Eigenfunctions and matrix elements for a class of eigenvalue problems with staggered ladder spectra

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We present an alternate solution to an eigenvalue problem which arises in the study of the Fokker-Planck equation for generalized Ornstein-Uhlenbeck processes. We obtain the staggered ladder spectra found previously but in addition we obtain the normalized eigenfunctions in terms of associated Laguerre polynomials. The representation of the eigenfunctions in this form greatly simplifies the evaluation of matrix elements required in calculating ensemble averages and correlation coefficients for various observables. The solution to the eigenvalue problem is given in the generic and general cases.

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In a recent paper [1] a generalized Ornstein-Uhlenbeck process was considered in which a particle in one dimension is subject to damping force and a random force $f(x,t)$ which depends on both the position of the particle x and the time t. This stochastic system can be shown to lead to a Fokker-Planck equation or a generalized diffusion equation for the probability density of the momentum $P(p,t)$ in which the momentum diffusion constant $D(p)$ is $\sim|p|^{-1}$. The general form of the Fokker-Planck equation for P is

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial p}\left(\gamma p+D(p)\frac{\partial}{\partial p}\right)P, \tag{1}$$

where γ is the damping term. This equation had been considered previously in, for example, modeling stochastic acceleration without damping [2].

For the case $D(p)=C|p|^{-1}$, where C is a constant, the above equation can be written in scaled variables t' and z as

$$\frac{\partial P}{\partial t'}=\frac{\partial}{\partial z}\left(z+\frac{1}{|z|}\frac{\partial}{\partial z}\right)P\equiv\hat{F}P. \tag{2}$$

To solve this partial differential equation Arvedsen *et al.* [1] calculate its kernel by first transforming the Fokker-Planck operator to a Hermitian form

$$\hat{H}=P_0^{-1/2}\hat{F}P_0^{1/2}=\frac{d}{dz}\frac{1}{|z|}\frac{d}{dz}+\frac{1}{2}-\frac{|z|^3}{4}, \tag{3}$$

where $P_0(z)\propto\exp\left(-\frac{|z|^3}{3}\right)$, and by then performing a spectral decomposition of this operator. All of the relevant moments and correlation coefficients can then be expressed in terms of the eigenvalues and eigenfunctions of the above operator. To find the eigenvalues and eigenfunctions Arvedsen *et al.* defined raising and lowering operators which are nonlinear second order differential operators. This approach leads to a rather complicated way of determining the necessary matrix elements central to evaluating the correlation coefficients. For example, to obtain the matrix element $Z_{mn}=\langle\psi_m^+|\hat{z}|\psi_n^-\rangle$ where ψ_m^+ and ψ_n^- are the even and odd eigenfunctions, respectively, one has to first evaluate Z_{0n}, then Z_{mn} for $m\leq n$, and finally Z_{mn} for $m>n$ [3]. In addition selection rules are not obvious in this procedure.

In this paper we present an alternative solution to the eigenvalue equation and in so doing we obtain explicit expressions for the eigenfunctions of the above operator in terms of associated Laguerre polynomials. We then show how the known properties of these polynomials can be used to evaluate matrix elements such as Z_{mn} above and any other matrix elements in a very straightforward way. The normalized eigenfunctions are given in terms of Laguerre polynomials for the case $D(p)\sim|p|^{-1}$ and the general case $D(p)\sim|p|^{-\alpha}$ where $\alpha>0$.

The eigenvalue problem for the operator $\hat{H}$ is

$$\frac{d}{dz}\frac{1}{|z|}\frac{d\psi}{dz}+\left(\frac{1}{2}-\frac{|z|^3}{4}\right)\psi=\lambda\psi \tag{4}$$

with $\psi\to 0$ as $|z|\to\infty$. The operator $\hat{H}$ commutes with the parity operator and so the eigenfunctions can be divided into even or odd with respect to $z\to -z$.

Consider Eq. (4) for $z>0$. [The above equation, in this case, can be related to the Schrödinger equation for a particle whose mass increases linearly with distance while moving in a binding potential $V(z)$.] As $z\to\infty$, the equation becomes

$$\frac{d^2\psi}{dz^2}-\frac{z^4}{4}\psi\approx 0 \tag{5}$$

and $\psi\sim\exp(-z^3/6)$ so a natural substitution to try is $x=z^3$.

