A CRITICAL STUDY OF A NEW DOUBLE BRIDGE ~~BRIDGE~~ METHOD FOR THE MEASUREMENT OF IMPEDANCE AT ULTRA HIGH FREQUENCIES USING A SYSTEM OF LECHER WIRES

A Thesis presented in candidature for the degree of Master of Science of the University of London.

By

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Introduction

The fundamental theory of transmission lines shows that the terminating impedance of a pair of lines may be expressed in terms of the current at any point in the lines, the potential difference across the lines at this point, and ~~other~~ certain characteristics of the lines. Therefore, if it were possible to measure the currents and voltages at various points along the lines, this would provide a method of measuring impedance. Unfortunately, when a bridge containing a current measuring instrument is placed across the lines, the current through the terminating impedance and the current through the bridge assume complicated forms. Also, the movement of the bridge along the lines changes the input impedance, and the input current varies with the position of the bridge. These difficulties render the direct method of measuring current and voltage at any point by means of a single bridge impracticable.

Flint and Williams (1941) eliminated the effect of variations in the input by measuring the ratio of the currents flowing in two bridges. One bridge is fixed at the end of a pair of Lecher wires, and contains the unknown impedance in series with a current meter; the other bridge is movable, and contains a current meter only. Graphs of the current ratio plotted against certain functions of the distance between the two bridges give values of two constants, from which the terminating impedance may be calculated. Since the terminating impedance consists of the unknown impedance and the impedance of the current meter, a subsidiary experiment must be performed to determine the impedance of the current meter. Also, the exact way in which the two impedances are combined is not known.

The theory of this method shows that the ratio of the currents in the two bridges when at a distance apart is

$$\frac{I_1}{I_2} = \frac{Z_2 \cosh Ps + Z_0 \sinh Ps}{Z_1},$$

where Z_1 is the impedance of the movable bridge, Z_2 is the terminating impedance, Z_0 is the characteristic impedance, and P is the propagation constant. This can be reduced to the form

$$\rho^2 = A(\sinh^2 a + \sin^2 \overline{b+\beta s})$$

where $\rho = \left|\frac{I_1}{I_2}\right|$, and $\beta = \frac{P}{i} = \frac{2\pi}{\lambda}$, attenuation in the wire being neglected. a and b are certain quantities introduced for convenience. The resistive component of Z_2 is

$$\frac{Z_0 \tanh a \sec^2 b}{1 + \tanh^2 a \tan^2 b},$$

and the reactive component is

$$\frac{Z_0 \operatorname{sech}^2 a \tan b}{1 + \tanh^2 a \tan^2 b}.$$

The sensitivity of this method was studied by Rogers, who found that the best results were obtained when b has the value $\frac{\pi}{4}$.

In a later method the disadvantage of connecting an unknown impedance with a current meter led to a modification by Williams (1944). The unknown impedance is still fixed at the end of a pair of Lecher wires, and the ratio of the currents flowing in two movable bridges is measured. In theory, the separation of the bridges may have any value, but the method becomes practicable only when this separation has a certain fixed value. A preliminary experiment must be performed to determine this critical separation. The value of the unknown impedance is again calculated from constants which are found from graphs similar to those plotted in the original method. This method can be used in the frequency region where resonance methods ~~fail~~ become difficult, (above 100 Mc/s.), and also has the advantage that the resistive and reactive components of the impedance are measured independently.

In the present experiment, a first investigation of the accuracy of this modification of the original method has been undertaken. It is evident that the symmetry of certain curves is the criterion for the application of the method, and it is thus worth while examining the accuracy with which this symmetry can be realised. This is the main object of the work. In investigating this point some new features came to light, and these have been examined practically and discussed theoretically. The final result is an improvement in the determination of the critical separation of the bridges, a knowledge of which is essential for this method of measurement.

SECTION I

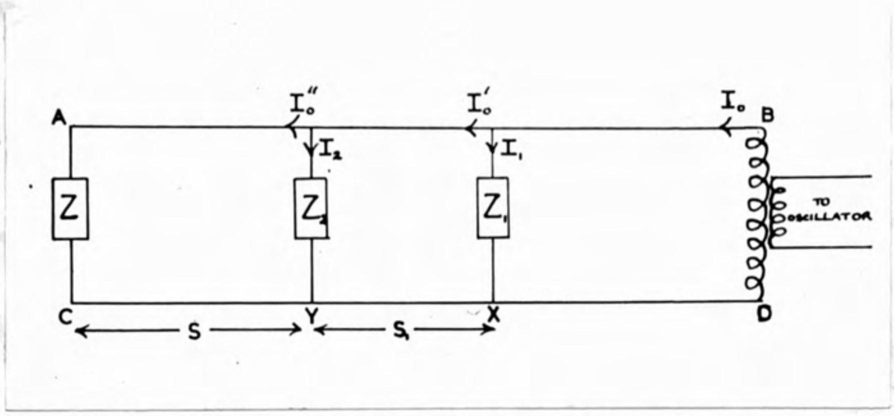


Fig. 1·1.

WILLIAMS' METHOD FOR THE MEASUREMENT OF IMPEDANCE

(a) THEORY OF THE METHOD

With reference to Fig1·1, AB and CD represent a pair of transmission lines which are loosely coupled to a line controlled oscillator. Z represents the unknown impedance fixed between A and C, and Z_1 and Z_2 are the impedances of the movable bridges, X and Y. The characteristic impedance of the lines is Z_0. An incident wave from the oscillator travels along the lines and is reflected at AC with a reflection coefficient

$$K = \frac{Z_0 - Z}{Z_0 + Z} \qquad (1.1)$$

A standing wave system is formed along the lines, of wavelength λ, and the resulting currents are as shown. The distance of Y from the end AC is s and the separation of X and Y is s_1.

If Z' is the impedance of the circuit beyond Y not including Z_2,

$$\frac{I_1}{I_2} = \frac{1}{Z_1}\left\{ Z_2 \cosh Ps_1 + Z_0 \sinh Ps_1 + \frac{Z_2}{Z'}.Z_0 \sinh Ps_1 \right\} \quad (1\cdot 2)$$

P represents the propagation constant of the lines, and is in general of the form ($\alpha + i\beta$), where α is the attenuation constant, and β is the phase constant. Since the lines are air-spaced, attenuation is assumed negligible, and P is an imaginary quantity $i\beta$, β being equal to $\frac{2\pi}{\lambda}$. As losses are negligible, Z_0 is assumed to be purely resistive; Z_2 is in general complex, and (1.1) may be written

$$K = e^{-2(a+ib)} \tag{1.3}$$

where a and b are real constants. Hence,

$$Z' = Z_0 \tanh[a + i(b+\beta s)]. \tag{1.4}$$

Splitting Z_2 into its resistive and reactive series components, R_2 and X_2, (1.1) becomes

$$\frac{I_1}{I_2} = \frac{1}{Z_1}\left\{A + iB + (C+iD)\coth[a + i(b+\beta s)]\right\}, \tag{1.5}$$

where

$$\left.\begin{aligned} A &= R_2 \cos\beta s_1 \; ; & B &= X_2 \cos\beta s_1 + Z_0 \sin\beta s_1 \\ C &= -X_2 \sin\beta s_1 \; ; & D &= R_2 \sin\beta s_1 \end{aligned}\right\} \tag{1.6}$$

In the measurement of the current ratio, it is the modulus, p, of this ratio which is determined; also, since vacuum thermo-junctions are used in the bridges, readings taken are proportional to the squares of the currents, so that it is more convenient to consider p^2.

$$\rho^2 = K_1 + \frac{K_2 + K_3 \sin 2(b+\beta s)}{\sinh^2 a + \sin^2(b+\beta s)}, \tag{1.7}$$

where

$$\left.\begin{aligned} K_1 &= \frac{1}{|Z_1|^2}\left[(A^2+B^2) - (C^2+D^2)\right] \\ K_2 &= \frac{1}{|Z_1|^2}\left[(C^2+D^2)\cosh 2a + (AC+BD)\sinh 2a\right. \\ K_3 &= \frac{1}{|Z_1|^2}\left[AD - BC\right] \end{aligned}\right\} \tag{1.8}$$

Equation (1.7) may be simplified by making $K_3 = 0$. This condition is satisfied when s_1 has the critical value which makes

$$\tan \beta s_1 = - \frac{Z_2^2}{Z_0 X_2} , \tag{1.9}$$

and then

$$\rho^2 = K_1 + \frac{K_2}{\sinh^2 a + \sin^2(b+\beta s)} . \tag{1.10}$$

If the impedance of Z_2 is known, the critical separation may be calculated from (1.9) since Z_0 may be found from the dimensions of the apparatus. It is not necessary to know Z_2, however, as the critical separation may be found experimentally.

From equations (1.1) and (1.3),

$$Z = Z_0 \tanh(a + ib). \tag{1.11}$$

hence,

real or resistive component of $Z = \dfrac{Z_0 \tanh a \sec^2 b}{1 + \tanh^2 a \tan^2 b}$, (1.12)

and imaginary or reactive component of $Z = \dfrac{Z_0 \operatorname{sech}^2 a \tan b}{1 + \tanh^2 a \tan^2 b}$. (1.13)

Hence the components of Z may be calculated if the values of a and b are found.

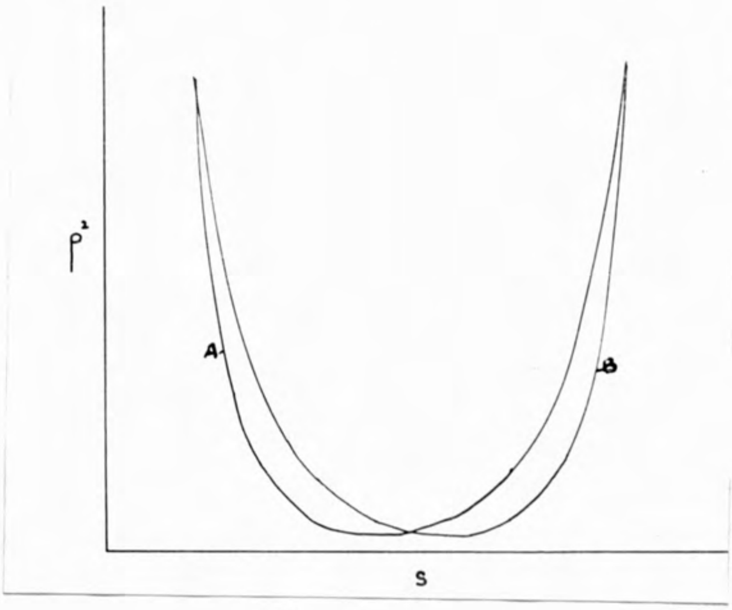


Fig. 1·2.

