ON A CLASS OF DISTANCE-REGULAR GRAPHS

by

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ABSTRACT

We prove that there is no distance-regular graph Γ having intersection array

$$i(\Gamma) = \left\{ \begin{matrix} * & 1 & 1 & \dots & 1 & c \\ 0 & 0 & 0 & \dots & 0 & k-c \\ k & k-1 & k-1 & \dots & k-1 & * \end{matrix} \right\}, \quad k > 2$$

with diameter $d > 13$.

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To my wife Aliki, I can only say you made it possible.

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Introduction

Definitions and elementary results

A simple graph Γ is a pair $(V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is a set $\{v_1, v_2, \ldots, v_n\}$ of distinct element called vertices and $E(\Gamma)$ is a finite set of distinct unordered pairs of distinct elements of $V(\Gamma)$ called edges.

Throughout this thesis the term graph will mean simple graph.

Definition 1. Two vertices v_i, v_j of a graph Γ are adjacent if $\{v_i, v_j\}$ is an edge.

Definition 2. A walk of length ℓ in Γ, joining v_i to v_j is a finite sequence of vertices of Γ

$$v_i = u_0, u_1, \ldots, u_\ell = v_j$$

such that u_t and u_{t+1} are adjacent for $0 \leq t \leq \ell-1$. A walk whose edges are all distinct is called a path.

Definition 3. A connected graph is a graph that contains a walk joining v_i, v_j for each pair v_i, v_j of vertices.

Definition 4. The degree of the vertex v_i is the number of edges having v_i as a vertex. A regular graph is a graph in which every vertex has the same degree.

Definition 5. Let Γ be a graph whose vertex set is $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of Γ is the $n \times n$ matrix $A = A(\Gamma)$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases} .$$

Some elementary consequences of this definition are:

1. The eigenvalues of A are real.
2. The sum of the eigenvalues is zero.
3. The sum of the entries in any row or column is equal to the degree of the corresponding vertex.
4. The number of walks of length ℓ in Γ joining v_i to v_j is the entry b_{ij} in A^ℓ. [5] p. 11

Definition 6. The spectrum of the graph Γ is the set of pairs

$$S = \{(\lambda_0, m(\lambda_0)), \ (\lambda_1, m(\lambda_1)), \ldots, (\lambda_v, m(\lambda_v))\}$$

where λ_i, $i = 0, 1, \ldots, v$ are the distinct eigenvalues of the matrix A and $m(\lambda_i)$ their multiplicities.

Proposition 1 [5] p.14 If Γ is a regular connected graph of degree k then

(1) k is a simple eigenvalue

(2) For any eigenvalue λ of $A(\Gamma)$, we have $|\lambda| \leq k$.

Definition 7. The number of edges in a shortest walk joining v_i to v_j is called the distance in Γ between v_i and v_j and is denoted by $\delta(v_i,v_j)$.

The diameter $d(\Gamma) = d$ of a graph Γ is the maximum of the distances i.e.

$$d = \max_{v_i,v_j \varepsilon V(\Gamma)} \delta(v_i,v_j) \quad .$$

Definition 8. The adjacency algebra of a graph is the algebra of polynomials, over the complex field, in the adjacency matrix A.

Proposition 2 [5] A connected graph Γ with n vertices and diameter d has at least d+1 and at most n distinct eigenvalues.

Distance-Regular Graphs.

Let Γ be a connected graph of diameter d. Then, for any vertex v, the vertex set $V(\Gamma)$ can be partitioned into disjoint subsets

$$\Gamma_0(v),\ \Gamma_1(v),\ \ldots,\ \Gamma_d(v)\ , \text{ where}$$

$$\Gamma_f(v) = \{u \ \varepsilon\ V(\Gamma) |\ \delta(u,v) = f\},\ f = 0,1,\ldots,d\ .$$

Graphs of small diameter can be drawn by arranging their vertices in rows according to distance from an arbitrary vertex v. For example the graph of figure 2 is the same as the one in figure 1 but drawn in the manner just described.

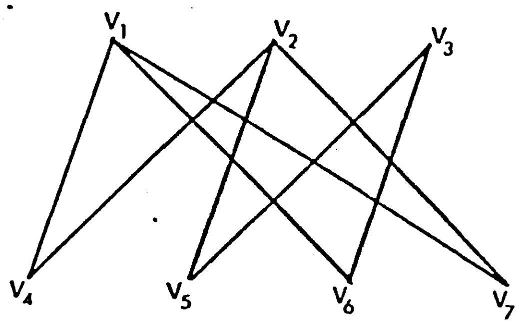

fig.1

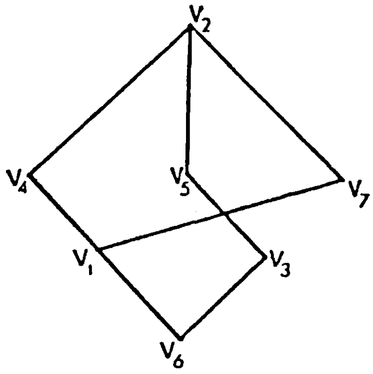

fig.2

For any connected graph Γ, and any vertices u,v of Γ, let

$$S_{h,i}(u,v) = |\{w \in V(\Gamma) \mid \delta(u,w) = h \text{ and } \delta(v,w) = i\}|;$$

that is, $S_{hi}(u,v)$ is the number of vertices of Γ whose distance from u is h and whose distance from v is i.

Definition 9. The connected graph Γ with diameter d is distance-h-regular if for all integers i and j $(0 \leq i,j \leq d)$ and for all pairs of vertices u,v with $\delta(u,v) = j$ the number

$$S_{hi}(u,v) = S_{hij} \quad \text{(say)}$$

depends only on h,i,j and not on the individual pair (u,v). Γ is distance-regular if it is distance-h-regular for all h, $0 \leq h \leq d$.

Theorem I [6] If Γ is distance-1-regular then Γ is distance-regular.

For a proof see [6].

For a fixed j, the number S_{1ij} counts the vertices w such that w is adjacent to u and $\delta(v,w) = i$, where $\delta(u,v) = j$. Now, if w is adjacent to u and $\delta(u,v) = j$, then $\delta(v,w)$ must be one of the numbers $j-1, j, j+1$; in other words

$$S_{1,i,j} = 0 \quad \text{if} \quad i \neq j-1, j, j+1.$$

We introduce the notation

$$a_j = S_{1,j,j}, \quad b_j = S_{1,j+1,j}, \quad c_j = S_{1,j-1,j}$$

where $0 \leqq j \leqq d$, except that c_o and b_d are undefined. Then for any arbitrary vertex $v \in V(\Gamma)$ and a vertex $u \in \Gamma_j(v)$ we have that u is adjacent to c_j vertices in $\Gamma_{j-1}(v)$, a_j vertices in $\Gamma_j(v)$ and b_j vertices in $\Gamma_{j+1}(v)$.

Definition 10. The intersection array of a distance-regular graph is the array

$$I(\Gamma) = \left\{ \begin{matrix} * & c_1 \ldots c_j \ldots c_d \\ a_0 & a_1 \ldots a_j \ldots a_d \\ b_0 & b_1 \ldots b_j \ldots * \end{matrix} \right\} \qquad 0.1$$

Now let Γ be a distance regular-graph of diameter d and vertex set $V(\Gamma) = \{v_1, \ldots, v_n\}$ and let A_i, $0 \leqq i \leqq d$ be the $n \times n$ matrix whose entries are

$$a_{rs} = \begin{cases} 1, & \text{if } \delta(v_r, v_s) = i \\ 0, & \text{otherwise} \end{cases} .$$

For those matrices we observe that $A_0 = I$, $A_1 = A(\Gamma)$ (The usual adjacency matrix) and $A_0 + A_1 + \ldots + A_d = J$ (J being the matrix whose all the entries are equal to 1).

Theorem II [5] p.136. Let Γ be a distance-regular graph of diameter d. Then $\{A_0, A_1, \ldots, A_d\}$ is a basis, for the adjacency algebra of the graph, described by the formula

$$A_h A_i = \sum_{j=0}^{d} S_{hij} A_j .$$

Proposition 3 [5] p 141 If Γ is a distance-regular graph of degree k and diameter d, then $A(\Gamma)$ has $d+1$ distinct eigenvalues $k = \lambda_0, \lambda_1, \ldots, \lambda_d$ which are the eigenvalues of the $(d+1) \times (d+1)$ matrix B whose, entries are $(B)_{ij} = S_{1ij}$, $i,j \in \{0,1,\ldots,d\}$.

Notice that B is a tridiagonal matrix with entries

$$\left.\begin{array}{lll} c_j, & \text{when } j-1 = i & \text{in the upper diagonal} \\ a_j, & \text{when } j = i & \text{in the main diagonal} \\ b_j, & \text{when } j+1 = i & \text{in the lower diagonal.} \end{array}\right\} \quad 0.2$$

Definition 11. The matrix B is called the intersection matrix of Γ.

Theorem III [5] p.143. Let Γ be a distance-regular graph with intersection matrix B, and suppose that u_i, v_i are left and right eigenvectors such that $(u_i)_0 = (v_i)_0 = 1$ corresponding to the eigenvalue λ_i of B. Then

1. $(v_i)_j = k_j(u_i)_j$ for all $i,j \in \{0,1,\ldots d\}$.
2. The multiplicity of λ_i as an eigenvalue of $A(\Gamma)$ is

$$m(\lambda_i) = \frac{N}{\sum_{j=0}^{d} k_j(u_i)_j^2} , \quad 0 \leqq i \leqq d$$

where N is the number of vertices.

The natural question to be asked is when an arbitrarily given array corresponds to a distance-regular graph?

The answer to this question is not yet known but the following theorem yields certain conditions which although not sufficient for the existence of the graph,related to a given array, they are nevertheless so restrictive that most known arrays satisfying these conditions correspond to a graph.

Theorem IV [5] p.144 If the array (0.1) is the intersection array of a distance-regular graph of diameter d then

1. $a_o = 0$, $b_o = k$ (the degree of the graph),
   $c_1 = 1$ and $k = a_i + b_i + c_i$ for $1 \leq i \leq d-1$
   $k = a_d + c_d$.

2. For $2 \leq i \leq d$ the numbers
   $k_i = (k\, b_1 \ldots b_{i-1})/(c_2 c_3 \ldots c_i)$ are integers.

3. $k \geqq b_1 \geqq \ldots \geqq b_{d-1}$; $1 \leqq c_2 \leqq \ldots \leqq c_d$.

4. If $N = 1 + k + k_2 + \ldots + k_d$ $1 \leq i \leq d-1$ then
   $N.k \equiv 0(\bmod 2)$ and $k_i a_i \equiv 0(\bmod 2)$.

5. For each eigenvalue λ_i of the matrix B, given by (0.2) with eigenvector u_i defined as in theorem III the number $N / \sum_{j=0}^{d} k_i (u_i)_j^2$, is an integer.

<u>Definition 12</u> An array of the form of (0.1) is said to be <u>feasible</u> if it satisfies the conditions of theorem IV.

In [16] Tutte showed that the number of vertices N, say, of a regular graph whose degree $(k \geq 3)$ and girth $(\gamma \geq 3)$ are given is greater than or equal to

$$1 + k + k(k-1) + \ldots + k(k-1)^{\frac{1}{2}(\gamma-3)} \quad \text{if } \gamma \text{ is odd}$$

and

$$1 + k + k(k-1) + \ldots + k(k-1)^{\frac{1}{2}\gamma-2} + (k-1)^{\frac{1}{2}\gamma-1} \quad \text{if } \gamma \text{ is even} \qquad (0.3)$$

Here the <u>girth</u> of a graph is the length of one of its shortest circuits.

A graph which attains the bound (0.3) is called a Moore graph if γ is odd and a Generalized Polygon if γ is even.

A lot of work has been done on the classification of such graphs. Generalized Polygons have been studied by Feit and Higman [9], Singleton [15] and Benson [3]. Moore graphs have been studied by Hoffman and Singleton [12], Vijayan [17], Bannai and Ito [1] and Damerell [6].

In his book [5] ch.23 Biggs considered both types of graphs as special cases of the distance-regular graph of diameter d whose intersection array is

$$\left\{\begin{matrix} * & 1 & 1 & \cdots & 1 & c \\ 0 & 0 & 0 & \cdots & 0 & k-c \\ k & k-1 & k-1 & \cdots & k-1 & * \end{matrix}\right\}, \quad k > 2 \qquad 0.4$$

The case $c = k$ is a Generalized Polygon and the case $c = 1$ is a Moore graph.

In this thesis we investigate the feasibility of the intersection array (0.4) using the methods and formulae given by Biggs [5] and we prove that this is not feasible if $d > 13$.

Supposing that a graph of this type exists, Biggs has derived a formula for the minimum polynomial of its adjacency matrix A.

In chapter one, by using that formula we calculate the multiplicity of each eigenvalue λ of the matrix A as a function of λ. Using that result (Theorem 1 below) Bannai and Ito [2] have proved that the characteristic polynomial splits into linear or quadratic factors.

In chapter two we prove that the characteristic polynomial splits into quadratic factors over GF(2) only if its degree (= d) is of the form $d = 2^t \times f$ or $2^t \times f + 1$ where f = 1,3 or 5. To get this result since the characteristic polynomial $G_d(x)$ alone does not always provide the answer, we construct another polynomial $H_d(Y)$ whose roots are linear or quadratic over the rationals whenever the roots of the characteristic polynomial are.

In chapter three the factorization of $G_d(x)$ and $H_d(Y)$ over GF(2), provides the information that the constant terms of these polynomials are divisible by a certain power of 2. But we observe that this happens only when $t \leqq 4$. From this it follows that $d \leqq 81$.

In chapter four we classify our graphs according to the value of f. When f = 1 the polynomial $H_d(Y)$, reduced modulo $Y^2 - 4Y$, provides the information that $t < 4$. In the case f = 3 the value of the polynomial $H_d(Y)$ at $Y = \theta$, where θ is any odd numbers, provides a necessary condition which enables us to eliminate t = 4.

