London University

(Bedford College)

THESIS

FOR THE DEGREE OF PH.D.

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CONTENTS

CHAPTER I

Introduction

In this chapter we give an historical outline of the main theoretical developments there have been in the formulation and interpretation of relativistic equations describing the behaviour of fundamental particles. Special attention is paid to the Dirac equation as it is the pattern on which the present particle equations are based.

An introduction to this subject tracing in more detail the interconnection of the theoretical developments with experimental discoveries will be found in Bhabha (44), while a comprehensive mathematical review dealing with particles of spin 0, $\frac{1}{2}$ and 1 is given in Pauli (41).

Units. Unless otherwise stated we shall always use atomic units with $\hbar = c = 1$, where $\hbar$ is Planck's constant divided by 2π and c is the velocity of light.

Notation. The general Bhabha matrices applicable to a particle of any spin will be denoted by α, while γ will denote the Dirac matrices and β the Kemmer matrices.

References. These are listed alphabetically at the end of the thesis (p. 121), the two figures after a name in the text referring to the year the paper or book was published.

(1) The Beginnings of Particle Equations.

A particle equation is an equation predicting the behavior and describing the properties of a fundamental particle within the framework of relativity and quantum theory.

It is mainly the "special" form of the first we are interested in, and this implies that the momenta, energy, and rest-mass of a free particle referred to a cartesian reference system must be related by the equation

$$p_x^2 + p_y^2 + p_z^2 + \kappa^2 = E^2 \qquad \text{--- 1.1}$$

where p_x, p_y and p_z are the three momenta components, E is the total energy and κ the rest-mass of the particle.

From quantum theory it is known that with the momenta and energy of any system are associated the operators

$$p_x = i\frac{\partial}{\partial x} \quad \text{etc.} \qquad E = -i\frac{\partial}{\partial t} \qquad \text{--- 1.2}$$

these acting on some wave function ψ.

These two results suggest that 1.1 and 1.2 should be combined into the equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right]\psi = \kappa^2\psi$$

or

$$\left[\Box_x^2 - \kappa^2\right]\psi = 0 \qquad \text{--- 1.3}$$

(where ψ is some scalar wave function), and that this equation should describe the behavior of any kind of particle.

As ψ is a scalar 1.3 can be transformed so as to hold for any orthogonal curvilinear system; it then becomes

$$\left[\Box^2 - \kappa^2 \right] \psi = 0 \qquad 1.4$$

(we use the notation $\Box^2$ for the general four-dimensional Laplacian; $\Box_x^2$ refers only to cartesian co-ordinates).

Equation 1.4 is the Klein-Gordon (26) relativistic wave equation and was the first such equation to be put forward. It was expected that it would hold for any particle, but in the end it was just this extreme generality which showed it incapable of describing the finer characteristics of any of the experimentally discovered particles.

Experimental investigation of the properties of the electron showed that as well as possessing the usual relativistic properties of all fast moving particles, it in addition possessed certain intrinsic properties, characteristic of itself alone, such as a spin angular momentum of magnitude $\frac{1}{2}$, and a magnetic moment, neither of which would be predicted from the Klein-Gordon equation.

What therefore was needed was some equation which would be consistent with 1.3, but yet possessed some additional structure which enabled it to explain these intrinsic electron properties.

Dirac (28 a and b) solved this problem with his famous electron equation. He gave up the restriction that ψ was a scalar, only imposing on it the condition that an equation of the form 1.3 must be satisfied by each of its components. Such an equation he showed was

$$\left.\begin{array}{l}\gamma^i \dfrac{\partial\psi}{\partial x^i} + \kappa\psi = 0 \\ \text{where } i^{*} \text{ runs } 1-4 \quad \& \; x_4 = it\end{array}\right\} \quad \text{——} \quad 1.5$$

Here ψ is a single-column matrix with four components, and the γ^i are four. 4 x 4 matrices satisfying the commutation relations

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij} \qquad \text{——} \quad 1.6$$

where δ^{ij} is the usual Kronecker symbol

$$\begin{aligned}\delta^{ij} &= 0 \quad \text{if } i \neq j \\ &= 1 \quad \text{if } i = j\end{aligned}$$

By multiplying 1.5 throughout with $\gamma^j \dfrac{\partial}{\partial x^j}$ and using 1.5 and 1.6 it can easily be verified that this equation does lead to equation 1.3 for each component of ψ. As ψ is no longer a scalar however, it is important to note we cannot pass from 1.3 to the general Klein-Gordon equation 1.4, valid for any curvilinear system of co-ordinates. As we shall see later (§ 14), the Dirac equation in any co-ordinate except cartesians does not yield a second order equation of the type 1.4

Dirac and others showed further that the equation 1.5 could be obtained from a Lagrangian principle, that it predicted both the half-integral spin and the proper magnetic moment of the electron, and also that expressions could be found for the energy-momentum tensor and current

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* Latin indices will run 1 - 4 throughout the thesis.

vector for the electron, which exhibited all the necessary conservation properties.

As a Lagrangian formulation was possible, the electromagnetic field could be introduced into 1.5 in the usual way by replacing $\frac{\partial}{\partial x^i}$ by $\frac{\partial}{\partial x^i} - ie\varphi_i$, where e is the electron charge and φ_i is the vector potential of the external electromagnetic field.

Now up to the time of the Dirac equation it had been thought that all relativistic equations must be expressible in tensor form. However, the Dirac equation 1.5 was shown to be relativistic yet it contained the quantities $\dot{\gamma}^i$ and ψ which certainly did not fall into the usual tensor classification. As Darwin (28) put it in 1928, "it is rather disconcerting to find that apparently something has slipped through the net."

The proper interpretation of these new quantities then opened up a field of investigation which aimed mainly at the solution of the following now closely related problems:-

a) What is the mathematical nature of the quantities $\dot{\gamma}^i$ and ψ , and how do these properties account for the invariance of 1.5? and

b) What is the correct generalization of 1.5 so it will apply to other co-ordinate systems apart from cartesians?

In the next section we shall consider the answers given to these questions.

(2) <u>Interpretations of the Dirac Equation</u>.

We first outline the answers given to question (a). These fall roughly into four groups.

(i) Firstly Weyl (31), and then Van der Waerden (32) showed that Dirac's four-component ψ could be split into two two-component half-vectors or spinors, which transformed under rotations of the co-ordinate axes just like a vector in the two-dimensional plane. Similarly the γ's split into two 2 x 2 matrices whose form could be obtained by considering the relation of the two, three and four-dimensional rotation groups. Van der Waerden developed this spinor analysis into a systematic mathematical theory, which in the hands of Uhlenbeck and Laporte (31), Dirac (36), Pauli (40) and others has led to many advances in theoretical physics, while on the mathematical side also it has led to discoveries through Carton (38), Weyl (39), Murnaghan (38), and many others.

There is little doubt that the spinor analysis gives a greater insight into the structure of the Dirac equation than any other interpretation, but its notation * is rather

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* In spinor calculus Dirac's equation is written (Van der Waerden 32, p. 97)

$$\begin{cases} h^{i}\sigma_{i}'^{\,\dot{\nu}\lambda}\psi_{\lambda} = \kappa\psi^{\dot{\nu}} \\ h^{i}\sigma_{i\lambda\dot{\nu}}\psi^{\dot{\nu}} = \kappa\psi_{\lambda} \end{cases}$$

where i runs 1–4, $\dot{\nu}$ & λ run 1–2, & $h^{i} = \frac{\partial}{\partial x^{i}}$

The σ_i & σ_i' are the Pauli 2 x 2 spin matrices (with $\sigma_4 = 1$), and the indices $\dot{\nu}$ and λ refer to the element of σ under consideration, and act on the corresponding index in ψ or $\dot{\psi}$, which are two two-component spinors. The dot over the ν distinguishes the behaviour of co- and contra-variant vectors under two-dimensional rotations.

complicated, and as we shall have no need for using spinors explicitly in this thesis, we will not stop to develop the notation here.

(ii) Dirac originally, and also Pauli (36), von Neumann (28) and Möglich (28) looked on all four components of ψ as behaving like one quantity, which transformed under a co-ordinate transformation.

$\bar{x}^{i} = a^{i}_{j} x^{j}$ where a^{i}_{j} are independent of x^{i}, in the manner

$$\bar{\psi} = S^{-1}\psi \qquad\text{—— } 2.1$$

where S^{-1} is some 4 x 4 matrix, independent of x^{i}. The γ^{i} are treated as scalars under the co-ordinate transformation.

Under such a transformation 1.4 goes over into

$$\gamma^{i} S^{-1} a_{i}^{j} \frac{\partial \bar{\psi}}{\partial \bar{x}^{j}} + \kappa S^{-1}\bar{\psi} = 0$$

and for relativistic invariance this must be equal to

$$\gamma^{i} \frac{\partial \bar{\psi}}{\partial \bar{x}^{i}} + \kappa \bar{\psi} = 0$$

We see this is only true if the matrix S satisfies the equation

$$a_{i}^{j}\gamma^{i} = S^{-1}\gamma^{j} S \qquad\text{—— } 2.2$$

The proof of the relativistic invariance of 1.5 therefore depends on the proof of the existence of a matrix S satisfying 2.2 Such proofs have been given by Dirac (47 p.258), and Pauli (36).

The important point in this treatment is that under the co-ordinate transformation ψ transforms, but the γ^{i} are considered as scalars.

Although this method lacks the insight provided by (i), yet its greater simplicity in presentation has led to it being used in nearly all applications of the Dirac equation to actual problems.

(iii) Schroedinger (32), Bargmann (33), and also Pauli (33a) gave another method of treating the invariance at 1.5. Again the four components of ψ are looked on as one quantity, but now under a co-ordinate transformation ψ remains fixed while $\gamma^{\dot{\sim}}$ varies like an ordinary vector (i.e. just the opposite treatment to (ii)). The invariance of 1.5 under co-ordinate transformations is then obvious.

If we make the transformation of the $\gamma^{\dot{\sim}}$

$$\bar{\gamma}^{\dot{\sim}} = S^{-1}\gamma^{\dot{\sim}} S \qquad \text{——— } 2.3$$

where S is any 4 x 4 matrix which in general can be a function of $x^{\dot{\sim}}$, we see from 1.6 that the $\bar{\gamma}^{\dot{\sim}}$ satisfy the same commutation relations as the $\gamma^{\dot{\sim}}$. Such a transformation is called a similarity or S transformation, and we link with it a transformation of ψ such that

$$\bar{\psi} = S^{-1}\psi \qquad \text{——— } 2.4$$

A quantity ψ which behaves in this way under co-ordinate and S transformations is called an undor.

As it is the commutation relations which are important and not the particular representations of the matrices satisfying them, we would expect that any S transformation (which leaves 1.6 unaltered), would have no effect on the results derivable from 1.5 -- or in other words 1.5 should be invariant for these S transformations.

This is put in the form of a new postulate and it is said that any physical equation must be invariant under

a) co-ordinate transformations

and b) S transformations.

From 2.3 and 2.4 it can be seen that as long as S is independent of x^ν, 1.5 is clearly invariant under S transformations. The case where S is some function of the x^ν is more difficult, but here again the invariance can be shown. (See § (10) where this demonstration is given in detail).

This undor presentation of the Dirac equation lacks both the simplicity of (ii) and the insight of (i); however when the initial steps have been taken it proves to have many advantages, especially from the point of view of generalization to arbitrary co-ordinate systems; ~~as we shall see later in this section~~.

(iv) Eddington (36, 46) has developed an entirely new calculus - the wave-tensor calculus - in order to interpret the Dirac equation. His starting point is not, as in the previous treatments, that the ψ and the γ^ν are new mathematical quantities whose transformation properties we must determine, but that they are just the ordinary tensors and vectors occurring in tensor calculus. Instead of building up the laws of transformation of such tensors from the properties of spacial vectors however, he begins with a more abstract frame, the ψ frame, and it is not until much later that spacial vectors and tensors are linked up with the scheme.

According to his point of view, the ψ in 1.5 is a wave-vector ψ_i, while $\gamma^i \frac{\partial}{\partial x^i} + \kappa$ can be written as a mixed wave-tensor operator, $H^i_{\ j}$ say. The whole equation becomes therefore

$$H^i_{\ j}\, \psi_i = 0 \qquad \text{—— } 2.5$$

and its invariance is clear from ordinary tensor considerations.

Eddington has deduced many remarkable results from his interpretation, but as again we shall have little contact with his theory in this thesis, we will not expand it further here, but will pass on to the corresponding answers which have been given to question (b) above.

(i) Infeld (32, 33) and Van der Waerden have developed a Dirac equation in terms of two two-component spinors which is valid for any form of co-ordinate system, but their treatment met with some difficulties and it was not until Weyl and Brauer (35) showed how spinors could be introduced into any Riemannian space that the theory became satisfactory. Recently Carton (38) has also given a spinor generalization to any co-ordinate system.

In actual calculations however, the complication of the formalism makes it much simpler to work with the other formulations of the equation.

(ii) In expanding this method the difficulty is met that the γ^i are scalars and must therefore always satisfy 1.6. In the general case this involves interpreting these at each point in space by means of a local cartesian system of axes (or "four legs"). Although this is always a possible procedure, yet again it involves much complication and is not so convenient as method (iii). Fock (29) has written papers from this point of view.

(iii) To generalize this method we note that the γ^i are to transform as vectors, and so they will not always satisfy 1.6, δ_{ij} not being a tensor. To amend this we change the definition of the γ matrices to

$$\left.\begin{array}{ll} & \gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij} \\ \text{or} & \gamma_i\gamma_j + \gamma_j\gamma_i = 2g_{ij} \end{array}\right\} \qquad 2.6$$

where $$ds^2 = g_{ij}\,dx^i\,dx^j \qquad 2.7$$

the g_{ij} being the usual matric tensor and ds the scalar interval for the geometry considered. Equations 2.6 and 2.7 were first given by Tetrode (28) and are called the Tetrode relations.

In proving the invariance of 1.5 under S transformations it is now found (see §(10)) that the equation must be generalized to

$$\gamma^i\left(\frac{\partial}{\partial x^i} - \Delta_i\right)\psi + \kappa\psi = 0 \qquad 2.8$$

where Δ_i is some 4 x 4 matrix transforming like a vector under co-ordinate transformations (see 10.7 for behaviour under S transformations).

This Δ_i is zero for a cartesian system, so 2.8 reduces to the customary Dirac equation 1.5, but in other systems the whole character of the equation depends on the value of the Δ_i. Schroedinger (32) and Bargmann (33) showed how the appropriate Δ_i for any co-ordinate system (i.e. the value of Δ_i corresponding to a particular g_{ij}) could always be found.

(iv) No general treatment has so far been given in terms of Eddington's theory, although it would not be difficult to devise one on the lines of (ii) or (iii) above.

It is strange that although these different forms for the necessary extensions to the Dirac equation have been known since about 1930, it was not until 1938 that any application of them was made. Then Schroedinger (38a) gave the treatment of the hydrogen atom beginning with the Dirac equation in spherical polar co-ordinates, and later applied the more general equations to cosmological problems. (Schroedinger (39, 40b)), as did also Taub (37) and Podolski and Branson (40).

(3) Equations for Particles of Other Spins.

Although the Dirac equation satisfactorily explained the properties of the electron, the experimental discovery of the proton and neutron and the hypothetical postulation of the neutrino led Dirac (36) in 1936 to try and find the form for an equation which would describe particles of any half-integral or integral spin.

Using the spinor notation, and again starting from the postulate that all components of the wave-equation must satisfy 1.3, he found a form for such an equation. Fierz and Pauli (39a, b & c) further extended his work and showed that not all the components of this new Dirac equation satisfied an equation of the form 1.5 (with of course generalixed matrices). The other components satisfied equations which had to be looked on as subsidiary conditions, these being nevertheless necessary to pass over to the second-order equation 1.3. Moreover the equations could not be derived from a Lagrangian principle and this led to inconsistencies appearing in the equations when the electromagnetic field was introduced in the usual way. Pauli and Fierz showed how this difficulty could logically be overcome, but only at the expense of introducing much complication into the theory, and it was this fact, together with the unsymmetric nature of the equations, which caused these investigations to lapse.

Meanwhile however Yukawa (35, 37, 38) had postulated the existence of particles of spin 0 and 1 in order to explain the mechanism of nuclear forces, and as similar particles were later experimentally found in cosmic rays, the theory of such particles, called mesons, became intensively studied. (Kemmer (38), Frohlich (38), Bethe (40), Möller & Rosenfeld (40)).

In 1939 Kemmer (39) showed that the behaviour of mesons could be described by an equation

$$\beta^i \frac{\partial \psi}{\partial x^i} + K\psi = 0 \qquad\text{—— 3.1}$$

where the β^i are matrices satisfying the commutation relations

$$\left.\begin{array}{ll} & \beta^i\beta^j\beta^k + \beta^k\beta^j\beta^i = \delta^{ij}\beta^k + \delta^{kj}\beta^i \\ \text{or} & \beta_i\beta_j\beta_k + \beta_k\beta_j\beta_i = \delta_{ij}\beta_k + \delta_{kj}\beta_i \end{array}\right\} \text{—— 3.2}$$

This equation shows a striking resemblance to the Dirac electron equation, and Kemmer showed that it explained all the intrinsic properties of the meson in the same way as Dirac's equation did for the electron. The possible representations of the β matrices are of degree 10, 5 or 1, the corresponding ψ being single-column matrices with 10, 5 or 1 components, and for each case it can be shown from 3.1 and 3.2 that every component of ψ satisfies a second-order equation of the form 1.3. Further the equation can be derived from a Lagrangian principle, the energy-momentum tensor and current vector formed, and the electromagnetic field simply introduced, all as in the Dirac electron case.

The great success of this Kemmer equation together with its similarity to the Dirac equation, led to further attempts being made to find an equation describing particles of arbitrary spin. Independently Lubanski (42a) and Bhabha (44, 45) showed that if we base our equations, not as Dirac did on the postulate that the second-order equation must be of the form 1.3, but on the postulate that the first-order equation must be of a form similar to the Dirac equation 1.5, i.e. of form

$$\alpha^i \frac{\partial \psi}{\partial x^i} + \varkappa \psi = 0 \qquad \text{—— } 3{\cdot}3$$

then we can build up a scheme which does describe particles of arbitrary spin and yet does not lead to any of the difficulties which beset Dirac's earlier attempt.

In equation 3.3 $\varkappa$ is not the rest-mass of the particle, but for the present just a constant. The α^i are square matrices which can satisfy a number of commutation relations, given in different but equivalent ways by Lubanski and Bhabha (see § (4) for actual forms of these relations). Each particular commutation relation leads to certain representations of the α matrices, the associated equations all relating to particles of a definite maximum* spin.

The Dirac and Kemmer commutation relations, 1.6 and 3.2, are special forms of the α matrices relating to particles with maximum spin $\frac{1}{2}$ or 1; for these cases therefore the constant $\varkappa$ is actually the rest-mass of the particle.

As in the Dirac and Kemmer cases, the equations can

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* for meaning of maximum see § (4) p. 26.

be derived from a Lagrangian principle, and the energy-momentum tensor, current vector, introduction of the electromagnetic field, etc., follow in the usual way with no difficulties.

The second-order equation obtainable from 3.3 now becomes however, (see § (4) p. 28.)

$$\left.\begin{array}{l} \{\kappa^2 - \Box_x^2\lambda^2\}\{\kappa^2 - \Box_x^2(\lambda-1)^2\} \cdots\cdots \kappa^2\psi = 0 \\ \quad \text{if } \lambda \text{ is integral} \\ \text{or } \{\kappa^2 - \Box_x^2\lambda^2\}\{\quad\} \cdots\cdots \{\kappa^2 - \frac{\Box_x^2}{4}\}\psi = 0 \\ \quad \text{if } \lambda \text{ is half-integral.} \end{array}\right\} \text{— } 3.4$$

Here λ is the maximum spin of the particle considered, and from 3.4 we see that equation 3.3 describes a particle which has as possible values of the rest-mass

$$\pm\kappa, \pm\frac{\kappa}{2}, \cdots\cdots \pm\frac{\kappa}{\lambda} \qquad \text{if } \lambda \text{ is integral}$$

$$\text{or } \pm 2\kappa, \pm\frac{2\kappa}{3}, \cdots \pm\frac{\kappa}{\lambda} \qquad \text{if } \lambda \text{ is half-integral}$$

i.e. for λ integral there are 2λ possible values of the rest-mass, while for λ half-integral there are $2\lambda+1$ possible values.

The negative values of the rest-mass were first encountered in the Dirac equation, where they were interpreted (Dirac (47, p.272)) as relating to the positron. In a similar way the negative values in 3.5 are taken as referring to the "anti-particle" of the more stable one with positive rest-mass. All the difficulties with regard to the theory of holes and the resultant infinities are therefore inherited in this more general theory from the Dirac case.

We see also that it is only for spin $\frac{1}{2}$ and 1 (i.e. for the Dirac and Kemmer cases), that 3.3. describes a particle which has a unique (apart from sign) value of its rest-mass. All particles with spin greater than one <u>must</u> be capable of existing in more than one state of rest-mass, and conceivably if such particles do exist, transitions between these states of rest-mass should be possible.

Recently Harish-Chandra (46) has investigated a problem which in a sense is the inverse of that solved by Lubanski and Bhabha. He does not specify any commutation relations for the α matrices, but considers the question, "If a particle equation must satisfy both the Dirac and the Bhabha postulates, then what possible commutation relations can the α matrices satisfy?"
i.e. the equations must be of the form

$$\alpha^{i} \frac{\partial \psi}{\partial x^{i}} + \kappa \psi = 0 \qquad \text{- Bhabha postulate}$$

and lead to the second-order equation

$$(\Box_{x}^{2} - \kappa^{2}) \psi = 0 \qquad \text{- Dirac postulate}$$

where K <u>is</u> here the rest-mass of the particle.

Harish-Chandra's results are not quite definite, but except in rather artificial cases they seem to verify the result found above from the Bhabha treatment, that the only possible commutation relations for the α_s are the Dirac and the Kemmer ones.

If it was considered necessary that a particle equation should satisfy both the conditions 3.6, then Harish-Chandra's result would point to the conclusion that particles of spin greater than one cannot exist in nature. There seems no theoretical reason why this should be so however, and until such a reason is forthcoming it seems better to interpret Harish-Chandra's result as meaning that all particles with spin greater than one must have several possible values of their rest-mass.

So far no work has been done on the generalization of these equations to arbitrary co-ordinate systems, although treatments for the Kemmer equation have been given by Lubanski and Rosenfeld (42b), using the "four-leg" procedure, and by Fuchs (40), using the spinor calculus as developed by Cartan (38).

Finally one might ask "Is there any experimental evidence for the existence of particles with spin greater than one?"

Although Bhabha has given some evidence in support of the idea that the proton should be described by the equation for a particle of spin $\frac{3}{2}$ (see § (4) p. 28), and Pauli has suggested that the gravitational field can be described by a particle of zero rest-mass and spin 2, yet at the present time the answer to the question must be in the negative.*

With the modern extremely high sources of energy however there is quite a possibility that such particles may soon be discovered, and in that case the theory of these equations of arbitrary spin, which at the present is only of academic interest, may become of great importance.

We began this chapter with the statement, "a particle equation is an equation predicting the behaviour and describing the properties of the fundamental particles within the framework of relativity and quantum theory". After the preceding discussion on the developments of these equations it is more satisfactory to put the statement in its inverse form.

A fundamental particle is one described by an irreducible particle equation. (i.e. an equation which cannot be split in a relativistically invariant way into a number of simpler equations).

This is the meaning we will attach to "fundamental particle" in the remainder of this thesis.

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* See also § (4) p. 27.

CHAPTER 2

The Bhabha Matrices

In the first part of this chapter we give the main results from Bhabha's formulation of the particle equations. A fuller account containing details of any proofs omitted here will be found in Bhabha (45a).

The second part deals with the "pentad" problem for the Bhabha matrices. It has been known since 1928 that it was possible to form a fifth matrix from the four Dirac matrices $\gamma_1 - \gamma_4$ which satisfied the same commutation relations, and it is this result which is the basis of Eddington's theory of E-numbers. Schroedinger (43) in 1943 was the first to show that a similar result held for the 10 x 10 representation of the Kemmer matrices, but not for the 5 x 5 representation, and different proofs of this were given by Kemmer (43) and Harish-Chandra (47a). These proofs all depend directly on the commutation relations* for the β_s however, and no general method has been given as to how one can tell in any particular case whether a pentad is possible. We give such a method for any of the Bhabha matrices here.