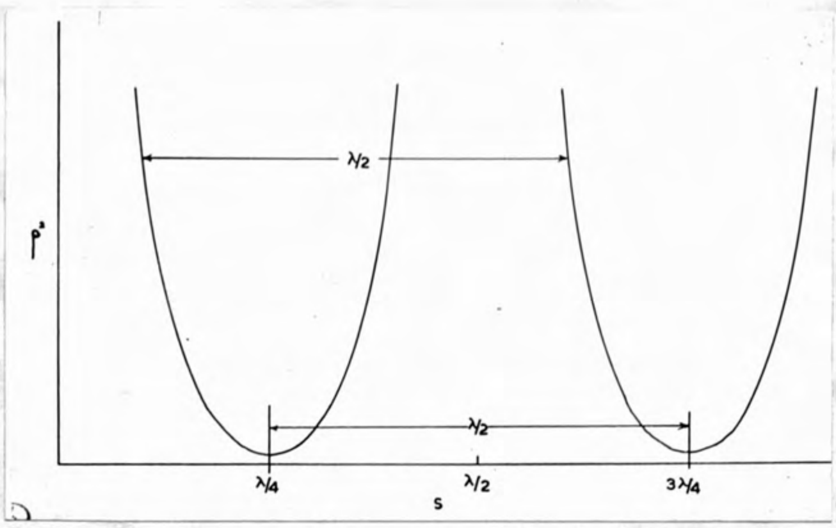


Fig. 1·3.

(b) DETERMINATION OF CONSTANTS

In order to determine the values of the constants a and b, it is necessary to set the bridges at the critical separation. Therefore the critical separation, s_0 must first be found.

Determination of s_0.

The unknown impedance is replaced by a polished copper shorting plate. This makes the reflection coefficient unity and a and b zero. Equation (1.7) then becomes,

$$\rho^2 = K_1 + K_2 \operatorname{cosec}^2\beta s + 2K_3 \cot\beta s. \qquad (1.14)$$

A graph of ρ^2 against s is in general of the form shown in Fig 1·2. Curve A is obtained if $s_1 > s_0$ and curve B if $s_1 < s_0$. When $s_1 = s_0$, i.e. $K_3 = 0$, the curve is symmetrical about lines parallel to the ρ^2 axis through its turning points and the minimum occurs where $s = \frac{\lambda}{4}$ (Fig 1·3). Curves are therefore plotted of ρ^2 against s near the minimum for various values of s_1, and the value of s_1 for which the minimum occurs at $s = \frac{\lambda}{4}$ is taken as the critical separation.

Determination of a.

Rewriting equation (1.10),

$$\frac{K_2}{\rho^2 - K_1} = \sinh^2 a + \sin^2(b + \beta s). \qquad (1.15)$$

Thus a graph of $\frac{1}{\rho^2 - K_1}$ against $\sin^2(b + \beta s)$ is a straight line with an intercept $\sinh^2 a$ on the $\sin^2(b+\beta s)$ axis. With the bridges set so that $K_3 = 0$, this graph is plotted, and a value of a determined.

Determination of K_1

K_1 is a constant independent of Z and can therefore be found with the lines shorted as in the determination of a, b, and K_3 are zero, and

$$\rho^2 = K_1 + K_2 \operatorname{cosec}^2 \beta s. \qquad (1.16)$$

Hence a graph of ρ^2 against $\operatorname{cosec}^2 \beta s$ is a straight line with an intercept K_1 on the ρ^2 axis.

Determination of λ and b.

Equation (1.10) is,

$$\rho^2 = K_1 + \frac{K_2}{\sinh^2 a + \sin^2(b+\beta s)}.$$

Therefore maxima of ρ^2 occur when

$$\sin^2(b+\beta s) = 0; \quad b + \beta s = 0, \pi, 2\pi, \text{----} \qquad (1.17)$$

and minima of ρ^2 occur when

$$\sin^2(b+\beta s) = 1; \quad b+\beta s = \frac{\pi}{2}, \frac{3\pi}{2}, \text{-----} \qquad (1.18$$

Thus the distance between two successive maxima or minima

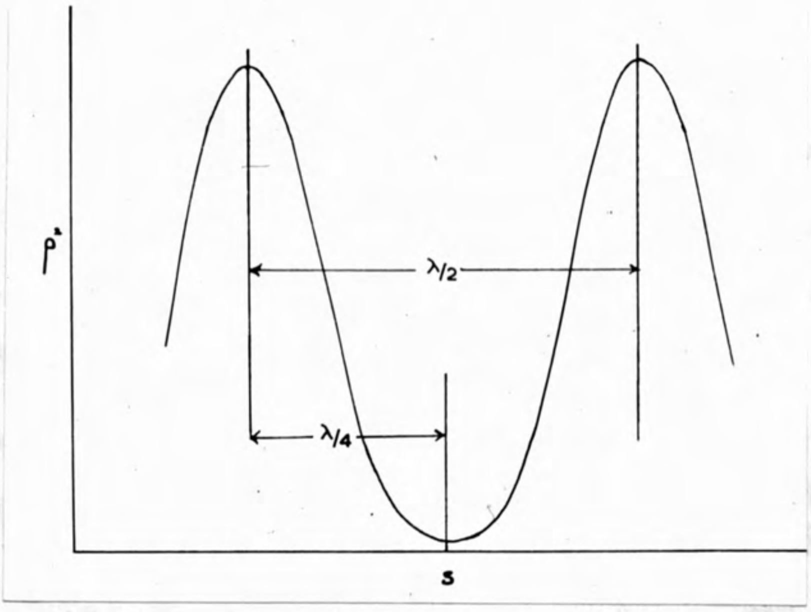


Fig. 1·4.

is . A graph is plotted of ρ^2 against s , as shown in Fig. 1.4, and a mean value of λ determined by measuring distances between corresponding points. This gives a value for β ($= \frac{2\pi}{\lambda}$) and by finding the values of s at the turning points a value for b may be calculated from equations (1.17) and (1.18).

<u>SECTION II</u>

THE DETERMINATION OF THE CRITICAL SEPARATION

(a) CRITICISM OF WILLIAMS' METHOD

In describing the determination of the critical separation in his paper, (1), Dr. Williams states:-

"Three values of ρ^2 are obtained, one for a value of s less than $\frac{\lambda}{4}$, say ($\frac{\lambda}{4}$ -5) cm; another when s = $\frac{\lambda}{4}$ cms; and a third value when s = ($\frac{\lambda}{4}$ + 5)cm.; and the separation of the bridges is adjusted until the value of ρ^2 at s = $\frac{\lambda}{4}$ is a minimum. The final adjustment can be made by obtaining the values of ρ^2 in more positions."

This statement is ambiguous. It may mean, as has been assumed in the foregoing theory, that s_1 remains fixed while values of ρ^2 are found for values of s in the region of $\frac{\lambda}{4}$ cm., and that this is repeated for different values of s_1 until the minimum occurs at s = $\frac{\lambda}{4}$; or it may mean that s remains fixed at $\frac{\lambda}{4}$, while values of ρ^2 are found for various values of s_1, and the value of s_1 for which ρ^2 is a minimum is taken as the critical separation.

The second method gives an incorrect value for the critical separation. For, although the minimum of the ρ^2/s curve occurs at s = $\frac{\lambda}{4}$ cm. when the separation is critical, the converse is not necessarily true. When $K_3 = 0$

~~(1.15)~~

$$\rho^2 = K_1 + K_2 \operatorname{cosec}^2 \beta s \qquad (1.16)$$

$$\therefore \frac{\partial \rho^2}{\partial s} = -2\beta K_2 \operatorname{cosec}^2 \beta s \cot \beta s$$

$$\frac{\partial \rho^2}{\partial s} = 0 \text{ when } -\operatorname{cosec}^2 \beta s \cot \beta s = 0.$$

As $\operatorname{cosec}\beta s \neq 0$, $\cot \beta s = 0$. $\therefore \beta s = \frac{\pi}{2}$ and $s = \frac{\pi}{2} \cdot \frac{\lambda}{2\pi} = \frac{\lambda}{4}$.

Therefore when the separation is critical and $K_3 = 0$, the minimum occurs at $s = \frac{\lambda}{4}$ cm.

But, if $s = \frac{\lambda}{4}$ and $s_1 \neq s_0$ and $K_3 \neq 0$, then

$$\rho^2 = K_1 + K_2 \operatorname{cosec}^2 \beta s + 2K_3 \cot \beta s \qquad (1.14)$$

$$= K_1 + K_2$$

$$= \frac{1}{|Z_1|^2}\left[(A^2+B^2)-(C^2+D^2)+(C^2+D^2)\cosh 2a + (AC+BD)\sinh 2a\right] \quad \text{from (1.8)}$$

$$= \frac{1}{|Z_1|^2}\left[A^2+B^2\right] \qquad \text{since } a = 0$$

$$= \frac{1}{|Z_1|^2}\left[R_2^2\cos^2\beta s_1 + X_2^2\cos^2\beta s_1 + Z_0^2\sin^2\beta s_1 + 2Z_0X_2\sin\beta s_1\cos\beta s_1\right] \quad \text{from (1.6)}$$

$$= \frac{1}{|Z_1|^2}\left[Z_2^2\cos^2\beta s_1 + Z_0^2\sin^2\beta s_1 + 2Z_0X_2\sin\beta s_1\cos\beta s_1\right]$$

$$\therefore \frac{\partial \rho^2}{\partial s_1} = \frac{1}{|Z_1|^2}\left[-2Z_2^2\beta\cos\beta s_1\sin\beta s_1 + 2Z_0^2\beta\sin\beta s_1\cos\beta s_1 + 2X_2Z_0(\cos^2\beta s_1 - \sin^2\beta s_1)\right]$$

$$\therefore \frac{\partial \rho^2}{\partial s_1} = 0 \text{ when } \frac{1}{2}(Z_2^2 - Z_0^2)\sin 2\beta s_1 = X_2 Z_0 \cos 2\beta s_1$$

$$\text{i.e.} \quad \tan 2\beta s_1 = -\frac{2X_2Z_0}{Z_0^2 - Z_2^2}. \qquad (2.1)$$

Now the value of s_1 which makes $K_3 = 0$ is given by

$$\tan \beta s_1 = -\frac{Z_2^2}{Z_0 X_2} \qquad (1.9)$$

from which (2.2)

$$\tan 2\beta s_1 = -\frac{2X_2Z_0}{Z_0^2 \cdot \frac{X_2^2}{Z_2^2} - Z_2^2}.$$

(2.1) and (2.2) do not give the same value for s_1 unless $X_2^2 = Z_2^2$. Consequently, this method gives an incorrect value for the critical separation. In practice, it may not be far from the correct value especially when the impedance is largely reactive.

The first method does give the correct value for s_0 but it requires many more sets of readings than Dr. Williams suggests to obtain an accurate value. A more satisfactory method is as follows.

(b) NEW METHOD FOR DETERMINATION OF CRITICAL SEPARATION

In this method, graphs of ρ^2 against s are plotted for values of s near the minimum, and the positions of the minima found, for varying values of s_1. The values of s at the minima are then used to plot a straight line graph from which the critical separation can be found.