When f = 5, the same polynomial, reduced modulo $Y^2 + \theta Y + \theta$ yields the conditions which prevent t from being 3 or 4. From these results we get that $d \leqq 25$.

Chapter five deals with graphs of diameters 25 and 24. There from the theorem of Dumas [7] we derive the conditions which forbid those diameters.

In conclusion I wish to state that the principal result contained in this thesis, which is described in the abstract, was obtained by my own research.

CHAPTER ONE

Eigenvalues and their multiplicities

We adopt the notation of Biggs [5] which is

$$h = k-1, \quad q = +\sqrt{k-1}, \quad \lambda = 2q\cos\alpha \qquad 1.1$$

and we regard the matrix

$$B = \begin{bmatrix} 0 & 1 & & & & & & \\ k & 0 & 1 & & & & & \\ & k-1 & 0 & \cdot & & & & \\ & & k-1 & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & 1 & \\ & & & & & & 0 & c \\ & & & & & & k-1 & k-c \end{bmatrix} \qquad 1.2$$

as the intersection matrix of a D-R graph. Then the characteristic equation of B is given by the following result. ([5] lemma 23.3).

The number λ is an eigenvalue of the matrix B if either

$$\lambda = k \qquad 1.3$$

or

$$\lambda = + 2q\cos\alpha \qquad 1.4$$

and $F_d(\cos\alpha) = 0$ where F_d is defined by

$$q\,\frac{\sin(d+1)\alpha}{\sin\alpha} + c\,\frac{\sin d\alpha}{\sin\alpha} + \frac{c-1}{q}\,\frac{\sin(d-1)\alpha}{\sin\alpha} \equiv F_d(\cos\alpha) \qquad 1.5$$

where d is the diameter of the putative graph.

In this chapter in §1 we calculate the recursion for $F_d(\cos\alpha)$ and we exclude $\lambda = -2q$ from being an eigenvalue. In §2 we express the multiplicity $m(\lambda)$ say of any eigenvalue λ of the adjacency matrix A of the assumed D-R graph as a function in λ. Finally we state the theorem of Bannai and Ito which is the starting point of our investigation.

§1. The characteristic equation of B.

Lemma 1.1. With F_d as defined in (1.5)

$$F_1 = 2q\cos\alpha + c, \quad F_2 = 4q\cos^2\alpha + 2c\cos\alpha + \frac{c-1}{q} - q \qquad 1.6$$

$$F_d = 2\cos\alpha\, F_{d-1} - F_{d-2}, \; d > 2 \quad . \qquad 1.7$$

Proof. The Tchebycev polynomials of the second kind are of the form ([8] (10.11.12))

$$U_\eta(\cos\alpha) = \frac{\sin(n+1)\alpha}{\sin\alpha} \qquad 1.8$$

and satisfy the following recurrence relation ([8]. (10.11.16))

$$U_{\eta+1}(\cos\alpha) = 2\cos\alpha\, U_\eta(\cos\alpha) - U_{\eta-1}(\cos\alpha) \quad . \qquad 1.9$$

Hence from 1.5 we get

$$F_d(\cos\alpha) = q\, U_d(\cos\alpha) + c\, U_{d-1}(\cos\alpha) + \frac{c-1}{q} U_{d-2}(\cos\alpha) \, . \qquad 1.10$$

Thus

$$F_d(\cos\alpha) = q\{2\cos\alpha\, U_{d-1} - U_{d-2}\} + c\{2\cos\alpha\, U_{d-2} - U_{d-3}\}$$

$$+ \frac{c-1}{q}\{2\cos\alpha\, U_{d-3} - U_{d-4}\} \qquad 1.11$$

$$= 2\cos\alpha\, F_{d-1} - F_{d-2} \quad . \qquad 1.12$$

Now putting $d = 1,2$ into 1.5 we get 1.6.

We note for future use the formula

$$F_d(\cos\alpha) = (2q\cos\alpha + c)\,U_{d-1}(\cos\alpha) + \frac{c-1-q^2}{q}\,U_{d-2}(\cos\alpha) \qquad 1.13$$

got by substituting for $U_d(\cos\alpha)$ in 1.10.

Proposition 1.1. $\lambda = -2q$ is never an eigenvalue.

Proof. Let q be irrational. Then if $-2q$ is an eigenvalue so is $+2q$. But $\lambda = +2q$ implies $\alpha = 0$ by 1.1 and in that case 1.5 gives, after using L'Hôpital's rule that

$$q(d+1) + cd + \frac{c-1}{q}(d-1) = 0 \qquad 1.14$$

which is impossible since this is strictly positive. Therefore q has to be rational and integral. Now if the integer $\lambda = -2q$ is an eigenvalue then $\alpha = \pi$ and again by L'Hôpital's rule from 1.5 we get

$$(-1)^{d-1}\{q(d+1) - cd + \frac{c-1}{q}(d-1)\} = 0 \qquad 1.15$$

$$\therefore \quad c-1 = \frac{-dq^2 + dq - q^2}{-dq + d - 1} = q + \frac{q^2 - q}{dq - d + 1} \qquad 1.16$$

$$\therefore \quad \frac{q^2 - q}{dq - d + 1} \text{ has to be an integer}$$

$$\therefore \quad \frac{q}{dq - d + 1} = q - \frac{d(q^2 - q)}{dq - d + 1} \text{ must be an integer}$$

$$\therefore \quad dq - d + 1 \leq q \qquad 1.17$$

which is not true when $q > 1$.

Now if $q = 1$ then $k = 2$ by 1.1, in which case our graph is a polygon with $2d + 1$ edges.

§2. The multiplicity of λ as an eigenvalue of A.

Theorem 1. Let λ be an eigenvalue of the intersection matrix B and $m(\lambda)$ its multiplicity, as an eigenvalue of the adjacency matrix A. Then

$$\frac{N}{m(\lambda)} = \frac{\lambda-h-1}{(h+1)(\lambda^2-4h)}\left[\frac{(h+1-c)(\lambda c+2h+2c-2)}{\lambda(c-1)+h+(c-1)^2} + 2d(\lambda+h+1)\right] \equiv \Sigma(\lambda) \qquad 1.18$$

(say) where N is the number of vertices of the supposed graph.

Proof. By [5] p.158 the multiplicity of λ as an eigenvalue of the matrix A is given by

$$\frac{N}{m(\lambda)} = \Sigma(\lambda) = \sum_{i=0}^{d} k_i u_i^2 \qquad 1.19$$

where

$$k_0 = 1, \quad k_i = kh^{i-1}, \ (1 \leq i < d), \quad k_d = c^{-1}k\,h^{d-1} \qquad 1.20$$

and $\underline{u} = (u_0=1, u_1, \ldots, u_d)$ is a left eigenvector of B corresponding to the eigenvalue λ. Now since the first d columns of B are the same as in the matrix B of [6] , the formulae derived there for $\underline{u}$ in terms of any eigenvalue λ hold.

Thus

$$u_i = C\left(\frac{\theta}{q}\right)^i + D\left(\frac{\theta^{-1}}{q}\right)^i \qquad 1.21$$

where $C = \dfrac{h\theta - \theta^{-1}}{k(\theta-\theta^{-1})}$, $D = \dfrac{h\theta^{-1} - \theta}{k(\theta^{-1}-\theta)}$ and $\theta = e^{i\alpha}$. 1.22

To simplify the resulting algebra we adopt the convention that if W is a rational function of θ , then $\hat{W}$ is the function got by replacing θ by θ^{-1} throughout.

Thus

$$k_0 u_0^2 = 1 = -h^{-1} + kh^{-1} (C^2 + 2C\hat{C} + \hat{C}^2)$$ 1.23

$$k_i u_i^2 = kh^{-1} (C^2 \theta^{2i} + 2C\hat{C} + \hat{C}^2 \theta^{-2i}), \quad 1 \leqq i < d$$ 1.24

and

$$k_d u_d^2 = c^{-1} k h^{-1} (C^2 \theta^{2d} + 2C\hat{C} + \hat{C}^2 \theta^{-2d}) \quad .$$ 1.25

Hence

$$\Sigma(\lambda) = \sum_{i=0}^{d} k_i u_i^2 = -h^{-1} + k h^{-1} c^{-1} \{Z + Y + \hat{Z}\}$$ 1.26

where

$$Y = 2C\hat{C} (dc + 1)$$ 1.27

and

$$Z = C^2 \{c \sum_{i=0}^{d} \theta^{2i} + (1-c) \theta^{2d}\} \quad .$$ 1.28

Consider Z. Sum the series and multiply by $k^2(q\theta + c-1).(\theta^2-1)^3$.

Then

$$k^2(\theta^2-1)^3 (q\theta+c-1).Z = (h\theta^2-1)(q\theta-1).$$

$$.\{(q\theta+c-1)(q\theta+1)(\theta^2+c-1).\theta^{2d}-c(q\theta+c-1)(q\theta+1)\} \; .$$ 1.29

Now if we put $\theta = e^{i\alpha}$ we have

$$\sin n\alpha = \frac{e^{in\alpha} - e^{in\alpha}}{2i} = \frac{\theta^n - \theta^{-n}}{2i}$$ 1.30

and equation 1.5 becomes

$$\theta^{2d} (q\theta + c - 1)(q\theta + 1) = (q + \theta) [q + (c-1)\theta]$$ 1.31

and substituting for θ^{2d} into 1.29 we get

$$k^2(\theta^2-1)^2 (q\theta+c-1). Z = (h\theta^2-1)(q\theta-1) [(c-1)\theta^2+qc\theta+ (c-1)(c-q^2)] \; .$$ 1.32

Then multiplying both sides of the above equation by $\theta^{-2}(q\theta^{-1} + c - 1)$ and arranging the R.H.S. terms in powers of θ we get

$$\begin{aligned} A.Z = {} & \theta^3 hq(c-1)^2 + \theta^2.h(c-1)\,[h(c+1) - (c-1)] \\ & + \theta q[h^2(3c-c^2-1) + h(c-1)(c^2-2c-1) - (c-1)^2] \\ & - h^3(c-1) + h^2(2c^2-4c+1) - h(c-1)(c^2+1) + (c-1)^2 \\ & + \theta^{-1}q[h^2(c-1) - h(2c-1) + (c-1)(-c^2+2c+1)] \\ & + \theta^{-2}[h^2(c-1) - h(2c^2-4c+1) + c(c-1)^2] + \theta^{-3}q(c-1)(c-h) \end{aligned} \quad 1.33$$

where

$$\begin{aligned} A = {} & k^2(\theta^2-1)^2\,(q\theta+c-1)(q\theta^{-1}+c-1).\theta^{-2} \\ = {} & k^2(\theta - \theta^{-1})^2\,[q(c-1)(\theta+\theta^{-1}) + h + (c-1)^2] = \hat{A}\ . \end{aligned} \quad 1.34$$

Thus

$$\begin{aligned} A(Z + \hat{Z}) = {} & (\theta^3 + \theta^{-3}).\ q(c-1)\,[h(c-2) + c] \\ & + (\theta^2 + \theta^{-2})\,[h^2(c^2+c-2) - h(3c^2-6c+2) + c(c-1)^2] \\ & + (\theta+\theta^{-1}).q[h^2(4c-c^2-2) + h(c-1)(c^2-2c-2) - hc + (c^2-1)(2-c)] \\ & - 2h^3(c-1) + 2h^2(2c^2-4c+1) - 2h(c-1)(c^2+1) + 2(c-1)^2\ . \end{aligned} \quad 1.35$$

Now $\theta = e^{i\alpha}$ thus

$$\theta + \theta^{-1} = \frac{\lambda}{q}\ , \quad \theta^2 + \theta^{-2} = \frac{\lambda^2-2h}{h}\ , \quad \theta^3 + \theta^{-3} = \frac{\lambda(\lambda^2-3h)}{q^3}$$

and $$(\theta - \theta^{-1})^2 = \frac{\lambda^2 - 4h}{h}\ . \quad 1.36$$

From which we get that

$$A = k^2(\lambda^2-4h)[\lambda.(c-1) + h + (c-1)^2].h^{-1} \quad 1.37$$

and

$$A(Z + \hat{Z}).\mathfrak{h} = \lambda^3(c-1)\,[h(c-2) + c]$$

$$+ \ \lambda^2[h^2(c^2+c-2) - h(3c^2-6c+2) + c(c-1)^2]$$

$$+ \ \lambda.h[h^2(-c^2+4c-2) + h(c-1)(c^2-5c+4)-hc + (c-1)(2-2c-c^2)]$$

$$- \ 2h\,[h^3(c-1) - h^2(c^2-5c+3) - h(-c^3+4c^2-7c+3) + (c-1)^3] \ . \qquad 1.38$$

Similarly by equation 1.27 we get

$$Y = 2C\hat{C}(dc+1) = \frac{h^2+1-h(\theta^2+\theta^{-2})}{k^2[2-(\theta^2+\theta^{-2})]}.2(dc+1)$$

$$= \frac{(h+1)^2 - \lambda^2}{k^2(4h-\lambda^2)}\ h.2(dc+1) \ . \qquad 1.39$$

Thus

$$\mathfrak{h}\cdot AY = [\lambda^2 - (h+1)^2]\,[\lambda.(c-1) + h + (c-1)^2].2h(dc+1) \ . \qquad 1.40$$

Now equation 1.26 can be written in the form

$$hc\ A\Sigma(\lambda) = -\,cA + kA\{Z+Y+\hat{Z}\} \qquad 1.41$$

$$= (h+1).c.h\{\lambda^3(c-1).2d + \lambda^2\,[h(c+2d) - c(c-1) + 2d(c-1)^2]\}$$

$$+ \ (h+1).c.h\ .\ \lambda\{h^2[2-c-2d\,(c-1)] + h[c^2-2c-4d\,(c-1)] + (c-1)\,(2-c-2d)\}$$

$$+ \ 2h(h+1).c\ \{-h^3(d+1) - h^2\,[d(c^2-2c+3) + 1] - h[d(2c^2-4c+3) - (c-1)^2]$$

$$+ \ (c-1)^2\,(1-d)\} \ . \qquad 1.42$$

Hence

$$(h+1)\,(\lambda^2-4h)\,[\lambda(c-1) + (c-1)^2 + h].\ \Sigma(\lambda) =$$

$$[\lambda - (h+1)]\,\{(h+1-c)[c\lambda+2(h+c-1)] + 2d(\lambda+h+1)[(c-1)\lambda + (c-1)^2 + h]\}. \qquad 1.43$$

From which we get 1.18.

