SIAM J. DISCRETE MATH.
Vol. 13, No. 2, pp. 179–193

CONVEX-ROUND AND CONCAVE-ROUND GRAPHS*

JØRGEN BANG-JENSEN†, JING HUANG‡, AND ANDERS YEO§

Abstract. We introduce two new classes of graphs which we call convex-round, respectively concave-round graphs. Convex-round (concave-round) graphs are those graphs whose vertices can be circularly enumerated so that the (closed) neighborhood of each vertex forms an interval in the enumeration. Hence the two classes transform into each other by taking complements. We show that both classes of graphs have nice structural properties. We observe that the class of concave-round graphs properly contains the class of proper circular arc graphs and, by a result of Tucker [*Pacific J. Math.*, 39 (1971), pp. 535–545] is properly contained in the class of general circular arc graphs. We point out that convex-round and concave-round graphs can be recognized in $O(n+m)$ time (here n denotes the number of vertices and m the number of edges of the graph in question). We show that the chromatic number of a graph which is convex-round (concave-round) can be found in time $O(n+m)$ ($O(n^2)$). We describe optimal $O(n+m)$ time algorithms for finding a maximum clique, a maximum matching, and a Hamiltonian cycle (if one exists) for the class of convex-round graphs. Finally, we pose a number of open problems and conjectures concerning the structure and algorithmic properties of the two new classes and a related third class of graphs.

Key words. proper circular arc graphs, round graphs, round enumeration, Hamiltonian cycle, coloring, maximum matching, maximum clique, linear algorithms, recognition

AMS subject classifications. 05C85, 05C38, 05C70, 68R10

PII. S0895480197322154

1. Introduction.

An *interval graph* is the intersection graph of a family of linear intervals; a *proper interval graph* is the intersection graph of an inclusion-free family of linear intervals. A *circular arc graph* is the intersection graph of a family of circular arcs; as above, it is called *proper* if the family can be chosen to be inclusion-free.

Algorithmic aspects of interval graphs and circular arc graphs have been intensively studied; cf. [13]. Interval graphs can be recognized in linear time (i.e., $O(n+m)$ where n and m are, respectively, the numbers of vertices and of edges of the input graph) [6, 19]. There exists an $O(n^2)$ algorithm to recognize circular arc graphs [10]. Many optimization problems such as finding a maximum independent set, a maximum clique, a minimum dominating set, and so on, can all be solved efficiently for both interval graphs and circular arc graphs [1, 4, 15]. It is worth noticing that the minimum coloring problem is solvable in polynomial time for proper circular arc graphs but NP-complete for general circular arc graphs [3, 11, 26].

An important feature of proper circular arc graphs is that the vertices can be circularly enumerated so that the neighbors of each vertex v, together with v itself,

*Received by the editors May 30, 1997; accepted for publication (in revised form) December 16, 1999; published electronically April 6, 2000.

http://www.siam.org/journals/sidma/13-2/32215.html

†Department of Mathematics and Computer Science, Odense University, Odense, DK-5230, Denmark (jbj@imada.sdu.dk). The research of this author was supported by the Curtin University of Technology, Perth, Australia, and NSERC grant 203191-98.

‡Department of Mathematics and Statistics, University of Victoria, Victoria, BC, V8W 3P4, Canada (jing@math.uvic.ca). The research of this author was supported by the Danish Research Council under grant 11-9743-1, the Australia Research Council under grant A49532661, and NSERC under grant 203191-98.

§Brics, Department of Computer Science, University of Aarhus, Ny Munkegade, Building 540, DK-8000 Aarhus C, Denmark (yeo@daimi.au.dk). The research of this author was supported by the Danish Research Council under grant 9800435.

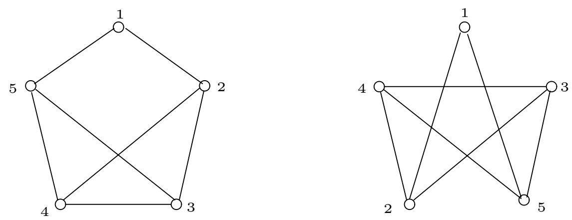

FIG. 1. *A graph which is concave-round and convex-round.*

appear consecutively in the enumeration (see [13]). More precisely, if G is a proper circular arc graph, then the vertices of G can be circularly enumerated, $v_1, v_2, \ldots, v_n$, so that the closed neighborhood of each vertex v_i forms a set $\{v_{i-l}, v_{i-l+1}, \ldots, v_{i+r}\}$ for some $l, r \geq 0$ (which depend on i), where the addition and subtraction are modulo n. However, not all circular arc graphs enjoy this property. Because of this, many problems can be solved much more efficiently for proper circular arc graphs than for general circular arc graphs. For instance, the recognition problem and the maximum clique problem for proper circular arc graphs can be solved in linear time (see [3, 9]), but no linear time algorithms are known for general circular arc graphs.

In order to better understand the above feature of proper circular arc graphs, we introduce two new classes of graphs as follows: A graph G is *concave-round* if the vertices of G can be circularly enumerated, $v_1, v_2, \ldots, v_n$, so that the closed neighborhood of each vertex v_i forms a set $\{v_{i-l}, v_{i-l+1}, \ldots, v_{i+r}\}$ for some $l, r \geq 0$. Here l, r depend on i and the addition is modulo n. We shall refer to the circular enumeration $v_1, v_2, \ldots, v_n$ as a *concave-round enumeration* and often denote it by $\mathcal{L}$. A graph is *convex-round* if it is the complement of a concave-round graph, that is, if the neighborhood of each vertex forms an interval in the enumeration. Within the class of bipartite graphs, convex-round graphs form a proper subclass of so-called *convex bipartite graphs*. Convex bipartite graphs have nice algorithmic properties and practical applications; see [7, 8, 12, 20, 22, 24, 29, 32]. In [20], a superclass of convex bipartite graphs was introduced under the name *circular convex bipartite graphs*. Even though this name seems to indicate that a circular convex bipartite graph must also be convex-round, this is not the case, as one can see from the definition in [20].

Let $\mathcal{L} = v_1, v_2, \ldots, v_n$ be a circular enumeration of the vertices of a graph G. A vertex v_i is *concave* with respect to $\mathcal{L}$ if the closed neighborhood of v_i forms an interval in $\mathcal{L}$. A vertex v_i is *convex* with respect to $\mathcal{L}$ if its neighborhood forms an interval in $\mathcal{L}$. A circular enumeration $\mathcal{L}$ of $V(G)$ is a *concave-round* (*convex-round*) enumeration of $V(G)$ if all vertices of G are concave (convex) with respect to $\mathcal{L}$. Figure 1 shows a concave-round and a convex-round enumeration of the same graph.

It is easy to see that every induced subgraph of a concave-round (convex-round) graph is concave-round (convex-round). It follows from the definition of a concave-round graph and an earlier remark on a feature of proper circular arc graphs that all proper circular arc graphs are concave-round graphs. But the converse is not true, that is, there exist concave-round graphs which are not proper circular arc graphs. So the class of concave-round graphs strictly contains the class of proper circular arc graphs. See Figure 2 for examples of graphs showing relationships between the above classes.