When $K_3 \neq 0$, $\rho^2 = K_1 + K_2 \operatorname{cosec}^2 \beta s + 2K_3 \cot \beta s$. (1.14)

$$\frac{\partial \rho^2}{\partial s} = -2K_2 \beta \operatorname{cosec}^2 \beta s \cot \beta s - 2K_3 \beta \operatorname{cosec}^2 \beta s.$$

$$\therefore \frac{\partial \rho^2}{\partial s} = 0 \text{ when } K_3 \operatorname{cosec}^2 \beta s = -K_2 \operatorname{cosec}^2 \beta s \cot \beta s.$$

Since $\operatorname{cosec} \beta s \neq 0$, $\cot \beta s_{min.} = -\dfrac{K_3}{K_2}$. (2.3)

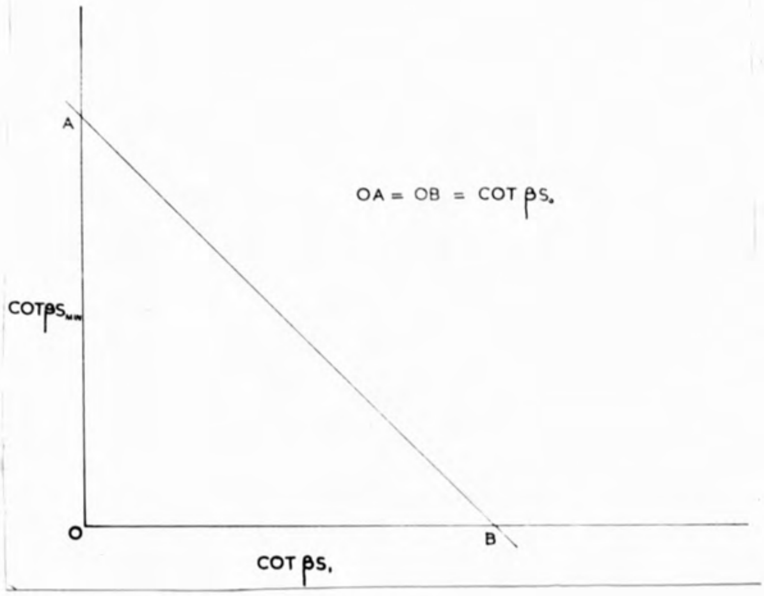


Fig. 2·1.

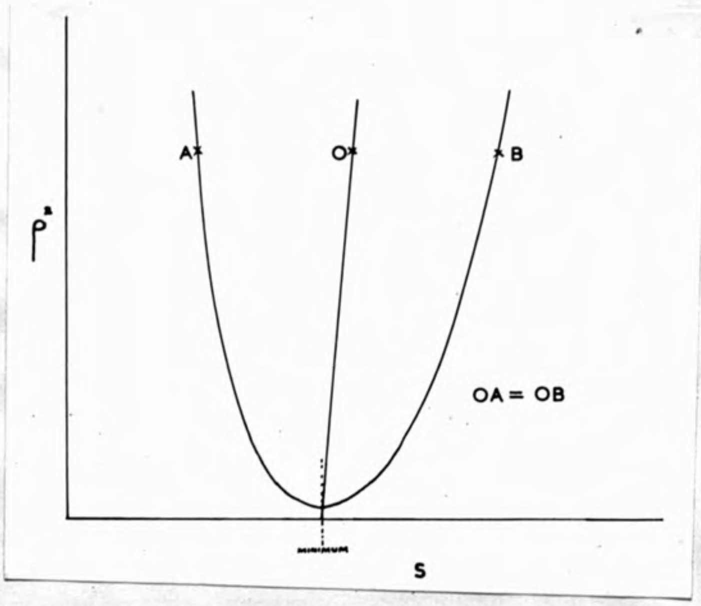


Fig. 2·2.

Substituting the values of K_3 and K_2 from (1.8) and (1.6), this condition becomes,

$$\cot \beta s_{min.} = - \frac{R_2^2 \cos\beta s_1 \sin\beta s_1 + X_2 \cos\beta s_1 \sin\beta s_1 + Z_0 X_2 \sin^2\beta s_1}{Z_2^2 \sin^2\beta s_1}.$$

$$= - \frac{1 + \frac{Z_0 X_2}{Z_2^2} \tan\beta s_1}{\tan\beta s_1}$$

$$= - \cot\beta s_1 - \frac{Z_0 X_2}{Z_2^2}$$

$$\underline{\cot \beta s_{min.} = - \cot\beta s_1 + \cot\beta s_0} \qquad (2.4)$$

In this equation, $s_{min.}$ ~~min.~~ refers to the value of s at the minimum of the ρ^2/s curve for a certain value of s_1. A graph of cot $\beta s_{min.}$ ~~min.~~ against $\cot\beta s_1$ is a straight line with two equal intercepts, which give a value for $\cot\beta s_0$. (fig.2.1)

This method gives a definite value for the critical separation from any five or more sets of readings for which $K_3 \neq 0$. Thus the critical separation can be found without having to set the bridges at the correct position by a process of trial and error. The only difficulty lies in locating the positions of the minima.

The positions of the minima can be found by plotting values of the mean abscissae of the ρ^2/s curve against s. This is a straight line (fig.2.2) which must pass through the minimum. It is parallel to the ρ^2 axis when $s_1 = s_0$ and thus gives an indication of the nearness of the separation to its critical value.

(c) EXPERIMENTAL WORK

Experimental work was carried out to test the sensitivity and accuracy of Dr. Williams' method. The apparatus used is described in the following paragraphs. Much of it was the same as was used by Dr. Williams.

Apparatus

The transmission lines were constructed of right-angle brass so that the system should be rigid and the separation uniform. The characteristic impedance for right-angle brass is difficult to calculate, and the value given by Dr. Williams, 191 ohms, was taken as correct. The lines were three metres long and the inner surfaces were 6 cms. apart. They were supported at intervals along their length by ebonite pillars mounted on wooden blocks. The unknown impedance was connected across one end of the lines and the other end was coupled to the oscillator.

The oscillator consisted of two E 1171 valves working in push-pull. A pair of brass rods formed the tuned anode

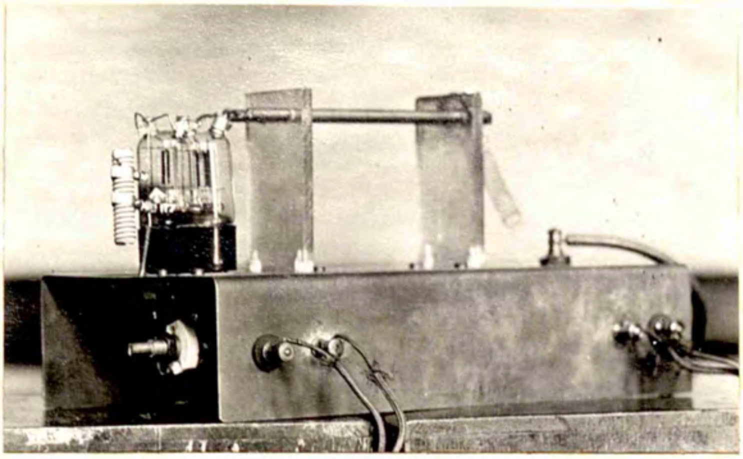


Fig. 2.3.

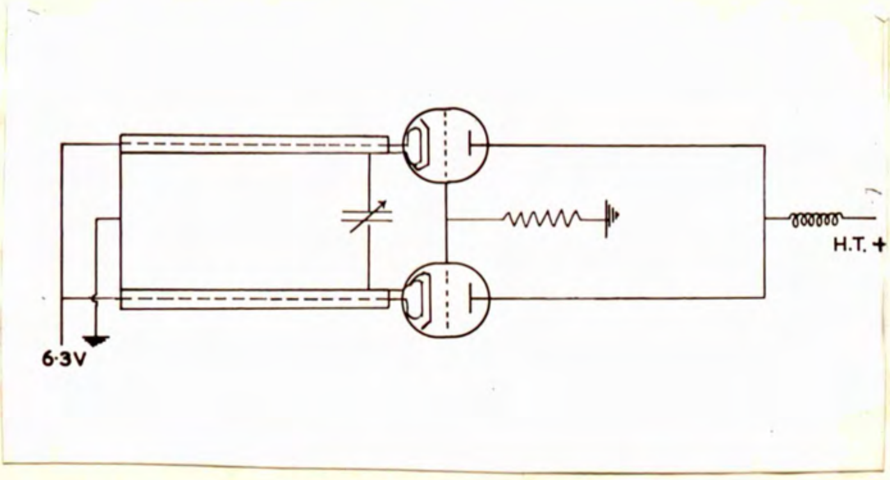


Fig. 2.4.

