\omega}$$

$$= \omega\omega^{\omega} = \omega^{1+\omega} = \omega^{\omega}$$

Suppose for I.H. that theorem 4.14 has been proved for all β' such that $\omega<\beta'<\beta$, for some $\beta>\omega+1$. Then if $\mathrm{Snc}(\beta)$, $\mathcal{U}_2(\beta';\omega)\prod_{\beta'<\beta}$

$$= (\mu\alpha<\omega_1)[(\forall\beta^*<\beta)[\mathcal{U}_2(\beta^*;\omega).\mathcal{U}_2(\beta';\omega)\prod_{\beta'<\beta^*}\leq\alpha]]$$

$$= \omega.\,\mathcal{U}_2(\beta';\omega)\prod_{\beta'<P\beta}$$

$$= \omega.\omega^{\omega} \text{ by induction hypothesis}$$

$$= \omega^{\omega}.$$

If $\mathrm{Lim}(\beta)$, then

$$\mathcal{U}_2(\beta';\omega)\prod_{\beta'<\beta} = \lim_{\beta^*<\beta}\{\omega.\,\mathcal{U}_2(\beta';\omega)\prod_{\beta'<\beta^*}\}$$

$$= \lim_{\beta^*<\beta}(\omega^{\omega})_{\beta^*} \text{ by induction hypothesis}$$

$$= \omega^{\omega}$$

Thus by transfinite induction, and since $(\forall\beta < \omega_1)_{>\omega}$ $\omega^\omega < \omega^\beta$, theorem 4.14 follows $\square$

The distinction between finite and infinite ordinals with respect to leading terms is instanced by the inequality $\omega^{\varepsilon_0^2} > \varepsilon_0 = \omega^{\omega^{\varepsilon_0}}$;

whereas $(\forall n, m, x < \omega)_{\geq 2}$ $m < n < x \Rightarrow$

$x^{m^n} < m^{x^n} < m^{n^x} \leq y^{y^x}$, where $y = \max(m, n)$.

SOME FUNCTIONS OF A FINITE OPERAND

It follows from definitions 3·1, 3·7c that for all numbers $x, y, n < \omega$, $\mathbb{D}_n^x(y) = D_n^x(y)$.

Theorem 4·1a2 to corollary 4·4a inclusive deal only with transfinite values of the operand γ.

THEOREM 4.15 (i) $(\forall\alpha < \omega_1)_{>0}$ $\mathbb{D}_\alpha^2(2) = 4$

(ii) $(\forall n < \omega)_{\geq 2}$ $\mathbb{D}_4^\omega(n; \mathbb{D}_4^2(\omega)) = \varepsilon_0$

PROOF (i) by t.i. For $\alpha = 1$, $\mathbb{D}_1^2(2) = 4$.

Suppose 4.15 has been proved for all $\alpha' < \alpha$, for some $\alpha > 1$. Then if $\mathrm{Suc}(\alpha)$

$$\mathbb{D}_\alpha^2(2) = \mathbb{D}_{P\alpha}^2(2), \text{ by } 3\cdot1\,(ii)$$
$$= 4 \text{ by I.H.}$$

If $\mathrm{Lim}(\alpha)$, $\mathbb{D}_\alpha^2(2) = \lim_{n<\omega} \mathbb{D}_{\Omega_\alpha(n)}^2(2)$

$$= \lim_{n<\omega} (4)_n \text{ by I.H.}$$
$$= 4.$$

(ii) $\Phi_4^{\omega}(n;\Phi_4^{2}(\omega)) = \lim[\omega^{\omega}, n^{\omega^{\omega}}, n^{n^{\omega^{\omega}}}, n^{n^{n^{\omega^{\omega}}}}, \dots]$

Now $n^{\omega^{\omega}} = n^{\omega^{(1+\omega)}} = n^{\omega.\omega^{\omega}} = (n^{\omega})^{\omega^{\omega}} = \omega^{\omega^{\omega}}$,

since $(\forall n<\omega)_{n>i}$ $n^{\omega} = \lim\limits_{k<\omega} n^{k} = \omega$.

Suppose for some $i<\omega$, $i>2$, that

$\Phi_4^{i}(\omega) = \Phi_4^{i-2}(n;\Phi_4^{2}(\omega))$. Then

$$\begin{aligned}
\Phi_4^{i-1}(n;\Phi_4^{2}(\omega)) &= n^{\Phi_4^{i-2}(n;\Phi_4^{2}(\omega))} \\
&= n^{\omega^{\Phi_4^{i-3}(n;\Phi_4^{2}(\omega))}} \\
&= n^{\omega^{1+\Phi_4^{i-3}(n;\Phi_4^{2}(\omega))}} \\
&= n^{\omega.\omega^{\Phi_4^{i-3}(n;\Phi_4^{2}(\omega))}} \\
&= (n^{\omega})^{\omega^{\Phi_4^{i-3}(n;\Phi_4^{2}(\omega))}} \\
&= \omega^{\omega^{\Phi_4^{i-3}(n;\Phi_4^{2}(\omega))}} \\
&= \Phi_4^{i+1}(\omega) \text{ by I.H.}
\end{aligned}$$

Therefore $\Phi_4^{\omega}(n;\Phi_4^{2}(\omega))$

$$\begin{aligned}
&= \lim_{i<\omega} \Phi_4^{i}(n;\Phi_4^{2}(\omega)) \\
&= \lim_{i<\omega} \Phi_4^{i}(\omega) = \varepsilon_0 \qquad \square \text{ 4.15}
\end{aligned}$$

But note that $\Phi_4^{\omega+1}(n;\Phi_4^{2}(\omega)) < \Phi_4^{\omega+1}(\omega)$, as

l.h.s $= n^{\varepsilon_0+1} = n^{\varepsilon_0}.n = n^{\omega.\varepsilon_0}.n$ as ε_0 is a principal number of multiplication

$$\begin{aligned}
&= (n^{\omega})^{\varepsilon_0}.n \\
&= \omega^{\varepsilon_0}.n \\
&= \varepsilon_0.n \\
&< \varepsilon_0.\omega = \omega^{\varepsilon_0}.\omega = \omega^{\varepsilon_0+1} = \text{r.h.s.}
\end{aligned}$$

Still, it can readily be shown that

$$n^{\omega^{\varepsilon_0+1}} = \omega^{\omega^{\varepsilon_0+1}} = \varepsilon_0^{\omega}.$$

SOME ARITHMETIC PROPERTIES OF THE OPERATIONS $\textcircled{\alpha}$

It is well-known that the operations of ordinal addition and multiplication are not commutative, but are associative:

$$(\forall\alpha,\beta,\gamma<\omega_1)\ \alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma\ ;$$

$$\alpha.(\beta.\gamma)=(\alpha.\beta).\gamma$$

and that ordinal multiplication is left distributive over ordinal addition:

$$(\forall\alpha,\beta,\gamma<\omega_1)\ \alpha.(\beta+\gamma)=\alpha.\beta+\alpha.\gamma$$

The operation of exponentiation is not even associative; however the following laws hold, expressed using the notation $\mathfrak{D}_\alpha^\beta(\gamma)=\gamma\,\textcircled{\alpha}\,\beta$:

$$\alpha\,\textcircled{3}\,(\beta\,\textcircled{2}\,\gamma)=(\alpha\,\textcircled{3}\,\beta)\,\textcircled{3}\,\gamma$$

$$\alpha\,\textcircled{3}\,(\beta\,\textcircled{1}\,\gamma)=(\alpha\,\textcircled{3}\,\beta)\,\textcircled{2}\,(\alpha\,\textcircled{3}\,\gamma)$$

It may be asked whether these and perhaps other laws apply to operations belonging to the class $\mathfrak{D}^{(2)}$ and which are more complex than exponentiation. It appears that this is not the case, and there is a curious distinction between the operations $\textcircled{i}$, $i\leq 3$, and the operations $\textcircled{\alpha}$, $4\leq\alpha$. In the first instance, only the restriction of these operations to the natural numbers need be considered.

By virtue of the inductive definition of operations $\textcircled{\alpha}$ from operations $\textcircled{\alpha'}$, $\alpha'<\alpha$, for $4 \leq \alpha$, none of the operations $\textcircled{\alpha}$ is either commutative or associative; and it can be proved by induction that $(\forall x, y, z < \omega)_{>2}$

$$
\begin{aligned}
& x \textcircled{4} (y \textcircled{3} z) \\
\gneqq\ & x \textcircled{4} (y \textcircled{2} z) \\
\gneqq\ & (x \textcircled{4} y) \textcircled{4} z \\
\gneqq\ & x \textcircled{4} (y \textcircled{1} z) \\
\gneqq\ & (x \textcircled{4} y) \textcircled{3} (x \textcircled{4} z) \\
\gneqq\ & (x \textcircled{4} y) \textcircled{2} (x \textcircled{4} z) \\
\gneqq\ & (x \textcircled{4} y) \textcircled{1} (x \textcircled{4} z)
\end{aligned}
$$

Similar inequalities hold for operations $\textcircled{k}$, $4<k<\omega$.

Now suppose $Suc(\beta, P\beta)$, and $3 \leq \alpha$. Then for each $\gamma < \omega_1$:

$$\gamma \textcircled{\alpha+1} (\beta+1) = \gamma \textcircled{\alpha} (\gamma \textcircled{\alpha+1} \beta) \quad \cdot \cdot \cdot \cdot \cdot \cdot \ (1)$$

For example, $\beta = \omega+1$, $\alpha = 3$, so that

$$\gamma \textcircled{4} (\omega+2) = \gamma \textcircled{3} (\gamma \textcircled{4} (\omega+1)) = \gamma^{\gamma \textcircled{4} (\omega+1)}$$

Now suppose $Suc(\beta)$, and $\alpha \leq 2$. Then

$$\gamma \textcircled{\alpha+1} \beta+1 = (\gamma \textcircled{\alpha+1} \beta) \textcircled{\alpha} \gamma \quad \cdot \cdot \cdot \cdot \cdot \cdot \ (2)$$

For example $\beta = \omega+1$, $\alpha = 2$ so that

$$\begin{aligned}\gamma \textcircled{3} (\omega+2) = (\gamma \textcircled{3} (\omega+1)) \textcircled{2} \gamma &= \gamma^{\omega+1} \cdot \gamma \\ &= \gamma^{\omega+1+1} \\ &= \gamma^{\omega+2}\end{aligned}$$

Now, had the expression $\gamma \textcircled{3} (\omega+2)$ been evaluated on the basis of formula (1) in place of formula (2), a different value would have been obtained:

$$\begin{aligned}\gamma \textcircled{3} (\omega+2) = \gamma \textcircled{2} (\gamma \textcircled{3} (\omega+1)) &= \gamma \cdot \gamma^{\omega+1} \\ &= \gamma^{1+\omega+1} \\ &= \gamma^{\omega+1} \\ &\neq \gamma^{\omega+2}\end{aligned}$$

Thus, definition 3.1, in generating the operations $\textcircled{\alpha}$, has to include formulas (1), (2) in accordance with the value of α.

§5. GENERALIZED EPSILON NUMBERS.

NORMAL FUNCTIONS Strictly increasing continuous functions (normal functions) defined on the countable ordinals, together with some of their elementary properties, were introduced in §1, p. 13. Veblen [20] has studied a number of properties of normal functions, and an operation which leads to a new normal function from a given normal function. Throughout the remainder of §5, each citation of this author is a reference to [20]. The operation is that of obtaining a first derived function f' from a normal function f, according to the following definition:

$$f'(0) = (\mu\beta)[f(\beta) = \beta]$$

$$f'(\alpha+1) = (\mu\beta)[f'(\alpha) < \beta \;\&\; f(\beta) = \beta],$$ and for

$\mathrm{Lim}(\alpha)$:

$$f'(\alpha) = \lim_{\alpha' < \alpha} f'(\alpha').$$

Thus by lemma 1.4, f' enumerates the fixed points of the function f. Furthermore, in theorem 1.3 it was proved that for f a normal function, the smallest fixed point of f, when $f(0) > 0$, is $f^{\omega}(0)$, and the smallest fixed point greater than some $\beta < \omega_1$ is the number $f^{\omega}(\beta+1)$. Thus, to evaluate f', write

$$f'(0) = f^{\omega}(0)$$

$$f'(\alpha+1) = f^{\omega}(f'(\alpha)+1),$$ and for $Lim(\alpha)$,

$$f'(\alpha) = \lim_{\alpha'<\alpha} f'(\alpha').$$

Thus by definition 1.5 it follows that for each normal f, $f^{F} = f'$. Now let f be the normal function $\lambda\alpha . 1+\alpha$. Then $(\forall\alpha<\omega_1)\ f'(\alpha) = \omega+\alpha$. This is readily proved by induction, for $f'(0) = \omega$. Suppose $f'(\alpha) = \omega+\alpha$. Then $f'(\alpha+1) = \lim_{n<\omega}[n+\omega+\alpha+1] = \omega+\alpha+1$. And if $Lim(\alpha)$, suppose $(\forall\alpha'<\alpha)\ f'(\alpha') = \omega+\alpha'$. Then $f'(\alpha) = \lim_{\alpha'<\alpha} f'(\alpha')$ $= \lim_{\alpha'<\alpha} \omega+\alpha' = \omega+\alpha$

As was observed in § 1, it follows from the strictly increasing property of f^{F}, and from lemma 1.4 that f normal $\Rightarrow f^{F}$ normal.

Tabulated below are some normal functions, and their corresponding first derived functions. For proofs of the exponential cases, see theorem 5.25.

f	f^F
$\lambda\beta . 1+\beta$	$\lambda\beta . \omega+\beta = \lambda\beta . \Phi_1^{\beta}(\omega)$
$\lambda\beta . \gamma+\beta$	$\lambda\beta . \gamma\omega+\beta = \lambda\beta . \Phi_1^{\beta}(\Phi_2^{\omega}(\gamma))$
$\lambda\beta . \gamma\beta$	$\lambda\beta . \gamma^{\omega}+\beta = \lambda\beta . \Phi_2^{\beta}(\Phi_3^{\omega}(\gamma))$
$\lambda\beta . \omega^{\beta}$	$\lambda\beta . \varepsilon_{\beta} = \lambda\beta . \Phi_4^{\omega(1+\beta)}(\omega)$
$\lambda\beta . \gamma^{\beta}$	$\lambda\beta . \varepsilon_{(\alpha'+\beta)} = \lambda\beta . \Phi_4^{\omega(\alpha'+\beta)}(\omega)$

for $\omega \leq \alpha'+\beta$, where $\alpha' = (\mu\eta)[\gamma \leq \varepsilon_\eta]$.

The notion of 1st derived function is developed by Veblen to that of α-th derived function f_α of a normal function $f (= f_0)$.

