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# Learning Stability in Economies with Heterogenous Agents*

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**Abstract**

An economy exhibits structural heterogeneity when the forecasts of different agents have different effects on the determination of aggregate variables. We study the important case of economies in which agents' behavior depends on forecasts of aggregate variables and show how different forms of heterogeneity in structure, forecasts, and adaptive learning rules affect the conditions for convergence of adaptive learning towards rational expectations equilibrium. Results are applied to a market model with speculative demand and a New Keynesian model of interest rate setting.

*Key words*: Adaptive learning, expectations formation, stability of equilibrium, market model, inflation, monetary policy
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# 1 Introduction

There has been a large amount of research into the implications of adaptive learning behavior in expectations formation for economic dynamics. Paralleling general macroeconomics, most of the research that uses adaptive learning has been carried out in models with representative agents, i.e. in economies with *structural homogeneity.* In studies of adaptive learning the assumption of a representative agent is usually interpreted to mean that expectations and learning rules are also identical. These kinds of assumptions are made mostly for analytical convenience rather than for their realism. In this paper we reconsider stability of rational expectations equilibrium (REE) under adaptive learning when the economy exhibits a particular type of *structural heterogeneity,* in which the basic characteristics differ across consumers (and firms) and they thus respond to expectations of economy-wide aggregate variables in different ways. (This terminology is introduced in Chapter 2 of (Evans and Honkapohja 2001).)

In this kind of setting it is natural to assume that expectations of different agents can also differ. We will make the further distinction that heterogeneity in learning can be *transient* (e.g. as a result of different initial beliefs) or *persistent* (when different agents use different learning algorithms).[1] Our goal is to consider the stability of REE when both structural and expectational heterogeneity is present. We will first show that transient heterogeneity in learning does not affect the conclusions drawn from the representative agent model, i.e. stability is entirely determined by the aggregate characteristics of the economy (in a sense defined below). The conclusion is markedly different when heterogeneity in learning is persistent. Details of agents' characteristics and learning rules influence the conditions for stability under learning, as was conjectured by (Grandmont 1998), Remark 2.3. We illustrate different possibilities that can arise using two economic examples, a market model with speculative demand and a New Keynesian model of monetary policy.

The basic framework will be a forward-looking multivariate linear model with two classes of agents. While the assumption of linearity is directly postulated for some models in the literature, it can be observed that most applied studies are in any case based on linearization.[2] The restriction to two classes of agents in the main analysis is done only for simplicity of exposition, and we will also state the stability conditions for economies with any finite number of different classes of agents.

Our analysis is focused on models where different agents need to forecast a common vector of aggregate variables, which often arises in the literature. In other words, we assume that information is symmetric between the agents. This is done for simplicity and brevity, though we conjecture that the approach can be generalized to models with informational asymmetries once the concept of equilibrium is suitably modified. We al-

[1] In independent work (Giannitsarou 2003b) considers similar forms of heterogeneity in learning under structural homogeneity of the economy.

[2] (Evans and Honkapohja 1995) and (Honkapohja and Mitra 2003a) show how learning stability in the linearized model implies stability in the original nonlinear model with sufficiently small shocks that are *iid* or a finite Markov chain, respectively.

low for some heterogeneity in the agents' learning algorithms, though we limit attention to econometric learning with infinite memory and examine the implications of agents using two well-known classes of econometric algorithms, versions of recursive least squares (RLS) or stochastic gradient (SG) updating rules. This is obviously restrictive, but it is a natural starting point since research has very often employed these schemes. The implications of many other forms of learning rules could be examined; these include bounded memory rules, algorithms for computational intelligence, just to mention a couple of other possibilities.[3] Our analysis does not cover these other possibilities, but the framework is still useful as a starting point as testified by the applications.

In the earlier literature, the bulk of work using econometric learning has assumed homogeneity in both expectations and structure, though there exist several studies that permit heterogenous expectations in a homogenous structure, see e.g. (Bray and Savin 1986), (Evans and Honkapohja 1997), (Evans, Honkapohja, and Marimon 2001) and (Giannitsarou 2003b). In a non-stochastic setting (Grandmont 1998), Remark 2.3 suggests the use of average expectations in models with heterogenous expectations and structure. Heterogenous expectations are also present in some of the other approaches to adaptive learning. Structural heterogeneity is permitted for a class of models in (Marcet and Sargent 1989a). Expectations are heterogenous in the Marcet and Sargent setup, but this arises solely from informational differences as different agents are assumed to use versions of recursive least squares (RLS) estimation.[4]

# 2 The Framework

## 2.1 The General Model

We consider a class of linear models where there are two types of agents (1 and 2) with different forecasts and with structural heterogeneity. $\zeta_i \geq 0$ denotes the mass of type $i$ agents. The model may be multi- or univariate. We will develop the algebra and basic results using the multivariate setting, but the matrices and vectors are sometimes interpreted as scalars. The multivariate model is needed in some applications, but stronger results can be obtained for the univariate model.

The formal model is given by

$$
\begin{aligned}
y_t &= \alpha + A_1 \hat{E}_t^1 y_{t+1} + A_2 \hat{E}_t^2 y_{t+1} + B w_t, && (1) \\
w_t &= F w_{t-1} + v_t. && (2)
\end{aligned}
$$

[3]Non-econometric approaches include the use of computational intelligence (see e.g. (Arifovic 1998)), models of discrete predictor choice (see e.g. (Brock and Hommes 1997) and (Brock and de Fontnouvelle 2000)) and eductive learning (see (Guesnerie 2002)). (Barucci 1999) and (Negroni 2003) consider heterogeneity in adaptive expectations.

[4]Marcet and Sargent employ a restrictive version of the stochastic approximation methodology by using the so-called projection facility, which has been criticized in (Grandmont and Laroque 1991), (Grandmont 1998) and (Moreno and Walker 1994). Ways to avoid a projection facility are discussed in (Evans and Honkapohja 1998a) and Chapter 6 of (Evans and Honkapohja 2001).

To shorten notation the mass $\zeta_i$ of agents of type $i$ are incorporated into the corresponding matrices $A_i$, but we will introduce them explicitly for some interpretations, in which case $A_i = \zeta_i \hat{A}_i$, where $\hat{A}_i$ describes how agents of type $i$ respond to their forecasts. Our main interest is in the *structurally heterogenous economy* for which $\hat{A}_1 \neq \hat{A}_2$. When $\hat{A}_i = A$ and $\zeta_1 + \zeta_2 = 1$ we have the case of a *structurally homogenous economy*, which has been studied by (Giannitsarou 2003b).

In the model $y_t$ is $n \times 1$ vector of endogenous variables and $w_t$ is $k$ dimensional vector of exogenous variables that is assumed to follow a stationary VAR, so that $v_t$ is white noise. For simplicity, it is assumed that $F$ is known to the agents (if not, it could be estimated) and that $M_w = \lim_{t\to\infty} E w_t w_t'$ is a positive definite matrix. As for the matrices, $A_i$, $i = 1, 2$, are $n \times n$ while $B$ is $n \times k$. The univariate case is $n = k = 1$.

We let $\hat{E}_t^i y_{t+1}, i = 1, 2$, denote the (in general non-rational) expectations by agent $i$ of the endogenous variables in the economy. Expectations without "ˆ" refer to rational expectations (RE). In our analysis we will keep track of individual expectations as they will be stacked into vectors. The stacking is useful since the general framework is both multivariate and stochastic, and agents can have different types of algorithms for parameter updating. For some results it is worth while to define and use the concept of average expectations as a benchmark, as suggested in (Grandmont 1998), Remark 2.3. (1) can clearly be written as

$$y_t = \alpha + A^M \hat{E}_t^{AV} y_{t+1} + B w_t, \tag{3}$$

where

$$\hat{E}_t^{AV} y_{t+1} = (A^M)^{-1}(A_1 \hat{E}_t^1 y_{t+1} + A_2 \hat{E}_t^2 y_{t+1})$$

can be called *average expectations* and

$$A^M = A_1 + A_2$$

*aggregate characteristics* or the *average economy.*

A key feature of model (1) is that both agents' characteristics and forecasts differ. If either agents or forecasts are identical, so that $A_i = \zeta_i A$ or $\hat{E}_t^1 y_{t+1} = \hat{E}_t^2 y_{t+1}$, the model can be aggregated. In the former case the evolution of $y_t$ depends only on average expectations $\zeta_1 \hat{E}_t^1 y_{t+1} + \zeta_2 \hat{E}_t^2 y_{t+1}$. In the latter case only the mean or aggregate characteristics $A^M$ matter.

We will focus attention on the learnability of the fundamental or minimal state variable (MSV) solution to the class of models (1)-(2).[5] The MSV REE takes the form

$$y_t = a + b w_t, \tag{4}$$

[5] As is well known, under certain conditions, known as indeterminacy of REE, there also exist other well behaved REE and these could also be studied for learnability. See e.g. (Evans and Honkapohja 2001), Part III for a discussion of the homogenous expectations case. The techniques developed in our paper can be extended to the study of learnability of the other types of REE under structural and/or expectational heterogeneity.

where the $n$ vector $a$ and $n \times k$ matrix $b$ are to be computed in terms of the structural parameters of the model. The MSV solution can be obtained by solving the following system of linear equations

$$\begin{aligned} a &= \alpha + A^M a \\ b &= A^M bF + B. \end{aligned}$$

This system has a unique solution under certain conditions:

**Proposition 1** *There exists a unique, symmetric equilibrium $(\bar{a}, \bar{b})$ of the model (1)-(2) if the matrices $I_n - A^M$ and $I_{nk} - F' \otimes A^M$ are invertible.*

The proofs of all Propositions are given in Appendix A. Here and in the rest of the paper $I_m$ denotes the $m-$dimensional identity matrix.