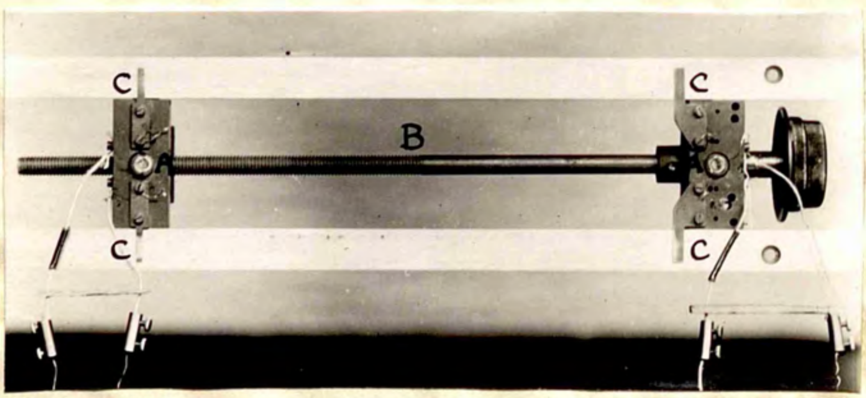


Fig. 2.5

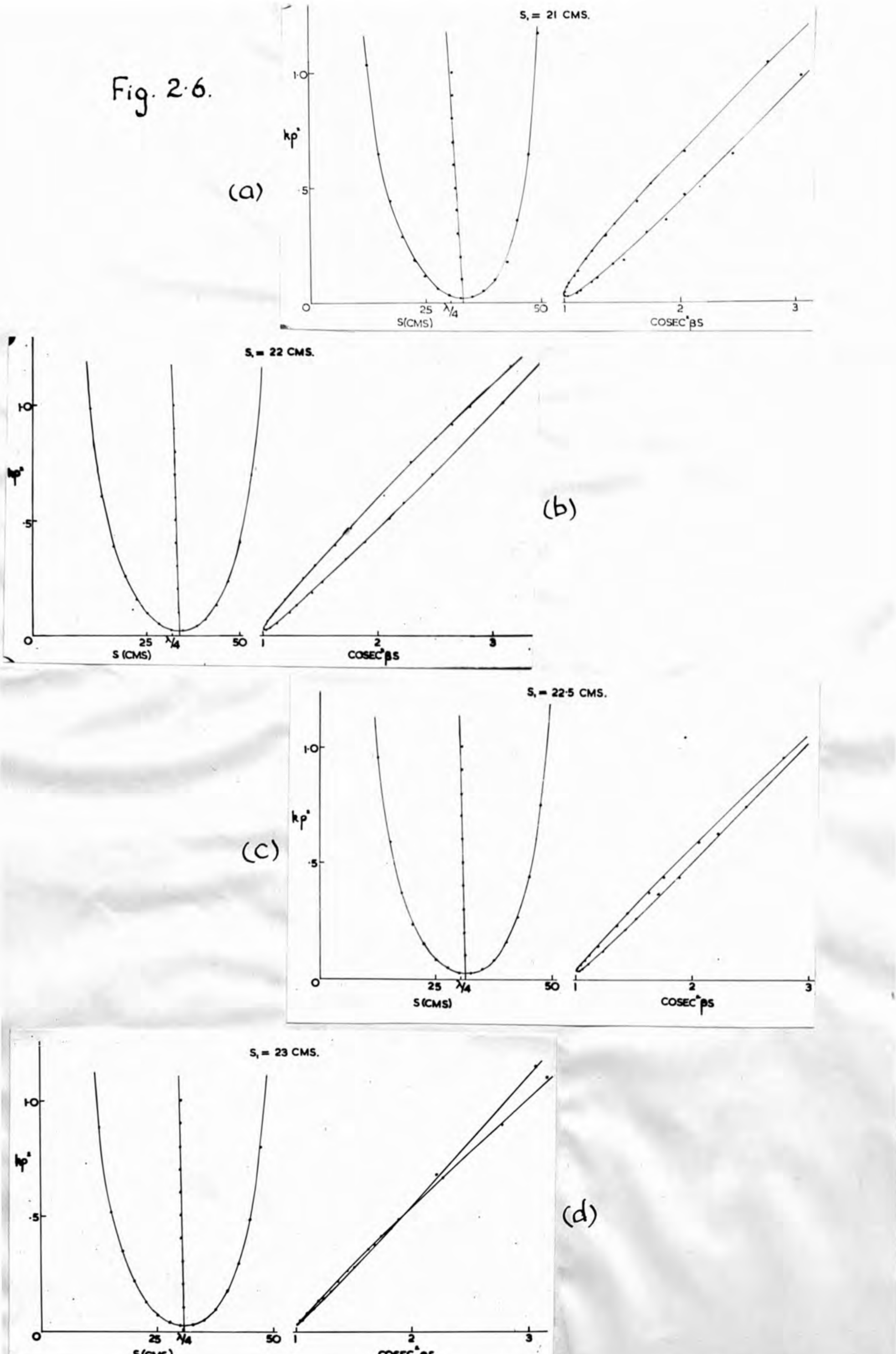
circuit, and two coaxial lines formed the tuned ~~grid~~ filament circuit. The transmission lines were coupled to the oscillator by means of a loop of thick copper wire supported above the anode lines. The amount of current flowing in the transmission lines could be adjusted by altering the tightness of the coupling of this loop. The whole oscillator was enclosed in a metal case to shield it from stray fields and external disturbances. The oscillator is shown in fig. 2.3 and the circuit diagram in fig. 2.4.

Power for the oscillator was supplied by a mains stabilised power pack. The combination was found to be sufficiently stable and the wavelength remained fairly constant. The value of the wavelength was 123.0 cms.

The bridges consisted of vacuo-thermojunctions (A) mounted in ebonite blocks, and are shown in fig. 2.5. A rod (B) of insulating material passed through the centres of the blocks. One block was fixed at the end of the rod and the other could be moved along the rod on a screw thread. The heater wires of the thermocouples were connected to brass knife edges (C) which rested on the lines. The knife edges were screwed on to the ebonite blocks, and good electrical contact was ensured by keeping the lines clean and smooth with emery paper. Springs underneath the lines also helped to maintain good contacts. The thermocouple wires were connected to screws in the ebonite blocks, from which connections were made to current measuring instruments.

The currents in the thermocouples were measured with sensitive microammeters. The wires leading from the thermocouples to the microammeters were kept, as nearly as possible, at right angles to the lines so that there should be no effect from stray fields due to the currents flowing in them. The resistance of the heater wires increases with frequency due to skin-effect, so that the heat produced is not always proportional to the squares of the currents flowing. For a given frequency, however, the resistance is constant, and so the heat produced is proportional to I_1^2 and I_2^2. Both thermo-junctions were calibrated with direct current and found to possess straight line characteristics, so that the microammeter readings could be taken to be proportional to I_1^2 and I_2^2. The ratio of these readings gave a value for $k\rho^2$. The actual value of is not needed as it is the position and not the value of the minimum that is required.

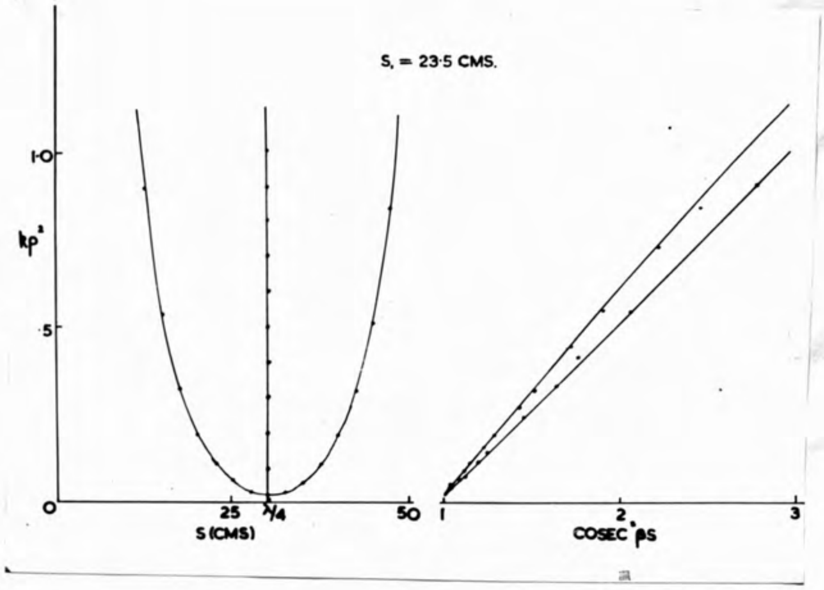
Fig. 2.6.



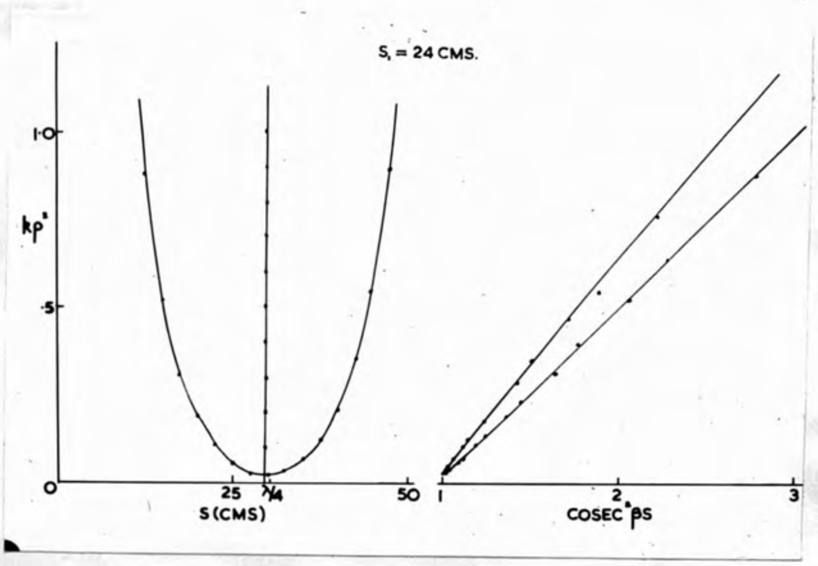
(a)
S_1 = 21 CMS.
$k\rho^2$
1.0
.5
0
25
λ/4
50
S(CMS)
1
2
3
$COSEC^2 \beta S$
(b)
S_1 = 22 CMS.
$k\rho^2$
1.0
.5
0
25
λ/4
50
S (CMS)
1
2
3
$COSEC^2 \beta S$
(c)
S_1 = 22.5 CMS.
$k\rho^2$
1.0
.5
0
25
λ/4
50
S(CMS)
1
2
3
$COSEC^2 \beta S$
(d)
S_1 = 23 CMS.
$k\rho^2$
1.0
.5
0
25
λ/4
50
S(CMS)
1
2
3
$COSEC^2 \beta S$

Fig. 2.6.

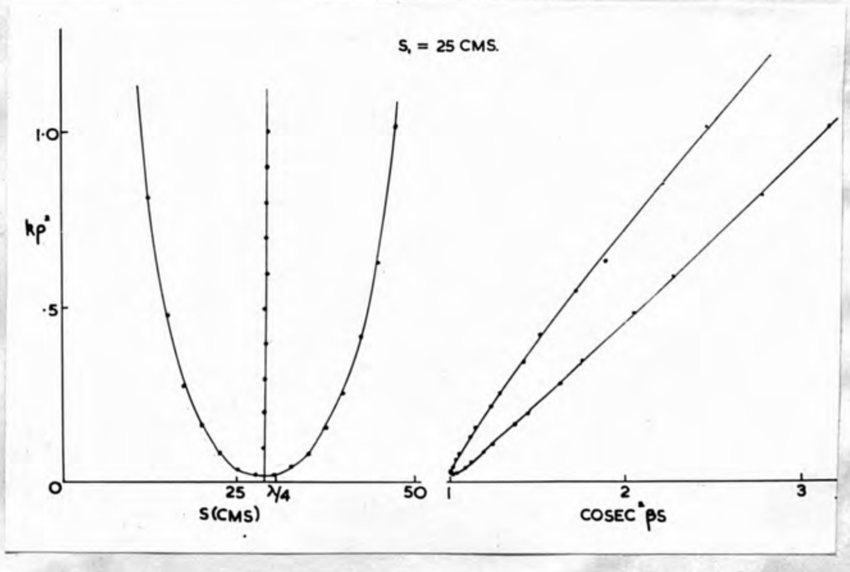
(e)


S, = 23·5 CMS.
kρ²
1·0
·5
0
25
λ/4
50
S (CMS)
1
2
3
COSEC²βS

(f)


S, = 24 CMS.
kρ²
1·0
·5
0
25
λ/4
50
S (CMS)
1
2
3
COSEC²βS

(g)


S, = 25 CMS.
kρ²
1·0
·5
0
25
λ/4
50
S (CMS)
1
2
3
COSEC²βS

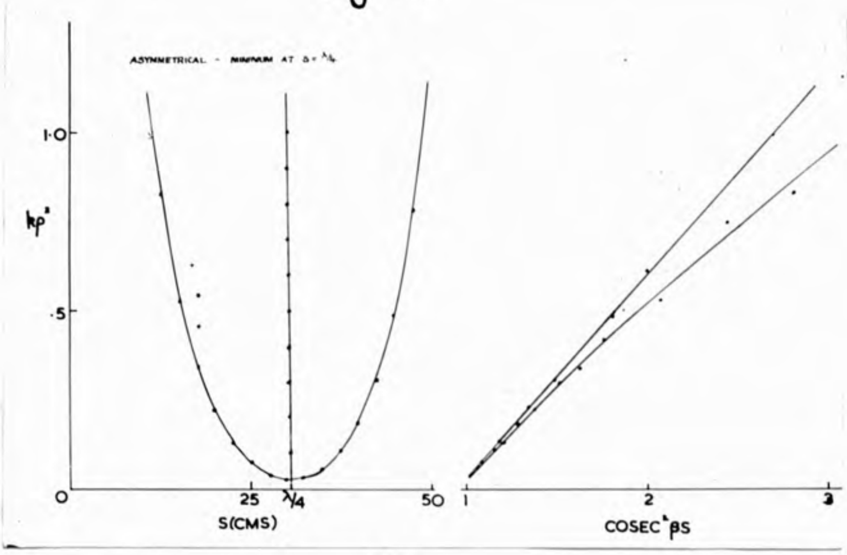
All readings were taken some distance from the lines in order to prevent the introduction of a varying capacity.