DEFINITION 5.1. For each non-zero $\alpha < \omega_1$, the α-th derived function f_α of a normal function f is defined as:

$$(\forall\beta < \omega_1)\ f_1(\beta) = f^F(\beta) \text{ and for } \alpha > 1,\ f_\alpha(\beta) =$$

$$(\mu\eta)\Big[(\forall\beta' < \beta)\ f_\alpha(\beta') < \eta \ \&$$

$$\eta \in \bigcap_{\substack{1 \leq \alpha' \\ \alpha' < \alpha}} \{\xi \mid f_{\alpha'}(\xi) = \xi\}\Big]$$

SOME VEBLEN FUNCTIONS ϕ. The notion of derived function of a normal function φ is extended still further in [20] to that of functions ϕ defined for a positive countable number of arguments such that

(i) each argument is non-zero

(ii) only finitely many arguments simultaneously take values exceeding unity

(iii) the number of arguments is a successor number.

In fact the definition given in [20] allows for the condition of countability to be replaced by restriction to ordinals $< X = \omega_\tau$, for some arbitrary ordinal τ. Thus, in the definition given here, $\tau = 1$:

Suppose φ is a normal function. Then $(\forall \beta, \rho_1 < \omega_1)_{\beta > 1}$

$$(1)\ (\forall \rho_1 < \omega_1)\ \phi((\rho_1)_1) =_{Df} \varphi(\rho_1)$$

$$(2)\ \phi((\rho_1)_1, (1)_2, (1)_3, \ldots, (1)_\beta) =_{Df}$$

$$(\mu\eta)_{<\omega_1}\Big[(\forall \rho_1' < \rho_1)\ \phi((\rho_1')_1, (1)_2, (1)_3, \ldots, (1)_\beta) < \eta$$

$$\& \ \eta \in \bigcap_{\gamma < \beta} \{\alpha \mid \phi((1)_1, (1)_2, \ldots, (\alpha)_\gamma) = \alpha\}\Big]$$

$$(3)\ (\forall \xi \leq \beta)_{>1} (\forall \rho_\xi, \rho_{\xi+1}, \ldots, \rho_\beta < \omega_1)_{>0,\ \text{sat. cond. (ii)}}$$

$$\phi((\rho_1), (1)_2, (1)_3, \ldots, (\rho_\xi)_\xi, \ldots, (\rho_\beta)_\beta) =_{Df}$$

$$(\mu\eta)_{<\omega_1} \Big[(\forall \rho_1' < \rho_1)\ \phi((\rho_1')_1, (1)_2, (1)_3, \ldots, (1)_\beta) < \eta$$

$$\& \ \eta \in \bigcap_{\gamma<\xi} \bigcap_{\rho_\xi' < \rho_\xi} \{\alpha \mid \phi((1)_1, (1)_2, \ldots, (\alpha)_\gamma, (1)_{\gamma+1}, \ldots$$

$$\ldots, (\rho_\xi')_\xi, \ldots, (\rho_\beta)_\beta) = \alpha\}\Big] .$$

Veblen goes on to show that $\lambda\beta \cdot \phi((1)_1, (1)_2, \ldots, (1)_\beta)$ is in fact normal, from which it follows that there exists a band $\{E(\sigma)\}_{\sigma<\omega_1}$ of solutions to the equation $\phi((1)_1, \ldots, (1)_{E(\sigma)}) = E(\sigma)$. Each number $E(\sigma)$, $\sigma<\omega_1$ is inexpressible in terms of ordinals smaller than $E(\sigma)$, and by means of a function symbol ϕ involving subscripts smaller than $E(\sigma)$.

Some of the properties of the α-th derived function of a normal function are examined below.

Suppose $f_0 : \omega_1 \to \omega_1$ is a normal function. Then the sequence of functions $\{f_\alpha\}_{\alpha<\omega_1}$ obtained in definition 5.1 is also obtained by the scheme of definition 5.1a to follow. The identity is proved in theorem 5.7.

DEFINITION 5.1a.

$(\forall\alpha<\omega_1)[\ f_{\alpha+1}(0) =_{Df} f^{\omega}_{\alpha}(0)$ &

$(\forall\eta<\omega_1)[\ f_{\alpha+1}(\eta+1) =_{Df} f^{\omega}_{\alpha}\{f_{\alpha+1}(\eta)+1\}$ &

$(\forall\lambda<\omega_1)_{Lim(\lambda)}[\ f_{\alpha}(\lambda) =_{Df} \lim_{\lambda'<\lambda} f_{\alpha}(\lambda')$ &

$f_{\lambda}(0) =_{Df} \lim_{\lambda'<\lambda} f_{\lambda'}(0)$ &

$f_{\lambda}(\eta+1) =_{Df} \lim_{\lambda'<\lambda} f_{\lambda'}\{f_{\lambda}(\eta)+1\}\ ]]]$

where, $(\forall\alpha,\eta<\omega_1)(\forall n<\omega)_{>0}$

$[\ f^{1}_{\alpha}(\eta) =_{Df} f_{\alpha}(\eta)$ &

$f^{n+1}_{\alpha}(\eta) =_{Df} f_{\alpha}\{f^{n}_{\alpha}(\eta)\}$ &

$f^{\omega}_{\alpha}(\eta) =_{Df} \lim_{n<\omega} f^{n}_{\alpha}(\eta)\ ]$ □5.1a

THEOREM 5.2 $(\forall\alpha<\omega_1)$ f_α is normal.

PROOF by nested transfinite induction.

By hypothesis, f_0 is normal.

Suppose for some $\alpha<\omega_1$ it has been proved $\forall\alpha'<\alpha$ $f_{\alpha'}$ is normal. Then suppose $(\exists\eta<\omega_1)(\forall\eta',\eta''<\eta)$ $[\eta''<\eta' \Rightarrow f_\alpha(\eta'') < f_\alpha(\eta')$ & $Lim(\eta') \Rightarrow$ $f_\alpha(\eta') = \lim_{\eta^*<\eta'} f_\alpha(\eta^*)$.

Now suppose Suc(α, η).

Then $f_\alpha(\eta) = f^{\omega}_{P\alpha}\{f_\alpha(P\alpha)+1\}$ by definition 5.1a

$\geq f_{P\alpha}\{f_\alpha(P\alpha)+1\}$, since $f_{P\alpha}$ is normal by I.H. on α

$> f_\alpha(P\alpha)$, also since $f_{P\alpha}$ is normal,

such that $(\forall \beta < \omega_1)\, f_{P\alpha}(\beta) \geq \beta$.

Thus f_α increases at the successor ordinal η.

Now suppose Suc(α), Lim(η).

Then $f_\alpha(\eta) = \lim\limits_{\eta' < \eta} f_\alpha(\eta')$ by definition 5.1a, and thus f_α is continuous at the limit ordinal η

$> f_\alpha(\eta')$, for each $\eta' < \eta$ by I.H. on η.

Thus f_α increases at the limit ordinal η.

Now suppose Lim(α). Then for Suc(η),

$f_\alpha(\eta) = \lim\limits_{\alpha' < \alpha} f_{\alpha'}\{f_\alpha(P\eta)+1\}$ by definition 5.1a

$\geq f_0\{f_\alpha(P\eta)+1\} \geq f_\alpha(P\eta)+1 > f_\alpha(P\eta)$,

thus f_α increases at successor ordinal α.

Furthermore, if Lim(η), then $f_\alpha(\eta) = \lim\limits_{\eta' < \eta} f_\alpha(\eta')$,

thus f_α is continuous at the limit ordinal η.

Also $f_\alpha(\eta) > f_\alpha(\eta')$, for each $\eta' < \eta$ by I.H. on η, and thus f_α increases at the limit ordinal η. Thus it follows by transfinite induction that the function f_α is normal, and from this it follows again by transfinite induction that

$(\forall \alpha < \omega_1)$ f_α is normal. $\square$5.2

THEOREM 5.3(a) $(\forall \alpha < \omega_1)$ $f_{\alpha+1}$ enumerates in increasing order (e.i.i.o.) the fixed points of f_α.

(b) $(\forall \alpha < \omega_1)$ $\mathrm{Lim}(\alpha) \Rightarrow f_\alpha$ e.i.i.o. the set of ordinal numbers

$$\bigcap_{\alpha' < \alpha} FP(f_{\alpha'})$$

where $FP(f_{\alpha'})$ = the set of fixed points of the function $f_{\alpha'}$, and f_0 is a nl. fn. satisfying $f_0(0) > 0$.

PROOF (a) For every countable α, the smallest fixed point of f_α is given by $f_\alpha^\omega(0) = f_{\alpha+1}(0)$. Suppose $f_{\alpha+1}(\eta)$ is the η-th fixed point of f_α. Then the least fixed point of f_α greater than this number is given by $f_\alpha^\omega\{f_{\alpha+1}(\eta)+1\} = f_{\alpha+1}(\eta+1)$

Now suppose $\{f_{\alpha+1}(\lambda')\}_{\lambda'<\lambda}$, for some $\lambda < \omega_1$

such that Lim (λ), is a segment of the sequence of fixed points of f_α. Then the smallest ordinal greater than any in the segment is $\lim_{\lambda'<\lambda} f_{\alpha+1}(\lambda')$, but by theorem 5.2, f_α is normal, therefore

$$f_\alpha\{\lim_{\lambda'<\lambda} f_{\alpha+1}(\lambda')\} = \lim_{\lambda'<\lambda} f_\alpha\{f_{\alpha+1}(\lambda')\} \text{ by congruence } (*)$$

$$= \lim_{\lambda'<\lambda} f_{\alpha+1}(\lambda'), \text{ as the numbers } f_{\alpha+1}(\lambda'), \lambda'<\lambda \text{ are fixed points of } f_\alpha$$

$$= f_{\alpha+1}(\lambda),$$ thus $f_{\alpha+1}(\lambda)$ is the smallest fixed point of f_α greater than any in the sequence $\{f_{\alpha+1}(\lambda')\}_{\lambda'<\lambda}$. □5.3(a)

(*) Suppose f is a normal function. Then for Lim (λ), $f(\lambda) = \lim_{\lambda'<\lambda} f(\lambda')$. Suppose $\lambda = \lim_{\mu'<\mu} \lambda_{\mu'}$, where Lim ($\mu$) and $\{\lambda_{\mu'}\}_{\mu'<\mu}$ is an increasing sequence. Then the sequences $\{f(\lambda')\}_{\lambda'<\lambda}$ and $\{f(\lambda_{\mu'})\}_{\mu'<\mu}$ are congruent, as f is strictly increasing, thus $f(\lambda) = \lim_{\mu'<\mu} f(\lambda_{\mu'})$.

PROOF (b) Suppose $\alpha < \omega_1$ is a limit ordinal. Then the smallest point belonging to $\bigcap_{\alpha'<\alpha} FP(f_{\alpha'})$ is the number $\lim_{\alpha'<\alpha} f_{\alpha'}(0) = f_\alpha(0)$.

Now suppose the η-th point belonging to $\bigcap_{\alpha'<\alpha} FP(f_\alpha)$ is $f_\alpha(\eta)$. Then the next largest ordinal number belonging to this set is the least which is a fixed point, greater than $f_\alpha(\eta)$, of each of the functions $f_{\alpha'}$, $\alpha' < \alpha$. Thus, it is the limit of the sequence $f_{\alpha'}\{f_\alpha(\eta)+1\}_{\alpha'<\alpha}$, and by definition 5.1(a), this limit is $f_\alpha(\eta+1)$.

Now suppose $\{f_\alpha(\eta')\}_{\eta'<\eta}$, for some $\eta < \omega_1$ such that $Lim(\eta)$ is a segment of the sequence of points in $\bigcap_{\alpha'<\alpha} FP(f_{\alpha'})$. Then the smallest ordinal greater than any in the segment is $\lim_{\eta'<\eta} f_\alpha(\eta')$, since by theorem 5.2, f_α is normal, thus strictly increasing. Furthermore,

$$(\forall \alpha' < \alpha)\ f_{\alpha'}\{\lim_{\eta'<\eta} f_\alpha(\eta')\} = \lim_{\eta'<\eta} f_{\alpha'}\{f_\alpha(\eta')\}$$

$$= \lim_{\eta'<\eta} f_\alpha(\eta'), \text{ as}$$

$(\forall \alpha' < \alpha)\ f_\alpha(\eta')$, each η', are fixed points of the function $f_{\alpha'}$, therefore
$f_{\alpha'}\{\lim_{\eta' < \eta} f_\alpha(\eta')\} = f_\alpha(\eta)$ by definition 5.1a.

Thus, part (b) is proved by transfinite induction. □5.3

THEOREM 5.4 $\lambda\alpha . f_\alpha(0)$ is normal, if $0 < f_0(0)$.

PROOF. $f_0(0) = \kappa > 0$. As f_0 is normal, $f_0(1) > f_0(0) = \kappa$, therefore $f_0(\kappa) > f_0(0) = \kappa$. Thus, as $f_1(0) = \lim\{f_0(0), f_0\{f_0(0)\}, \ldots\}$, $f_1(0) > \kappa$, therefore $f_1(0) > f_0(0)$.

The proof proceeds by transfinite induction. Suppose for I.H. that it has been proved that for some non-zero $\alpha < \omega_1$ $(\forall \beta < \alpha)_{\beta > 0}$

$$Suc(\beta) \Rightarrow f_\beta(0) > f_{P\beta}(0)\ \&$$
$$Lim(\beta) \Rightarrow f_\beta(0) > f_{\beta'}(0),\ (\forall \beta' < \beta).$$

Then if $Suc(\alpha)$,

$f_{P\alpha}(0) = \kappa^* \geq \kappa$, for some $\kappa^* < \omega_1$ by I.H., and as $f_{P\alpha}$ is normal, $f_{P\alpha}(1) > f_{P\alpha}(0) = \kappa^*$, therefore $f_{P\alpha}(\kappa^*) > f_{P\alpha}(0) = \kappa^*$. Thus, as

$f_\alpha(0) = \lim \{f_{P\alpha}(0), f_{P\alpha}\{f_{P\alpha}(0)\}, \ldots\}$

$f_\alpha(0) > \kappa^*$, therefore $f_\alpha(0) > f_{P\alpha}(0)$. Thus $\lambda\alpha. f_\alpha(0)$ is strictly increasing at α.

If $\text{Lim}(\alpha)$, by definition 5.1a, $f_\alpha(0) = \lim\limits_{\alpha' < \alpha} f_{\alpha'}(0)$, thus $\lambda\alpha. f_\alpha(0)$ is continuous at α.

But by I.H., $\{f_{\alpha'}(0)\}_{\alpha' < \alpha}$ is strictly increasing, thus it's limit, $f_\alpha(0)$, is greater than any of it's terms, thus $\lambda\alpha. f_\alpha(0)$ is s.i. at α.