The proof of this method depends on some properties

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* Schroedinger's proof really depends on the form of the representations of the β matrices, and not directly on the commutation relations.

of the representations of the orthogonal group. All results stated here, together with further details and missing proofs will be found in Murnaghan (38 - chapters 9 & 10). We have furthermore found it no more difficult to treat the more general problems:-

"If $\alpha_1 - \alpha_n$ are the Bhabha matrices in a space of n dimensions, for what irreducible representations of the α_s is it possible to find an $(n+1)$th matrix satisfying the same commutation relations as $\alpha_1 - \alpha_n$?"

(4) <u>Properties of the Bhabha matrices.</u>

<u>Definitions</u>

Any fundamental particle is described by an equation of the form

$$\alpha^k \frac{\partial \psi}{\partial x^k} + \chi \psi = 0 \qquad \text{—— } 4.1$$

where k runs 1 - 4, the x^k are cartesian co-ordinates, χ is some constant yet to be interpreted, ψ is a single-column matrix in which the number of components depends on the degree of the α^k, and the α^k are irreducible matrices satisfying the commutation relations.*

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* In Lubanski's formulation the α^k are built up from the Dirac γ_s in the way $\alpha^k = \sum_{m=1}^{N} \gamma^k_{(m)}$

where $\gamma^k_{(m)} = I_{(1)} \times \text{--} \times \gamma^k_{(m)} \times \text{--} I_{(N)}$

and $\therefore$ $\begin{cases} \gamma^k_{(m)} \gamma^i_{(m)} + \gamma^i_{(m)} \gamma^k_{(m)} = 2\delta^{ki} \\ \gamma^k_{(m)} \gamma^i_{(n)} = \gamma^i_{(n)} \gamma^k_{(m)} \qquad m \neq n \end{cases}$

Here the Is represent the unit matrix, and N is an integer. e.g. $N=1$ $\alpha^i = \gamma^i$ and get Dirac case

cont. on p.22.

$$[\alpha^i, I^{jk}] = \alpha^i I^{jk} - I^{jk}\alpha^i$$
$$= \delta^{ij}\alpha^k - \delta^{ik}\alpha^j \qquad \text{—— 4.2}$$

where $$I^{jk} = \alpha^j\alpha^k - \alpha^k\alpha^j \qquad \text{—— 4.3}$$

Equivalent forms of 4.2 are clearly

$$\left.\begin{aligned} &[\alpha_i, I^{jk}] = \delta_i^{\,j}\alpha^k - \delta_i^{\,k}\alpha^j \\ \text{and}\quad &[\alpha_i, I_{jk}] = \delta_{ij}\alpha_k - \delta_{ik}\alpha_j \end{aligned}\right\} \qquad \text{—— 4.2}$$

Equation 4.2 is satisfied by $\frac{\gamma_i}{2}$ where γ_i are the Dirac matrices and also by the Kemmer βs.

From 4.3 it follows that the I^{jk} satisfy the commutation relations

$$[I^{jk}, I^{lm}] = -\delta^{jl}I^{km} + \delta^{jm}I^{kl} + \delta^{kl}I^{jm} - \delta^{km}I^{jl} \qquad \text{—— 4.4}$$

Equation 4.4 is (Pauli 33b, p.180) just the commutation relation of the infinitesimal transformations of the restricted orthogonal group in four dimensions (i.e. the group of all rotations without reflections). Therefore any six matrices satisfying 4.4 can be used to build a representation of the restricted orthogonal group; or conversely, the representations of the restricted orthogonal group give representations of the matrices satisfying 4.4

We now introduce a further suffix 0, and put

$$I^{j0} = \alpha^j \qquad \text{—— 4.5}$$

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continuation of note from p.21.
$N = 2 \quad \alpha^i = \gamma^i_{(1)} \times I_{(2)} + I_{(1)} \times \gamma^i_{(2)}$ and this is an alternative form for the β_s (Kemmer (39)).

One can show that the Lubanski , do satisfy 4.2 above, and so the theories are identical, as can also be seen by comparing the subsequent developments in the papers cited.

It is then found that 4.2 and 4.4 can be combined into the one relation

$$[I^{JK}, I^{LM}] = -\delta^{JL} I^{KM} + \delta^{JM} I^{KL} + \delta^{KL} I^{JM} - \delta^{KM} I^{JL} \qquad 4.6$$

where J,K now run 0 - 4.

These are now the commutation relations of the infinitesimal transformations of the restricted orthogonal group in <u>five</u> dimensions, and in the same way as above we see that the possible representations of the matrices I^{JK} (and thus of the α^{j}) are given by the well-known representations of the restricted five-dimensional orthogonal group.

<u>Representations of the Restricted Orthogonal Group in Five Dimensions.</u>

We will denote the restricted group in five dimensions by R_5. Any irreducible representation of R_5 is labelled by two numbers λ_1 & λ_2, with

$$\lambda_1 \geqslant \lambda_2 \geqslant 0 \qquad 4.7$$

The λ_1 & λ_2 are both integers (including 0), or half-integers (excluding 0). Such an irreducible representation is denoted by $R_5\{\lambda_1, \lambda_2\}$ and its degree written; $d_5\{\lambda_1, \lambda_2\}$.

The corresponding quantities in four dimensions can be similarly denoted e.g. R_4, $R_4\{\lambda_1, \lambda_2\}$, & $d_4\{\lambda_1, \lambda_2\}$.

The explicit formulae for d_4 & d_5 in terms of λ_1 & λ_2 are

$$d_5\{\lambda_1, \lambda_2\} = \tfrac{2}{3}(\lambda_1+\tfrac{3}{2})(\lambda_2+\tfrac{1}{2})(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2+2) \qquad 4.8$$

& $$d_4\{\lambda_1, \lambda_2\} = (\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2+1) \qquad 4.9$$

Using 4.7 the following table of possible irreducible representations can then be drawn up.

λ_1	λ_2	$d_5\{\lambda_1,\lambda_2\}$	$d_4\{\lambda_1,\lambda_2\}$
0	0	1	1
$\frac{1}{2}$	$\frac{1}{2}$	4	2
1	0	5	6
1	1	10	3
$\frac{3}{2}$	$\frac{1}{2}$	16	6
$\frac{3}{2}$	$\frac{3}{2}$	20	4
2	0	14	14
2	1	35	8
2	2	35	5

— 4.10

and so on.

The $R_5\{\frac{1}{2},\frac{1}{2}\}$ corresponds to the Dirac case, while $R_5\{1,1\}$ and $R_5\{1,0\}$ are the Kemmer cases.

It can further be shown that the representations given by 4.7, 4.8 and 4.9 are the <u>only</u> irreducible representations of the restricted group in four or five dimensions.

Also the four matrices $\alpha_1 - \alpha_4$ corresponding through 4.5 to any irreducible representation of R_5 of degree d_5, generate a complete set of d_5^2 matrices (i.e. any matrix of degree d_5 can be expressed as a linear sum of these d_5^2 matrices).

e.g. in the Dirac case, $R_5\{\frac{1}{2},\frac{1}{2}\}$, the complete set consists of $4^2=16$ matrices, a well-known result.

Commutation Relations.

Although all the matrices with degrees given by 4.8 or 4.9 satisfy the commutation relations 4.2, yet the various representations also obey more specific commutation relations. These are determined from the fact that λ_1 denotes the maximum eigen-value of the matrices in the representation $R_5\{\lambda_1,\lambda_2\}$. Therefore if X denotes any of the ten matrices I^{jk} from 4.6, then X (multiplied sometimes by i) satisfies an equation of the type

$$\left.\begin{array}{ll} \{X^2-\lambda_1^2\}\{X^2-(\lambda_1-1)^2\}\ ---\ X^2=0 & \text{if } \lambda_1 \text{ integral} \\ \text{or } \{X^2-\lambda_1^2\}\{X^2-(\lambda_1-1)^2\}\ --\ \{X^2-\frac{1}{4}\}=0 & \text{if } \lambda_1 \text{ is half-integral.} \end{array}\right\} -4.11$$

The specific relations for any λ_1 can then be found by combining 4.11 with 4.2

e.g. in the Dirac case $\lambda_1=\frac{1}{2}$, and 4.11 gives just

$$X^2-\frac{1}{4}=0$$

Now taking case where $X=\alpha_i$, this gives

$$\alpha_i^2=\frac{1}{4} \qquad \text{—— (a)}$$

and from 4.2 we know

$$\alpha_i(\alpha_j\alpha_k-\alpha_k\alpha_j)-(\alpha_j\alpha_k-\alpha_k\alpha_j)\alpha_i=\delta_{ij}\alpha_k-\delta_{ik}\alpha_j \qquad \text{—— (b)}$$

In (b) let $i=j\neq k$ and use (a). We get

$$-\alpha_i\alpha_k\alpha_i=\frac{\alpha_k}{4}$$ and multiplying on left by α_i and again using (a) we deduce

$$\alpha_i \alpha_k + \alpha_k \alpha_i = 0 \qquad \text{--- (c)}$$

(a) and (c) are then the commutation relations of $\frac{\gamma_i}{2}$.

This method can be used also to deduce Kemmer's commutation relations ($\lambda_1 = 1$), and Rao (42) has applied it to find the commutation relations for $\lambda_1 = \frac{3}{2}$ & $\lambda_1 = 2$.

From 4.11 it can be seen that all the representations with the same λ_1 obey the same commutation relations.

Spin

The question of the spin of a particle described by an equation of the type 4.1 is linked up with the possible eigen-values of the operators I^{ik} . We will discuss how this arises in §(16) p 63 , but for the present we will say an equation of the form 4.1 describes a particle of maximum spin λ , if λ is the greatest eigen-value of I^{ik} , i.e. from the previous section if $\lambda = \lambda_1$. Further we see that all representations referring to a particle of maximum spin λ will satisfy the same commutation relations.

Bhabha (45) discusses the question of whether this definition of spin coincides with the one usually held. He says that under different circumstances a particle can exhibit different spin characteristics. In the relativistic region the spin of a particle with the representation $R_5\{\lambda_1, \lambda_2\}$ is actually λ_1 , but in the non-relativistic limit it tends to λ_2 . For example, the representation $R_5\{1, 0\}$ refers to what is usually termed the scalar meson ~~and~~

considered to have zero spin, but at high energies it has the magnetic moment of a particle of spin one. On the other hand the vector meson $R_5\{1,1\}$ always shows spin one. For a particle to always exhibit some definite spin it must therefore have the representation $R_5\{\lambda,\lambda\}$, that is $\lambda_1 = \lambda_2$.

From the table 4.10, we see there are two possible representations for a particle of maximum spin $\frac{3}{2}$, and three for a particle of maximum spin 2. Generally, for a particle of maximum spin λ , there are

$\lambda + 1$ possible representations if λ is integral

or $\lambda + \frac{1}{2}$ possible representations if λ is half-integral.

We are now in a position to amplify a little the answer given to the question in §(3), p. 18 .

The list* of particles considered as fundamental at the present time are the electron, proton, neutron and meson. The representations relating to the meson are as mentioned above, $R_5\{1,1\} + R_5\{1,0\}$. The electron, proton and neutron are usually considered as all being described by equations of the Dirac type, i.e. with representations $R_5\{\frac{1}{2},\frac{1}{2}\}$. There seems no reason to doubt the correctness of this for the electron, but the properties of the neutron and proton are not nearly so well

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* the photon can be brought into the above list as a case of the vector meson, $R_5\{1,1\}$ with zero rest-mass.

explained by the Dirac equation, and Bhabha suggests that for the proton the equation may be that related to the representation $R_5\{\frac{3}{2},\frac{1}{2}\}$. This particle would still behave like a particle of spin $\frac{1}{2}$ in the non-relativistic approximation, but for high energies (e.g. in the value of the magnetic moment) it would behave as a particle of spin $\frac{3}{2}$. Evidence is not conclusive enough yet to decide the question (see however p. 29).

The same possibility exists for the neutron, but other suggestions have been made by Temple (34) and Eddington (47) in which the neutron is not looked on as a fundamental particle at all, but is a combination of a proton and an electron, its equation being the limiting one of the hydrogen atom (i.e. with the azimuthal quantum number j = 0). Again however evidence is inconclusive.

Second-Order Equation.

Let us denote the operator $\alpha^k \frac{\partial}{\partial x^k} = \alpha^k p_k$ by P. As we saw in equation 4.11, the α^k matrices satisfy a certain algebraic equation, and therefore P must also satisfy an equation of the same degree in which the occur as coefficients. Further, relativistic invariance requires that the p_k should occur only in the combination $p^2 = p^k p_k = \Box_x^2$ (from 1.3). If we now consider a system with the particle at rest we have $p_1 = p_2 = p_3 = 0$ $p_4 \neq 0$, so P reduces to $p_4\alpha^4$, and as α^4 satisfies 4.11, P must satisfy the equation

$$\{P^2 - \lambda_1^2 h^2\}\{P^2 - (\lambda_1 - 1)^2 h^2\}\{\quad\} \text{ -- } = 0$$

Operating on the wave-function ψ with this equation, and continually replacing P by $-\varkappa$ from 4.1, we find every component of ψ satisfies the equation

$$\left.\begin{array}{l} \{\varkappa^2 - \square_x^2 \lambda_1^2\}\{\varkappa^2 - \square_x^2(\lambda_1 - 1)^2\} \text{ --- } \{\varkappa^2 - \square_x^2\}\varkappa^2 \psi = 0 \\ \qquad\qquad \text{if } \lambda_1 \text{ is integral} \\ \text{or } \{\varkappa^2 - \square_x^2 \lambda_1^2\}\{\varkappa_2^2 - \square_x^2(\lambda_1 - 1)^2\} \text{ -- } \{\varkappa^2 - \frac{\square_x^2}{4}\}\psi = 0 \end{array}\right\} \text{— } 4.12$$

if λ_1 is half-integral.

These are the equations discussed in §(3) p.16.

The equation with the representation $R_5\{\frac{3}{2}, \frac{1}{2}\}$ would, from 4.12, have two possible values for its rest-mass, either $\frac{2\varkappa}{3}$ or $2\varkappa$, i.e. mass M or 3M. The first value is the one we would expect to find at low energies, but for high energies, for example in cosmic-rays, the existence of particles with mass 3M should be possible. If experimental evidence was found in such cases for a proton of mass 3M, it would give great support to Bhabha's contention of the proper form for the proton equation.

From 4.12 we see that the number of possible states of rest-mass of any particle depends again only on λ_1.

<u>Summary of the Main Results.</u> —————— 4.13

For future reference we give here a summary of the main results which have been given in this section concerning the properties of the Bhabha matrices.

(a) <u>All</u> the irreducible representations of the α_5 in four dimensions are given by the irreducible representations of the restricted orthogonal group in five dimensions.

(b) Each irreducible representation is labelled by two numbers λ_1, λ_2 with $\lambda_1 \geqslant \lambda_2 \geqslant 0$. The $\lambda_1 + \lambda_2$ are both integers or both half-integers.

(c) The four α_5 in any irreducible representation of degree d_5, generate a complete set with d_5^2 elements.

(d) All the irreducible representations with the same value of $\lambda_1 = \lambda$

(i) obey the same commutation relations,

(ii) refer to particles with maximum spin λ,

and (iii) refer to particles with 2λ or $2\lambda+1$ possible values of the rest-mass, according as λ is integral or half-integral. ——— 4.13

(5) <u>The Pentad Problem.</u>

The following treatment of the Dirac matrices will illustrate the general method we will use in solving this problem.

In four dimensions the Dirac matrices $\gamma_1 - \gamma_4$ are introduced by the commutation relations (see 1.6)

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} \qquad \text{——— } 5.1$$

with i, j running 1-4.

It can be shown algebraically from 5.1, or seen directly from the table 4.10 together with 4.13, that

(a) the only irreducible representation of the γ_s is of degree four

and (b) the matrices $\gamma_1 - \gamma_4$ generate a complete set with 16 elements. — 5.2

We can similarly introduce Dirac matrices $\gamma_1 - \gamma_5$ in five dimensions by the relations

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\delta_{\alpha\beta} \qquad \text{—— } 5.3$$

where α, β run 1 - 5.

Again it can be proved algebraically (or seen from the table 9.2), that the only irreducible representations of $\gamma_1 - \gamma_5$ are two inequivalent representations of degree four. Let us denot the matrices belonging to one of these by $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3, \bar{\gamma}_4$ & $\bar{\gamma}_5$.

Now the first four of these, $\bar{\gamma}_1 - \bar{\gamma}_4$ are 4 x 4 matrices, and they certainly satisfy 5.1. But from 5.2(a) we know $\gamma_1 - \gamma_4$ is the <u>only</u> irreducible representation of the commutation relation 5.1; therefore $\bar{\gamma}_1 - \bar{\gamma}_4$ must be equivalent to $\gamma_1 - \gamma_4$ and thus we can put

$$\gamma_1 \to \bar{\gamma}_1, \quad \gamma_2 \to \bar{\gamma}_2, \quad \gamma_3 \to \bar{\gamma}_3, \quad \gamma_4 \to \bar{\gamma}_4$$

From 5.2(b) we know $\gamma_1 - \gamma_4$ and therefore $\bar{\gamma}_1 - \bar{\gamma}_4$ generate a complete set. That is any 4 x 4 matrix can be expressed in terms of $\bar{\gamma}_1 - \bar{\gamma}_4$ and their products. But $\bar{\gamma}_5$ is a 4 x 4 matrix, and it definitely obeys the commutation relations 5.1.

Therefore in this case it must be possible to find a fifth matrix from $\gamma_1 - \gamma_4$, to wit $\bar{\gamma}_5$, satisfying the same commutation relations, or in other words it must be possible to form a pentad.

In the case of the general Bhabha matrices it can be seen that a necessary* condition for the above argument to hold is that the degree of one of the irreducible representations of the matrices in four dimensions must equal the degree of one of the representations of the matrices obeying the same commutation relations in five dimensions.

As another rather more complicated example we can take the Kemmer β matrices.

The degrees of the matrices in four dimensions are those given by Kemmer (39) or see table 4.10.

They are 10, 5 and 1.

In five dimensions the degrees have been given by Lubanski and Rosenfeld (42b) - or see table 9.2.

They are 15, 10, 6 and 1.

Apart therefore from the trivial 1 x 1 representation, we can see the only possibility for a pentad is the 10x10 representation. This is just the result of Shroedinger, Kemmer and Harish-Chandra.

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* This condition is however not sufficient as it may happen that the five-dimensional matrices will yield a representation of the four-dimensional matrices which is reducible - see § (6) p.35.

From the preceding it can be seen that the possibility of pentads can be determined from a knowledge of the representations of the Bhabha matrices in five dimensions. As the representations of these matrices in four dimensions necessitated a study of the five-dimensional orthogonal group, we can see that the study of the representations in five dimensions will lead to the six dimensional orthogonal group.

In the next section therefore we state some results from Murnaghan (38) relating to the n dimensional orthogonal group, and later apply them to the more general problem stated on p. 21, of which the pentad problem is a particular case.

(6) Representations of the n-dimensional Orthogonal Group.

In the following we will be dealing with both the full and the restricted orthogonal groups. We will denote the full group by R_n' and the restricted group by R_n.

Any possible irreducible representation of R_n' (or R_n) is now labelled by k numbers, $\lambda_1 - \lambda_k$ where

$$\left.\begin{array}{ll} n = 2k & n \text{ even} \\ n = 2k+1 & n \text{ odd} \end{array}\right\} \qquad \text{——} \qquad 6.1$$

and $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \text{--} \geqslant \lambda_k \geqslant 0$ —— 6.2

e.g. if $n = 4$ or 5, $k = 2$ and there are only the two numbers λ_1 & λ_2 used in §(4).

A possible irreducible representation is denoted by

$R_n'\{\lambda_1,\lambda_2 -- \lambda_k\}$ and its degree by $d_n'\{\lambda_1,\lambda_2, --\lambda_k\}$. Similar designations hold for R_n.

The explicit expressions for $d_n'\{\lambda_1 - \lambda_k\}$ differ according as n is odd or even. They are

(a) n odd $\quad n = 2k+1$

$$d_n'\{\lambda_1 - \lambda_k\}$$
$$= \frac{2^k}{(2k-1)!\,--3!\,1!}\,(l_1+\tfrac{1}{2})(l_2+\tfrac{1}{2})--(l_k+\tfrac{1}{2})\prod_{h<q}^{k}\{(l_h+\tfrac{1}{2})^2-(l_q+\tfrac{1}{2})^2\} \qquad \text{— 6.3}$$

and (b) n even $\quad n = 2k$

$$\left.\begin{aligned} d_n'\{\lambda_1-\lambda_k\} &= \frac{2^k}{(2k-2)!\,--4!\,2!}\prod_{h<q}^{k}(l_h^2-l_q^2) \quad l_k \neq 0 \\ &= \frac{2^{k-1}}{(2k-2)!\,--4!\,2!}\prod_{h<q}^{k}(l_h^2-l_q^2) \quad l_k = 0 \end{aligned}\right\} \text{— 6.4}$$

where

$$l_1 = \lambda_1 + (k-1),\ l_2 = \lambda_2 + (k-2),\ ---\ l_k = \lambda_k \qquad \text{— 6.5}$$

e.g.

$n = 5,\ k = 2$

$l_1 = \lambda_1 + 1,\quad l_2 = \lambda_2$ & 6.3 gives

$$d_5'\{\lambda_1,\lambda_2\} = \frac{2^2}{3!\,1!}(\lambda_1+\tfrac{3}{2})(\lambda_2+\tfrac{1}{2})(\lambda_1+\lambda_2+2)(\lambda_1-\lambda_2+1)$$

which is the formula 4.8 for $d_5\{\lambda_1,\lambda_2\}$ - see however equation 6.6 for proof that $d_5' = d_5$.

Equations 6.3 and 6.4 give the <u>only</u> irreducible representations of R_n'.

<u>Irreducible Representations of R_n induced by R_n'.</u>

It is clear that by leaving out reflections any irreducible representation of R_n' will induce a representation of R_n. This representation may or may not be reducible, the situation differing according as n is odd or even.

n odd   Representation is irreducible and we can denote the reduction by

$$\left.\begin{array}{l} R_n'\{\lambda_1-\lambda_2\} \to R_n\{\lambda_1-\lambda_2\} \\ \text{the corresponding equation for degrees being} \\ d_n'\{\lambda_1-\lambda_2\} = d_n\{\lambda_1-\lambda_2\} \end{array}\right\} \quad \text{—— 6.6}$$

n even

(i) $\lambda_2 = 0$ Representation is again irreducible

(ii) $\lambda_2 \neq 0$ Representation is reducible and the reduction is given by

$$\left.\begin{array}{l} R_n'\{\lambda_1-\lambda_2\} \to R_{\underline{n}}\{\lambda_1-\lambda_2\} + R_{\overline{n}}\{\lambda_1-\lambda_2\} \\ \text{and} \quad d_n'\{\lambda_1-\lambda_2\} = 2\,d_n\{\lambda_1-\lambda_2\} \end{array}\right\} \text{— 6.7}$$

We shall call $R_{\underline{n}}\{\lambda_1-\lambda_2\}$ & $R_{\overline{n}}\{\lambda_1-\lambda_2\}$ the two twin* irreducible representations of R_n induced by R_n' - they are both of the same degree $d_n\{\lambda_1-\lambda_2\}$.

Again the irreducible representations of R_n formed in this way are the <u>only</u> irreducible representations of R_n.

<u>Irreducible Representations of R_{n-1}' (R_{n-1}) induced by R_n' (R_n).</u>

By keeping one of the axes in R_n' fixed, we obtain from it a representation of R_{n-1}', which in general is reducible. The situation again differs if n is odd or even

<u>n odd.</u> $n = 2k+1$

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*The usual terminology is to call these adjoint representations. The word "twin" was first used by Kemmer (43), and suits our purpose better.

Also $n-1$ is even and $n-1 = 2h$. Therefore the "h" for n equals the "h" for $n-1$, and the representations in R_n' and R_{n-1}' are labelled by the same number of numbers.

The formula giving the irreducible representations of R_{n-1}' induced by R_n' is

$$R_n'\{\lambda_1 - \lambda_h\} \to (1-\xi_1)^{-1}(1-\xi_2)^{-1} --- (1-\xi_h)^{-1} R_{n-1}'\{\lambda_1 - \lambda_h\} \quad \text{— } 6\cdot 8$$

where $\xi_1 - \xi_h$ are operators which reduce the corresponding λ_h by one - but they must be able to operate on <u>all</u> terms to the right of them.

e.g. let $n = 5$, $h = 2$ and consider the reduction of $R_5'\{3,2\}$ to R_4'

Using 6.8 we have

$$R_5'\{3,2\} \to (1-\xi_1)^{-1}(1-\xi_2)^{-1} R_4'\{3,2\}$$

$$\to (1-\xi_1)^{-1}(1+\xi_2+\xi_2^2+--) R_4'\{3,2\}$$

$$\to (1-\xi_1)^{-1}\left[R_4'\{3,2\} + R_4'\{3,1\} + R_4'\{3,0\}\right]$$

$$\to (1+\xi_1+\xi_1^2+--)\left[\quad '' \quad\right] \quad 6\cdot 9$$

$$\to R_4'\{3,2\} + R_4'\{3,1\} + R_4'\{3,0\} + R_4'\{2,2\} + R_4'\{2,1\} + R_4'\{2,0\}$$

it being impossible to operate with ξ_1^2 on the large bracket as $\xi_1^2 R_4'\{3,2\} \to R_4'\{1,2\}$

which contradicts $\lambda_1 \geqslant \lambda_2$.