It should be noted that the framework is restrictive in that the model (1)-(2) is purely forward-looking. This is done for brevity. The same general approach can be used for models with lags (some of our results do not, however, generalize). Another extension is to have $S > 2$ classes of agents and the model becomes

$$y_t = \alpha + \sum_{s=1}^{S} A_s \hat{E}_t^s y_{t+1} + \sum_{s=1}^{S} C_s \hat{E}_t^s y_t + Bw_t, \tag{5}$$

with $w_t$ following (2). (5) also incorporates expectations of current endogenous variables that will appear in one of the applications. For most part we will assume $S = 2$ and $C_s = 0$ for all $s$, but we will summarize the convergence conditions for (5) in Section 5.3.

## 2.2 Economic Examples

We outline two economic models that fit our general setup.

**Example 1** (Speculative Demand with Externality) The supply function for a single good is assumed to be linear and upward sloping, that is

$$s_t = l + kp_t + \varepsilon_t.$$

Here $k, l$ are positive parameters and $\varepsilon_t$ is a shock that follows the AR(1) process

$$\varepsilon_t = r\varepsilon_{t-1} + \tilde{\varepsilon}_t,$$

where $\tilde{\varepsilon}_t$ is white noise with variance $\sigma_\varepsilon^2$ and $|r| < 1$.

There are $S$ classes of demanders with different linear demand functions that depend on expected change in the market price due to a speculative motive. The possibility of externality is also assumed, so that the demand of person $i$ depends on the aggregate demand in the market. Formally,

$$d_t^i = \alpha_i - \beta_i p_t + \kappa_i(\hat{E}_t^i p_{t+1} - p_t) + \eta_i(\sum_{s=1}^{S} d_t^s),\ i = 1, \dots, S. \tag{6}$$

where $\alpha_i, \beta_i, \kappa_i$ are positive parameters and $\hat{E}_t^i(p_{t+1}) - p_t$ denotes (possibly non-rational) expectation of the perceived change in market price of demander $i$. From (6)

$$\begin{aligned}
\sum_{i=1}^{S} d_t^i &= (1-\eta)^{-1}\left[\alpha - (\beta+\kappa)p_t + \sum_{j=1}^{S} \kappa_j \hat{E}_t^j p_{t+1}\right], \\
\eta &= \sum_{i=1}^{S} \eta_i, \alpha = \sum_{i=1}^{S} \alpha_i, \beta = \sum_{i=1}^{S} \beta_i, \kappa = \sum_{i=1}^{S} \kappa_i.
\end{aligned}$$

From market clearing $s_t = \sum_{i=1}^{S} d_t^i$ we obtain the reduced form

$$\begin{aligned}
p_t &= \psi[(1-\eta)^{-1}\alpha - l] + \psi(1-\eta)^{-1}\sum_{j=1}^{S} \kappa_j \hat{E}_t^j p_{t+1} - \psi\varepsilon_t, && (7) \\
\psi &= [(k + (1-\eta)^{-1}(\beta+\kappa)]^{-1}.
\end{aligned}$$

We make the regularity assumptions $\rho\psi(1-\eta)^{-1}\kappa_i \neq 1$ and $\psi(1-\eta)^{-1}\kappa_i \neq 1$ for all $i$. Model (7) is of the form (5) with $C_s = 0$ for all $s$.

**Example 2.** (Model of Monetary Policy) Recent studies of monetary policy are often based on a model with representative consumer, monopolistic competition in product market and stickiness in price setting. We consider the bivariate linearized model suggested e.g. in (Clarida, Gali, and Gertler 1999):

$$\begin{aligned}
x_t &= -\phi(i_t - \hat{E}_t^P \pi_{t+1}) + \hat{E}_t^P x_{t+1} + g_t, && (8) \\
\pi_t &= \lambda x_t + \beta \hat{E}_t^P \pi_{t+1} + u_t, && (9)
\end{aligned}$$

where $x_t$ is the "output gap" i.e. the difference between actual and potential output, $\pi_t$ is the inflation rate and $i_t$ is the nominal interest rate. $\hat{E}_t^P \pi_{t+1}$ and $\hat{E}_t^P x_{t+1}$ denote *private sector* expectations of inflation and output gap next period. All the parameters in (8) and (9) are positive. $0 < \beta < 1$ is the discount rate of the representative firm.

$u_t$ and $g_t$ denote observable shocks that follow first order autoregressive processes:

$$\begin{pmatrix} u_t \\ g_t \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} u_{t-1} \\ g_{t-1} \end{pmatrix} + \begin{pmatrix} \hat{u}_t \\ \hat{g}_t \end{pmatrix}, \tag{10}$$

where $0 < |\mu| < 1, 0 < |\rho| < 1$ and $\hat{g}_t \sim iid(0, \sigma_g^2), \hat{u}_t \sim iid(0, \sigma_u^2)$. $g_t$ represents shocks to government purchases as well as shocks to potential GDP. $u_t$ represents any cost push shocks to marginal costs other than those entering through $x_t$.

The model is complete once an interest rate rule for the central bank is specified. In the literature both optimal reaction functions and instrument, i.e. non-optimal, rules have been considered. (Evans and Honkapohja 2003a) review the recent results on determinacy and stability of REE under learning for both types of rules. An interesting hybrid case is the approximate targeting rule proposed by (McCallum and Nelson 2000), where the policy-maker adjusts the current interest rate in response to the discrepancy from the commitment optimality condition anticipated for the next period.

Let $\alpha$ denote the relative weight of output in a commonly used quadratic objective function of the central bank. Formally, the McCallum-Nelson rule is

$$i_t = \hat{E}_t^{CB}\pi_{t+1} + \theta[\hat{E}_t^{CB}\pi_{t+1} + (\alpha/\lambda)(\hat{E}_t^{CB}x_{t+1} - \hat{E}_t^{CB}x_t)]. \tag{11}$$

This rule is forward-looking, i.e. it depends on forecasts of inflation and outputs. If private expectations can be observed only with large errors, then the central bank might try to substitute its own forecasts in place of private expectations, which is indicated by notation $\hat{E}_t^{CB}(.)$. Rule (11) yields stability of REE under learning only for sufficiently small values of the adjustment parameter $\theta$ when the central bank can use private expectations; see (Evans and Honkapohja 2003c). Below we will consider the implications of using internal central bank forecasts in the rule (11).

# 3 Econometric Learning

We now formulate econometric learning by agents in real time when agents use a standard econometric procedure for estimating and updating the parameters of the PLM. In the literature it is often assumed that agents use a version of recursive least squares (RLS). Another possible procedure is to assume that some agents use stochastic gradient (SG) estimation. We now introduce the structure of learning using an abstract formulation. After this we indicate how RLS and SG algorithms fit this formulation.

## 3.1 The Mapping from Perceptions to Outcomes

A mapping from the perceptions of the economic agents to the resulting temporary equilibrium of the economy has turned out to be a key relationship in the study of convergence of adaptive learning dynamics. The form of this mapping in the structurally heterogenous economy with heterogenous expectations is developed as follows.

The two types of agents are assume to have their own forecast functions, which take the same parametric form. During the learning dynamics the agents have different beliefs about the parameters they are estimating, and these beliefs are adjusted over time. For given values of the parameters of the forecast function of each agent $i$, called the perceived law of motion (PLM) of agent $i$, one computes the actual law of motion (ALM) implied by the structure of the economy.

Define the vector of state variables $z_t = (1, w_t)'$ and the matrix of parameters $\varphi'_{i,t} = (a_{i,t}, b_{i,t})$ with $a_{i,t}$ being an $n$ dimensional vector and $b_{i,t}$ being an $n \times k$ matrix. The time subscript $t$ indicates the parameter estimates of agent $i$ at time $t$. Formally, we assume that the two agents have PLMs

$$y_t = a_{i,t} + b_{i,t} w_t = \varphi'_{i,t} z_t,\ i = 1, 2 \tag{12}$$

with corresponding forecast functions $\hat{E}^i_t y_{t+1} = a_{i,t} + b_{i,t} F w_t$, $i = 1, 2$. Note that the PLMs have the same form as the MSV solution (4), but in general $a_{i,t}, b_{i,t}$ are not at their RE values. Inserting the forecasts into the model (1), one obtains the ALM

$$\begin{aligned} y_t &= [\alpha + A_1 a_{1,t} + A_2 a_{2,t}, (A_1 b_{1,t} + A_2 b_{2,t})F + B] \begin{bmatrix} 1 \\ w_t \end{bmatrix} \\ &= T(\varphi'_{1,t}, \varphi'_{2,t}) z_t. \end{aligned} \tag{13}$$

We can usefully interpret the $T-$mapping using average expectations and the average (or aggregate) economy. Defining the average PLM

$$a^M_t + b^M_t w_t = (A^M)^{-1} \sum_{i=1}^{2} A_i (a_{i,t} + b_{i,t}) w_t,$$

we have

$$a_{i,t} \to \alpha + A^M a^M_t,\ b_{i,t} \to A^M b^M_t F + B,\ i = 1, 2,$$

which is the mapping from the PLM into the ALM in the average economy formulated in (3). In other words, each parameter in the different PLMs is mapped into its ALM value corresponding to the average PLM in the average economy.

## 3.2 The General Learning Algorithm

The second step is to describe how agents update the parameters $a_{i,t}$ and $b_{i,t}$ of the PLMs. We will use a general formulation of the learning algorithms, of which RLS and SG learning are special cases. The learning algorithm may involve further parameters in addition to $a_{i,t}$ and $b_{i,t}$ of the PLM, so we define $\theta_{i,t} = (vec(\varphi'_{i,t}), \Phi'_{i,t})$, where $\Phi_{i,t}$ is a column vector of the possible additional parameters. Agent $i$ updates $\theta_{i,t}$ according to

$$\theta_{i,t} = \theta_{i,t-1} + \gamma_{i,t} N_i(\theta_{i,t-1}, X_t), i = 1, 2, \tag{14}$$

where $X_t$ is a vector of relevant state variables in parameter updating.