Thus it follows by transfinite induction that $\lambda\alpha. f_\alpha(0)$ is normal. $\square$ 5.4

COROLLARY 5.5 $(\forall \beta, \lambda < \omega_1)(\forall \alpha < \beta)$

(i) $\text{Lim}(\lambda) \Rightarrow \lim\limits_{\lambda' < \lambda} f_\alpha(\lambda') \in \text{Rge}(f_\alpha)$

(ii) $\text{Rge}(f_\alpha) \subseteq \omega_1$

(iii) $\overline{\text{Rge}(f_\alpha)} = \omega_1$

(iv) $(\forall \alpha' < \alpha)$ $\text{Rge}(f_\alpha) \subseteq \text{Rge}(f_{\alpha'})$

(v) $\text{Rge}(f_\beta) \subseteq \bigcap\limits_{\alpha < \beta} \text{Rge}(f_\alpha)$

(vi) $\text{Lim}(\lambda) \Rightarrow \lim\limits_{\lambda' < \lambda} f_\beta(\lambda') \in \text{Rge}(f_\beta)$

(vii) $\overline{\text{Rge}(f_\beta)} = \omega_1$,

where f_0 is a normal function.
PROOF. (i) from definition 5.1a, (ii) by definition 5.1a, as f_α is only defined by the two operations of taking successor and countable limit, (iii) $(\forall \eta, \eta' < \omega_1)\ \eta \neq \eta' \Rightarrow f_\alpha(\eta) \neq f_\alpha(\eta')$, by theorem 5.2, (iv), (v) from theorem 5.3, (vi) from definition 5.1a, (vii) as for (iii). □ 5.5

COROLLARY 5.6

$FP(\lambda\alpha . f_\alpha(0)) \subseteq FP(f_0)$, if $f_0(0) > 0$.

PROOF By corollary 5.5, (iv), $(\forall \alpha < \omega_1)_{\alpha > 0}\ f_\alpha(0) \in Rge(f_1)$. Therefore $(\forall \alpha < \omega_1)_{>0}$ $f_\alpha(0)$ is a fixed point of f_0. Furthermore, $f_0(0) = K > 0$, and by theorem 5.4 $\lambda\alpha . f_\alpha(0)$ is normal, therefore $f_1(0) > f_0(0) = \kappa$. Therefore the smallest fixed point μ of $\lambda\alpha . f_\alpha(0)$ which satisfies $f_\mu(0) = \mu$ is therefore itself a fixed point of f_0, since $\mu > 0$.

Also, since $FP(\lambda\alpha . f_\alpha(0)) \subseteq Rge(\lambda\alpha . f_\alpha(0))$, corollary 5.6 follows. □

Now let $(\forall \beta < \omega_1)\ \lambda\vec{\alpha} . \phi((\alpha_1)_1, (\alpha_2)_2, \ldots, (\alpha_\beta)_\beta)$ be the Veblen functions. Let $(\forall \alpha_1 < \omega_1)\ \phi((\alpha_1)_1) = f_0(\alpha_1)$.

Then:

$\emptyset((\alpha_1)_1,(1)_2) = \alpha_1$-th number α such that

$$\emptyset((\alpha_1)_1)=\alpha$$

$\emptyset((\alpha_1)_1,(2)_2) = \alpha_1$-th number α such that

$$\emptyset((\alpha)_1,(1)_2)=\alpha$$

$\emptyset((\alpha_1)_1,(\alpha_2)_2) = \alpha_1$-th number belonging to

$$\bigcap_{\alpha_2'<\alpha_2}\{\alpha \mid \emptyset((\alpha)_1,(\alpha_2')_2)=\alpha\}$$

Thus:

THEOREM 5.7 $(\forall\alpha_1,\alpha_2<\omega_1)$

$\emptyset((\alpha_1)_1) = f_0(\alpha_1)$ & $\emptyset((\alpha_1)_1,(\alpha_2)_2) = f_{\alpha_2}(\alpha_1)$

□ 5.7.

COROLLARY 5.8 $(\forall\beta<\omega_1)$ the sequence of sets $\{\mathrm{Rge}\ f_\alpha\}_{\alpha<\beta}$, and the set $\mathrm{Rge}\ f_\beta$ have the properties of the sets $\{S_\alpha\}$ and of the set S of theorem 5 of [20], p. 284.

PROOF. Immediate from corollary 5.5 □ 5.8

Thus it follows that $(\forall\beta<\omega_1)$

$f_1(\beta) = f^F(\beta)$

$f_{\alpha+1}(\beta) = f_\alpha^F(\beta)$, and for Lim (α)

$$f_\alpha(\beta) = \left\{ \begin{array}{ll} \lim\limits_{\alpha'<\alpha} f_{\alpha'}(0) \text{ if } \beta = 0 \\ \lim\limits_{\beta'<\beta} f_\alpha(\beta') \text{ if } \mathrm{Lim}(\beta) \\ \lim\limits_{\alpha'<\alpha} f_{\alpha'}\{f_\alpha(P\beta)+1\} \text{ if } \mathrm{Suc}(\beta). \end{array} \right.$$

For simple values of f, f_α can be computed inductively, without introducing the higher arithmetic operations. For example, if $f = \lambda\beta.1+\beta$, then $f^F = \lambda\beta.\omega+\beta$. Thus $f_1(\beta) = \omega+\beta$; if $f_\alpha(\beta) = \omega^\alpha+\beta$, for some α and all $\beta<\omega_1$, then

$$f_{\alpha+1}(\beta) = \left\{ \begin{array}{l} \lim\limits_{\beta'<\beta} \omega^{\alpha+1}+\beta' = \omega^{\alpha+1}+\beta \text{ if } \mathrm{Lim}(\beta) \\ \lim\limits_{n<\omega} [\underbrace{\omega^\alpha+\omega^\alpha+\ldots+\omega^\alpha}_{n}+(\omega^{\alpha+1}+P\beta)+1] \\ \qquad = \omega^{\alpha+1}+\beta \text{ if } \mathrm{Suc}(\beta) \\ \lim\limits_{n<\omega} [\underbrace{\omega^\alpha+\omega^\alpha+\ldots+\omega^\alpha}_{n}] \\ \qquad = \omega^\alpha\cdot\omega = \omega^{\alpha+1} \text{ if } \beta = 0 \end{array} \right. ;$$

and if Lim (α), and for $\alpha'<\alpha$, $f_{\alpha'}(\beta) = \omega^{\alpha'}+\beta$, then:

$$f_\alpha(\beta) = \begin{cases} \lim\limits_{\alpha' < \alpha} \omega^{\alpha'} + 0 = \omega^\alpha \text{ if } \beta = 0 \\ \lim\limits_{\beta' < \beta} \omega^\alpha + \beta' = \omega^\alpha + \beta \text{ if } \mathrm{Lim}(\beta) \\ \lim\limits_{\alpha' < \alpha} \omega^{\alpha'} + \{\omega^\alpha + P_\beta + 1\} = \omega^\alpha + P_\beta + 1 \\ \qquad\qquad\qquad\qquad \text{if } \mathrm{Suc}(\beta), \end{cases}$$

therefore it follows by transfinite induction that

$$(\forall \alpha, \beta < \omega_1)_{\alpha > 0}\ f_\alpha(\beta) = \omega^\alpha + \beta.$$

There are, of course, functions definable from the function Φ which are strictly increasing but not continuous and therefore non-normal. For example, let $(\forall \alpha < \omega_1)(\forall x < \omega)\ g(\omega\alpha + x) =_{Df} \Phi^{\beta}_{2+\alpha}(\omega)$, for $\beta = \omega\alpha + x$. Then

$$g(x) = \omega \cdot x, \quad x < \omega$$

$$g(\omega + x) = \omega^{(\omega + x)}$$

$$g(\omega.2) = \overbrace{\cdot\cdot\cdot\,\omega^{\cdot^{\cdot^{\cdot^{\omega^{\omega}}}}}}^{\omega.2} = \varepsilon_1$$

$$g(\omega.2 + x) = \overbrace{\omega^{\cdot^{\cdot^{\omega}}}}^{x}{}^{\varepsilon_1 + 1}$$

$$g(\omega.3) = \Phi^{\omega.3}_{5}(\omega)$$

THEOREM. The function g is strictly increasing, but is discontinuous.

PROOF. Suppose $\beta' < \beta < \omega_1$. Then for $\beta' = \omega\alpha_1 + x_1$, $\beta = \omega\alpha_2 + x_2$, if $\alpha_1 = \alpha_2 = \alpha$

$$g(\beta) = \Phi^{\beta}_{2+\alpha}(\omega) > \Phi^{\beta'}_{2+\alpha}(\omega) \text{ by theorems 4.1}$$
$$= g(\beta')$$

Thus without loss of generality, let $\alpha_1 < \alpha_2$. Then

$$\begin{aligned} g(\beta) &= \Phi^{\beta}_{2+\alpha_2}(\omega) \\ &\geqslant \Phi^{\beta}_{2+\alpha_1}(\omega) \text{ by corollary 4.2} \\ &> \Phi^{\beta'}_{2+\alpha_1}(\omega) \text{ by theorem 4.1} \\ &= g(\beta'). \end{aligned}$$

Therefore g is strictly increasing, but from the examples given above, can be seen to be discontinuous :

e.g. $\beta = \omega.2$

$$\begin{aligned} g(\omega.2) &= \Phi_4^{\omega.2}(\omega) \\ &= \varepsilon_1 \\ &> \omega^{\omega.2} \\ &= \lim_{x<\omega} \omega^{\omega+x} \\ &= \lim_{x<\omega} \Phi_{2+1}^{\omega+x}(\omega) = \lim_{\beta'<\omega.2} g(\beta'). \end{aligned}$$

e.g. $\beta = \omega^2$

$$g(\omega^2) = \Phi_{2+\omega}^{\omega^2}(\omega);$$

$(\forall x < \omega)$

$$g(\omega . x) = \Phi_{2+x}^{\omega . x}(\omega)$$

$$\leq \Phi_{2+x}^{\Phi_{2+x+1}^{2}(\omega)}(\omega), \text{ by theorem 4.1, as } \omega . x \leq \Phi_{2+x+1}^{2}(\omega)$$

$$= \Phi_{2+x+1}^{3}(\omega) \text{ by definition 3.1}$$

$$< \Phi_{2+x+1}^{\omega}(\omega) \text{ by theorem 4.1}$$

$$= \Phi_{2+x+1+1}^{2}(\omega) \text{ by definition 3.1}$$

$$= \Phi_{x+4}^{2}(\omega)$$

therefore

$$\lim_{x < \omega} g(\omega . x) \leq \lim_{x < \omega} \Phi_{x+4}^{2}(\omega)$$

$$= \Phi_{\omega}^{2}(\omega)$$

$$< \Phi_{\omega}^{\omega^2}(\omega) \text{ by theorem 4.1}$$

$$= g(\omega^2)$$

□

EPSILON NUMBERS DERIVED FROM GENERALIZED ARITHMETIC OPERATIONS

In the first instance, the sequence of countable epsilon numbers $\{\varepsilon_\mu\}_{\mu<\omega_1}$ is examined. These numbers can be completely characterized in terms of the operations Φ_α.

THEOREM 5.11. Each of the countable ε-numbers is of the form $\varepsilon_\beta = \Phi_4^{\omega(1+\beta)}(\omega)$.

PROOF, by transfinite induction. Firstly, suppose $\beta=0$. Then

$$\Phi_4^{\omega}(\omega) = \lim_{n<\omega} \Phi_4^{n}(\omega)$$

$$= \lim\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\} = \varepsilon_0$$

Now suppose that β is such that $0<\beta<\omega_1$, and that $(\forall\beta'<\beta)\ \varepsilon_{\beta'} = \Phi_4^{\omega(1+\beta')}(\omega)$. Then if Suc$(\beta)$, the smallest ε-number greater than $\varepsilon_{P\beta}$ is the limit of the increasing sequence

$$\left\{ \overbrace{\omega^{\omega^{\cdot^{\cdot^{\omega}}}}}^{n}{}^{(\varepsilon_{P\beta}+1)} \right\}_{n<\omega}$$

Thus $$\varepsilon_\beta = \lim_{n<\omega} \left\{ \overbrace{\Phi_3{}^{\Phi_3{}^{\cdot^{\cdot^{\Phi_3}}}}}^{n}{}^{\Phi_4^{\omega(1+P\beta)}(\omega)+1}(\omega) \cdots (\omega)(\omega) \right\}_{n<\omega}$$

$$= \lim_{n<\omega} \left\{ \Phi_4^{\omega(1+\beta)+n}(\omega) \right\}$$

$$= \Phi_4^{\omega(1+P\beta)+\omega}(\omega) = \Phi_4^{\omega(1+\beta)}(\omega).$$

If $\mathrm{Lim}(\beta)$, then the smallest ε-number greater than any of the form $\varepsilon_{\beta'}$, $\beta' < \beta$, is

$$\lim_{\beta' < \beta} \varepsilon_{\beta'} = \lim_{\beta' < \beta} \Phi_4^{\omega(1+\beta')}(\omega) \text{ by I.H.}$$

$$= \lim_{n < \omega} \Phi_4^{\omega(1+\Omega_\beta(n))}(\omega) \text{ by theorem 4.1}$$

$$= \Phi_4^{\omega(1+\beta)}(\omega)$$

Thus theorem 5.11 is proved by transfinite induction □

Since for transfinite β, $1+\beta=\beta$, theorem 5.11 can be propounded as:

$$(\forall \beta < \omega_1) \qquad \varepsilon_\beta = \begin{cases} \Phi_4^{\omega(\beta+1)}(\omega) \text{ if } \beta < \omega \\ \Phi_4^{\omega\beta}(\omega) \text{ if } \omega \leq \beta \end{cases}$$

THEOREM 5.12 $(\forall \beta < \omega_1)_{>1} [\beta \notin \{\varepsilon_\mu\}_{\mu < \omega_1} \Rightarrow \varepsilon_0^\beta \notin \{\varepsilon_\mu\}_{\mu < \omega_1}]$

PROOF. Suppose $\beta = n < \omega$. Then for $n > 1$:

$$\omega^{\varepsilon_0^n} = \omega^{\varepsilon_0 \cdot \varepsilon_0^{n-1}}$$

$$= (\omega^{\varepsilon_0})^{\varepsilon_0^{n-1}} \text{ by multiplication law for powers of an ordinal}$$

$$= \varepsilon_0^{\varepsilon_0^{n-1}}$$

$$\geq \varepsilon_0^{\varepsilon_0}$$

$> \varepsilon_0^n$, therefore ε_0^n is not an ε-number.

Now suppose $\beta \geq \omega$. Then $1+\beta = \beta$, therefore

$$\omega^{\varepsilon_0^{\beta}} = \omega^{\varepsilon_0^{1+\beta}} = \omega^{\varepsilon_0 . \varepsilon_0^{\beta}}$$

$$= (\omega^{\varepsilon_0})^{\varepsilon_0^{\beta}} \text{ by multiplication law}$$

$$= \varepsilon_0^{\varepsilon_0^{\beta}}$$

$$\geq \varepsilon_0^{\omega^{\beta}}$$

$$> \varepsilon_0^{\beta}, \text{ as } \beta < \omega^{\beta} \text{ by assumption}.$$

Therefore ε_0^{β} is not an ε-number. □ THEOREM 5.12

G. Hessenberg introduces the terminology 'δ-numbers' for ordinals $\alpha > 0$ satisfying $\omega . \alpha = \alpha$. These numbers, as non-zero fixed points of $\lambda\alpha . \omega\alpha$, are the numbers $\omega^{\omega} . \alpha$, $\alpha > 0$, so that $\delta_0 = \omega^{\omega}$; every epsilon number is a delta number, but not vice-versa.

Since $(\forall \mu < \omega_1)(\forall \alpha < 4)\ \Phi_\alpha^{\varepsilon_\mu}(\omega) = \varepsilon_\mu$, for each $\mu < \omega_1$ the notation $C_\mu^{(4)}$ is adopted for the number ε_μ.