The check by degrees of the above reduction can be obtained using 4.8 and 4.9 (together with 6.6 and 6.7), and gives

105 $=$ 24 $+$ 30 $+$ 16 $+$ 10 $+$ 16 $+$ 9

The reduction for $R_n \to R_{n-1}$ also comes from 6.8, using in addition the results 6.6, and 6.7. As n is odd R_n' gives an irreducible representation of R_n, but $n-1$ is even so the representations $R_{n-1}'\{\lambda_1 - \lambda_k\}$ with $\lambda_k \neq 0$ split into the two twin irreducible representations of 6.7. ——— 6.10

e.g. in case treated above we obtain

$$R_5'\{3,2\} \to R_5\{3,2\}$$

$$\to R_{\underline{4}}\{3,2\} + R_{\overline{4}}\{3,2\} + R_{\underline{4}}\{3,1\} + R_{\overline{4}}\{3,1\} + R_4\{3,0\} + R_{\underline{4}}\{2,2\}$$

$$+ R_{\overline{4}}\{2,2\} + R_{\underline{4}}\{2,1\} + R_{\overline{4}}\{2,1\} + R_4\{2,0\} \qquad \text{——— } 6.11$$

The check by degrees now being

105 = 12 + 12 + 15 + 15 + 16 + 5 + 5 + 8 + 8 + 9

<u>n even</u> $\quad n = 2k$

Also $n-1$ is odd and $n-1 = 2(k-1)+1$ .Therefore the "k" for $n-1$ is one less than the "k" for n and a representation of R_{n-1}' is labelled by one number less than that of R_n'.

The rule here is initially to write down the reduction of $R_n'\{\lambda_1 - \lambda_k\}$ as given by 6.8. Then if on the right side

$$\left.\begin{array}{lll} \lambda_k > 1 & \text{put term} & = 0 \\ \lambda_k = 1 & \text{put term} & = \varepsilon R_{n-1}'\{\lambda_1 - \lambda_{k-1}\} \\ \lambda_k = 0 & \text{put term} & = R_{n-1}'\{\lambda_1 - \lambda_{k-1}\} \end{array}\right\} - 6.12$$

Here ε is an operator which equals +1 for an element evaluated in R_{n-1}, and equals -1 for an element in R_{n-1}' but not in R_{n-1}.

As an example we take the case treated above for $R_4' \to R_3'$. We get from 6.9 using 6.12

$$R_4'\{3,2\} \to \varepsilon R_3'\{3\} + R_3'\{3\} + \varepsilon R_3'\{2\} + R_3'\{2\} \qquad 6.13$$

The check by degrees is, using 4.9, 6.7 and the footnote[*]

$$24 = 7 + 7 + 5 + 5$$

The reduction $R_n \to R_{n-1}$ follows also from 6.12, although now $\varepsilon = +1$. As n is now even and $n-1$ odd, we see from 6.6 and 6.7 that R_n' splits into the twin representations while R_{n-1}' is irreducible. — 6.14

For the case above we get from 6.13

$$R_4'\{3,2\} \to R_4\{3,2\} + R_{\bar{4}}\{3,2\}$$

$$\to R_3\{3\} + R_3\{3\} + R_3\{2\} + R_3\{2\}$$

The check by degrees being

$$24 = 12 + 12 = 7 + 7 + 5 + 5$$

We notice finally that the irreducible representations of R_{n-1}' (or R_{n-1}) induced in this way by R_n' (or R_n) are the <u>only</u> irreducible representations induced by R_n' in R_{n-1}'. — 6.15

<u>Summary of the Main Results for the Bhabha Matrices in n Dimensions.</u> — 6.16

Using the results established in the preceding part of this section we now tabulate the main properties of the Bhabha matrices in n dimensions. This list is analogous

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[*] from 6.3 we find for $n = 3$

$$d_3'\{\lambda\} = 2\lambda + 1$$

to that for the four-dimensional matrices given in 4.13, and the results stated here can be arrived at by similar arguments to those used there.

(a) <u>All</u> the irreducible representations of the α_s, in n dimensions are given by the irreducible representations of the restricted orthogonal group in $(n+1)$ dimensions.

(b) Each irreducible representation is labelled by k numbers, $\lambda_1 - \lambda_k$ where

$$n = 2k \qquad \text{if n is even}$$

$$n = 2k+1 \qquad \text{if n is odd}$$

and $\qquad \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \dots \geqslant \lambda_k \geqslant 0$

The λ_s are all integers (including 0), or all half-integers (excluding 0).

(c) The n α_s in any irreducible representation of degree d_n generate a complete set with d_n^2 elements.

(d) All irreducible representations with the same value of $\lambda_1 = \lambda$ (<u>and no matter what the value of n</u>)

(i) obey the same commutation relations,

(ii) refer to particles with maximum spin λ,

and (iii) refer to particles with 2λ or $2\lambda+1$ possible values of the rest-mass, according as λ is integral or half-integral. ——— 6.16

(7) Solution of the Problem.

We now apply the results of the last section to the general problem stated on p. 21, "If $\alpha_1 - \alpha_n$ are the Bhabha matrices in n dimensions, for what irreducible representations of the α_s is it possible to find an $(n+1)$th matrix satisfying the same commutation relations?"

Consider the reduction of the representation $\underset{k\text{ factors}^*}{R_n'\{\lambda,\lambda,\lambda,--\lambda\}}$ into R_{n-1}', where n is even and

$$\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_k$$

Using 6.8 and 6.2 we find

$$\underset{k\text{ factors}}{R_n'\{\lambda-\lambda\}} \rightarrow \Sigma\, \underset{k-1\text{ factors}}{R_{n-1}'\{\lambda-\lambda\}} + \underset{k-1\text{ factors}}{R_{n-1}'\{\lambda-\lambda\}} \qquad 7.1$$

From 6.14 we obtain the corresponding reduction of R_n

$$\underset{k\text{ factors}}{R_n\{\lambda-\lambda\}} + \underset{k\text{ factors}}{R_{\bar n}\{\lambda-\lambda\}} \rightarrow \underset{k-1\text{ factors}}{R_{n-1}\{\lambda-\lambda\}} + \underset{k-1\text{ factors}}{R_{n-1}\{\lambda-\lambda\}} \qquad 7.2$$

The check by degrees of this reduction can be obtained directly from 6.3 and 6.4, and the calculation is given in the appendix, p. 47.

The result is however

$$\underset{k\text{ factors}}{d_n\{\lambda-\lambda\}} + \underset{k\text{ factors}}{d_n\{\lambda-\lambda\}} = \underset{k-1\text{ factors}}{d_{n-1}\{\lambda-\lambda\}} + \underset{k-1\text{ factors}}{d_{n-1}\{\lambda-\lambda\}}$$

or

$$\underset{k\text{ factors}}{d_n\{\lambda-\lambda\}} = \underset{k-1\text{ factors}}{d_{n-1}\{\lambda-\lambda\}} \qquad 7.3$$

---

* The notation "k factors" means only that there are k numbers λ within the bracket.

Now clearly any irreducible representation of R_n will induce a representation which may or may not be reducible in R_{n-1}. In particular $R_{\bar{n}}\{\lambda-\lambda\}$ will induce some representation in R_{n-1}. If this representation is reducible it must induce irreducible representations of degree less than $d_n\{\lambda-\lambda\}$. But this would mean that from $R_{\bar{n}}\{\lambda-\lambda\}+R_{\underline{n}}\{\lambda-\lambda\}$ we had obtained an irreducible representation of R_{n-1} of degree less than d_n, and this contradicts 6.15 and 7.3 which show that the <u>only</u> irreducible representations induced by $R_{\bar{n}}\{\lambda-\lambda\}+R_{\underline{n}}\{\lambda-\lambda\}$ in R_{n-1} are of degree $d_n\{\lambda-\lambda\}$

$R_{\bar{n}}\{\lambda-\lambda\}$ must therefore induce an irreducible representation of degree $d_n\{\lambda-\lambda\}=d_{n-1}\{\lambda-\lambda\}$ in R_{n-1}, and the reduction must in fact be -

$$\left.\begin{array}{l} R_{\bar{n}}\underbrace{\{\lambda-\lambda\}}_{h\text{ factors}} \rightarrow R_{n-1}\underbrace{\{\lambda-\lambda\}}_{h-1\text{ factors}} \\ \text{and similarly} \\ R_{\underline{n}}\underbrace{\{\lambda-\lambda\}}_{h\text{ factors}} \rightarrow R_{n-1}\underbrace{\{\lambda-\lambda\}}_{h-1\text{ factors}} \end{array}\right\} \quad \text{——} \; 7.4$$

together with 7.3, i.e.

$$d_n\underbrace{\{\lambda-\lambda\}}_{h\text{ factors}} = d_{n-1}\underbrace{\{\lambda-\lambda\}}_{h-1\text{ factors}} \quad \text{——} \; 7.3$$

The results 7.3 and 7.4 are those we need for our problem.

With n even we can write 7.3 and 7.4 as

$$R_{\overline{n+2}}\underbrace{\{\lambda-\lambda\}}_{h+1\text{ factors}} \rightarrow R_{n+1}\underbrace{\{\lambda-\lambda\}}_{h\text{ factors}} \quad \text{——} \; 7.5$$

$$\left.\begin{array}{l} \underset{h+1 \text{ factors}}{R_{n+2}\{\lambda - \lambda\}} \rightarrow \underset{h \text{ factors}}{R_{n+1}\{\lambda - \lambda\}} \\ \text{and} \\ \underset{h+1 \text{ factors}}{d_{n+2}\{\lambda - \lambda\}} = \underset{h \text{ factors}}{d_{n+1}\{\lambda - \lambda\}} \end{array}\right\} \quad \text{— 7.5 (cont)}$$

Now from 6.16(a), $R_{\overline{n+2}}\{\lambda - \lambda\}$ gives an irreducible representation of the Bhabha matrices in $(n+1)$ dimensions, and from 7.5 this induces an irreducible representation <u>of the same degree</u> of the Bhabha matrices in n dimensions. Further, from 6.16(d), these representations satisfy the same commutation relations.

We can now use the same argument as that for the Dirac case in §(5).

Let $\alpha_1 - \alpha_{n+1}$ be the $(n+1)$ Bhabha matrices in $(n+1)$ dimensions corresponding to $R_{\overline{n+2}}\{\lambda - \lambda\}$. Any n of these, $\alpha_1 - \alpha_n$ say, yield an irreducible representation of the Bhabha matrices in n dimensions of the same degree and satisfying the same commutation relations. Also from 6.16(c) these n matrices generate a complete set. But α_{n+1} is a matrix of the same degree as $\alpha_1 - \alpha_n$ and satisfies the same commutation relations.

Therefore in this case it must be possible from $\alpha_1 - \alpha_n$ to find an $(n+1)$th matrix satisfying the same commutation relations.

So our result is:- "<u>It is always possible to find</u>

<u>an extra matrix for those Bhabha matrices in n dimensions whose representations are given by</u> $R_{n+1}\underbrace{\{\lambda - \lambda\}}_{h \text{ factors}}$ <u>with n even.</u>" —— 7.6

The other twin representation, $R_{\underline{n+2}}\{\lambda - \lambda\}$ leads to the same result 7.6. If $\alpha_1 - \alpha_n$ represent the matrices in n dimensions, then $R'_{\overline{n+2}}$ and $R_{\underline{n+2}}$ differ only by having α_{n+1} with a positive or negative sign.

The result 7.6 does not preclude the possibility of an extra matrix existing for some other form of $R_{n+1}\{\lambda_1, \lambda_2, \dots \lambda_h\}$. Such a case could only occur however when an irreducible representation of R_{n+2} induces an irreducible representation of R_{n+1} of the same degree and with the same λ_1, (for same commutation relations) and apart from the case 7.6 this would only occur in very fortuitous circumstances.

(8) <u>Special Cases of the Result.</u>

(a) $n = 2, \quad n+1 = 3 \qquad h = 1$

Any representation of the α_3 in two dimensions are therefore given by $R_3\{\lambda\}$, and from 7.6 triads are possible in all cases.

A special case of interest is when $\lambda = \frac{1}{2}$. From 6.16(d) we see these matrices satisfy the Dirac commutation relations, and from the footnote on p. 38 their degree is two. They are just the Pauli spin matrices σ_x and σ_y and it is well-known that the triad is formed by putting $\sigma_z = i\sigma_x\sigma_y$.

(b) $n = 4, \quad n+1 = 5 \qquad h = 2$

This is the pentad problem, and 7.6 tells us that pentads will be possible for the representations

$$\begin{array}{lll} R_5\{\tfrac{1}{2},\tfrac{1}{2}\} & \text{of degree} & 4 \\ R_5\{1,1\} & \text{of degree} & 10 \\ R_5\{\tfrac{3}{2},\tfrac{3}{2}\} & \text{of degree} & 20 \\ R_5\{2,2\} & \text{of degree} & 35 \end{array} \qquad \text{—— } 8\cdot 1$$

and so on.

The first is the Dirac case and the second that relating to the Kemmer matrices, but the other results have not so far been given.

We see that in general there is only one representation associated with a particle of any particular maximum spin for which it is possible to form a pentad, and moreover by referring to a remark on p. 27 , this particular representation is just the one for which the particle manifests the same value of the spin under all circumstances.

In the cases where pentads are possible a scheme can be set up analogous to the pentad, triad pattern of Eddington's E numbers. (Eddington (36) p.22).

We introduce a further index 5, and put

$$\alpha^1 = \Gamma^{01},\ \alpha^2 = \Gamma^{02},\ \text{---}\ \alpha^5 = \Gamma^{05} \qquad \text{—— } 8\cdot 2$$

The commutation relation of these five αs together with their ten commutators can be combined in the form

$$[I^{\alpha\beta}, I^{\gamma\delta}] = -\delta^{\alpha\gamma} I^{\beta\delta} + \delta^{\alpha\delta} I^{\beta\gamma} + \delta^{\beta\gamma} I^{\alpha\delta} - \delta^{\beta\delta} I^{\alpha\gamma} \quad 8.3$$

where α, β now run 0 - 5.

If we introduce also the unit elements by $I^{16} = 1$, the sixteen matrices have, from 8.3, the following properties:

(i) Any five matrices with one index in common form a pentad.

e.g. $I^{10}, I^{12}, I^{13}, I^{14}$ & I^{15} are a pentad.

(ii) Any matrix commutes with eight others in the set (including itself and I^{16})

e.g. I^{12} commutes with

$I^{34}, I^{35}, I^{30}, I^{45}, I^{40}, I^{50}, I^{12}$ & I^{16}

(iii) An anti-triad of mutually commuting matrices is given by

$I^{\mu\nu}, I^{\sigma\tau}$ & $I^{\lambda\rho}$ with $\mu, \nu, \sigma, \tau, \lambda$ & ρ all different e.g. I^{12}, I^{34} & I^{50} are an anti-triad.

The relation of any matrix to the other eight not given by (ii), e.g. the relation of I^{12} to $I^{10}, I^{13}, I^{14}, I^{15}, I^{20}, I^{23}, I^{24}, I^{25}$ is however (except in the Dirac case) not the relation "to anti-commute", but depends on the particular commutation relations of the α matrices under consideration.

(9) The Representations of the Bhabha matrices in five dimensions.

We conclude this chapter by tabulating the possible irreducible representations of the Bhabha matrices in

five dimensions, that is the representations $R_6\{\lambda_1,\lambda_2,\lambda_3\}$
These may be of importance in studying the five-dimensional theories of Klein, Pauli, Flint and others, and also give a simple check on the possibility of forming pentads.
From 6.4 and 6.7 it is found that the degree of $R_6\{\lambda_1,\lambda_2,\lambda_3\}$ is given by

$$d_6\{\lambda_1,\lambda_2,\lambda_3\} = \tfrac{1}{12}(\lambda_1+\lambda_2+3)(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2+2)\times$$
$$\times(\lambda_1-\lambda_3+2)(\lambda_2+\lambda_3+1)(\lambda_2-\lambda_3+1) \qquad \text{—— } 9{\cdot}1$$

We therefore get the following table for the possible irreducible representations of R_6 & R_6' (using 6.7).

λ_1	λ_2	λ_3	$d_6'\{\lambda_1,\lambda_2,\lambda_3\}$	$d_6\{\lambda_1,\lambda_2,\lambda_3\}$	
0	0	0	1	1	
1/2	1/2	1/2	8	4 *	
1	0	0	6	6	
1	1	0	15	15	
1	1	1	20	10 *	
3/2	1/2	1/2	40	20 *	
3/2	3/2	1/2	72	36 *	—— 9·2
3/2	3/2	3/2	40	20 *	
2	0	0	20	20	
2	1	0	64	64	
2	1	1	90	45 *	
2	2	1	140	70 *	
2	2	2	70	35 *	

etc.

The x indicates there are two twin representations in these cases. When $\lambda_1 = 1$ the representations are those of the β matrices in five-dimensions studied by Lubanski and Rosenfeld (see p. 32).

Appendix.

We wish to prove that when n is even

$$d_n\{\underbrace{\lambda - \lambda}_{h \text{ factors}}\} = d_{n-1}\{\underbrace{\lambda - \lambda}_{h-1 \text{ factors}}\}$$

If $n = 2h$, then $n-1 = 2(h-1)+1$ so the "h" for R_{n-1} is one less than the "h" for R_n.

(a) We first consider $d_{n-1}\{\lambda - \lambda\}$

From 6.3 and 6.6 we have

$$d_{n-1}\{\lambda - \lambda\} = \frac{2^{h-1}}{(2h-3)!\ldots 3!1!}(l_1+\tfrac{1}{2})(l_2+\tfrac{1}{2})\ldots(l_{h-1}+\tfrac{1}{2}) \times$$

$$\times \prod_{\substack{p<q\\1}}^{h} \{(l_p+\tfrac{1}{2})^2 - (l_q+\tfrac{1}{2})^2\}$$

$$= \frac{2^{h-1}}{(2h-3)!\ldots 3!1!} \prod_{q=1}^{h-1}(l_q+\tfrac{1}{2}) \prod_{q=2}^{h-1}\{(l_1+\tfrac{1}{2})^2 - (l_q+\tfrac{1}{2})^2\} \times$$

$$\times \prod_{q=3}^{h-1}\{(l_2+\tfrac{1}{2})^2 - (l_q+\tfrac{1}{2})^2\} \times \ldots \times \prod_{q=h-1}^{h-1}\{(l_{h-2}+\tfrac{1}{2})^2 - (l_q+\tfrac{1}{2})^2\}$$

where

$l_1 = \lambda + h - 2,$ $\qquad$ $l_1 + \tfrac{1}{2} = \lambda + h - \tfrac{3}{2}$

$l_2 = \lambda + h - 3$ $\qquad$ $l_2 + \tfrac{1}{2} = \lambda + h - \tfrac{5}{2}$

– – – – – $\qquad$ – – – – – –

$l_{h-1} = \lambda$ $\qquad$ $l_{h-1} + \tfrac{1}{2} = \lambda + \tfrac{1}{2}$

so $l_q = \lambda + h - q - 1$ $\qquad$ so $l_q + \tfrac{1}{2} = \lambda + h - q - \tfrac{1}{2}$

$\therefore\ d_{n-1}\{\lambda - \lambda\}$

$$= \frac{2^{h-1}}{(2h-3)!\ldots 3!1!} \prod_{q=1}^{h-1}(\lambda + h - q - \tfrac{1}{2}) \prod_{q=2}^{h-1}(2\lambda + 2h - q - 2)(q-1) \times$$

$$\times \prod_{q=3}^{h-1}(2\lambda + 2h - q - 3)(q-2) \times \ldots \times (2\lambda+2)1$$

$$\doteq \frac{(h-2)!\,(h-3)!\;\cdots\;2!\,1!}{(2h-3)!\;\cdots\;3!\,1!}\prod_{q=1}^{h-1}(2\lambda+2h-2q-1)\prod_{q=2}^{h-1}(2\lambda+2h-q-2)\,\times$$

$$\times\prod_{q=3}^{h-1}(2\lambda+2h-q-3)\times\cdots\times(2\lambda+2).$$

(b) We next treat $d_n\{\lambda-\lambda\}$.

From 6.4 and 6.7 we obtain

$$d_n\{\lambda-\lambda\} = \frac{2^{h-1}}{(2h-2)!\;\cdots\;4!\,2!}\prod_{1\le n<q}^{h}(l_n^{2}-l_q^{2})$$

where

$$l_1 = \lambda+h-1$$
$$l_2 = \lambda+h-2$$
$$\cdots$$
$$l_h = \lambda h$$
$$\text{so } l_q = \lambda+h-q$$

$$\therefore\; d_n\{\lambda\;\cdots\;\lambda\}$$

$$= \frac{2^{h-1}}{(2h-2)!\;\cdots\;4!\,2!}\prod_{q=2}^{h}\{(2\lambda+2h-q-1)(q-1)\}\,\times$$

$$\times\prod_{q=3}^{h}\{(2\lambda+2h-q-2)(q-2)\}\times\cdots\times\{(2\lambda+3)1\times(2\lambda+2)2\}\{(2\lambda+1)1\}$$

$$= \frac{2^{h-1}(h-1)!\,(h-2)!\;\cdots\;1!}{(2h-2)!\,(2h-4)!\;\cdots\;4!\,2!}\prod_{q=1}^{h-1}(2\lambda+2h-2q-1)\,\times$$

$$\times\prod_{q=3}^{h}(2\lambda+2h-q-1)\prod_{q=4}^{h}(2\lambda+2h-q-2)\times\cdots\times(2\lambda+2)$$

$$= \frac{(h-2)!\,(h-3)!\;\cdots\;1!}{(2h-3)!\,(2h-5)!\;\cdots\;3!\,1!}\prod_{q=1}^{h-1}(2\lambda+2h-2q-1)\,\times$$

$$\times\prod_{q=2}^{h-1}(2\lambda+2h-q-2)\prod_{q=3}^{h-1}(2\lambda+2h-q-3)\times\cdots\times(2\lambda+2)$$

$$= d_{n-1}\{\lambda-\lambda\}.$$

CHAPTER 3

The Generalized Bhabha Equation

Using a method analogous to that of Schroedinger (32) and Bargman (33) for the Dirac equation, Bhabha's equation is in this chapter put in a form valid for any co-ordinate system. This equation is then determined explicitly for any orthogonal* co-ordinate system; that is for a system whose metric is given by

$$ds^2 = h_1{}^2(dx^1)^2 + h_2{}^2(dx^2)^2 + h_3{}^2(dx^3)^2 + h_4{}^2(dx^4)^2$$

where $h_1 - h_4$ are any functions of the x^i.

The equation is then put into a Lagrangian form and expressions for the energy-momentum tensor and current-vector calculated.

Finally the bearing of this generalized equation in terms of spherical polar co-ordinates on the question of spin is discussed.

(10) The Metric, Undors and Covariant Differentiation.

We shall consider any geometry whose metric is given by

$$ds^2 = g_{ij}\, dx^i dx^j \qquad \text{—— } 10{\cdot}1$$

where ds represents the scalar interval, g_{ij} is the metric tensor (in general a function of x^i) and i, j runs 1 - 4.

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* This is a more general use of the word "orthogonal" than that in the previous chapter. There the meaning is really "orthogonal cartesian", and the metrics considered are the special core of the above with $h_1 = h_2 = h_3 = h_4 = 1$ Both meanings of the word are however in general use.

We are concerned with two types of transformation within this geometry (see p. 8)

(a) co-ordinate transformation of the type

$$\bar{x}^i = a^i_{\ j} x^j \qquad \text{— } 10.2$$

where the $a^i_{\ j}$ are in general functions of x^i, and (b) S transformations which leave the commutation relations of the α matrices unaffected.

The behaviour of the matrices α^i and the undors ψ under (a) and (b) is defined by

(a) if $\bar{x}^i = a^i_{\ j} x^j$

$$\left.\begin{array}{ll} \text{then} & \bar{\alpha}^i = a^i_{\ j} \alpha^j \\ \text{and} & \bar{\psi} = \psi. \end{array}\right\} \qquad \text{— } 10.3$$

and (b)

$$\left.\begin{array}{ll} & \bar{\alpha}^i = S^{-1} \alpha^i S \\ \text{and} & \bar{\psi} = S^{-1}\psi \end{array}\right\} \qquad \text{— } 10.4$$

where S is a square matrix, in general a function of x^i.