Results and Discussion

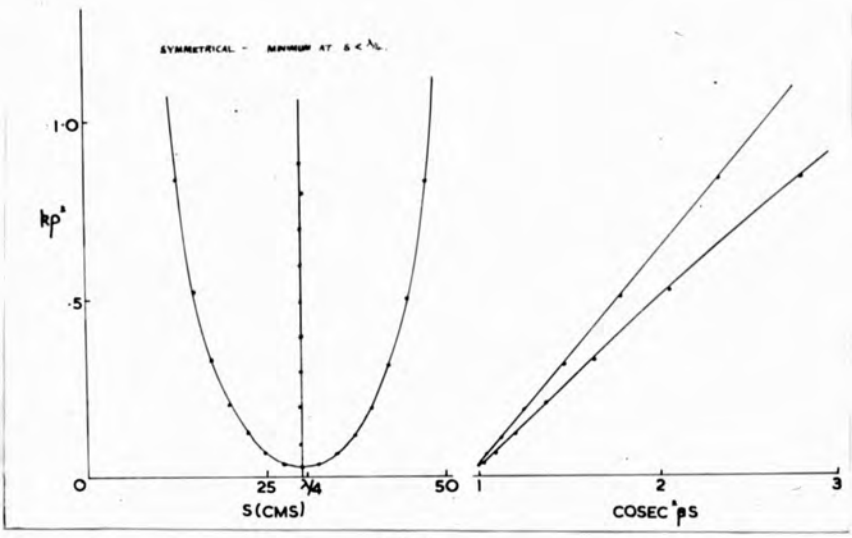
Graphs were plotted of $k\rho^2$ against s for separations varying from 21-25 cms. These are shown in fig. 2.6. The corresponding $\rho^2/\mathrm{cosec}^2\beta s$ curves are shown with them. It will be seen that the ρ^2/s curves pass through various stages of asymmetry to symmetry, and to asymmetry again, and that the position of the minimum moves from a value of s greater than $\frac{\lambda}{4}$ to $s = \frac{\lambda}{4}$ and to $s < \frac{\lambda}{4}$. Also the $\rho^2/\mathrm{cosec}^2\beta s$ curves gradually close up to an approximately straight line and open out again. The reason for the crossing over of these curves was discovered later and will be discussed subsequently.

K_3 is zero only when the ρ^2/s curve is symmetrical, and not when the minimum occurs at $s = \frac{\lambda}{4}$. The symmetrical curve should have its minimum at $s = \frac{\lambda}{4}$, but in practice it was found that the minimum was always slightly to the left of $s = \frac{\lambda}{4}$. Using a pair of thermojunctions of lower resistance, this effect was much more noticeable. The shift of the minimum indicated that b was not zero but had a finite value, owing to the fact that the lines were not properly shorted.

Fig. 2·7.



(a)



(b)

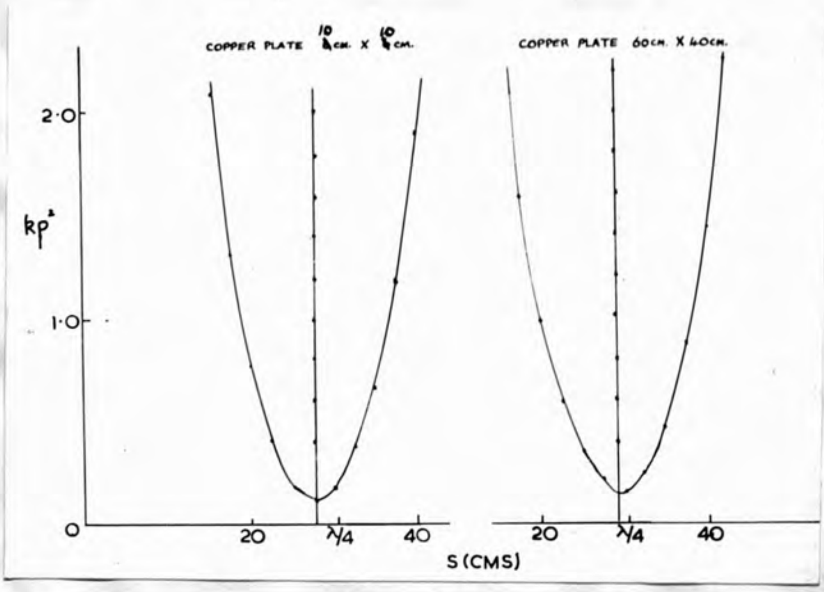


Fig. 2·8.

The $\rho^2/\mathrm{cosec}^2\beta s$ curve is only a straight line for the symmetrical ρ^2/s curve having a minimum at $s = \frac{\lambda}{4}$ cm. An asymmetrical ρ^2/s curve with a minimum at $s = \frac{\lambda}{4}$ gives a curve for the $\rho^2/\mathrm{cosec}^2\beta s$ graph because of its asymmetry, and a symmetrical ρ^2/s curve with a minimum at $s \neq \frac{\lambda}{4}$ gives a curve for the $\rho^2/\mathrm{cosec}^2\beta s$ graph because it is symmetrical about the wrong point. This is illustrated in fig. 2.7.

Theoretically, in order to short circuit a pair of transmission lines it is necessary to have a copper plate stretching to infinity in all directions. In practice, however, it is sufficient to use a plate whose dimensions are large compared with the separation of the lines. (2). The plate used in the first experiments was 10 cm. square compared with a line separation of 6 cm. This is obviously unsatisfactory. The small plate was later replaced by a copper plate about 60 cm. x 40 cm., with marked improvements. The position of the minimum of the ρ^2/s curve was much nearer to $s = \frac{\lambda}{4}$, and the $\rho^2/\mathrm{cosec}^2\beta s$ curve approximated more nearly to a straight line. (fig.2.8).

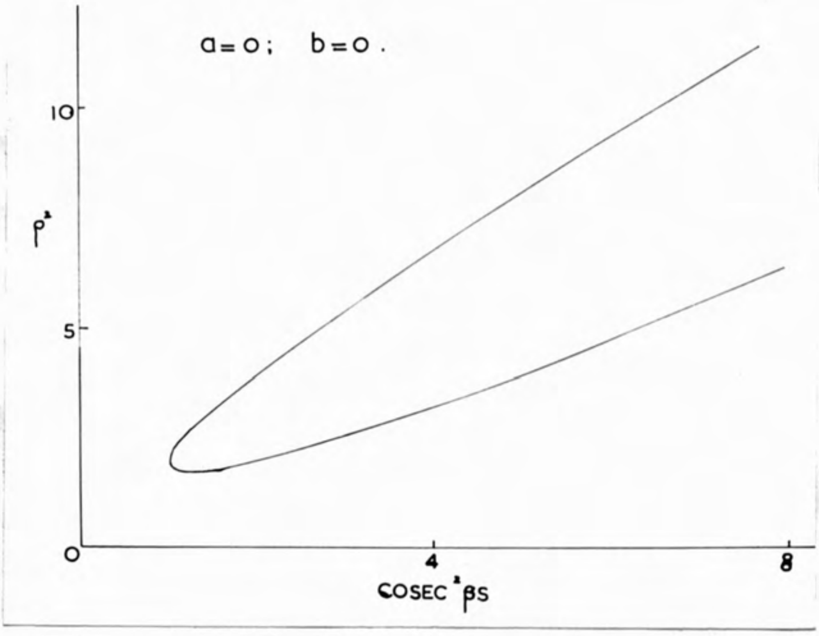


Fig. 2·9.

The finite value of the impedance of the shorting plate also accounts for the crossing over of the $\rho^2/\mathrm{cosec}^2\beta s$ curves. The general equation is

$$\rho^2 = K_1 + \frac{K_2 + K_3 \sin 2(b+\beta s)}{\sinh^2 a + \sin^2(b+\beta s)}. \qquad (1.7)$$

When the shorting plate has zero impedance, $a = b = 0$, and equation (1.7) becomes

$$\rho^2 = K_1 + K_2 \,\mathrm{cosec}^2\beta s + 2K_3 \cot\beta s. \qquad (1.14)$$

This can be written

$$\rho^2 = K_1 + K_2 x^2 \pm \sqrt{1-x^2} \qquad (2.5)$$

where $x = \mathrm{cosec}\,\beta s$, which shows that equation (1.14) represents in general a parabola, closing up to a straight line when $K_3 = 0$. By making $K_1 = K_2 = 2K_3 = 1$, and plotting values of ρ^2 calculated from equation (1.14) against $\mathrm{cosec}^2\beta s$, the curve shown in fig. 2.9 is obtained.

Equation (1.7) may be written

$$\rho^2(1 + \sinh^2 a \,\mathrm{cosec}^2\,\overline{b+\beta s}) = (K_1 \sinh^2 a + K_2)\,\mathrm{cosec}^2\,\overline{b+\beta s} + K_1 + K_3 \cot \overline{b+\beta s}$$

so that equation (1.14) is replaced by

$$\rho^2(1 + A\,\mathrm{cosec}^2\,\overline{b+\beta s}) = K_1 + K_2'\,\mathrm{cosec}^2(b+\beta s) + 2K_3\cot(b+\beta s) \qquad (2.6)$$

where $A = \sinh^2 a$ and $K_2' = K_2 + K_1 \sinh^2 a$.

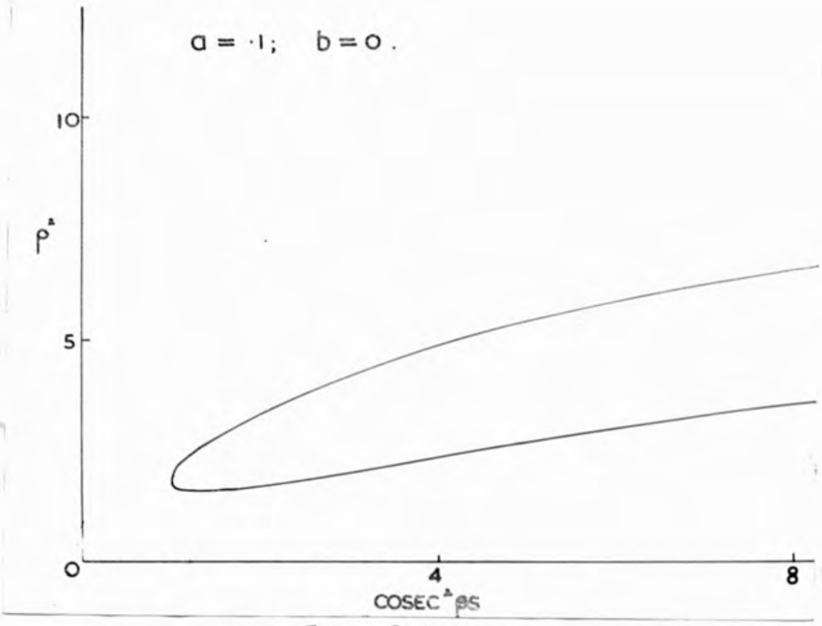


Fig. 2·10.