Theorem 1 shows that the multiplicity $m(\lambda)$ of each eigenvalue of the incidence matrix A is a quotient of two cubic polynomials in λ. This together with the fact that the multiplicity $m(\lambda)$ has to be rational implies that all the eigenvalues of the assumed D-R graph must be of degree ≤ 3 over the rationals.

This result has been improved by Bannai and Ito. In [2] they prove the following .

<u>Theorem 2</u>. If Γ is a D-R graph with intersection matrix B and valency $k > 2$, then the roots of the characteristic polynomial of B are all of degree ≤ 2 over the rationals.

For proof see [2] theorem A.

CHAPTER TWO

The factorization of the characteristic polynomial of the matrix B over GF(2).

If a D-R graph with intersection matrix B exists, then, from the theorem of Bannai and Ito, we know that the roots of its characteristic polynomial will be all of degree less than or equal to 2, over the rationals. Consequently, if we reduce that polynomial modulo 2, this must have all its roots in GF(4).

Definition 2.1. Let

$$G_d(x) = q^{d-1} F_d(\cos\alpha) \qquad 2.1$$

where

$$x = 2q \cos\alpha \qquad 2.2$$

and $F_d(\cos\alpha)$ is here regarded as a polynomial in $\cos\alpha$.

By lemma 1.1 the polynomial $G_d(x)$ satisfies the following recurrence relation.

$$G_d(x) = xG_{d-1}(x) - hG_{d-2}(x) , \quad d > 2 \qquad 2.3$$

where

$$G_1(x) = x + c, \quad G_2(x) = x^2 + cx + c-k . \qquad 2.4$$

The polynomial $(x-k)\, G_d(x)$ is monic and of degree $d + 1$. Its roots are those given by [5] lemma 23.3. Thus $(x-k)G_d(x)$ is the characteristic polynomial of the matrix B. Therefore if the graph exists, $G_d(x)$, reduced modulo 2, will have all its roots in GF(4).

Now when h is odd this is not always possible. But when h is even the polynomial $G_d(x)$ factorizes trivially over GF(2).

To overcome this difficulty we construct another polynomial in a new variable Y, whose roots have to be rational or quadratic whenever the roots of $G_d(x)$ are. Moreover, this new polynomial reduced modulo 2, has all its roots in GF(4) only when certain conditions for d are satisfied.

The main result of this chapter is

Theorem 3. If a D-R graph with intersection matrix B exists then its diameter has to be of the form

$$d = 2^t \times f \quad \text{or} \quad 2^t \times f + 1, \quad \text{where} \quad f = 1 \text{ or } 3 \text{ or } 5 .$$

§1. Preliminary results

Definition 2.2. Let

$$H_d(Y) = (-1)^d \, h \, F_d(\cos\alpha) \, F_d(-\cos\alpha) \qquad 2.5$$

where

$$Y = \frac{x^2}{h} = 4 \cos^2\alpha = 2 \cos 2\alpha + 2. \qquad 2.6$$

The so defined $H_d(Y)$ is a polynomial in Y of degree d. The degree of each root of $H_d(Y)$, over the rationals, is less than or equal to the degree of the corresponding eigenvalue x of the intersection matrix B.

Now since

$$F_d(-\cos\alpha) = F_d(\cos\beta) \quad \text{where} \quad \beta = \pi - \alpha \qquad 2.7$$

and since

$$\frac{\sin n\beta}{\sin\beta} = (-1)^{n+1} \frac{\sin n\alpha}{\sin\alpha} \qquad 2.8$$

we get that

$$H_d(Y) = h\{[q\frac{\sin(d+1)\alpha}{\sin\alpha} + \frac{c-1}{q}\ \frac{\sin(d-1)\alpha}{\sin\alpha}]^2 - c^2\ \frac{\sin^2 d\alpha}{\sin^2\alpha}\} \qquad 2.9$$

Putting $q^2 = h$ we get

$$4\ \sin^2\alpha\ H_d(Y) = -2(c-1)^2\ \cos2(d-1)\alpha + 2h(c^2-2c+2)\ \cos2d\alpha$$
$$- 2h^2\ \cos2(d+1)\alpha + 2h^2 + 2(c-1)^2 - 2hc^2 + 4h(c-1)\ \cos 2\alpha \qquad 2.10$$

Definition 2.3 Define

$$K_d(Y) = -2(c-1)^2\cos2(d-1)\alpha + 2h(c^2-2c+2)\cos2d\alpha - 2h^2\cos2(d+1)\alpha \qquad 2.11$$

$$M(Y) = 2h^2 + 2(c-1)^2 - 2hc^2 + 4h(c-1)\ \cos2\alpha$$
$$= 2[h(c-1)Y + (h-c+1)^2 - hc^2] \qquad 2.12$$

Then

$$(4-Y)\ H_d(Y) = K_d(Y) + M(Y) \quad . \qquad 2.13$$

Lemma 2.1

$$K_o(Y) = -\{h^2 + (c-1)^2\}Y + 2\{(h-c+1)^2 + hc^2\} \qquad 2.14$$

$$K_1(Y) = -h^2Y^2 + h\{c^2-2(c-1) + 4h\}\ Y - 2\ \{(h-c+1)^2 + hc^2\} \qquad 2.15$$

$$K_d(Y) = (Y-2)\ K_{d-1}(Y) - K_{d-2}(Y) \quad . \qquad 2.16$$

Proof The polynomials $\cos\eta\theta = T_\eta(\cos\theta)$ are the Tchebycev polynomials of the first kind and satisfy the following recurrence relation [[8], (10.11.16)] .

$$T_{\eta+1}(\cos\theta) = 2\cos\theta\ T_\eta(\cos\theta) - T_{\eta-1}(\cos\theta) \quad . \qquad 2.17$$

Now if we put $\theta = 2\alpha$ 2.11 can be written on the form

$$K_d(Y) = -2(c-1)^2 \; T_{d-1} + 2h(c^2-2c+2) \; T_d - 2h^2 \; T_{d+1} \quad . \tag{2.18}$$

Hence by 2.17

$$\begin{aligned} K_d(Y) &= 2\cos 2\alpha \; [-2(c-1)^2 \; T_{d-2} + 2h(c^2-2c+2) \; T_{d-1} - 2h^2 \; T_d] \\ &\quad - [-2(c-1)^2 \; T_{d-3} + 2h(c^2-2c+2) \; T_{d-2} - 2h^2 \; T_{d-1}] \\ &= 2\cos 2\alpha \; K_{d-1}(Y) - K_{d-2}(Y) \quad . \end{aligned} \tag{2.19}$$

Thus from 2.6 we get 2.16 . Next putting $d = 0,1$ into 2.11 we get 2.14, 2.15.

Corollary 2.1. $H_d(Y) = (Y-2) \; H_{d-1}(Y) - H_{d-2}(Y) + M(Y)$ 2.20

$$H_o(Y) = (k-c)^2 \; , \qquad H_1(Y) = h^2Y - hc^2 \quad . \tag{2.21}$$

Proof. From 2.13

$$K_d(Y) = (4-Y) \; H_d(Y) - M(Y), \; d = 0,1, \; \ldots \; . \tag{2.22}$$

Thus for $d \geqq 2$ 2.16 becomes

$$(4-Y) \; H_d(Y) - M(Y) = (4-Y) \; [(Y-2) \; H_{d-1}(Y) - H_{d-2}(Y)] + (3-Y) \; M(Y)$$

from which we get 2.20.

Now if we put $d = 0,1$ into 2.9 we get the stated values of $H_0(Y), H_1(Y)$.

Definition 2.4 Put $c-1 = e$. Then we define r,s such that $2^r \| h$ and $2^s \| e$. Also define $p = \min(r,2s)$ and $L_d(Y) = 2^{-p} \; H_d(Y)$. Denote reduction modulo 2 by an asterisk .

Lemma 2.2.

I. $L_d(Y)$ has all its coefficients integral and $2^{-p} M(Y) \equiv 0 (\mathrm{mod}\ 2)$.

II. Let ρ,σ be the roots of the equation

$$\rho^2 + Y\rho + 1 = 0 \qquad 2.23$$

in characteristic 2. Then $\rho = \sigma^{-1}$ and

$$L_d^*(Y) = \frac{\rho L_1^* + L_0^*}{\rho^2 + 1} \cdot \rho^d + \frac{\rho^2 L_0^* + \rho L_1^*}{\rho^2 + 1} \cdot \sigma^d \qquad 2.24$$

Proof By (2.12) and (2.21) $\frac{1}{2}M(Y) = h\, e\, Y + (h-e)^2 - h(e+1)^2$ $H_o(Y) = (h - e)^2$, $H_1(Y) = h^2 Y - h(e+1)^2$. By inspection we see that each term of $H_o(Y)$ or $H_1(Y)$ or $\frac{1}{2}M(Y)$ is divisible by h or e and hence by 2^p .

II. From 2.20 and the definition of $L_d(Y)$ we have

$$L_d(Y) = (Y-2)\, L_{d-1}(Y) - L_{d-2}(Y) + 2^{-p}M(Y),\ d \geqq 2 . \qquad 2.25$$

Reducing modulo 2 we get

$$L_d^*(Y) = Y\, L_{d-1}^*(Y) - L_{d-2}^*(Y) ,\quad d \geqq 2 . \qquad 2.26$$

Now the solution to 2.26 is

$$L_d^* = A\rho^d + B\sigma^d \qquad 2.27$$

where ρ,σ are the roots of the equation 2.23 and A,B the solution of the system

$$A + B = L_0^*$$
$$A\rho + B\sigma = L_1^* \qquad 2.28$$

from which we get 2.24.

§2. Factorization of $H_d(Y)$ over GF(2).

We next consider how the polynomial $L_d^*(Y)$ factorizes in GF(2) . To do that we first write 2.21 in the form

$$H_0(Y) = (h - e)^2, \quad H_1(Y) = h^2 - h(e + 1)^2 \qquad 2.29$$

and separate the following five cases.

I. $2s > r = 0$

II. $2s = r > 0$

III. $2s = r = 0$

IV. $2s > r > 0$

V. $r > 2s \geqq 0$

where r,s are as in definition 2.4 .

Lemma 2.3. If ρ,σ are the roots of the equation

$$\rho^2 + Y\rho + 1 = 0 \qquad 2.30$$

in characteristic 2, then

$$L_d^*(Y) = \begin{cases} \frac{1}{(\rho+1)\rho^d}(\rho^{2d+1} + 1) & \text{in case I} \\ \frac{1}{(\rho+1)\rho^{d-1}}(\rho^{2d-1} + 1) & \text{II} \\ \frac{1}{\rho^d}(\rho^d + 1)^2 & \text{III} \\ \frac{1}{(\rho^2+1)\rho^{d-1}}(\rho^d + 1)^2 & \text{IV} \\ \frac{1}{(\rho^2+1)\rho^{d-2}}(\rho^{d-1} + 1)^2 & \text{V} \end{cases} \qquad 2.30^b$$

<u>Proof</u> Call the RHS of 2.30^b $N_d(\rho)$

Now $L_0 = 2^{-p} (h-e)^2$ 2.31

$L_1 = 2^{-p} [h^2 Y - h(e+1)^2]$ 2.32

Thus

Case I. $L_0^* \equiv h^2 \equiv 1$ (mod 2)

$L_1^* \equiv (h^2 Y - h.1) \equiv Y + 1 = \rho + \sigma + 1$

and 2.24 becomes

$$L_d^* = \frac{\rho(\rho+\rho^{-1}+1) + 1}{\rho^2 + 1} \rho^d + \frac{\rho^2 + \rho(\rho+\rho^{-1}+1)}{\rho^2 + 1} \rho^{-d}$$

$$= \frac{\rho^2 + \rho}{\rho^2+1} \rho^d + \frac{\rho + 1}{\rho^2+1} \rho^{-d}$$

$$= \frac{1}{(\rho+1)\rho^d} \cdot (\rho^{2d+1} + 1) \qquad 2.33$$

Case II. $L_0^* \equiv 2^{-2s} e^2 \equiv 1$ (mod 2)

$L_1^* \equiv 2^{-2s} . h(e+1)^2 \equiv 1$

Hence

$$L_d^* = \frac{\rho + 1}{\rho^2+1} \rho^d + \frac{\rho^2 + \rho}{\rho^2+1} \rho^{-d} = \frac{1}{\rho+1} \rho^d + \frac{\rho}{\rho+1} \cdot \frac{1}{\rho^d}$$

$$= \frac{1}{(\rho+1)\rho^{d-1}} (\rho^{2d-1} + 1) \qquad 2.34$$

Case III. $L_0^* \equiv h^2 + e^2 \equiv 1 + 1 \equiv 0$ (mod 2)

$L_1^* \equiv h^2 Y \equiv Y = \rho + \sigma$.

