The number of graphs not containing $K_{3,3}$ as a minor

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Abstract

We derive precise asymptotic estimates for the number of labelled graphs not containing $K_{3,3}$ as a minor, and also for those which are edge maximal. Additionally, we establish limit laws for parameters in random $K_{3,3}$-minor-free graphs, like the number of edges. To establish these results, we translate a decomposition for the corresponding graphs into equations for generating functions and use singularity analysis. We also find a precise estimate for the number of graphs not containing the graph $K_{3,3}$ plus an edge as a minor.

1 Introduction

We say that a graph is $K_{3,3}$-minor-free if it does not contain the complete bipartite graph $K_{3,3}$ as a minor. In this paper we are interested in the number of simple labelled $K_{3,3}$-minor-free and maximal $K_{3,3}$-minor-free graphs, where maximal means that adding any edge to such a graph yields a $K_{3,3}$-minor. It is known that there exists a constant c, such that there are at most $c^n n!$ $K_{3,3}$-minor-free graphs on n vertices. This follows from a result of Norine et al. [13], which prove such a bound for all proper graph classes closed under taking minors. This gives also an upper bound on the number of maximal $K_{3,3}$-minor-free graphs as they are a proper subclass of $K_{3,3}$-minor-free graphs.

In [11], McDiarmid, Steger and Welsh give conditions where an upper bound of the form $c^n n!$ on the number of graphs $|\mathcal{C}_n|$ on n vertices in a graph class $\mathcal{C}$ yields that $(|\mathcal{C}_n|/n!)^{\frac{1}{n}} \to c > 0$ as $n \to \infty$. These conditions are satisfied for $K_{3,3}$-minor-free graphs, but not in the case of maximal $K_{3,3}$-minor-free graphs as one condition requires that two disjoint copies of a graph of the class in question form again a graph of the class.

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Thus we know that there exists a growth constant c for $K_{3,3}$-minor-free graphs, but not its exact value. For maximal $K_{3,3}$-minor-free graphs we only have an upper bound. Lower bounds on the number of graphs in our classes can be obtained by considering (maximal) planar graphs. Due to Kuratowski's theorem [10] planar graphs are $K_{3,3}$- *and* K_5-minor-free. Hence, the class of (maximal) planar graphs is contained in the class of maximal $K_{3,3}$-minor-free graphs and we can use the number of planar graphs and the number of triangulations as lower bounds. Determining the number (of graphs of sub-classes) of planar graphs has attracted considerable attention [1, 7, 2, 3] in recent years. Giménez and Noy [7] obtained precise asymptotic estimates for the number of planar graphs. Already in 1962, the asymptotic number of triangulations was given by Tutte [15]. Investigating how much the number of planar graphs (triangulations) differs from (maximal) $K_{3,3}$-minor-free graphs was also a main motivation for our research.

In this paper we derive precise asymptotic estimates for the number of simple labelled $K_{3,3}$-minor-free and maximal $K_{3,3}$-minor-free graphs on n vertices, and we establish several limit laws for parameters in random $K_{3,3}$-minor-free graphs. More precisely, we show that the number g_n, c_n, and b_n of not necessarily connected, connected and 2-connected $K_{3,3}$-minor-free graphs on n vertices, and the number m_n of maximal $K_{3,3}$-minor-free graphs on n vertices satisfy

$$\begin{aligned} g_n &\sim \alpha_g\, n^{-7/2}\, \rho_g^{-n}\, n!, \\ c_n &\sim \alpha_c\, n^{-7/2}\, \rho_c^{-n}\, n!, \\ b_n &\sim \alpha_b\, n^{-7/2}\, \rho_b^{-n}\, n!, \\ m_n &\sim \alpha_m\, n^{-7/2}\, \rho_m^{-n}\, n! \end{aligned}$$

where $\alpha_g \doteq 0.42643 \cdot 10^{-5}$, $\alpha_c \doteq 0.41076 \cdot 10^{-5}$, $\alpha_b \doteq 0.37074 \cdot 10^{-5}$, $\alpha_m \doteq 0.25354 \cdot 10^{-3}$, and $\rho_c^{-1} = \rho_g^{-1} \doteq 27.22935$, $\rho_b^{-1} \doteq 26.18659$, and $\rho_m^{-1} \doteq 9.49629$ are analytically computable constants. Moreover, we derive limit laws for $K_{3,3}$-minor-free graphs, for instance we show that the number of edges is asymptotically normally distributed with mean κn and variance λn, where $\kappa \doteq 2.21338$ and $\lambda \doteq 0.43044$ are analytically computable constants. Thus the expected number of edges is only slightly larger than for planar graphs [7].

To establish these results for $K_{3,3}$-minor-free graphs, we follow the approach taken for planar graphs [1, 7]: we use a well-known decomposition along the connectivity structure of a graph, i.e. into connected, 2-connected and 3-connected components, and translate this decomposition into relations of our generating functions. This is possible as the decomposition for $K_{3,3}$-minor-free graphs which is due to Wagner [16] fits well into this framework. Then we use singularity analysis to obtain asymptotic estimates and limit laws for several parameters from these equations.

For maximal $K_{3,3}$-minor-free graphs the situation is quite different, as the decomposition which is again due to Wagner has further constraints (it restricts which edges can be used to merge two graphs into a new one). The functional equations for the generating functions of *edge-rooted* maximal graphs are easy to obtain but in order to go to unrooted graphs, special integration techniques based on rational parametrization of rational curves are needed. This is the most innovative part of the paper with respect to previous work, specially with respect to the techniques developed in [7]. As a result, we can derive equations for the generating functions which involve the generating function for triangulations derived by Tutte [15], and deduce precise asymptotic estiamates.

In the subsequent sections, we proceed as follows. First, we turn to maximal $K_{3,3}$-minor-free and $K_{3,3}$-minor-free graphs in Sections 2 and 3 respectively. In each of these sections, we will first derive relations for the generating functions based on a decomposition of the considered graph class and then apply singularity analysis to obtain asymptotic estimates for the number (and properties) of the graphs in these classes. The last section contains the enumeration of graphs not containing $K_{3,3}^+$ as a minor, where $K_{3,3}^+$ is the graph obtained from $K_{3,3}$ by adding an edge.