Different initial beliefs can be accommodated by different initial conditions for the dynamics. Heterogeneity in learning rules can be introduced through differences in the updating functions $N_i(.)$ and below we will specify $N_i(.)$ either as RLS or SG type algorithm. Another type of heterogeneity arises when the agents have different degrees of responsiveness to the updating function as indicated by the gain sequences $\gamma_{i,t} > 0$. We allow for $\gamma_{1,t} \neq \gamma_{2,t}$ for the gain parameters of the learning rules and make the following assumption:

**Assumption A:** The gain sequences satisfy $\gamma_{i,t} = \hat{\gamma}_{i,t}\vartheta_{i,t}\xi_{i,t}$, where $\hat{\gamma}_{i,t}$ are deterministic and positive. $\vartheta_{i,t}$ is random and is assumed to be positive, bounded and *iid* over time. $\xi_{i,t}$ is an *iid* over time Bernoulli random variable equal to 0 with probability $\rho_i \in [0, 1)$ and equal to 1 with probability $1 - \rho_i$. $\xi_{i,t}$ is independent of $\vartheta_{i,t}$. In addition, $\lim_{t\to\infty} E(\hat{\gamma}_{i,t}\vartheta_{i,t}\xi_{i,t}/\gamma_t) = \delta_i > 0$, where the deterministic, decreasing and positive sequence $\gamma_t$ satisfies:
(i) $\hat{\gamma}_{i,t} \leq K_i\gamma_t$ for some constant $K_i > 0$,
(ii) $\sum \gamma_t = \infty$ and $\sum \gamma_t^p < \infty$ for some $p \geq 2$, and
(iii) $\limsup(1/\gamma_{t+1} - 1/\gamma_t) < \infty$.

This condition allows for significant amounts of heterogeneity in the adaption speeds of the different agents, including both inertia and random variation across agents as specified by $\xi_{i,t}$ and $\vartheta_{i,t}$, respectively. $\hat{\gamma}_{i,t}$ specifies how the mean of $\gamma_{i,t}$ moves over time.

Heterogeneity in the formation of expectations is observed in experimental data, see for instance (Marimon and Sunder 1993) and (Evans, Honkapohja, and Marimon 2001). A similar formulation of heterogeneity was suggested in (Evans, Honkapohja, and Marimon 2001). Assumption A can allow various weighting schemes for data in later periods relative to early ones, see e.g. (Ljung and Söderström 1983) and (Marcet and Sargent 1989b) for further discussion. We remark that the conditions (i)–(iii) on $\gamma_t$ are commonly assumed in the literature.[6]

The formulation of parameter updating (14) by each agent is formally similar to general adaptive algorithms that have been employed in the literature; see e.g. (Evans and Honkapohja 2001). As we will see below, RLS and SG learning can be cast as special cases of (14). An important feature of (14) is that it incorporates infinite memory, i.e. more and more data is being used in parameter updating over time. Bounded memory learning rules cannot be cast in this formulation; see (Grandmont 1998), (Honkapohja and Mitra 2003b) and references cited therein for learning with bounded memory.

The analysis of learning proceeds by defining a stacked algorithm, which is a standard recursive stochastic algorithm, and uses standard techniques for such systems; see e.g. Chapters 6 and 7 of (Evans and Honkapohja 2001) for an exposition. Let $\theta_t' = (\theta_{1,t}', \theta_{2,t}')$ and write

$$\theta_t = \theta_{t-1} + \gamma_t H(\theta_{t-1}, X_t) + \gamma_t^2 \rho_t(\theta_{t-1}, X_t), \tag{15}$$

where

$$H(.) = \begin{pmatrix} N_1(.) \\ N_2(.) \end{pmatrix}, \; \rho_t(.) = \begin{pmatrix} \frac{\hat{\gamma}_{1,t}-\gamma_t}{\gamma_t^2} N_1(.) \\ \frac{\hat{\gamma}_{2,t}-\gamma_t}{\gamma_t^2} N_2(.) \end{pmatrix}.$$

In our setting $X_t' = (1, w_t', w_{t-1}', \vartheta_{1,t}\xi_{1,t}, \vartheta_{2,t}\xi_{2,t})$. The state variable $X_t$ dynamics are linear and can be written as

$$X_t = AX_{t-1} + BW_t,$$

[6] We note that one can assume $K_i \leq 1$ without loss of generality. If $\gamma_t$ satisfies Assumption A for $K_i > 1$, then one can construct another sequence $\tilde{\gamma}_t$ satisfying assumption A with $K_i \leq 1, \forall i$.

where

$$A=\begin{pmatrix}0&0&0&0&0\\0&F&0&0&0\\0&0&F&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},B=\begin{pmatrix}1&0&0&0&0\\0&I&0&0&0\\0&0&I&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},W_t=\begin{pmatrix}1\\v_t\\v_{t-1}\\\vartheta_{1,t}\xi_{1,t}\\\vartheta_{2,t}\xi_{2,t}\end{pmatrix}.$$

It is well known that conditions for convergence of $\theta_t$ to an equilibrium $\bar{\theta}$ are determined by defining an associated ordinary differential equation (ODE)

$$\frac{d\theta}{d\tau}=h(\theta),\text{ where }h(\theta)=\lim_{t\to\infty}EH(\theta,X_t). \tag{16}$$

Learning converges to $\bar{\theta}$ if $\bar{\theta}$ is a locally stable fixed point of the associated ODE. See (Evans and Honkapohja 2001) for formal details, including the appropriate notions of stochastic convergence. Next we introduce the RLS or SG algorithms. The explicit conditions for convergence in the different cases will be given in Sections 4 and 5.

### 3.3 The RLS Algorithm

If agent $i$ uses a (generalized) RLS learning algorithm the explicit form of (14) can be derived as follows. RLS algorithm has the form

$$\varphi_{i,t} = \varphi_{i,t-1}+\gamma_{i,t}R_{i,t}^{-1}z_{t-1}(y_{t-1}-\varphi'_{i,t-1}z_{t-1})', \tag{17}$$
$$R_{i,t} = R_{i,t-1}+\gamma_{i,t}(z_{t-1}z'_{t-1}-R_{i,t-1}), \tag{18}$$

for $i=1,2$.[7] In the system (17)-(18) $R_{i,t}$ is the matrix of second moments of the state vector, which are needed to write down the estimation of the PLM parameters $\varphi_{i,t}$, when a version of least squares is employed. Since contemporaneous $R_{i,t}$ appears in the right-hand side of (17), it is necessary to making a timing change $S_{i,t-1}=R_{i,t}$, after which the RLS algorithm is clearly a particular case of (14) with $\theta'_{i,t}=(vec(\varphi'_{i,t}),vec(S_{i,t})')$.

In the Appendix it is shown that the associated ODE (16) of the RLS algorithm (17)-(18) is

$$d\varphi_i/d\tau = \delta_i S_i^{-1}M_z(T(\varphi'_1,\varphi'_2)'-\varphi_i), \tag{19}$$
$$dS_i/d\tau = \delta_i(M_z-S_i), i=1,2 \tag{20}$$

where

$$lim_{t\to\infty}Ez_{t-1}z'_{t-1}=M_z=\begin{pmatrix}1&0\\0&M_w\end{pmatrix}. \tag{21}$$

From (20) it is evident that we have $S_i\to M_z$ and so asymptotically (19) becomes

$$\frac{d\varphi_i}{d\tau}=\delta_i(T(\varphi'_1,\varphi'_2)'-\varphi_i),\ i=1,2. \tag{22}$$

Stability conditions for (22) provide the conditions for convergence of RLS learning.

[7] In this formulation the parameter estimates are assumed to depend on data up to $t-1$, but current observation on exogenous variables are allowed to be used in the forecasts. (This is typically done in the learning literature.)

### 3.4 The SG Algorithm

An alternative statistical learning algorithm, known as stochastic gradient (SG) algorithm, has occasionally been employed in the literature in place of RLS. (SG algorithm is also called the least mean squares algorithm in the technical literature.) The SG algorithm is computationally much simpler than the RLS algorithm; however, the latter is more efficient from an econometric viewpoint since it uses information on the second moments of the variables. For parameter estimation of fixed exogenous stochastic processes, both the RLS and SG algorithms yield consistent estimates of parameters but the RLS, in addition, possesses some optimality properties. For instance, if the underlying shock process is *iid* normal, then the RLS estimator is minimum variance unbiased.[8]

Formally, agents of type $i$ update the PLM parameters using a (generalized) stochastic gradient (SG) algorithm

$$\varphi_{i,t} = \varphi_{i,t-1} + \gamma_{i,t} z_{t-1}(y_{t-1} - \varphi'_{i,t-1} z_{t-1})', \tag{23}$$

where the gain parameters $\gamma_{i,t}$ satisfy Assumption A and Condition 1 above. The SG algorithm is a particular case of (14) with $\theta'_{i,t} = vec(\varphi'_{i,t})$, i.e. there are no additional parameters $\Phi_{i,t}$ in the SG case.

It is shown in the Appendix that the associated ODE of (23) takes the form

$$d\varphi_i/d\tau = \delta_i M_z (T(\varphi'_1, \varphi'_2)' - \varphi_i) \tag{24}$$

or

$$\dot{\varphi}'_i = \delta_i[\alpha + A_1 a_1 + A_2 a_2 - a_i, \{(A_1 b_1 + A_2 b_2)F + B - b_i\} M_w] \tag{25}$$

for $i = 1, 2$ and where $M_z$ is defined in (21). Stability conditions for (24) or (25) provide the convergence conditions for SG learning.