Thus

$$L_d^* = \frac{\rho(\rho+\rho^{-1}) + 0}{\rho^2 + 1} . \rho^d + \frac{\rho(\rho + \rho^{-1})}{\rho^2 + 1} . \frac{1}{\rho^d} = \rho^d + \frac{1}{\rho^d}$$

$$= \frac{1}{\rho^d} (\rho^d + 1)^2 . \qquad 2.35$$

Case IV. $L_0^* \equiv 2^{-r}.e^2 \equiv 0$ (mod 2)

$L_1^* \equiv 2^{-r}.h(e+1)^2 \equiv 1$

$$\therefore \quad L_d^* = \frac{\rho}{\rho^2+1} . \rho^d + \frac{\rho}{\rho^2 + 1} \frac{1}{\rho^d} = \frac{1}{\rho^2+1} (\rho^{d+1} + \frac{1}{\rho^{d-1}})$$

$$= \frac{1}{(\rho^2+1)\rho^{d-1}} \left(\rho^d + 1 \right)^2 . \qquad 2.36$$

Case V. $L_0^* \equiv 2^{-2s}.e^2 \equiv 1$ (mod 2)

$L_1^* \equiv 2^{-2s}.h \equiv 0$.

$$\therefore \quad L_d^* = \frac{1}{\rho^2+1} \rho^d + \frac{\rho^2}{\rho^2+1} . \frac{1}{\rho^d} = \frac{1}{\rho^2+1} (\rho^d + \frac{1}{\rho^{d-2}})$$

$$= \frac{1}{(\rho^2+1)\rho^{d-2}} (\rho^{d-1} + 1)^2 . \qquad 2.37$$

Proposition 2.1 Suppose that all the roots of $L_d^*(Y)$ lie in GF(4). Then d must satisfy the following conditions.

$$d \leq 2 \qquad \text{in case I}$$
$$d \leq 3 \qquad \text{II}$$
$$d = 2^t \times f \qquad \text{III and IV} \qquad 2.38$$
$$d = 2^t \times f + 1 \qquad \text{V}$$

where $f = 1$ or 3 or 5.

Proof. Let $0, 1, w, w^2$ be the elements of GF(4), where w, w^2 denote cube roots of 1. Now from the equation 2.30 we see that

$Y = 0$ corresponds to $\rho = 1$

$Y = 1$ corresponds to $\rho = w$ or w^2

and

$Y = w$ or w^2 corresponds to ρ = a root of

the equation

$$\rho^2 + w\rho + 1 = 0 \quad \text{or} \quad \rho^2 + w^2\rho + 1 = 0 \ . \qquad 2.39$$

These two equations do not have roots in GF(4). But

$$(x^2+wx+1)(x^2+w^2x+1) = x^4+x^3+x^2+x+1 = \frac{x^5+1}{x+1} \ . \qquad 2.40$$

So $Y = w$ or w^2 corresponds to ρ = a primitive 5th root of 1, ε say. Therefore $L_d^*(Y)$ has roots in GF(4) if and only if the equation $N_d(\rho) = 0$ has its roots in the set

$$R = \{1, w, w^2, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4\} \ . \qquad 2.41$$

Now take $\ell = 2^t \times f$, where f is odd. Then $\rho^\ell+1 = (\rho^f+1)^{2t}$, and the roots of $\rho^f + 1 = 0$ are all distinct and form a cyclic group. A generator of this group is a primitive f^{th} root of unity. Therefore the roots of the equation $\rho^f + 1 = 0$ are not all in R unless $f \leqq 5$. From this we get 2.38.

Corollary 2.2. The diameter d of a D- R graph with intersection matrix B must satisfy conditions 2.38 .

Proof. By theorem 2 all the eigenvalues of the assumed graph must be of degree $\leqq 2$ over the rationals. Therefore the same must be true over GF(2). Now if such a graph exists the roots of the polynomial $H_d(Y)$ must be rational or quadratic. Thus the polynomial $[2^{-p} H_d(Y)]^* = L_d^*(Y)$ (regarded as a function of ρ) must have all its roots in the set R .
This, by proposition 2.1, is true only when d satisfies the conditions 2.38
This proves theorem 3.

CHAPTER THREE

An upper bound of d

In this chapter we deal with the cases III, IV and V, aiming to obtain a bound for the diameter d of the assumed D- R graphs. We prove that there cannot be D- R graphs with intersection matrix B of diameter $d > 81$. To obtain that result we consider each of those three cases separately.

In case III, by looking into the polynomial $G_d(x)$ we get that $t \leq 3$.

In cases IV and V the polynomial $H_d(Y)$ provides the information that $t \leq 4$.

Those two results, together with the result of theorem 3, give rise to the following .

Theorem 4. The only possible D- R graphs with intersection matrix B are those of diameter

d = 1,2 in case I , d = 1,2,3 in case II

d = 1,2,4,8,3,6,12,24,5,10,20,40 in case III

d = 1,2,4,8,16,3,6,12,24,48,5,10,20,40,80 in case IV

d = 2,3,5,9,17,4,7,13,25,49,6,11,21,41,81 in case V .

Throughout this chapter the order of any integer I say, is defined to be the maximum power of 2 which divides I and it is denoted as ord(I) .

With this notation we have

$$\text{ord}(h) = r, \quad \text{ord}(e) = s .$$

§1 Case III $r = s = 0$.

From equations (2.3) and (2.4) we have

$$G^*_d(x) = xG^*_{d-1}(x) + G^*_{d-2}(x) \ , \quad d > 2 \qquad 3.1$$

$$G^*_1(x) = x \ , \quad G^*_2(x) = x^2 \quad . \qquad 3.2$$

Suppose that ρ,σ where $\rho^{-1} = \sigma$ are the roots of the equation

$$\rho^2 + x\rho + 1 = 0 \qquad 3.3$$

in characteristic 2.

Then the solution to the recurrence 3.1 is

$$G^*_d(x) = A\rho^d + B\sigma^d \qquad 3.4$$

where

$$A + B = G^*_0(x)$$
$$A\rho + B\sigma = G^*_1(x) \qquad 3.5$$

from which we get that

$$A = B = 1 \quad . \qquad 3.6$$

Lemma 3.1. If $G_d(x)$ has all its roots rational or quadratic, then $2^{2^{t-1}}$ divides $G_d(a)$ for any even number a.

Proof. From equations 3.4 and 3.6 we get that

$$G^*_d(x) = \rho^d + \rho^{-d} = (\rho^f + \rho^{-f})^{2^t} \qquad 3.7$$

Now, for any odd f, $x = \rho + \rho^{-1}$ divides $\rho^f + \rho^{-f}$.

Therefore x^{2^t} divides $G^*_d(x)$.

By hypothesis $G_d(x)$ is a product of at least $f \times 2^{t-1}$ quadratic factors over the rationals. Reduced modulo 2 this product is divisible by x^{2^t} . Therefore at least 2^{t-1} factors of $G_d^*(x)$ have constant term zero. Hence at least 2^{t-1} factors of $G_d(x)$ have even constant term. Thus $2^{2^{t-1}}$ divides the constant term of $G_d(x)$. Now let a be any even number. Then by the same argument $2^{2^{t-1}}$ divides the constant term of $G_d(x+a)$, which equals $G_d(a)$.

Corollary 3.1. ord $(c-k) \geqq 2^{t-1}$.

Proof. By equations 2.3 and 2.4 we have that the constant term of $G_d(x)$ is equal to

$$G_d(0) = (-h)^{\frac{d-2}{2}} . (c-k) \quad . \qquad 3.8$$

In this case h is odd

$\therefore$ ord $(c-k) = \text{ord}(G_d(0)) \geqq 2^{t-1}$, by lemma 3.1.

Lemma 3.2. Let ϕ,ψ be the roots of the equation

$$u^2 - xu + h = 0 \quad . \qquad 3.9$$

Then for any $d \geqq 2$

$$G_d(x) = (x+c)\frac{\phi^d - \psi^d}{\phi - \psi} + (c-k)\frac{\phi^{d-1} - \psi^{d-1}}{\phi - \psi} \qquad 3.10$$

provided $\phi \neq \psi$.

Proof. The roots of the equation 3.9 for $x = 2q\cos\alpha$, are $\phi,\psi = q.e^{\pm i\alpha}$.

Thus for any integer n we have

$$\phi^n - \psi^n = 2i\, q^n.\sin n\alpha \quad . \qquad 3.11$$

Now by 2.1 and 1.13 we get

$$G_d(x) = q^{d-1} F_d(\cos\alpha) = (2q\cos\alpha + c).q^{d-1} U_{d-1}(\cos\alpha) + (c-1-q^2)q^{d-2} U_{d-2}(\cos\alpha) \quad 3.12$$

But by 1.8 and 3.11 for every integer n, we have

$$q^n . U_n(\cos\alpha) = q^n \frac{\sin(n+1)\alpha}{\sin\alpha} = \frac{\phi^{n+1} - \psi^{n+1}}{\phi - \psi} . \quad 3.13$$

Thus since $q^2 = k - 1$, from 3.12, we get 3.10 .

<u>Lemma 3.3.</u> Let $x = \pm 2$ and ϕ, ψ be as in lemma 3.2.

Let $d = 2^t \times f$, where f is odd.

Then

(i) $\phi^d + \psi^d \equiv 2 \bmod 4.$ 3.14

(ii) $\operatorname{ord} \dfrac{\phi^d - \psi^d}{\phi - \psi} = t$. 3.15

<u>Proof</u>.

(i) We first note that in this case $\phi . \psi = h$.

Also

$$\phi + \psi = \pm 2 \equiv 2 \bmod 4$$

and

$$\phi^2 + \psi^2 = (\phi + \psi)^2 - 2\phi\psi = 4 - 2\phi\psi \equiv 2 \bmod 4 .$$

Hence we have result for $d = 1,2$.

Suppose now that result holds for given $d + 1$ and d .

Then

$$\phi^{d+2} + \psi^{d+2} = (\phi+\psi)(\phi^{d+1}+\psi^{d+1}) - h(\phi^d+\psi^d) \equiv 2(\text{mod} 4) . \quad 3.16$$

Hence 3.14 holds by induction for every d.

(ii) $$\frac{\phi^d - \psi^d}{\phi - \psi} = \frac{\phi^f - \psi^f}{\phi - \psi} . \prod_{i=1}^{t} (\phi^{d/2^i} + \psi^{d/2^i}) . \quad 3.17$$

Now

$$\text{ord} \prod_{i=1}^{t} (\phi^{d/2^i} + \psi^{d/2^i}) = \sum_{i=1}^{t} \text{ord} (\phi^{d/2^i} + \psi^{d/2^i}) = t \qquad 3.18$$

and

$$\frac{\phi^f - \psi^f}{\phi - \psi} = (\phi\psi)^{\frac{f-1}{2}} + \sum_{k=0}^{\frac{f-3}{2}} (\phi^{f-2k-1} + \psi^{f-2k-1}).(\phi\psi)^k \quad . \qquad 3.19$$

Now $\phi\psi$ is odd and by (i) $\sum_{k=0}^{\frac{f-3}{2}} (\phi^{f-2k-1} + \psi^{f-2k-1})$

is even $\therefore$ $\frac{\phi^f - \psi^f}{\phi - \psi}$ is odd.

Hence

$$\text{ord} \left(\frac{\phi^d - \psi^d}{\phi - \psi}\right) = \text{ord} \left(\frac{\phi^f - \psi^f}{\phi - \psi}\right) + \text{ord} \prod_{i=1}^{t} (\phi^{d/2^i} + \psi^{d/2^i})$$

$$= 0 + t = t \ .$$

<u>Proposition 3.1.</u> Let $r = s = 0$, $d = 2^t \times f$.

Then $t \leqq 3$.

<u>Proof.</u> From lemma 3.2 for $x = \pm 2$ we get

$$G_d(\pm 2) = (\pm 2 + c) \frac{\phi^d - \psi^d}{\phi - \psi} + (c-k) \frac{\phi^{d-1} - \psi^{d-1}}{\phi - \psi} \quad . \qquad 3.20$$

By lemma 3.1 $G_d(\pm 2)$ must be divisible by $2^{2^{t-1}}$.

Now $\dfrac{\phi^{d-1} - \psi^{d-1}}{\phi - \psi}$ is an integer and by corollary 3.1

$2^{2^{t-1}}$ divides c-k. Thus $2^{2^{t-1}}$ must divide the first term of 3.20 which, by lemma 3.3, is equal to

$$(\pm 2 + c) \times 2^t \times \text{odd number} = T, \text{ say}$$

and since c is even

either 4 divides c, which implies $\pm 2 + c \equiv 2 \pmod 4$

or 4 does not divide c, which implies $\pm 2 + c \equiv 4 \pmod 8$.

Thus $\operatorname{ord}(T) \leqq t + 2$. 3.21

Hence $G_d(\pm 2)$ will be divisible by $2^{2^{t-1}}$ only if T is.

This implies

$$2^{t-1} \leqq \operatorname{ord}(T) \leqq t+2 \qquad 3.22$$

which is true only when $t \leqq 3$.

Since $d = 2^t \times f$, f = 1 or 3 or 5 the above result proves Theorem 4 in case III.

§2. Cases IV and V

For the rest of this chapter we deal with case IV, where

$2s > r > 0, \quad d = 2^t \times f$

and

Case V, where $r > 2s \geq 0, \quad d = 2^t \times f + 1.$

Lemma 3.4 In cases IV and V $L_d^*(Y)$ is divisible by Y^{2^t} and $2^{-p} H_d(a)$ is divisible by $2^{2^{t-1}}$ for every even a.

Proof. From lemma 2.3 we have

$$L_d^*(Y) = \begin{cases} \dfrac{1}{\rho + \rho^{-1}} [\rho^d + \rho^{-d}] & \text{in case IV} \\ \\ \dfrac{1}{\rho + \rho^{-1}} [\rho^{d-1} + \rho^{-(d-1)}] & \text{in case V .} \end{cases} \qquad 3.23$$

Call the R.H.S. of 3.23 $N_d(\rho)$. Then the equation $L_d^*(Y) = N_d(\rho)$ holds even if ρ is transcentendal over GF(2).

Thus 3.23 holds identically if we put

$$Y = \rho + \rho^{-1} .$$

But $N_d(\rho) = \dfrac{(\rho^f + \rho^{-f})^{2^t}}{\rho + \rho^{-1}}$, and since f is odd we have that $(\rho + \rho^{-1})^{2^t - 1}$ divides $N_d(\rho)$.

Therefore

$$Y^{2^t - 1} \text{ divides } L_d^*(Y) = [2^{-p} H_d(Y)]^* .$$

Now, by theorem 2, $2^{-p} H_d(Y)$ is a product of at least $f \times 2^{t-1}$ quadratic factors. Reduced modulo 2, this product is divisible by Y^{2^t-1}. Therefore at least 2^{t-1} factors of $[2^{-p} H_d(Y)]^*$ have constant term zero. Hence at least 2^{t-1} factors of $2^{-p} H_d(Y)$ have an even constant term. Thus $2^{2^{t-1}}$ divides the constant term of $2^{-p} H_d(Y)$.