Throughout the paper variable x marks vertices and variable y marks edges. Unless we specify the contrary, the graphs we consider are labelled and the corresponding generating functions are exponential. We often need to distinguish an atom of our combinatorial objects; for instance we want to mark a vertex in a graph as a root vertex. For the associated generating function this means taking the derivative with respect to the corresponding variable and multiplying the result by this variable. To simplify the formulas, we use the following notation. Let $G(x,y)$ and $G(x)$ be generating functions, then we abbreviate $G^\bullet(x,y) = x\frac{\partial}{\partial x}G(x,y)$ and $G^\bullet(x) = x\frac{\partial}{\partial x}G(x)$. Additionally, we use the following standard notation: for a graph G we denote by $V(G)$ and $E(G)$ the vertex- and edge-set of G.

2 Maximal $K_{3,3}$-minor-free graphs

Already in the 1930s, Wagner [16] described precisely the structure of maximal $K_{3,3}$-minor-free graphs. Roughly speaking his theorem states that a maximal graph not containing $K_{3,3}$ as a minor is formed by gluing planar triangulations (different from K_5^-) and the exceptional graph K_5 along edges, in such a way that no two different triangulations are glued along an edge. Before we state the theorem more precisely, we introduce the following notation (similar to [14], see also Section 3.1).

Definition 2.1. *Let G_1 and G_2 be graphs with disjoint vertex-sets, where each edge is either colored blue or red. Let $e_1 = (a,b) \in E(G_1)$ and $e_2 = (c,d) \in E(G_2)$ be an edge of G_1 and G_2 respectively. For $i = 1,2$ let G_i' be obtained by deleting edge e_1 and coloring edge e_2 blue if e_1 and e_2 were both colored blue and red otherwise. Let G be the graph obtained from the union of G_1' and G_2' by identifying vertices a and b with c and d respectively. Then we say that G is a* strict 2-sum *of G_1 and G_2. We say that a strict 2-sum is* proper *if at least one of the edges e_1 and e_2 is blue.*

Theorem 2.2 (Wagner's theorem [16]). *Let $\mathcal{T}$ denote the set of all labelled planar triangulations (excluding the graph obtained by removing one edge from K_5) where each edge is colored red. Let each edge of the complete graph K_5 be colored blue. A graph is maximal $K_{3,3}$-minor-free if and only if it can be obtained from planar triangulations and K_5 by proper, strict 2-sums.*

Let $\mathcal{A}$ be the family of maximal graphs not containing $K_{3,3}$ as a minor. Let $\mathcal{H}$ be the family of edge-rooted graphs in $\mathcal{A}$, where the root belongs to a triangulation, and let $\mathcal{F}$ be edge-rooted graphs in $\mathcal{A}$, where the root does not belong to a triangulation.

Let $T_0(x,y)$ be the generating function (GF for short) of edge-rooted planar triangulations (the root-edge is included), and let $K_0(x,y)$ be the GF of edge-rooted K_5 (the root-edge is not included). Let $A(x,y), F(x,y), H(x,y)$ be the GFs associated respectively to the families $\mathcal{A}, \mathcal{F}, \mathcal{H}$. In all cases the two endpoints of the root edge do not bear labels, and the root edge is oriented; this amounts to multiplying the corresponding GF by $2/x^2$. Notice that

$$K_0 = \frac{2}{x^2}\frac{\partial}{\partial y}\left(y^{10}\frac{x^5}{5!}\right) = y^9\frac{x^3}{6}.$$

Since edge-rooted graphs from $\mathcal{A}$ are the disjoint union of $\mathcal{H}$ and $\mathcal{F}$, we have

$$H(x,y) + F(x,y) = \frac{2}{x^2}y\,\frac{\partial A(x,y)}{\partial y}. \tag{2.1}$$

The fundamental equations that we need are the following:

$$H = T_0(x,F) \tag{2.2}$$

$$F = y\exp\left(K_0(x,H+F)\right) \tag{2.3}$$

The first equation means that a graph in $\mathcal{H}$ is obtained by substituting every edge in a planar triangulation by an edge-rooted graph whose root does not belong to a triangulation (because of the statement of Wagner's theorem). The second equation means that a graph in $\mathcal{F}$ is obtained by taking (an unordered) set of K_5's in which each edge is substituted by an edge-rooted graph either in $\mathcal{H}$ or in $\mathcal{F}$.

Eliminating H we get the equation

$$F = y\exp\left(K_0(x, F + T_0(x,F))\right). \tag{2.4}$$

Hence, for fixed x,

$$\psi(u) = u\exp\left(-K_0(x, u+T_0(x,u)\right) = u\exp\left(-\frac{x^3}{6}(u+T_0(x,u))^9\right) \tag{2.5}$$

is the functional inverse of $F(x,y)$.

In order to analyze F using Equation (2.3) we need to know the series $T_0(x,y)$ in detail. Let T_n be the number of (labelled) planar triangulations with n vertices. Since a triangulation has exactly $3n-6$ edges, we see that

$$T(x,y) = \sum T_n y^{3n-6}\frac{x^n}{n!}$$

is the GF of triangulations. And since

$$T_0(x,y) = \frac{2}{x^2}y\frac{\partial T(x,y)}{\partial y},$$

it is enough to study T.

Let now t_n be the number of rooted (unlabelled) triangulations with n vertices in the sense of Tutte and let $t(x) = \sum t_n x^n$ be the corresponding *ordinary* GF. We know (see [15]) that $t(x)$ is equal to

$$t = x^2\theta(1-2\theta)$$

where $\theta(x)$ is the algebraic function defined by

$$\theta(1-\theta)^3 = x.$$

It is known that the dominant singularity of θ is at $R = 27/256$ and $\theta(R) = 1/4$.