None of the operations Φ_α, $\alpha < 4$, when applied to ordinals $\beta, \gamma < \varepsilon_0$ are sufficient to generate any ordinal $\geq \varepsilon_0$. Thus theorems 5.11, 5.12 emphasize that the definition of epsilon numbers requires a new arithmetical operation, Φ_4.

For $\beta = 0$, $\varepsilon_\beta = \Phi_4^\omega(\omega) = \Phi_5^2(\omega)$. This corresponds with the standard notation for certain smaller ordinals:

$\omega+2 = \Phi_1^2(\omega)$

$\omega.2 = \Phi_2^2(\omega)$

$\omega^2 = \Phi_3^2(\omega)$

$\omega^\omega = \Phi_4^2(\omega) = \delta_0$

$\varepsilon_0 = \Phi_5^2(\omega)$

DEFINITION 5.12 a $(\forall\alpha<\omega_1)_{\alpha\geqslant 4}$ $\{\varepsilon_\mu^{(\alpha)}\}_{\mu<\omega_1}$ denotes the sequence* of numbers ξ which are solutions to the equation $\Phi_{P\alpha}^{\xi}(\omega) = \xi$, if Suc$(\alpha)$, or $(\forall\alpha'<\alpha)$ to the equations $\Phi_{\alpha'}^{\xi}(\omega) = \xi$, if Lim$(\alpha)$.

The alternative notation already adopted for ε-numbers thus satisfies definition 5.12a.

Note that every $\varepsilon^{(\alpha)}$-number, for α such that $4 \leqslant \alpha < \omega_1$, is a limit number. For suppose otherwise, and let $\beta+1$ be such an $\varepsilon^{(\alpha)}$-number. Then if Suc(α), $(\forall\beta<\omega_1)_{>0}$ $\Phi_{P\alpha}^{\beta+1}(\omega) > \beta+1$ by theorem 4.1, but Lim(l.h.s.) and Suc(r.h.s.) by theorem 4.1a2, therefore $\beta+1 < \Phi_{P\alpha}^{\beta+1}(\omega)$ $\divideontimes$

---

* ordered according to magnitude

If Lim (α), then $(\forall\beta<\omega_1)$

$\Phi_\alpha^{\beta+1}(\omega) \geqslant \beta+1$ by theorem 4.1, but

$$\Phi_\alpha^{\beta+1}(\omega) = \lim_{\alpha'<\alpha} \Phi_{\alpha'}^{\Phi_\alpha^\beta(\omega)+1}(\omega),$$

therefore Lim (l.h.s.) by theorem 4.1a2 and corollary 4.2, and so $\beta+1 < \Phi_\alpha^{\beta+1}(\omega)$ ⋇

Thus every $\varepsilon^{(\alpha)}$-number, for $4\leqslant\alpha<\omega_1$, is a limit number.

THEOREM 5.13 $(\forall\alpha<\omega_1)_{\alpha\geqslant4}$ $\mathrm{Suc}(\alpha)\Rightarrow$

a) $\varepsilon_0^{(\alpha)} = \lim_{n<\omega} \Phi_\alpha^n(\omega)$

b) $(\forall\beta<\omega_1)$ $\mathrm{Suc}(\beta)\Rightarrow$ the least $\varepsilon^{(\alpha)}$-number greater than $\varepsilon_{P\beta}^{(\alpha)}$ is the limit of the sequence

$$\left\{\varepsilon_{P\beta}^{(\alpha)}+1,\ \Phi_{P\alpha}^{\varepsilon_{P\beta}^{(\alpha)}+1}(\omega),\ \Phi_{P\alpha}^{\Phi_{P\alpha}^{\varepsilon_{P\beta}^{(\alpha)}+1}(\omega)}(\omega), \ldots\right\}$$

c) $(\forall\beta<\omega_1)$ $\mathrm{Lim}(\beta)\Rightarrow$ $\varepsilon_\beta^{(\alpha)} = \lim_{\beta'<\beta} \varepsilon_{\beta'}^{(\alpha)}$.

PROOF. In the case $\alpha=4$, the cases a), b), c) are standard properties of the epsilon numbers which are identical with the $\varepsilon^{(4)}$-numbers.

Suppose α is a successor number satisfying

$4 < \alpha < \omega_1$. By theorem 4.1, $\lambda\beta . \underline{\Phi}_\alpha{}^\beta(\omega)$ is strictly increasing. Thus, in the various cases:

(a) Consider the sequence

$$\left\{\omega, \underline{\Phi}_{P_\alpha}^{\omega}(\omega), \underline{\Phi}_{P_\alpha}^{\underline{\Phi}_{P_\alpha}^{\omega}(\omega)}(\omega), \underline{\Phi}_{P_\alpha}^{\underline{\Phi}_{P_\alpha}^{\underline{\Phi}_{P_\alpha}^{\omega}(\omega)}(\omega)}(\omega), \ldots\right\}$$

$= \left\{\underline{\Phi}_\alpha^n(\omega)\right\}_{n<\omega}$, since by definition 3.1,

$$\underline{\Phi}_{P_\alpha}^{\omega}(\omega) = \underline{\Phi}_\alpha^2(\omega), \text{ and } (\forall n<\omega)_{\geq 2}$$

$$\underline{\Phi}_{P_\alpha}^{\underline{\Phi}_\alpha^n(\omega)}(\omega) = \underline{\Phi}_\alpha^{n+1}(\omega).$$

Since $\omega < \underline{\Phi}_{P_\alpha}^{\omega}(\omega)$, the sequence is strictly increasing, and it's limit is denoted by

$\varepsilon = \lim\limits_{n<\omega} \underline{\Phi}_\alpha^n(\omega)$. Now $\underline{\Phi}_{P_\alpha}^{\varepsilon}(\omega)$

$$= \lim_{n<\omega} \underline{\Phi}_{P_\alpha}^{\underline{\Phi}_\alpha^n(\omega)}(\omega)$$

$$= \lim_{n<\omega} \underline{\Phi}_\alpha^{n+1}(\omega)$$

$$= \underline{\Phi}_\alpha^{\omega}(\omega) = \varepsilon .$$

Thus ε is a fixed point of $\lambda\beta . \underline{\Phi}_{P_\alpha}^{\beta}(\omega)$. It is the smallest such number, for if $\underline{\Phi}_{P_\alpha}^{\beta}(\omega) = \beta$, then $\beta > 1$, so that $\omega < \underline{\Phi}_{P_\alpha}^{\beta}(\omega)$.

Now suppose $\underline{\Phi}_\alpha^n(\omega) < \beta$, for some n. Then

$\Phi_{P\alpha}^{\Phi_\alpha^n(\omega)}(\omega) < \Phi_{P\alpha}^{\beta}(\omega)$, therefore

$\Phi_\alpha^{n+1}(\omega) <\cdot\ \Phi_{P\alpha}^{\beta}(\omega) = \beta$, by theorem 4.1, thus by induction $(\forall n < \omega)\ \Phi_\alpha^n(\omega) < \beta$, so $\lim_{n<\omega} \Phi_\alpha^n(\omega) = \Phi_\alpha^\omega(\omega) \leq \beta$. Thus $\varepsilon_{\varepsilon_0}^{(\alpha)} =_{Df} \varepsilon$.

Now let β be a successor number, and suppose that $(\forall \beta' < \beta)$ numbers $\varepsilon_{\beta'}^{(\alpha)}$ have been defined according to 5.13 a), b), c). Then for case b) consider the number $\varepsilon_{P\beta}^{(\alpha)}$. Let $\gamma_1 =_{Df} \varepsilon_{P\beta}^{(\alpha)} + 1$, and for each n such that $0 < n < \omega$, let $\gamma_{n+1} =_{Df} \Phi_{P\alpha}^{\gamma_n}(\omega)$.

Now $\varepsilon_{P\beta}^{(\alpha)} + 1 \leq \Phi_{P\alpha}^{\varepsilon_{P\beta}^{(\alpha)}+1}(\omega)$; but since Suc(l.h.s.) and Lim (r.h.s.) by theorem 4.1a2,

$$\varepsilon_{P\beta}^{(\alpha)} + 1 \lneq \Phi_{P\alpha}^{\varepsilon_{P\beta}^{(\alpha)}+1}(\omega),$$

and so by induction $(\forall n < \omega)\ \gamma_n < \gamma_{n+1}$. Thus, as in case a) it follows:

$$\Phi_{P\alpha}^{\lim_{n<\omega}\gamma_n}(\omega) = \lim_{n<\omega} \Phi_{P\alpha}^{\gamma_n}(\omega)$$

$$= \lim_{n<\omega} \gamma_{n+1}$$

$$= \lim_{n<\omega} \gamma_n.$$

This is the smallest

$\varepsilon^{(\alpha)}$- number greater than $\varepsilon^{(\alpha)}_{P\beta}$, for if for some δ, $\Phi^{\delta}_{P\alpha}(\omega) = \delta$ and $\varepsilon^{(\alpha)}_{P\beta} < \delta$ $< \lim_{n<\omega} \gamma_n$, then since Lim (δ), $\varepsilon^{(\alpha)}_{P\beta}+1 \lneqq \delta$, and $\gamma_n < \delta \Rightarrow \Phi^{\gamma_n}_{P\alpha}(\omega) < \Phi^{\delta}_{P\alpha}(\omega)$, therefore $\gamma_{n+1} < \delta$, thus $\lim_{n<\omega} \gamma_n \leq \delta$, and so $\varepsilon^{(\alpha)}_{\beta} =_{df} \lim_{n<\omega} \gamma_n$.

Now let β be a limit number, and suppose that $(\forall \beta' < \beta)$, numbers $\varepsilon^{(\alpha)}_{\beta'}$ have been defined according to 5.13a), b), c). Then for case c):

c) Let $\varepsilon = \lim_{\beta'<\beta} \varepsilon^{(\alpha)}_{\beta'}$.

Then $\Phi^{\varepsilon}_{P\alpha}(\omega) = \lim_{\beta'<\beta} \Phi^{\varepsilon^{(\alpha)}_{\beta'}}_{P\alpha}(\omega)$ by corollary 4.1a1

$= \lim_{\beta'<\beta} \varepsilon^{(\alpha)}_{\beta'}$

$= \varepsilon$; thus ε is an $\varepsilon^{(\alpha)}$-number. Since $\varepsilon = \lim_{\beta'<\beta} \varepsilon^{(\alpha)}_{\beta'}$, it is the smallest $\varepsilon^{(\alpha)}$- number greater than any of the numbers $\varepsilon^{(\alpha)}_{\beta'}$, $\beta' < \beta$, so $\varepsilon = \varepsilon^{(\alpha)}_{\beta}$, thus $\varepsilon^{(\alpha)}_{\beta} =_{df} \lim_{\beta'<\beta} \varepsilon^{(\alpha)}_{\beta'}$.

☐ THEOREM 5.13.

COROLLARY 5.14 $(\forall \alpha < \omega_1)\ Suc(\alpha) \Rightarrow$ the least $\varepsilon^{(\alpha)}$ number is $\Phi_{\alpha+1}^{2}(\omega)$.

PROOF. From theorem 5.13(a),

$$\begin{aligned} \varepsilon_0^{(\alpha)} &= \lim_{n<\omega} \Phi_\alpha^{n}(\omega) \\ &= \Phi_\alpha^{\omega}(\omega) \text{ by definition 3.1} \\ &= \Phi_{\alpha+1}^{2}(\omega) \text{ by definition 3.1} \quad \square \end{aligned}$$

COROLLARY 5.15 $(\forall \alpha, \beta < \omega_1)_{3<\alpha,\ Suc(\alpha)}$
$\varepsilon_\beta^{(\alpha)} = \Phi_\alpha^{\omega(1+\beta)}(\omega)$

PROOF by transfinite induction on index β.

For $\beta = 0$, $\Phi_\alpha^{\omega}(\omega) = \varepsilon_0^{(\alpha)}$ by theorem 5.13 a).

Suppose the cases for $\beta' < \beta$ have already been proved, for some β such that $0 < \beta < \omega_1$. Then if $Suc(\beta)$, by theorem 5.13 b)

$$\begin{aligned} \varepsilon_\beta^{(\alpha)} &= \lim \left\{ \Phi_\alpha^{\omega(1+P\beta)}(\omega)+1,\ \Phi_{P\alpha}^{\Phi_\alpha^{\omega(1+P\beta)}(\omega)+1}(\omega), \ldots \right\} \\ &= \lim \left\{ \Phi_\alpha^{\omega(1+P_\beta)}(\omega)+1,\ \Phi_\alpha^{\omega(1+P\beta)+1}(\omega),\ \Phi_\alpha^{\omega(1+P_\beta)+2}(\omega) \ldots \right\} \\ &= \Phi_\alpha^{\omega(1+P\beta)+\omega}(\omega) \\ &= \Phi_\alpha^{\omega(1+P\beta+1)}(\omega) \\ &= \Phi_\alpha^{\omega(1+\beta)}(\omega). \end{aligned}$$

If $Lim(\beta)$, then $\varepsilon_\beta^{(\alpha)} = \lim_{\beta'<\beta} \varepsilon_{\beta'}^{(\alpha)}$ by theorem 5.13 c), therefore $\mathcal{E}_{\beta}^{(\alpha)} = \lim_{\beta' < \beta} \Phi_{\alpha}^{\omega(1+\beta')}(\omega)$ by I.H.

$$= \Phi_{\alpha}^{\omega(1+\beta)}(\omega)$$

Thus corollary 5.15 is proved. $\square$

Sierpinski [18], p. 328, has shown that numbers $\beta > \omega$ satisfying $2^{\beta} = \beta$ are each ε-numbers. Since it is not in general true that $\Phi_{\alpha+1}^{\Phi_{\alpha}^{\beta}(\omega)}(2) = \Phi_{\alpha+1}^{\beta}(\Phi_{\alpha+1}^{\omega}(2))$, Sierpinski's method of proof cannot be directly generalized to the $\varepsilon^{(\alpha)}$-numbers. But the theorem itself can be used to obtain a similar result for $\varepsilon^{(5)}$-numbers. Thus

LEMMA 5.16 $(\forall \beta < \omega_1)_{>0}\ \Phi_3^{\beta}(2) \leq \Phi_4^{\beta}(2)$.

PROOF by T.I. on β.

It is true that $\Phi_4^{1}(2) \geq \Phi_3^{1}(2) = 2^1$.

Suppose for some β such that $1 < \beta < \omega_1$, that $(\forall \beta' < \beta)\ \Phi_4^{\beta'}(2) \geq \Phi_3^{\beta'}(2)$.