In this paper we study the structure of concave-round and convex-round graphs. We prove that the chromatic number problem can be solved in time $O(n^2)$ for both

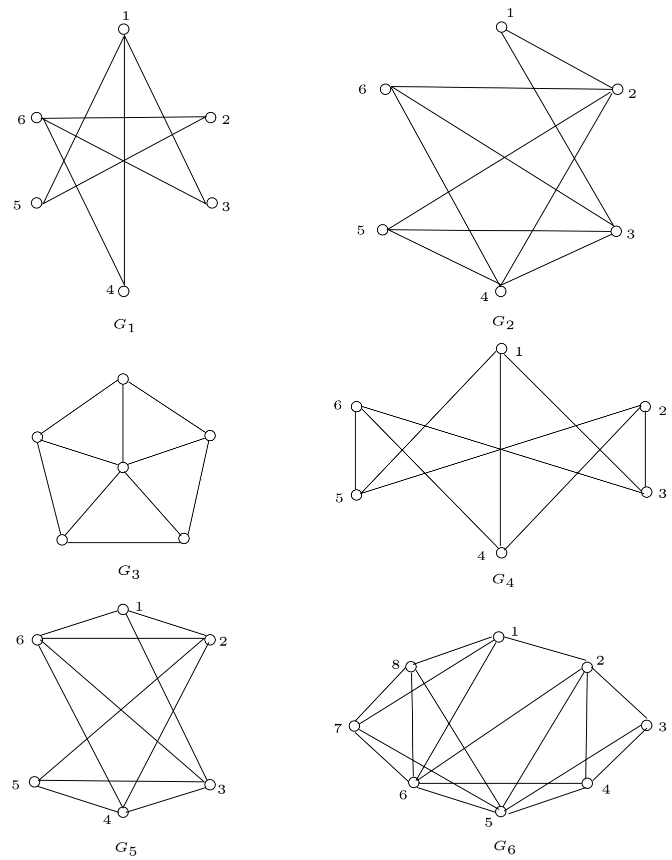

FIG. 2. *Graphs showing the relations between concave-round, convex-round graphs, and circular arc graphs: G_1, G_4 are convex-round but not circular arc graphs; G_3 is a circular arc graph but is neither concave-round nor convex-round; G_5 is a proper circular arc graph and concave-round; G_6 is concave-round but not a proper circular arc graph; G_2 is neither concave-round nor convex-round, but with respect to the labelling shown, every vertex is either concave or convex.*

of these classes. (The reason why our algorithms are $O(n^2)$ is that the operation of taking complementary graphs is used.) We describe optimal linear time algorithms for finding a maximum clique, a maximum matching, and a Hamiltonian cycle (if one exists) for convex-round graphs. Since every concave-round graph is a circular arc graph (see Theorem 3.1) and since the above three problems can be solved efficiently for circular arc graphs (see [4, 23, 27]), these three problems are efficiently solvable for concave-round graphs. Due to this and space considerations, we shall concentrate more on the algorithmic aspects of convex-round graphs. However, we wish to point out that none of the algorithms in [4, 23, 27] achieves optimal running time and we believe this is possible for each of the above problems when restricted to concave-round graphs.

2. Terminology and notation. We assume all graphs are finite and simple, i.e., they contain no loops or multiple edges. For standard terminology, we refer to [5].

Let G be a graph. We shall use $V(G)$ (resp., $E(G)$) to denote the set of vertices (resp., the set of edges) in G. We shall always use n (resp., m) to denote the number of vertices (resp., edges) of G. Two vertices $x, y \in V(G)$ are *adjacent* and y is a *neighbor* of x if $xy \in E(G)$. If $xy \notin E(G)$, then y is a *nonneighbor* of x. The *neighborhood* of x, denoted by $N(x)$, is the set of all neighbors of x. The *closed neighborhood* of x, denoted by $N[x]$, is defined as $N(x) \cup \{x\}$.

A graph is *bipartite* if the vertex set $V(G)$ can be partitioned into two sets A and B such that every edge of G has one endvertex in A and the other in B. We shall often use the terminology (A, B) for a bipartite graph with vertex set $A \cup B$ and we only specify the set of edges E when this is necessary.

A graph G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. If, in addition, $E(G') = \{xy \in E|\ x, y \in V(G')\}$, then G' is an *induced subgraph* of G. For each $S \subseteq V(G)$, the subgraph of G *induced by* S, denoted by $G\langle S\rangle$ or simply S, is the unique induced subgraph of G with vertex set S. A *clique* of G is a subgraph G' of G for which every vertex in $V(G')$ is adjacent to every other vertex in $V(G')$. The size of a largest clique in a graph G is denoted by $\omega(G)$.

Suppose that G is a graph and $S \subseteq V(G)$ is a set of vertices of G. We shall use $G - S$ to denote the subgraph induced by $V(G) - S$. We shall write $G - x$ instead of $G - \{x\}$. If S contains no adjacent vertices, then S is called an *independent set* of G. The size of a largest independent set is denoted by $\alpha(G)$.

Let G be a graph. The *complement* of G, denoted by $\overline{G}$, is a graph such that the vertex set of $\overline{G}$ is $V(G)$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

A *path* P of length k is a graph with the vertex set $\{x_1, x_2, \ldots, x_{k+1}\}$ and the edge set $\{x_1x_2, x_2x_3, \ldots, x_kx_{k+1}\}$, such that all the vertices shown are distinct. We shall call such a path an (x_1, x_{k+1})-path and denote it by $x_1x_2x_3 \ldots x_kx_{k+1}$. A *cycle* C of length k is a graph with the vertex set $\{v_1, v_2, \ldots, v_k\}$ and the edge set $\{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k, x_kx_1\}$. A *Hamiltonian* path (cycle) in a graph G is a path (cycle) with vertex set $V(G)$.

Suppose that $x_1x_2x_3 \ldots x_k$ is a path (resp., a cycle). Then vertices v_i and v_{i+1} are called *consecutive vertices*. (The subscript addition is modulo k in the case of cycle.) A path or a cycle is *chordless* (in a graph G) if nonconsecutive vertices are not adjacent (in G).

A graph G is an *interval graph* if there is a one-to-one correspondence between $V(G)$ and a family $\mathcal{I}$ of intervals on the real line so that two vertices in G are adjacent if and only if the two corresponding intervals intersect. If the family $\mathcal{I}$ can be chosen so that no interval is contained in another, then G is called a *proper interval graph*. A graph G is a *circular arc graph* if there is a one-to-one correspondence between $V(G)$ and a family $\mathcal{F}$ of circular arcs on a circle so that two vertices are adjacent in G if and only if the corresponding two circular arcs intersect. As above, if the family $\mathcal{F}$ can be chosen so that no circular arc is contained in another, then G is called a *proper circular arc graph*.

A matrix whose entries belong to $\{0, 1\}$ has the *circular* 1*'s property* for rows (columns) if its columns (rows) can be permuted in such a way that the 1's in each row occur in a circular consecutive order. The *augmented adjacency matrix* of a graph G is obtained from the adjacency matrix of G by adding 1's along the main diagonal.

It is helpful to visualize a concave-round enumeration of the vertices of G by putting the vertices $v_1, v_2, \ldots, v_n$ clockwise around a circle. If $a = v_i$ and $b = v_j$ are two vertices, then we define the interval $[a, b]$ as the set $\{v_i, v_{i+1}, \ldots, v_j\}$, with the subscription calculated modulo n. Thus in our visualized representation, the interval extends from a to b clockwise. Similarly, we define $(a, b) = [a, b] - \{a, b\}$,

$[a, b) = [a, b] - \{b\}$, and $(a, b] = [a, b] - \{a\}$. It is easy to see from the remark above that a graph is concave-round if and only if its augmented adjacency matrix has the circular 1's property for columns (and rows). Similarly, a graph is convex-round if and only if its adjacency matrix has the circular 1's property for columns (and rows).