From 10.3 we see that α^i transforms as a vector under co-ordinate transformations. This implies that the α^i are defined by a tensor equation - but 4.2 is not such an equation as δ_{ij} is not a tensor. We therefore generalize the commutation relations to

$$[\alpha_i, I_{jk}] = g_{ij}\alpha_k - g_{ik}\alpha_j \qquad \text{— } 10.5$$

where the g_{ij} are defined by 10.1

In the case of cartesian co-ordinates $g_{ij} = \delta_{ij}$ and 10.5 is equivalent to 4.2.

As S is now a function of x^i, we see that the usual spacial covariant derivative of α^i will not transform like α^i under an S transformation; nor will the derivative of ψ transform like ψ under an S transformation. In the following we will reserve the term "covariant derivative", for a derivative which (a) transforms like a tensor under co-ordinate transformations, i.e.

$$\left.\begin{aligned} \bar{x}^i &= a^i_{\;j}\, x^j \\ \bar{\alpha}^i_{\;;k} &= a^i_{\;j}\, a^l_{\;k}\, \alpha^j_{\;;l} \\ \text{and} \quad \bar{\psi}_{;k} &= a^j_{\;k}\, \psi_{;j} \end{aligned}\right\} \quad \text{— 10.6}$$

and (b) transforms like α^i or ψ under S transformations

i.e.
$$\left.\begin{aligned} \bar{\alpha}^i_{\;;j} &= S^{-1}\alpha^i_{\;;j}\, S \\ \text{and} \quad \bar{\psi}_{;j} &= S^{-1}\psi_{;j} \end{aligned}\right\} \quad \text{— 10.7}$$

where the symbol ; denotes covariant differentiation.

In order to see the modification 10.7 induces in the usual covariant derivative of α^i we consider the effect of an infinitesimal S transformation on α^i.

Let $S = 1 + \varepsilon$ where ε is small

then $S^{-1} = 1 - \varepsilon$

and $\bar{\alpha}^i = (1-\varepsilon)\,\alpha^i\,(1+\varepsilon)$

$$= \alpha^i - \varepsilon\alpha^i + \alpha^i\varepsilon$$

$$\therefore\ \bar{\alpha}^i - \alpha^i = \delta\alpha^i = -\varepsilon\alpha^i + \alpha^i\varepsilon$$

If we put $\varepsilon = \Delta_j\, \delta x^j$ where Δ_j is some square matrix, we see that

$$\frac{\delta \alpha^i}{\delta x^j} = -\Delta_j \alpha^i + \alpha^i \Delta_j$$

This suggests we define the covariant derivative of α^i by the formula

$$\alpha^i{}_{;j} = \frac{\partial \alpha^i}{\partial x^j} + \Gamma^i_{jl}\alpha^l - \Delta_j \alpha^i + \alpha^i \Delta_j \quad \text{—— } 10{\cdot}8$$

where the Γ^i_{jl} are the affine connections for the geometry considered (see Eddington(37)p.213), and Δ_j is some square matrix whose value and transformation properties are yet to be determined.

From 10.6 it follows that under co-ordinate transformations Δ_j must transform as a covariant vector i.e.

$$\left.\begin{aligned} \bar{x}^i &= a^i{}_k x^k \\ \bar{\Delta}_j &= a_j{}^k \Delta_k \end{aligned}\right\} \text{—— } 10{\cdot}9$$

From 10.4 we find after a little manipulation that Δ_j must transform under S transformations in the manner

$$\left.\begin{aligned} \bar{\Delta}_j &= S^{-1}\Delta_j S + \frac{\partial S^{-1}}{\partial x^j} S \\ &= S^{-1}\Delta_j S - S^{-1}\frac{\partial S}{\partial x^j} \end{aligned}\right\} \text{—— } 10{\cdot}10$$

the two forms arising from the identity $S^{-1}S = 1$.

In a similar way the covariant derivative of ψ can be investigated, and it is found that the definition

$$\psi_{;j} = \frac{\partial \psi}{\partial x^j} - \Delta_j \psi \quad \text{—— } 10{\cdot}11$$

(where the Δ_j are the same as in 10.8), satisfies both the conditions 10.6 and 10.7.

(11) The General Bhabha Equation

Our fundamental equation must satisfy the two postulates (see p. 9)

(1) It is invariant under co-ordinate transformations and (2) It is invariant under S transformations.

Using the results of the last section we see that the simplest generalisation of equation 4.1 satisfying both these requirements is

$$\left.\begin{array}{l} \alpha^i \psi_{;i} + \varkappa \psi = 0 \\ \text{or} \quad \alpha^i \left(\frac{\partial}{\partial x^i} - \Delta_i \right) \psi + \varkappa \psi = 0 \end{array}\right\} \quad \text{--- } 11.1$$

The invariance properties follow directly from 10.3 - 10.7 and we will now take 11.1 as the fundamental equation relating to particles of arbitrary spin in any co-ordinate system.

It can be seen that the character of this equation in any particular co-ordinate system depends upon the value of Δ, and in the next section we show how this value can be found in all cases of any physical interest. In a cartesian system $\Delta_i = 0$ (see p 59) so 11.1 reduces to the usual Bhabha equation 4.1.

(12) The Determination of the Δ_i

The only condition we have for the determination of the Δ_i is that the value of $\frac{\partial g_{ij}}{\partial x^l}$ obtained by differentiating 10.5 w.r.t. x^l and substituting from 10.8, should be the same as that given by the ordinary tensor calculus from the relation

$$g_{ij;l} = \frac{\partial g_{ij}}{\partial x^l} - \Gamma^m_{il} g_{mj} - \Gamma^m_{lj} g_{im} \qquad \text{— 12.1}$$

In order to use this condition for determining the Δ_i we first introduce the matrices $\overset{\circ}{\alpha}_i$ which satisfy the commutation relations 4.2, that is

$$\left.\begin{aligned} [\overset{\circ}{\alpha}_i, \overset{\circ}{I}_{jk}] &= \delta_{ij}\overset{\circ}{\alpha}_k - \delta_{ik}\overset{\circ}{\alpha}_j \\ \text{where} \quad \overset{\circ}{I}_{jk} &= \overset{\circ}{\alpha}_j\overset{\circ}{\alpha}_k - \overset{\circ}{\alpha}_k\overset{\circ}{\alpha}_j \end{aligned}\right\} \qquad \text{— 12.2}$$

These matrices $\overset{\circ}{\alpha}_i$ are therefore the usual Bhabha ones whose algebra we discussed in chapter 2.

We now confine ourselves to those geometries for which the α_i defined by 10.5 can be expressed in the form

$$\alpha_i = b^m{}_i \overset{\circ}{\alpha}_m \qquad \text{— 12.3}$$

where the $\overset{\circ}{\alpha}_m$ are defined by 12.2 and the $b^m{}_i$ are in general functions of x^i. Such a solution for the α_i is not possible for all values of g_{ij}, but as we shall see later, it covers all cases of physical interest.

The inverse relation to 12.3 can be written

$$\left.\begin{aligned} \overset{\circ}{\alpha}_m &= a^i{}_m \alpha_i \\ \text{where} \quad a^i{}_m b^m{}_j &= \delta^i{}_j \end{aligned}\right\} \qquad \text{— 12.4}$$

From 12.3 and 12.4 we find

$$\left.\begin{aligned} \frac{\partial \alpha_i}{\partial x^l} &= \frac{\partial b^m{}_i}{\partial x^l}\overset{\circ}{\alpha}_m = \frac{\partial b^m{}_i}{\partial x^l} a^n{}_m \alpha_n = \mathscr{C}^n_{il}\alpha_n \\ \text{where} \quad \mathscr{C}^n_{il} &= \frac{\partial b^m{}_i}{\partial x^l} a^n{}_m \end{aligned}\right\} \qquad \text{— 12.5}$$

Now on differentiating 10.5 w.r.t. x^e and substituting for $\frac{\partial \alpha_i}{\partial x^e}$ and $\frac{\partial g_{ij}}{\partial x^e}$ from 12.5 and 12.1 we obtain

$$(\mathcal{C}^m_{ie} g_{mj} + \mathcal{C}^m_{je} g_{im} - g_{ij;e} - T^m_{ie} g_{mj} - T^m_{je} g_{im})\alpha_k$$
$$-(\mathcal{C}^m_{ie} g_{mk} + \mathcal{C}^m_{ke} g_{im} - g_{ik;e} - T^m_{ie} g_{mk} - T^m_{ke} g_{im})\alpha_j = 0$$

For this equation to be satisfied the coefficients of α_j & α_k must be zero, so

$$(\mathcal{C}^m_{ie} - T^m_{ie}) g_{mj} + (\mathcal{C}^m_{je} - T^m_{je}) g_{im} = g_{ij;e} \qquad \text{— } 12.6$$

We now put

$$A_{jie} = (\mathcal{C}^m_{ie} - T^m_{ie}) g_{mj} \qquad \text{— } 12.7$$

so 12.6 can be written

$$A_{jie} + A_{ije} = g_{ij;e} \qquad \text{— } 12.8$$

From 12.5 we now have

$$\frac{\partial \alpha_i}{\partial x^e} = \mathcal{C}^m_{ie}\alpha_m = (A^m_{ie} + T^m_{ie})\alpha_m \quad \text{using 12.7}$$
$$= (A_{mie} + T_{mie})\alpha^m$$
$$\therefore \frac{\partial \alpha_i}{\partial x^e} - T^m_{ie}\alpha_m = A_{mie}\alpha^m \qquad \text{— } 12.9$$

Now 12.9 can be put in the form

$$\frac{\partial \alpha_i}{\partial x^e} - T^m_{ie}\alpha_m - \Delta_e \alpha_i + \alpha_i \Delta_e = \tfrac{1}{2} g_{im;e}\alpha^m \qquad \text{— } 12.10$$

if Δ_e * $= \frac{1}{2} A_{rse} I^{rs}$ — 12.11

For $\Delta_e \alpha_i - \alpha_i \Delta_e$

$$= \tfrac{1}{2}\{A_{rse} I^{rs}\alpha_i - \alpha_i A_{rse} I^{rs}\}$$
$$= \tfrac{1}{2} A_{rse}\{g^s{}_i \alpha^r - g^r{}_i \alpha^s\} \quad \text{using 10.5}$$

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* A term independent of the α's can be added to Δ_e and has been used by Schroedinger (32) to introduce the electromagnetic field.

$$= \frac{1}{2}\{A_{rie}\alpha^r - A_{ise}\alpha^s\}$$

$$= \frac{1}{2}\{A_{rie} - A_{ire}\}\alpha^r$$

$$= -\frac{1}{2} g_{ir;e}\alpha^r \qquad \text{using 12.8}$$

Also from 10.8 equation 12.10 can be written

$$\alpha_{i;e} = \frac{1}{2} g_{in;e}\alpha^n \qquad \text{—— 12.12}$$

Conversely we can begin our argument by defining 12.12 to give the relation between the covariant derivative of α_i and that of g_{ij}. A solution for Δ_e can then be obtained from this equation of the form 12.11

i.e.
$$\Delta_e = \frac{1}{2} A_{rse} I^{rs} \qquad \text{—— 12.11}$$

where as in 12.8
$$A_{jie} + A_{ije} = g_{ij;e} \qquad \text{—— 12.8}$$

These two relations then lead back to equation 12.9

i.e.
$$\frac{\partial \alpha_i}{\partial x^e} - T^n_{ie}\alpha_n = A_{nie}\alpha^n \qquad \text{—— 12.9}$$

and by direct substitution of this into the differentiated version of 10.5, it can be verified we obtain equation 12.1

i.e.
$$g_{ij;e} = \frac{\partial g_{ij}}{\partial x^e} - T^m_{ie} g_{mj} - T^m_{ej} g_{im} \qquad \text{—— 12.1}$$

The definition 12.12 together with its solution 12.11 and 12.8 is therefore completely consistent with the usual definition of covariant differentiation given by 12.1.

Furthermore we are dealing with a geometry in which 12.5 holds. i.e.
$$\frac{\partial \alpha_i}{\partial x^e} = \mathcal{C}^n_{ie}\alpha_n \qquad \text{—— 12.5}$$

and by combining 12.5 and 12.9 we again find 12.7

$$A_{jil} = (b^m_{il} - \Gamma^m_{il})\, g_{mj} \qquad \text{— } 12.7$$

We shall therefore adopt 12.12 as the equation defining the relation between the undor and the tensor covariant differentiation; and its solution giving Δ_l in any co-ordinate system of the type 12.3, we take as given by 12.11 together with 12.7.

(13) Calculation of Δ_l for an Orthogonal Metric.

We now consider a Riemannian metric, that is one for which

$$g_{ij;l} = 0 \qquad \text{— } 13.1$$

From 12.12 and 12.8 we find

$$\left.\begin{aligned} &\alpha_{i;l} = 0 \\ \text{and}\quad &A_{jil} + A_{ijl} = 0 \end{aligned}\right\} \text{— } 13.2$$

The affine connection Γ^i_{jk} is now given by the Christoffel brackets (see Eddington (37) p.58).

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} = \tfrac{1}{2} g^{im} \left\{ \frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right\} \text{— } 13.3$$

We shall further limit our metric to be of the form

$$\left.\begin{aligned} ds^2 &= h_1^2 (dx^1)^2 + h_2^2 (dx^2)^2 + h_3^2 (dx^3)^2 + h_4^2 (dx^4)^2 \\ &= h_m^2 (dx^m)^2 \end{aligned}\right\} - 13.4$$

where the h_m are any functions of the x^i.

In what follows the usual summation convention will not be used. Whether a suffix is summed or not will be clear from the context.

With the metric 13.4 we have the following results (where in 13.5-13.9 i, j and l are three <u>different</u> suffixes).

$$\left.\begin{array}{ll} g_{ii} = h_i^{\,2} & g^{ii} = \frac{1}{h_i^{\,2}} \\ g_{ij} = g^{ij} = 0 & \end{array}\right\} \quad \text{—— } 13.5$$

Also a solution for 10.5 of the form 12.3 is given by

$$\alpha_i = h_i \overset{0}{\alpha}_i \qquad \alpha^i = \frac{1}{h_i} \overset{0}{\alpha}_i \quad \text{—— } 13.6$$

and from 12.3 - 12.5 we obtain

$$\left.\begin{array}{ll} b^i_{\ i} = h_i & a^i_{\ i} = \frac{1}{h_i} \\ b^i_{\ j} = a^i_{\ j} = 0 & \end{array}\right\} \quad \text{—— } 13.7$$

and

$$\left.\begin{array}{ll} c^i_{\ ii} = \frac{1}{h_i}\frac{\partial h_i}{\partial x^i} & c^i_{\ il} = \frac{1}{h_i}\frac{\partial h_i}{\partial x^l} \\ c^i_{\ li} = c^i_{\ ll} = c^i_{\ jl} = 0 & \end{array}\right\} \quad \text{—— } 13.8$$

The values for the Christoffel bracket, 13.3, become

$$\left.\begin{array}{l} \Gamma^l_{ll} = \frac{1}{2} g^{ll} \frac{\partial g_{ll}}{\partial x^l} = \frac{1}{h_l}\frac{\partial h_l}{\partial x^l} \\ \Gamma^l_{jl} = \Gamma^l_{lj} = \frac{1}{2} g^{ll}\frac{\partial g_{ll}}{\partial x^j} = \frac{1}{h_l}\frac{\partial h_l}{\partial x^j} \\ \Gamma^i_{ll} = -\frac{1}{2} g^{ii}\frac{\partial g_{ll}}{\partial x^i} = -\frac{h_l}{h_i^2}\frac{\partial h_l}{\partial x^i} \end{array}\right\} \quad \text{—— } 13.9$$

$$\Gamma^i_{jl} = 0$$

From 12.11 and 12.7 we wish to calculate the value of

$$\Delta_l = \frac{1}{2} A_{rsl} I^{rs}$$

$$= \frac{1}{2}\left(c^n_{\ sl} - \Gamma^n_{sl}\right) g_{nr} I^{rs} \quad \text{—— } 13.10$$

(a) Consider the first term

$$\frac{1}{2} c^n_{\ sl}\, g_{nr} I^{rs}$$

$$= \frac{1}{2} c^s_{\ sl}\, g_{sr} I^{rs} \qquad \text{using 13.8}$$

$$= 0$$

as g_{sr} is symmetrical in r & s while I^{rs} is antisymmetrical.

(b) The second term gives

$$-\tfrac{1}{2}\,\Gamma^{m}_{s\ell}\,g_{mm}\,I^{\ell s} \qquad \text{(no summation over } \ell \text{)}$$

$$= -\tfrac{1}{2}\,\Gamma^{\ell}_{s\ell}\,g_{\ell\ell}\,I^{\ell s} - \tfrac{1}{2}\,\Gamma^{m}_{\ell\ell}\,g_{mm}\,I^{m\ell} \qquad \text{using 13.9 and 13.5}$$

$$= -\tfrac{1}{2}\left[\Gamma^{\ell}_{s\ell}\,g_{\ell\ell} - \Gamma^{s}_{\ell\ell}\,g_{ss}\right] I^{\ell s}$$

$$= -\tfrac{1}{2}\left[\frac{1}{h_\ell}\frac{\partial h_\ell}{\partial x^s}(h_\ell)^2 + \frac{h_\ell}{(h_s)^2}\frac{\partial h_\ell}{\partial x^s}(h_s)^2\right] I^{\ell s} \qquad \text{from 13.9}$$

$$= -h_\ell\,\frac{\partial h_\ell}{\partial x^s}\,I^{\ell s}$$

From 13.6 we find

$$I^{\ell s} = \frac{1}{h_\ell h_s}\,\overset{\circ}{I}_{\ell s}$$

So

$$\Delta_\ell = -\frac{1}{h_s}\,\frac{\partial h_\ell}{\partial x^s}\,\overset{\circ}{I}_{\ell s} \qquad \text{—— } 13{\cdot}11$$

(14) The Bhabha Equation in Orthogonal Co-ordinates

From 11.1 the generalized Bhabha equation is

$$\alpha^i\left(\frac{\partial}{\partial x^i} - \Delta_i\right)\psi + \chi\psi = 0 \qquad \text{—— } 14{\cdot}1$$

so using 13.6 and 13.11 this becomes

$$\frac{\overset{\circ}{\alpha}_i}{h_i}\left[\frac{\partial}{\partial x^i} + \frac{1}{h_s}\frac{\partial h_i}{\partial x^s}\,\overset{\circ}{I}_{is}\right]\psi + \chi\psi = 0 \qquad \text{—— } 14{\cdot}2$$

which is therefore the explicit form for the Bhabha equation, valid in any orthogonal co-ordinate system.

With $h_i = 1$, 14.2 gives just

$$\overset{\circ}{\alpha}_i\,\frac{\partial\psi}{\partial x^i} + \chi\psi = 0 \qquad \text{—— } 14{\cdot}3$$

which is the usual Bhabha equation in cartesian co-ordinates.

We now consider some special cases for the $\overset{\circ}{\alpha}$

matrices in 14.2.

(a) Dirac Case.

From 4.11 and 4.12 we have the relations

$$\left.\begin{aligned} \overset{\circ}{\alpha}_i &= \frac{\gamma_i}{2} \qquad \alpha = \frac{\kappa}{2} \\ \gamma_i\gamma_j &+ \gamma_j\gamma_i = 2\delta_{ij} \end{aligned}\right\} \qquad \text{— 14.4}$$

$$\therefore \quad \overset{\circ}{\Gamma}_{is} = \tfrac{1}{2}\gamma_i\gamma_s$$

and the equation 14.2 becomes

$$\frac{\gamma_i}{h_i}\left[\frac{\partial}{\partial x^i} + \frac{1}{2h_s}\frac{\partial h_i}{\partial x^s}\underset{i\neq s}{\gamma_i\gamma_s}\right]\psi + \kappa\psi = 0$$

or

$$\gamma_i\left[\frac{1}{h_i}\frac{\partial}{\partial x^i} + \underset{i\neq s}{\frac{1}{2h_ih_s}}\frac{\partial h_s}{\partial x^i}\right]\psi + \kappa\psi = 0 \qquad \text{— 14.5}$$

where we have used 14.4.

This is then the generalized form of the Dirac equation.

If we restrict our metric further by putting

$$h_4 = 1, \qquad x_4 = i4 \qquad \text{— 14.6}$$

then 14.5 gives

$$\gamma_a\left[\frac{1}{h_a}\frac{\partial}{\partial x^a} + \underset{a\neq b}{\frac{1}{2h_ah_b}}\frac{\partial h_b}{\partial x^a}\right]\psi - i\gamma_4\frac{\partial\psi}{\partial 4} + \kappa\psi = 0 \qquad \text{— 14.7}$$

where a, b, run 1 - 3, and this can be written

$$\gamma_a\left[\frac{1}{h_a}\frac{\partial}{\partial x^a} + \frac{1}{2h_a}\frac{\partial}{\partial x^a}\log\frac{h_1h_2h_3}{h_a}\right]\psi - i\gamma_4\frac{\partial\psi}{\partial 4} + \kappa\psi = 0 \qquad \text{— 14.8}$$

In this form it can be compared with an equation given by Frenkel (34) p.366. He has made a slip however and the tarm in the bracket there occurs with an incorrect sign.

It can be seen from 14.8 that the Dirac equation only leads to a second-order equation of the Klein-Gordon form when the h_i are constants, that is in the cartesian case. In general the Dirac equation does not lead to a second-order equation independent of the γ_i, and it is for this reason that the solution of the Dirac equation is extremely difficult in all but the simplest problems.

(b) Kemmer case.

Here we have

$$\left.\begin{array}{l} \overset{\circ}{\alpha}_i = \beta_i \qquad \alpha = \kappa \\ \beta_i\beta_j\beta_k + \beta_k\beta_j\beta_i = \delta_{ij}\beta_k + \delta_{kj}\beta_i \end{array}\right\} \quad - 14.9$$

and 14.9 gives

$$\beta_i \overset{\circ}{I}_{ij} = \beta_i(\beta_i\beta_j - \beta_j\beta_i) = \beta_i^2\beta_j$$

Equation 14.2 therefore yields

$$\frac{\beta_i}{h_i}\frac{\partial\psi}{\partial x^i} + \frac{1}{\underset{i\neq s}{h_i h_s}}\frac{\partial h_i}{\partial x^s}\beta_i^2\beta_s\psi + \kappa\psi = 0 \qquad — 14.10$$

This generalized form of the Kemmer equation has not so far been stated.

(15) Lagrangian Form of the Equations

In this section we restrict ourselves to metrics of the form 13.4 with $h_4 = 1$.

The equation 11.1 can then be derived from the Lagrangian

$$\mathcal{L} = \psi^{\times} D\alpha^i\psi_{;i} + \psi^{\times} D\alpha\psi \qquad — 15.1$$

where $\psi^{\times}$ is the complex conjugate of ψ and has similar properties except that under an S transformation

$$\bar{\psi}^{\times} = \psi^{\times} S \qquad \text{—— } 15.2$$

D is the Hermitian* matrix $\quad \text{const. } e^{i\pi\overset{\circ}{\alpha}_4}$ —— 15.3

and is introduced so that the two equations deducible from 15.1 by varying ψ and $\psi^{\times}$ should be consistent. As $\overset{\circ}{\alpha}_4$ satisfies the equation 4.11, we see that for a particle of maximum spin λ, D reduces to a polynomial in odd or even powers of $\overset{\circ}{\alpha}_4$ with the leading term $(\overset{\circ}{\alpha}_4)^{2\lambda}$.
e.g. in the Dirac case $\lambda = \frac{1}{2}$ and ~~11.1~~ 14.4 gives $(\overset{\circ}{\alpha}_4)^2 = \frac{1}{4}$

$$\therefore D = \text{const.} \left[1 + i\pi\overset{\circ}{\alpha}_4 - \frac{\pi^2}{4} - \frac{i\pi\overset{\circ}{\alpha}_4}{4} + \cdots \right]$$

and by suitable adjusting the constant this gives just $\overset{\circ}{\alpha}_4$, which is the usual result.

On varying 15.1 w.r.t. $\psi^{\times}$ we get

$$\alpha^{i}\psi_{;i} + \chi\psi = 0 \qquad \text{—— } 15.4$$

and by varying w.r.t. ψ we get

$$(\psi^{\times} D)_{;i}\,\alpha^{i} - \chi\psi^{\times} D = 0 \qquad \text{—— } 15.5$$

Bhabha (45b) shows that it is only when D has the form 15.3 that it is possible to derive 15.5 by taking the complex conjugate of 15.4, and thus it is only for this value of D that 15.4 and 15.5 are consistent.