There is a direct relation between the numbers T_n and t_n. An unlabelled rooted triangulation can be labelled in $n!$ ways, and a labelled triangulation ($n \geq 4$) can be rooted in $4(3n-6)$ ways, since we have $3n-6$ possibilities for choosing the root edge, two for orienting the edge, and two for choosing the root face. Hence we have

$$t_n n! = 4(3n-6)T_n, \quad n \geq 4, \qquad t_3 = T_3 = 1.$$

The former identity implies easily the following equation connecting the exponential GF $T(x,y)$ and the ordinary GF $t(x)$:

$$y\frac{\partial T}{\partial y} = y^3\frac{x^3}{4} + \frac{t(xy^3)}{4y^6}.$$

Hence we have

$$T_0(x,y) = \frac{2}{x^2}y\frac{\partial T}{\partial y} = y^3\frac{x}{2} + \frac{t(xy^3)}{2x^2y^6}.$$

The last equation is crucial since it expresses T_0 in terms of a known algebraic function. It is convenient to rewrite it as

$$T_0(x,y) = y^3\frac{x}{2} + \frac{1}{2}L(x,y)(1-2L(x,y)), \qquad \text{where } L(x,y) = \theta(xy^3). \tag{2.6}$$

The series $L(x,y)$ is defined through the algebraic equation

$$L(1-L)^3 - xy^3 = 0. \tag{2.7}$$

This defines a rational curve, i.e. a curve of genus zero, in the variables L and y (here x is taken as a parameter) and admits the rational (actually polynomial) parametrization

$$L = \lambda(t) = -\frac{t^3}{x^2}, \qquad y = \xi(t) = -\frac{t^4 + x^2 t}{x^3}. \tag{2.8}$$

This is a key fact that we use later.

We have set up the preliminaries needed in order to analyze the series $A(x,y)$, which is the main goal of this section. From (2.1) it follows that

$$A(x,y) = \frac{x^2}{2}\int_0^y \frac{H(x,t)}{t}\,dt + \frac{x^2}{2}\int_0^y \frac{F(x,t)}{t}\,dt.$$

The following lemma expresses $A(x,y)$ directly in terms of H and F *without* integrals.

Lemma 2.3. *The generating function $A(x,y)$ of maximal graphs not containing $K_{3,3}$ as a minor can be expressed as*

$$A(x,y) = \frac{-x^2}{60}\left(27(H+F)\log\left(\frac{F}{y}\right) + 10L + 20L^2 + 15\log(1-L) - 30F - 5xF^3\right) \tag{2.9}$$

where $L = L(x, F(x,y))$, $H = H(x,y)$ and $F = F(x,y)$ are defined through (2.7), (2.2) and (2.3).

Proof. Integration by parts gives

$$A(x,y) = \frac{x^2}{2}\int_0^y \frac{H(x,t)+F(x,t)}{t}\,dt = \frac{x^2}{2}(H+F)\log(y) - \frac{x^2}{2}I \tag{2.10}$$

where

$$I = \int_0^y \log(t)\left(H'(x,t)+F'(x,t)\right)\,dt$$

and derivatives are with respect to the second variable. Because of (2.5), the change of variable $s = F(x,t)$ gives $t = \psi(s)$ and

$$\log(t) = \log(s) - \frac{x^3}{6}\left(s + T_0(x,s)^9\right).$$

Since $H = T_0(x,F)$ we have $H' = T_0'(x,F)F'$ and so

$$\begin{aligned} I &= \int_0^F \left(\log(s) - \frac{x^3}{6}(s+T_0(x,s))^9\right)(1+T_0'(x,s))\,ds \\ &= -\frac{x^3}{6}\frac{(F+T_0(x,F))^{10}}{10} + \int_0^F \log(s)\,(1+T_0'(x,s))\,ds \\ &= -\frac{1}{10}(H+F)\log\left(\frac{F}{y}\right) + \int_0^F \log(s)\,(1+T_0'(x,s))\,ds \end{aligned}$$

where the last equality follows from Equation (2.3).

It remains to compute the last integral. From (2.6) it follows easily that

$$T_0' = \frac{3y^2x}{2}\left(1 + \frac{1}{(1-L)^2}\right). \tag{2.11}$$

Now we change variables according to (2.8) taking $s = \xi(t)$, so that $L = \lambda(t)$. Let ζ be the inverse function of ξ, so that $t = \zeta(s)$. Observe that $\zeta(s)$ satisfies

$$\zeta^4 + x^2\zeta + x^3 s = 0.$$

Then we have

$$\begin{aligned}
&\int_0^F \log(s)\,(1+T_0'(x,s))\;ds\\
=\;&\int_0^{\zeta(F)} \log(\xi(t))\left(1+\frac{3\xi(t)^2x}{2}\left(1+\frac{1}{(1-\lambda(t))^2}\right)\right)\xi'(t)\,dt
\end{aligned}$$

After substituting the expressions for $\xi(t)$ and $\lambda(t)$, the integrand in the last integral is equal to

$$f(x,t) = -\frac{1}{2x^8}\,\left(4\,t^3+x^2\right)\left(2\,x^5+3\,t^8+6\,t^5x^2+6\,t^2x^4\right)\ln\left(-\frac{t^4+x^2t}{x^3}\right).$$

The function $f(x,t)$ can be integrated in elementary terms, resulting in

$$\begin{aligned}
\int_0^{\zeta(F)} f(x,t)dt = &\left(-\frac{5\zeta^6}{2x^4}-\frac{\zeta^{12}}{2x^8}-\frac{\zeta^3}{x^2}-\frac{\zeta^4}{x^3}-\frac{\zeta}{x}-\frac{3\zeta^9}{2x^6}\right)\log\left(-\frac{\zeta^4+x^2\zeta}{x^3}\right)\\
&+\frac{7\zeta^6}{6x^4}-\frac{\zeta^3}{6x^2}+\frac{\zeta}{x}+\frac{\zeta^4}{x^3}+\frac{\zeta^9}{2x^6}+\frac{\zeta^{12}}{6x^8}-\frac{1}{2}\log\left(1+\frac{\zeta^3}{x^2}\right),
\end{aligned}$$

where $\zeta=\zeta(F)$. All terms with ζ are powers of either of the two forms

$$-\frac{\zeta^4+x^2\zeta}{x^3}=\xi(\zeta(F))=F, \qquad -\frac{\zeta^3}{x^2}=\lambda(\zeta(F))=L(x,F),$$

so we can write the integral of $f(x,t)$ explicitly in terms of F and $L=L(x,F)$,

$$\left(-\frac{1}{2}L^4+\frac{3}{2}L^3-\frac{5}{2}L^2+L+F\right)\log(F)+\frac{L^4}{6}-\frac{L^3}{2}+\frac{7L^2}{6}+\frac{L}{6}+\frac{\log(1-L)}{2}-F.$$

We simplify this expression further using that, according to Equations (2.2), (2.6) and (2.7),

$$H=T_0(x,F)=\frac{1}{2}\left(xF^3+L(1-2L)\right)=\frac{1}{2}(-L^4+3L^3-5L^2+2L). \tag{2.12}$$

Obtaining the final expression for $A(x,y)$ is just a matter of going back to Equation (2.10) and adding up all terms. □

Summarizing, we have an explicit expression for A in terms of x, y, $H(x,y)$ and $F(x,y)$ which involves only elementary functions and the algebraic function $L(x,y)$. Moreover, note that $H(x,y)$ can be written in terms of $L(x,F)$ by Equation (2.12). Our goal is to carry out a full singularity analysis of the univariate GF $A(x)=A(x,1)$. To this end we first perform singularity analysis on $F(x)=F(x,1)$.