Let G be a convex-round graph with a convex-round enumeration $\mathcal{L} = v_1, v_2, \ldots, v_n$. Suppose that G is bipartite. Then G is said to be *bipartite with respect to* $\mathcal{L}$ if there exist i and j with $i \neq j$ such that every edge of G has an endvertex in $A = [v_i, v_j)$ and an endvertex in $B = [v_j, v_i)$. We shall refer to (A, B) as a *bipartition of* G *with respect to* $\mathcal{L}$.

3. Structure of concave-round graphs and convex-round graphs. In this section we shall describe several structural results on graphs which are concave-round or convex-round. These results will be used in the later sections. As we observed earlier, concave-rounds graphs form a superclass of the class of proper circular arc graphs. We will now show that they form a subclass of the class of circular arc graphs.

THEOREM 3.1. *If a graph is concave-round, then it is a circular arc graph.*

Proof. By a theorem of Tucker, see [13, Theorem 8.16, p. 190], a graph is a circular arc graph if its augmented adjacency matrix has the circular 1's property for columns. As we remarked earlier the augmented adjacency matrix of any concave-round graph has this property. ∎

THEOREM 3.2. *There is a linear time algorithm for recognizing concave-round (convex-round) graphs and finding (if it exists) a concave-round (convex-round) enumeration.*

Proof. Checking whether a graph is convex-round can be done using PQ-trees [6]: we are checking whether the adjacency matrix has the circular 1's property for columns. Instead of working with the adjacency matrix (whose size may not be $O(n+m)$ if m is not large) we can perform the checking by just giving the adjacency lists of G (i.e., entries which correspond to a one). For details on PQ-trees, see [6, 13]. The same argument shows that we can decide whether a graph is concave-round in time $O(n+m)$. A convex-round (concave-round) enumeration can be found from the resulting circular 1's ordering. ∎

A *tournament* is an orientation of a complete graph, i.e., an oriented graph in which every pair of distinct vertices are joined by an arc. A tournament is *transitive* if and only if it has no directed cycles. An oriented graph $D = (V, A)$ is a *local tournament* (*local transitive tournament*) if the set of out-neighbors as well as the set of in-neighbors of each vertex $v \in V$ forms a tournament (a transitive tournament) [2, 14, 16].

The following lemma can be found in [16]; see also [28].

LEMMA 3.3. *Let G be an undirected graph; then each of the following are equivalent:*

1. *G is a proper circular arc graph.*
2. *G can be oriented as a local tournament digraph.*
3. *G can be oriented as a local transitive tournament digraph.*

LEMMA 3.4. *Let G be a concave-round graph with a concave-round enumeration $\mathcal{L} = v_1, v_2, \ldots, v_n$. Then G is a proper circular arc graph if G satisfies the following three properties:*

1. *Every vertex of G has a nonneighbor.*
2. *G has no pair of vertices v_i, v_j such that $(v_i, v_j] \subseteq N(v_i)$ and $(v_j, v_i] \subseteq N(v_j)$.*
3. *G has no pair of vertices v_i, v_j such that $[v_j, v_i) \subseteq N(v_i)$ and $[v_i, v_j) \subseteq N(v_j)$.*

Proof. By the assumption above, for each vertex v_i we can distinguish those neighbors of v_i that are after v_i and those that are before v_i, namely, v_j is after (before) v_i if and only if v_i is adjacent to every vertex in $[v_{i+1}, v_j]$ ($[v_j, v_{i-1}]$). Now we orient G by orienting an edge v_iv_j from v_i to v_j if v_j is after v_i and from v_j to v_i otherwise. It follows from the assumption in the lemma that this orientation is well defined. Furthermore, if $v_i \rightarrow v_j$ is an arc then v_i is adjacent to every vertex in $[v_{i+1}, v_j]$ and since v_j has a nonneighbor in $[v_{j+1}, v_i]$ (by the assumption of the lemma), we also have that v_j is adjacent to every vertex in $[v_i, v_{j-1}]$. This shows that the orientation is a local transitive tournament orientation of G. Hence, by Lemma 3.3, G is a proper circular arc graph. □

In [16], the second author characterized all possible orientations of a proper circular arc graph as local tournaments. The following problem may be seen as an attempt to generalize that result.

Problem 3.5. *Given a graph G which is concave-round, what are the possible concave-round enumerations of G?*

We now turn our attention to convex-round graphs.

Lemma 3.6. *Let $G = (A, B)$ be a connected bipartite graph. Suppose that G is convex-round with a convex-round enumeration $\mathcal{L}$. Then G is bipartite with respect to $\mathcal{L}$.*

Proof. Let $[x, y] \subseteq A$ be an interval (with respect to $\mathcal{L}$) which is maximal (in terms of cardinality of $[x, y]$). If $A \subseteq [x, y]$, then we are done; otherwise let w be a vertex in $A - [x, y]$ and let $P = u_1u_2 \dots u_l$ be a shortest path from $[x, y]$ to w. Then $u_2 \in B$, and we must have $N(u_2) \subseteq [x, y]$ since u_2 is adjacent to neither x^- nor y^+ (here x^- is the vertex "next" to x in counterclockwise direction and y^+ is the vertex "next" to y in clockwise direction). Therefore $u_3 \in [x, y]$, contradicting the minimality of P. □

Lemma 3.7. *Let G be a connected convex-round graph with a convex-round enumeration $\mathcal{L} = v_1, v_2, \dots, v_n$. If $v_iv_{i+k} \notin E(G)$ for some i, then either G is bipartite, or at least one of $[v_i, v_{i+k}]$, $[v_{i+k}, v_i]$ is independent. Moreover, it can be decided in time $O(n + m)$ which of the above properties holds for G.*

Proof. Since $v_iv_{i+k} \notin E(G)$ and G is convex-round, we have $N(v_i) \subseteq (v_i, v_{i+k})$ or $N(v_i) \subseteq (v_{i+k}, v_i)$ and, similarly, $N(v_{i+k}) \subseteq (v_i, v_{i+k})$ or $N(v_{i+k}) \subseteq (v_{i+k}, v_i)$.

Suppose first that $N(v_i) \subseteq (v_i, v_{i+k})$ and $N(v_{i+k}) \subseteq (v_i, v_{i+k})$. We shall show that $[v_{i+k}, v_i]$ is independent. Assume that $[v_{i+k}, v_i]$ is not independent, i.e., there exists an edge between two vertices in $[v_{i+k}, v_i]$, say one of them is x_1. Clearly, x_1 can be chosen such that $x_1 \in (v_{i+k}, v_i)$. Let $P = x_1x_2 \dots x_l$ be a shortest path from x_1 to $[v_i, v_{i+k}]$. Then x_{l-1} is adjacent to $x_l \in (v_i, v_{i+k})$ and a vertex in (v_{i+k}, v_i). Since x_{l-1} is convex, we must either have $x_{l-1}v_i \in E(G)$ or $x_{l-1}v_{i+k} \in E(G)$. However, this contradicts the assumption that $N(v_i) \subseteq (v_i, v_{i+k})$ and $N(v_{i+k}) \subseteq (v_i, v_{i+k})$. Therefore $[v_{i+k}, v_i]$ is independent. Analogously, if $N(v_i) \subseteq (v_{i+k}, v_i)$ and $N(v_{i+k}) \subseteq (v_{i+k}, v_i)$, then we obtain that $[v_i, v_{i+k}]$ is independent.