Then if $\left\{\begin{matrix}\text{Suc}(P\beta)\\ \text{Suc}(\beta)\end{matrix}\right\}$, $\Phi_4^{\beta}(2) \geq 2^{\Phi_4^{P\beta}(2)}$

$$\geq 2^{2^{P\beta}} \text{ by I.H.}$$

$$\geq 2^{P\beta}.2 \quad \text{since Suc}(P\beta)$$

$$= 2^{\beta} = \Phi_3^{\beta}(2).$$

If $\text{Suc}(\beta)$ and $\text{Lim}(P\beta)$

$$\Phi_4^{\beta}(2) = 2^{\Phi_4^{P\beta}(2)+1}$$

So $\Phi_4^{\beta}(2) \geqslant 2^{\Phi_3^{P\beta}(2)+1}$ by I.H.

$$= 2^{(2^{P\beta}+1)}$$

$$= 2^{2^{P\beta}}.2$$

$$\geqslant 2^{P\beta}.2 = 2^{\beta} = \Phi_3^{\beta}(2).$$

And if $Lim(\beta)$,

$$\Phi_4^{\beta}(2) = \lim_{\beta'<\beta} \Phi_4^{\beta'}(2)$$

$$\geqslant \lim_{\beta'<\beta} 2^{\beta'} \text{ by I.H.}$$

$$= 2^{\beta} = \Phi_3^{\beta}(2)$$

therefore $(\forall\beta<\omega)_{>0}\ \Phi_4^{\beta\cdot}(2) \geqslant \Phi_3^{\beta}(2)$ $\square$ 5.16

LEMMA 5.17

$(\forall\beta<\omega_1)_{>0}\quad \Phi_4^{\omega\cdot\beta}(2) \geqslant \Phi_4^{\beta}(\omega)$.

PROOF. The relation is true for $\beta=1$. Suppose each case true for $\beta'<\beta$, for some β such that $1<\beta<\omega_1$. Then if $Suc(\beta)$:

$$\Phi_4^{\omega\beta}(2) = \Phi_4^{\omega P\beta+\omega}(2) = \lim_{n<\omega} \Phi_4^{\omega P\beta+n}(2)$$

$$\geqslant$$

$$\lim\left\{\Phi_4^{P\beta}(\omega)+1,\ 2^{\Phi_4^{P\beta}(\omega)+1},\ 2^{2^{\Phi_4^{P\beta}(\omega)+1}},\ldots\right\} \text{ by I.H.}$$

Let $\gamma+1 = \Phi_4^{P\beta}(\omega)+1$, and consider

$$\delta = 2^{2^{2^{2^{\gamma+1}}}}.$$

Now $2^{\gamma+1} = 2^{\gamma} \cdot 2$

$$\geqslant \gamma + 2$$

$$2^{\gamma+2} = 2^{\gamma+1} . 2$$

$$2^{2^{\gamma+1} . 2} = \left(2^{2^{\gamma+1}}\right)^2 = 2^{2^{\gamma+1}} . \; 2^{2^{\gamma+1}}$$

$$\geqslant \omega . 2^{2^{\gamma+1}}$$

$$2^{\omega . 2^{2^{\gamma+1}}} = (2^{\omega})^{2^{2^{\gamma+1}}}$$

$$= \omega^{2^{2^{\gamma+1}}} \geqslant \omega^{\gamma+1}$$

Therefore $\delta \geqslant \omega^{\beta+1}$; but $\Phi_4^{\omega\beta}(2) \geqslant \delta$, therefore $\Phi_4^{\omega\beta}(2) \geqslant \omega^{\Phi_4^{\beta}(\omega)} = \Phi_4^{\beta}(\omega)$.

If $\mathrm{Lim}(\beta)$, then $\Phi_4^{\omega\beta}(2) = \lim_{\beta' < \beta} \Phi_4^{\omega\beta'}(2)$

$$\geqslant \lim_{\beta' < \beta} \Phi_4^{\beta'}(\omega) \text{ by I.H.}$$

$$= \Phi_4^{\beta}(\omega)$$

Thus the lemma is proved. $\square$ 5.17

Now suppose $\beta > \omega$ and $\beta = \Phi_4^{\beta}(2)$. Therefore by lemma 5.16, $\beta \geqslant \Phi_3^{\beta}(2)$, thus by Sierpinski's theorem cited above, β is an epsilon number, so $\omega . \beta = \beta$. Therefore $\beta = \Phi_4^{\omega\beta}(2)$

and therefore $\beta \geq \Phi_4^{\beta}(\omega)$ by lemma 5.15. But by theorem 4.1, strict inequality cannot hold, therefore $\beta = \Phi_4^{\beta}(\omega)$, and so β is an $\varepsilon^{(3)}$- number.

DEFINITION Let $\{\varepsilon_\beta^{(2)}\}_{\beta<\omega_1}$, $\{\varepsilon_\beta^{(3)}\}_{\beta<\omega_1}$ denote the γ- and δ- numbers respectively.

THEOREM 5.18. Suppose $\alpha = \varepsilon_\beta^{(\mu)}$, for some $\beta, \mu < \omega_1$ such that $2 \leq \mu$. Then $(\forall \mu' < \mu)_{\geq 2}$ $(\exists \beta_{\mu'})$ $\alpha = \varepsilon_{\beta_{\mu'}}^{(\mu')}$

PROOF. In the cases Lim(μ), by definition $\varepsilon_\beta^{(\mu)}$ is a solution ξ to every equation $\Phi_{\mu'}^{\xi}(\omega) = \xi$, for $\mu' < \mu$, therefore in those cases there exist the required indices $\beta_{\mu'}$.

Thus, suppose Suc(μ). Then by hypothesis, $\Phi_{P\mu}^{\alpha}(\omega) = \alpha$. Let μ' be any number such that $2 \leq \mu' \leq P\mu$. Then

$\Phi_{\mu'}^{\alpha}(\omega) \geq \alpha$ by theorem 4.1.

Also $\Phi_{\mu'}^{\alpha}(\omega) \leq \Phi_{P\mu}^{\alpha}(\omega)$ by corollary 4.2, therefore $\Phi_{\mu'}^{\alpha}(\omega) = \alpha$, thus for some $\beta_{\mu'}$,

$\alpha = \varepsilon_{\beta_{\mu'}}^{(\mu')}$. □ 5.18

Thus :

COROLLARY 5.19 $(\forall \alpha, \beta < \omega_1)_{\geq 2}$ $\beta < \alpha \Rightarrow$ every $\varepsilon^{(\alpha)}$-number is an $\varepsilon^{(\beta)}$-number. □

Thus, for example, it is proved that every epsilon number is also a delta and a gamma number. Corollary 5.15 characterizes the $\varepsilon^{(\alpha)}$-numbers, for α a countable successor number. Now the case for α a countable limit is dealt with.

THEOREM 5.20 $(\forall \mu, \alpha < \omega_1)_{Lim(\mu),\, 4 \leq \alpha}$ let $\{\beta_{\mu'}\}_{\mu' < \mu}$ be a strictly increasing sequence, and let $\{\varepsilon^{(\alpha)}_{\beta_{\mu'}}\}_{\mu' < \mu}$ be a μ-sequence of $\varepsilon^{(\alpha)}$-numbers. Then the number $\varepsilon = \lim_{\mu' < \mu} \varepsilon^{(\alpha)}_{\beta_{\mu'}}$ is an $\varepsilon^{(\alpha)}$-number, of index $\lim_{\mu' < \mu} \beta_{\mu'}$.

PROOF Since the sequence of $\varepsilon^{(\alpha)}$-numbers is strictly increasing, and so $\{\varepsilon^{(\alpha)}_{\beta_{\mu'}}\}_{\mu' < \mu}$ and $\{\varepsilon^{(\alpha)}_{\beta}\}_{\beta < \lim_{\mu' < \mu} \beta_{\mu'}}$ are congruent, in the case $Suc(\alpha)$ it follows from theorem 5.13 c) that

$$\lim_{\mu' < \mu} \varepsilon^{(\alpha)}_{\beta_{\mu'}} = \lim_{\beta < \lim_{\mu' < \mu} \beta_{\mu'}} \varepsilon^{(\alpha)}_{\beta} = \varepsilon^{(\alpha)}_{\lim_{\mu' < \mu} \beta_{\mu'}}.$$

Suppose now Lim (α). Then $(\forall\mu'<\mu)(\forall\alpha'<\alpha)$

$$\Phi_{\alpha'}^{\varepsilon^{(\alpha)}_{\beta_{\mu'}}}(\omega) = \varepsilon^{(\alpha)}_{\beta_{\mu'}} .$$

Therefore $(\forall\alpha'<\alpha)$

$$\Phi_{\alpha}^{\lim\limits_{\mu'<\mu}\varepsilon^{(\alpha)}_{\beta_{\mu'}}}(\omega) = \lim_{\mu'<\mu}\Phi_{\alpha}^{\varepsilon^{(\alpha)}_{\beta_{\mu'}}}(\omega)$$ by theorem 4.1a1

$$= \lim_{\mu'<\mu}\varepsilon^{(\alpha)}_{\beta_{\mu'}} .$$

Furthermore, by the congruence just indicated, this expression denotes the least $\varepsilon^{(\alpha)}$- number greater than any $\varepsilon^{(\alpha)}_{\beta}$, $\beta < \lim\limits_{\mu'<\mu}\beta_{\mu'}$, therefore

$$\lim_{\mu'<\mu}\varepsilon^{(\alpha)}_{\beta_{\mu'}} = \varepsilon^{(\alpha)}_{\lim\limits_{\mu'<\mu}\beta_{\mu'}} .$$ □ THEOREM 5.20

THEOREM 5.21. $(\forall\alpha<\omega_1)$ Lim $(\alpha) \Rightarrow$
$\varepsilon_0^{(\alpha)} = \Phi_\alpha^2(\omega)$.

PROOF, by transfinite induction on the limit ordinals smaller than ω_1.

Let $\alpha=\omega$. Then $\varepsilon_0^{(\omega)}$ is the smallest ordinal $\xi<\omega_1$ such that $(\forall n<\omega)\ \Phi_n^{\xi}(\omega)=\xi$.

Thus $\varepsilon_0^{(\omega)} \geq \lim\limits_{n<\omega}\Phi^2_{n+1}(\omega)$ by corollary 5.14

$= \Phi_\omega^2(\omega)$ by definition 3.1

But $(\forall n^*<\omega)\ \Phi_\omega^2(\omega) = \lim\limits_{n<\omega}\Phi^2_{n+n^*}(\omega)$

by lemma 4.1d, therefore $(\forall n^* < \omega)$

$$\Phi_{n^*}^{\Phi_\omega^2(\omega)}(\omega) = \Phi_{n^*}^{\lim\limits_{n<\omega} \Phi^2_{n+n^*+1}(\omega)}(\omega)$$

$$= \Phi_\omega^2(\omega)$$ by corollary 5.14, corollary 5.19 and theorem 5.20.

Therefore $\varepsilon_0^{(\omega)} = \Phi_\omega^2(\omega)$.

Now suppose for induction hypothesis that, for some α such that $\omega < \alpha < \omega_1$ and $\mathrm{Lim}(\alpha)$, $(\forall \alpha' < \alpha)\ \mathrm{Lim}(\alpha') \Rightarrow \varepsilon_0^{(\alpha')} = \Phi^2_{\alpha'}(\omega)$. Let $\lambda\beta . T(\beta)$ be the characteristic function of the predicate Suc, so that

$$T(\beta) = \begin{cases} 1 \text{ if } \mathrm{Suc}(\beta) \\ 0 \quad \text{o.w.} \end{cases}$$

Then $\varepsilon_0^{(\alpha)} \geq \lim\limits_{\alpha'<\alpha} \Phi^2_{\alpha'+T(\alpha')}(\omega)$ by corollary 5.14 and by induction hypothesis

$$= \lim_{\alpha'<\alpha} \Phi^2_{\alpha'+1}(\omega)$$ by lemma 4.1d

$$= \Phi^2_\alpha(\omega).$$

But $(\forall \alpha^* < \alpha)$ $\Phi_\alpha^2(\omega) = \lim\limits_{\alpha'<\alpha} \Phi^2_{\alpha'+\alpha^*}(\omega)$ by lemma 4.1d, therefore $(\forall \alpha^* < \alpha)$

$$\Phi_{\alpha^*}^{\Phi_\alpha^2(\omega)}(\omega) = \Phi_{\alpha^*}^{\lim\limits_{\alpha'<\alpha} \Phi^2_{\alpha'+\alpha^*+T(\alpha^*)}(\omega)}(\omega)$$

Therefore $\Phi_{\alpha*}^{\Phi_{\alpha}^{2}(\omega)}(\omega) = \lim_{\alpha'<\alpha} \Phi_{\alpha'}^{2}(\omega)$ by corollaries 5.14, 5.19, and by theorem 5.20, and by the inductive hypothesis.

Thus $\Phi_{\alpha*}^{\Phi_{\alpha}^{2}(\omega)}(\omega) = \Phi_{\alpha}^{2}(\omega) = \varepsilon_0^{(\alpha)}$.

□ THEOREM 5.21

LEMMA 5.22 $(\forall \varepsilon, \alpha < \omega_1)_{\geqslant 4}$ ε an $\varepsilon^{(\alpha)}$-number $\Rightarrow (\forall \alpha' < \alpha)_{\geqslant 3}$ $\Phi_{\alpha'}^{\varepsilon+1\,'}(\omega) = \omega^{\varepsilon+1}$.

PROOF, by transfinite induction. Let α be any ordinal in the given range. For $\alpha' = 3$, the equality is true by definition. Suppose it has been proved to hold for all α' such that $3 \leqslant \alpha' < \alpha^* < \alpha$, for some α^*. Then if Suc (α^*)

$$\Phi_{\alpha*}^{\varepsilon+1\,'}(\omega) = \Phi_{P\alpha*}^{\Phi_{\alpha*}^{\varepsilon}(\omega)+1}(\omega) \text{ by definition 3.1}$$

$$= \Phi_{P\alpha*}^{\varepsilon+1\,'}(\omega) \text{ by corollary 5.19, as } \varepsilon \text{ is an } \varepsilon^{(\alpha)}\text{-number}$$

$$= \omega^{\varepsilon+1} \text{ by induction hypothesis.}$$

If Lim (α^*), then

$$\Phi_{\alpha*}^{\varepsilon+1}(\omega) = \lim_{\alpha'<\alpha^*} \Phi_{\alpha'}^{\Phi_{\alpha*}^{\varepsilon}(\omega)+1}(\omega)$$

$$= \lim_{\alpha'<\alpha^*} \Phi_{\alpha'}^{\varepsilon+1\,'}(\omega) \text{ by corollary 5.19,}$$

as ε is an $\varepsilon^{(\alpha)}$- number, therefore

$$\Phi_{\alpha^*}^{\varepsilon+1}(\omega) = \lim_{\alpha'<\alpha^*} \left(\omega^{\varepsilon+1}\right)_{\alpha'}$$ by induction hypothesis

$= \omega^{\varepsilon+1}$, therefore it follows by transfinite induction that $(\forall \alpha' < \alpha)$

$$\Phi_{\alpha'}^{\varepsilon+1}(\omega) = \omega^{\varepsilon+1}$$

□ LEMMA 5.22

LEMMA 5.23 $(\forall \alpha, \varepsilon < \omega_1)(\forall n < \omega)_{\geqslant 3}$
$\mathrm{Lim}(\alpha) \Rightarrow (\exists \alpha' < \alpha)$ such that:

$$\Phi_{\alpha'}^{\omega^{\varepsilon+1}+1}(\omega) > \overbrace{\Phi_{\Omega_\alpha(n)}^{\Phi_{\Omega_\alpha(n)}^{\cdot^{\cdot^{\cdot^{\Phi_{\Omega_\alpha(n)}^{\varepsilon+1}(\omega)}}}\cdots(\omega)}(\omega)}}^{n},$$

where Ω is an f.s.a.
PROOF. Suppose $n \geqslant 3$, and choose $\alpha' = \Omega_\alpha(n)+1$.
Then l.h.s. $> \Phi_{\alpha'}^{\varepsilon+n+2}(\omega)$

$$= \Phi_{\Omega_\alpha(n)}^{\Phi_{\alpha'}^{\varepsilon+n+1}(\omega)}(\omega)$$

$$= \overbrace{\Phi_{\Omega_\alpha(n)}^{\Phi_{\Omega_\alpha(n)}^{\cdot^{\cdot^{\cdot^{\Phi_{\Omega_\alpha(n)}^{\varepsilon+1}(\omega)}}}\cdots(\omega)}(\omega)}}^{n+1}$$

$\geqslant$ r.h.s.