The expression for the current vector follows in the usual way and is

$$j^{i} = i\,\psi^{\times} D\alpha^{i}\psi \qquad \text{—— } 15.6$$

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* This is so if $\overset{\circ}{\alpha}_4$ is taken to be anti-Hermitian, a procedure which is always possible.

The "canonical" energy-momentum tensor also follows simply from 15.1 and gives

$$T_i^{\ j} = \psi^\times D \alpha^j \psi_{;i} \qquad \text{——} \quad 15.7$$

The expression 15.7 satisfies all necessary conditions of conservation, but it is not symmetric. Pais (41) has given a method of forming the symmetric energy-momentum tensor in the Dirac case, and his method can easily be generalized to hold here also. We leave the actual calculation to the appendix (p.66) but the final result is

$$\Theta_{ij} = \tfrac{1}{2}\psi^\times D\left[\alpha_i \psi_{;j} + \alpha_j \psi_{;i}\right] + \tfrac{1}{2}\left\{\psi^\times D\left[\alpha_i I_j^{\ m} + \alpha_j I_i^{\ m}\right]\psi\right\}_{;m} \qquad \text{——} \quad 15.8$$

It can be verified that

$$\int T_{ij}\, d\tau = \int \Theta_{ij}\, d\tau \qquad \text{——} \quad 15.9$$

The introduction of the electromagnetic field follows straightforwardly by replacing

$$\frac{\partial}{\partial x^i} \quad \text{by} \quad \frac{\partial}{\partial x^i} - ie\varphi_i \qquad \text{——} \quad 15.10$$

It is possible to set up an analogous Lagrangian form of the equations in metrics more complicated then 13.4, but I can see no general way of determining the appropriate form for the matrix D in these cases.

(16) Expressions for the Spin Operator.

The following is the usual way of determining this operator.

The Bhabha equation in cartesian co-ordinates is given by 4.1, or

$$\alpha^a \frac{\partial\psi}{\partial x^a} - i\overset{\circ}{\alpha}_4 \frac{\partial\psi}{\partial t} + \varkappa\psi = 0 \qquad \text{--- } 16.1$$

where a run 1 - 3.

Using the commutation relations 4.2 we can find from 16.1 the three following first-order mutually commuting operators

$$\left.\begin{array}{ll} \text{(a)} & \overset{\circ}{\alpha}{}^a \dfrac{\partial}{\partial x^a} - i\overset{\circ}{\alpha}_4 \dfrac{\partial}{\partial t} \\ \text{(b)} & -i\dfrac{\partial}{\partial t} \\ \text{and (c)} & x^1 \dfrac{\partial}{\partial x^2} - x^2 \dfrac{\partial}{\partial x^1} + \overset{\circ}{I}_{12} \end{array}\right\} \qquad \text{--- } 16.2$$

We interpret (a) as the "rest-mass" operator, (b) as the Hamiltonian of the system or the total energy, while (c) is the operator relating to the z component of angular momentum. The first two terms of this operator are the ordinary orbital angular momentum, and the third is taken to represent the spin, showing that the spin operator is $\overset{\circ}{I}_{12}$ *, a result we presupposed in § (4) p. 26 .

This same problem can also be treated using the Bhabha equation in spherical-polar co-ordinates.

Here $\quad ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 - dt^2 \qquad$ --- 16.3

$$\left.\begin{array}{ll} \text{so} & x^1 = r, \quad x^2 = \theta, \quad x^3 = \varphi, \quad x^4 = it \\ \text{and} & h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta, \quad h_4 = 1 \end{array}\right\} \qquad \text{--- } 16.4$$

Substituting these values in 14.2 we obtain

$$\Big\{ \overset{\circ}{\alpha}_1 \frac{\partial}{\partial r} + \frac{1}{r}\left[\overset{\circ}{\alpha}_2 \overset{\circ}{I}_{21} + \overset{\circ}{\alpha}_3 \overset{\circ}{I}_{31}\right] + \frac{1}{r}\left[\overset{\circ}{\alpha}_2 \frac{\partial}{\partial\theta} + \cot\theta\, \overset{\circ}{\alpha}_3 \overset{\circ}{I}_{32}\right] + \overset{\circ}{\alpha}_3 \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} - i\overset{\circ}{\alpha}_4 \frac{\partial}{\partial t} \Big\}\psi + \varkappa\psi = 0 \qquad \text{--- } 16.5$$

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* multiplied perhaps by $-i$ to give real eigen-values.

The operators analogous to those in 16.2 are now

(a) operator in large brackets on left-side of 16.5

(b) $-i\frac{\partial}{\partial \lambda}$

and (c) $\frac{\partial}{\partial \varphi}$

The z component of angular momentum is now denoted merely by $\frac{\partial}{\partial \varphi}$, the usual result, and there is no explicit mention of the spin. This apparent contradiction with 16.2 arises because here the α matrices are vectors, which are continually changing in direction as they rotate with the r, θ, φ axes. To make the treatment comparable with that of 16.2 we must find the form of 16.6(c) referred to fixed axes. From §(4) p. 22 we know that the $\overset{o}{I}_{\rho\sigma}$ are the transformations of the rotation group in four dimensions, so the operator necessary to return to fixed α matrices is the transformation in the "φ" plane, that is $\overset{o}{I}_{12}$, which is the extra term found in 16.2.

From this point of view the concept of spin arises as a consequence of always referring the description of events to a fixed cartesian reference frame, instead of to one exhibiting the natural symmetry of the system under discussion. Or, in other words, when we say a particle has maximum spin λ we mean no more than that it is described by a Bhabha equation containing matrices $\overset{o}{\alpha}_i$ which have the maximum eigen-value of λ . This interpretation of spin in the Dirac case has previously been pointed out by Eddington (36) p.121.

Appendix

The form of the symmetric energy-momentum tensor given by Pais is

$$\Theta^{i}{}_{j} = \frac{\partial \mathcal{L}}{\partial Q_{\alpha;i}} Q_{\alpha;j} - \mathcal{L}\,\delta^{i}{}_{j} + \mathcal{R}^{ni}{}_{j;n} \quad \text{—— (1)}$$

where the Q_α are the undor variables occurring in the description of the field, $\mathcal{L}$ is the Lagrangian and $\mathcal{R}$ is defined by

$$\mathcal{R}^{ni}{}_{j} = \tfrac{1}{2} g_{jm}\left[\mathcal{D}^{n;im} - \mathcal{D}^{m;ni} + \mathcal{D}^{i;mn}\right] \quad \text{—— (2)}$$

where $$\mathcal{D}^{n;im} = \frac{\partial \mathcal{L}}{\partial Q_{\alpha;n}} I^{im} Q_\alpha \quad \text{—— (3)}$$

In our case (15.1)

$$\mathcal{L} = \psi^\times D\,\alpha^i \psi_{;i} + \psi^\times D\,\varkappa\,\psi \quad \text{—— (4)}$$

and the Q_α are ψ and $\psi^\times$.

Varying ψ & $\psi^\times$ gives the field equations 15.4 and 15.5

$$\left.\begin{aligned} &\alpha^i \psi_{;i} + \varkappa\psi = 0 \\ &(\psi^\times D)_{;i}\,\alpha^i - \varkappa\,\psi^\times D = 0 \end{aligned}\right\} \text{—— (5)}$$

Also from (3)

$$\mathcal{D}^{n;im} = \psi^\times D\,\alpha^n I^{im}\psi \quad \text{—— (6)}$$

and $$\mathcal{R}^{ni}{}_{j} = \tfrac{1}{2} g_{jm}\,\psi^\times D\left[\alpha^n I^{im} - \alpha^m I^{ni} + \alpha^i I^{mn}\right]\psi$$

$$= \tfrac{1}{2}\,\psi^\times D\left[\alpha^n I^{i}{}_{j} - \alpha_j I^{ni} + \alpha^i I_j{}^{n}\right]\psi \quad \text{—— (7)}$$

& $$\mathcal{R}^{ni}{}_{j;n} = \tfrac{1}{2}\left\{\psi^\times D\left[\alpha^n I^{i}{}_{j} - \alpha_j I^{ni} + \alpha^i I_j{}^{n}\right]\psi\right\}_{;n} \quad \text{—— (8)}$$

The first term of this is

$$\tfrac{1}{2}\left\{\psi^\times D\,\alpha^n I^{i}{}_{j}\,\psi\right\}_{;n}$$

$$= \tfrac{1}{2}(\psi^\times D)_{;m}\alpha^m I^i{}_j\psi + \tfrac{1}{2}\psi^\times D\alpha^m I^i{}_j\psi_{;m}$$

$$= \tfrac{1}{2}(\psi^\times D)_{;m}\alpha^m I^i{}_j\psi + \tfrac{1}{2}\psi^\times D[I^i{}_j\alpha^m + g^{mi}\alpha_j - g^m{}_j\alpha^i]\psi_{;m}$$

using 10.5

$$= \tfrac{\alpha}{2}\psi^\times D I^i{}_j\psi - \tfrac{\alpha}{2}\psi^\times D I^i{}_j\psi$$

$$+ \tfrac{1}{2}\psi^\times D[g^{mi}\alpha_j - g^m{}_j\alpha^i]\psi_{;m} \quad \text{—(9)}$$

where we have used (5)

$\therefore$ (1) gives

$$\Theta_{ij} = \psi^\times D\alpha_i\psi_{;j} + \tfrac{1}{2}\psi^\times D[g^m{}_i\alpha_j - g^m{}_j\alpha_i]\psi_{;m}$$

$$- \tfrac{1}{2}\{\psi^\times D[\alpha_j I^m{}_i - \alpha_i I_j{}^m]\psi\}_{;m}$$

$$= \tfrac{1}{2}\psi^\times D[\alpha_i\psi_{;j} + \alpha_j\psi_{;i}]$$

$$+ \tfrac{1}{2}\{\psi^\times D[\alpha_i I_j{}^m + \alpha_j I_i{}^m]\}_{;m} \quad \text{——— (10)}$$

CHAPTER 4

The Solution of the Dirac Equation

The usual method of solving the Dirac equation is to take some special representation of the γ matrices and write down explicitly the four first-order equations so resulting. These can be grouped into two pairs of equations in two unknown wave-functions, ψ_1 and ψ_2 say, and the solutions for these functions are obtained either by

(a) the method of simultaneous power series applied to the two equations (see Dirac (47) p.268 and Weyl (31) p.234).

(b) eliminating ψ_1 (or ψ_2) from the two equations, so obtaining a second-order equation in one dependent variable which can be solved by the ordinary theory of differential equations (see Taub (37), and Podolski (40)).

More satisfying methods not involving any particular representations of the γ matrices but depending only on their commutation relations have been given by Sauter and Temple. Sauter (30) uses an ingenious device which leads to two first-order equations analogous to those found above, and his treatment then follows the methods (a) or (b). Temple (30, 31, 34) has given two procedures. The first (30) applies the method of power series to the first-order equation involving the γ matrices while the second (31,34) involved the application of the algebraic theorem, * "any two mutually commuting operators must have a common eigen-function", in order to pass from the first-order equation to a second-order equation independent of the γ matrices.

In this chapter we will use the second method of Temple's, but will combine with it not the usual theory of second-order differential equations, but a recent treatment given by Schroedinger (40a, 41) for the determination of eigen-functions and eigen-values. The combination of these two procedures gives, I think, the simplest method of dealing with the Dirac equation.

The first problem treated is that of a particle moving in a homogeneous magnetic field. Solutions in terms of cartesian co-ordinates have been given by Huff (31) and von Laue (34), and the problem is of importance in discussions of the paths of electrons under such conditions, for example in the deflection experiments for the determination of $\frac{e}{m}$.

We shall give a solution using the natural cylindrical symmetry of the problem.

The second problem concerns the value for the energy-levels of the hydrogen atom in the hypersphere. Schroedinger (40a) has given the solution starting from

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* for a proof of this see Eddington (36) p.45.

the Schroedinger equation, but so far the proper relativistic treatment has not been given. As might be anticipated the extra effects due to space-curvature are very small, but the solution introduces questions of considerable theoretical interest. We shall solve the problem using the type of spherical co-ordinates applicable to the hypersphere.

(17) Notation for the Dirac Equation

We shall find it more convenient to use the notation of Eddington's E numbers (with squares $=-1$), instead of the usual Dirac matrices γ , as they enable commutation properties to be more easily recognized.

These fifteen E numbers have the properties (Eddington (46) p.107)

$$\left.\begin{aligned} &E_{\mu\nu} = -E_{\nu\mu} \\ &E_{\mu\nu}E_{\mu\nu} = -1 \\ &E_{\mu\sigma}E_{\nu\sigma} = -E_{\nu\sigma}E_{\mu\sigma} = E_{\mu\nu} \\ &E_{\mu\nu}E_{\sigma\tau} = E_{\sigma\tau}E_{\mu\nu} = iE_{\lambda\rho} \end{aligned}\right\} \quad — 17\cdot 1$$

where μ, ν run 0-5, and $\mu,\nu,\sigma,\tau,\lambda,\rho$ are an even permutation of 0,1,2,3,4,& 5.

We take the relation between these E numbers and the γ matrices to be given by

$$\gamma_1 = iE_{15} \quad \gamma_2 = iE_{25} \quad \gamma_3 = iE_{35} \quad \gamma_4 = iE_{45} \quad — 17\cdot 2$$

The Dirac equation for any orthogonal metric in the presence of an electromagnetic field then becomes from 14.5 and 15.10

$$iE_{i5}\left[\frac{1}{h_i}\left(\frac{\partial}{\partial x_i} - ie\varphi_i\right) + \frac{1}{2h_ih_s}\frac{\partial h_s}{\partial x_i}\right]\psi + \kappa\psi = 0 \qquad \text{— } 17.3$$
$$i \neq s$$

where e is the charge on the electron, and φ_i is the four-potential of the external electromagnetic field.

(18) <u>The Motion of an Electron in a Homogeneous Magnetic Field.</u>

In cylindrical co-ordinates the line-element is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 - dt^2 \qquad \text{— } 18.1$$

$$\therefore \left.\begin{array}{llll} x^1 = r & x^2 = \theta & x^3 = z & x^4 = it \\ h_1 = 1 & h_2 = r & h_3 = 1 & h_4 = 1 \end{array}\right\} \qquad \text{— } 18.2$$

The field is taken to be along the z axis - it is therefore described by the cartesian components

$$\varphi_x = \tfrac{1}{2}Hy \qquad \varphi_y = -\tfrac{1}{2}Hx \qquad \text{— } 18.3$$

where H is the strength of the magnetic field.

Using the relations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

we get from 18.3

$$\varphi_r = \varphi_1 = 0 \qquad \varphi_\theta = \varphi_2 = -\tfrac{1}{2}Hr^2 \qquad \varphi_z = \varphi_3 = 0 \qquad \text{— } 18.4$$

The Dirac equation 17.3 therefore becomes

$$iE_{15}\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right)\psi + iE_{25}\frac{1}{r}\left(\frac{\partial}{\partial\theta} + \frac{i\eta}{2}r^2\right)\psi$$
$$+ iE_{35}\frac{\partial\psi}{\partial z} + E_{45}\frac{\partial\psi}{\partial t} = -\kappa\psi \qquad \text{— } 18.5$$

where

$$\eta = He \qquad \text{— } 18.6$$

Now Eddington (36),p.140 has shown that the solution of the Dirac equation depends essentially on being able to find four mutually commuting operators for the problem in hand. As in quantum mechanics commutability and non-interference are almost synonymous, this means we can find four constants of the motion for our problem, to wit the four eigen-values of these operators, and the various solutions to the problem can then be classified in terms of these constants. This process is the analogue of the solution by separation of variables in the ordinary theory of partial differential equations.

From 18.5 the following four first-order mutually-commuting operators can be obtained.

$$\left.\begin{array}{ll} \text{(i)} & iE_{15}\left(\frac{\partial}{\partial r}+\frac{1}{2r}\right)+iE_{25}\frac{1}{r}\left(\frac{\partial}{\partial \theta}+\frac{i\eta}{2}r^2\right)+iE_{35}\frac{\partial}{\partial z}+E_{45}\frac{\partial}{\partial t} \\ \text{(ii)} & -i\frac{\partial}{\partial t} \\ \text{(iii)} & -i\frac{\partial}{\partial \theta} \\ \text{(iv)} & -i\frac{\partial}{\partial z} \end{array}\right\} \quad -18.7$$

The corresponding eigen-values will be denoted by

$$-\kappa \qquad \varepsilon \qquad m \qquad k \qquad — 18.8$$

The first of these relates to the rest-mass of the electron, the second to its total energy, the third to the z component of the angular momentum while the fourth represents the z component of the ordinary momentum.

We now deal in turn with the possible values of these quantities.

(a) <u>Value of κ</u> This value is unique and is given in the initial data to the problem.

(b) <u>Values of k</u> From 18.7 the equation to be satisfied

is
$$-i\frac{\partial\psi}{\partial z} = h\psi$$
$$\therefore \psi = e^{ihz}$$

and h can have any value whatsoever — 18.9

(c) Values of m.

We have
$$-i\frac{\partial\psi}{\partial\theta} = m\psi$$
$$\therefore \psi = e^{im\theta} \qquad \text{— 18.10}$$

The physics of the problem implies that the wave-function of ψ must be either single or double-valued, and we must now treat these two cases separately

(i) ψ single-valued

m must be integral — 18.11

(ii) ψ double-valued

m must be half-integral — 18.12

In both cases m may be either positive or negative.

(d) Values of ε

On substituting from 18.8 into 18.5 we obtain — 18.13
$$\left\{ iE_{15}\left(\frac{d}{dr}+\frac{1}{2r}\right) - E_{25}\frac{1}{r}\left(m+\frac{\eta r^2}{2}\right) - E_{35}h + iE_{45}\,\varepsilon \right\}\psi = -\kappa\psi$$

We now apply the operator within the large brackets on the left side of this equation (that is the operator 18.7(i)), to both sides of 18.13.

The result of this is the equation (where we have used 17.1)
$$\left\{ \left(\frac{d}{dr}+\frac{1}{2r}\right)^2 + \frac{1}{r^2}\left(m+\frac{\eta r^2}{2}\right)^2 + \varepsilon^2 - h^2 - \kappa^2 + iE_{12}\left(\frac{m}{r^2}-\frac{\eta}{2}\right)\right\}\psi = 0 \qquad \text{— 18.14}$$

Now the operator in large brackets on the left of

18.14 clearly commutes with E_{12} , so from the algebraic theorem (see p. 69) these two operators have a common eigen-function. Let this common eigen-function be Φ where

$$\left.\begin{aligned} \Phi &= (E_{12}+i)\,\psi \\ \text{or} \qquad \Phi &= (E_{12}-i)\,\psi \end{aligned}\right\} \qquad \text{—— } 18.15$$

according as E_{12} has the eigen-value $\pm i$.

Inserting Φ and the two possible eigen-values of E_{12} into 18.14, the following two second-order equations, independent of the E numbers, are obtained.

$$\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr} + \left\{ \varepsilon^2 - \kappa^2 - k^2 \pm \frac{\eta}{2} - m\eta - \frac{m^2 \pm m + \frac{1}{4}}{r^2} - \frac{\eta^2 r^2}{4} \right\}\Phi = 0 \qquad \text{—— } 18.16$$

In the next section we shall apply Schroedinger's method to find the possible values of ε allowed by the equation 18.16.

(19) Schroedinger's Method of Determining Eigen-Values.*

We write 18.16 in the form

$$r^2\frac{d^2\Phi}{dr^2} + r\frac{d\Phi}{dr} + \left\{ \lambda r^2 - (m^2 \pm m + \tfrac{1}{4}) - \frac{\eta^2 r^4}{4} \right\}\Phi = 0 \qquad \text{—— } 19.1$$

where $\lambda = \varepsilon^2 - \kappa^2 - k^2 \pm \frac{\eta}{2} - m\eta$ —— 19.2

and shall consider the equation as an eigen-value problem for λ , returning to ε through equation 19.2.

We first consider the equation with the positive signs in 19.1 and 19.2.

It can be verified that 19.1 can be written in the

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* A variant of this method has been given by Infeld(41)

two forms

$$\text{I} \quad \left(r\frac{d}{dr} + a - \frac{\eta r^2}{2}\right)\left(r\frac{d}{dr} - a + \frac{\eta r^2}{2}\right)\Phi + \left[a^2 - (m+\tfrac{1}{2})^2\right]\Phi = 0 \qquad \text{— } 19.3$$

and

$$\text{II} \quad \left(r\frac{d}{dr} - \overline{a+2} + \frac{\eta r^2}{2}\right)\left(r\frac{d}{dr} + \overline{a+2} - \frac{\eta r^2}{2}\right)\Phi + \left[(a+2)^2 - (m+\tfrac{1}{2})^2\right]\Phi = 0 \qquad \text{— } 19.4$$

where

$$\lambda = \eta(a+1) \qquad \text{— } 19.5$$

It can be seen that we pass from the factorization I to the factorization II by reversing the order of the factors and replacing a by $a+2$.

Now suppose we have found some solution Φ from 19.3 corresponding to a particular value of a, and consider the function

$$\Theta = \left(r\frac{d}{dr} + \overline{a+2} - \frac{\eta r^2}{2}\right)\Phi \qquad \text{— } 19.6$$

On multiplying 19.4 throughout by the operator $\left(r\frac{d}{dr} + \overline{a+2} - \frac{\eta r^2}{2}\right)$, we find Θ satisfies the equation

$$\left(r\frac{d}{dr} + \overline{a+2} - \frac{\eta r^2}{2}\right)\left(r\frac{d}{dr} - \overline{a+2} + \frac{\eta r^2}{2}\right)\Theta + \left[(a+2)^2 - (m+\tfrac{1}{2})^2\right]\Theta = 0$$

so, from 19.3 Θ must be the eigen-function corresponding to the value $a+2$ of a.

That is, starting with any solution Φ corresponding to a particular value of a, we can find solutions corresponding to the values $a+2, a+4, a+6$ etc., by operating on Φ with

$$r\frac{d}{dr} + \overline{a+2} - \frac{\eta r^2}{2} \qquad \text{— } 19.7$$

Similarly it can be shown that solutions corresponding to the values $a-2, a-4$ etc., are obtained by operating on Φ with

$$r\frac{d}{dr} - a + \frac{\eta r^2}{2} \qquad \text{—— } 19.8$$

We call 19.7 and 19.8 the two "manufacturing" operators for the differential equation 19.1

A solution of 19.3 to begin with can be obtained by putting *

$$a = m + \tfrac{1}{2} \qquad \text{—— } 19.9$$

The corresponding Φ then satisfies

$$\left(r\frac{d}{dr} - a + \frac{\eta r^2}{2}\right)\Phi = 0$$

so

$$\Phi = r^a e^{-\frac{\eta r^2}{2}} \qquad \text{—— } 19.10$$

By repeated operation on 19.10 with the manufacturing operator 19.7, we therefore find a series of eigen-functions of equation 19.1 corresponding to the following eigen-values of a.

$$\left.\begin{array}{l} a = m+\tfrac{1}{2},\ m+\tfrac{5}{2},\ m+\tfrac{9}{2},\ \text{---}\quad m+2l+\tfrac{1}{2},\ \text{--} \\ \text{where } l \text{ is integral. } (a+ve) \end{array}\right\} \text{— } 19.11$$

We want now to show that these are the <u>only</u> eigen-functions and eigen-values of the equation 19.1.

Let us suppose there is some eigen-function corresponding to a value of a not included in 19.11. By repeated application with the operator 19.8 we can obtain the eigen-functions corresponding to $a-2$, $a-4$ etc., and this downward ladder of eigen-values will not stop unless at some step the application of 19.8 annihilates the last bracket in 19.3 - that is if $a^2 = (m+\frac{1}{2})^2$. But this is just the case corresponding to our ladder of eigen-values 19.11. Therefore if we can show that the

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* using the fact that $\int \Phi^2 d\tau$ must be finite, it can be shown that the other initial solution with $a = -(m+\frac{1}{2})$ leads to the same energy-levels.

downward ladder of eigen-values <u>must</u> stop somewhere, then the only possibility is that the ladder is that in 19.11.