By the same argument, for any even a, $2^{2^{t-1}}$ divides the constant term of $2^{-p} H_d(Y+a)$, which equals $2^{-p} H_d(a)$.

Lemma 3.5.

(i) If 4 divides d then

$$H_d(0) = H_d(2) = (c-k)^2 = (h-e)^2 \; . \qquad 3.24$$

(ii) If 4 divides d-1 then

$$H_d(0) = -hc^2 \quad \text{and} \quad H_d(2) = -h(c^2-2h) \; . \qquad 3.25$$

Proof. From definition 2.3

$$K_d(Y) = -2(c-1)^2 \cos 2(d-1)\alpha + 2h(c^2-2c+2)\cos 2d\alpha - 2h^2 \cos 2(d+1)\alpha$$

$$M(Y) = 2[h^2+(c-1)^2+2h(c-1)\cos 2\alpha - hc^2]$$

$$(4-Y)\, H_d(Y) = M(Y) + K_d(Y) \; . \qquad 3.26$$

Now by 2.6

$$Y = 0 \Rightarrow \cos 2\alpha = -1 \Rightarrow \alpha = \frac{\pi}{2}$$

Thus

$$K_d(0) = \begin{cases} 2e^2 + 2h(e^2+1) + 2h^2 & \text{if } d \equiv 0(\text{mod}4) \\ -2e^2 - 2h(e^2+1) - 2h^2 & \text{if } d \equiv 1(\text{mod } 4) \end{cases}$$

and

$$M(0) = 2h^2 + 2e^2 - 4he - 2h(e+1)^2 .$$

∴ by 3.26 we get

$$H_d(0) = \begin{cases} (h - e)^2 & \text{if } d \equiv 0(\text{mod}4) \\ -h(e + 1)^2 & \text{if } d \equiv 1(\text{mod}4) . \end{cases}$$

Next $Y = 2 \Rightarrow \cos 2\alpha = 0 \Rightarrow \alpha = \frac{\pi}{4}$.

Thus

$$K_d(2) = \begin{cases} 2h(e^2 + 1) & \text{if } d \equiv 0(\text{mod}4) \\ 2(h^2 - e^2) & \text{if } d \equiv 1(\text{mod}4) \end{cases}$$

and

$$M(2) = 2h^2 + 2e^2 - 2h(e + 1)^2 .$$

∴ by 3.26 we get

$$H_d(2) = \begin{cases} (h - e)^2 & \text{if } d \equiv 0(\text{mod}4) \\ 2h^2 - h(e+1)^2 & \text{if } d \equiv 1(\text{mod}4) . \end{cases}$$

<u>Lemma 3.6</u> $H_d(4) = [d(h+e) + (h-e)]^2 - d^2 c^2 h$. 3.27

<u>Proof</u> By equation 2.6, $Y = 4 \Rightarrow \cos^2\alpha = 1 \Rightarrow \alpha = 0$.

From 2.9 we have

$$H_d(Y) = h[(q\frac{\sin(d+1)\alpha}{\sin\alpha} + \frac{c-1}{q}.\frac{\sin(d-1)\alpha}{\sin\alpha})^2 - c^2 \frac{\sin^2 d\alpha}{\sin^2\alpha}] .$$

On letting $\alpha \to 0$ and applying L'Hopital's rule we get 3.27.

Case IV. $2s > r > 0$. Here $p = \min(2s,r) = r$.

By lemma 3.4 $2^{-r} H_d(a)$, where a is any even number, is divisible by $2^{2^{t-1}}$. Thus

$$\text{ord}\,(H_d(a)) \geqq r + 2^{t-1} \quad . \tag{3.28}$$

Also by lemmas 3.5 and 3.6

$$H_d(2) = (h-e)^2 \qquad , d \equiv 0 \pmod 4) \tag{3.29}$$

$$H_d(4) = [d(h+e) + (h-e)]^2 - d^2(e+1)^2 h \quad . \tag{3.30}$$

Lemma 3.7. Let $r > s$. Then $t \leqq 4$.

Proof. Since $r > s > 0$ from 3.29 and 3.30 we get

$$\text{ord}\,(H_d(2)) = 2s \tag{3.31}$$

$$\text{ord}\,(H_d(4)) \geqq \min(2s,\ 2t + r) \tag{3.32}$$

where the equality holds only if $2s \neq 2t + r$.

Thus if $2s \neq 2t + r$ we have

$$2t + r \geqq \text{ord}\,(H_d(4)) \quad . \tag{3.33}$$

Now let $2s = 2t + r$ then 3.34

$$2t + r = \text{ord}\,(H_d(2)) \quad \text{(by 3.31)} \ . \tag{3.35}$$

$\therefore\ 2t + r \geqq r + 2^{t-1}$ by 3.28 $\therefore\ t \leqq 4$.

Lemma 3.8. Let $s > r$. Then $t \leqq 4$.

Proof. In this case since $s > r > 0$

$$\text{ord}\,(H_d(2)) = 2r \tag{3.36}$$

and

$$\text{ord}\,(H_d(4)) \geqq \min\,(2r,\ 2t + r) \tag{3.37}$$

where the equality holds only if $r \neq 2t$.

Thus when $r \neq 2t$

$$2t + r \geqq ord(H_d(4)) \quad . \tag{3.38}$$

When $r = 2t$, then by 3.36

$$r+2t = 2r = ord(H_d(2)) \quad . \tag{3.39}$$

Thus from 3.28 we get that

$$2t + r \geqq r + 2^{t-1} \quad ; \quad \text{for any } r.$$

$$\therefore \quad t \leqq 4 \; .$$

<u>Lemma 3.9</u> Let $s = r$. Then $t \leqq 4$

<u>Proof</u>. Let $ord\,(h+e) = r + u$, and $ord\,(h-e) = r + v$.

Now by 3.28 both $H_d(4)$ and $H_d(2)$ are of order greater than or equal to $r + 2^{t-1}$. Thus the same will be true for the number

$$H_d(4) - H_d(2) = d^2[(h+e)^2 - h(e+1)^2] + 2d(h^2-e^2) \tag{3.40}$$

$$= T_1 + T_2 \; , \quad \text{say} \; .$$

Now, $ord(T_1) = 2t+r$ and $ord(T_2) = t+2r+1+u+v$.

Therefore if $ord(T_1) \neq ord(T_2)$ then

$$ord(H_d(4) - H_d(2)) \leqq 2t + r \; . \tag{3.41}$$

Thus

$$2t + r \geqq r + 2^{t-1} \qquad \therefore \; t \leqq 4.$$

Next suppose that $ord(T_1) = ord(T_2)$, then

$$t - 1 = r + u + v \quad . \tag{3.42}$$

But by 3.28

$$r + 2^{t-1} \leqq ord(H_d(2)) = 2r + 2v \quad , \tag{3.43}$$

$\therefore \quad r + 2^{r+u+v} \leqq 2r + 2v < 2(r + u + v)$ which is impossible. Thus $ord(T_1) \neq ord(T_2)$.

From lemmas 3.7, 3.8 and 3.9 we have that the order of the diameter d in case IV is less than or equal to 4. That proves theorem 4 in case IV.

Case V $r > 2s \geq 0$. In this case $p = \min(r,2s) = 2s$. For any even number a we have, by lemma 3.4 that $2^{-2s} H_d(a)$ is divisible by $2^{2^{t-1}}$. Thus

$$ord(H_d(a)) \geq 2s + 2^{t-1} \quad . \tag{3.44}$$

By lemmas 3.5 and 3.6 we have

$$H_d(0) = -hc^2 = -h(e+1)^2, \quad d \equiv 1 (\mathrm{mod}\ 4)$$

$$\begin{aligned} H_d(4) &= [d(h+e) + (h-e)]^2 - d^2.hc^2 \\ &= [h(d+1) + e(d-1)]^2 + d^2 H_d(0) \quad . \end{aligned} \tag{3.45}$$

Here $d-1 = 2^t \times f$ and we will prove that $t \leq 4$.
We suppose that $t \geq 2$. Then $ord(d+1) = 1$.

Lemma 3.10. Let $r > 2s > 0$. Then $t \leq 4$.

Proof. Since $s > 0$, $e + 1$ is odd. Thus

$$ord\ (H_d(0)) = r \quad . \tag{3.46}$$

Thus by 3.44 we get that

$$r \geq 2s + 2^{t-1} \quad . \tag{3.47}$$

Now since both $H_d(4)$ and $d^2 H_d(0)$ have order greater than or equal to $2s + 2^{t-1}$ the same will be true for

$$H_d(4) - d^2 H_d(0) = [h(d+1) + e(d-1)]^2 . \qquad 3.48$$

But $\text{ord}\,(H_d(4) - d^2H_d(0)) \geqq 2 \times \min\,(r+1, t+s)$ where inequality holds only when $r+1 = t+s$. In which case from 3.47 we get that

$$r \geqq 2s + 2^{r-s} \quad \text{which is impossible.}$$

Thus always $r + 1 \neq t + s$. Hence

$$\text{ord}(H_d(4) - d^2H_d(0)) = 2 \times \min(r+1,t+s) \leqq 2t + 2s$$

$$\therefore \quad 2^{t-1} + 2s \leqq 2t + 2s \qquad 3.49$$

which is true only when $t \leqq 4$.

<u>Lemma 3.11.</u> Let $r > 2s = 0$. Then $t \leqq 4$.

<u>Proof.</u> Let $\text{ord}(e+1) = u$. Then

$$\text{ord}(H_d(0)) = r + 2u \quad . \qquad 3.50$$

Since e is odd

$$\text{ord}[H_d(4) - d^2 H_d(0)] \geqq 2 \times \min\,(r + 1, t) \quad . \qquad 3.51$$

If $r \nmid t-1$, then equality holds in (3.51) so $\text{ord}[H_d(4) - d^2H_d(0)] \leqq 2t$

Thus from 3.51 and 3.44, for $s = 0$, we have

$$2t \geqq 2^{t-1} \quad \therefore \quad t \leqq 4 .$$

Now if $r = t-1$, we consider the number

$$H_d(4) - d^2 H_d(2) = h^2[(d+1)^2 - 2d^2] + 2h(d^2-1).e + (d-1)^2 e$$

whose order must be greater than or equal to 2^{t-1} .

Now the orders of the terms in the R H S when $r = t-1$ are

$$2t - 1, \quad 2t + 1, \quad 2t$$

$$\therefore \quad \text{ord}(H_d(4) - d^2 H_d(2)) = 2t - 1$$

$$\therefore \quad 2t-1 \geqq 2^{t-1} \qquad \therefore \quad t \leqq 3 \ .$$

From lemmas 3.10 and 3.11 we have that in case V t is less than or equal to 4.

This proves theorem 4 in case V .

CHAPTER FOUR

Further Results

In the previous chapters we have proved that the only possible D-R graphs with intersection matrix B are those of diameter $d \leq 81$.

In this chapter we classify the assumed graphs according to the value of f and we prove.

Theorem 5. The only possible D-R graphs with intersection matrix B are those of diameter

d = 1,2,3,4,5,8,9	when f = 1
d = 3,4,6,7,12,13,24,25	when f = 3
d = 5,6,10,11	when f = 5 .

To obtain this result we consider each one of those classes separately. For each one we obtain certain arithmetical conditions which are necessary for the existence of the corresponding graph.

Those conditions alone provide the answer for f = 1. For the rest of the cases, we use the computer-algebra system Reduce 2 [10] (implemented on the computer at Newcastle). For each value of d we calulate $H_d(Y)$ reduced to various moduli. Then we observe that not all those polynomials satisfy the necessary conditions.

Definition 4.1. We define the order of the linear polynomial $AY + B$ to be the min(ord(A), ord(B)).

$\oplus$ means reduction modulo $Y^2 - Y - 1$.

We also define

$u = \text{ord}(h - e)$

$v = \text{ord}(h+e)$

$w = \text{ord}(e+1)$

$p = \min(r, 2s)$

$r = \text{ord}(h)$

$s = \text{ord}(e)$

$t = \text{ord}(d)$ or $\text{ord}(d-1)$ according to case.

Throughout this chapter the Greek letters ε and θ mean even numbers and odd numbers respectively.

§1. Graphs of diameter $d = 2^t$ or 2^t+1.

In chapter three we have proved that $t \leqq 3$ in case III and $t \leqq 4$ in cases IV and V. Here we prove that $t = 4$ is impossible.

Lemma 4.1. If $H_d(Y)$ is a product of rational or quadratic factors, then reduced modulo $Y^2 + \varepsilon Y$ will be divisible by $2^{2^{t-1}+p}$.

Proof. By lemma 2.3 for $f = 1$ in cases IV and V we have

$$L_d^*(Y) = Y^{2^t-1} \qquad 4.1$$

where $L_d(Y) = 2^{-p} H_d(Y)$. By hypothesis the polynomial $L_d(Y)$ is a product of at least 2^{t-1} quadratic factors.

Reduced modulo 2 this product has the form 4.1. Thus $L_d(Y)$ will be a product of 2^{t-1} factors of the type $Y^2 + \varepsilon Y + \varepsilon$ over the rationals. But

$$Y^2 + \varepsilon Y + \varepsilon \equiv (\varepsilon Y + \varepsilon)(\mathrm{mod}(Y^2 + \varepsilon Y)) . \qquad 4.2$$

Thus

$$L_d(Y) = P(Y).(Y^2 + \varepsilon Y) + 2^{2^{t-1}}.Q(Y) \qquad 4.3$$

where $Q(Y)$ is a polynomial of degree 2^{t-1} .

Therefore

$$L_d(Y) \equiv 2^{2^{t-1}} (aY + b)(\mathrm{mod}(Y^2+\varepsilon Y)) . \qquad 4.4$$

From which we get the stated result.