□ 5.23

Now suppose again that Ω is an f.s.a.

LEMMA 5.23a $(\forall \alpha, \varepsilon < \omega_1)_{\geq\omega} (\forall n < \omega)_{n \geq 3}$ $\mathrm{Lim}(\alpha) \Rightarrow$

$$\overbrace{\Phi_{\Omega_\alpha(n)}^{\Phi_{\Omega_\alpha(n)}^{\cdot^{\cdot^{\cdot^{\Phi_{\Omega_\alpha(n)}^{\varepsilon+1}(\omega)+1}}}\cdots(\omega)+1}}}^{k}(\omega)+1$$

$$< \overbrace{\Phi_{\Omega_\alpha(n)}^{\Phi_{\Omega_\alpha(n)}^{\cdot^{\cdot^{\cdot^{\Phi_{\Omega_\alpha(n)}^{\varepsilon+1}(\omega)}}}\cdots(\omega)}}}^{k+1}(\omega)$$

PROOF by induction. For $k=0$, $\varepsilon+1 < \Phi_{\Omega_\alpha(n)}^{\varepsilon+1}(\omega)$, as by theorem 4.1 $\lambda\beta.\Phi_{\Omega_\alpha(n)}^{\beta}(\omega)$ is strictly increasing, and Suc (l.h.s.) and Lim (r.h.s.) by corollary 4.1a4. Suppose the inequality of lemma 5.23a has been proved for all cases of continued exponents of index $< k$, for some $k > 0$. Then

$$\overbrace{\Phi_{\Omega_\alpha(n)}^{\Phi_{\Omega_\alpha(n)}^{\cdot^{\cdot^{\cdot^{\Phi_{\Omega_\alpha(n)}^{\varepsilon+1}(\omega)+1}}}\cdots(\omega)+1}}}^{k-1}(\omega)+1$$

$$\leq \overbrace{\Phi_{\Omega_\alpha(n)}^{\Phi_{\Omega_\alpha(n)}^{\cdot^{\cdot^{\cdot^{\Phi_{\Omega_\alpha(n)}^{\varepsilon+1}(\omega)}}}\cdots(\omega)}}}^{k}(\omega) \text{ by I.H.}$$

But Suc(l.h.s.), and Lim(r.h.s.) by theorem 4.1a2, therefore strict inequality holds, and the lemma is proved by induction. □5.23a

Theorems 5.20, 5.21 and lemmas 5.22, 5.23, 5.23a enable a characterization of the $\varepsilon^{(\alpha)}$- numbers for limit α to be given:

THEOREM 5.24. Suppose μ is a countable ordinal uniquely expressed as $\mu = \omega\beta + n$, $n < \omega$. Then

$$\mathrm{Lim}(\alpha) \Rightarrow \varepsilon_\mu^{(\alpha)} = \begin{cases} \Phi_\alpha^{(n+1).2}(\omega) \text{ when } \mu < \omega \\ \\ \Phi_\alpha^{\omega\beta+n-2}(\omega) \\ \text{when } \omega \leq \mu < \omega_1 \end{cases}$$

PROOF, by transfinite induction on μ. The cases for $\mu < \omega$ are identical with the general cases given below, except that $\varepsilon_0^{(\alpha)} = \Phi_\alpha^2(\omega)$,[*] and therefore in the cases $\mu < \omega$, there is an increment of 2 in the exponent of the expression defining $\varepsilon_\mu^{(\alpha)}$.

For each α such that Lim(α) and $\alpha < \omega_1$, suppose the expression for $\varepsilon_{\mu'}^{(\alpha)}$ has been proved to hold for all $\mu' < \mu$, for some non-zero $\mu < \omega_1$. Then if Suc(μ) such that $\mu = \omega\beta + n$:

(*) By theorem 5.21.

$$\mathcal{E}_\mu^{(\alpha)} = \mathrm{LIM}\ \Big[\ \lim\Big\{ \Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+1,\ \Phi_3^{\Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+-1}(\omega),\ \Phi_3^{\Phi_3^{\Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+1}(\omega)}(\omega),\ \dots\Big\}$$

$$\lim\Big\{ \Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+1,\ \Phi_4^{\Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+1}(\omega),\ \Phi_4^{\Phi_4^{\Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+1}(\omega)}(\omega),\ \dots\Big\}$$

$$\vdots$$

$$\lim\Big\{ \Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+1,\ \Phi_\omega^{\Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+-1}(\omega),\ \Phi_\omega^{\Phi_\omega^{\Phi_\alpha^{\omega\beta+(n-1).2}(\omega)+1}(\omega)}(\omega),\ \dots\Big\}$$

$$\vdots$$

$$\Big]$$

$$= \underset{\alpha'<\alpha}{\mathrm{LIM}}\ \Big[\lim\Big\{ \Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+1,\ \Phi_{\alpha'}^{\Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+1}(\omega),\ \Phi_{\alpha'}^{\Phi_{\alpha'}^{\Phi_\alpha^{\omega\beta+(n-1).2}(\omega)+1}(\omega)}(\omega),\ \dots\Big\}\Big]$$

, by theorem 5.13 b), and by theorem 5.20.

This expansion has the property that $(\forall k<\omega)(\forall\alpha';\alpha''<\alpha)_{\geq 3}$

(i) $$\overbrace{\Phi_{\alpha'}^{\Phi_{\alpha'}^{\cdots\Phi_{\alpha'}}}}^{k}{}^{\Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+1}(\omega)\cdots(\omega)(\omega) < \overbrace{\Phi_{\alpha'}^{\Phi_{\alpha'}^{\cdots\Phi_{\alpha'}}}}^{k+1}{}^{\Phi_\alpha^{\omega\beta+(n-1).2}(\omega)+1}(\omega)\cdots(\omega)(\omega)$$

(ii) $$\alpha''<\alpha' \Rightarrow \overbrace{\Phi_{\alpha''}^{\Phi_{\alpha''}^{\cdots\Phi_{\alpha''}}}}^{k}{}^{\Phi_\alpha^{\omega\beta+(n-1).2}(\omega)+1}(\omega)\cdots(\omega)(\omega) \leq \overbrace{\Phi_{\alpha'}^{\Phi_{\alpha'}^{\cdots\Phi_{\alpha'}}}}^{k}{}^{\Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+1}(\omega)\cdots(\omega)(\omega)$$

Thus the double limit expression can be evaluated by choosing an f.s.a. Ω and taking the limit of a diagonal ω-sequence (i.e. a sequence whose coordinates x, $\Omega_\alpha(y)$ determine ultimately increasing functions of x, y).

Thus

$$\mathcal{E}_\mu^{(\alpha)} = \lim_{3 \leq x < \omega} \left\{ \Phi_{\Omega_\alpha(x)}^{\Phi_{\Omega_\alpha(x)}^{\cdot^{\cdot^{\cdot^{\Phi_{\Omega_\alpha(x)}^{\omega^{\Phi_\alpha^{\omega\beta+(n-1)2}(\omega)+1}}(\omega)}}}\cdot(\omega)}(\omega)} \right\}$$

by Lemma 5.22. Furthermore, by Lemma 5.23a, the converse to Lemma 5.23 holds, i.e. that $(\forall \alpha' < \alpha)$ there exists an $n < \omega$ such that the direction of the inequality of 5.23 may be reversed; therefore

$$\mathcal{E}_\mu^{(\alpha)} = \lim_{\alpha' < \alpha} \Phi_{\alpha'}^{\left\{\omega^{\left\{\mathcal{E}_{\omega\beta+n-1}^{(\alpha)}+1\right\}}+1\right\}}(\omega)$$

$$= \lim_{\alpha' < \alpha} \Phi_{\alpha'}^{\lim_{\alpha'' < \alpha} \Phi_{\alpha''}^{\Phi_\alpha^{\omega\beta+(n-1)\cdot 2}(\omega)+1}(\omega)+1}(\omega)$$

by induction hypothesis, and by Lemma 5.22, therefore

$$\mathcal{E}_\mu^{(\alpha)} = \lim_{\alpha' < \alpha} \Phi_{\alpha'}^{\Phi_\alpha^{\omega\beta+(n-1)\cdot 2+1}(\omega)}(\omega)$$

$$= \Phi_\alpha^{\omega\beta+n\cdot 2}(\omega) \text{ by definition } 3.1 .$$

In the case $\mathrm{Lim}(\mu)$, $\mathcal{E}_\mu^{(\alpha)} = \lim_{\mu' < \mu} \mathcal{E}_{\mu'}^{(\alpha)}$

$$= \lim_{\mu' < \mu} \Phi_\alpha^{\omega\beta'+n'\cdot 2}(\omega), \quad \mu' = \omega\beta' + n'$$

$$= \Phi_\alpha^\mu(\omega),$$ thus theorem

5.24 is proved by transfinite induction. □

The sequence of $\varepsilon^{(\alpha)}$-numbers enumerates the fixed points common to all functions $\lambda\beta.\ \Phi_{\alpha'}^{\beta}(\omega)$, $\alpha' < \alpha$, by definition and by corollary 5.19. Thus, taken with corollary 5.15, theorem 5.24 gives: $(\forall\alpha < \omega_1)[\mathrm{Suc}(\alpha) \Rightarrow$

$$[f = \lambda\beta.\ \Phi_{P\alpha}^{\beta}(\omega) \Rightarrow f^{F} = \lambda\beta.\ \Phi_{\alpha}^{\omega(1+\beta)}(\omega)]\ \&$$

$$\alpha > \omega\ \&\ \mathrm{Lim}(\alpha) \Rightarrow$$

$$[f = \lambda\beta.\ \lim_{\alpha' < \alpha} \Phi_{\alpha'}^{\beta}(\omega) \Rightarrow \Phi_{\alpha}^{\omega\beta' + n.2}(\omega),\ \text{for } \beta = \omega\beta' + n,$$

is the β-th fixed point of the function f]]

The function defined here in the limit case is continuous and ultimately increasing, but it is not strictly increasing.

Theorem 5.24 indicates the effect of a slight alteration in definition 3.1. Suppose that in part (iv) of this definition, the fourth segment is replaced by

$$(\forall\alpha < \omega_1)(\forall\beta < \omega_1)_{\beta > 2}\ \mathrm{Lim}(\alpha)\ \&\ \mathrm{Suc}(\beta) \cancel{\Rightarrow}\ \&\ \mathrm{Suc}(I\beta) \Rightarrow$$

$$\Phi_{\alpha}^{\beta+1}(\omega) = \lim_{n < \omega} \Phi_{\Omega_\alpha(n)}^{\Phi_{\alpha}^{\beta}(\omega)}(\omega),$$ that is, in the cases indicated, the unit increment is omitted from

the exponents of the r.h.s. Call this definition 3.1′

Now consider the number $\Phi_\omega^\omega(\omega) = \lim_{n<\omega} \Phi_\omega^n(\omega)$.

$(\forall n<\omega)_{n>3}$ write $\Phi_\omega^n(\omega) = \lim_{m_n<\omega} \Phi_{m_n}^{\Phi_\omega^{n-1}(\omega)}(\omega)$,

by definition 3.1′. Then

$$\Phi_\omega^\omega(\omega) =$$

$$\underset{\substack{n<\omega\\3\leq n}}{\mathrm{LIM}}\left[\lim_{m_n<\omega}\left\{\Phi_{m_n}^{\lim_{m_{n-1}<\omega}\Phi_{m_{n-1}}^{\cdot^{\cdot^{\cdot^{\lim_{m_3<\omega}\Phi_{m_3}^{\Phi_\omega^2(\omega)+1}(\omega)}}}}\cdots(\omega)}(\omega)\right\}\right]$$

Now suppose it has been proved that $(\forall n<\omega)_{>0}$

$$\Phi_\omega^{n+2}(\omega) = \lim_{m<\omega}\Big[\overbrace{\Phi_m^{\cdot^{\cdot^{\cdot^{\Phi_m}}}}}^{n}{}^{\Phi_\omega^2(\omega)+1}(\omega)\cdots(\omega)\Big]$$

Then

$$\Phi_\omega^{n+3}(\omega) = \lim_{m_{n+3}<\omega}\Phi_{m_{n+3}}^{\Phi_\omega^{n+2}(\omega)}(\omega) \text{ by definition 3.1′}$$

$$= \lim_{m_{n+3}<\omega}\Big[\lim_{m<\omega}\Big\{\Phi_{m_{n+3}}^{\overbrace{\Phi_m^{\cdot^{\cdot^{\cdot^{\Phi_m}}}}}^{n}{}^{\Phi_\omega^2(\omega)+1}(\omega)\cdots(\omega)}(\omega)\Big\}\Big]$$

$$= \lim_{m<\omega}\overbrace{\Phi_m^{\cdot^{\cdot^{\cdot^{\Phi_m}}}}}^{n+1}{}^{\Phi_\omega^2(\omega)+1}(\omega)\cdots(\omega)$$ by choosing a diagonal from the double limit. Thus it is proved by induction that:

$$\Phi_\omega^\omega(\omega) = \lim_{n<\omega}\left\{ \overbrace{\Phi_n^{\cdot^{\cdot^{\cdot^{\Phi_n^{\Phi_\omega^2(\omega)+1}(\omega)}}}}}^{n}(\omega)\right\}$$

by definition 3.1'

$= \varepsilon_1^{(\omega)}$, by substitution into the expression derived for the $\varepsilon^{(\alpha)}$- numbers on p.

$= \Phi_\omega^4(\omega)$, in the notation of definition 3.1, by theorem 5.24.