To show this we multiply 19.1 on the left by $\frac{\Phi}{r}$ and integrate from $0-\infty$. After performing a partial integration we obtain

$$\left[r\Phi\frac{d\Phi}{dr}\right]_0^\infty - \int_0^\infty r\left(\frac{d\Phi}{dr}\right)^2 dr + \int_0^\infty \frac{1}{r}\left[\lambda r^2 - (m+\tfrac{1}{2})^2 - \frac{\eta^2 r^4}{4}\right]\Phi^2 dr = 0 \qquad \text{—— } 19.12$$

Now the boundary conditions imply that $\Phi=0$ when $r=\infty$ so the first term is zero, while the second is clearly negative. The third term must then be positive, so the expression in the brackets there must be positive for some value of r,

i.e. $\lambda^2 > 4\frac{\eta^2}{4}(m+\frac{1}{2})^2$

or $\lambda^2 > \eta^2(m+\frac{1}{2})^2$

or $\eta^2(a+1)^2 > \eta^2(m+\frac{1}{2})^2$ (from 19.5)

so $(a+1)^2 > (m+\frac{1}{2})^2$ —— 19.13

This means that (except perhaps for $m=0$) the downward ladder for a <u>must</u> stop, and it stops therefore with $a=m+\frac{1}{2}$, or with the eigen-function 19.10. The only possible eigen-values to the problem are then those given by 19.11. To cover the case $m=0$ one must unfortunately refer to the ordinary power series solution about $r=0$ in order to show the exhaustiveness.

From 19.2 and 19.11 we find for the possible values of ε

$\lambda = \varepsilon^2-\kappa^2-h^2+\frac{\eta}{2}-m\eta = \eta(m+2l+\frac{1}{2}+1)$

or $\frac{\varepsilon^2-\kappa^2-h^2}{\eta} = 2m+2l+1$ —— 19.14

We must now combine 19.14 with the possible values for m in 18.11 and 18.12.

We notice first that the physics of the problem implies that $\int \Phi^2 d\tau$ must be finite.

Using 19.10 this gives the result

$$\left.\begin{array}{l} a+1>0 \\ \text{or} \quad m+\frac{3}{2}>0 \end{array}\right\} \qquad 19.15$$

so the smallest value m can have is -1, all other negative values being impossible.

There are now two cases to consider

(i) ψ single-valued, m integral and ≥ -1

From 18.16 and 19.14 the possible energy values are

$$\varepsilon^2 - \kappa^2 - \ell^2 = 4e(2s+1) \qquad 19.16$$

where $s = m + \ell$ and has the values -1,0,1,2 etc.

Although the case $s=-1$ corresponding to $m=-1$, $\ell=0$ is mathematically possible, yet physically it refers to negative values for $\varepsilon^2-\kappa^2$ which would seem impossible for this problem.

(ii) ψ double-valued, m half-integral and $\geq -\frac{1}{2}$.

The energy values are here

$$\varepsilon^2 - \kappa^2 - \ell^2 = 4e \cdot 2s \qquad 19.17$$

where $s = m + \ell + \frac{1}{2}$ and has the values 0,1,2, etc.

The possibility of the solutions corresponding to the double-valued wave-functions was first pointed out by Temple (31) p.106,131, and by Eddington(36) p.60,150, and later Schroedinger (38b) showed that these two sets of solutions could not co-exist in any particular problem.

These alternative solutions introduce an ambiguity into the solution of quantum-mechanical problems which up to the present, theoretical considerations have been unable to remove, the only criterion in choosing between them being the practical one of comparison with experiment. Schroedinger compares the situation with regard to this ambiguity with the similar one which until recently existed in the association of the Fermi-Dirac or Bose-Einstein statistics with assemblies of identical particles. In that case Pauli (40) has now shown that a knowledge of the spin of the particles under consideration is sufficient to uniquely determine the form of the statistics describing them. Here however the ambiguity seems to concern something innate in the form of the Dirac equation itself. Moreover in addition to the ambiguity mentioned above, we shall see in §(21), p. 87, that by solving the Dirac equation in one co-ordinate system using the single-valued wave-functions, we get the same energy-values as when we solve the equation in a different co-ordinate system and yet use the double-valued functions. There is presumably some reason for this contained in the structure of the Dirac equation, but asyet it has not been pointed out.

For our problem the experimental evidence is inconclusive and either of the sets 19.16 or 19.17 (which only differ by a constant) explain the results equally well. The result found by Huff (31) coincides with that from the single-valued wave functions, that is 19.16. A full account of the applications of these results to the deflection experiments will be found in the papers of Huff and von Laue (34), and also in a paper by Leigh-Page (30), where the treatment using the Schroedinger equation is given and discussed.

The treatment of the other equation contained in 19.1 with the negative signs has still to be given, but using identical methods to those above, it can be shown that it leads to exactly the same eigen-functions and eigen-values and so yields nothing new.

(20) The Dirac Equation for the Hydrogen Atom in the Hypersphere.

We shall now discuss the problem of finding the energy-levels of the hydrogen atom in the hypersphere. We take axes co-moving with the proton in the hydrogen atom, and take as line element (Eddington (37) p.156).

$$ds^2 = R^2 d\chi^2 + R^2 \sin^2\chi \, d\theta^2 + R^2 \sin^2\chi \sin^2\theta \, d\varphi^2 - dt^2 \qquad \text{— 20.1}$$

Here χ is an angular variable ranging from $0 - \pi$; its relation to the usual "flat" radial variable r is given by

$$\left.\begin{matrix} R\sin\chi \\ R\chi \end{matrix}\right\} \to r \quad \text{as} \quad R \to \infty \qquad \text{— 20.2}$$

R (a constant) is the radius of the hypersphere. From 20.1 we obtain

$$\left.\begin{aligned} & x^1 = \chi \qquad x^2 = \theta \qquad x^3 = \varphi \qquad x^4 = it \\ & g_{11} = R^2 \quad g_{22} = R^2\sin^2\chi \quad g_{33} = R^2\sin^2\chi\sin^2\theta \quad g_{44} = 1 \\ & g^{11} = \frac{1}{R^2} \quad g^{22} = \frac{1}{R^2\sin^2\chi} \quad g^{33} = \frac{1}{R^2\sin^2\chi\sin^2\theta} \quad g^{44} = 1 \\ & g = g_{11}g_{22}g_{33}g_{44} = R^6\sin^4\chi\sin^2\theta \end{aligned}\right\} \text{— 20.3}$$

$$\sqrt{g} = R^3\sin^2\chi\sin\theta$$

$$h_1 = R \quad h_2 = R\sin\chi \quad h_3 = R\sin\chi\sin\theta \quad h_4 = 1 \qquad \text{— 20.3 (cont.)}$$

We must next determine the proper form for the potential between the proton and the electron in the metric 20.1.

This potential is assumed to be static and symmetrical, that is, it has the form

$$\left.\begin{array}{l} \varphi_1 = \varphi_2 = \varphi_3 = 0 \\ \varphi_4 \text{ a function only of } \chi \;, \end{array}\right\} \qquad \text{— 20.4}$$

Denoting the electromagnetic field strengths by φ_{ij} where

$$\varphi_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i} \qquad \text{— 20.5}$$

we find from 20.4 that the only surviving components are

$$\varphi_{41} = -\varphi_{14} = \frac{\partial \varphi_4}{\partial \chi} = \varphi_4' \quad \text{(say)} \qquad \text{— 20.6}$$

Now the field equations are (Eddington (37) p.173)

$$\frac{\partial(\varphi^{ij}\sqrt{g})}{\partial x^j} = \mathfrak{J}^i \qquad \text{— 20.7}$$

where $\varphi^{ij}\sqrt{g}$ is the tensor-density corresponding to φ^{ij} and $\mathfrak{J}^i$ is the tensor-density of the current vector.

Using 20.3 and 20.6 we have

$$\varphi^{41} = g^{44} g^{11} \varphi_{41} = \frac{\varphi_4'}{R^2}$$

$$\therefore \quad \varphi^{41}\sqrt{g} = R\sin^2\chi\sin\theta\,\varphi_4'$$

Therefore, from 20.7, the condition there will be no charge or current (except at the proton in the origin) is that

$$\frac{\partial(\varphi^{41}\sqrt{g})}{\partial x^1} = \frac{\partial}{\partial \chi}\left(R\sin^2\chi\sin\theta\,\varphi_4'\right) = 0$$

so

$$\varphi_4 = \frac{e}{R \sin^2 \chi}$$

or

$$\varphi_4 = -\frac{e}{R} \cot \chi \qquad \text{--- } 20.8$$

where e is some constant, and as in the limit $R \to \infty$, $\varphi_4 \to -\frac{e}{r}$, it can be seen e is the usual charge on the proton.

Substituting now from 20.8 and 20.3 into the Dirac equation 17.3 we obtain

$$\Big\{ \frac{-iE_{15}}{R} \left(\frac{\partial}{\partial \chi} + \cot \chi \right) + \frac{iE_{25}}{R \sin \chi} \left(\frac{\partial}{\partial \theta} + \tfrac{1}{2} \cot \theta \right)$$

$$+ \frac{-iE_{35}}{R \sin \chi} \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} + iE_{45} \left(-i \frac{\partial}{\partial t} + \frac{\alpha \cot \chi}{R} \right) \Big\} \psi = -\kappa \psi \qquad \text{--- } 20.9$$

where

$$\alpha = e^2 = \frac{1}{137} \qquad \text{--- } 20.10$$

and is the fine-structure constant ($= \frac{e^2}{\hbar c}$ in usual units).

The equation 20.9 admits of the following four first-order mutually commuting operators.

(i) complete operator on the left-side of 20.9

(ii) $-E_{03} \left(\frac{\partial}{\partial \theta} + \frac{1}{2} \cot \theta \right) + E_{02} \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi}$

(iii) $-i \frac{\partial}{\partial \varphi}$

(iv) $-i \frac{\partial}{\partial t}$

--- 20.11

The corresponding eigen-values are denoted by

$$-\kappa, \quad j \quad m \quad \& \quad \mathcal{E} \qquad \text{--- } 20.12$$

The first relates to the rest-mass, the second to the total angular momentum, the third to the z component of angular momentum, while the last is the total energy of the system.

In the next sections we shall discuss the possible values of these quantities.

(21) The Eigen-Values of κ, m & γ.

(i) Value of κ This value is unique and is a given constant of the problem.

(ii) Values of m The equation to be satisfied is

$$-i\frac{\partial\psi}{\partial\varphi} = m\psi$$

so

$$\psi = e^{-im\varphi}$$

and as in the previous problem there are two cases to be considered

(a) ψ single-valued, m integral — 21.1

and (b) ψ double-valued, m half-integral. — 21.2

(iii) Values of γ

From 20.11 we find the equation

$$\left[-iE_{03}\left(\frac{\partial}{\partial\theta}+\frac{1}{2}\cot\theta\right) - mE_{02}\frac{1}{\sin\theta}\right]\psi = i\gamma\psi \qquad \text{— 21.3}$$

Schroedinger (41) has shown that this equation is a slightly more general one than that considered by Weyl (31) p.230, for his "spherical harmonics with spin". We shall however treat the equation by our method.

Applying the operator in the large bracket to both sides of 21.3 we obtain, after some simplification, the second-order equation

$$\left\{\frac{d^2}{d\theta^2} + \cot\theta\frac{d}{d\theta} + \left[\gamma^2 - \frac{1}{4} - \frac{m^2 + iE_{23}m\cos\theta + \frac{1}{4}}{\sin^2\theta}\right]\right\}\psi = 0$$

— 21.4

~~(21) Eigen-Values of ________.~~

~~(i) The value of is unique and is a given piece of data for the problem.~~

~~(ii)~~ The operator in the large bracket in 21.4 commutes with E_{23}, so these two operators must have a common eigen-function Φ, where

$$\left.\begin{aligned} \Phi &= (E_{23}+i)\psi \\ \text{or} \quad \Phi &= (E_{23}-i)\psi \end{aligned}\right\} \qquad \text{—— } 21.5$$

according as E_{23} has the eigen-value $\pm i$.

Inserting Φ, and the two eigen-values $\pm i$ for E_{23}, we obtain from 21.4 the two equations

$$\sin^2\theta \frac{d^2\Phi}{d\theta^2} + \sin\theta\cos\theta \frac{d\Phi}{d\theta} + \left[(j^2-\tfrac{1}{4})\sin^2\theta - (m^2 \pm m\cos\theta + \tfrac{1}{4})\right]\Phi = 0 \qquad \text{—— } 21.6$$

where we have multiplied throughout by $\sin^2\theta$.

These are now the equations for which we must determine the possible values of j.

The following properties of 21.6 are apparent:

(1) Any solution corresponding to a value $+j$ is identical with that for $-j$.

(2) A solution of one of the equations 21.6 corresponding to a value $+m$ is identical with that of the other equation corresponding to the value $-m$. —— 21.7

We shall therefore investigate the properties of both of the equations 21.6 and restrict ourselves to positive values of m and j. The negative solutions can then be deduced from the relations 21.7 above.

(a) <u>"Positive" equation from 21.6</u>

The factorizations of this equation are

I $\left[\sin\theta\frac{d}{d\theta}+(\gamma-\tfrac{1}{2})\cos\theta+b_1\right]\left[\sin\theta\frac{d}{d\theta}-(\gamma-\tfrac{1}{2})\cos\theta+b_1\right]\Phi$

$$+\left[(\gamma-\tfrac{1}{2})^2+b_1^2-m^2-\tfrac{1}{4}\right]\Phi=0 \qquad \text{—— } 21.8$$

and

II $\left[\sin\theta\frac{d}{d\theta}-(\gamma+\tfrac{1}{2})\cos\theta-b_2\right]\left[\sin\theta\frac{d}{d\theta}+(\gamma+\tfrac{1}{2})\cos\theta+b_2\right]\Phi$

$$+\left[(\gamma+\tfrac{1}{2})^2+b_2^2-m^2-\tfrac{1}{4}\right]\Phi=0 \qquad \text{—— } 21.9$$

where $b_1=\dfrac{m}{2(\gamma-\frac{1}{2})}$ $\qquad b_2=\dfrac{m}{2(\gamma+\frac{1}{2})}$ —— 21.10

The factorization II is obtained from I by changing the order of the factors and replacing γ by $\gamma+1$. The manufacturing operator which changes an eigen-function corresponding to some value of γ, to an eigen-function corresponding to the value $(\gamma+1)$ is

$$\sin\theta\frac{d}{d\theta}+(\gamma+\tfrac{1}{2})\cos\theta+\frac{m}{2(\gamma+\frac{1}{2})} \qquad \text{—— } 21.11$$

A solution to begin with is obtained by putting

$$(\gamma-\tfrac{1}{2})^2+b_1^2-m^2-\tfrac{1}{4}=0$$

i.e. $\qquad (\gamma-\tfrac{1}{2})^2+\dfrac{m^2}{4(\gamma-\frac{1}{2})^2}-m^2-\tfrac{1}{4}=0$

so $\qquad \gamma-\tfrac{1}{2}=m$ —— 21.12

<u>unless</u> $\quad m=0 \quad$ when $\quad \gamma=0$. —— 21.13

The first allowed value of γ is therefore $\gamma=m+\frac{1}{2}$ and the corresponding eigen-function is given by

$$\sin\theta\frac{d\Phi}{d\theta}-m\cos\theta\,\Phi-\tfrac{1}{2}\Phi=0$$

so $\qquad \Phi=(\tan\tfrac{\theta}{2})^{\frac{1}{2}}\sin^m\theta$ —— 21.14

except when $m=0$ when the solution is

$$\Phi = \left(\sin\frac{\theta}{2}\right)^{-\frac{1}{2}} \qquad \text{—— } 21.15$$

By operating on 21.14 with 21.11 we find the following ladder of eigen-values for j ,

$$j = m+\tfrac{1}{2},\ m+\tfrac{3}{2},\ m+\tfrac{5}{2},\ -- \qquad \text{—— } 21.16$$

together with $j = 0, 1, 2$ etc. relating to $m=0$

It must next be proved that this is the <u>only</u> ládder of eigen-values satisfying the equation 21.6. To do this we must prove that the downward ladder of eigen-functions starting from any function Φ <u>must</u> stop, for this can only happen when the manufacturing operator annihilates the last bracket in 21.8 - but if this is so the last eigen-function obtained must be 21.14, and so the ladder must be our ladder.

If we multiply 21.~~8~~6 on the left by $\frac{\Phi}{\sin\theta}$ and integrate from $\theta = 0 - \pi$, we find* in a similar way to that of 19.12 that

$$j^2 > \frac{m^2}{2} + \frac{3}{8} \qquad \text{—— } 21.17$$

So the downward ladder must stop and the only possible eigen-values are those given in 21.16.

(b) <u>"Negative" equation in 21.6</u>

The procedure can be carried out exactly as for the positive case. The eigen-values for j are again those given in 21.16 but now the manufacturing operator is

$$\sin\theta\frac{d}{d\theta} + (j+\tfrac{1}{2})\cos\theta - \frac{m}{2(j+\tfrac{1}{2})} \qquad \text{—— } 21.18$$

* This integration implies that $(j^2-\frac{1}{4})\sin^2\theta - (m^2 + m\cos\theta + \frac{1}{4})$ should be +ve for some value of θ. This leads to the result 21.17 , also precludes the solution $j=0$, $m=0$.

and the initial eigen-function satisfies the equation

$$\sin\theta \frac{d\Phi}{d\theta} - m\cos\theta\,\Phi + \tfrac{1}{2}\Phi = 0$$

and so is

$$\Phi = \left(\cot\tfrac{\theta}{2}\right)^{\frac{1}{2}} \sin^m\theta \qquad \text{——— } 21.19$$

Using 21.1, 21.2 and 21.16 we can now discuss the possible combinations of m and j.

(1) <u>m integral, j half-integral.</u> ——— 21.20

For any particular value of j, m can have any of the $2j$ values

$$m = j-\tfrac{1}{2},\ j-\tfrac{3}{2},\ \text{---}\quad -j+\tfrac{1}{2}$$

These solutions, together with $j=0$, $m=0$, give the possible eigen-values corresponding to the single-valued wave-functions, and from above we see they split into j pairs of solutions with positive and negative values of m. (there is no pair however for $j=0, m=0$).

(2) <u>m half-integral, j integral.</u> ——— 21.21

These correspond to the double-valued wave functions, and again we see there are j pairs of solutions for any particular value of j.

In this problem experimental evidence distinguishes decisively between 21.20 and 21.21. The erroneous Klein-Gordon discussion for the hydrogen atom had quantum numbers with the combinations 21.20, while the verified Dirac discussion has the combination 21.21 corresponding to the double-valued wave functions. This result is surprising as the usual treatment of the Dirac equation in cartesian co-ordinates (e.g. see Eddington

(46) p.222) obtain the correct Sommerfeld formula using single-valued wave functions. This is the circumstance discussed on p. 79 , and as stated there seems to point to some ambiguity in the form of the Dirac equation.

(22) <u>The Values of ε .</u>

From 20.9 we obtain the equation

$$\left\{ iE_{15}\left(\frac{d}{d\chi}+\cot\chi\right)+\frac{E_{14}\,j}{\sin\chi}+iE_{45}(\bar{\varepsilon}+\alpha\cot\chi)\right\}\psi=-\bar{\kappa}\psi \qquad \text{—— } 22.1$$

where $\quad \bar{\varepsilon}=R\varepsilon \qquad \bar{\kappa}=R\kappa \qquad$ —— 22.2

We first change the dependent variable in 22.1 from ψ to $\bar{\psi}$, where

$$\bar{\psi}=\sin^{\frac{1}{2}}\chi\cdot\psi \qquad \text{—— } 22.3$$

Equation 21.1 then becomes on multiplying throughout by $\sin\chi$.

$$\left\{ iE_{15}\left(\sin\chi\frac{d}{d\chi}+\tfrac{1}{2}\cos\chi\right)+E_{14}\,j+iE_{45}(\bar{\varepsilon}\sin\chi+\alpha\cos\chi)\right\}\bar{\psi}$$
$$=-\bar{\kappa}\sin\chi\cdot\bar{\psi} \qquad \text{—— } 22.4$$

Now operating in the usual way with the function in the large bracket in 22.4, we obtain the following second-order equation

$$\sin^2\chi\frac{d^2\bar{\psi}}{d\chi^2}+2\sin\chi\cos\chi\frac{d\bar{\psi}}{d\chi}+\sin^2\chi\left(-\tfrac{3}{4}+\bar{\varepsilon}^2-\alpha^2-\bar{\kappa}^2+E_{14}\alpha\right)\bar{\psi}$$
$$+\sin\chi\cos\chi\left(2\alpha\bar{\varepsilon}-\bar{\varepsilon}E_{14}+i\bar{\kappa}E_{15}\right)\bar{\psi}+\left(\tfrac{1}{4}-j^2+\alpha^2\right)\bar{\psi}=0$$
$$\text{—— } 22.5$$

The term in 21.26 which causes difficulty is the $E_{14}\alpha$ term in the first bracket. However as $\alpha=\frac{1}{137}$ and E_{14} has modulus unity, we see that compared to the terms $\bar{\varepsilon}^2$ and $\bar{\kappa}^2$ which contain a factor R^2

(with $R \approx 10^4$ in our units), its effect will be negligible. For the present we will therefore neglect this term along with the term $\alpha^2 = \frac{1}{(137)^2}$, in the same bracket, and shall consider the possible errors involved later (p. 95).

On neglecting the term $E_{14}\alpha$ it can be seen that the operator on the left side of 22.5 commutes with $i\bar{\kappa}E_{15} - \bar{\varepsilon}E_{14}$, so these two operators must have a common eigen-function Φ, where

$$\Phi = \{i\bar{\kappa}E_{15} - \bar{\varepsilon}E_{14} \pm (\bar{\kappa}^2 - \bar{\varepsilon}^2)^{\frac{1}{2}}\}\bar{\Psi} \qquad \text{—— } 22.6$$

On inserting Φ and the two possible eigen-values of $i\bar{\kappa}E_{15} - \bar{\varepsilon}E_{14}$ in equation 22.5 we obtain

$$\sin^2\chi \frac{d^2\Phi}{d\chi^2} + 2\sin\chi\cos\chi\frac{d\Phi}{d\chi} + \{\lambda\sin^2\chi + 2\mu\sin\chi\cos\chi \pm \eta\}\Phi = 0 \qquad \text{—— } 22.7$$

where

$$\left.\begin{aligned} \lambda &= -\tfrac{3}{4} + \bar{\varepsilon}^2 - \bar{\kappa}^2 \\ 2\mu &= 2\alpha\bar{\varepsilon} \pm (\bar{\kappa}^2 - \bar{\varepsilon}^2)^{\frac{1}{2}} \\ \eta &= \tfrac{1}{4} - j^2 + \alpha^2 \end{aligned}\right\} \qquad \text{—— } 22.8$$

We shall only consider the case with the positive sign in 22.8 - it can be verified that the negative sign leads to the same eigen-values for the energy.

The solution * to the equation 22.7 has been given

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* A solution in terms of the theory of differential equations has also been given by Stevenson (41).

by Schroedinger (40a). The appropriate factorisations are

I $\left(\sin\chi \frac{d}{d\chi} + a\cos\chi - b\sin\chi\right)\left(\sin\chi \frac{d}{d\chi} - \overline{a-1}\cos\chi + b\sin\chi\right)\Phi$

$$+ \left[a(a-1) + \eta\right]\Phi = 0 \qquad \text{—— } 22.9$$

and

II $\left(\sin\chi \frac{d}{d\chi} - a\cos\chi + b\sin\chi\right)\left(\sin\chi \frac{d}{d\chi} + \overline{a+1}\cos\chi - b\sin\chi\right)\Phi$

$$+ \left[(a+1)a + \eta\right]\Phi = 0 \qquad \text{—— } 22.10$$

where $\left.\begin{aligned} \lambda &= (a-1)(a+1) - b^2 \\ \mu &= ab \end{aligned}\right\}$ —— 22.11

In a similar way to our previous examples it can be shown that the manufacturing operator is

$$\sin\chi \frac{d}{d\chi} + (a+1)\cos\chi - b\sin\chi \qquad \text{—— } 22.12$$

and a solution to begin with occurs when

$$a(a-1) + \eta = 0$$

i.e. from 22.8 when

$$a(a-1) = \gamma^2 - \alpha^2 - \tfrac{1}{4}$$

and if we put* $\rho^2 = \gamma^2 - \alpha^2 \qquad \rho = +(\gamma^2 - \alpha^2)^{\frac{1}{2}}$ —— 22.13

then $a(a-1) = \rho^2 - \tfrac{1}{4} = (\rho + \tfrac{1}{2})(\rho - \tfrac{1}{2})$

so $a = \rho + \tfrac{1}{2}$ —— 22.14

The corresponding eigen-function satisfies

$\sin\chi \frac{d\Phi}{d\chi} - (a-1)\cos\chi\,\Phi + b\sin\chi\,\Phi = 0$ with $a = \rho + \tfrac{1}{2}$

and is therefore

$$\Phi = \sin^{\rho - \frac{1}{2}}\chi\, e^{-b\chi} \qquad \text{—— } 22.15$$

By operating on 22.15 with 21.33 we obtain the series

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* the negative value of the square-root is not possible as $\int \Phi^2 d\tau$ must be finite.

of eigen-values for a ,

$$\left.\begin{array}{l} a = \rho+\tfrac{1}{2},\ \rho+\tfrac{3}{2},\ \rho+\tfrac{5}{2},\ --\quad \rho+n+\tfrac{1}{2} \\ \text{where } n \text{ is integral and has values } 0,1,2 \text{ etc.} \end{array}\right\} \quad \text{——} 22.16$$

In a similar way to the previous examples it can again be shown that these are the only eigen-values for the problem.