Lemma 2.4. *The dominant singularity of $F(x)$ is the unique $\rho > 0$ such that $\rho F(\rho)^3 = 27/256$. The approximate value is $\rho \approx 0.10530385$. The value $F(\rho) \approx 1.0005216$ is the solution of*

$$t = \exp\left(\frac{27^3}{6 \cdot 256^3}\left(1 + \frac{59}{512t}\right)^9\right). \tag{2.13}$$

Proof. The function $F(x)$ satisfies

$$\Phi(x, F) = \exp\left(\frac{x^3}{6}\left(F + T_0(x, F)\right)^9\right) - F. \tag{2.14}$$

Thus the dominant singularity ρ of $F(x)$ may come from T_0 or from a branch point when solving (2.14). Assume that there is no such branch point. Then, since $L(x, y) = \theta(xy^3)$ and the dominant singularity of θ is at $27/256$, we have that $L(\rho, F(\rho)) = 1/4$ and $\rho F(\rho)^3 = 27/256$. Substituting in $\Phi(x, F) = 0$ we obtain Equation (2.13), where t stands for $F(\rho)$. The approximate value is $t \approx 1.0005216$, which gives $\rho \approx 0.10530385$, slightly smaller than $R = 27/256 = 0.10546875$.

We now prove that there is no branch point when solving Φ. If this were the case, then there would exist $\tilde{\rho} \leq \rho$ such that $\partial_F \Phi(\tilde{\rho}, F(\tilde{\rho})) = 0$, where

$$\frac{\partial}{\partial F}\Phi(x, F(x)) = \frac{3}{1024}(-3L^2 + 3L + 2F + 3xF^3)x^3(2F + xF^3 + L - 2L^2)^8 - 1. \tag{2.15}$$

follows by differentiating Equation (2.14), by using $\Phi(x, F(x)) = 0$ and Equations (2.7), (2.11), and (2.12).

Consider $\partial_F \Phi(x, F, L)$ as a function of three independent variables, where $x \geq 0$, $F \geq 1$ and $0 \leq L \leq 1/4$. It follows easily that it is an increasing function on any of them. Hence

$$0 = \partial_F \Phi(\tilde{\rho}, F(\tilde{\rho}), L(\tilde{\rho}, F(\tilde{\rho}))) \leq \partial_F \Phi(\rho, F(\tilde{\rho}), 1/4),$$

since, by assumption, $\tilde{\rho} \leq \rho$. On the other hand $\partial_F \Phi(\rho, t, 1/4) \approx -0.9939$, so by comparing this with $\partial_F \Phi(\rho, F(\tilde{\rho}), 1/4)$ we deduce that $t < F(\tilde{\rho})$. Since $1 = F(0) < t$, by continuity of $F(x)$ there exists a value $x \in (0, \tilde{\rho})$ such that $F(x) = t$, and by the monotonicity of $\Phi(x, F)$ for fixed F there is a unique solution x to $\Phi(x, t) = 0$. This solution is precisely $x = \rho$, contradicting $\tilde{\rho} \leq \rho$. □

Proposition 2.5. *Let ρ and t be as in Lemma 2.4. The singular expansions of $F(x)$ at ρ is*

$$F(x) = t + F_2 X^2 + F_3 X^3 + \mathcal{O}(X^4),$$

where $X = \sqrt{1 - x/\rho}$, and F_2 and F_3 are given by

$$F_2 = \frac{12t(128t + 71)\log(t)}{Q}, \qquad F_3 = \frac{96\sqrt{6}\; t\log(t)M^{3/2}}{Q^{5/2}}$$

$$M = 531\log(t) + 512t + 59, \qquad Q = 9(225 + 512t)\log(t) - 512t - 59.$$

Proof. To obtain the coefficients of the singularity expansion, including the fact that $F_1 = 0$, we apply indeterminate coefficients F_i, L_i of X^i to Equations (2.14) and

$$L(x)(1-L(x))^3 - xF(x)^3 = 0,$$

where $X = \sqrt{1 - x/\rho}$, so that $x = \rho(1 - X^2)$. These calculations are tedious, but can be done with a computer algebra system such as Maple. □

Proposition 2.6. *Let ρ and t be as in Lemma 2.4. The dominant singularity of $A(x)$ is ρ, and its singular expansion at ρ is*

$$A(x) = A_0 + A_2X^2 + A_4X^4 + A_5X^5 + \mathcal{O}(X^6),$$

where $X = \sqrt{1 - x/\rho}$ and A_0, A_2, A_4 and A_5 are given by

$$\begin{aligned}
A_0 &= -\frac{3C}{20t^6}\left(4608\log(t)t + 531\log(t) + 2560\log(3/4) - 5120t + 550\right)\\
A_2 &= \frac{C}{4t^6}\left(4608\log(t)t + 531\log(t) + 3072\log(3/4) - 6144t + 542\right)\\
A_4 &= \frac{3C}{t^6}\left(16Q^{-1}\log(t)(128t+71)^2 + 59\log(t) + 2^9(\log(t)t - 2t + \log(3/4)) + 26\right)\\
A_5 &= \frac{40\sqrt{6}C}{3t^6}\left(\frac{M}{Q}\right)^{5/2}
\end{aligned}$$

where $C = 3^5/2^{25}$, and M and Q are as in Proposition 2.5.

Proof. We just compute the singular expansion

$$A(x) = \sum_{k\geq 0} A_k X^k,$$

by plugging the expansions for $F(x)$ and $L(x)$ of Proposition 2.5 in (2.9). Again, the computations are performed with Maple. □

Theorem 2.7. *The number A_n of maximal graphs with n vertices not containing $K_{3,3}$ as a minor is asymptotically*

$$A_n \sim a \cdot n^{-7/2}\gamma^n n!,$$

where $\gamma = 1/\rho \approx 9.49629$ and $a = -15A_5/8\pi \simeq 0.25354 \cdot 10^{-3}$.