A relation is now established between the α-th derived function of an exponential function, and the $\varepsilon^{(\alpha)}$-numbers, as a corollary to the following

THEOREM 5.25 . Suppose $f = \lambda\beta \cdot \omega^\beta$. Then $(\forall \alpha, \beta < \omega_1)_{\alpha > 0}$

$$f_\alpha(\beta) = \begin{cases} \Phi_{3+\alpha}^{\omega(1+\beta)}(\omega) & \text{if } \mathrm{Suc}(\alpha) \\ \Phi_\alpha^{2+\beta}(\omega) & \text{if } \mathrm{Lim}(\alpha) \end{cases}$$

PROOF by nested transfinite induction. Let $\alpha = 1$. Then $(\forall \beta < \omega_1)$ $f_1(\beta) = f^F(\beta) = \Phi_4^{\omega(1+\beta)}(\omega)$, by theorem 5.11. Suppose for outer inductive hypothesis (O.I.H.) that the relation of 5.25 is true for all $\beta < \omega_1$, and for all $\alpha' < \alpha$, for some $\alpha < \omega_1$. Suppose for inner inductive hypothesis (i.i.h.) that $(\forall \beta' < \beta)$

for some $\beta < \omega_1$

$$f_\alpha(\beta') = \begin{cases} \Phi_{3+\alpha}^{\omega(1+\beta')}(\omega) & \text{if } Suc(\alpha) \\ \Phi_\alpha^{2+\beta'}(\omega) & \text{if } Lim(\alpha) \end{cases}$$

Now, by theorem 5.7, for every normal function $f = f_0$, $(\forall \alpha < \omega_1)$ f_α is the α-th derived function of f, according to the formulation of f_α given in definition 5.1a. For $\beta = 0$:

$$f_\alpha(0) = \begin{cases} \lim\limits_{\alpha' < \alpha} f_{\alpha'}(0) \text{ if } Lim(\alpha) \\ = \lim\limits_{\alpha' < \alpha} \Phi_{\alpha'}^{2}(\omega) \text{ by O.I.H.} \\ = \Phi_\alpha^{2}(\omega) \\ \qquad \text{OR} \\ f_{P\alpha}^{\omega}(0) \text{ if } Suc(\alpha) \\ = \lim\{\omega, \Phi_{P\alpha}^{\omega}(\omega), \Phi_{P\alpha}^{\Phi_{P\alpha}^{\omega}(\omega)}(\omega), \dots\} \\ = \lim\limits_{n < \omega} \Phi_\alpha^{n}(\omega) = \Phi_\alpha^{\omega}(\omega) \end{cases}$$

Now suppose that $Suc(\alpha, \beta)$. Then

$$f_\alpha(\beta) = f_{P\alpha}^{\omega}\{f_\alpha(P\beta) + 1\} \text{ by definition 5.1a}$$

$$= f_{P\alpha}^{\omega}\{\Phi_{3+\alpha}^{\omega(1+P\beta)}(\omega) + 1\} \text{ by i.i.h.}$$

Therefore $f_\alpha(\beta)$

$$= \lim \left\{ \Phi_{3+\alpha}^{\omega(1+P\beta)}(\omega)+1,\ \Phi_{3+P\alpha}^{\omega\left(1+\Phi_{3+\alpha}^{\omega(1+P\beta)}(\omega)+1\right)}(\omega),\ \ldots \right\}$$

$$= \lim \left\{ \Phi_{3+\alpha}^{\omega(1+P\beta)}(\omega)+1,\ \Phi_{3+P\alpha}^{\omega\left(\Phi_{3+\alpha}^{\omega(1+P\beta)}(\omega)+1\right)}(\omega),\ \ldots \right\} \ldots (1)$$

but

$$\Phi_{3+P\alpha}^{\omega\left(\Phi_{3+\alpha}^{\omega(1+P\beta)}(\omega)+1\right)}(\omega) \leq \Phi_{3+P\alpha}^{\Phi_{3+P\alpha}^{\Phi_{3+\alpha}^{\omega(1+P\beta)}(\omega)+1}(\omega)}(\omega)$$

and the suitable induction hypothesis on the number $n < \omega$ of applications of operation $\Phi_{3+P\alpha}$ gives the n-th term in limit (1) as:

$$\leq \overbrace{\Phi_{3+P\alpha}^{\cdots\, \Phi_{3+P\alpha}^{\Phi_{3+\alpha}^{\omega(1+P\beta)}(\omega)}(\omega)}}^{n} \cdots (\omega)$$, therefore

$$f_\alpha(\beta) \leq \lim \left\{ \Phi_{3+\alpha}^{\omega(1+P\beta)}(\omega)+1,\ \Phi_{3+P\alpha}^{\Phi_{3+\alpha}^{\omega(1+P\beta)}(\omega)+1}(\omega),\ \ldots \right\} \ldots (2)$$

but by direct comparison of the corresponding terms of the two non-decreasing sequences of limits (1), (2), strict inequality cannot hold, thus

$$f_\alpha(\beta) = \text{limit } (2) = \lim_{n<\omega} \Phi_{3+\alpha}^{\omega(1+P\beta)+n}(\omega)$$

$$= \Phi_{3+\alpha}^{\omega(1+P\beta)+\omega}(\omega) = \Phi_{3+\alpha}^{\omega(1+\beta)}(\omega) .$$

Now suppose Suc(α) and Lim(β). Then

$$f_\alpha(\beta) = \lim_{\beta'<\beta} f_\alpha(\beta') \text{ by definition 5.1a}$$

$$= \lim_{\beta'<\beta} \Phi_{3+\alpha}^{\omega(1+\beta')}(\omega) \text{ by i.i.h.}$$

$$= \Phi_{3+\alpha}^{\lim_{\beta'<\beta}\omega(1+\beta')}(\omega) \text{ by corollary 4.1a1}$$

$$= \Phi_{3+\alpha}^{\lim_{\beta'<\beta}\omega\beta'}(\omega), \text{ since } (\forall\beta'<\beta)\ 1+\beta' \le \beta'+1 \le \beta$$

$$= \Phi_{3+\alpha}^{\omega\beta}(\omega) \text{ as } \lambda\beta.\,\omega\beta \text{ is normal}$$

$$= \Phi_{3+\alpha}^{\omega(1+\beta)}(\omega) \text{ as } \beta \text{ is transfinite.}$$

Now suppose Lim(α, β). Then

$$f_\alpha(\beta) = \lim_{\beta'<\beta} f_\alpha(\beta') \text{ by definition 5.1a}$$

$$= \lim_{\beta'<\beta} \Phi_{3+\alpha}^{2+\beta'}(\omega) \text{ by i.i.h.}$$

$$= \Phi_{3+\alpha}^{\beta}(\omega)$$

$$= \Phi_{3+\alpha}^{2+\beta} \text{ as } \beta \text{ is transfinite.}$$

Only one case remains: suppose Lim(α) and Suc(β).

Then $f_\alpha(\beta) = \lim_{\alpha'<\alpha} f_{\alpha'}\{f_\alpha(P\beta)+1\}$ by definition 5.1a therefore $f_\alpha(\beta)$

$$= \lim_{\alpha' < \alpha} f_{\alpha'} \{ \Phi_{3+\alpha}^{2+P\beta}(\omega) + 1 \} \text{ by i.i.h.}$$

$$= \lim_{\alpha' < \alpha} \Phi_{3+\alpha'}^{2+\Phi_{3+\alpha}^{2+P\beta}(\omega)+1}(\omega) \text{ by O.I.H.}$$

$$= \Phi_{\alpha}^{2+P\beta+1}(\omega), \text{ as } \Phi_{3+\alpha}^{2+P\beta}(\omega)+1 \text{ is transfinite}$$

$$= \Phi_{\alpha}^{2+\beta}(\omega).$$

Thus the case for α is proved for every $\beta < \omega_1$, so the theorem is proved by transfinite induction.

□ 5.25.

COROLLARY 5.26 $(\forall \alpha, \beta < \omega_1)\, Suc(\alpha)$ & $f =_{df} \lambda\beta . \omega^\beta \Rightarrow f_\alpha(\beta) = \varepsilon_\beta^{(3+\alpha)}$.

PROOF follows from corollary 5.15 □

§6. NORMAL FORM THEOREMS.

The object here of defining normal form expressions is to assign to certain ordinals $\alpha \geqslant \varepsilon_0$ indices smaller than α such that α can be expressed arithmetically in terms of those indices.

THEOREM 6·8. $(\forall \alpha < \underline{\Phi}_6^2(\omega))_{>0} (\exists ! n < \omega)_{>0}$ $(\forall i \leq n)_{>0} (\exists ! \alpha_i < \omega_1)_{<\alpha} (\exists ! a_i < \omega)(\exists ! \beta < \alpha)$

$$\alpha = \begin{cases} \sum\limits_{1 \leq i \leq n} \omega^{\alpha_i} \cdot a_i \ \text{ if } (\exists \mu < \omega_1)\ \varepsilon_\mu < \alpha < \varepsilon_{\mu+1} \\ \\ \underline{\Phi}_4^{\omega(1+\beta)}(\omega) \ \text{ if } \ \omega^\alpha = \alpha \ (\text{In this case, } n =_{\text{df}} 1,\ a_1 =_{\text{df}} 1) . \end{cases}$$

PROOF. The first case holds as it is simply the Cantor normal form applied to ordinals $< \underline{\Phi}_6^2(\omega)$ which are not epsilon numbers, as in the example below. The relation in the second case is proved from the fact that if $\alpha = \underline{\Phi}_4^{\omega\beta}(\omega)$, and $\beta = \alpha$, then by theorem 4.1 $\beta \leqslant \underline{\Phi}_4^{\beta}(\omega) \leqslant \underline{\Phi}_4^{\omega\beta}(\omega) \leqslant \beta$, therefore $\underline{\Phi}_4^{\beta}(\omega) = \beta$, and thus $\beta = \alpha$ would be an $\varepsilon^{(5)}$-number, and the least such number is, by corollary 5.14, $\underline{\Phi}_6^2(\omega)$, which contradicts the assumption that $\alpha < \underline{\Phi}_6^2(\omega)$.

$\square$ 6.8

Thus, for example if $\alpha = \varepsilon_0 + \omega^\omega + \omega^2 \cdot 5$, then $\alpha = \omega^{\varepsilon_0} + \omega^\omega + \omega^2 \cdot 5$ $= \omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_2} \cdot a_2 + \omega^{\alpha_3} \cdot a_3 + a_4$, such that $\alpha_1, \alpha_2, \alpha_3 < \alpha$.

Now suppose $\alpha < \underline{\Phi}_6^2(\omega)$ and $\text{Lim}(\alpha)$. Then a fundamental sequence $\{\Omega_\alpha(x)\}_{x<\omega}$ for α can be inductively defined by:

$$\Omega_{\alpha}(x) = \begin{cases} \displaystyle\sum_{i=1}^{n-1} \omega^{\alpha_i} \cdot a_i + \omega^{\alpha_n} \cdot (a_n - 1) + \begin{cases} \omega^{P\alpha_n} \cdot x & \text{if } Suc(\alpha_n) \\ \omega^{\Omega_{\alpha_n}(x)} & \text{if } Lim(\alpha_n) \end{cases} \\ \text{If } \alpha < \omega^{\alpha}; \\ \text{otherwise} \\ = \begin{cases} \Phi_4^{\omega\beta + x}(\omega) & \text{if } Suc.(\beta),\ \beta < \omega \\ \Phi_4^{\omega \cdot P\beta + x}(\omega) & \text{if } Suc(\beta),\ \omega < \beta \\ \Phi_4^{\omega \cdot \Omega_\beta(x)}(\omega) & \text{if } Lim(\beta) \end{cases} \\ \text{If } \alpha \text{ is of the form } \alpha = \Phi_4^{\omega(1+\beta)}(\omega) \end{cases}$$

THE NUMBER θ_0.

In expressing an ordinal α arithmetically in terms of smaller indices, certain operations have been used, each of which belongs to the class of functions $\Phi_\Omega^{(2)} = \Phi^{(2)}$ of definition 3.1, and is thus indexed by an ordinal. E.g. $\omega^{\alpha_i} = \Phi_3^{\alpha_i}(\omega) = \omega \textcircled{3} \alpha_i$. Arbitrary operations $\textcircled{\alpha}$ might be applied to obtain a normal form, but only those whose index α is smaller than a certain fixed upper bound θ_0 will ensure a proper decomposition of a segment of the ordinals $<\omega_1$; in fact the ordinals smaller than θ_0.

θ_0 is the least fixed point of $\lambda\alpha.\,\Phi^2_\alpha(\omega)$.

Let
$$\begin{aligned} \tilde\theta_1 &=_{Df} \omega \\ \tilde\theta_{n+1} &=_{Df} \Phi^2_{\tilde\theta_n}(\omega) \\ \theta_0 &=_{Df} \lim_{n<\omega} \tilde\theta_n . \end{aligned}$$

By theorem 4.1b, $\lambda\alpha.\,\Phi^2_\alpha(\omega)$ is strictly increasing, therefore since $\tilde\theta_1 = \omega < \Phi^2_\omega(\omega) = \tilde\theta_2$, and
$$\begin{aligned} (\forall n<\omega)\ \tilde\theta_n < \tilde\theta_{n+1} \Rightarrow \tilde\theta_{n+1} &< \Phi^2_{\tilde\theta_{n+1}}(\omega) \\ &= \tilde\theta_{n+2}, \end{aligned}$$
it follows by induction that the sequence $\{\tilde\theta_n\}_{n<\omega}$ is strictly increasing. Furthermore,

$$\begin{aligned} \Phi^2_{\theta_0}(\omega) &= \Phi^2_{\lim_{n<\omega}\tilde\theta_n}(\omega) \\ &= \lim_{n<\omega} \Phi^2_{\tilde\theta_n}(\omega) \quad \text{by definition } 3.1 \\ &= \lim_{n<\omega} \tilde\theta_{n+1} \\ &= \theta_0 . \end{aligned}$$

θ_0 is the least such fixed point; for suppose otherwise, such that $(\exists \theta' < \theta_0)\ \Phi^2_{\theta'}(\omega) = \theta'$. Then $\theta' < \omega$ or $(\exists n < \omega)\ \tilde{\theta}_n < \theta'$. Clearly $\omega < \theta'$, so suppose for some n such that $1 < n < \omega$, $\tilde{\theta}_n < \theta'$. Then

$$\begin{aligned} \Phi^2_{\tilde{\theta}_n}(\omega) &= \tilde{\theta}_{n+1} \\ &< \Phi^2_{\theta'}(\omega), \text{ by theorem 4.1b} \\ &= \theta', \text{ so by induction } (\forall n < \omega)_{>0} \end{aligned}$$

$\tilde{\theta}_n < \theta'$, therefore $\theta_0 \leq \theta'$, a contradiction.

The notation $\{\theta_\mu\}_{\mu < \omega_1}$ is adopted for the sequence of countable solutions, ordered according to magnitude, of the equation

$$\Phi^2_{\theta}(\omega) = \theta .$$

By theorem 4.1b, θ_0 is greater than any $\Phi^2_\alpha(\omega)$, $\alpha < \theta_0$, and is thus, by corollary 5.14, greater than any $\varepsilon_0^{(\alpha)}$-number, for $\alpha < \theta_0$.

The process of applying $\varepsilon^{(\beta)}$-numbers to the determination of normal form decompositions of ordinals α, for $\beta < \alpha < \theta_0$ is now considered.