We now wish to find the values of ε corresponding to the above values of a . From 22.11, 22.16 and 22.8 we obtain

$$2\,ab = 2b(\rho+n+\tfrac{1}{2}) = 2\alpha\bar{\varepsilon} + (\bar{\kappa}^2-\bar{\varepsilon}^2)^{\frac{1}{2}}$$

so $$4b^2(\rho+n+\tfrac{1}{2})^2 = \{2\alpha\bar{\varepsilon} + (\bar{\kappa}^2-\bar{\varepsilon}^2)^{\frac{1}{2}}\}^2$$

Also $$\lambda = (a-1)(a+1) - b^2$$

so $$-\tfrac{3}{4} + \bar{\varepsilon}^2 - \bar{\kappa}^2 = (\rho+n-\tfrac{1}{2})(\rho+n+\tfrac{3}{2}) - \frac{1}{4(\rho+n+\tfrac{1}{2})^2}\{2\alpha\bar{\varepsilon} + (\bar{\kappa}^2-\bar{\varepsilon}^2)^{\frac{1}{2}}\}^2$$

or $$4(\rho+n+\tfrac{1}{2})^2\{\bar{\kappa}^2-\bar{\varepsilon}^2+(\rho+n)(\rho+n+1)\}$$
$$= \{2\alpha\bar{\varepsilon} + (\bar{\kappa}^2-\bar{\varepsilon}^2)^{\frac{1}{2}}\}^2$$

Using now 22.2 and dividing throughout by R^2 we get

$$4(\rho+n+\tfrac{1}{2})^2\left\{\kappa^2-\varepsilon^2+\frac{(\rho+n)(\rho+n+1)}{R^2}\right\}$$
$$= \{2\alpha\varepsilon + (\kappa^2-\varepsilon^2)^{\frac{1}{2}}\}^2. \qquad \text{——} 22.17$$

This equation therefore contains the solutions determining the possible energy values for the hydrogen atom in the hypersphere.

(23) <u>Discussion of the Result</u>.

We take some special cases of 22.17.

(a) $R \to \infty$. That is we neglect curvature, and the result should be that given by the Sommerfeld formula in the flat case.

From 22.17 we obtain

$$4(\rho+n+\tfrac{1}{2})^2(\kappa^2-\varepsilon^2) = \{2\alpha\varepsilon + (\kappa^2-\varepsilon^2)^{\frac{1}{2}}\}^2$$

or $$2(\rho+n+\tfrac{1}{2})(\kappa^2-\varepsilon^2)^{\frac{1}{2}} = 2\alpha\varepsilon + (\kappa^2-\varepsilon^2)^{\frac{1}{2}}$$

so $$2(\rho+n+\tfrac{1}{2}) = 2\rho+2n+1 = \frac{2\alpha\varepsilon}{(\kappa^2-\varepsilon^2)^{\frac{1}{2}}} + 1$$

and substituting for ρ from 22.13 this gives

$$(j^2-\alpha^2)^{\frac{1}{2}} + n = \frac{\alpha\varepsilon}{(\kappa^2-\varepsilon^2)^{\frac{1}{2}}} \qquad \text{——— 23.1}$$

which together with the values of j given in 21.21, and those for n in 22.16, is the usual Sommerfeld result.

(b) $\alpha \to 0$ That is we neglect the force between the proton and the electron and our result should be that for the possible energy-levels of a free electron in the hypersphere.

Equation 22.17 now gives

$$4(\rho+n+\tfrac{1}{2})^2\left\{\kappa^2-\varepsilon^2 + \frac{(\rho+n)(\rho+n+1)}{R^2}\right\} = \kappa^2-\varepsilon^2$$

so $$(\varepsilon^2-\kappa^2)4(\rho+n)(\rho+n+1) = \frac{4(\rho+n+\frac{1}{2})^2(\rho+n)(\rho+n+1)}{R^2}$$

or $$(\varepsilon^2-\kappa^2)^{\frac{1}{2}} = \frac{\rho+n+\frac{1}{2}}{R}$$

and as here $\rho = j$ from 22.13, we obtain

$$(\varepsilon^2-\kappa^2)^{\frac{1}{2}} = \frac{j+n+\frac{1}{2}}{R} \qquad \text{——— 23.2}$$

This result has been given previously by Schroedinger (38a) and by Taub (37).

(c) $\rho + n < R$. That is we consider the shift in the usual hydrogen energy-levels brought about by the curvature.

We put $\quad \rho + n = \bar{n}$ — 23.3

Here $\bar{n}$ is not strictly integral but for large quantum numbers where α^2 can be neglected $\bar{n} \to j + n$ and is the usual principal quantum number of the Schroedinger theory.

Equation 21.38 now yields

$$\left.\begin{aligned} &\frac{\alpha \varepsilon}{(\kappa^2 - \varepsilon^2)^{\frac{1}{2}}} \approx \bar{n} + x \\ &\text{where} \quad x = \frac{(\bar{n} + \frac{1}{2})\, \bar{n}\, (\bar{n} + 1)}{2R^2 (\kappa^2 - \varepsilon^2)} \end{aligned}\right\} \qquad \text{— 23.4}$$

After some simplification we find from 23.4 for the energy-levels W, where $\varepsilon = W + \kappa$

$$W \approx \kappa \left\{ -\frac{\alpha^2}{2\bar{n}^2} + \frac{\alpha^2 x}{\bar{n}^3} \right\} \qquad \text{— 23.5}$$

From 23.4 a first approximation to $\frac{1}{\kappa^2 - \varepsilon^2}$ is given by $\frac{\bar{n}^2}{\alpha^2 \kappa^2}$ and inserting this in 23.5 we obtain

$$W \approx -\frac{\kappa \alpha^2}{2\bar{n}^2} + \frac{\bar{n}^2}{2\kappa R^2}$$

or in usual units

$$W \approx -\frac{m_e \alpha^2 c^2}{2\bar{n}^2} + \frac{\bar{n}^2 \hbar^2}{2 m_e R^2} \qquad \text{— 23.6}$$

where m_e is the electron-mass in grams.

The orders of magnitude of the quantities occurring in 23.6 are

$$\left.\begin{array}{lll} & \alpha \approx 10^{-2} & R \approx 10^{36} \\ c \approx 10^{10} & m_e \approx 10^{-28} & \hbar \approx 10^{-27} \end{array}\right\} \quad -23.7$$

Now the first term in 23.6 is that leading to the usual Sommerfeld formula, while the second represents the perturbation of these levels due to the curvature. This perturbation slightly decreases the magnitude of each energy-level, but from 23.7 it can be seen that for small (i.e. $< 10^{15}$) quantum numbers (and therefore for all those of spectroscopic interest), the effect is quite negligible.

As $\tilde{n}$ increases from 0, the spectrum passes through the usual Sommerfeld levels until $\tilde{n} \approx 10^{20}$, when the effect of the curvature begins to take effect, and after a short period where the two effects are comparable, the spectrum passes continuously into that of the discrete levels corresponding to the natural frequencies of a free electron in the hypersphere. The distinction in the flat case between the discrete negative energy-levels and the continuous spectrum for the positive levels is completely obliterated.

We see also from 21.38 that the two fundamental spectra, that of the hydrogen atom and that of the electron in the hypersphere, are inextricably linked together in the proper relativistic treatment. This is in contrast with the results of Schroedinger's (40a) treatment (using the Schroedinger equation) in which these two spectra combined additively and so existed without any interference. This intimate relation between the two spectra is a state of affairs one would expect to hold, as it is impossible to completely isolate any system from the effects of space-curvature, no matter how small this effect may prove to be.

It only remains now to verify that the terms we neglected in 22.5 really are negligible. The $\epsilon_{14}\alpha$ term would have the effect of introducing an extra factor of the order $\frac{\alpha}{R^2}$ into equation 21.38, the second bracket becoming $\kappa^2 - \varepsilon^2 + \frac{(\rho+m)(\rho+m+1)}{R^2} + \frac{h\alpha}{R}$ where h is a constant of the order of magnitude unity.

For small quantum numbers this last term is negligible, as shown above, and for large values of it becomes negligible compared with $\frac{m^2}{R^2}$. In all cases therefore it can safely be neglected and the same holds clearly for the α^2 term.

CHAPTER 5

Unified Field Theories and the Bhabha Equation

In this chapter we shall formulate the Bhabha equation for the world geometrics of Weyl (22) and of Flint (40,42,45), and so investigate the electromagnetic terms thus introduced. It is well-known that Weyl's unified field theory is a special case of the more general one due to Eddington (37), but we shall use Weyl's theory here because of its greater simplicity. It can be verified that Eddington's theory leads to an identical result. Flint (42) has already calculated the extra terms arising in the Dirac equation on his five-dimensional theory, and similar results have been given in the closely-related "projective" theories of Pauli (33a) and of Veblen and Hoffmann (30). Only a brief review of the calculation following from the general theory in Chapter 3 will therefore be given here.

As throughout our main interest lies in the electromagnetic terms we shall consider the Bhabha equation in cartesian* co-ordinates - the modifications necessary to

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* In cartesian co-ordinates there is no distinction between co- and contra-variant tensors so suffices can be placed up or down.

extend the results obtained to general co-ordinate systems can then be seen from simple tensor considerations.

(24) The Weyl Case

In Weyl's geometry (see Eddington (37) p.136 and (36) p.136) the affine connection is given by

$$\Gamma^{h}_{ij} = \left\{ {h \atop ij} \right\} + P^{h}_{ij} \qquad \text{—— } 24.1$$

where $\left\{ {h \atop ij} \right\}$ is the usual Christoffel bracket and

with

$$\left.\begin{aligned} P_{hij} &= g_{hn} P^{n}_{ij} = K_{hij} - K_{ihj} - K_{jih} \\ K_{hij} &= i\bar{\varphi}_h \, g_{ij} \end{aligned}\right\} \text{ — } 24.2$$

Here $\bar{\varphi}_h$ is the electromagnetic potential, but it is expressed in terms of some unknown unit.

From 24.1 it follows that

$$g_{ij;h} = 2K_{hij} = 2i\bar{\varphi}_h g_{ij}$$

so from 12.12 and 12.8 the definition of undor differentiation becomes

$$\left.\begin{aligned} \alpha_{i;j} &= -i\bar{\varphi}_j \, g_{in} \alpha^{n} \\ \& \quad A_{jil} + A_{ijl} &= 2i\bar{\varphi}_l \, g_{ij} \end{aligned}\right\} \text{ — } 24.3$$

Now from 12.11

$$\Delta_l = \tfrac{1}{2} A_{rsl} I^{rs} = \tfrac{1}{2} g_{mr} \left(C^{m}_{sl} - \Gamma^{m}_{sl} \right) I^{rs}$$

and as in cartesian co-ordinates

$$\left.\begin{aligned} g_{ij} &= \delta_{ij} \\ \left\{ {h \atop ij} \right\} &= 0 \\ \& \quad \alpha_i &= \overset{0}{\alpha}_i \end{aligned}\right\} \text{ ———— } 24.4$$

we find

$$\Delta_l = -\tfrac{1}{2} P_{rsl} \overset{0}{I}{}^{rs}$$

$$= i\bar{\varphi}_r \overset{0}{I}_{rl} \qquad \text{(using 24.2)}$$

The Bhabha equation, 11.1, therefore becomes

$$\overset{\circ}{\alpha}_i \left[\frac{\partial}{\partial x_i} - i \bar{\varphi}_s \overset{\circ}{I}_{is} \right] \psi + \chi \psi = 0 \qquad \text{—— } 24.5$$

This result becomes much more familiar in the Dirac case. That is from 4.11 and 4.12, when

$$\left. \begin{array}{ll} \overset{\circ}{\alpha}_i = \frac{\gamma_i}{2} & \chi = \frac{\kappa}{2} \\ \text{and} \quad \gamma_i \gamma_j + \gamma_j \gamma_i = 2 \delta_{ij} & \end{array} \right\}$$

Equation 24.5 is then

$$\gamma_i \left[\frac{\partial}{\partial x_i} - \frac{3}{2} i \bar{\varphi}_i \right] \psi + \kappa \psi = 0 \qquad \text{—— } 24.6$$

Comparing this with the rule 15.10 it can be seen that for equation 24.6 to be equivalent to the usual Dirac equation we must have

$$\frac{3}{2} \bar{\varphi}_i = e \varphi_i$$

or in ordinary units

$$\frac{3}{2} \bar{\varphi}_i = \frac{e}{\hbar c} \varphi_i \qquad \text{—— } 24.7$$

This equation therefore fixes the unit in which the electromagnetic potential is expressed in Weyl's theory. Such an identification however raises a number of important questions.

The first is that the relation 24.7 is completely at variance with that obtained originally from Weyl's theory by considering the expression for the total energy-momentum tensor. Weyl's expression for this was (cf. Eddington (37) p.210)

$$\frac{8\pi G}{c^2} T_{ij} + \frac{1}{\lambda} E_{ij} + \text{ small terms } = 0 \qquad \text{—— } 24.8$$

where $T_{\mu\nu}$ is the total energy-momentum tensor, $E_{\mu\nu}$ the electromagnetic tensor, G the Newtonian constant of gravitation, and $\lambda = \frac{3}{R^2}$ (R radius of universe) is the cosmical constant. Except for the electromagnetic potential expressed in terms of $\bar{\varphi}$, all quantities are in the usual units.

The first two terms are interpreted as the difference between the total and electromagnetic energy tensors, and so must be in the same unit. Hence

$$\bar{\varphi} = \sqrt{\frac{G\lambda}{c^2}}\,\varphi \approx \frac{G^{\frac{1}{2}}}{Rc}\,\varphi \qquad \text{--- } 24.9$$

where φ is the electromagnetic potential in C.G.S.units.

As $R \approx 10^{36}$, $G \approx 10^{-8}$ and $c \approx 10^{10}$ it can be seen that from this point of view the unit is negligibly small, a result in contradiction to equation 24.7.

This circumstance has been discussed by Eddington (47) p.239, where he has derived a result analagous* to that in 24.7, and he explains it by saying that the association of the λ in 24.8 with the cosmical constant is incorrect, and that a proper treatment must take into account the deep-seated distinction between microscopic and macroscopic theory. It is this distinction with which the greater part of Eddington's "Fundamental Theory" is concerned, and it seems that the contradiction contained in 24.7 and 24.9 can only be explained by some investigation such as Eddington has carried out into the

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* Eddington's result is obtained by a method quite different to that used here.

division which is normally made in physics between the macroscopic and the microscopic.

The second point arising from 24.7 is that although we have arranged things so that the Dirac equation in Weye's theory is identical with the usual one, the equations for the other particles contained in 24.5 are not those obtained by simply replacing $\frac{\partial}{\partial x_i}$ by $\frac{\partial}{\partial x_i} - ie\varphi_i$ This in turn suggests the more general but related question "What is the justification for introducing the electromagnetic field, as is usually done, by the substitution $\frac{\partial}{\partial x_i} \rightarrow \frac{\partial}{\partial x_i} - ie\varphi_i$?"

Most derivations of this result (cf. Möller and Rosenfeld (43)), assume that the equations for any body in the presence of an electromagnetic field can be derived from a Lagrangian principle, and then postulate that these equations must be invariant for gauge-transformations of the type

$$\varphi_i \rightarrow \varphi_i + \frac{\partial \xi}{\partial x_i} \qquad \psi \rightarrow e^{-ie\xi}\psi \qquad \text{——— } 24.10$$

where ξ is any constant.

If the further condition is imposed that the variational derivative of the Lagrangian w.r.t. the potential φ_i must yield a current-vector satisfying the usual conservation properties, then it can be shown that the only possible substitution is the one given above.

The gauge transformations 24.10 are however only a special case of the whole concept of variation in gauge introduced by Weyl in his unified theory, and one would expect that any equation derived directly from Weyl's geometry would automatically satisfy any conditions of guage invariance which could possibly be imposed upon it.

Again however the question seems in need of further investigation before a satisfactory answer can be given to it.

(25) Notation for Flint's Five-Dimensional Theory.

The geometry is Riemannian but now in five dimensions. The line element is given by

$$d\sigma^2 = \gamma_{\mu\nu}\, dx^\mu dx^\nu \qquad \text{— } 25\cdot 1$$

where μ, ν run 1 - 5; and also

$$\gamma_{\mu\nu;\lambda} = 0 \qquad \text{— } 25\cdot 2$$

The affine connection is the five-dimensional Christoffel bracket

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\gamma^{\alpha\lambda}\left\{ \frac{\partial \gamma_{\lambda\mu}}{\partial x^\nu} + \frac{\partial \gamma_{\lambda\nu}}{\partial x^\mu} - \frac{\partial \gamma_{\mu\nu}}{\partial x^\lambda} \right\} \qquad \text{— } 25\cdot 3$$

The electromagnetic field is introduced into the metric by the relations

$$\left. \begin{array}{l} \gamma_{5i} = h\,\varphi_i \\ \text{with } \; h = \frac{e}{\kappa} \end{array} \right\} \qquad \text{— } 25\cdot 4$$

where i goes 1 - 4, e is the charge and κ the rest-mass of the particle under consideration, and φ_i is the electromagnetic potential in the usual units.

The possible transformations in five-space are limited to those of the form

$$\left.\begin{aligned}\bar{x}^i &= f(x^i)\\ \bar{x}^5 &= x^5 + g(x^i)\end{aligned}\right\} \quad \text{--- } 25.5$$

and this implies it is possible to associate a five-vector C^{μ} with a four-vector A^i and a four-dimensional scalar A by the relations

$$\left.\begin{aligned}C^i &= A^i\\ C_5 &= A^5 = A\\ C^5 &= A - h\varphi_i A^i\\ C_i &= A_i + h\varphi_i A\end{aligned}\right\} \quad \text{--- } 25.6$$

and

Relations for tensors of the second and higher ranks can be obtained by multiplication from 25.6, and a particular case is that of the relation between the five-dimensional metric tensor $\gamma_{\mu\nu}$ and the four-dimensional metric tensor g_{ij}. These relations are (see Flint (40b))

$$\left.\begin{aligned}\gamma^{ij} &= g^{ij}\\ \gamma^{i5} &= -h g^{ij}\varphi_j\\ \gamma_{ij} &= g_{ij} + h^2\varphi_i\varphi_j\\ \gamma_{5i} &= \gamma_{i5} = h\varphi_i\\ \gamma_{55} &= 1\end{aligned}\right\} \quad \text{--- } 25.7$$

Using 25.7 the following relations are found between the Christoffel brackets $\Gamma^{\lambda}_{\mu\nu}$ and the usual four-dimensional brackets $\left\{ {i \atop jk} \right\}$.

$$\left.\begin{aligned}\Gamma^{\alpha}_{55} &= 0\\ \Gamma^{i}_{j5} &= \frac{h}{2}\varphi^{i}{}_{j}\\ \Gamma^{i}_{jk} &= \left\{ {i \atop jk} \right\} + \frac{h^2}{2}\left(\varphi_k\varphi^{i}{}_{j} + \varphi_j\varphi^{i}{}_{k}\right)\\ \Gamma^{5}_{jk} &= -h\varphi_i\Gamma^{i}_{jk} + \frac{h}{2}\left(\frac{\partial\varphi_j}{\partial x^k} + \frac{\partial\varphi_k}{\partial x^j}\right)\\ \Gamma^{5}_{j5} &= -\frac{h^2}{2}\varphi_i\varphi^{i}{}_{j}\end{aligned}\right\} \quad \text{--- } 25.8$$

where $$\varphi_{ij} = \frac{\partial\varphi_i}{\partial x^j} - \frac{\partial\varphi_j}{\partial x^i} \quad \text{--- } 25.9$$

All the above formulae hold in the original Klein-Kaluza five-dimensional theory and in the projective theory of Veblen and Hoffmann as well as in Flint's theory. These theories differ however in the interpretation which is placed on $\frac{\partial}{\partial x^5}$ - that is on differentiation w.r.t. the fifth co-ordinate. The two schemes are given by

Klein and Veblen	Flint	
$\frac{\partial \psi}{\partial x^5} = 0$	$\frac{\partial \psi}{\partial x^5} = i\kappa\psi \qquad \frac{\partial \psi^*}{\partial x^5} = -i\kappa\psi^*$	— 25.10

and the corresponding Bhabha equations are

$$\alpha^\mu \psi_{;\mu} + \kappa\psi = 0 \qquad\qquad \alpha^\mu \psi_{;\mu} = 0 \qquad — 25.11$$

where the α^μ now satisfy the commutation relations

$$[\alpha_\mu, I_{\nu\lambda}] = \gamma_{\mu\nu}\alpha_\lambda - \gamma_{\mu\lambda}\alpha_\nu \qquad — 25.12$$

(26) The Bhabha Equation in Flint's theory.

In four-dimensional cartesian co-ordinates, we find from 25.12 and 25.7

$$\left.\begin{array}{l} [\alpha_\mu, I_{\nu\lambda}] = (g_{\mu\nu} + h^2\varphi_\mu\varphi_\nu)\alpha_\lambda - (g_{\mu\lambda} + h^2\varphi_\mu\varphi_\lambda)\alpha_\nu \\ \text{with} \quad g_{ij} = \delta_{ij} \qquad g_{i5} = g_{55} = 0 \end{array}\right\} — 26.1$$

A solution to this equation of the form 12.3 is given by

$$\left.\begin{array}{l} \alpha_i = \overset{\circ}{\alpha}_i + h\varphi_i\overset{\circ}{\alpha}_5 \\ \alpha_5 = \overset{\circ}{\alpha}_5 \end{array}\right\} — 26.2$$

where the $\overset{\circ}{\alpha}_i, \overset{\circ}{\alpha}_5$ are the usual Bhabha matrices whose degrees are now given by table 9.2; not by the four-dimensional table 4.10.

The determination of Δ_i and Δ_5 now follows from 12.11 together with the formulae in the last section, and after a lengthy but straightforward calculation we obtain the following results

and
$$\left.\begin{aligned} \Delta_i &= -\frac{h^2}{4}\varphi_i \varphi_{kj} \overset{\circ}{I}_{kj} + \frac{h}{2}\varphi_{ki}\overset{\circ}{I}_{5k} \\ \Delta_5 &= -\frac{h}{4}\varphi_{jk}\overset{\circ}{I}_{jk} \end{aligned}\right\} \quad — 26.3$$

which are identical in the Dirac case with those found by Flint.

Using 25.4, 25.6, 25.10 and 25.11 we find for the Bhabha equation

$$\overset{\circ}{\alpha}_i \left[\frac{\partial}{\partial x_i} - ie\varphi_i - \frac{h}{2}\varphi_{ki}\overset{\circ}{I}_{5k} \right] \psi$$
$$+ \overset{\circ}{\alpha}_5 \left[-i\alpha + \frac{h}{4}\varphi_{jk}\overset{\circ}{I}_{jk} \right] \psi = 0 \qquad —— 26.4$$

The extra terms appearing are the usual one $-ie\varphi_i$ together with terms containing h .

Now $h = \frac{e}{\kappa} = \frac{e}{mc^2}$ (in usual units)
which for the electron $\approx 10^{-4}$

The additional terms introduced by Flint's theory are therefore in all cases quite negligible, and the theory leads to the customary rule for the introduction of the electromagnetic field.

In the Dirac case equation 26.4 can be further simplified. It becomes

$$\gamma_i \left\{ \frac{\partial}{\partial x_i} - ie\varphi_i \right\} \psi + \gamma_5 \left\{ -i\kappa - \frac{h}{16}\varphi_{jk}\overset{\circ}{I}_{jk} \right\} \psi = 0 \qquad —— 26.5$$

in which form it can be compared with the Dirac equations given by Pauli (33a) and Taub (34).

CHAPTER 6

The Interaction between Nucleons in Five Dimensions.

Recently Möller (41) has evaluated the force existing between two nucleons in the meson theory of nuclear forces, using a five-dimensional theory which is almost identical with that of Klein. He showed that the vector form of this theory led directly to a potential of Möller-Rosenfeld type consisting in a mixture of the potentials due to the four-dimensional vector and pseudo-scalar theories. As mentioned in § (25) Flint's theory differs from the other five-dimensional theories in the interpretation placed on $\frac{\partial}{\partial x^5}$, and in this chapter we shall show that the different forms of the Dirac and Remmer equations thus resulting lead to quantitatively different results for the nuclear force. The method used here to evaluate the interaction follows that of Yukawa (38) and Bethe(40).