<u>Lemma 4.2.</u> Let $t \geq 2$. Then

$$H_d(Y) \equiv \begin{cases} (h-e)^2 \; (\mathrm{mod}(Y^2 - 2Y)) & \text{if} \quad d = 2^t \\ (h^2.Y-h(e+1)^2)(\mathrm{mod}(Y^2-2Y) & \text{if} \quad d = 2^t+1 \end{cases}$$

<u>Proof</u> $H_d(Y) \equiv (AY + B) \bmod Y^2-2Y$ 4.5

where

$$\begin{aligned} 2A + B &= H_d(2) \\ B &= H_d(0) \end{aligned} \qquad 4.6$$

By lemma 3.5

$$H_d(0) = H_d(2) \equiv (h-e)^2; \quad \text{when} \quad d \equiv 0(\mathrm{mod}\ 4)$$

and

$$\left.\begin{aligned} H_d(0) &= -h(e + 1)^2 \\ H_d(2) &= 2h^2 - h(e+1)^2 \end{aligned}\right\} \quad \text{when} \quad d \equiv 1(\mathrm{mod}\ 4) .$$

Now by 4.6

$$A = 2^{-1}\,[H_d(2) - H_d(0)]\ , \quad B = H_d(0) \qquad 4.7$$

from which we get the stated results .

<u>Definition 4.2</u> ⊛ means reduction modulo $Y^2 - 4Y$.

<u>Lemma 4.3.</u> Let $\mathrm{ord}(H_d(Y)(\mathrm{mod}\ Y^2-2Y)) \geqq 8 + p$.

Then

I $\mathrm{ord}(H^{⊛}_{16}(Y)) \leqq 6 + r$

II $\mathrm{ord}(H^{⊛}_{17}(Y)) \leqq 6 + 2s$.

<u>Proof.</u> $H_d(Y) \equiv CY + D \bmod Y^2 - 4Y$

where $4C + D = H_d(4)$

4.8

and $D = H_d(0)$.

Thus

$$C = 2^{-2}[H_d(4) - H_d(0)]\ , \quad D = H_d(0) \qquad 4.9$$

$$\therefore \quad H^{⊛}_d(Y) = 2^{-2}\,[H_d(4) - H_d(0)].Y + H_d(0) \ . \qquad 4.10$$

By lemma 3.6

$$H_d(4) = [d(h+e) + (h-e)]^2 - d^2h(e+1)^2$$

and by lemma 3.5

$$H_d(0) = (h-e)^2, \text{ when } d \equiv 0(\mathrm{mod}4)$$

$$H_d(0) = -h(e+1)^2, \text{when } d \equiv 1(\mathrm{mod}4)\ .$$

Thus when $d = 16$

$$H_{16}^{\Theta}(Y) = 8[8(h-1)(h-e^2) + (h^2 - e^2)]Y + (h-e)^2$$

$$= C_1 Y + D_1 \qquad 4.11$$

and when $d = 17$

$$H_{17}^{\Theta}(Y) = [81h^2 + 64e^2 - 72h(e^2+1)]Y - h(e+1)^2$$

$$= C_2 Y + D_2 \ . \qquad 4.12$$

I. $d = 16$ corresponds to the case IV, where $2s > r > 0$.

By hypothesis and by lemma 4.2 we have

$$8 + r = 8 + p \leqq \mathrm{ord}[H_d(Y) \bmod Y^2 - 2Y] = \mathrm{ord}(h-e)^2 = 2u \qquad 4.13$$

where
$$u = \begin{cases} r & \text{when } s > r \\ s & \text{when } r > s \\ r + q,\ q \geqq 1 & \text{when } r = s \ . \end{cases} \qquad 4.14$$

Now the orders of the terms in C_1 for $d = 16$ are

$$6 + r,\ 3 + u + v$$

where $u + v = 2u \geqq 8 + r$ when $r \neq s$

and $u + v \geqq 2r + 3$ when $r = s$.

Thus always $6 + r < 3 + u + v$

$$\therefore \quad \mathrm{ord}(C_1) = 6 + r \ . \qquad 4.16$$

II. $d = 17$ corresponds to the case V where $r > 2s \geqq 0$.

Now if $s > 0$ by hypothesis and lemma 4.2 we have that

$$r = \mathrm{ord}(B) \geqq 8 + 2s \ . \qquad 4.17$$

and since for $s > 0$ the orders of the terms in C_2 are

$$2r,\ 6 + 2s,\ 3 + r$$

we get that $\text{ord}(C_2) = 6 + 2s$. 4.18

Next let $s = 0$. Then by lemma 4.2

$$2r = \text{ord}(A) \geqq 8$$ 4.19

thus $r \geqq 4$. The orders of the terms in C_2 when $s = 0$ are

$$2r,\ 6,\ 4 + r$$ 4.20

thus since $r \geqq 4$ $\text{ord}(C_2) = 6$. 4.21

Hence by 4.16, 4.18, 4.21 and definition 4.1 we get the stated results.

Corollary 4.1. There is no graph of diameter $d = 16$ or 17.

Proof. Suppose that such graphs exist. Then the corresponding polynomials $H_{16}(Y)$ and $H_{17}(Y)$ will factorize into quadratic factors. Then by lemma 4.1 $H_{16}(Y)$ and $H_{17}(Y)$ reduced modulo $Y^2 + \varepsilon Y$ for any even ε will be of order greater than or equal to $8 + p$. This contradicts lemma 4.3.

Thus we have proved theorem 5 in the case $f = 1$.

§2. Graphs of diameter $d = 2^t \times 3$ or $2^t \times 3 + 1$

From theorem 4 we have that the maximal graphs of this class who possibly exist are those of diameter $d = 48$ in case IV and $d = 49$ in case V. Here we prove that no graphs of these diameters exist.

Lemma 4.4. If the D-R graph with diameter $d = 2^t \times 3$ or $2^t \times 3 + 1$ exists.
Then $L_d^*(Y)$ is divisible by $(Y^2+1)^{2^t}$ and $2^{-p}H_d(\theta)$ is divisible by 2^{2^t} for every odd θ.

Proof. From lemma 2.3 for $f = 3$ we have

$$L_d^*(Y) = \begin{cases} (\rho^3+\rho^{-3})^{2^t} & \text{in case III} \\ (\rho^3+\rho^{-3})^{2^t} . (\rho+\rho^{-1})^{-1} . & \text{in case IV and V .} \end{cases} \qquad 4.22$$

Call the R.H.S of 4.22 $N_d(\rho)$. Then the equation $L_d^*(Y) = N_d(\rho)$ holds even if ρ is transcendental over GF(2). Thus 4.22 holds identically if we put $Y = \rho + \rho^{-1}$.
Now $N_d(\rho)$ is divisible by $(\rho^2+\rho^{-2}+1)^{2^t}$ and since $\rho^2+\rho^{-2} = Y^2$ (in GF(2)) then

$$(Y^2+1)^{2^t} \text{ divides } L_d^*(Y) = [2^{-p} H_d(Y)]^* .$$

By theorem 2 if the graph of diameter d exists the polynomial $2^{-p} H_d(Y)$ will be a product of at least $f \times 2^{t-1}$ quadratic factors over the rationals. Reduced modulo 2, this product when $f = 3$ is divisible by $(Y^2+1)^{2^t}$. Therefore at least 2^t factors of $[2^{-p} H_d(Y)]^*$ will have the form $Y^2 + Y$ or $Y^2 + 1$.

Hence at least 2^t quadratic factors of $2^{-p}H_d(Y)$ will have the form $Y^2 + \theta Y + \varepsilon$ or $Y^2 + \varepsilon Y + \theta$. Thus since the value of each one of those terms at $Y = \theta$ is even we get the stated result.

Lemma 4.5

$$H_d(1) = \begin{cases} (h-e)^2 & , \text{ when } d \equiv 0(\text{mod}3) \\ h^2 - h(e+1)^2 & , \text{ when } d \equiv 1(\text{mod}3) \end{cases} .$$

Proof. By definition 2.2 $Y = 2\cos 2\alpha + 2$.

Thus for $Y = 1$, $\alpha = \frac{2\pi}{3}$. By definition 2.3

$$M(1) = 2he + 2(h-e)^2 - 2h(e+1)^2 \qquad 4.23$$

$$K_d(1) = -2e^2\cos(d-1)\frac{4\pi}{3} + 2h(e^2+1)\cos d\frac{4\pi}{3} - 2h^2\cos(d+1)\frac{4\pi}{3} \qquad 4.24$$

$$3.H_d(1) = K_d(1) + M(1) \qquad 4.25$$

Let $d \equiv 0(\text{mod}3)$. Then

$$K_d(1) = -2e^2\cos\frac{4\pi}{3} + 2h(e^2+1) - 2h^2\cos\frac{4\pi}{3}$$

$$= e^2 + 2h(e^2+1) + h^2 \qquad 4.26$$

and from 4.25 we get

$$H_d(1) = (h-e)^2, \quad d \equiv 0(\text{mod } 3) .$$

Let $d \equiv 1(\text{mod}3)$. Then

$$K_d(1) = -2e^2 + 2h(e^2+1)\cos\frac{4\pi}{3} - 2h^2\cos\frac{2\pi}{3}$$

$$= -2e^2 - h(e^2+1) + h^2 . \qquad 4.27$$

Thus

$$H_d(1) = h^2 - h(e+1)^2, \quad d \equiv 1(\text{mod}3) .$$

Lemma 4.6. Let $2s > r > 0$. Let $d = 48$ (so $t = 4$) and $\mathrm{ord}(H_{48}(1)) \geqq 16 + r$.
Then $\mathrm{ord}(H_{48}(-3)) = 12 + r$.

Proof. By hypothesis and lemma 4.5 we have

$$2u = \mathrm{ord}(h-e)^2 = \mathrm{ord}(H_{48}(1)) \geqq 16+r \qquad 4.28$$

Now by computer calculations we get

$$H_{48}(-3) = 2^{13}.5.h^2 + 2^{12}.13.e^2 + 2^6.47.(h^2-e^2)+2^{12}.h(e^2+1)+(h-e)^2 \ . \qquad 4.29$$

This result is reduced modulo 2^{16} since otherwise the result would be intolerable. The orders of the terms in 4.29 are

$$13+2r, \quad 12+2s, \quad 6+u+v, \quad 12+r, \quad 2u \quad . \qquad 4.30$$

Now if $r \nmid s$ $v = \mathrm{ord}(h+e) = \mathrm{ord}(h-e) = u \geqq 8 + \frac{r}{2}$ by (4.28) and if $r = s$ then $u \geqq 8 + \frac{r}{2}$ by (4.28) and $v > r$

Thus always $12 + r < 6 + u + v$ and all the other terms of 4.30 are greater than $12 + r$.

From which we get that $H_{48}(-3)$ is of order $12 + r$.

Lemma 4.7. Let $r > 2s \geqq 0$. Let $d = 49$ (so $t = 4$) and $\mathrm{ord}(H_{49}(1)) \geqq 16 + 2s$. Then $\mathrm{ord}\ (H_{49}(-3)) = 12 + 2s$.

Proof. We first consider the case $s > 0$. Here by hypothesis and lemma 4.5$'$ we have

$$r = \mathrm{ord}\ (H_{49}(1)) \geqq 16 + 2s \ . \qquad 4.31$$

Now by computer we calculate $H_{49}(-3)$ reduced modulo 2^{16} and we get

$$H_{49}(-3) = -43715h^2 - 2^6.687.h(e^2+1) - h(e+1)^2 - 2^{12}.11.e^2 \quad . \qquad 4.32$$

The orders of the terms here are

$$2r, \quad 6+r, \quad r, \quad 12+2s \qquad 4.33$$

and since (by 4.31) $r \geqq 16 + 2s$ we have that

$$\mathrm{ord}(H_{49}(-3)) = 12+2s \ .$$

Now let $s = 0$.

Consider

$$H_{49}(-3) - H_{49}(1) = -2^2.10929h^2 - 2^6.687h(e^2 + 1) - 2^{12}.11.e^2 \quad . \qquad 4.34$$

Here the orders are

$$2+2r, \quad 7+r, \quad 12 \quad . \qquad 4.35$$

Now

$$12 < 7 + r < 2 + 2r \quad ; \text{ when } r > 5$$

$$12 = 7 + r = 2 + 2r \quad , \text{ when } r = 5$$

$$2 + 2r < 7 + r < 12 \quad , \text{ when } r < 5 \ .$$

Thus

$$\mathrm{ord}\ (H_{49}(-3) - H_{49}(1)) \leqq 12 \qquad 4.36$$

and since by hypothesis $\mathrm{ord}(H_{49}(1)) \geqq 16$ we have that $\mathrm{ord}(H_{49}(-3)) \leqq 12$.

Corollary 4.2. There is no graph of diameter $d = 48$ or 49.

Proof. Suppose that such graphs exist. Then by lemma 4.4 $H_{48}(\theta)$ and $H_{49}(\theta)$ will have orders greater than or equal to $16 + p$ for any odd numbers θ. By lemmas 4.6 and 4.7 when $\theta = -3$ this is not so.

This proves theorem 5 in the case $f = 3$.

Graphs of diameter $d = 2^t \times 5$ or $2^t \times 5 + 1$.

In chapter three we have proved that possible graphs of this class are those of diameter

$$5, 10, 20, 40, 80 \qquad \text{when} \quad d = 2^t \times 5$$

and

$$6, 11, 21, 41, 81 \qquad \text{when} \quad d = 2^t \times 5 + 1.$$

Here we prove that the only possible graphs are those of diameter $d = 5, 6, 10$ or 11.

§3. Necessary conditions for graphs of diameter $d = 2^t \times 5$ or $2^t \times 5 + 1$.

Lemma 4.8. If a graph of diameter $d = 2^t \times 5$ or $d = 2^t \times 5 + 1$ exists then the polynomial $H_d(Y)$ reduced modulo $Y^2 + \theta Y + \theta$ will be divisible by $2^{2^{t-1}+p}$.