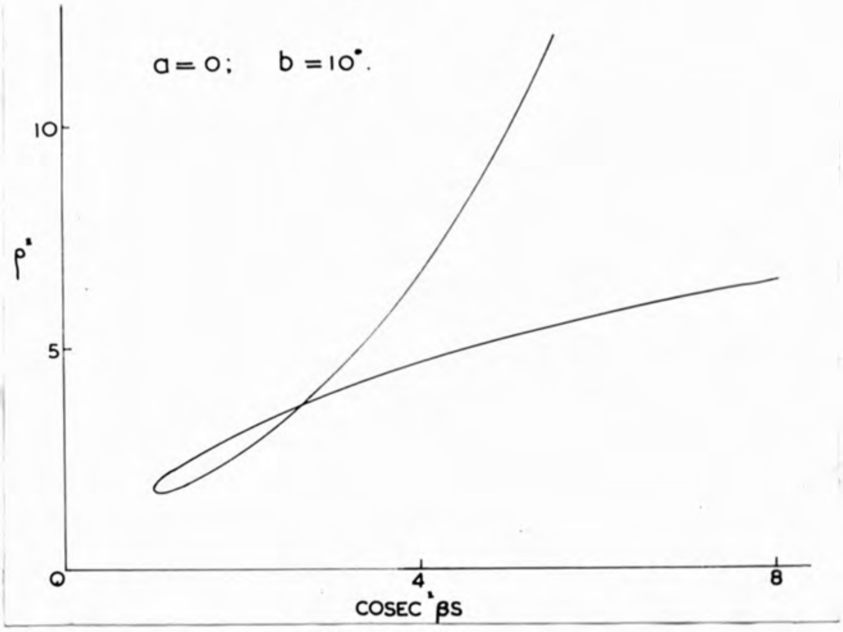


Fig. 2·11.

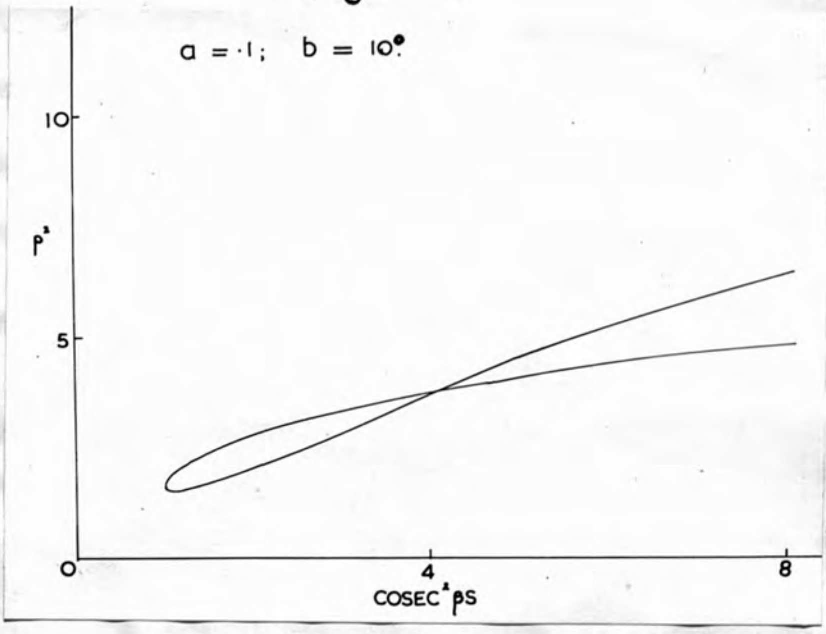


Fig. 2·12.

If $\sinh^2 a = \cdot 1$, $b = 0$, and $K_1 = K_2' = 2K_3 = 1$, equation (2.6) becomes,

$$\rho^2\left(1 + \frac{\operatorname{cosec}^2\beta s}{16}\right) = 1 + \operatorname{cosec}^2\beta s + \cot\beta s. \qquad (2.7)$$

Values of ρ^2 are calculated from equation (2.7) and plotted against $\operatorname{cosec}^2\beta s$, and the curve shown in fig. 2.10 is obtained. A finite value of a tends to draw the curve of fig. 2.9 downwards.

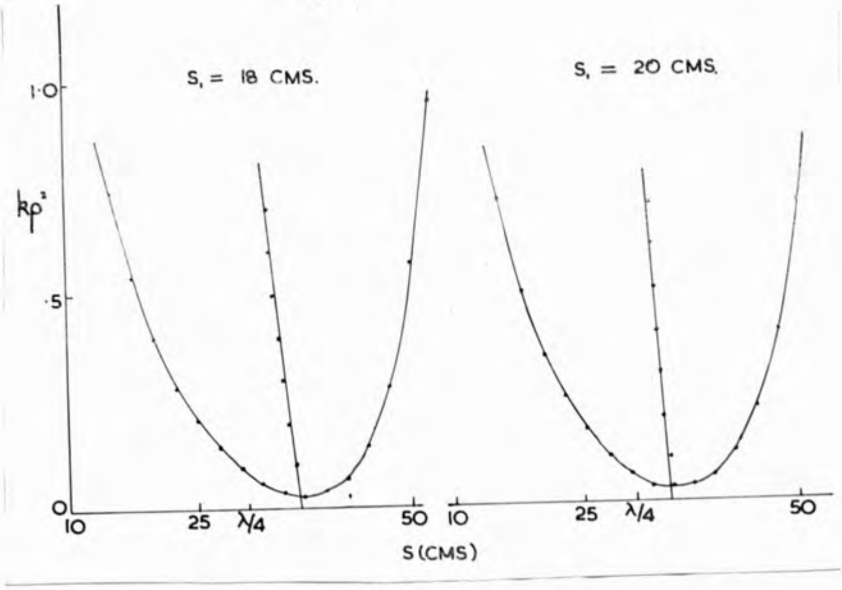
If a is zero and b is finite, equation (1.7) becomes

$$\rho^2 = K_1 + K_2 \operatorname{cosec}^2(b+\beta s) + 2K_3 \cot(b+\beta s) \qquad (2.8)$$

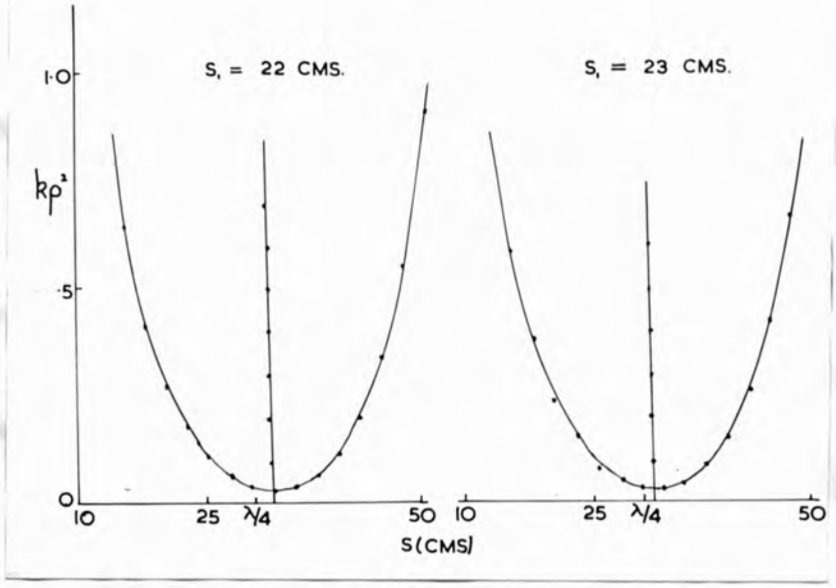
The curve shown in fig. 2.11 is obtained by making $b = 10^\circ$, $K_1 = K_2 = 2K_3 = 1$, and plotting values of ρ^2 calculated from equation (2.8) against $\operatorname{cosec}^2\beta s$. Thus a finite value of b causes the curve to cross over.

The effect of finite values of both a and b is shown in fig. 2.12. As before, $K_1 = K_2' = 2K_3 = 1$, $\sinh^2 a = \cdot 1$ and $b = 10^0$; values of ρ^2 are calculated from equation (2.6) and plotted against $\operatorname{cosec}^2\beta s$. This curve agrees very well with the experimental curves, showing that their form was due to a finite value of the impedance of the shorting plate.

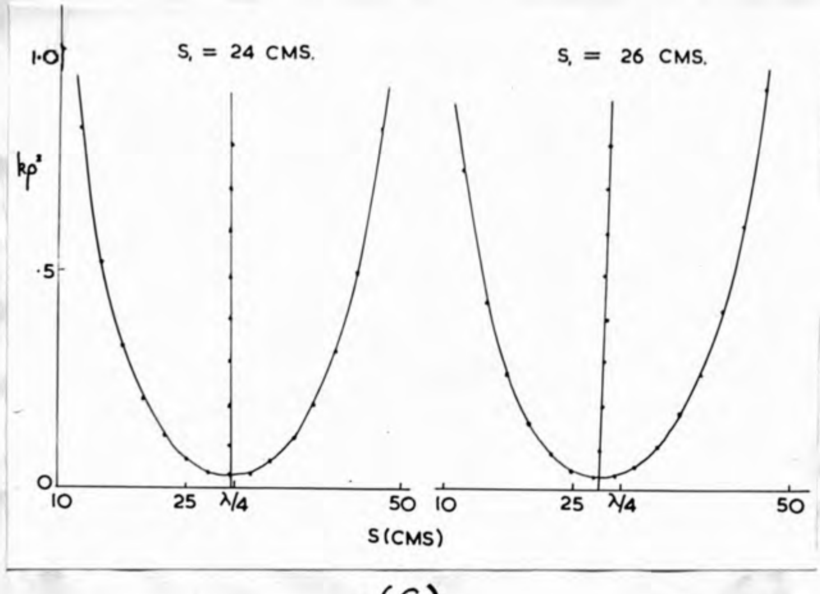
Fig. 3·1


S, = 18 CMS.
S, = 20 CMS.
kρ²
1·0
·5
0
10
25
λ/4
50
10
25
λ/4
50
S (CMS)

(a)

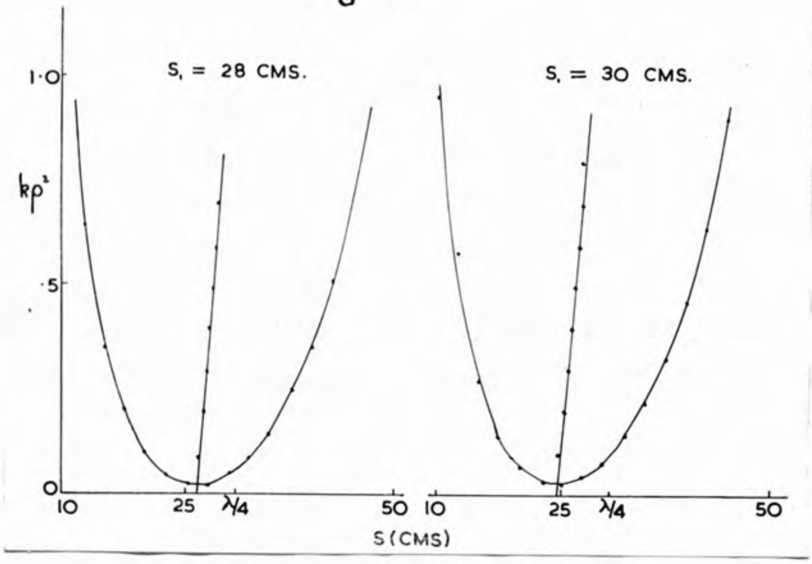

S, = 22 CMS.
S, = 23 CMS.
kρ²
1·0
·5
0
10
25
λ/4
50
10
25
λ/4
50
S (CMS)