Proof. This is a standard application of singularity analysis (see for example Corollary VI.1 of [6]) to the singular expansion of $A(x)$ obtained in the previous lemma. The singular exponent $5/2$ gives rise to the subexponential term $n^{-7/2}$, and the multiplicative constant is $A_5\Gamma(-5/2)$. □

3 $K_{3,3}$-minor-free graphs

In this section, we derive the asymptotic number of $K_{3,3}$-minor-free graphs and properties of random $K_{3,3}$-minor-free graphs.

3.1 Decomposition and Generating Functions

Let $G(x, y)$, $C(x, y)$ and $B(x, y)$ denote the exponential generating functions of simple labelled *not necessarily connected*, *connected* and 2-*connected* $K_{3,3}$-minor-free graphs respectively. We will use the additional variable q to mark the number of K_5's used in the "construction process" of a $K_{3,3}$-minor-free graph (see below for a more precise explanation), but we won't give it explicitly in the argument list of our generating functions to simplify expressions.

We want to apply singularity analysis to derive asymptotic estimates for the number of $K_{3,3}$-minor-free graphs. To achieve this, we first present a decomposition of this graph class which is due to Wagner [16]. Then we will translate it into relations of our generating functions.

As seen in Theorem 2.2 above, Wagner [16] characterized the class of maximal $K_{3,3}$-minor-free graphs. As a direct consequence we also obtain a decomposition for $K_{3,3}$-minor-free graphs. We will present here a more recent formulation of it, given by Thomas, Theorem 1.2 of [14]. Roughly speaking the theorem states that a graph has no minor isomorphic to $K_{3,3}$ if and only if it can be obtained from a planar graph or K_5 by merging on an edge, a vertex, or taking disjoint components. To state the theorem more precisely, we need the following definition of [14].

Definition 3.1. *Let G_1 and G_2 be graphs with disjoint vertex-sets, let $k \geq 0$ be an integer, and for $i = 1, 2$ let $X_i \subseteq V(G_i)$ be a set of pairwise adjacent vertices of size k. For $i = 1, 2$ let G'_i be obtained by deleting some (possibly none) edges with both ends in X_i. Let $f : X_1 \to X_2$ be a bijection, and let G be the graph obtained from the union of G'_1 and G'_2 by identifying x with $f(x)$ for all $x \in X_1$. In those circumstances we say that G is a k-*sum *of G_1 and G_2.*

Now, we can state the theorem as a consequence of Wagner's theorem in the following way.

Theorem 3.2 ([16], see also Theorem 1.2 of [14]). *A graph has no minor isomorphic to $K_{3,3}$ if and only if it can be obtained from planar graphs and K_5 by means of* 0-*,* 1-*, and* 2-*sums.*

Observe that for 2-connected $K_{3,3}$-minor-free graphs we only have to take 2-sums into account as 0- and 1-sums do not yield a 2-connected graph. In this way the decomposition of Wagner fits perfectly well into a result of Walsh [17] which delivers us – similarly to the case of planar graphs (see [1]) – with the necessary relations for our generating functions.

The second ingredient for obtaining relations for our generating functions is to use a well-known decomposition of a graph along its connectivity-structure, i.e. into connected, 2-connected, and 3-connected components. Eventually, we obtain the following Lemma.

Lemma 3.3. *Let $G(x,y)$, $C(x,y)$ and $B(x,y)$ denote the bivariate exponential generating functions for not necessarily connected, connected and 2-connected $K_{3,3}$-minor-free graphs. Then we have*

$$G(x,y) = \exp\left(C(x,y)\right) \quad \textit{and} \quad C^\bullet(x,y) = x\exp\left(\frac{\partial}{\partial x}B\left(C^\bullet(x,y),y\right)\right). \tag{3.1}$$

Moreover, let $M(x,y)$ denote the bivariate generating function for 3-connected planar maps which satisfies

$$M(x,y) = x^2y^2\left(\frac{1}{1+xy} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right), \tag{3.2}$$

where $U(x,y)$ and $V(x,y)$ are algebraic functions given by

$$U = xy(1+V)^2, \quad V = y(1+U)^2, \tag{3.3}$$

then we have

$$\frac{\partial}{\partial y}B(x,y) = \frac{x^2}{2}\left(\frac{1+D(x,y)}{1+y}\right), \tag{3.4}$$

where $D(x,y)$ is defined implicitly by $D(x,0)=0$ and

$$\frac{M(x,D)}{2x^2D} + \frac{qx^3D^9}{6} - \log\left(\frac{1+D}{1+y}\right) + \frac{xD^2}{1+xD} = 0, \tag{3.5}$$

where q marks the monomial for K_5.

Proof. Equations (3.1) are standard and encode that a not necessarily connected graph consists of a set of connected graphs and a connected graph can be decomposed at a vertex into a set of 2-connected graphs whose vertices can again be replaced by rooted connected graphs. For more detailed proofs see for example [6](p.95) and [9](p.10).

Using Euler's polyhedral formula, Equations (3.2) and (3.3) follow from [12], where Mullin and Schellenberg derived the generating function for rooted 3-connected planar maps according to the number of vertices and faces.

Next, to derive the connection between 2-connected and 3-connected graphs, we can use a result of Walsh. More precisely, by Proposition 1.2 of [17] we obtain Equations (3.4) and (3.5), where we have to add only a monomial for K_5 compared to the class of planar graphs. For more details we refer to [1]. □

3.2 Singular Expansions and Asymptotic Estimates

We use the relations of the generating functions obtained so far to derive singular expansions for the generating functions of the different connectivity levels. We start from 3-connected $K_{3,3}$-minor-free graphs and then go up the connectivity hierarchy level by level. Eventually, this will allow us to derive asymptotic estimates for the number of and properties of not necessarily connected $K_{3,3}$-minor-free graphs in the subsequent sections.

We start with 3-connected $K_{3,3}$-minor-free graphs. We have to introduce only a slight modification into the formulas already known for planar graphs ([1, 7]).