Suppose now that $N(v_i) \subseteq (v_i, v_{i+k})$ and $N(v_{i+k}) \subseteq (v_{i+k}, v_i)$ (the case when $N(v_i) \subseteq (v_{i+k}, v_i)$ and $N(v_{i+k}) \subseteq (v_i, v_{i+k})$ can be handled analogously). Let $x \in [v_{i+k}, v_i)$ be arbitrary. Observe that x cannot have edges to vertices in both (x, v_i) and in $[v_i, x)$, as this would imply that x is adjacent to v_i, a contradiction. Thus we can partition $[v_{i+k}, v_i)$ into three sets A, L, and R as follows: A consists of vertices x which have a neighbor in $[v_i, v_{i+k})$, L consists of vertices x with $N(x) \subseteq (x, v_i)$, and R consists of vertices x with $N(x) \subseteq [v_{i+k}, x)$. Note that $L \neq \emptyset$ as it contains v_{i+k}. Since G is connected, we have that $A \neq \emptyset$.

Let $a \in A$ be chosen so that $[v_{i+k}, a) \cap A = \emptyset$. We claim that $[a, v_i)$ must be independent. Assume that this is not the case and let $x_1 \in [a, v_i)$ be chosen such that x_1 has an edge to a vertex in (x_1, v_i). Let $P = x_1 x_2 \ldots x_l$ be a shortest path from x_1 to $[v_i, a)$. Clearly, $N(x_{l-1}) \cap (x_{l-1}, v_i) = \emptyset$, as otherwise x_{l-1} would be adjacent to v_i, a contradiction. This implies that $l \geq 3$ and $x_{l-1} \notin L$. If $x_{l-2} \in L$, then x_{l-1} must be adjacent to a, since x_{l-1} is convex, $x_{l-2} \in [a, x_{l-1})$, and $x_l \in [v_i, a)$. However, this implies that a is adjacent to v_i, which is a contradiction. Hence $x_{l-2} \notin L$, which implies that $x_{l-2} \in R$ (the minimality of l ensures $x_{l-2} \notin A$). However, this implies that $N(x_{l-1}) \cap (x_{l-1}, v_i) \neq \emptyset$, a contradiction. Therefore $[a, v_i)$ is independent.

Now suppose that $[v_{i+k}, a) \cap R \neq \emptyset$. Let $r \in [v_{i+k}, a) \cap R$ be chosen such that $[v_{i+k}, r) \subseteq L$. We will show that $[r, a) \subseteq R$. If this is not true, then there is a vertex $l \in [r, a) \cap L$ such that $[r, l) \subseteq R$. Let $P = x_1 x_2 \ldots x_m$ be a shortest path from $r = x_1$ to $[l, v_{i+k})$. As $[v_{i+k}, a) \cap A = \emptyset$ we must have $x_m \in [l, v_i)$, $x_{m-1} \in L$, and $x_{m-2} \in R$. However this implies that x_{m-1} and l is adjacent, a contradiction against $l \in L$. Therefore $[r, a) \subseteq R$.

Let $w \in L \cap [v_{i+k}, a)$ be chosen such that $(w, a) \cap L = \emptyset$. The choice of w, together with the fact that $[a, v_i)$ is independent, implies that (w, v_i) is independent. Using the above arguments, we see that $[v_{i+k}, w]$ is also independent and, furthermore, the vertices in $[v_{i+k}, w]$ can have edges only into (w, v_i).

Repeating all the above arguments for $[v_i, v_{i+k})$, we can analogously obtain a vertex w' (in $[v_i, v_{i+k})$), such that both (w', v_{i+k}) $[v_i, w']$ are independent and vertices in $[v_i, w']$ can have edges only into (w', v_{i+k}). This implies that both $(w, w']$ and $(w', w]$ are independent, which proves G is bipartite. ∎

By inspecting the proof above, we see that the following holds.

COROLLARY 3.8. *Let G be a connected convex-round graph with a convex-round enumeration $\mathcal{L} = v_1, v_2, \ldots, v_n$. Suppose that $v_i v_{i+k} \notin E(G)$. Then the following statements hold:*

1. *If $N(v_i) \subseteq (v_i, v_{i+k})$ and $N(v_{i+k}) \subseteq (v_{i+k}, v_i)$, then G is bipartite.*
2. *If $N(v_i) \subseteq (v_{i+k}, v_i)$ and $N(v_{i+k}) \subseteq (v_i, v_{i+k})$, then G is bipartite.*
3. *If $N(v_i) \subseteq (v_i, v_{i+k})$ and $N(v_{i+k}) \subseteq (v_i, v_{i+k})$, then $[v_{i+k}, v_i]$ is independent.*
4. *If $N(v_i) \subseteq (v_{i+k}, v_i)$ and $N(v_{i+k}) \subseteq (v_{i+k}, v_i)$, then $[v_i, v_{i+k}]$ is independent.*

The following lemma is easy to prove; we leave the details to the reader.

LEMMA 3.9. *Let $C = v_1 v_2 \ldots v_n v_1$ be a chordless cycle with $n \geq 5$.*

- *If $n = 2k$ with $k \geq 3$, then (up to cyclic permutations and full reversal) there is a unique concave-round enumeration $\mathcal{L}$ of C, namely, $\mathcal{L} = v_1, v_2, \ldots, v_{2k}$. Furthermore, there is no convex-round enumeration of C.*
- *If $n = 2k + 1$ with $k \geq 1$, then (up to cyclic permutations and full reversal) there is precisely one concave-round enumeration of C, namely, $\mathcal{L}_1 = v_1, v_2, \ldots, v_{2k+1}$. Furthermore, there is precisely one convex-round enumeration of C, namely, $\mathcal{L}_2 = v_1, v_3, \ldots, v_{2k+1}, v_2, v_4, \ldots, v_{2k}$.*

COROLLARY 3.10. *If a convex-round digraph is not bipartite, then it is connected.*

Proof. Suppose that G is convex-round with a convex-round enumeration $\mathcal{L}$. Assume that G contains an odd cycle $C = v_1 v_2 \ldots v_{2k+1}$ ($k \geq 1$). By Lemma 3.9, the vertices of C must occur in the order $v_1, v_3, \ldots, v_{2k+1}, v_2, v_4, \ldots, v_{2k}$ in $\mathcal{L}$. Let v be a vertex not on C. Then v must be between v_i and v_{i+2} for some $i \leq 2k + 1$ (the addition is modulo $2k + 1$). Since $\mathcal{L}$ is a convex-round enumeration, we see that v is adjacent to v_{i+1}. Hence every vertex of G is adjacent to at least one vertex of C and therefore G is connected. ∎

LEMMA 3.11. *If G is concave-round and $\overline{G}$ is not bipartite, then G is a proper circular arc graph.*

Proof. Let $\mathcal{L} = v_1, v_2, \ldots, v_n$ be a concave-round enumeration of G and let C be an arbitrary odd cycle in $\overline{G}$. Since C is odd it is easy to check that for every edge v_iv_j in G, there is some edge v_kv_r in C such that v_k, v_r are either both in (v_i, v_j), or both in (v_j, v_i). Suppose, without loss of generality, that v_k, v_r are both in (v_j, v_i) and that, according to $\mathcal{L}$, the vertices come in the order v_j, v_k, v_r, v_i. Now apply Lemma 3.7 to $\overline{G}$ and the edge v_iv_j. It follows from the lemma that $[v_i, v_j]$ is an independent set in $\overline{G}$ and hence a clique in G.

By Corollary 3.10, every vertex of G has a nonneighbor. Above we observed that for every edge v_iv_j in G, either $[v_i, v_j]$, or $[v_j, v_i]$ is a clique. Combining these two facts we see that for every edge v_iv_j in G, either v_i, v_j both have a nonneighbor in $[v_i, v_j]$, or they both have a nonneighbor in $[v_j, v_i]$. It follows that G satisfies 1–3 in Lemma 3.4 and hence G is a proper circular arc graph. ∎

4. Hamiltonian cycles in convex-round graphs.

LEMMA 4.1. *The Hamiltonian cycle problem is solvable in linear time for bipartite graphs which are convex-round.*

Proof. Since this follows from a more general result for circular convex bipartite graphs (using a reduction to circular arc graphs) [20], we give here only the main idea which is to reduce the problem to the same problem for interval graphs.