(b)


S, = 24 CMS.
S, = 26 CMS.
kρ²
1·0
·5
0
10
25
λ/4
50
10
25
λ/4
50
S (CMS)

(c)

Fig. 3·1



(d)

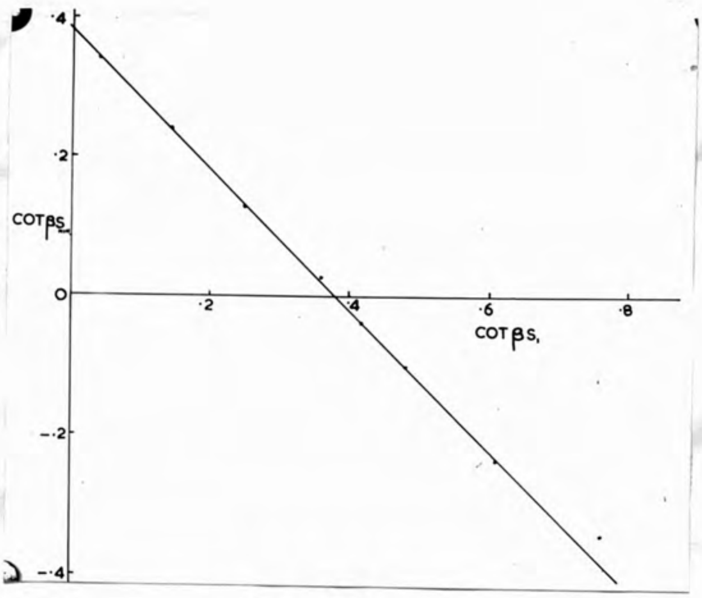


Fig. 3·2.

MEASUREMENT OF AN IMPEDANCE

Using the apparatus previously described, the impedances of two pairs of brass rods were measured. The rods were screwed into the ends of the original lines and shorted with the small copper plate. The pairs were of different lengths, but were similar in every other respect, so that the error introduced at the join could be eliminated. The actual impedances could be easily calculated, and thus served as a check on the accuracy of the method.

Throughout the experiment, microammeter readings were taken at intervals of ~~2-5~~ 2·5 cms. along the lines. Values of s and s_1 were measured with a metre rule. The coupling to the oscillator was arranged so that readings on both microammeters covered the whole scale as nearly as possible.

With the lines shorted with the small copper plate, sets of readings of $k\rho^2$ were taken for varying values of the separation, s_1. Graphs of $k\rho^2$ against s were plotted, (fig.3.1) and the positions of their minima found by plotting the line of mean abscissae. A mean value for the wavelength, was also found from these graphs. Values of $\cot\beta s_{min}$. ~~min.~~ and were calculated, and a graph of $\cot\beta s_{min.}$ against $\cot\beta s_1$ $\cot\beta s_1$ λ was plotted. This was a straight line having equal intercepts, (fig.3.2) and from this s_0 was found to be 23.6 cms.

Fig. 3·3.

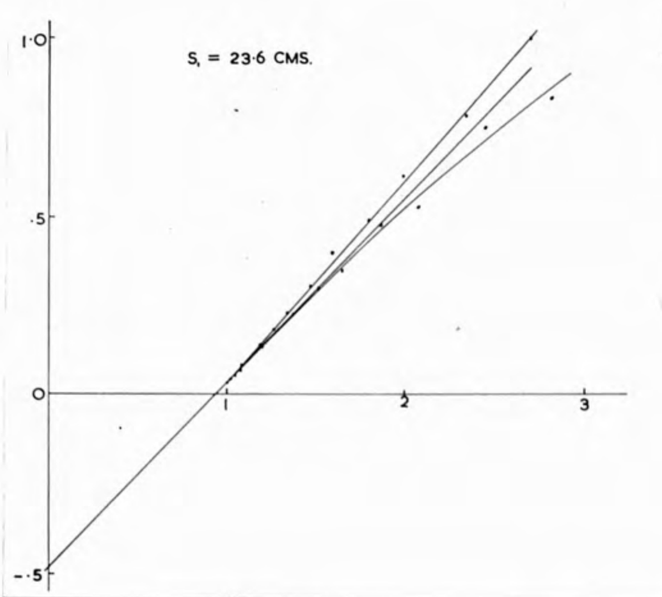


Fig. 3·4.

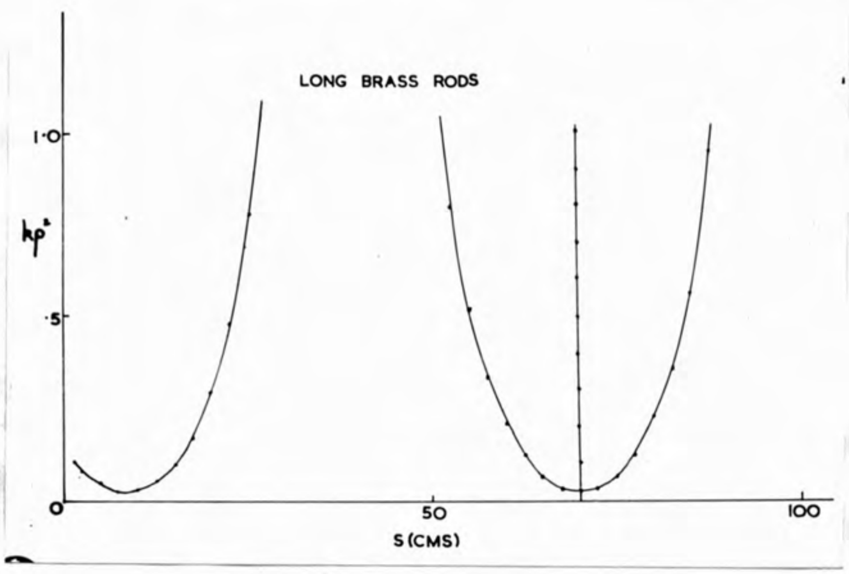
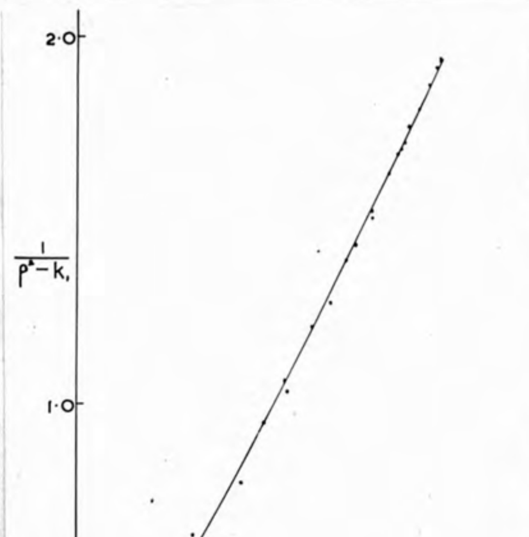


Fig. 3·5.



It will be seen from fig. 3.1 that the graph most nearly symmetrical is that for which $s_1 = 24$ cms., and therefore 23.6 cms. is not the correct value for s_0. This value for s_0 was taken, however, as it was the mean between the most nearly symmetrical curve, and the curve having its minimum at $s = \frac{\lambda}{4}$. The graph of $k\rho^2$ against $\operatorname{cosec}^2\beta s$ for $s_1 = 23.6$ cms. is shown in fig. 3.3. It was, of course, not a straight line. The value of K_1 obtained from the intercept was -.49.

The longer pair of brass rods was then screwed on to the lines and shorted with the small copper plate. The bridges were set 23.6 cms. apart and a set of readings of $k\rho^2$ was taken. (fig.3.4). The positions of the minima were found as before and a mean value for b was calculated. This was 64.8^0. It was then possible to calculate $\frac{1}{\rho^2 - K_1}$ and $\sin^2(b+\beta s)$ and the graph of $\frac{1}{\rho^2 - K_1}$ against $\sin^2(b+\beta s)$ was plotted. (fig.3.5). This, again, was a curve instead of a straight line, due to the fact that the separation was not quite the critical value. The intercept, $\sinh^2 a$, was small, and therefore difficult to measure accurately. Also, the points are rather scattered in this part of the curve, and this adds to the inaccuracy. The value for a was found to be .17. From these values of a and b the components of the

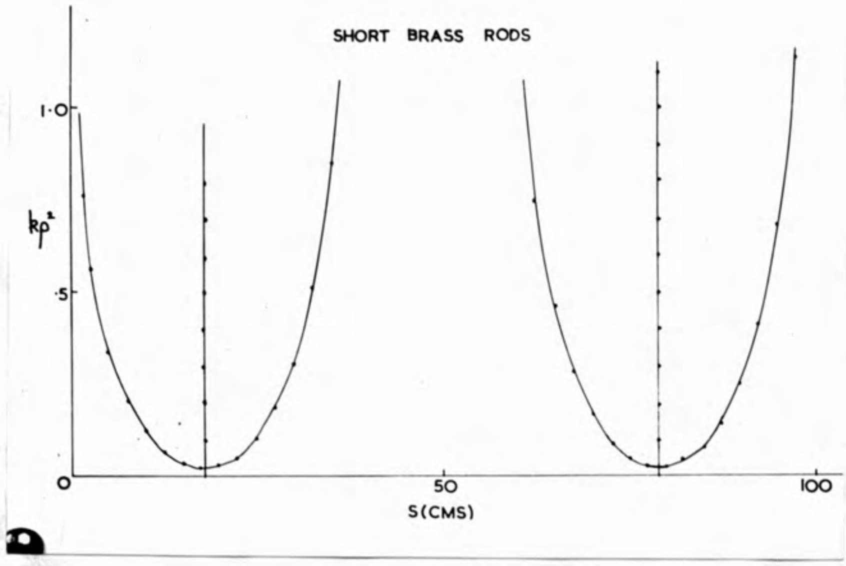

SHORT BRASS RODS
1·0
·5
$k\rho^2$
0
50
100
S(CMS)

Fig. 3·6.

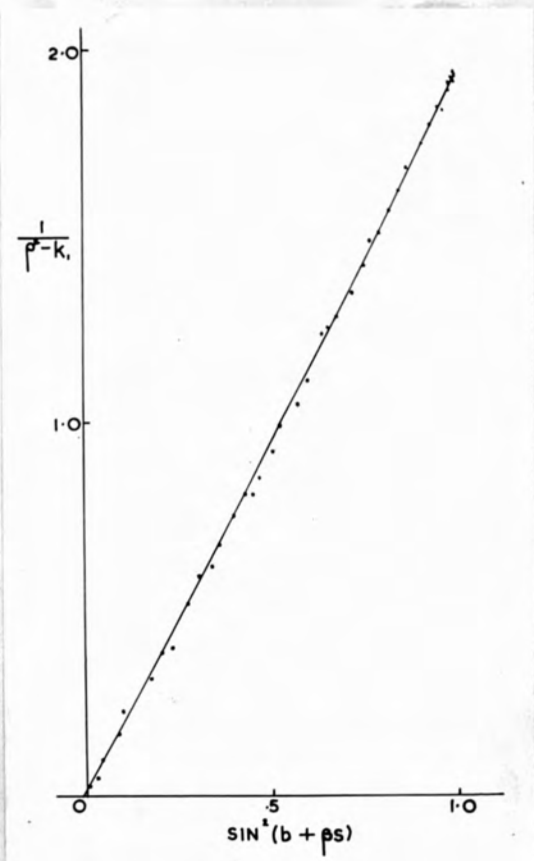

2·0
1·0
$\frac{1}{\rho^2 - k_1}$
0
·5
1·0
$\mathrm{SIN}^2(b+\beta s)$

Fig. 3·7.

apparent impedance were calculated and it was found that

resistive component = 157.3 ohms.

reactive component = 349.6 ohms (positive ∴ inductive)

and the impedance = 383.4 ohms.