Proof. By lemma 2.3 for $f = 5$ we have

$$L_d^* = \begin{cases} (\rho^5 + \rho^{-5})^{2^t} & \text{in case III} \\ (\rho^5 + \rho^{-5})^{2^t} (\rho + \rho^{-1})^{-1} & \text{in cases IV and V} \end{cases} \qquad 4.37$$

where $\rho + \rho^{-1} = Y$. Now in GF(2)

$$\rho^5 + \rho^{-5} = (\rho + \rho^{-1}) \; [(\rho + \rho^{-1})^2 + (\rho + \rho^{-1}) + 1]^2$$

$\therefore$

$$L_d^*(Y) = \begin{cases} Y^{2^t} (Y^2 + Y + 1)^{2^{t+1}} & \text{in case III} \\ Y^{2^t-1} (Y^2 + Y + 1)^{2^{t+1}} & \text{in cases IV and V .} \end{cases} \qquad 4.38$$

By theorem 2, if the assumed graph exists, the polynomial $L_d(Y) = 2^{-p} H_d(Y)$ will be a product of linear or quadratic factors over the rationals. Reduced modulo 2 this product has 2^{t+1} irreducible quadratic factors of the form $Y^2 + Y + 1$. Hence the polynomial $2^{-p} H_d(Y)$ has at least 2^{t+1} irreducible (over the rationals) quadratic factors of the form $Y^2 + \theta Y + \theta$. Now for each one of these factors we have

$$Y^2 + \theta Y + \theta \equiv \varepsilon Y + \varepsilon \pmod{Y^2 + \theta Y + \theta} \qquad 4.39$$

Thus

$$2^{-p} H_d(Y) = P(Y).(Y^2 + \theta Y + \theta) + 2^{2^{t+1}}.Q(Y) \qquad 4.40$$

where $Q(Y)$ is a polynomial of degree $\leq 2^t$.

Therefore

$$2^{-p}\, H_d(Y) \equiv 2^{2^{t+1}}\ (AY + B)(\text{mod}\, Y^2 + \theta Y + \theta) \quad . \qquad 4.41$$

Now since $Y^2 + \theta Y + \theta$ is monic and $Q(Y)$ has integer coefficients we have that A, B are integers. Thus $2^{-p}H(Y)$ reduced modulo $Y^2 + \theta Y + \theta$ is divisible by $2^{2^{t+1}}$. Hence result.

Lemma 4.9.

$$H_d(Y) = \begin{cases} (h-e)^2 \ (\text{mod}\ Y^2 - 3Y + 1) & \text{if}\ \ d \equiv 0(\text{mod}5) \\ (h^2Y - h(e+1)^2)(\text{mod}\ Y^2 - 3Y + 1) & \text{if}\ \ d \equiv 1(\text{mod}5) \end{cases}$$

Proof. Let Y_1, Y_2 be the roots of $Y^2 - 3Y + 1 = 0$.

Then

$$H_d(Y) = P(Y).(Y^2 - 3Y + 1) + AY + B \qquad 4.42$$

where

$$\begin{aligned} AY_1 + B &= H_d(Y_1) \\ AY_2 + B &= H_d(Y_2) \quad . \end{aligned} \qquad 4.43$$

Now $Y_1 = \dfrac{3 + \sqrt{5}}{2}$, $Y_2 = \dfrac{3 - \sqrt{5}}{2}$. By definition 2.2 equation (2.6) $Y = 2\cos 2\alpha + 2$. Thus

$$\cos 2\alpha = \frac{-1 + \sqrt{5}}{4} \quad \text{or} \quad \cos 2\alpha = -\frac{1 + \sqrt{5}}{4} \quad .$$

Hence

Y_1 corresponds to $\alpha = \frac{\pi}{5}$

Y_2 corresponds to $\alpha = \frac{2\pi}{5}$.

By definition 2.3

$K_d(Y) = -2e^2\cos(d-1).2\alpha+2h(e^2+1)\cos d.2\alpha-2h^2\cos(d+1)2\alpha$

$M(Y) \;=\; 2[h\ e\ Y + (h - e)^2 - h(e + 1)^2]$.

Thus when $d \equiv 0(\text{mod}5)$

$K_d(Y_1) = 2h(e^2+1) - 2\cos\frac{2\pi}{5}(h^2 + e^2)$

$K_d(Y_2) = 2h(e^2+1) - 2\cos\frac{4\pi}{5}(h^2 + e^2)$

and

$M(Y_1) = 2h^2 + 2e^2 + 4he.\cos\frac{2\pi}{5} - 2h(e+1)^2$

$M(Y_2) = 2h^2 + 2e^2 + 4he.\cos\frac{4\pi}{5} - 2h(e+1)^2$.

Now for every $Y \not\equiv 4$

$$H_d(Y) = [K_d(Y) + M(Y)](4-Y)^{-1} .$$

Thus

$$H_d(Y_1) = H_d(Y_2) = (h-e)^2 \qquad 4.44$$

and from 4.43 we have

$$A = 0, \quad B = (h-e)^2 \quad . \qquad 4.45$$

In the case $d \equiv 1(\text{mod}5)$, working similarly we have

$$A = h^2, \; B = -h(e+1)^2 \quad . \qquad 4.46$$

4.45 and 4.46 prove lemma 4.9.

Corollary 4.1. The only possible D-R graphs with intersection matrix B and diameter $d = 2^t \times 5$ or $2^t \times 5 + 1$ are those for which

$$2 \times \text{ord}(h-e) \geqq 2^{t+1} + p, \quad \text{when} \quad d \equiv 0(\text{mod}5)$$

or

$$\text{ord}\,(h^2Y - h(e+1)^2) \geqq 2^{t+1} + p, \text{ when } \quad d \equiv 1(\text{mod}5)\ .$$

Proof. By lemma 4.8 if such a graph exists then the corresponding polynomial $H_d(Y)$ reduced modulo $Y^2 + \theta Y + \theta$ for any odd numbers θ must be divisible by $2^{2^{t+1}+p}$. By lemma 4.9.

$$H_d(Y) \equiv \begin{cases} (h-e)^2 \ (\text{mod}Y^2-3Y+1)\ , & d \equiv 0(\text{mod}5) \\ (h^2Y-h(e+1)^2)(\text{mod}Y^2-3Y+1)\ , & d \equiv 1(\text{mod}5) \end{cases}$$

Hence result.

§4. Graphs with diameter $d = 2^t \times 5$.

These Graphs correspond to the cases III where $s = r = 0$ and IV where $2s > r > 0$.

By corollary 4.1 we have that the only possible graphs of this class are those for which

$$2u = 2 \times \text{ord}(h-e) \geqq 2^{t+1} + r. \tag{4.47}$$

Here we note that

$$u = v = \begin{cases} r, & \text{when } s > r \\ s, & \text{when } s < r \end{cases} \qquad 4.48$$

and

$$u + v > 2^t + r \,, \quad \text{when } s = r \quad . \qquad 4.49$$

Since $u \geq 2^t + \frac{r}{2}$ by (4.47) and $v \geq \min(r,s) > \frac{r}{2}$

In the following we assume that condition 4.47 holds.
We also recall that $\oplus$ means reduction modulo $Y^2 - Y - 1$

Lemma 4.10. Let $t = 2,3$ or 4. Then

$$\text{ord}(H_d^{\oplus}(Y)) \leq 2(t+1) + r \,.$$

Proof We calculate $H_{20}^{\oplus}$, $H_{40}^{\oplus}$ and $H_{80}^{\oplus}$ on computer. This gives intolerably large numbers. To avoid this we have reduced all numerical coefficients modulo 2^{16} and we get.

$$\begin{aligned} H_{20}^{\oplus}(Y) &= [2^6.191.h(e^2+1) + 2^7.83.e^2 + 2^5.667(h^2-e^2)]Y \\ &\quad + 2^6.25h(e^2+1) + 2^6.883e^2 + (h-e)^2 - 2^3.2101\,(h^2-e^2) \\ &= A_1 Y + B_1 \quad . \end{aligned} \qquad 4.50$$

$$\begin{aligned} H_{40}^{\oplus}(Y) &= [2^8.111\,h(e^2+1) - 2^9.29.e^2 + 2^6.513(h^2-e^2)]Y \\ &\quad - 2^8.87.h(e^2+1) - 2^8.29.e^2 + (h-e)^2 - 2^4.2441(h^2-e^2) \\ &= A_2 Y + B_2 \end{aligned} \qquad 4.51$$

and

$$H^{\oplus}_{80}(Y) = [-2^{10}.17.h(e^2+1) + 2^{11}.3.e^2 - 2^7.115\ (h^2-e^2)]Y$$

$$+ 2^{10}.41.h(e^2+1) - 2^{10}.29.e^2 + (h-e)^2 + 2^5.1807(h^2-e^2)$$

$$= A_3 Y + B_3 \ . \qquad 4.52$$

Now if $2s > r > 0$ the orders of the terms in the constant term B_i, $i = 1,2,3$ are

$$2t + 2 + r, \quad 2t + 2 + 2s, \quad 2u, \quad t + 1 + u + v \qquad 4.53$$

and since $2u \geqq 2^{t+1} + r$ and $u + v > 2^t + r$ for $t = 2,3$ or 4 (so $d = 20, 40, 80$) we get that

$$\mathrm{ord}(B_i) = 2t + 2 + r \ . \qquad 4.54$$

Next if $s = r = 0$ the orders in the constant term become

$$2t + 3, \ 2t + 2, \ 2u, \ t + 1 + u + v \qquad 4.55$$

where $2u \geqq 2^{t+1}$ and $u + v > 2^t$.

Thus

$$\mathrm{ord}(B_i) = 2t + 2 \ .$$

<u>Corollary 4.2.</u> If a graph of diameter $d = 2^t \times 5$ exists then $t \leqq 1$.

<u>Proof.</u> By lemma 4.8 the corresponding polynomial $H_d(Y)$ reduced modulo $Y^2 + \theta Y + \theta$ where θ is any odd numbers will be of order greater than or equal to $2^{t+1} + r$.

By lemma 4.10 for $t = 2,3,4$

$$2(t+1) + r \geqq \mathrm{ord}(H^{\oplus}_d(Y)) \geqq 2^{t+1} + r$$

which is impossible.

§5. Graphs with diameter $d = 2^t \times 5 + 1$

Graphs of this diameter correspond to the case V where $r > 2s \geqq 0$.

By corollary 4.1 there is no graph of this type unless $\mathrm{ord}(h^2 Y - h(e+1)^2) \geqq 2^{t+1} + 2s$. From this we get that

$$r \geqq 2^{t+1} + 2s, \qquad \text{when } s \neq 0 \tag{4.56}$$

and

$$\min(2r, r + 2w) \geqq 2^{t+1} \qquad \text{when } s = 0 . \tag{4.57}$$

Lemma 4.11. Let $t = 2, 3$ or 4. Then

$$\mathrm{ord}(H_d^{\oplus}(Y)) \leqq 2(t+1) + 2s .$$

Proof For the same reason as in lemma 4.10 the polynomials $H_{21}^{\oplus}(Y)$, $H_{41}^{\oplus}(Y)$ and $H_{81}^{\oplus}(Y)$ have been calculated modulo 2^{16}. These calculations give

$$\begin{aligned} H_{21}^{\oplus}(Y) &= [-25927h^2 - 2^5.667\, h(e^2+1) - 2^6.191\, e^2]Y \\ &\quad + 2^4.3535\, h^2 - h(e+1)^2 + 2^3.2101 h(e^2+1) - 2^6.25.e^2 \\ &= A_1 Y + B_1 . \end{aligned} \tag{4.58}$$

$$\begin{aligned} H_{41}^{\oplus}(Y) &= [-43471.h^2 - 2^6.513\, h(e^2+1) - 2^8.111\, e^2]Y \\ &\quad + 2^5.723\, h^2 - h(e^2+1) + 2^4.2441\, h(e^2+1) + 2^8.87.e^2 \\ &= A_2 Y + B_2 \end{aligned} \tag{4.59}$$

and

$$H^{\oplus}_{81}(Y) = [55137\ h^2 + 2^7.115\ h(e^2+1) + 2^{10}.17.e^2]Y$$

$$+ 2^6.357\ h^2 - h(e+1)^2 - 2^5.1807\ h(e^2+1) - 2^{10}.41.e^2$$

$$= A_3Y + B_3 \quad . \qquad 4.60$$

Now if $s \neq 0$ the orders of the terms in the coefficient A_i $(i = 1,2,3)$ of Y are

$$2r,\ t + 3 + r,\ 2t + 2 + 2s \qquad 4.61$$

where $t = 2,3$ or 4 (so $d = 21$, 41 or 81). Thus by 4.56 we have that

$$\text{ord}(A_i) = 2t + 2 + 2s\ , \quad \text{if} \quad s > 0\ . \qquad 4.62$$

Next let $s = 0$. Then the orders of the terms in A_i, $i = 1,2,3$ become for $t = 2,3$ or 4

$$2r,\ t + 4 + r,\ 2t + 2 \qquad 4.63$$

and by 4.57 we have that $r > 2^t$.

Then $\text{ord}(A_i) = 2t + 2$.

<u>Corollary 4.3</u>. If a graph of diameter $d = 2^t \times 5 + 1$ exists then $t \leq 1$.

<u>Proof</u>. By lemmas 4.8 and 4.11 if such a graph exists then for $t = 2,3,4$

$$2(t+1) + 2s \geq \text{ord}(H^{\oplus}_d(Y)) \geq 2^{t+1} + 2s$$

which is impossible.

Corollaries 4.2 and 4.3 prove theorem 5 in the case $f = 5$.

CHAPTER FIVE

The impossibility of graphs with $d = 25$ or 24.

In this chapter we deal with graphs of diameter $d = 25$ or 24, and we suppose that these graphs exist. This assumption together with the theorems of Bannai and Ito [2] and Dumas [7] implies that the coefficients of the polynomials $H_{25}(z)$ and $H_{24}(z)$ where $z = Y-1$ must be divisible by certain powers of 2.
But, by computing the actual coefficients, we see that this is not always so.
Thus we prove
Theorem 6. The only possible D-R graphs with intersection matrix B are those of diameter $1 \leqq d \leqq 13$.

§1. Newton Polygons

Consider the polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_o, \quad a_n a_0 \neq 0$$

with rational coefficients.
For the prime number q, this polynomial can be written in the form

$$P(x) = \sum_{i=0}^{n} q^{\alpha_i} \frac{m_i}{\ell_i} x^i$$

where $(m_i, \ell_i) = 1$, $(m_i, q) = 1$ and $(\ell_i, q) = 1$, $i = 0, \dots, n$.