From Lemma 3.3 we can derive analogously to [1] a singular expansion for $D(x, y)$. It will turn out that the singularity of $D(x, y)$ changes only slightly compared to the case of 2-connected planar graphs, but yields a larger exponential growth rate.

To simplify expressions, we will use the following notation. The equation $Y(t) = y$ has a unique solution in $t = t(y)$ in a suitable small neighbourhood of 1. Then we denote by $R(y) = \zeta(t(y))$. See Appendix A for expressions for $Y(t)$ and ζ.

Lemma 3.4. *For fixed y in a small neighbourhood of 1, $R(y)$ is the unique dominant singularity of $D(x, y)$. Moreover, $D(x, y)$ has a branch-point at $R(y)$, and the singular expansion at $R(y)$ is of the form*

$$D(x, y) = D_0(y) + D_2(y)X^2 + D_3(y)X^3 + O(X^4)$$

where $X = \sqrt{1 - x/R(y)}$ and the $D_i(y)$, $i = 0, \ldots, 3$ are given in Appendix A.

To prove this lemma, one can mimic the proof of Lemma 3 in [1]. Although we slightly changed the equations by adding a monomial for K_5, one can check that the same arguments still hold.

Next, we solve Equation (3.4) for $B(x, y)$ by integrating according to y. We omit the proof as it follows closely the lines of proof of Lemma 5 in [7].

Lemma 3.5. *Let $W(x, z) = z(1+U(x, z))$. The generating function $B(x, y)$ of 2-connected $K_{3,3}$-minor-free graphs admits the following expression as a formal power series:*

$$B(x, y) = \beta(x, y, D(x, y), W(x, D(x, y))) + \frac{qx^5 D(x, y)^{10}}{120}, \tag{3.6}$$

where

$$\beta(x, y, z, w) = \frac{x^2}{2}\beta_1(x, y, z) - \frac{x}{4}\beta_2(x, z, w),$$

and

$$\begin{aligned}
\beta_1(x, y, z) &= \frac{z(6x - 2 + xz)}{4x} + (1 + z)\log\left(\frac{1 + y}{1 + z}\right) - \frac{\log(1 + z)}{2} + \frac{\log(1 + xz)}{2x^2} \\
\beta_2(x, z, w) &= \frac{2(1 + x)(1 + w)(z + w^2) + 3(w - z)}{2(1 + w)^2} - \frac{1}{2x}\log(1 + xz + xw + xw^2) \\
&+ \frac{1 - 4x}{2x}\log(1 + w) + \frac{1 - 4x + 2x^2}{4x}\log\left(\frac{1 - x + wz - xw + xw^2}{(1 - x)(z + w^2 + 1 + w)}\right).
\end{aligned}$$

We can use this lemma to obtain the singular expansion for $B(x, y)$.

Lemma 3.6. *For fixed y in a small neighbourhood of 1, the dominant singularity of $B(x, y)$ is equal to $R(y)$. The singular expansion at $R(y)$ is of the form*

$$B(x, y) = B_0(y) + B_2(y)X^2 + B_4(y)X^4 + B_5(y)X^5 + O(X^6) \tag{3.7}$$

where $X = \sqrt{1 - x/R(y)}$, and the $B_i(y)$, $i = 0, \ldots, 5$ are analytic functions in a neighbourhood of 1.

Proof. From Equation (3.6) we can see that for y close to 1 the only singularities come from the singularities of $D(x,y)$; hence the first claim of the theorem follows.

The singular expansion for $B(x,y)$ can be obtained using Equation (3.6) and the singular expansion for $D(x,y)$. We substitute the singular expansion for $D(x,y)$, $U(x,D(x,y))$ in (3.6). Then we set $x=\zeta(t)(1-X^2)$ and $y=Y(t)$ and expand the resulting expression. Now, collecting and simplifying the coefficients of the X^i for $i=1,\ldots,5$ is a tedious calculation, but can be done with a computer algebra system such as MAPLE. This yields the expressions for the $B_i(y)$ given in the appendix. □

For connected and not necessarily connected $K_{3,3}$-minor-free graphs, we can derive singular expansions by carrying out an analogous calculation as in the proof of Theorem 1 in [7]. We only have to adapt for the different $D_i(y)$ and $B_i(y)$. One can easily check that the intermediate step of Claim 1 in [7] still holds and the rest of the calculations stays the same. The coefficients of the expansions, which we obtain in this way, can be found in Appendix A.

Lemma 3.7. *For fixed y in a small neighbourhood of 1, the dominant singularity of $C(x,y)$ and $G(x,y)$ is equal to $\rho(y)$. The singular expansions at $\rho(y)$ are of the form*

$$C(x,y)=C_0(y)+C_2(y)X^2+C_4(y)X^4+C_5(y)X^5+O(X^6) \tag{3.8}$$

and

$$G(x,y)=G_0(y)+G_2(y)X^2+G_4(y)X^4+G_5(y)X^5+O(X^6) \tag{3.9}$$

where $X=\sqrt{1-x/\rho(y)}$, and the $C_i(y)$ and $G_i(y)$, $i=0,\ldots,5$, are analytic functions in a neighbourhood of 1.

From Lemmas 3.6 and 3.7 we obtain the following asymptotic estimates using the "transfer theorem", Corollary VI.1 of [6].

Theorem 3.8. *Let g_n, c_n, and b_n denote the number of not necessarily connected, connected and 2-connected resp. $K_{3,3}$-minor-free graphs on n vertices. Then it holds*

$$g_n \sim \alpha_g\, n^{-7/2}\, \rho_g^{-n}\, n!, \tag{3.10}$$

$$c_n \sim \alpha_c\, n^{-7/2}\, \rho_c^{-n}\, n!, \tag{3.11}$$

$$b_n \sim \alpha_b\, n^{-7/2}\, \rho_b^{-n}\, n!, \tag{3.12}$$

where and $\alpha_g \doteq 0.42643\cdot 10^{-5}$, $\alpha_c \doteq 0.41076\cdot 10^{-5}$, $\alpha_b \doteq 0.37074\cdot 10^{-5}$, $\rho_c^{-1}=\rho_g^{-1}\doteq 27.22935$, and $\rho_b^{-1}\doteq 26.18659$ are analytically computable constants.