(27) The Dirac Equation in Flint's Theory.

We shall consider the equation in cartesian co-ordinates in the absence of any electromagnetic field.

In a similar way to §(15), the equation can be derived from a Lagrangian

$$\mathcal{L} = \psi^{\times} D \gamma^{\mu} \frac{\partial \psi}{\partial x^{\mu}} \qquad \text{—— } 27\cdot 1$$

As the γ_5 form a pentad (§(5)), we shall take

$\gamma_1 - \gamma_4$ as the usual Dirac matrices in four dimensions, and put

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 \qquad \text{--- } 27.2$$

From 27.1, by variation w.r.t. $\psi^{\times} D \psi$ we obtain the equations

$$\left.\begin{aligned} &\gamma^{\mu} \frac{\partial \psi}{\partial x^{\mu}} = 0 \\ \text{or} \quad &\gamma^{i} \frac{\partial \psi}{\partial x^{i}} + i\kappa \gamma_5 \psi = 0 \end{aligned}\right\} \qquad \text{--- } 27.3(a)$$

$$\left.\begin{aligned} \text{and} \quad &\frac{\partial (\psi^{\times} D)}{\partial x^{\mu}} \gamma^{\mu} = 0 \\ \text{or} \quad &\frac{\partial (\psi^{\times} D)}{\partial x^{i}} \gamma^{i} - i\kappa \psi^{\times} D \gamma_5 = 0 \end{aligned}\right\} \qquad \text{--- } 27.3(b)$$

It can be shown that the two equations 27.3 are only consistent if

$$D = \gamma_4 \qquad \text{--- } 27.4$$

From 27.1 we can form a scalar

$$P = \psi^{\times} D \psi \qquad \text{--- } 27.5$$

a current-vector

$$J_{\mu} = i \psi^{\times} D \gamma_{\mu} \psi \qquad \text{--- } 27.6$$

and also an antisymmetric tensor of the second rank.

$$S_{\mu\nu} = \frac{i}{2} \psi^{\times} D I_{\mu\nu} \psi = i \psi^{\times} D \gamma_{\mu} \gamma_{\nu} \psi \qquad \text{--- } 27.7$$

In five dimensions it is impossible to form pseudo-scalars, pseudo-vectors, etc. from the γ matrices, the sixteen possible combinations of $\gamma_1 - \gamma_5$ being accounted for by P, J & S . Pais (42) has shown that these pseudo-quantities can be formed with the help of an operator ε depending only on the structure of the five-space, and such that εJ is the pseudo-vector corresponding to J etc. He also shows however that the meson field equations resulting from these "pseudo" theories always lead to the same expressions for the interaction as do the usual theories, so we will not consider them here.

We wish now to determine the values of ρ, J_μ & $S_{\mu\nu}$ in the static case; that is in the extreme non-relativistic case when the particle (electron or proton) is considered at rest. The spin of the particle remains finite even in the static case, and we will denote the spatial components of the spin-operator $I_{\mu\nu}$ by $\vec{\sigma}$ where

$$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = \frac{i}{2}(I_{23}, I_{31}, I_{12}) \qquad \text{—— } 27.8$$

the σs being the ordinary Pauli spin matrices.

We can write equation 27.3(b) in the form

$$\left.\begin{aligned} & i\frac{\partial\psi}{\partial t} = H\psi = i\gamma_4\gamma_a h_a\psi + i\gamma_4\gamma_5 K\psi \\ \text{where} \quad & h_a = -i\frac{\partial}{\partial x_a} \end{aligned}\right\} \text{—— } 27.9$$

and a runs 1 - 3.

From 27.9 it can be seen that the "velocity" and "rest-mass" operators are $i\gamma_4\gamma_a$ and $i\gamma_4\gamma_5$ respectively and in the static case these must have the values

$$\left.\begin{aligned} i\gamma_4\gamma_a &= 0 \\ i\gamma_4\gamma_5 &= 1 \end{aligned}\right\} \text{—— } 27.10$$

Using these results we can determine the static values of ρ, J & $S_{\mu\nu}$ which become

	Flint	Möller
P	0	1
J_a	0	0
J_4	i	i
J_5	1	0
$S_{ab} = (S_{23}, S_{31}, S_{12})$	0	$\vec{\sigma}$
$S_{a4} = (S_{14}, S_{24}, S_{34})$	$-i\vec{\sigma}$	0
$S_{a5} = (S_{15}, S_{25}, S_{35})$	$\vec{\sigma}$	$\vec{\sigma}$
S_{45}	0	0

— 27.11

where we have also tabulated the corresponding results from Möller for comparison.

It is this difference in the static values of the quantities P, J & S which causes the interaction potential to differ in the two cases, and this difference depends entirely on the form of the two Dirac equations used, which are

$$\gamma^a \frac{\partial \psi}{\partial x^a} - i\gamma_4 \frac{\partial \psi}{\partial t} + i\kappa\gamma_5\psi = 0 \qquad \text{Flint}$$

and $$\gamma^a \frac{\partial \psi}{\partial x^a} - i\gamma_4 \frac{\partial \psi}{\partial t} + \kappa\psi = 0 \qquad \text{Möller}$$

It can be seen that Möller's γ^a is equivalent to the $-i\gamma^5\gamma^a$ of Flint's and this relation is sufficient to reconcile the two columns in 27.11.

(28) The Equations for the Meson.

The meson is a particle of maximum spin one, and the equations describing it are therefore the Kemmer equations. Harish-Chandra (46) has shown that for such a particle it is always possible to pass from the particle picture in terms of the Kemmer equation, to the wave picture in terms of ordinary tensor equations; the type of equation differing according to the particular representation of the β matrices.

From 9.2 we find the following possibilities for the representations of the β matrices describing particles of spin one, and give their associated tensor wave equations according to Harish-Chandra and Flint.

(a) Representation of degree six (scalar theory).

The wave-function ψ is composed of a vector and a scalar, $\psi = (V_\mu, A)$, and the wave equations are

$$\left.\begin{aligned} V_\mu &= -\frac{\partial A}{\partial x^\mu} \\ \text{and} \quad \frac{\partial V_\mu}{\partial x_\mu} &= 0 \end{aligned}\right\} \quad \text{——— } 28.1$$

(b) Representations of degree ten (tensor theory).

We shall combine these two twin representations, and then is composed of an antisymmetric tensor of rank two and one of rank three, so $\psi = (B_{\mu\lambda\tau}, A_{\sigma\tau})$ and the wave equations are

$$\left.\begin{aligned} \frac{\partial B_{\mu\lambda\tau}}{\partial x^\tau} &= 0 \\ B_{\mu\lambda\tau} &= \frac{\partial A_{\mu\lambda}}{\partial x^\tau} + \frac{\partial A_{\lambda\tau}}{\partial x^\mu} + \frac{\partial A_{\tau\mu}}{\partial x^\lambda} \\ \text{and} \quad \frac{\partial A_{\mu\lambda}}{\partial x^\lambda} &= 0 \end{aligned}\right\} \text{— } 28.2$$

(c) Representation of degree fifteen (vector theory).

Here ψ is composed of an anti-symmetric tensor of rank two and a vector, $\psi = (T_{\mu\nu}, f_\mu)$ and the wave equations are

$$\left.\begin{aligned} \frac{\partial T_{\mu\nu}}{\partial x^\nu} &= 0 \\ T_{\mu\nu} &= \frac{\partial f_\mu}{\partial x^\nu} - \frac{\partial f_\nu}{\partial x^\mu} \\ \text{and} \qquad \frac{\partial f_\mu}{\partial x^\mu} &= 0 \end{aligned}\right\} \qquad 28.3$$

It can be seen that the equations 28.3 are the five-dimensional analogue of the Maxwell equations in four dimensions, and the last equation in 28.3 (as also the last in 28.2) resembles the normalizing condition

$\operatorname{div} A + \frac{1}{c}\dot{\phi} = 0$ in electromagnetic theory, and arises because only fourteen of the fifteen equations in 28.3 are independent.

In the remainder of this chapter we shall derive the form of the nuclear interaction for the meson theories whose equations are given by 28.1, 28.2 and 28.3.

(29) The Formulation of the Interaction Problem.

It was first suggested by Yukawa (35) that the force between two nucleons could be described with the aid of a tensor field (or in the particle picture through the intermediary of a particle of spin 0 or 1). To explain the exchange nature of the nuclear force these intermediary particles or mesons were considered to be charged, but neutral mesons had later to be also considered in connection with the force between two protons. The most satisfactory way of combining these three particles, two charged and one neutral, into one mathematical scheme was given by Kemmer (38b) with his symmetric meson theory and it is this form of the theory we shall consider here.

In Kemmer's symmetric theory the interaction is described by three real fields, one referring to each of the particles. However in Flint's theory there is nothing corresponding to a real field as the mass always enters into the field variables through the factor e^{ikx^5}. We shall call a "real" field therefore one which is real in the four variables $x_1 - x_4$.

i.e. if $V = V(x^i)e^{ikx^5}$ then $V^* = V(x^i)e^{-ikx^5}$ — 29.1

We now consider two nucleons, one at a point $r^{(1)}$ and the other at the point $r^{(2)}$. Associated with the particle at $r^{(i)}$ say, we can form the following source densities from the quantities ρ, J & S of 25.7-27.7.

$$\left.\begin{aligned} \bar{\rho} &= g_1 \tau^{(i)} \psi^* D \psi\, \delta(r - r^{(i)}) \\ \bar{J}_\mu &= i g_2 \tau^{(i)} \psi^* D \gamma_\mu \psi\, \delta(r - r^{(i)}) \\ \bar{S}_{\mu\nu} &= i g_3 \tau^{(i)} \psi^* D \gamma_\mu \gamma_\nu \psi\, \delta(r - r^{(i)}) \end{aligned}\right\} \quad \text{— 29.2}$$

where g_1, g_2 & g_3 are constants, τ is the isotopic-spin operator, and $\delta(r - r^{(i)})$ is the Dirac delta-function. The static values of the quantities in 29.2 can be found directly from 27.11.

In order to calculate the interaction between the nucleon at $r^{(1)}$ and that at $r^{(2)}$, we first find the form of the interaction potential between any meson field and the nucleon at $r^{(2)}$, and then insert in this operator the value for the meson field generated by the nucleon at $r^{(1)}$, this being calculated from field equations of the type 28.1 - 28.3.

This procedure will now be carried out in turn for the three different types of field.

(30) <u>The Scalar Field.</u>

We shall derive the field equations for the system of a nucleon at $r^{(2)}$ plus its mesom field by using a Lagrangian principle.

We take

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial A^{\times}}{\partial x_{\mu}}\frac{\partial A}{\partial x_{\mu}}\right) - \frac{1}{2}\left(J_{\mu}^{\times}\frac{\partial A}{\partial x_{\mu}} + \frac{\partial A^{\times}}{\partial x_{\mu}}J_{\mu}\right) - \frac{1}{2}\left(P^{\times}A + A^{\times}P\right) + \psi^{\times}\gamma_4\gamma_{\mu}\frac{\partial \psi}{\partial x_{\mu}} \qquad \text{— 30.1}$$

and we define

$$V_{\mu} = -\frac{\partial A}{\partial x_{\mu}} \qquad \text{— 30.2}$$

Here A is a "real" variable and the J_{μ} & P refer for the moment to a nucleon at the point $r^{(i)}$.

From 30.1 by varying w.r.t. $A^{\times}$ & $\psi^{\times}$ we obtain the equations

$$\frac{\partial}{\partial x_{\mu}}\left(V_{\mu} + J_{\mu}\right) = P \qquad \text{— 30.3}$$

and
$$\gamma_4\gamma_{\mu}\frac{\partial \psi}{\partial x_{\mu}} + \frac{i}{2} g_2 \tau^{(i)} \delta(r - r^{(i)}) \left\{(\gamma_4\gamma_{\mu})^{\times} V_{\mu} + V_{\mu}^{\times}\gamma_4\gamma_{\mu}\right\}\psi - \frac{1}{2} g_3 \tau^{(i)} \delta(r - r^{(i)}) \left\{\gamma_4^{\times} A + A^{\times}\gamma_4\right\}\psi = 0 \qquad \text{— 30.4}$$

Equation 30.3 together with 30.2 give the generalized form of the equations 28.1 for the scalar meson field in the presence of nuclear sources, while equation 30.4 is the generalized form for the Dirac equation describing the nucleon at a point $r^{(i)}$. From 30.4 we obtain as the interaction operator for the meson field and the particle at $r^{(2)}$..

$$H_{int.} = \tau^{(2)}\,\delta(r - r^{(2)}) \left[\frac{-ig_2}{2} \left\{ (\gamma_4\gamma_\mu)^* V_\mu + V_\mu^* \gamma_4\gamma_\mu \right\} - \frac{g_3}{2} \left(\gamma_4^* A + A^* \gamma_4 \right) \right] \qquad \text{— } 30.5$$

In the static case $\frac{\partial}{\partial x_4} = 0$ so $V_4 = 0$ and the source densities take up the values given in 27.11. The operator then becomes just

$$H_{int.} = \tau^{(2)}\,\delta(r - r^{(2)})\,\frac{g_2}{2}\left[iKA^* - iKA \right] \qquad \text{— } 30.6$$

The meson potential generated by the nucleon at can be found from 30.3. In the static case these equations give

$$(\nabla^2 - K^2) A = iK\tau^{(1)} g_2\, \delta(r - r^{(1)})$$

so $$A = iK\tau^{(1)} g_2\, \phi(r^{(1)}) \qquad \text{— } 30.7$$

where $$\phi(r^{(1)}) = e^{\frac{-K(r-r')}{r-r'}} \qquad \text{— } 30.8$$

and also $$A^* = -iK\tau^{(1)} g_2\, \phi(r^{(1)}) \qquad \text{— } 30.7(a)$$

Therefore substituting 30.7 and 30.7(a) in 30.6 we find

$$\left.\begin{aligned} H_{int.} &= \tau^{(1)}\cdot\tau^{(2)}\, g_2^2 K^2 \phi(r) \\ \text{where} \quad r &= r^{(1)} - r^{(2)} \end{aligned}\right\} \qquad \text{— } 30.9$$

This is then the result for the potential of the interaction between the two nucleons on Flint's scalar meson theory, and it is just the one derived from the four-dimensional scalar theory, and is the same as that in Möller's five-dimensional theory.

(31) <u>The Vector Field.</u>

We apply the same method as that used in the last section for the system of nucleon plus vector meson field. The appropriate source densities here are J_μ & $S_{\mu\nu}$, so we put

$$\mathcal{L} = \frac{1}{4}\left(\frac{\partial f_\mu^*}{\partial x_\nu} - \frac{\partial f_\nu^*}{\partial x_\mu}\right)\left(\frac{\partial f_\mu}{\partial x_\nu} - \frac{\partial f_\nu}{\partial x_\mu}\right) + \frac{1}{4}\left[\left(\frac{\partial f_\mu^*}{\partial x_\nu} - \frac{\partial f_\nu^*}{\partial x_\mu}\right) S_{\mu\nu} + S_{\mu\nu}^*\left(\frac{\partial f_\mu}{\partial x_\nu} - \frac{\partial f_\nu}{\partial x_\mu}\right)\right] + \frac{1}{2}\left(f_\mu^* J_\mu + f_\mu J_\mu^*\right) + \psi^* \gamma_4 \gamma_\mu \frac{\partial \psi}{\partial x_\mu} \quad \text{--- } 31.1$$

and we define

$$T_{\mu\nu} = \frac{\partial f_\mu}{\partial x_\nu} - \frac{\partial f_\nu}{\partial x_\mu} \quad \text{--- } 31.2$$

and postulate the normalizing condition for the vector potential

$$\frac{\partial f_\mu}{\partial x_\mu} = 0 \quad \text{--- } 31.3$$

By varying 31.1 w.r.t. f_μ^* we obtain the equation describing the meson field in the presence of sources, which is

$$\frac{\partial}{\partial x_\nu}\left(T_{\mu\nu} + S_{\mu\nu}\right) = J_\mu \quad \text{--- } 31.4$$

and by varying w.r.t. ψ^* we obtain the generalized Dirac equation for the nucleon in the presence of the meson field

$$\gamma_4 \gamma_\mu \frac{\partial \psi}{\partial x_\mu} + \frac{i g_2 \tau^{(i)}}{2} \delta(r - r^{(i)}) \left\{ (\gamma_4 \gamma_\mu)^* f_\mu + f_\mu^* \gamma_4 \gamma_\mu \right\} \psi + \frac{i g_3}{8} \tau^{(i)} \delta(r - r^{(i)}) \left\{ (\gamma_4 I_{\mu\nu})^* T_{\mu\nu} + \gamma_4 I_{\mu\nu} T_{\mu\nu}^* \right\} \psi = 0 \quad \text{--- } 31.5$$

From 31.5 the expression for the interaction potential between the meson field and the nucleon at $r^{(2)}$ is given by

$$H_{int.} = \tau^{(2)} \delta(r - r^{(2)}) \left\{ \frac{i g_2}{2} \left[(\gamma_4 \gamma_\mu)^{\times} f_\mu + f_\mu^{\times} \gamma_4 \gamma_\mu \right] + \frac{i g_3}{8} \left[(\gamma_4 I_{\mu\nu})^{\times} T_{\mu\nu} + T_{\mu\nu}^{\times} \gamma_4 I_{\mu\nu} \right] \right\} \quad \text{— 31.6}$$

Inserting the values for the source densities from 27.11 and putting $\frac{\partial}{\partial x_4} = 0$ we find for $H_{int.}$ in the static case

$$H_{int.} = \tau^{(2)} \delta(r - r^{(2)}) \left\{ \frac{g_2}{2} \left[-i f_4^{\times} - i f_4 + f_5^{\times} + f_5 \right] + \frac{g_3}{2} \vec{\sigma} \cdot \left[-i\kappa \vec{f} - i\kappa \vec{f}^{\times} - i \nabla f_4^{\times} + i \nabla f_4 - \nabla f_5^{\times} - \nabla f_5 \right] \right\} \quad \text{— 31.7}$$

where we have used equation 31.2 and denoted f_a by $\vec{f}$ and $\frac{\partial}{\partial x_a}$ by ∇, these being three-dimensional vector notations.

Equations 31,2, 31.3 and 31.4 are the generalized form of the set given in 28.3, and from them and the static values of the source densities given in 27.11, we find the following equations for the static vector potential of the meson field generated by a nucleon at $r^{(1)}$

$$\left.\begin{aligned} (\nabla^2 - \kappa^2) \vec{f} &= -i\kappa \tau^{(1)} \delta(r - r^{(1)}) g_3 \vec{\sigma}^{(1)} \\ (\nabla^2 - \kappa^2) f_4 &= i \tau^{(1)} \delta(r - r^{(1)}) \{ g_2 + g_3 \operatorname{div} \vec{\sigma}^{(1)} \} \\ (\nabla^2 - \kappa^2) f_5 &= \tau^{(1)} \delta(r - r^{(1)}) \{ g_2 + g_3 \operatorname{div} \vec{\sigma}^{(1)} \} \end{aligned}\right\} \text{— 31.8}$$

From 31.8 we obtain the relations

$$f_4 = i f_5 \qquad f_5^{\times} = f_5 \qquad f_4^{\times} = -i f_5 \qquad \vec{f}^{\times} = -\vec{f} \quad \text{— 31.9}$$

so inserting these values in 31.7 the interaction potential becomes

$$H_{int.} = \tau^{(2)}\delta(r - r^{(2)})\{2g_2 f_5 + g_3\vec{\sigma}(-i\kappa\vec{f} - 2\nabla f_5)\} \qquad -31.10$$

The solutions of 31.8 for $\vec{f}$ and f_5 are

$$\left.\begin{aligned} \vec{f} &= -i\kappa\, g_3\, \tau^{(1)} \varphi(r^{(1)}) \\ \text{and} \quad f_5 &= \tau^{(1)}\{g_2\varphi(r^{(1)}) + g_3\, \mathrm{div}[\vec{\sigma}^{(1)}\varphi(r^{(1)})]\} \end{aligned}\right\} \quad -31.11$$

where $\varphi(r)$ is defined in 30.8.

On substituting from 31.11 into 31.10 we then find for the interaction potential between the nucleons at $r^{(1)}$ that at $r^{(2)}$,

$$\begin{aligned} H_{int.} = \tau^{(1)}.\tau^{(2)}[2g_2^2 + g_3^2\kappa^2\vec{\sigma}^{(1)}.\vec{\sigma}^{(2)} - 2g_3^2(\vec{\sigma}^{(1)}.\nabla)(\vec{\sigma}^{(2)}.\nabla) \\ - 2g_2g_3((\vec{\sigma}^{(1)} + \vec{\sigma}^{(2)}).\nabla]\varphi(r) \end{aligned} \qquad -31.12$$

where

$$r = r^{(1)} - r^{(2)} \qquad -31.13$$

In deriving both 31.11 and 31.12 we have used the vector formulae

$$\left.\begin{aligned} \mathrm{div}[\vec{\sigma}\varphi(r)] &= -\vec{\sigma}.\mathrm{grad}\,\varphi(r) \\ \text{and} \quad \vec{\sigma}^{(1)}.\mathrm{grad}\,\mathrm{div}[\vec{\sigma}^{(2)}\varphi(r)] &= (\vec{\sigma}^1.\nabla)(\vec{\sigma}^2.\nabla)\varphi(r) \end{aligned}\right\} \quad -31.14$$

which are valid when $\vec{\sigma}$ is a constant vector.

The first two terms in 31.12 are those given by Möller's theory and represent the Möller-Rosenfeld combination of a four-dimensional vector and pseudo-scalar field. The third term is similar to that appearing in the four-dimensional theory and it yields a potential with a term $\frac{1}{r^3}$ which strictly speaking prevents there being any bound states of the deuteron and also leads to all the infinities which the Möller - Rosenfeld theory tried to eradicate. The last term is a rather peculiar one which represents a kind of cross-interaction between the two meson "charges", g_2 & g_3 . A nuclear potential of this form seems never to have been suggested before, and it involves no direct spin-spin interaction but a coupling of each spin-vector with the space co-ordinate between the two nucleons.

(32) <u>The Tensor Field.</u>

The calculation for this case can be carried out in a similar way to that for the scalar and vector fields in the last two sections, so here we shall only quote the result.

$$H_{int.} = \tau^{(1)}\cdot\tau^{(2)}\, g_3^2 \left[(2-\kappa^2)\, \vec{\sigma}^{(1)}\cdot\vec{\sigma}^{(2)} + 2(\vec{\sigma}^1\cdot\nabla)(\vec{\sigma}^2\cdot\nabla) \right] \varphi(r) \qquad \text{--- } 32.1$$

Again we see the potential involves the "infinite" term $(\vec{\sigma}^1\cdot\nabla)(\vec{\sigma}^2\cdot\nabla)\varphi(r)$, and the expression again differs from that of Möller's.

Although the nuclear interactions deduced from Flint's theory are not so well in accord with experiment as those derived from the other forms of the five-dimensional theory, yet the present state of meson theory suggests that the difficulties lie foremost in the meson theory itself, and that no definite conclusions can be drawn by comparisons of different forms for this theory. The only importance of the investigation carried out in this chapter lies in the demonstration that the different forms of the five-dimensional theories do lead to quantitatively different results, and when the theory of the nucleus is in a more advanced state, these differences may lead to some direct test of the validity of either of these five-dimensional theories.

Summary of the Contributions of the Thesis to the Theory of the Fundamental Particles.

We conclude this thesis with a summary of the main contributions contained in this thesis to the theory of fundamental particles.

(1) A solution is given to the pentad problem (p. 30-48).

(2) The Bhabha equation is put in a form valid for any system of co-ordinates (p. 49-61).

(3) A combination of two procedures due to Temple and Schroedinger is suggested as the simplest method of treating the Dirac equation (p. 68-95).

(4) The energy-levels for a hydrogen atom in the hypersphere are determined (p. 80-95).

(5) The Bhabha equation is formulated in Weyl's unified field theory (p. 96-101).

(6) The interaction between two nucleons is evaluated for the different forms of Flint's five-dimensional meson theory (p. 105-118).

In conclusion I would like to thank Professor Flint for the great interest he has shown during the course of this work and also Mr. R.K.Dalitz for numerous stimulating conversations. I would also like to express my indebtedness to the Department of Scientific and Industrial Research for a grant which has enabled this work to be carried out.

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Z.P. Zeitschrift für Physik.

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