Let G be a connected bipartite graph which is convex-round with a convex-round enumeration $\mathcal{L} = v_1, v_2, \ldots, v_n$. By Lemma 3.6, there is a bipartition (A, B) of G with respect to $\mathcal{L}$. We have $A = [v_i, v_j)$ and $B = [v_j, v_i)$ for some i and j. We assume that $|A| = |B|$ as otherwise G has no Hamiltonian cycles. Let G' be a graph obtained from G by adding edges xy if $x, y \in A$ and $N(x) \cap N(y) \neq \emptyset$. The new graph G' is an interval graph: the interval family which represents G' consists of one interval containing just one point for each $v_k \in B = [v_j, v_i)$, and one interval which represents the neighborhood of v_s for each $v_s \in A = [v_i, v_j)$. It is easy to see that G is Hamiltonian if and only if G' is. Note that, given G and a convex-round enumeration $\mathcal{L}$ of G, we can construct in time $O(n+m)$ the interval representation of G' (here n and m are, respectively, the numbers of vertices and edges of G). Thus the claim follows from the existence of an $O(n)$ algorithm for finding a Hamiltonian cycle (if one exists) in an interval graph with a given interval representation [18]. ∎

THEOREM 4.2. *The Hamiltonian cycle problem is solvable in time $O(n+m)$ for convex-round graphs.*

Proof. Let $\mathcal{L} = v_1, v_2, \ldots, v_n$ be a convex-round enumeration of G. By Corollary 3.2, such an enumeration can be found in time $O(n+m)$. If G is not connected, then G is not Hamiltonian. So assume that G is connected.

If n is odd, then do the following. Let $k = \frac{1}{2}(n-1)$. If $v_iv_{i+k} \in E(G)$ for all $v_i \in V(G)$, then the subgraph containing precisely the edge set $\{v_iv_{i+k} : i = 1, 2, \ldots, 2k+1\}$ is a Hamiltonian cycle. If there is some $v_i \in V(G)$ such that $v_iv_{i+k} \notin E(G)$, then Lemma 3.7 implies that either $\alpha(G) \geq k+1 = \frac{1}{2}(n+1)$ or G is bipartite, which in both cases implies that G is not Hamiltonian.

If n is even, then do the following. Let $k = \frac{1}{2}n$. If there is some $v_i \in V(G)$ such that $v_iv_{i+k-1} \notin E(G)$, then Lemma 3.7 implies that either $[v_i, v_{i+k-1}]$ is independent, or $[v_{i+k-1}, v_i]$ is independent, or G is bipartite with respect to $\mathcal{L}$. If $[v_{i+k-1}, v_i]$ is independent, then G is not Hamiltonian, because $\alpha(G) > \frac{1}{2}n$. If $[v_i, v_{i+k-1}]$ is independent, then by deleting all edges in G with both endpoints in (v_{i+k-1}, v_i) we obtain a new convex-round graph, G'. Now G' is bipartite with respect to $\mathcal{L}$ and, furthermore, G' is Hamiltonian if and only if G is Hamiltonian. By Lemma 4.1, we can decide whether G' is Hamiltonian in time $O(n+m)$, and thereby also whether G is Hamiltonian in time $O(n+m)$. If G is bipartite with respect to $\mathcal{L}$, then by Lemma 4.1 we can decide whether it is Hamiltonian in time $O(n+m)$. Therefore we may assume that $v_iv_{i+k-1} \in E(G)$ for all $v_i \in V(G)$. As G is convex-round, we must have $\{v_{i+k-1}, v_{i+k}, v_{i+k+1}\} \subseteq N(v_i)$. Now we see that G is Hamiltonian: If k is even, then G contains the following Hamiltonian cycle:

$$v_1v_{k+2}v_3v_{k+4}\dots v_{2k}v_kv_{2k-1}v_{k-2}v_{2k-3}\dots v_{k+1}v_1.$$

If k is odd, then G contains the following Hamiltonian cycle:

$$v_1v_{k+2}v_3v_{k+4}v_5v_{k+6}\dots v_kv_{2k}v_{k-1}v_{2k-2}v_{k-3}v_{2k-4}\dots v_2v_{k+1}v_1. \qquad \square$$

Note that there exist highly connected convex-round graphs which contain a factor (a spanning collection of disjoint cycles) but not a Hamiltonian cycle. Such a graph can be obtained from the graph $G = (\{a,b,c,d\}, \{ab,bc,bd,ca\})$ by substituting for each vertex an independent set of size $k > 1$.

5. Maximum matchings in convex-round graphs. The following lemma is due to Glover [12]. For the sake of completeness and since the paper [12] may not be easily accessible, we give a short outline of Glover's (greedy) algorithm.

LEMMA 5.1. *The maximum matching problem is solvable in time $O(n+m)$ for bipartite graphs which are convex-round.* $\square$

Description of Glover's algorithm. Let $G = (A,B,E)$ be a bipartite graph. We may assume that G is connected, as otherwise we consider the components of G separately. According to Lemma 3.6, G has a convex-round enumeration $\mathcal{L} = a_1,\dots a_s, b_1,\dots,b_t$ where A consists of the first s vertices and B consists of the next t vertices. Such an enumeration can be found in time $O(n+m)$ ensured by Theorem 3.2. Now sort the vertices of A according to their highest numbered neighbor in B. The sorting is done in increasing order. Let $a'_1,\dots,a'_s$ denote this order. This takes time $O(n+m)$ since we are sorting numbers in the range $1,2,\dots,t$ (see [25, pp. 127–128]).

Perform the following greedy algorithm $\mathcal{A}$:

1. Let $M = \emptyset$ and let $i = 1$.
2. If a'_i has a neighbor that is not yet matched by M, then let b_j be the smallest numbered nonmatched neighbor of a'_i and let $M := M + a'_ib_j$.
3. If $i < s$, then let $i := i+1$ and goto 2.

Glover proved that the matching M, produced by $\mathcal{A}$, is a maximum matching of G. The algorithm $\mathcal{A}$ can be performed in time $O(n+m)$ and hence the total time to find M is $O(n+m)$ as claimed.

Let G be a convex-round graph with convex-round enumeration $\mathcal{L} = v_1, v_2, \dots, v_n$. An interval $[v_i, v_j]$ is called *potentially independent* if either $N(v_i) \cap [v_i, v_j] = \emptyset$ or $N(v_j) \cap [v_i, v_j] = \emptyset$. In general, a largest potentially independent interval can be found as follows: Treat the convex-round enumeration $\mathcal{L}$ as a circular order. For each vertex x, find the furthest vertex y in clockwise order with $N(x) \cap [x,y] = \emptyset$ and the furthest vertex z in counterclockwise order with $N(x) \cap [z,x] = \emptyset$. Then both $[x,y]$ and $[z,x]$ are potentially independent intervals. Compare the size of all these intervals to obtain a largest potentially independent interval. It is clear that these procedures can be conducted in time $O(n+m)$.

THEOREM 5.2. *The maximum matching problem is solvable in time $O(n+m)$ for convex-round graphs.*

Proof. Let G be a convex-round graph with a convex-round enumeration $\mathcal{L} = v_1, v_2, \ldots, v_n$. We may assume that G is connected, as otherwise we consider the components of G separately. We claim that the following algorithm will find a maximum matching in G in time $O(n+m)$.