3.3 Structural Properties

If we consider a random $K_{3,3}$-minor-free graph, i.e. drawing a $K_{3,3}$-minor-free graph on n vertices uniformly at random from all such graphs on n vertices, we can derive the following properties using the algebraic singularity schema (Theorem IX.10) of [6].

Theorem 3.9. *The number of edges in a not necessarily connected and connected random $K_{3,3}$-minor-free graph is asymptotically normally distributed with mean μ_n and variance σ_n^2, which satisfy*

$$\mu_n \sim \kappa n \quad \text{and} \quad \sigma_n^2 \sim \lambda n,$$

where $\kappa \doteq 2.21338$ and $\lambda \doteq 0.43044$ are analytically computable constants.

Recall that we introduced the variable q in the equations of the generating functions above to mark the monomial for K_5. We can use this variable to obtain a limit law for the number of K_5 used in the construction process (following the above decomposition, see Theorem 3.2) of a random $K_{3,3}$-minor-free graph. The next theorem shows that a linear number of K_5 is used, but the constant is very small; this is exactly what one would expect as the expected number of edges is only slightly larger than for planar graphs (see Theorem 3.9 and [7]).

Theorem 3.10. *Let $\mathrm{C}(G)$ denote the number of K_5 used in the construction of a random $K_{3,3}$-minor-free graph G according to Theorem 3.2. Then $\mathrm{C}(G)$ is asymptotically normally distributed with mean μ_n and variance σ_n^2, which satisfy*

$$\mu_n \sim \kappa n \quad \text{and} \quad \sigma_n^2 \sim \lambda n,$$

where $\kappa \doteq 0.92391 \cdot 10^{-4}$ and $\lambda \doteq 0.92440 \cdot 10^{-4}$ are analytically computable constants. The same holds for a random connected $K_{3,3}$-minor-free graph.

4 Graphs not containing $K_{3,3}^+$ as a minor

In this brief section we give estimates for the number of graphs not containing $K_{3,3}^+$ (the graph obtained from $K_{3,3}$ by adding one edge) as a minor. For this we use the following recent result from [4].

Theorem 4.1. *A 3-connected graph not containing $K_{3,3}^+$ as a minor is either planar or isomorphic to $K_{3,3}$ or K_5.*

The analogous result to Lemma 3.5 holds, that is we get $B(x,y)$ with an additional term (now we set $q=1$)

$$10D(x,y)^9\frac{x^6}{6!}.$$

This is because $K_{3,3}$ has 6 vertices, 9 edges, and 10 different labellings. The equation for $D(x,y)$ in Lemma 3.3 has to be modified too in order to take into account $K_{3,3}$, and one has to add a term (again we set $q=1$) $x^4D^8/4$.

There is a corresponding expression for the singular coefficients B_i at the dominant singularity $R(y)$, which we do not write down in detail in order to avoid repetition. We obtain the dominant singularity $\rho(y)$ for the generating functions $C(x,y)$ and $G(x,y)$, compute the corresponding singular expansions and obtain the following result.

Theorem 4.2. *Let g_n, c_n, and b_n denote the number of not necessarily connected, connected and 2-connected resp. $K_{3,3}^+$-minor-free graphs on n vertices. Then it holds*

$$g_n \sim \alpha_g \, n^{-7/2} \, \rho_g^{-n} \, n!, \tag{4.1}$$

$$c_n \sim \alpha_c \, n^{-7/2} \, \rho_c^{-n} \, n!, \tag{4.2}$$

$$b_n \sim \alpha_b \, n^{-7/2} \, \rho_b^{-n} \, n!, \tag{4.3}$$

where $\rho_c^{-1} = \rho_g^{-1} \doteq 27.22948$, and $\rho_b^{-1} \doteq 26.18672$ are analytically computable constants.

Here is a table showing the approximate values of the growth constants for planar, $K_{3,3}$-minor-free and $K_{3,3}^+$-minor-free graphs.

Class of graphs	Growth constant	Growth constant for 2-connected
Planar	27.22688	26.18486
$K_{3,3}$-minor free	27.22935	26.18659
$K_{3,3}^+$-minor free	27.22948	26.18672

It is natural to ask if one can go further and treat the case where the forbidden minor is obtained from $K_{3,3}$ by adding *two* edges. If the two edges share a vertex, then the resulting graph is $K_{1,2,3}$; if the two edges do not share a vertex, let us denote denote by L the resulting graph.

Enumeration when we forbid $K_{1,2,3}$ is in principle feasible along the previous lines because of the following theorem due to Halin [8] (see also [5, Section 6.1]). The Wagner graph W consists of a cycle of length 8 in which opposite vertices are adjacent.

Theorem 4.3. *A 3-connected graph not containing $K_{1,2,3}$ as a minor is either planar or isomorphic to K_5, W, L or to nine sporadic non-planar graphs, or to a 3-connected subgraph of these.*

This means that the generating function for 3-connected graphs not containing $K_{1,2,3}$ as a minor is obtained by adding a finite number of monomials to the generating function of 3-connected planar graphs, which is already known to us, exactly as in the previous section. However performing all the computations corresponding to this case means: first to get the full list of exceptional 3-connected graphs up to isomorphism from the previous theorem, and in each case to compute the automorphism group in order to determine the number of different labellings in each case; and then to compute the singular expansions of all the generating functions involved. We have refrained from doing these computations, which would be along the same lines as before but likely very cumbersome.

However, forbidding L as a minor is another story and definitely we cannot solve this problem at this stage. The reason is that in this case one needs to take 3-sums (gluing along triangles) of graphs in order to describe the family of 3-connected graphs not containing L as a minor [5], and we do not have the necessary machinery to translate it into equations satisfied by the generating functions. This problem already appears if we try to count graphs not containing K_5 as a minor (notice that L contains K_5 as a proper minor). We believe this is a fascinating open problem that, if solved, will no doubt require new ideas and techniques.

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A Appendix

Here, we give the expressions for the coefficients of the singular expansions of $D(x,y)$, $U(x,y)$, $B(x,y)$, $C(x,y)$ and $G(x,y)$ as well as the expressions for the singularities. We use the same approach as in [1] and parametrize the expressions in the complex variable t.