# 4 Learning Under Transient Heterogeneity

In this section both types of agents are assumed to use the same general type of learning rule, either the RLS or SG rule, for updating of PLM parameters. Learning can start with different initial beliefs about the parameters but heterogeneity in the learning rules is only transient in the sense that, asymptotically, the gain sequences converge at the same rate:

**Condition 1:** (Asymptotically identical adaption speeds) $\delta_1 = \delta_2$.

The main result for this section can be stated as:

[8] See Section 3.5 of (Evans and Honkapohja 2001) and (Evans, Honkapohja, and Williams 2003) for discussion and references on the SG algorithm.

**Result I:** *Neither structural heterogeneity nor transient RLS (or SG) learning heterogeneity affects the conditions for learning stability obtained from the model with structural differences aggregated to $A^M$ and a single RLS (or SG) learning rule.*

We remark that, with transient (or permanent) heterogeneity, the economy may be stable even if individual behavior in some part of the economy is conducive to instability and the aggregate economy may be unstable under learning even if individual characteristics satisfy the stability conditions.[9]

## 4.1 The RLS Case

To facilitate statement of results we introduce some terminology for the different cases of the heterogenous learning that we study. We say that *transiently heterogenous RLS learning* occurs when initial conditions of the different types of agents are different and all agents use RLS learning rules (17)-(18) that satisfy Condition 1. In this case we have the following result:

**Proposition 2** *Assume that Assumption A and Condition 1 hold in the structurally heterogenous economy (1)-(2). Transiently heterogenous RLS learning converges (almost surely) to the MSV REE from any initial conditions if and only if the matrices $A^M - I_n$ and $F' \otimes A^M - I_{nk}$ have eigenvalues with negative real parts, i.e. there is convergence in the average economy.*[10]

The important implication of this result is that when learning is only transiently heterogenous in the sense defined above, stability of equilibrium depends only on the aggregate characteristics of the economy, i.e. matrix $A^M$.

We note that the initial conditions $\theta_{i,0}, i = 1, 2$ can take any value, except that for the moment matrices initial conditions should naturally be non-negative semidefinite matrices with positive diagonal elements.[11] Naturally, the cases of heterogenous initial beliefs with identical learning algorithms and of transiently heterogenous RLS learning in structurally homogenous economy are covered by Proposition 2. We give the proof of Proposition 2 in the Appendix.

Under transient heterogeneity stability of (22) is determined by the so-called *E-stability conditions* for the MSV REE. We also note that under some (mild) regularity conditions, the RLS algorithm will converge to an E-unstable symmetric (MSV) solution with probability zero; see (Evans and Honkapohja 2001) for a discussion of E-stability and instability.

[9] In fact, the latter possibility can arise even in a structurally homogenous economy, as shown by (Giannitsarou 2003b).

[10] Throughout the paper we ignore the non-generic cases where one or more relevant eigenvalues has a zero real part.

[11] While this theorem and many subsequent results are formally concerned with global convergence, it should be borne in mind that in specific applications the model may be a linearization around a steady state, and the study of learning is necessarily local in such settings.

The relationship between stability or instability in the associated ODE and the convergence or non-convergence of the algorithm also applies to other settings below and we will in part conduct our discussion using the ODEs. Below we will only state stability and convergence results, but it should be kept in mind that corresponding instability/non-convergence results also exist.

## 4.2 The SG Case

*Transiently heterogenous SG learning* arises when all agents use SG algorithms but have different initial beliefs and possibly transiently different gains (i.e. the gains satisfy Condition 1). We have the result:

**Proposition 3** *Assume that Assumption A holds and consider the economy (1)-(2). If the REE is a locally asymptotically stable fixed point of the average economy $A^M$ under homogenous SG learning, then transiently heterogenous SG learning converges globally (almost surely) to the MSV REE.*

We remark that Proposition 3 resolves the open issue raised by (Giannitsarou 2003b) in her Proposition 1(ii) and discussion after it, whether convergence of SG learning with different initial perceptions (in a structurally homogenous economy) is implied by stability of identical SG learning.

In general, convergence of SG learning is not dictated by E-stability conditions.[12] We emphasize that this phenomenon is not due to heterogeneity in expectations or economic structure. It is instead associated with SG learning itself, see (Evans, Honkapohja, and Williams 2003). In special cases E-stability suffices for convergence of SG learning. One important case is:[13]

**Corollary 4** *Assume that Assumption A and Condition 1 hold. If the exogenous variable is scalar ($k = 1$), then transiently heterogenous SG learning converges if the REE is E-stable.*

In the general case the measurements and specification of the exogenous variables influences the conditions for convergence of SG learning, as discussed in (Evans, Honkapohja, and Williams 2003). In contrast, RLS learning is not subject to this "scaling problem". Considering model (2), we introduce the Cholesky decomposition $M_w = PP'$, which exists for any positive definite matrix. $P$ is triangular and nonsingular. Next, we transform exogenous variables to

$$\tilde{w}_t = P^{-1} w_t.$$

[12]This possibility was first noted by (Barucci and Landi 1997). (Giannitsarou 2003a) provides an economic example.

[13]Another case is the static model, in which expectations of only current endogenous variables appear; see (Evans and Honkapohja 1998b). The Muth model is a classic example of the static model.

The PLMs become $y_t = a_i + \tilde{b}_i \tilde{w}_t$, where $\tilde{b}_i = b_i P$ and $\tilde{w}_t = \tilde{F}\tilde{w}_{t-1} + \tilde{e}_t$, where $\tilde{F} = P^{-1}FP$ and $\tilde{e}_t = P^{-1}e_t$. Noting that $\lim E\tilde{w}_t\tilde{w}_t' = M_{\tilde{w}} = I_k$, the associated ODE (24) becomes just the E-stability equation (apart from the unimportant scalar $\delta$) and thus E-stability is sufficient for convergence of SG learning:

**Remark 5** *Assume that Assumption A and Condition 1 hold. If exogenous variables $w_t$ are rescaled to $\tilde{w}_t = P^{-1}w_t$, where $M_w = PP'$, then transiently heterogenous SG learning converges to the REE if and only if the REE is E-stable.*

# 5 Persistent Heterogeneity in Learning

We now consider settings in which the agents are using different algorithms in their updating schemes. One type of heterogeneity in learning arises when the different agents are using different types of algorithms and we will specifically assume that agents use either RLS or SG updating rules. Another milder type of heterogeneity arises when the different agents are using the same type of algorithm but with asymptotically different adaption speeds (i.e. Condition 1 does not hold).

The main conclusion of this section can summarized as:

**Result II:** *Convergence of persistently heterogenous learning is no longer determined by aggregate characteristics alone. The stability conditions are in general affected by individual adaption speeds and the individual characteristics of the economy.*

We no longer have the previous conclusion (Result I) that only the aggregate characteristics matter for convergence of heterogenous learning. When heterogeneity in learning is persistent, the stability conditions are affected by $\delta_1$ and $\delta_2$ and the structure of the economy, that is the matrices $A_1$, $A_2$, $F$ and $M_w$.

## 5.1 Mixed RLS/SG Learning

Persistently heterogenous learning automatically arises when the different agents use different types of learning algorithms. The broad aim is to consider settings where one class of agents is using a learning algorithm that is either more or less sophisticated than the algorithm used by the other class of agents. Specifically, we assume that there are two possible types of learning algorithms, the RLS and the stochastic gradient (SG) algorithms that the agents might use.

We say that *mixed RLS/SG learning* takes place when initial conditions of the different types of agents are different, type 1 agents use RLS and type 2 agents use SG learning rules. For agent 1 the algorithm is given by (17)-(18), while for agent 2 it is given by (23). The gains $\gamma_{it}$ are assumed to satisfy Assumption A. However, Condition 1 is not imposed, so that mean gains of the agents can differ asymptotically.

Stability is determined by the ODE

$$\begin{aligned} d\varphi_1/d\tau &= \delta_1(T(\varphi_1', \varphi_2')' - \varphi_1), \\ d\varphi_2/d\tau &= \delta_2 M_z(T(\varphi_1', \varphi_2')' - \varphi_2), \end{aligned}$$

since (22) applies to agent 1 and (24) to agent 2. The system for $\dot{a}_1$ and $\dot{a}_2$ is

$$\begin{aligned} \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} &= D_1 \Omega \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \text{ where} & (26) \\ D_1 &\equiv \begin{pmatrix} \delta_1 I_n & 0 \\ 0 & \delta_2 I_n \end{pmatrix}, \Omega = \begin{pmatrix} A_1 - I_n & A_2 \\ A_1 & A_2 - I_n \end{pmatrix} & (27) \end{aligned}$$

and where the inessential constant term has been dropped. The vectorized system for $\dot{b}_1$ and $\dot{b}_2$ becomes (ignoring constant terms)

$$\begin{aligned} \begin{pmatrix} vec\dot{b}_1 \\ vec\dot{b}_2 \end{pmatrix} &= D_w \Omega_F \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix}, \text{ where } D_w = \begin{pmatrix} \delta_1 I_{nk} & 0 \\ 0 & \delta_2 (M_w \otimes I_n) \end{pmatrix} & (28) \\ \Omega_F &= \begin{pmatrix} F' \otimes A_1 - I_{nk} & F' \otimes A_2 \\ F' \otimes A_1 & F' \otimes A_2 - I_{nk} \end{pmatrix}. & (29) \end{aligned}$$

We can then prove the following result:

**Proposition 6** *In the economy (1)-(2) mixed RLS/SG learning converges globally (almost surely) to the MSV solution if the matrices $D_1\Omega$ and $D_w\Omega_F$ have eigenvalues with negative real parts.*

The Proposition shows that in general stability under learning depends in a subtle way on the interaction of structural heterogeneity of the economy and persistent heterogeneity in learning. Further conditions on structural heterogeneity that achieve convergence of learning are available, but they are restrictive.