Algorithm $\mathcal{B}$:

1. If $v_i v_{i+\lfloor n/2 \rfloor} \in E(G)$ for all $i = 1, 2, \ldots, \lfloor n/2 \rfloor$, then these edges form a matching of size $\lfloor n/2 \rfloor$ which clearly is maximum. Otherwise find a largest potentially independent interval in $\mathcal{L}$. Without loss of generality, let the interval be $[v_1, v_l]$ for some $l \geq 1$. (Since $v_i v_{i+\lfloor n/2 \rfloor} \notin E(G)$ for some i, either $N(v_i) \cap [v_i, v_{i+k}] = \emptyset$ or $N(v_i) \cap [v_{i+k}, v_i] = \emptyset$. This means that either $[v_i, v_{i+\lfloor n/2 \rfloor}]$ or $[v_{i+\lfloor n/2 \rfloor}, v_i]$ is a potentially independent interval. Hence a largest potentially independent interval must contain at least $\lfloor n/2 \rfloor$ vertices and, in particular, $l \geq \lfloor n/2 \rfloor$.)
2. If G is bipartite, then use Lemma 5.1 to find a maximum matching in G and return this.
3. If G is not bipartite, then let G' be the bipartite graph obtained from G by deleting all edges in $[v_{l+1}, v_n]$. (We will show this is bipartite below.) Now find a maximum matching in G' using Lemma 5.1 and return this matching.

The complexity of the algorithm is $O(n+m)$ by Lemma 5.1 and the remarks in step 1 of $\mathcal{B}$. To prove it is correct, it is enough to show that, when G is not bipartite, the graph G' defined in step 3 is in fact bipartite and convex-round and, furthermore, the size of a maximum matching in G equals the size of a maximum matching in G'. So we assume below that G is not bipartite.

It is easy to see that G' is convex-round. Since G is connected, either v_1 or v_l has an edge into $[v_l, v_1]$ and thus $[v_l, v_1]$ is not independent. However, since G is not bipartite, Lemma 3.7 now ensures that $[v_1, v_l]$ must be independent, which implies that G' (in step 3) is bipartite.

It remains to show that the size of a maximum matching in G equals the size of a maximum matching in G'. To do this, we shall construct a maximum matching in G, which is also a matching in G'. Define

$$
\begin{aligned}
j &= \max\{j \ : \ j \leq l \text{ and } v_{l+i} v_i \in E(G) \text{ for all } i = 1, 2, \ldots, j-1\}, \\
k &= \max\{k \ : \ k \leq l \text{ and } v_{n-i} v_{l-i} \in E(G) \text{ for all } i = 0, 1, \ldots, k-1\}.
\end{aligned}
$$

If $l + j \geq n - k + 1$ (i.e., $k - 1 \geq n - l - j$), then we obtain the following matching of size $n - l$, which is in both G and G':

$$\{v_1 v_{l+1}, v_2 v_{l+2}, \ldots, v_{j-1} v_{l+(j-1)}, v_l v_n, v_{l-1} v_{n-1}, v_{l-2} v_{n-2}, \ldots, v_{l-(n-l-j)} v_{n-(n-l-j)}\}.$$

Since $[v_1, v_l]$ is independent, a matching of G contains at most $n - l$ edges and thus the above matching is maximum in G. So assume that $l + j \leq n - k$. We will now make some useful observations.

(i) For all $v_r \in [v_{l+1}, v_n]$, we have $N(v_r) \cap [v_1, v_l] \neq \emptyset$: Assume $N(v_r) \cap [v_1, v_l] = \emptyset$ for some $v_r \in [v_{l+1}, v_n]$. If $N(v_r) \subseteq [v_r, v_n]$, then $[v_1, v_r]$ is a potentially independent interval of size greater than that of $[v_1, v_l]$, a contradiction. If $N(v_r) \subseteq [v_{l+1}, v_r]$, then $[v_r, v_l]$ is a potentially independent interval of size greater than that of $[v_1, v_l]$, a contradiction.

(ii) $j \geq 2$ and $k \geq 1$: As G is not bipartite, there exists an edge $v_a v_b$ in G with $l + 1 \leq a < b \leq n$. By (i), we have that $v_a v_n \in E(G)$ and $v_b v_{l+1} \in E(G)$. Using (i) again, we get that $v_{l+1} v_1 \in E(G)$ and $v_n v_l \in E(G)$. Therefore $j \geq 2$ and $k \geq 1$.

(iii) $N(v_{l+j}) \cap [v_1, v_j] = \emptyset$ and $N(v_{n-k}) \cap [v_{l-k}, v_l] = \emptyset$: The definition of j implies that $v_j v_{l+j} \notin E(G)$. If $N(v_{l+j}) \cap [v_1, v_j] \neq \emptyset$, then $N(v_{l+j}) \cap [v_j, v_{l+j}] = \emptyset$ (as G is convex-round). However, this implies that $[v_j, v_{l+j}]$ is a potentially independent interval of size $l+1$, a contradiction. An analogous argument shows that $N(v_{n-k}) \cap [v_{l-k}, v_l] = \emptyset$.

(iv) $N(v_q) \cap ([v_1, v_{j-1}] \cup [v_{l-k+1}, v_l]) = \emptyset$ for all $v_q \in [v_{l+j}, v_{n-k}]$: Assume that $N(v_q) \cap [v_1, v_{j-1}] \neq \emptyset$ for some $v_q \in [v_{l+j}, v_{n-k}]$ (the case when $N(v_q) \cap [v_{l-k+1}, v_l] \neq \emptyset$ can be handled analogously). If $v_q v_{j-1} \in E(G)$, then $v_{j-1} v_{l+j} \in E(G)$ (as $v_{j-1} v_{l+j-1} \in E(G)$) which contradicts (iii). Therefore $v_q v_{j-1} \notin E(G)$, which implies that $[v_{j-1}, v_q]$ is a potentially independent interval of size $q-(j-1)+1 \geq (l+j)-j+2 = l+2$, a contradiction.

Combining (i), (ii), and (iv), we have that $N(v_q) \subseteq [v_j, v_{l-k}]$ for all $v_q \in [v_{l+j}, v_{n-k}]$ (in particular, $[v_{l+j}, v_{n-k}]$ is independent). Now let M' be a maximum matching in the bipartite graph $G'' = G\langle [v_{l+j}, v_{n-k}] \cup [v_j, v_{l-k}] \rangle$. It is clear that any matching in G can have at most $|M'|$ edges with an endpoint in $[v_{l+j}, v_{n-k}]$ and at most $(j-1)+k$ edges with one endpoint in $[v_{l+1}, v_{l+j-1}] \cup [v_{n-k+1}, v_n]$. Since $[v_1, v_l]$ is independent, the following edges form a maximum matching in G (and G'):

$$M' \cup \{v_1 v_{l+1}, v_2 v_{l+2}, \ldots, v_{j-1} v_{l+(j-1)}, v_l v_n, v_{l-1} v_{n-1}, \ldots, v_{l-k+1} v_{n-k+1}\}. \qquad \square$$

6. Maximum cliques in convex-round graphs. In this section, we shall present a linear time algorithm for solving the maximum clique problem for convex-round graphs. We observe that an $O(n^2)$ algorithm may be obtained by adapting the algorithm in [15], for solving the maximum independent set problem for circular arc graphs. (Although the algorithm in [15] is linear, we obtain only an $O(n^2)$ algorithm, as we have to take the complement of the input graph.) The algorithm below is more efficient and in fact it applies for a larger class of graphs, which is of independent interest.