DEFINITION 6.8a. $(\forall\alpha<\omega_1)$ α is an *exact $\varepsilon^{(\beta)}$-number* iff α is an $\varepsilon^{(\beta)}$-number and $(\forall\mu<\omega_1)_{\geq\beta}$ $\alpha<\Phi_\mu^{\alpha}(\omega)$. A number which is an exact $\varepsilon^{(4)}$-number is called an exact epsilon number. $\square$ 6.8a

In general, for each β such that $1<\beta<\omega_1$ there are many exact $\varepsilon^{(\beta)}$-numbers, and by theorem 4.1b, the smallest $\varepsilon^{(\beta)}$-number is always an exact $\varepsilon^{(\beta)}$-number. Thus in the case $\beta=4$, the numbers $\varepsilon_0, \varepsilon_1=\lim\{\varepsilon_0+1, \omega^{\varepsilon_0+1}, \ldots\}$ are exact epsilon numbers. The smallest epsilon number which is not exact is

$$\begin{aligned}\varepsilon_0^{(5)} &= \Phi_6^{2}(\omega) = \Phi_4^{\Phi_6^2(\omega)}(\omega)\\ &= \Phi_4^{\omega^{\Phi_6^2(\omega)}}(\omega)\\ &= \varepsilon_{\varepsilon_0^{(5)}}.\end{aligned}$$

THEOREM $(\forall\mu<\omega_1)$ ε_μ is exact iff μ is not an $\varepsilon^{(\alpha)}$-number, for some α such that $4<\alpha<\omega_1$.

PROOF. Suppose $\mu=\varepsilon_\beta^{(\alpha)}$, for some $\alpha,\beta<\omega_1$ such that $4<\alpha$. Then

$$\varepsilon_\mu = \Phi_4^{\omega(1+\varepsilon_\beta^{(\alpha)})}(\omega) = \Phi_4^{\varepsilon_\beta^{(\alpha)}}(\omega)$$

$$= \varepsilon_\beta^{(\alpha)}, \text{ as } 4 < \alpha \text{ and}$$

by theorem 5.18. Therefore ε_μ is not exact.

Now suppose ε_μ is not exact. Then $(\exists \alpha, \mu < \omega_1)_{\alpha > 4}$

$$\varepsilon_\mu = \varepsilon_\beta^{(\alpha)} = \Phi_4^{\varepsilon_\beta^{(\alpha)}}(\omega), \text{ as } 4 < \alpha$$

$$= \Phi_4^{\omega(1+\varepsilon_\beta^{(\alpha)})}(\omega)$$

$$= \varepsilon_{\varepsilon_\beta^{(\alpha)}}, \text{ therefore } \mu = \varepsilon_\beta^{(\alpha)} \quad \square$$

Note that $(\forall \varepsilon < \omega_1)$ ε is an exact $\varepsilon^{(\alpha)}$- number $\Rightarrow$ α is unique. For suppose otherwise, such that ε is an exact $\varepsilon^{(\alpha)}, \varepsilon^{(\alpha')}$- number, where $\alpha \neq \alpha'$. Then without loss of generality suppose $\alpha' < \alpha$. Then by definition ε cannot be an exact $\varepsilon^{(\alpha')}$- number.

The following lemmas are used:

LEMMA 6.9 $(\forall \alpha < \theta_0)(\exists! \mu < \omega_1)_{4 \leq \mu < \alpha}$ $[\alpha < \omega^\alpha$ or α is an exact $\varepsilon^{(\mu)}$- number$]$.

PROOF. Suppose $\alpha < \theta_0$, and that α is an ε- number. Let $\{\alpha_{\eta'}\}_{\eta' < \eta}$ be the strictly increasing sequence of countable ordinals such that $\Phi_{\alpha_{\eta'}}^{\alpha}(\omega) = \alpha$.

Then by theorem 5.18, each ordinal ξ greater than zero and less than some $\alpha_{\eta'}$ is such that

$$\Phi_{\xi}^{\alpha}(\omega) = \alpha,$$

and so $\{\alpha_{\eta'}\}_{1<\eta'<\eta} = \{\lambda'\}_{\lambda'<\lambda}$, where $\lambda =_{\mathrm{df}} \lim_{\eta'<\eta} \alpha_{\eta'}$. Suppose $\lambda > 2$ is a successor number. Then by definition of λ, α is an exact $\varepsilon^{(\lambda+1)}$- number, and by theorem 4.1c, $\lambda \leq \alpha$.

Suppose λ is a limit number. Then, since for all $\lambda' < \lambda$, α is an $\varepsilon^{(\lambda'+1)}$- number by definition 5.12a, it is true that $(\forall \lambda' < \lambda)_{\lambda'>1}$ α is an $\varepsilon^{(\lambda')}$- number, and therefore by theorem 5.20, α is an $\varepsilon^{(\lambda)}$- number. But by definition of λ, $(\forall \lambda^* < \omega_1)_{\lambda \leq \lambda^*}$ $\alpha < \Phi_{\lambda^*}^{\alpha}(\omega)$, therefore α is an exact $\varepsilon^{(\lambda)}$- number; and by theorem 4.1c, $\lambda \leq \alpha$.

LEMMA 6.10. The least solution λ to the equation $\varepsilon_0^{(\lambda)} = \lambda$ is the number θ_0

PROOF. Clearly $\omega < \lambda$. Firstly, suppose $Suc(\lambda)$. Then $\varepsilon_0^{(\lambda)} = \Phi_{\lambda}^{\omega}(\omega) = \Phi_{\lambda+1}^{2}(\omega)$

$\geqslant \lambda+1$ by theorem 4.1c,

therefore $\lambda < \mathcal{E}_0^{(\lambda)}$, and so λ cannot be a successor number.
But for lim(λ), by definition 5.12a, $\lambda = \mathcal{E}_0^{(\lambda)} \Rightarrow$

$\lambda = (\mu\psi)[(\forall\lambda' < \lambda)_{>0}\ \Phi^{\psi}_{\lambda'}(\omega) = \psi]$. Therefore

$\lambda = \Phi^{2}_{\lambda}(\omega)$, by theorem 5.21

$= \mathcal{E}_0^{(\lambda)}$. Thus $\lambda = \theta_0$ ☐LEMMA 6.10

Returning now to the proof of lemma 6.9, suppose α is an exact $\varepsilon^{(\mu)}$-number, such that $\alpha = \varepsilon_\beta^{(\mu)}$, for some $\beta < \omega_1$, and also suppose $\mu = \alpha$.
Then $\beta > 0 \Rightarrow \varepsilon_\beta^{(\mu)} > \varepsilon_0^{(\mu)} \geq \mu = \alpha$, therefore $\beta = 0$.
Thus, since $\alpha = \varepsilon_0^{(\mu)}$, and $\alpha < \theta_0$ by assumption, then by lemma 6.10, $\mu < \alpha$. Furthermore, by the remark on p. ,$(\forall\alpha < \theta_0)$ α is an exact $\varepsilon^{(\mu)}$-number $\Rightarrow$ μ is unique.

☐LEMMA 6.9.

LEMMA 6.10a. Suppose $\alpha < \omega_1$ is an exact $\varepsilon^{(\mu)}$-number, such that $\alpha = \varepsilon_\beta^{(\mu)}$, for some $\mu, \beta < \omega_1$, $\mu > 2$. Then $\beta < \alpha$.

PROOF. Suppose otherwise, so that $\alpha = \varepsilon_{\alpha}^{(\mu)}$. Then by corollary 5.15 and by theorem 5.24

$$\alpha = \begin{cases} \Phi_{\mu}^{\omega(1+\alpha)}(\omega) \text{ if } Suc(\mu) \\ \Phi_{\mu}^{\alpha}(\omega) \text{ if } Lim(\mu), \text{ as } \alpha \text{ is a limit number.} \end{cases}$$

But as $\omega^{\alpha} = \alpha$, $\Phi_{\mu}^{\omega(1+\alpha)}(\omega) = \Phi_{\mu}^{\alpha}(\omega)$, therefore in each case $\alpha = \Phi_{\mu}^{\alpha}(\omega) \Rightarrow \alpha$ is an $\varepsilon^{(\mu+1)}$- number, contrary to exactness assumption. Therefore $\beta < \alpha$. ☐ LEMMA 6.10a

Lemmas 6.9, 6.10a enable a normal form expression to be assigned to each ordinal number smaller than θ_0.

THEOREM 6.11

$$(\forall \alpha < \omega_1)_{\leq_0 \theta_0} (\exists ! \, n < \omega)_{>0} (\forall i \leq n)_{1 \leq i} (\exists ! \, \alpha_i < \omega_1)_{< \alpha}$$
$$(\exists ! \, a_i < \omega)_{>0} (\exists ! \, \beta, \mu < \omega_1)_{< \alpha}$$

$$\alpha = \begin{cases} \sum_{i=1}^{n} \omega^{\alpha_i} \cdot a_i & \text{if } \alpha < \omega^{\alpha} \\ \text{otherwise} = \begin{cases} \Phi_{\mu}^{\omega(1+\beta)}(\omega) & \text{if } Suc(\mu) \\ Z(\mu, \beta', n) & \text{if } Lim(\mu) \end{cases} \end{cases}$$

where

$$Z(\mu, \beta', n) = \begin{cases} \Phi_{\mu}^{(n+1)\cdot 2}(\omega) & \text{when } \beta < \omega \\ \Phi_{\mu}^{\omega\beta' + n \cdot 2}(\omega) & \text{when } \omega \leq \beta < \omega_1 \\ \text{where } \beta = \omega\beta' + n,\ n < \omega \end{cases}$$

and where μ is the unique ordinal smaller than α such that α is an exact $\varepsilon^{(\mu)}$- number.

PROOF. The first clause follows immediately from the Cantor normal form and the definition of the ε- numbers. The relations of the remaining clauses follow from corollary 5.15, and from theorem 5.24. The strict inequalities quantifying the indices follow from lemmas 6.9, 6.10a

□ THEOREM 6.11.

Now suppose $\alpha < \theta_0$, and $Lim(\alpha)$. Then a fundamental sequence for α can be defined by the following scheme, where $\alpha = \varepsilon_\beta^{(\mu)}$, such that α is an exact $\varepsilon^{(\mu)}$- number, if α is an epsilon number:

$$\Omega_\alpha(x) = \begin{cases} \sum_{i=1}^{n-1} \omega^{\alpha_i} a_i + \omega^{\alpha_n}(a_n - 1) \\ \qquad\qquad + \begin{cases} \omega^{P\alpha_n} \cdot x \text{ if } Suc(\alpha_n) \\ \omega^{\Omega_{\alpha_n}(x)} \text{ if } Lim(\alpha_n) \end{cases} \\ \text{If } \alpha < \omega^\alpha, \\ \text{otherwise} \\ = \begin{cases} \underline{\overline{\Phi}}_\mu^{\,\omega\beta + x}(\omega) \text{ if } \beta < \omega \\ \underline{\overline{\Phi}}_\mu^{\,\omega \cdot P\beta + x}(\omega) \text{ if } Suc(\beta),\ \omega < \beta \\ \underline{\overline{\Phi}}_\mu^{\,\omega \cdot \Omega_\beta(x)}(\omega) \text{ if } Lim(\beta) \\ \text{If } Suc(\mu), \text{ otherwise} \\ = Z(\mu, \beta', n), \text{ for } Lim(\mu) \end{cases} \end{cases}$$

where

$$Z(\mu, \beta', n) = \begin{cases} \begin{cases} \underline{\overline{\Phi}}^{\,2}_{\Omega_\mu(x)}(\omega) \text{ if } n = 0 \\ \underline{\overline{\Phi}}^{\,\underline{\overline{\Phi}}_\mu^{n \cdot 2 + 1}(\omega) + 1}_{\Omega_\mu(x)}(\omega) \text{ if } 0 < n < \omega \\ \text{when } \beta = n < \omega \end{cases} \\ \begin{cases} \underline{\overline{\Phi}}^{\,\underline{\overline{\Phi}}_\mu^{\omega\beta' + (n-1) \cdot 2 + 1}(\omega)}_{\Omega_\mu(x)}(\omega) \\ \text{when } \beta = \omega\beta' + n,\ 0 < n, \beta' \end{cases} \\ \begin{cases} \underline{\overline{\Phi}}_\mu^{\,\Omega_\beta(x)}(\omega) \\ \text{when } Lim(\beta) \end{cases} \end{cases}$$

and where $\beta = \omega\beta' + n$, $n < \omega$.

CONCLUSION

With regard to the unsolved problem stated in §2, it is clear that the class $\mathfrak{D}_{\Omega}^{(2)} = \mathfrak{D}^{(2)}$, or indeed any class containing only countably many normal functions, will not provide a means of expressing an arbitrary limit number $\lambda \in Z(\aleph_0)$ as an arithmetical expression containing ordinal indices α all of which satisfy $\alpha < \lambda$.

Consider however, the class $\mathcal{D}_{\Omega}^{(1)}$, for some fundamental sequence assignment Ω. This class is extensive by comparison with $\mathfrak{D}^{(1)}$, the subclass of its members which are primitive recursive, since Ω enables new functions to be defined within $\mathcal{D}_{\Omega}^{(1)}$ by diagonalization processes.

The class $\mathfrak{D}^{(2)}$ may be viewed as providing a framework of primitive recursive functions $\mathfrak{D}_{\alpha}^{\beta} : \omega_1 \to \omega_1$, and this view suggests the possibility of defining a class $\mathcal{D}_{\Omega^*}^{(2)}$ which contains the members of $\mathfrak{D}^{(2)}$, and which also contains functions defined on W_{ω_1} which are defined by diagonalization processes of type ω_1, these processes being enabled by a fundamental sequence assignment Ω^* defined for limit numbers of $Z(\aleph_1)$. (c.f. Bachmann [3]).

Such a notion perhaps suggests that the problem as it applies to the second number class is unlikely to become resolved in isolation.

APPENDIX I

THE ACKERMANN FUNCTION.

$(\forall n, x, y < \omega)$

$$g_0(x, y) = y + 1$$

$$g_1(x, y) = x + y$$

$$g_2(x, y) = x \cdot y$$

$$g_{n+3}(0, y) = 1$$

$$g_{n+3}(1, y) = y$$

$$g_{n+3}(2, y) = g_{n+2}(y, y)$$

$$g_{n+3}(x+1, y) = g_{n+2}(g_{n+3}(x, y), y)$$

More than the minimum number of equations are given here, in order to show the initial behaviour of the function.

$(\forall n < \omega)$ $\lambda x, y \cdot g_n(x, y)$ is primitive recursive, but $\lambda n, x, y \cdot g_n(x, y)$ is not. cf. Peter [15].

APPENDIX II

EXPONENTIAL NOTATION.

It is sometimes convenient to write out the expression $\Phi_\alpha^\beta(\gamma)$, when β is a successor number, in repeated exponential form. For example

$$\Phi_\alpha^\beta(\gamma) = \Phi_{P_\alpha}^{\Phi_\alpha^{\beta-1}(\gamma)}(\gamma) =$$

$$\Phi_{(P_\alpha)_1}^{\Phi_{(P_\alpha)_2}^{\cdots^{\Phi_{(P_\alpha)_n}^{\Phi_\alpha^{(\beta-n)}(\gamma)}((\gamma)_n)}\cdots}((\gamma)_2)}((\gamma)_1)$$

where n is such that $1 \leq n \leq \omega$, and β of the form $\beta = \beta' + n + 2$.

The indices $(\,)_i$, $1 \leq i \leq n$, are applied only in this instance, to indicate which operands γ are to be paired with which indices P_α.

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