From this expression of $P(x)$ we form the ordered pairs (α_i, i) and we plot them on a rectangular coordinate system, omitting points corresponding to zero coefficients. From these points we construct the Newton Polygon Π which is the convex line enclosing all the points from below. For these polygons Dumas [7] has proved the following

<u>Theorem</u>. The polygon of a product is obtained from the polygons of the factors by joining their sides end to end according to non decreasing slope.

(For the proof of this theorem see [7]).

For example for the prime 3 the factors

$$x^2 + 3x + 9 \ , \quad x^3 - 3x + 27$$

have the polygons shown in figures 3 and 4 respectively which are combined in order of increasing slopes to form the polygon shown in figure 5 for the product

$$(x^2+3x+9)(x^3-3x+27) = x^5 + 3x^4 + 6x^3 + 18x^2 + 54x + 243 \ .$$

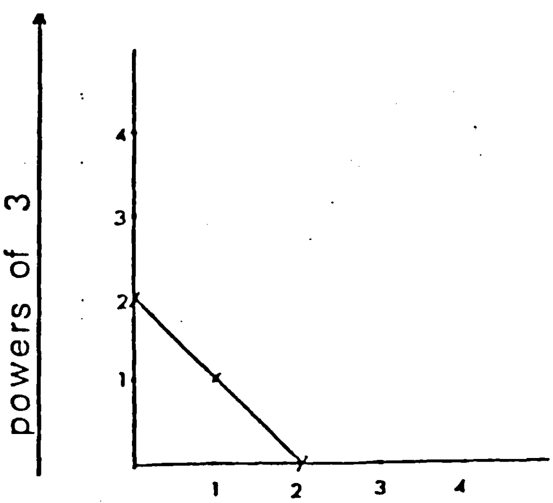

fig. 3

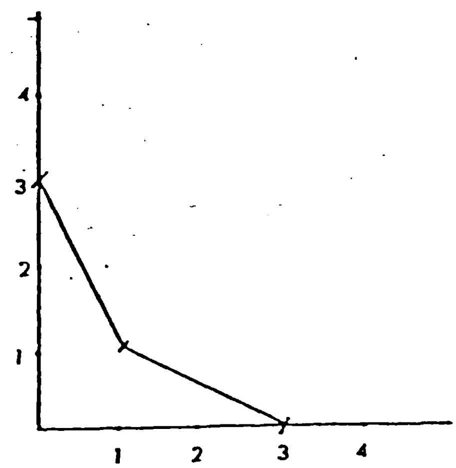

fig. 4

powers of X

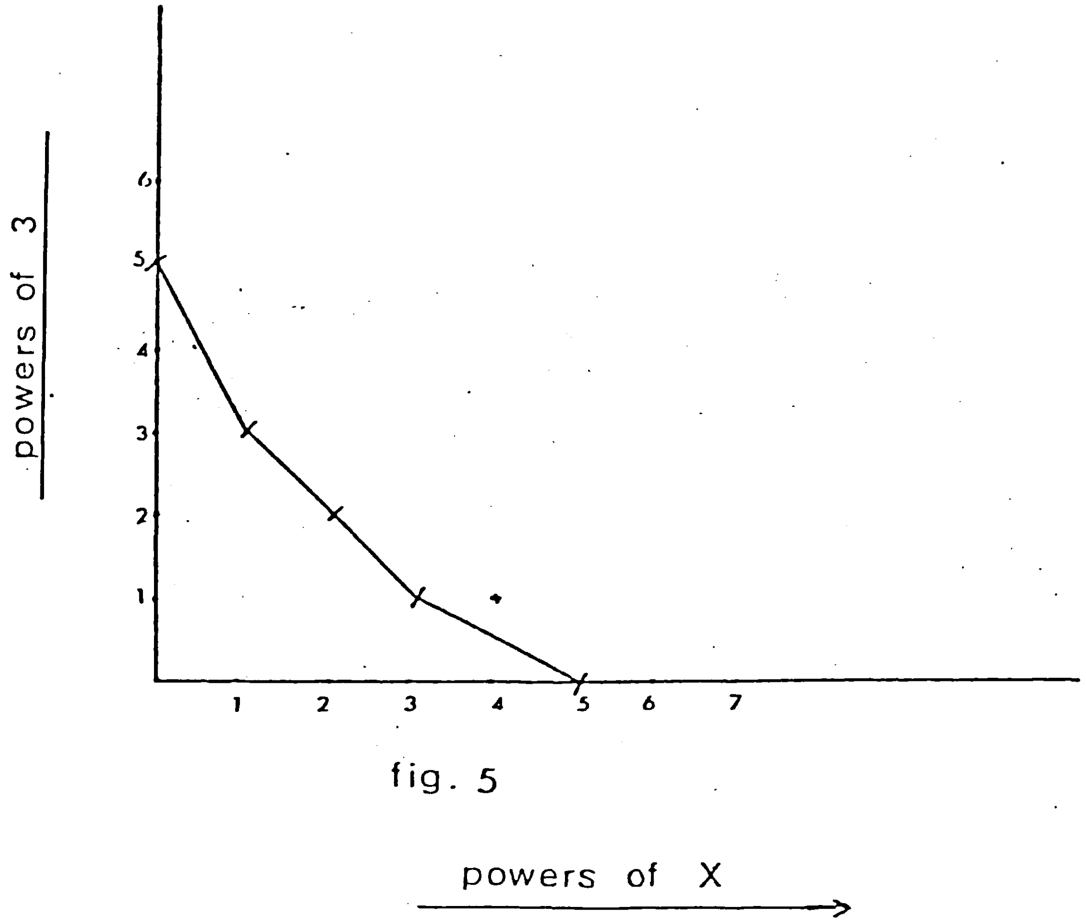

fig. 5

<u>Corollary 5.1.</u> If the polynomial p(x) with rational coefficients is a product of quadratic factors, over the rationals, then its Newton Polygon is combined of sections of (horizontal) length 2. These sections are flat or have slope $\frac{1}{2}$ an integer.

§2. Proof of theorem 6

We now apply the method of Newton Polygons to prove theorem 6. We recall that by theorem 4

$d = 25$ is impossible unless $r > 2s \geqq 0$ (case V)

$d = 24$ is impossible unless $2s > r > 0$ (case IV)

or $r = s = 0$ (case III)

and we shall prove that even these are impossible.

where $r = \text{ord}(h)$, $s = \text{ord}(e)$

Also $p = \min(r,2s)$, $u = \text{ord}(h-e)$,

$v = \text{ord}(h+e)$, $w = \text{ord}(e+1)$

and

$$L_d(Y) = H_d(Y)\, 2^{-p} .$$

Supposing that our graphs with diameters $d = 25$ or 24 exist from lemmas 4.4 and 4.5 we have that

$$2u \geqq 8 + 2r \quad \text{when} \quad d = 24 \qquad 5.1$$

and

$$\min(2r, r+2w) \geqq 8 + 2s \quad \text{when} \quad d = 25 . \qquad 5.2$$

Now for $d = 25$ or 24 and for $Y = z+1$ lemma 2.3 gives

$$2^{-p} H_d(z) \equiv \begin{cases} (z+1)^8.z^{16} & \text{when } r = s = 0 \\ (z+1)^7.z^{16} & \text{when } 2s > r > 0 \\ & \text{or } r > 2s \geqq 0 . \end{cases} \pmod 2$$

Thus in the polynomial $2^{-p} H_d(z)$, $d = 24,25$ the coefficient of z^m for any m such that $0 \leq m < 16$ is even while the coefficient of z^{16} is odd. Thus for the prime 2 we have a family of Newton Polygons with parameters r and s. All the members of this family have the point (16,0) in common. For r = 18, s = 3 a representative of this family is shown in figure 6.

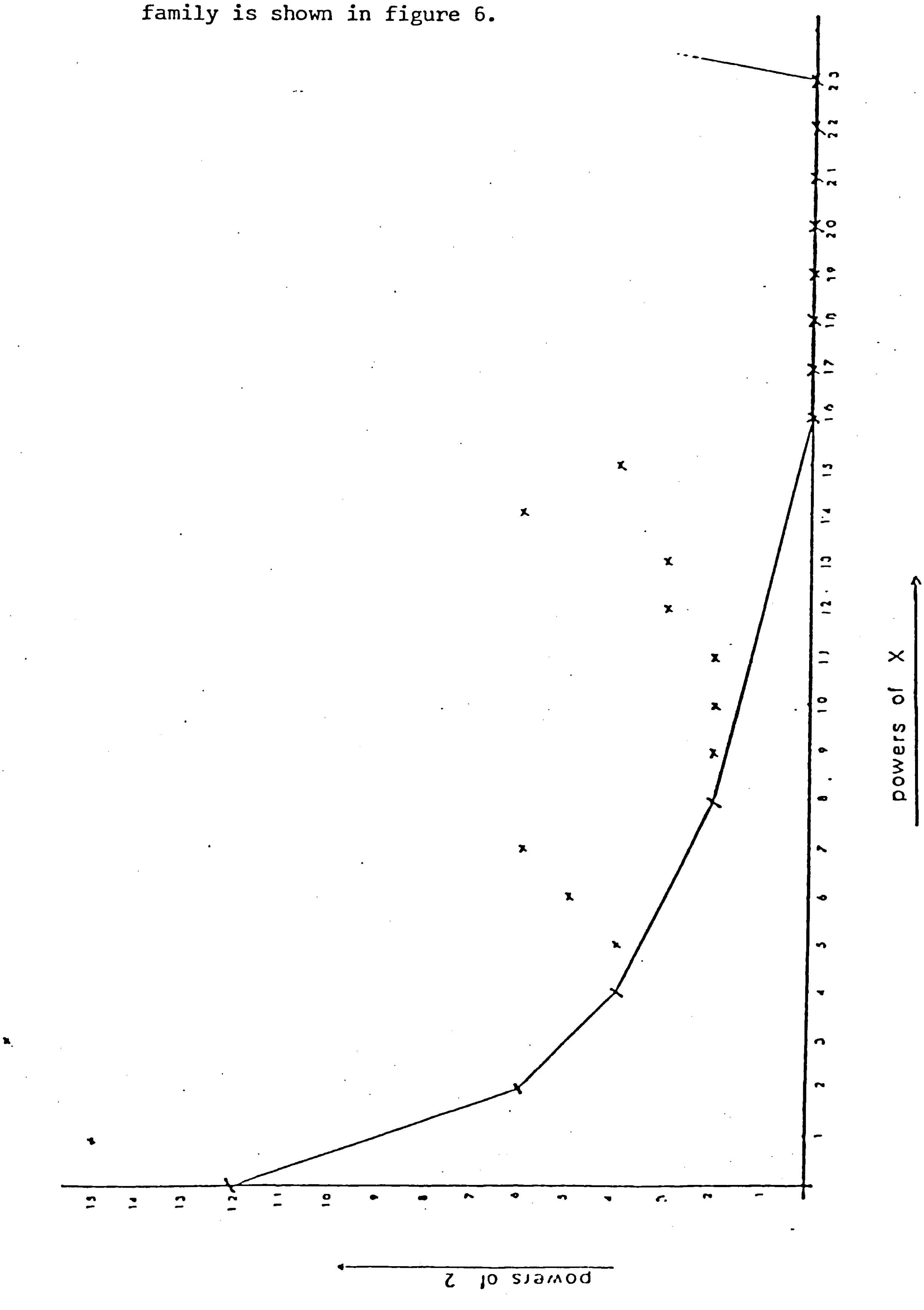

fig. 6

Lemma 5.1. If the D-R graphs with diameters $d = 25$ or 24 exist, then the coefficient of z^8 in $2^{-p} H_d(z)$ will be divisible by 2^4.

Proof. By hypothesis the polynomial $2^{-p} H_d(z)$, $d = 25, 24$ is a product of quadratic factors, over the rationals. Thus by corollary 5.1 the non flat section of its Newton Polygon will have slope greater than or equal $\frac{1}{2}$. Therefore the coefficient of z^8 will have order greater than or equal to $\frac{1}{2}\, 8 = 4$.

We now prove theorem 6 by showing that there are no D-R graphs with intersection matrix B of diameters $d = 25$ or 24.

Proof. By lemma 5.1 the coefficient of z^8 is divisible by 2^4. But, by computing coefficients, we get that the actual coefficient is

$$3.[10505\, h^2 - 407\, h(e^2+1) - 5716\, e^2] \qquad \text{if } d = 25 \qquad 5.3$$

$$3.[5716\, h(e^2+1) + 4372.e^2 + 407\,(h^2-e^2)] \qquad \text{if } d = 24 \;. \qquad 5.4$$

Now when $d = 25$ the orders of the terms of the coefficient of z^8 in $2^{-2s} H_{25}(z)$ are

$$2r-2s, \quad r-2s, \quad 2 \quad \text{when } s \neq 0 \qquad 5.5$$

$$2r, \quad r+1, \quad 2 \quad \text{when } s = 0 \;. \qquad 5.6$$

Let $s \neq 0$. Then $w = \text{ord}(e+1) = 0$ thus from 5.2 we get that $r \geq 8 + 2s$ $\therefore$ by 5.5 we get that the coefficient of z^8 in $2^{-2s} H_{25}(z)$ has order 2 which contradicts to lemma 5.1

Also when $s = 0$ since $r \geqq 1$ we have that the order of the coefficient of z^8 in $H_{25}(z)$ is again 2.

Next when $d = 24$ the orders of the terms of the coefficient of z^8 in $2^{-r} H_{24}(z)$ are

$$2, \quad 2s - r + 2, \quad u + v \quad \text{when} \quad 2s > r > 0$$

$$3, \quad 2, \quad u + v \quad \text{when} \quad r = s = 0 .$$

Now by 5.1 $u \geqq 4$ for any $r \geqq 0$ $\therefore$ $u + v > 2$. Thus the coefficient of z^8 in $2^{-r} H_{24}(z)$ is not divisible by 2^4 which contradicts to lemma 5.1.

This proves theorem 6.

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