We say that a graph G is *interval enumerable* if there exists a linear enumeration $\mathcal{I} = v_1, v_2, \ldots, v_n$ of $V(G)$ such that, for each vertex v_i, there exists numbers $1 \leq \ell_i,\ h_i \leq i-1$ such that $N(v_i) \cap\ [v_1, v_{i-1}] = [v_{\ell_i}, v_{h_i}]$, where $\ell_i > h_i$ is used to indicate that v_i has no neighbor v_i with a label less than i. That is, we require that for each v_i the neighbors with labels less than i form an interval. (Note that there is no restriction on the neighbors of v_i with index higher than i.) We shall refer to the enumeration $\mathcal{I}$ as an *interval enumeration* of G.

THEOREM 6.1. *The maximum clique problem is solvable in time $O(n+m)$ for interval enumerable graphs, provided that we are given an interval enumeration of the input graph.*

Proof. Given an interval enumeration $\mathcal{I} = v_1, \ldots, v_n$ of G, we can find all the numbers $\ell_1, \ldots, \ell_n, h_1, \ldots, h_n$ in time $O(n+m)$ just by scanning the neighbors of each vertex once. So assume below that all the numbers $\ell_1, \ldots, \ell_n, h_1, \ldots, h_n$ have been computed.

Now execute the following algorithm $\mathcal{C}$:

```
For i = 1, ..., n do begin
        C(i) := {v_i};
        s(i) := 1;
        p(i) := i;
end;
For i := 1 to n do
        For j := ℓ_i to h_i do
                If p(j) ≤ h_i then begin
```

```
9.                  C(j) := C(j) ∪ {v_i};
10.                 p(j) := i;
11.                 s(j) := s(j) + 1
12.             end;
13. t := 0; (* t denotes the size of a maximum clique found so far *)
14. For i := 1 to n do
15.         If s(i) > t then begin
16.             t := s(i);
17.             q := i
18.         end;
```

It should be clear from the description of the algorithm above that $C(i)$ is the set of vertices of a clique whose lowest (highest) labeled vertex is v_i ($v_{p(i)}$) and whose size is $s(i)$.

We claim that the algorithm $\mathcal{C}$ returns a maximum clique, namely, the one starting at the vertex v_q and whose vertices are stored in the set $C(q)$. In fact, we claim that the clique represented by $C(q)$ is the lexicographically smallest maximum clique[1] according to the ordering of $V(G)$. Suppose this is not the case and let $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq n$, denote the vertices of the lexicographically smallest maximum clique Ω of G. Then $C(q) \neq \Omega$. It is clear from lines 14–18 that the clique $C(q)$ returned by $\mathcal{C}$ has the lowest numbered first vertex v_q among all the $C(i)$ of the same size as $C(q)$. Consider $C(i_1)$ which is one of the possible choices for $\mathcal{C}$ in steps 14–18. By definition, v_{i_1} belongs to $C(i_1)$ and by the assumption that $C(q) \neq \Omega$, there is some $j \leq k$ so that v_{i_j} does not belong to $C(i_1)$, but each of the vertices $v_{i_1}, \ldots, v_{i_{j-1}}$ belong to $C(i_1)$.

Now, the reason v_{i_j} could not be added to $C(i_1)$ in step 9 above must have been that at this point there was already another vertex v_b with $i_{j-1} < b < i_j$ that had been added to $C(i_1)$ after $v_{i_{j-1}}$ with the property that v_{i_j} and v_b are not adjacent. On the other hand, since every v_{i_r}, $j < r \leq k$ is adjacent to $v_{i_{j-1}}$ and to v_{i_j}, it follows that v_{i_r} is also adjacent to v_b. However, this contradicts the fact that Ω is lexicographically smallest, since v_b is adjacent to each of $v_{i_1}, \ldots, v_{i_{j-1}}$ (implying that $\{v_{i_1}, \ldots, v_{i_{j-1}}, v_b, v_{i_{j+1}}, \ldots, v_{i_k}\}$ is also a clique of the same size as Ω). Hence, we have shown that $C(i_1)$ is precisely the clique Ω (it cannot contain more vertices since Ω is maximum) and by the previous remark this is exactly the clique returned by $\mathcal{C}$.

The complexity claim follows from the description of $\mathcal{C}$. Note that the total work of the loop 6–12 is $O(n+m)$, since v_i is adjacent to all vertices in the interval $[v_{\ell_i}, v_{h_i}]$ and hence scanning that interval in the loop 7–12 corresponds to scanning the neighbors of v_i once. □

If a graph is convex-round with convex-round enumeration $v_1, v_2, \ldots, v_n$, then the same enumeration shows that G is interval enumerable. Thus the class of convex-round graphs form a subclass of the interval enumerable graphs. (The inclusion is proper, since the graph obtained from an induced cycle of length 5 by adding any number of new vertices adjacent to all 5 vertices on the cycle is interval enumerable but not convex-round.) Hence we have the following corollary.

COROLLARY 6.2. *The maximum clique problem is solvable in time $O(n+m)$ for convex-round graphs.*

We remark that the complement of an interval enumerable graph may not be a circular arc graph. An example of such a graph is the 5-cycle C with two extra vertices joined to all 5 vertices on C.

[1] We say that an ordered set of vertices $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ is *lexicographically smaller* than another ordered set $\{v_{j_1}, v_{j_2}, \ldots, v_{j_r}\}$ if either $i_s = j_s$ for $s = 1, 2, \ldots, k$, or there exist p such that $i_q = j_q$ for $1 \leq q \leq p-1$ and $i_p < j_p$.

7. Chromatic number of concave-round and convex-round graphs. Recall that the *chromatic number* $\chi(G)$ is the least number of sets in a partition $V(G) = V_1 \cup \cdots \cup V_t$ such that each of the graphs $G\langle V_i \rangle$ has no edges.

Note that we can have that $\chi(G) > \omega(G)$ for convex-round graphs. The complement of any odd cycle is such an example and, in general, complements of powers of Hamiltonian cycles (on an odd number of vertices) provide such examples. For each of these examples we have $\chi(G) = \omega(G) + 1$. Through the correspondence between convex-round and concave-round graphs (by taking complements) and the fact that concave-round graphs are circular arc graphs, we obtain the following results. (Note again that the $O(n^2)$ algorithm is the best we can get because we need to consider the complement of the input graph.)

THEOREM 7.1 (see [15]). *The chromatic number of a convex-round graph can be found in time $O(n^2)$.*

Translating the proof of [15, Theorem 3.2] we get that the chromatic number of a convex-round graph is closely related to the size of a largest clique.

THEOREM 7.2 (see [15]). *For every convex-round graph G: $\chi(G) \leq \omega(G) + 1$.*

There exist concave-round graphs for which $\chi(G) = \lceil \frac{3}{2}\omega(G) \rceil$. Namely, $\chi(C_{3i-1}^{i-1}) = \lceil \frac{3i-1}{2} \rceil$ and $\omega(C_{3i-1}^{i-1}) = \lceil \frac{3i-1}{3} \rceil$. Here C_k^r denotes the rth power of a k-cycle. It follows from this example that the bound in the following theorem is best possible even for concave-round graphs.

THEOREM 7.3 (see [17]). *For every circular arc graph G, $\chi(G) \leq \lceil \frac{3}{2}\omega(G) \rceil$.*

THEOREM 7.4. *For concave-round graphs, the minimum coloring problem is solvable in time $O(n^2)$.*

Proof. We can construct $\overline{G}$ in time $O(n^2)$ and then check in linear time whether $\overline{G}$ is bipartite. Suppose first that $\overline{G}$ is not bipartite. Using the linear algorithm from [9] for recognizing and representing proper circular arc graphs we obtain a representation of G by inclusion-free circular arcs. Now we apply the $O(n^{1.5})$ algorithm of [26] for coloring a proper circular arc graph to find an optimal coloring of G.