We introduce the notions of $D-$ and $S-$stability. A matrix $K$ is said to be *D-stable* if the matrix $DK$ has all eigenvalues with negative real parts for any positive diagonal matrix $D$. A matrix $K$ is said to be *$S-$stable* if $SK$ is stable for any positive definite matrix $S$. See e.g. (Arrow and McManus 1958) and (Horn and Johnson 1991) for these definitions. Since $M_w$ is a positive definite matrix, we have:

**Corollary 7** *Consider the economy (1)-(2) with mixed RLS/SG learning. If $\Omega$ is $D-$stable and $\Omega_F$ is $S-$stable, then the learning dynamics converges globally to the MSV REE, for all $\delta_i$, $i = 1, 2$.*

We remark that if $F$ is diagonal $D-$stability of $\Omega_F$ is clearly sufficient.

An important case not covered by Corollary 7 is that the economy may be stable under learning even if the characteristics of one class of agents contribute to instability (which violates $D-$ stability). On the other hand, if the weight of the agents contributing to instability is sufficiently large, then instability of the economy can arise for some values of $\delta_i$. We will later see an economic example of this phenomenon.

A natural question concerns learning stability in a structurally homogenous economy when there is persistent heterogeneity in learning. The answer is affirmative for scalar economies, as discussed in the next section (see Corollary 9). We conjecture that the result also holds for the multivariate economy, but we have not been successful in proving it or finding a counter example.

### 5.1.1 The Scalar Case

For the scalar model $n = k = 1$ further results are obtainable. One result is:

**Corollary 8** *Consider the scalar model $n = k = 1$ and assume that (i) the aggregate economy is E-stable, i.e. $A^M < 1$ and $FA^M < 1$, and (ii) that the economies with only type $i = 1, 2$ agents are also E-stable, i.e. $A_i < 1$ and $FA_i < 1$ for $i = 1, 2$. Then mixed RLS/SG learning converges globally to the MSV REE, for all $\delta_i$, $i = 1, 2$.*

Note that Corollary 8 requires more than E-stability of the aggregate economy since the characteristics of each agent type must fulfill the E-stability requirements. Corollary 8 does not hold if there are more than two types of agents or for multivariate models (counter examples are available on request).

Though Corollary 8 does not generalize to economies with more than two types of agents, a further result can be obtained for scalar economies in which all parameters $A_i$ have the same sign.

**Corollary 9** *Consider the $S$ agent scalar (i.e. $n = k = 1$) economy (5) and assume that (i) the parameters $A_i$ have the same sign, (ii) $\sum_{s=1}^{S} A_s < 1$, $\sum_{s=1}^{S} FA_s < 1$ and (iii) $C_s = 0, \forall s$.[14] If the different agents use either RLS or SG learning rules, the economy converges to the MSV REE.*

A key difference between Corollaries 8 and 9 is that the former does not impose the restrictions on signs of individual responses. The sign restriction (i) means that all agents in the economy respond to forecasts qualitatively in the same manner. (ii) is automatically satisfied if $A_s < 0$ for all $s$, but in the case $A_s > 0$ (ii) is a restriction on the aggregate response (note that (ii) is formally one of the E-stability conditions for the average economy). Example 1 is an economic model that illustrates the role of these features.[15]

We also remark that Corollary 9 covers the case of a structurally homogenous scalar economy, since then $A_i = \zeta_i A$. This explains several examples of (Giannitsarou 2003b), where E-stability is sufficient for heterogenous learning.

## 5.2 Persistently Heterogenous RLS Learning

Here we briefly consider another case of persistent heterogeneity in learning, which arises when all agents use the same type of learning algorithm but the gain sequences differ even asymptotically in the sense that $\delta_1 \neq \delta_2$. For concreteness it is assumed that all agents are using RLS type algorithms. We say that *persistently heterogenous RLS learning* occurs when initial conditions of the different types of agents are different, the agents use RLS learning rules and $\delta_1 \neq \delta_2$.

[14] Assumption (iii) is made for simplicity.

[15] The role of a similar sign condition also emerges in the analysis of heterogenous adaptive expectations by (Negroni 2003).

In this case the dynamics continues to be given by the system (17)-(18). Stability is governed by the ODE (22) for $i = 1, 2$. The explicit form of (22) (without the constant terms) for $a_1$ and $a_2$ continues to be (26), and for $b_1$ and $b_2$ we get

$$\begin{pmatrix} vec\dot{b}_1 \\ vec\dot{b}_2 \end{pmatrix} = D_2\Omega_F \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix},\ D_2 \equiv \begin{pmatrix} \delta_1 I_{nk} & 0 \\ 0 & \delta_2 I_{nk} \end{pmatrix}, \tag{30}$$

where $\Omega_F$ is defined as in (29). The next remark provides the analogues of Proposition 6 as well as of Corollaries 7, 8 and 9:

**Remark 10** *(i) If the matrices $D_1\Omega$ and $D_2\Omega_F$ have eigenvalues with negative real parts then persistently heterogenous RLS learning converges globally (almost surely) to the MSV REE.*
*(ii) A sufficient condition for convergence for all values of $\delta_i$, $i = 1, 2$ is that the matrices $\Omega$ and $\Omega_F$ are D-stable.*
*(iii) Assume that the conditions stated in Corollary 8 hold. Then persistently heterogenous RLS learning converges globally to the MSV equilibrium, for all $\delta_i$, $i = 1, 2$.*
*(iv) Assume that the conditions stated in Corollary 9 hold. Then persistently heterogenous RLS learning converges globally to the MSV equilibrium, for all $\delta_i$, $i = 1, ..., S$.*

We note that if $\delta_1 = \delta_2$ then the stability conditions obtained from (22) would be identical to the E-stability conditions, which proves Proposition 2.

## 5.3 Extensions to $S > 2$ Classes of Agents

Here we note some extensions of the results to economies with more than two classes of agents and to global convergence of learning. Consider the model (5) with $S$ classes of agents and contemporaneous expectations. The PLMs of the agents are as before; see (12). The $T-$map is easily constructed to be

$$a_i \rightarrow (A^M + C^M)a^M,\ b_i \rightarrow A^M b^M F + C^M b^M, \tag{31}$$

where $C^M = \sum_{s=1}^{S} C_s$. Considering the case of RLS/SG learning, the associated ODE is

$$\dot{a}_i = \delta_i \left( \alpha + \sum_{j=1}^{S} (A_j + C_j)a_j - a_i \right),\ i = 1, ..., S$$

for the $a_i$ components and

$$\dot{b}_i = \delta_i \left[ \sum_{j=1}^{S} (A_j b_j F + C_j b_j) - b_i \right] M_w$$

for the $b_i$ components of SG learners. Convergence requires that the matrices

$$D_1\bar{\Omega} \equiv \begin{pmatrix} \delta_1 I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_n \end{pmatrix} \begin{pmatrix} A_1 + C_1 - I_n & \cdots & A_S + C_S \\ \vdots & \ddots & \vdots \\ A_1 + C_1 & \cdots & A_S + C_S - I_n \end{pmatrix},$$

$$Q\bar{\Omega}_F \equiv \begin{pmatrix} Q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_S \end{pmatrix} \begin{pmatrix} F' \otimes A_1 + I_k \otimes C_1 - I_{nk} & \cdots & F' \otimes A_S + I_k \otimes C_S \\ \vdots & \ddots & \vdots \\ F' \otimes A_1 + I_k \otimes C_1 & \cdots & F' \otimes A_S + I_k \otimes C_S - I_{nk} \end{pmatrix}$$

have eigenvalues with negative real parts, where $Q_s = \delta_s I_{nk}$ or $\delta_s(M_w \otimes I_n)$ if agent $s$ is using RLS or SG, respectively. Natural extensions of Proposition 6 as well as of Corollary 7, and Corollary 9 and part (i), (ii) and (iv) of Remark 10 hold.

# 6 Economic Examples Continued

## 6.1 The Model of Speculative Demand

We now consider stability of the REE under learning in the model of speculative demand and externalities; see Example 1. Stability can be checked by considering the matrices $\bar{\Omega}$ and $\bar{\Omega}_F$ defined in Section 5.3. For this model $A_s = r\psi(1-\eta)^{-1}\kappa_s$ and $C_s = 0$ for all $s$, and the matrices are

$$\bar{\Omega}_F = \begin{pmatrix} r\psi(1-\eta)^{-1}\kappa_1 - 1 & \cdots & r\psi(1-\eta)^{-1}\kappa_S \\ \vdots & \ddots & \vdots \\ r\psi(1-\eta)^{-1}\kappa_1 & \cdots & r\psi(1-\eta)^{-1}\kappa_S - 1 \end{pmatrix}$$

and $\bar{\Omega} = \bar{\Omega}_F$ with $r = 1$.