This was repeated for the shorter rods (figs. 3.6 & 3.7) and it was found that $a = .1$ and $b = 38.1^{0}$. Hence,

resistive component = 23.9 ohms

reactive component = 146.7 ohms

and impedance = 150.6 ohms.

The actual impedance can be calculated from

$$Z = Z_0 \tanh Pl \tag{3.1}$$

where Z_0 is the characteristic impedance, P is the propagation constant, $(\alpha + i\beta)$, and l is the length of the rods. Since the rods are air-spaced, attenuation is negligible, and

$$\begin{aligned} Z &= Z_0 \tanh i\beta l \\ &= iZ_0 \tan\beta l. \end{aligned} \tag{3.2}$$

Hence the actual impedances are purely reactive. For the rods used, $Z_0 = 331.4$, and therefore, for the larger rods

$$Z_L = 554.0$$

and for the shorter rods,

$$Z_S = 188.0.$$

The measured impedances are both smaller than the calculated impedances. This indicates the error introduced at the join may be regarded as an impedance, Z', in parallel with the known impedance. If Z_L' and Z_s' are the measured impedances,

$$\frac{1}{Z_L'} = \frac{1}{Z_L} + \frac{1}{Z'} \tag{3.3}$$

and

$$\frac{1}{Z_s'} = \frac{1}{Z_s} + \frac{1}{Z'} \tag{3.4}$$

Eliminating Z'

$$\frac{1}{Z_s'} - \frac{1}{Z_L'} = \frac{1}{Z_s} - \frac{1}{Z_L} \tag{3.5}$$

Putting this in terms of the resistive and reactive series components,

$$\frac{1}{R_s' + iX_s'} - \frac{1}{R_L' + iX_L'} = \frac{1}{R_s + iX_s} - \frac{1}{R_L + iX_L} . \tag{3.6}$$

Now $R_s = R_L = 0$. Substituting values in (3.6), it is found that the real part of the L.H.S. is 1.018×10^{-5} ohms which should be equal to the real part of thw R.H.S. which is zero; and the imaginary part of the L.H.S. is 4.27×10^{-3} ohms which should be equal to the imaginary part of the R.H.S., 3.52×10^{-3} ohms. The error still present is due to the difficulty of ~~ding~~ finding K_1 and a accurately, because the short circuit was not perfect.

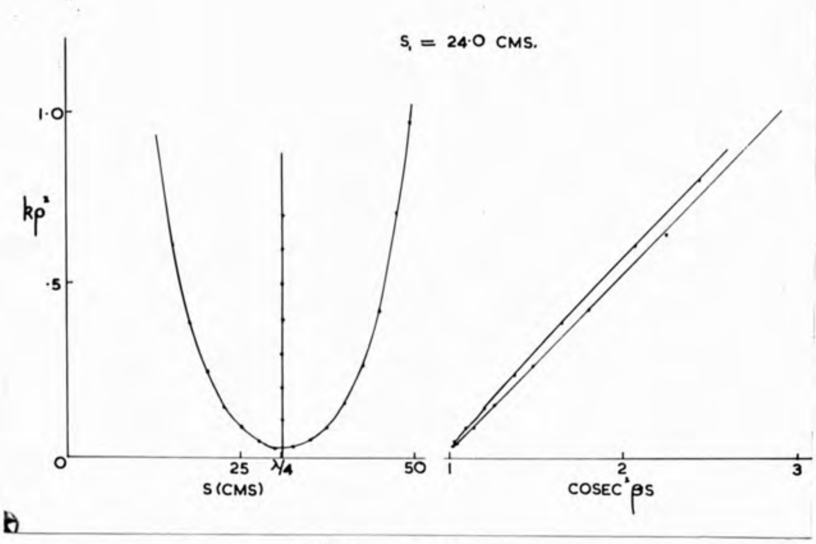


Fig. 3·8.

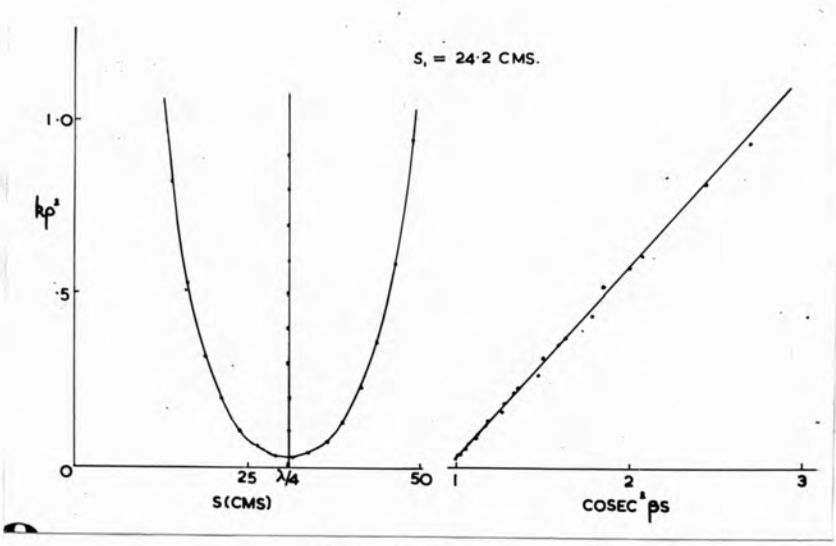


Fig. 3·9.

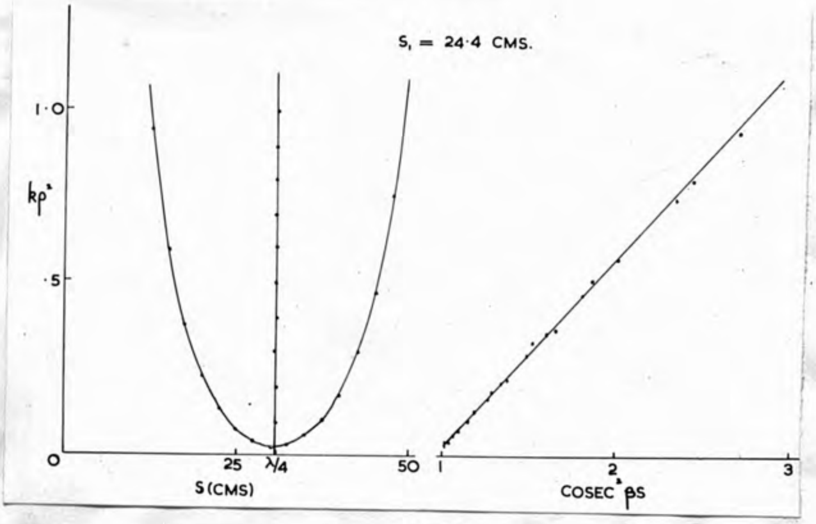


Fig. 3·10.

The large copper plate (40 cms. x 60 cms.) was then used to short the transmission lines. It was soldered instead of screwed on to the end of the lines. A set of readings of $k\rho^2$ was taken with s_1 = 24 cms., and the graph of $k\rho^2$ against s was plotted. (fig. 3.8). The minimum occurred at a value of s greater than $\frac{\lambda}{4}$, and the $\rho^2/\operatorname{cosec}^2\beta s$ graph was curved (fig.3.8). This showed that the critical separation was greater than 24 cms. Two sets of readings were taken at 24.2 cms. and 24.4 cms. and the ρ^2/s and $\rho^2/\operatorname{cosec}^2\beta s$ graphs were plotted. These are shown in figs. 3.9 and 3.10. The actual value of $\frac{\lambda}{4}$ = 30.75 cms. The minimum of the ρ^2/s graph for s_1 = 24.2 cms occured at 30.85 cms., and for s_1 = 24.4 cms. it occurred at 30.65 cms. Both the $\rho^2/\operatorname{cosec}^2\beta s$ curves were as near straight lines as was possible within the limits of experimental error, and both gave the value -.53 for K_1 . The critical separation was most probably 24.3 cms.

Conclusion

The accuracy of this method of measurement depends upon the accuracy with which the constants λ , b , K_1 , and a can be determined.

λ can be found with an accuracy of .1% by selecting a large number of values from the ρ^2/S graphs.

The actual values of a and b affect the accuracy of their measurement. For an impedance having a small resistive component, a is small and difficult to measure; similarly, for an impedance having a small reactive component, b is small. The measurement of a has been discussed by Rogers, (3), and he has shown that a has a maximum value when $b = 45^\circ$. This condition can be satisfied, when possible, by making the modulus of the characteristic impedance equal to the modulus of the unknown impedance.

Even when the impedances are matched in this way and a and b have large values, there will still be slight inaccuracies due to the experimental method of measurement. These can be reduced to a minimum by an accurate setting of the bridges. When the ρ^2/S curve is symmetrical, an accurate value of b can easily be found by plotting the line of mean abscissae. a is found from the $\frac{1}{\rho^2 - K_1} / \sin^2(b+\beta S)$ graph, and thus its accuracy depends on the accuracy with which K_1 can be found. K_1 is found from the $\rho^2/\operatorname{cosec}^2\beta S$ curve, which is a straight line only when the ρ^2/s curve is symmetrical about a line parallel to the ρ^2 axis passing through $s = \frac{\lambda}{4}$. This is achieved by setting the bridges accurately at the critical separation which involves correct shorting of the transmission lines. Any slight deviation from this condition causes the $\rho^2/\mathrm{cosec}^2\beta s$ graph to be curved, and it is thus essential to ensure correct shorting of the lines, in which case the critical separation can be found to .4% and the $\rho^2/\mathrm{cosec}^2\beta s$ and $\frac{1}{\rho^2 - K_1} \Big/ \sin^2(b+\beta s)$ graphs are straight lines within the limits of experimental error.

The wavelength used in these experiments was less than that used by Dr. Williams (120 cms. as compared with 220 cms.) The smaller the wavelength, the more difficult it becomes to obtain a good short circuit; and so in extending this method to smaller wavelengths, particular attention must be paid to this point.

In conclusion, I wish to thank Miss E.M. Williamson, M.Sc. for her help throughout this work.

References

(1) Williams, 1944. "A new method for the measurement of impedance at ultra-high frequencies using a system of Lecher wires." Proc. Phys. Soc., 56, 63.

(2) Willis Jackson. High frequency transmission lines, p.66.

(3) Rogers, 1944. "A discussion of the sensitivity of a method for measuring impedance by means of transmission lines." Proc. Phys. Soc., 56, 1.