The variable q used for counting the number of K_5 appears explicitly only in the expression for h; the reason is that this is the only place where it is needed for computing the radius of convergence, which in turn is needed for estimating the expected value and variance in Theorem 3.10.

$$\begin{aligned}
h &= \frac{t^2}{8192(3t+1)^6(2t+1)(t+3)}\big(13122qt^9+45927qt^8-1658880t^7+19683qt^7\\
&\quad -12496896t^6-8847360t^5+6832128t^4+10399744t^3+4739072t^2+958464t\\
&\quad +73728)\\
Y(t) &= -\frac{2t+1}{(3t+1)(t-1)}e^{-h}-1\\
\zeta &= -\frac{(t-1)^3(3t+1)}{16t^3}\\
Q &= 78732t^9-1328940t^8-26889705t^7-153744066t^6-415828997t^5-522964992t^4\\
&\quad -342073344t^3-121237504t^2-22151168t-1638400\\
K &= 78732t^{11}+472392t^{10}-2668221t^9-816345t^8+92026557t^7+562023429t^6\\
&\quad +1040556032t^5+926367744t^4+455663616t^3+127336448t^2+19005440t\\
&\quad +1179648\\
U_0 &= \frac{1}{3t}\\
U_1 &= -\left(-\frac{2}{27}\frac{(3t+1)K}{t^3(t+1)Q}\right)^{\frac{1}{2}}\\
U_2 &= -\frac{(3t+1)^2}{54t^2(t+1)^2Q^2}\big(6198727824t^{20}+180231719760t^{19}+891036025560t^{18}\\
&\quad -12902936763600t^{17}-197722264231071t^{16}-1821396525148269t^{15}\\
&\quad -13816272361145022t^{14}-79424397121737354t^{13}-324711461744767867t^{12}\\
&\quad -931873748086896665t^{11}-1881275802907541504t^{10}-2713502925437276160t^9\\
&\quad -2843653010633469952t^8-2190731661037666304t^7-1246514524950953984t^6\\
&\quad -521994799964094464t^5-158674913803108352t^4-34025665074298880t^3\\
&\quad -4876321721155584t^2-418948289921024t-16312285790208\big)\\
D_0 &= -\frac{3t^2}{(3t+1)(t-1)}\\
D_1 &= 0
\end{aligned}$$

$$D_2 = -\frac{t(2t+1)^2}{(3t+1)(t-1)Q}\big(19683t^8+118098t^7-1592325t^6-10616832t^5-30670848t^4 +7602176t^3+24444928t^2+9830400t+1179648\big)$$

$$D_3 = \frac{131072}{9Q^2}\left(\left(-\frac{(3t+1)K}{t^3(t+1)Q}\right)^{\frac{1}{2}}\sqrt{6}t^2(3t+1)(t+3)^2(2t+1)^2K\right)$$

$$\begin{aligned} P_1 = {} & 1549681956t^{19}-60580022472t^{18}-965388262815t^{17}-2822075181459t^{16} \\ & -63004687280883t^{15}-1326793976317287t^{14}-11608693177471470t^{13} \\ & -55082955555464994t^{12}-157459666865762304t^{11}-279393068914421760t^{10} \\ & -323288788914892800t^9-254483996115259392t^8-139939270751358976t^7 \\ & -54299625067175936t^6-14753365577572352t^5-2718756694392832t^4 \\ & -314310035243008t^3-18285655490560t^2-5905580032t+40265318400 \end{aligned}$$

$$\begin{aligned} P_2 = {} & -472392t^{12}-2991816t^{11}+15064542t^{10}+10234512t^9-550526652t^8 \\ & -3556193688t^7-7367383050t^6-7639318528t^5-4586717184t^4-1675345920t^3 \\ & -368705536t^2-45088768t-2359296 \end{aligned}$$

$$\begin{aligned} B_0 = {} & \frac{1}{4}\ln(3+t)-\frac{(3t+1)^2(-1+t)^6\ln(2t+1)}{1024t^6}-\frac{(3t^4-16t^3+6t^2-1)\ln(3t+1)}{32t^3} \\ & -\frac{1}{2}\ln(t)-\frac{3}{2}\ln(2)+\frac{(3t-1)^2(1+t)^6\ln(1+t)}{512t^6}-\frac{(t-1)^2}{41943040t^4\,(3t+1)^5\,(t+3)} \\ & \big(19683t^{13}-131220t^{12}-183708t^{11}+360921744t^{10}+2005423731t^9 \\ & +3887177580t^8+5603033310t^7+4821770240t^6+2013921280t^5+229048320t^4 \\ & -97157120t^3-31436800t^2-2048000t+122880\big) \end{aligned}$$

$$B_1 = 0$$

$$\begin{aligned} B_2 = {} & -\frac{(3t-1)(3t+1)(1+t)^3(-1+t)^3\ln(1+t)}{256t^6}+\frac{(3t+1)^2(-1+t)^6\ln(2t+1)}{512t^6} \\ & +\frac{(3t+1)(-1+t)^3\ln(3t+1)}{32t^3}+\frac{(t-1)^4}{8388608t^4\,(t+3)\,(3t+1)^5}\big(19683t^{11}-13122t^{10} \\ & -190269t^9+122862096t^8+626914188t^7+555393024t^6+28803072t^5 \\ & -163438592t^4-81084416t^3-14852096t^2-720896t+49152\big) \end{aligned}$$

$$B_3 = 0$$

$$B_4 = -\frac{(-1+t)^5P_1}{8388608t^4(t+3)(3t+1)^5Q}-\frac{9(t+\frac{1}{3})^2(t-1)^6(-2\ln(t+1)+\ln(2t+1))}{1024t^6}$$

$$B_5 = -\frac{\sqrt{\frac{3P_2}{t^3(t+1)Q}}P_2^2(t-1)^6}{2880(3t+1)^5(t+1)tQ^2}$$

$$
\begin{aligned}
C_0 &= R + B_0 + B_2 & G_0 &= \exp(C_0) \\
C_1 &= 0 & G_1 &= 0 \\
C_2 &= -R & G_2 &= \exp(C_0) C_2 \\
C_3 &= 0 & G_3 &= 0 \\
C_4 &= -\frac{1}{2}\left(R + \frac{R^2}{2B_4 - R}\right) & G_4 &= \exp(C_0)\left(C_4 + \frac{C_2^2}{2}\right) \\
C_5 &= B_5\left(1 - \frac{2B_4}{R}\right)^{-\frac{5}{2}} & G_5 &= \exp(C_0) C_5
\end{aligned}
$$