If there is no externality or if the positive externality is not very large, then $\eta < 1$ and it is easily verified that $sgn(A_s) = sgn(r)$ for all $s$, $\sum A_s < 1$ and $\sum rA_s < 1$, so that conditions of Corollary 9 are satisfied. The MSV REE is stable under persistently heterogenous learning. However, if the externality satisfies $\eta > 1$ the sign of $\psi$ could be negative or positive. If $\psi < 0$ and $r > 0$ it is possible that $\sum A_s > 1$. (A similar situation can also arise when $\psi > 0$ and $r < 0$.) $\sum A_s > 1$ violates one of the E-stability conditions and so cases of instability under learning can arise.[16]

We collect the observations:

**Proposition 11** *(A) If the aggregate externality $\eta$ is non-positive or only weakly positive (i.e. $\eta < 1$), the MSV solution to the model of speculative demand is stable under transiently or persistently heterogenous learning.*
*(B) If the externality is sufficiently strongly positive, the MSV REE can become unstable under learning.*

[16] Alternatively, these results can be established from Corollary 7 by considering, respectively, sufficient and necessary conditions for $D-$stability of matrices $\bar{\Omega}$ and $\bar{\Omega}_F$.

From the economic viewpoint, the key properties for result (A) are that, with weak externality, the behavior of all individuals satisfy the sign restriction (i) and the aggregate economy satisfies condition (ii) in Corollary 9. The case in part (B) verifies the intuition of (Grandmont 1998), Remark 2.3 that a small amount of instability in individual behavior (possibly in just one type of individual) can make the aggregate economy unstable.

## 6.2 Monetary Policy

Example 2 introduced a currently widely-used New Keynesian model of monetary policy and the forecast-based interest rate rule suggested by (McCallum and Nelson 2000). This model with internal central bank forecasting leads to a reduced form that is of type (5), where $S = 2$ and the two agents are the private sector and the central bank. There is necessarily structural heterogeneity.[17]

Introducing the notation $z_t' = (x_t, \pi_t)$ and $w_t' = (g_t, u_t)$, the reduced form is

$$\begin{aligned} z_t &= A_P \hat{E}_t^P z_{t+1} + A_{CB} \hat{E}_t^{CB} z_{t+1} + C_{CB} \hat{E}_t^{CB} z_t + B w_t, \text{ where} \\ A_P &= \begin{pmatrix} 1 & \phi \\ \lambda & \beta + \lambda\phi \end{pmatrix}, A_{CB} = \begin{pmatrix} -\alpha\phi\theta\lambda^{-1} & -\phi(1+\theta) \\ -\alpha\phi\theta & -\phi\lambda(1+\theta) \end{pmatrix} \\ C_{CB} &= \begin{pmatrix} \alpha\phi\theta\lambda^{-1} & 0 \\ \alpha\phi\theta & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}. \end{aligned} \tag{32}$$

The first key observation about model (32) is that private agents' learning contributes to instability. This is evident from computing $\det(A_P - I) < 0$, which implies that any policy of interest rates responding only to $w_t$ would lead to instability under learning; see (Evans and Honkapohja 2003b). Thus, monetary policy has an important role to play in this model: it must be designed to offset the tendency toward instability from private agents' learning.

If there is only transient heterogeneity in learning it is sufficient for policy to ensure that we have stability in the aggregate economy $A^M = A_P + A_{CB}$, $C^M = C_{CB}$, as indicated by the $T-$map (31). The forward-looking version of the (McCallum and Nelson 2000) approximate targeting rule can achieve stability of the economy under learning when the central bank can observe private expectations and $\theta$ is set appropriately. (Evans and Honkapohja 2003c) have shown that, depending on the model parameters $\beta, \lambda$ and $\phi$ as well as the policy weight $\alpha$, there is a bound $\theta^U$ such that we have stability under learning when $\theta < \theta^U$.

We now examine the sensitivity of this result to persistent heterogeneity in learning when the central bank does not observe private expectations and instead uses its own internal forecasts in the McCallum-Nelson rule (11). The results will depend on the parameter values and we select a particular calibration for $\beta, \lambda$ and $\phi$ suggested by (Clarida, Gali, and Gertler 2000):

[17]The companion paper (Honkapohja and Mitra 2003c) studies the performance of other forecast-based interest rate rules when the central bank uses internal forecasts.

**Calibration CGG:** $\beta = 0.99$, $\varphi = 1$ and $\lambda = 0.3$.

We also set $\alpha = 0.1$, $\mu = 0.35$ and $\rho = 0.35$.

A new phenomenon for the McCallum Nelson rule emerges when learning by the private sector and the central bank is persistently heterogenous. There can also exist a lower bound $\theta^L$ such that $\theta < \theta^L$ implies instability with heterogenous learning. We compute the lower and upper bounds $\theta^L$ and $\theta^U$ in the case where the private sector and the central bank both use RLS algorithms but with different asymptotic gain parameters $\delta_P$ and $\delta_{CB}$ (the normalization $\delta_P = 1$ is used without loss of generality). We consider this case for brevity as the case of RLS/SG learning would be qualitatively similar. Table 1 reports the critical values $\theta^L$ and $\theta^U$ for which $\theta^L \leq \theta \leq \theta^U$ is required to achieve stability under learning.

**Table 1: Critical values $\theta^U$ and $\theta^L$**

| $\delta_{CB}$ | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1 | 1.1 | 1.2 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta^U$ | 11.6 | 11.6 | 11.6 | 11.6 | 11.6 | 11.6 | 11.6 | 11.6 | 11.6 | 11.4 | 11.1 |
| $\theta^L$ | 10.8 | 5.9 | 3.4 | 2 | 1.2 | 0.6 | 0.3 | 0 | 0 | 0 | 0 |

Two important conclusions emerge from Table 1. First, with low values of $\delta_{CB}$ a too low value of $\theta$ will lead to instability. In fact, if $\delta_{CB}$ is very low, then instability always obtains. This would happen when $\delta_{CB} = 0.1$ (as we would then have $\theta^L > \theta^U$). Second, the upper threshold value $\theta^U$ is relatively insensitive to variations in $\delta_{CB}$, though it is seen that higher values of $\delta_{CB}$ do imply a slightly lower value for $\theta^U$. Further computations (not given explicitly) show that even very high values of $\delta_{CB}$ imply positive values for $\theta^U$. These results verify the suggestion of the intuition of (Grandmont 1998), Remark 2.3 that heterogeneity in structure and in learning can lead to very different outcomes relative to the cases where the properties of the aggregate economy determine stability of the REE under learning.

# 7 Concluding Remarks

Most macroeconomic models are based on the assumption of structural homogeneity, i.e. of the representative agent, and in the literature on learning this assumption is usually extended to include the learning rules of the agents. In this paper we have considered the significance of this assumption for stability of learning dynamics by studying the implications of structural heterogeneity, which is captured by the differential effect of the expectations of the different agents on the economy. Some central cases of structural and expectational heterogeneity were analyzed.

We first showed that introducing heterogeneity in beliefs or only transiently in learning rules has no significant consequences, as the convergence conditions are the same as in the corresponding model with homogenous expectations. This result was then reconsidered by analyzing the implications of heterogeneity in learning rules (and not only forecasts).

In general, the stability conditions for learning are affected by this kind of heterogeneity, but this is not always the case. Some standard models, which have been found to converge to REE under homogenous expectations and learning, continue to do so in the presence of heterogenous expectations and learning rules. We illustrated this point using a market model with speculative demand and there are other models such as the Muth market and Cagan models that share this feature. The assumption of homogenous expectations and learning rules is not always as restrictive as it may seem at first sight.

On the other hand, there are models for which heterogenous learning affects the conditions for convergence. An important case is the basic forward-looking model of monetary policy commonly used in the New Keynesian literature. We have considered this model for two cases. In this paper we analyzed the properties of a forecast-based interest rate rule proposed by McCallum and Nelson when internal central bank forecasts are used. The companion paper (Honkapohja and Mitra 2003c) examines to what extent heterogeneity can affect the desirability of different Taylor-type and optimal interest rate rules advocated in the literature.

The analysis and the results in this paper are based on the assumption of symmetric information, so that agents observe and make forecasts on the same set of "macro" variables in the economy. This setting is natural in many models, but extensions to our analysis are needed for many specific settings. For example, we have not considered the learnability of non-MSV REE. Perhaps more importantly, we stress that adaptive learning in economies with asymmetric information or when different agents are concerned with different local variables should be considered further as the existing literature is far from comprehensive.

# A Appendix: Proofs

**Proof of Proposition 1:** For the first equation the solution is evidently unique if and only if $I - A^M$ is invertible. The second equation for $b$ must be vectorized and we get

$$vecb = (F' \otimes A^M)vecb + vecB.$$

The determinant of this matrix is easily seen to be non-zero if and only if the matrix $I - F' \otimes A^M$ is invertible. Q.E.D.

**Derivation of the Associated ODEs in Sections 3.3 and 3.4:** For notational concreteness, we derive the associated ODE of the RLS algorithm for agent 1. (17)-(18), with the timing change, define $N_1(.)$. Here $y_{t-1} = T(\varphi'_{1,t-1}, \varphi'_{2,t-1})z_{t-1}$. The $\varphi_1$ components of the function $H(\theta_{t-1}, X_t)$ are

$$H_{\varphi 1}(z_{t-1}, \varphi_{1,t-1}, \varphi_{2,t-1}, S_{1,t-1}) = S^{-1}_{1,t-1} z_{t-1} z'_{t-1} (T(\varphi'_{1,t-1}, \varphi'_{2,t-1})' - \varphi_{1,t-1}). \tag{33}$$

Regarding the second order in $\gamma_t$ term in (15), we have

$$\rho_{\varphi 1,t}(\theta_{t-1}, X_t) = \frac{\hat{\gamma}_{1,t} - \gamma_t}{\gamma_t^2} S_{1,t-1}^{-1} z_{t-1}(T(\varphi'_{1,t-1}, \varphi'_{2,t-1}) z_{t-1} - \varphi'_{1,t-1} z_{t-1})',$$

and the validity of the method requires that this be bounded in $t$. This is easily established as by Assumption A (with $K_i \leq 1$ without loss of generality) we have $\hat{\gamma}_{1,t}/\gamma_t \leq 1$ $\Rightarrow \hat{\gamma}_{1,t}/\gamma_t \leq 1 + K\gamma_t$ for any $K > 0 \Rightarrow \frac{\hat{\gamma}_{1,t}-\gamma_t}{\gamma_t^2} \leq 1$.