Suppose now that $\overline{G}$ is bipartite with bipartition (A, B), where $|A| \geq |B|$. It is not difficult to check that $\chi(G) = n - k$, where k is the size of a maximum matching in $\overline{G}$ (recall that A and B induce cliques in G and hence all vertices inside these sets must receive different colors). Second, if we are given a maximum matching M of $\overline{G}$, then we can construct an optimal coloring with $n - |M|$ colors in linear time as follows: Color each vertex in A with its own color. Let $M = a_1b_1, \ldots, a_kb_k$. Color b_i with the color of a_i for $i = 1, 2, \ldots, k$. Finally give each vertex in $B - \{b_1, \ldots, b_k\}$ a new color. By Lemma 5.1, we can find a maximum matching in $\overline{G}$ in linear time and hence the whole algorithm is $O(n^2)$. □

Recall that the minimum coloring problem is NP-complete for general circular arc graphs [11] and polynomially solvable for proper circular arc graphs [3, 26].

8. Conjectures and open problems.

CONJECTURE 8.1. *For concave-round graphs, the maximum clique problem, the maximum matching problem, and the Hamiltonian cycle problem can all be solved in time $O(n + m)$.*

PROBLEM 8.2. *Find a characterization in terms of forbidden subgraphs of concave-round (convex-round) graphs.*

CONJECTURE 8.3. *Interval enumerable graphs can be recognized in polynomial time.*

CONJECTURE 8.4. *The Hamiltonian cycle problem is solvable in polynomial time for interval enumerable graphs.*

Acknowledgments. We would like to thank Stephan Brandt for stimulating discussions, in particular, for his insight which lead to the results on concave-round graphs in section 7. We would also like to thank the anonymous referees for several remarks that have improved on the presentation. The third author also wishes to acknowledge the hospitality of the Department of Mathematics and Statistics, University of Victoria, where part of this research was carried out.

REFERENCES

[1] A. Appostolico and S.E. Hambrush, *Finding maximum cliques on circular arc graphs*, Inform. Process. Lett., 26 (1987), pp. 209–215.

[2] J. Bang-Jensen, *Locally semicomplete digraphs: A generalization of tournaments*, J. Graph Theory, 14 (1990), pp. 371–390.

[3] B. Bhattacharya, P. Hell, and J. Huang, *A linear algorithm for maximum weight cliques in proper circular arc graphs*, SIAM J. Discrete Math., 9 (1996), pp. 274–289.

[4] B.K. Bhattacharya and D. Kaller, *An $O(m + n \log n)$ algorithm for the maximum clique problem in circular arc graphs*, J. Algorithms, 35 (1997), pp. 336–358.

[5] J.A. Bondy and U.S.R. Murty, *Graph Theory and Applications*, American Elsevier, New York, 1976.

[6] K.S. Booth and G.S. Lueker, *Testing for the consecutive ones property using PQ-tree algorithm*, J. Comput. System Sci., 13 (1976), pp. 335–379.

[7] A. Czumaj, K. Diks, and T.M. Przytycka, *Parallel maximum independent set in convex bipartite graphs*, Inform. Process. Lett., 59 (1996), pp. 289–294.

[8] P. Damaschke, H. Müller, and D. Kratch, *Domination in convex and chordal bipartite graphs*, Inform. Process. Lett., 36 (1990), pp. 231–236.

[9] X. Deng, P. Hell, and J. Huang, *Linear time representation for proper circular-arc graphs and proper interval graphs*, SIAM J. Comput., 25 (1996), pp. 390–403.

[10] E.M. Eschen and J. Spinrad, *$O(n^2)$ recognition and isomorphism algorithms for circular arc graphs*, in Proceedings of the Fourth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), Austin, TX, SIAM, Philadelphia, 1993, pp. 128–137.

[11] M.R. Garey, D.S. Johnson, G.L. Miller, and C.H. Papadimitriou, *The complexity of coloring circular arcs and chords*, SIAM J. Alg. Discrete Methods, 1 (1980), pp. 216–227.

[12] F. Glover, *Maximum matching in convex bipartite graphs*, Naval. Res. Logist., 14 (1967), pp. 313–316.

[13] M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.

[14] P. Hell, J. Bang-Jensen, and J. Huang, *Local tournaments and proper circular arc graphs*, in Algorithms, Lecture Notes in Comput. Sci. 450, T. Asano, T. Ibaraki, H. Imai, and T. Nishizeki, eds., Springer, Berlin, 1990, pp. 101–108.

[15] W.L. Hsu and K.H. Tsai, *Linear time algorithms on circular arc graphs*, Inform. Process. Lett., 40 (1991), pp. 123–129.

[16] J. Huang, *On the structure of local tournaments*, J. Combin. Theory Ser. B, 63 (1995), pp. 200–221.

[17] I.A. Karapetjan, *Coloring of arc graphs*, Akad. Nauk Armyan. SSR Dokl., 70 (1980), pp. 306–311.

[18] J.M. Keil, *Finding Hamiltonian circuits in interval graphs*, Inform. Process. Lett., 20 (1985), pp. 201–206.

[19] N. Korte and R.H. Möhring, *An incremental linear-time algorithm for recognizing interval graphs*, SIAM J. Comput., 18 (1989), pp. 68–81.

[20] Y.D. Liang and N. Blum, *Circular convex bipartite graphs: Maximum matching and Hamiltonian circuits*, Inform. Process. Lett., 56 (1995), pp. 215–219.

[21] Y.D. Liang, G.K. Manacher, C. Rhee, and T.A. Mankus, *A linear algorithm for Hamiltonian circuits in circular arc graphs*, in Proceedings of the Southeast ACM Conference, Tuscaloosa, AL, 1994, pp. 15–22.

[22] Y.D. Liang and Maw-Shang Chang, *Minimum feedback vertex sets in cocomparability graphs and convex bipartite graphs*, Acta Inform., 34 (1997), pp. 337–346.

[23] Y.D. Liang and C. Rhee, *Finding a maximum matching in a circular-arc graph*, Inform. Process. Lett., 45 (1993), pp. 185–190.

[24] W. Lipski and F.P. Preparata, *Efficient algorithms for finding a maximum matching in convex bipartite graphs and related problems*, Acta Inform., 15 (1981), pp. 329–346.

[25] U. Manber, *Introduction to Algorithms*, Addison-Wesley, Reading, MA, 1989.
[26] W.K. Shih and W.L. Hsu, *An $O(n^{1.5})$ algorithm to color proper circular arcs*, Discrete Appl. Math., 25 (1989), pp. 321–323.
[27] W.K. Shih, T. C. Chern, and W.L. Hsu, *An $O(n^2 \log n)$ algorithm for the Hamiltonian cycle problem on circular-arc graphs*, SIAM J. Comput., 21 (1992), pp. 1026–1046.
[28] D.J. Skrien, *A relationship between triangulated graphs, comparability graphs, proper interval graphs, proper circular arc graphs, and nested interval graphs*, J. Graph Theory, 6 (1982), pp. 309–316.
[29] G. Steiner and J.S. Yeomans, *A linear time algorithm for maximum matchings in convex bipartite graphs*, Comput. Math. Appl., 31 (1996), pp. 91–96.
[30] K.E. Stouffers, *Scheduling of traffic lights–A new approach*, Transportation Res., 2 (1968), pp. 199–234.
[31] A. Tucker, *Matrix characterizations of circular-arc graphs*, Pacific J. Math., 39 (1971), pp. 535–545.
[32] C.W. Yu and G.H. Chen, *Efficient parallel algorithms for doubly convex-bipartite graphs*, Theoret. Comput. Sci., 147 (1995), pp. 249–265.