From (18) the $S_1$ components of the function $H(\theta_{t-1}, X_t)$ are given by

$$H_{S1}(z_t, S_{1,t-1}) \equiv z_t z'_t - S_{1,t-1} \tag{34}$$

while the second order in $\gamma_t$ term

$$\rho_{S1,t}(\theta_{t-1}, X_t) = (\frac{\gamma_{1,t+1} - \gamma_t}{\gamma_t^2})(z_t z'_t - S_{1,t-1})$$

is bounded in $t$ again by Assumption A. Now

$$\lim_{t\to\infty} EH_{\varphi 1}(z_{t-1}, \varphi_1, \varphi_2, S_1) = \delta_1 S_1^{-1} M_z (T(\varphi'_1, \varphi'_2)' - \varphi_1).$$

where $M_z$ is defined in (21). Similarly

$$\lim_{t\to\infty} EH_{S1}(z_{t-1}, S_1) = \delta_1(M_z - S_1).$$

For notational concreteness, we derive the associated ODE of the SG algorithm for agent 2. We get from (23) the $\varphi_2$ components of the function $H(\theta_{t-1}, X_t)$, which for future use we denote by $H_{\varphi 2}(t, z_{t-1}, \varphi_{1,t-1}, \varphi_{2,t-1})$. (Note that there is no component $S_{2,t}$ under SG learning.) Thus

$$\lim_{t\to\infty} EH_{\varphi 2}(z_{t-1}, \varphi_1, \varphi_2) = \delta_2 M_z (T(\varphi'_1, \varphi'_2)' - \varphi_2).$$

**Proof of Proposition 2:** With $\delta_1 = \delta_2 = \delta$ the differential equations (22) have the following explicit form:

$$\begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = \delta\Omega \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \tag{35}$$

$$\begin{pmatrix} vec\dot{b}_1 \\ vec\dot{b}_2 \end{pmatrix} = \delta\Omega_F \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix} + \begin{pmatrix} vecB \\ vecB \end{pmatrix}, \tag{36}$$

where $\Omega$ is defined in (27) and $\Omega_F$ in (29). The constant $\delta$ does not affect stability and so these equations are locally stable at the equilibrium if and only if the eigenvalues of the matrices on the right hand sides of (27) and (29) have negative real parts.

The determinant for computing the eigenvalues of (27), $|\Omega - mI_{2n}|$, may be simplified as follows:

$$\begin{aligned}
|\Omega - mI_{2n}| &= \begin{vmatrix} A_1 - I_n(1+m) & A_2 \\ A_1 & A_2 - I_n(1+m) \end{vmatrix} \\
&= \begin{vmatrix} -I_n(1+m) & I_n(1+m) \\ A_1 & A_2 - I_n(1+m) \end{vmatrix} \\
&= \begin{vmatrix} -I_n(1+m) & 0 \\ A_1 & A_1 + A_2 - I_n(1+m) \end{vmatrix} \\
&= (-(1+m))^n \left| A^M - I_n(1+m) \right| .
\end{aligned}$$

The computation shows that $\Omega$ has $n$ eigenvalues equal to $-1$ and the remaining eigenvalues are those of $A^M - I_n$. Hence, $\Omega$ has eigenvalues with negative real parts if and only if $A^M - I_n$ has the same property.

Analogous computations show that $\Omega_F$ has $nk$ eigenvalues equal to $-1$ and the rest are the eigenvalues of $F' \otimes A^M - I_{nk}$.

**Proof of Proposition 3:** Writing (25) explicitly when $\delta_i = \delta$, we have

$$\begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \end{pmatrix} = \delta \begin{pmatrix} A_1 - I_n & A_2 \\ A_1 & A_2 - I_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

for the $a_i$ subsystem. Apart from $\delta > 0$ that does not affect stability, this is the same as (35). Vectorizing and dropping the constant $B$, the $b_i$ subsystem is

$$\begin{pmatrix} vec\dot{b}_1 \\ vec\dot{b}_2 \end{pmatrix} = \delta M \begin{pmatrix} F' \otimes A_1 - I_{nk} & F' \otimes A_2 \\ F' \otimes A_1 & F' \otimes A_2 - I_{nk} \end{pmatrix} \begin{pmatrix} vecb_1 \\ vecb_2 \end{pmatrix},$$

where

$$M = \begin{pmatrix} M_w \otimes I_n & 0 \\ 0 & M_w \otimes I_n \end{pmatrix}$$

and $\delta > 0$ does not affect stability. The the characteristic equation of coefficient matrix can be written as

$$\begin{aligned}
0 &= \begin{vmatrix} M_w F \otimes A_1 - M_w \otimes I_n - mI_{nk} & M_w F \otimes A_2 \\ M_w F \otimes A_1 & M_w F \otimes A_2 - M_w \otimes I_n - mI_{nk} \end{vmatrix} \\
&= \begin{vmatrix} -(M_w \otimes I_n) - mI_{nk} & 0 \\ M_w F \otimes A_1 & M_w F \otimes A^M - M_w \otimes I_n - mI_{nk} \end{vmatrix}.
\end{aligned}$$

The eigenvalues of the coefficient matrix thus solve

$$\begin{aligned}
|-(M_w \otimes I_n) - mI_{nk}| &= 0 \\
\left| M_w F \otimes A^M - M_w \otimes I_n - mI_{nk} \right| &= 0
\end{aligned}$$

The eigenvalues of $-(M_w \otimes I_n)$ are negative as $M_w$ is positive definite. For the latter we can write

$$\left|(M_w \otimes I_n)(F \otimes A^M - I_n) - mI_{nk}\right| = 0,$$

which is just the convergence condition for SG learning in the average economy. Q.E.D.

**Proof of Corollary 4:** The result follows for considering the $b_i$ component in (25). Since $b_i$ is now $n \times 1$ and $F$ and $M_w$ are scalars we can write this component as

$$\begin{pmatrix} \dot{b}_1 \\ \dot{b}_2 \end{pmatrix} = \delta M_w \left[ F \begin{pmatrix} A_1 & A_2 \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right].$$

Here $\delta M_w > 0$ is a scalar that does not affect stability while computation of the eigenvalues for the system inside the square brackets is a special case of the same computation for (29). Q.E.D.

**Proof of Proposition 6 and (i) of Remark 10:** Since the ODE for $S_1$ is globally stable with $S_1 \to M_z$ from any starting point, stability is determined entirely by the smaller dimensional system

$$\begin{aligned} d\varphi_1/d\tau &= \delta_1(T(\varphi_1', \varphi_2')' - \varphi_1), \\ d\varphi_2/d\tau &= \delta_2 M_z(T(\varphi_1', \varphi_2')' - \varphi_2). \end{aligned}$$

The explicit form of this ODE is given in the main text in equations (26) and (28). To prove that convergence is in fact global and takes place almost surely, we first note that the associated ODE is linear and globally stable. Second, it is easy to verify that the conditions of Theorem 6.10 in (Evans and Honkapohja 2001) or Theorem 2 in (Evans and Honkapohja 1998a) are satisfied, so that almost sure global convergence obtains.

To prove (i) of Remark 10, one proceeds as in the proof of Proposition 6, but for both agents the details are as for agent 1 in the earlier proof. Q.E.D.

**Proofs of Corollary 8 and Part (iii) of Remark 10:** We only consider the former for brevity (setting $M_w = 1$ proves the latter). Computing the trace and determinant of the matrix of $D_2\Omega_F$ in (28) in the scalar case yields

$$\begin{aligned} tr(D_1 A) &= \delta_1(FA_1 - 1) + \delta_2 M_w(FA_2 - 1) < 0 \\ \det(D_1 A) &= \delta_1 \delta_2 M_w(1 - F(A_1 + A_2)) > 0 \end{aligned}$$

under the made hypotheses. For $D_1\Omega$ in the scalar case we have the same formulas but where $F = 1$. Q.E.D.

**Proof of Corollary 9 and Part (iv) of Remark 10:** Consider the subsystem for $a_i$ and thus the matrix $\Omega$ in the $S$ agent case. Let

$$I - \Omega = \begin{pmatrix} 1 - A_1 & -A_2 & \cdots & -A_S \\ -A_1 & 1 - A_2 & \cdots & -A_S \\ \vdots & \vdots & \ddots & \vdots \\ -A_1 & -A_2 & \cdots & 1 - A_S \end{pmatrix}, \tag{37}$$

where the different $A_i$ have the same sign. We assume that $1 - \sum_{i=1}^{S} A_i > 0$.[18]

To compute $\det(I - \Omega)$ add all other columns to the first column and then subtract the first row from all other rows. This yields the determinant

$$\det(I-\Omega) = \begin{vmatrix} 1 - \sum_{i=1}^{S} A_j & -A_2 & \cdots & -A_S \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1 - \sum_{i=1}^{S} A_j > 0$$

Consider first any principal minor of $I - \Omega$ and denote it by $\hat{M}$. All rows of $\hat{M}$ have an element of the form $1 - A_j$ then $\hat{M}$ has the same general form as $I - \Omega$, except that $\hat{M}$ has some arbitrary collection of columns $J = \{j_1, ..., j_K\}$. In this case

$$\det(\hat{M}) = \pm \left(1 - \sum_{i \in J} A_i\right), \tag{38}$$

since a permutation of rows may be needed to get it in the form where the elements of type $1 - A_j$ are on the main diagonal.

Next, choose any minor of $I - \Omega$ and denote it by $M$. Note that each row of $M$ has at most one element of the form $1 - A_j$ and all other elements are of the form $-A_j$.

We want to compute the value of $\det(M)$. First, we note that if $M$ has two or more rows without an element of the the form $1 - A_j$ then it has two identical rows and then $\det(M) = 0$. Second, we note that if all rows of $M$ have an element of the form $1 - A_j$ then we are in the same case as with the principal minors, which was discussed above.

The remaining case is that $M$ has exactly one row without an element of the form $1 - A_i$. Note that for different rows these elements are in different columns. Note also that each column has at most one element of the form $1 - A_j$ and there is exactly one column with element of the form $1 - A_j$. We permute the rows and columns so that the row without the element of form $1 - A_j$ becomes the first row and so that the other rows have the elements $1 - a_j$ in a symmetric order:

$$\det(M) = \pm \begin{vmatrix} -A_{j_1} & -A_{j_2} & \cdots & -A_{j_K} \\ -A_{j_1} & 1 - A_{j_2} & \cdots & -A_{j_K} \\ \cdots & \cdots & \ddots & \vdots \\ -A_{j_1} & -A_{j_2} & \cdots & 1 - A_{j_K} \end{vmatrix}. \tag{39}$$

Then subtract the first row from all other rows, which yields

$$\det(M) = \pm \begin{vmatrix} -A_{j_1} & -A_{j_s} & \cdots & -A_{j_K} \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}.$$

[18] We write the matrices in this way since we will prove "positive stability", as is often done in the mathematics literature; see e.g. Chapter 2 of (Horn and Johnson 1991).

which we expand along the first column and obtain $\det(M) = \pm A_j$.

In the final step we need to consider a minor $M(\alpha, \beta)$, where $\alpha = \{j_1, ..., j_K\}$ are the selected rows and $\beta = \{i_1, ..., i_K\}$ are the selected columns, and the corresponding minor $M(\beta, \alpha)$. We make the following observations:

1. If $M(\alpha, \beta)$ has two rows with element of the form $1 - A_s$ then $M(\beta, \alpha)$ also has two columns without this type of element and both minors are zero.

2. If all rows of $M(\alpha, \beta)$ have an element of type $1 - A_s$ then all columns of $M(\beta, \alpha)$ have an element of that type. Moreover, when we do the permutations on rows that make $M(\alpha, \beta)$ into form (37), the same permutations on the columns of $M(\beta, \alpha)$ and taking the transpose will also lead to the form (37) but in general with different elements $-a_j$. Hence

$$\begin{aligned} \det(M(\alpha, \beta)) &= \pm(1 - \sum_{i \in \beta} A_i) \text{ and} \\ \det(M(\beta, \alpha)) &= \pm(1 - \sum_{i \in \alpha} A_i), \end{aligned}$$

where the plus or minus signs apply at the same time. It follows that

$$\det(M(\alpha, \beta)) \det(M(\beta, \alpha)) \geq 0. \quad (40)$$

3. If $M(\alpha, \beta)$ has exactly one row without element of type $1 - A_i$, the minor $M(\beta, \alpha)$ has one column with that type of element. We do the required row permutations to get $M(\alpha, \beta)$ into form (39) and the same number of column permutations and a transposition on $M(\beta, \alpha)$ will lead to form (39) and so

$$\begin{aligned} \det(M(\alpha, \beta)) &= \pm A_j \text{ and} \\ \det(M(\beta, \alpha)) &= \pm A_s, \end{aligned}$$

where the plus or minus signs apply at the same time. Again it follows that inequality (40) holds.

In all cases we see that the product of the symmetric minors of $I - \Omega$ is non-negative. We can then apply the criterion in (Carlson 1974) or criterion (10) in (Johnson 1974) and conclude that $I - \Omega$ is positively $D-$stable, i.e. $\Omega - I$ is $D-$stable.

The proof is complete once the following observations are made. First, because of $D-$stability a natural generalization of Corollary 7 can be exploited as in the scalar case. Second, the same argument can be used for subsystem for $b_i$ with $\Omega_F$ where $|F| < 1$ is a scalar. Q.E.D.

# References


Arifovic, J. (1998): "Stability of Equilibria under Genetic Algorithm Adaption: an Analysis," *Macroeconomic Dynamics*, 2, 1–21.

Arrow, K. J., and M. McManus (1958): "A Note on Dynamic Stability," *Econometrica*, 26, 448–454.

Barnett, W., B. Cornet, C. D'Aspremont, J. Gabszewicz, and A. Mas-Colell (eds.) (1991): *Equilibrium Theory and Applications, Proceedings of the Sixth International Symposium in Economic Theory and Econometrics*. Cambridge University Press, Cambridge.

Barucci, E. (1999): "Heterogeneous Beliefs and Learning in Forward-Looking Models," *Journal of Evolutionary Economics*, 9, 453–464.

Barucci, E., and L. Landi (1997): "Least Mean Squares Learning in Self-Referential Stochastic Models," *Economics Letters*, 57, 313–317.

Bray, M., and N. Savin (1986): "Rational Expectations Equilibria, Learning, and Model Specification," *Econometrica*, 54, 1129–1160.

Brock, W. A., and P. de Fontnouvelle (2000): "Expectational Diversity in Monetary Economies," *Journal of Economic Dynamics and Control*, 24, 725–759.

Brock, W. A., and C. H. Hommes (1997): "A Rational Route to Randomness," *Econometrica*, 65, 1059–1095.

Carlson, D. (1974): "A Class of Positive Stable Matrices," *Journal of Research of the National Bureau of Standards -B- Mathematical Sciences*, 78.

Clarida, R., J. Gali, and M. Gertler (1999): "The Science of Monetary Policy: A New Keynesian Perspective," *Journal of Economic Literature*, 37, 1661–1707.

——— (2000): "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics*, 115, 147–180.

Evans, G. W., and S. Honkapohja (1995): "Local Convergence of Recursive Learning to Steady States and Cycles in Stochastic Nonlinear Models," *Econometrica*, 63, 195–206.

——— (1997): "Least Squares Learning with Heterogeneous Expectations," *Economic Letters*, 52, 197–201.

——— (1998a): "Economic Dynamics with Learning: New Stability Results," *Review of Economic Studies*, 65, 23–44.

——— (1998b): "Stochastic Gradient Learning in the Cobweb Model," *Economic Letters*, 61, 333–337.

——— (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press, Princeton, New Jersey.

——— (2003a): "Adaptive Learning and Monetary Policy Design," *Journal of Money, Credit and Banking, forthcoming.*

——— (2003b): "Expectations and the Stability Problem for Optimal Monetary Policies," *Review of Economic Studies*, 70, 807–824.

——— (2003c): "Monetary Policy, Expectations and Commitment," mimeo, www.valt.helsinki.fi/RAKA/seppo.hmtl and http://darkwing.uoregon.edu/~gevans/.

Evans, G. W., S. Honkapohja, and R. Marimon (2001): "Convergence in Monetary Inflation Models with Heterogeneous Learning Rules," *Macroeconomic Dynamics*, 5, 1–31.

Evans, G. W., S. Honkapohja, and N. Williams (2003): "Stochastic Gradient Learning," work in progress.

Giannitsarou, C. (2003a): "E-Stability Does Not Imply Learnability," mimeo.

——— (2003b): "Heterogenous Learning," *Review of Economic Dynamics, forthcoming.*

Grandmont, J.-M. (1998): "Expectations Formation and Stability of Large Socioeconomic Systems," *Econometrica*, 66, 741–781.

Grandmont, J.-M., and G. Laroque (1991): "Economic Dynamics with Learning: Some Instability Examples," in (Barnett, Cornet, D'Aspremont, Gabszewicz, and Mas-Colell 1991), chap. 11, pp. 247–273.

Guesnerie, R. (2002): "Anchoring Economic Predictions in Common Knowledge," *Econometrica*, 70, 439–480.

Honkapohja, S., and K. Mitra (2003a): "Adaptive Learning in Nonlinear Models When Shocks Follow a Markov Chain," manuscript.

——— (2003b): "Learning with Bounded Memory in Stochastic Models," *Journal of Economic Dynamics and Control*, 27, 1437–1457.

——— (2003c): "Performance of Monetary Policy with Internal Central Bank Forecasting," *Journal of Economic Dynamics and Control,* forthcoming.

Horn, R., and C. Johnson (1991): *Topics in Matrix Analysis*. Cambridge University Press, Cambridge.

Johnson, C. R. (1974): "Sufficient Conditions for D-Stability," *Journal of Economic Theory*, 9, 53–62.

Ljung, L., and T. Söderström (1983): *Theory and Practice of Recursive Identification*. MIT Press, Cambridge Mass.

Marcet, A., and T. J. Sargent (1989a): "Convergence of Least-Squares Learning in Environments with Hidden State Variables and Private Information," *Journal of Political Economy*, 97, 1306–1322.

——— (1989b): "Convergence of Least-Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337–368.

Marimon, R., and S. Sunder (1993): "Indeterminacy of Equilibria in a Hyperinflationary World: Experimental Evidence," *Econometrica*, 61, 1073–1107.

McCallum, B. T., and E. Nelson (2000): "Timeless Perspective vs. Discretionary Monetary Policy in Forward-Looking Models," mimeo, Carnegie Mellon University and Bank of England.

Moreno, D., and M. Walker (1994): "Two Problems in Applying Ljung's "Projection Algorithms" to the Analysis of Decentralized Learning," *Journal of Economic Theory*, 62, 420–427.

Negroni, G. (2003): "Adaptive Expectations Coordinaiton in an Economy with Heterogenous Agents," *Journal of Economic Dynamics and Control*, 28, 117–140.