On substituting one gets

$$9x\frac{d^2\psi}{dx^2}+3\frac{d\psi}{dx}-\left(\frac{x}{4}+\lambda-\frac{1}{2}\right)\psi=0 \tag{6}$$

with the boundary condition $\psi\to 0$ as $x\to\infty$. As $x\to\infty$

$$\frac{d^2\psi}{dx^2}-\frac{1}{36}\psi\approx 0 \tag{7}$$

so the asymptotic form obeying the boundary condition is $\psi\sim\exp(-x/6)$. Factoring out the asymptotic behavior by letting $\psi(x)=g(x)\exp\left(-\frac{x}{6}\right)$, we obtain an equation for $g(x)$,

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$$x\frac{d^2g}{dx^2}+\left(\frac{1}{3}-\frac{1}{3}x\right)\frac{dg}{dx}-\frac{\lambda}{9}g=0. \tag{8}$$

This is the Kummer or confluent hypergeometric equation in the variable $x/3$ and the general solution can be written as [4]

$$g(x)=A\,{}_1F_1\left(\frac{\lambda}{3},\frac{1}{3},\frac{x}{3}\right)+Bx^{2/3}{}_1F_1\left(\frac{2}{3}+\frac{\lambda}{3},\frac{5}{3},\frac{x}{3}\right), \tag{9}$$

where A and B are arbitrary constants and ${}_1F_1$ is the confluent hypergeometric function. From Eq. (9) above, the even and odd solutions for all x can be written down directly as

$$\psi^+(x)=N^+{}_1F_1\left(\frac{\lambda}{3},\frac{1}{3},\frac{|x|}{3}\right)\exp\left(-\frac{|x|}{6}\right) \tag{10}$$

and

$$\psi^-(x)=N^-|x|^{2/3}{}_1F_1\left(\frac{2}{3}+\frac{\lambda}{3},\frac{5}{3},\frac{|x|}{3}\right)\exp\left(-\frac{|x|}{6}\right), \tag{11}$$

for $x\geq 0$ and $-\psi^-$ for $x<0$, which guarantees continuity at $x=0$. $N^\pm$ are normalization constants.

Now since ${}_1F_1\sim\exp(x/3)$ as $x\to\infty$ the only way the boundary condition $\psi\to 0$ as $|x|\to\infty$ is satisfied is if ${}_1F_1$ is a polynomial. This implies the first argument of ${}_1F_1$ must be zero or a negative integer. This gives for the even case

$$\frac{\lambda_n^+}{3}=-n\Rightarrow\lambda_n^+=-3n,\quad n=0,1,2,\ldots, \tag{12}$$

which gives the even eigenvalues and the odd eigenvalues are given by

$$\frac{2}{3}+\frac{\lambda_n^-}{3}=-n\Rightarrow\lambda_n^-=-3n-2,\quad n=0,1,2,\ldots. \tag{13}$$

The even and odd eigenvalues separately are evenly spaced like those of a harmonic oscillator but they are shifted relative to one another giving the staggered ladder spectra found in Refs. [1,3].

From Eqs. (10) and (11) we can write explicit expressions for the eigenfunctions in terms of associated Laguerre polynomials. For example, the even eigenfunctions are

$$\psi_n^+=N_n^+{}_1F_1\left(-n,\frac{1}{3},\frac{|x|}{3}\right)\exp\left(-\frac{|x|}{6}\right)$$
$$=N_n^+\frac{\Gamma\left(\frac{1}{3}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{1}{3}\right)}L_n^{-2/3}\left(\frac{|x|}{3}\right)\exp\left(-\frac{|x|}{6}\right). \tag{14}$$

Normalizing the eigenfunction to one, i.e., requiring

$$\int_{-\infty}^{\infty}\psi_n^{+2}dz=\frac{2}{3}\int_0^{\infty}x^{-2/3}\psi_n^{+2}(x)dx=1 \tag{15}$$

and using the orthogonality of the associated Laguerre polynomials over a weight function, namely [5],

$$\int_0^{\infty}e^{-x}x^{\alpha}L_n^{\alpha}L_m^{\alpha}dx=\frac{\Gamma(\alpha+n+1)}{\Gamma(n+1)}\delta_{nm}, \tag{16}$$

gives

$$N_n^+=\frac{3^{1/3}}{\Gamma\left(\frac{1}{3}\right)}\sqrt{\frac{1}{2}\frac{\Gamma\left(n+\frac{1}{3}\right)}{\Gamma(n+1)}}. \tag{17}$$

From Eq. (14), the normalized even eigenfunctions are then

$$\psi_n^+=3^{1/3}\sqrt{\frac{1}{2}\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{3}\right)}}L_n^{-2/3}\left(\frac{|x|}{3}\right)\exp\left(-\frac{|x|}{6}\right) \tag{18}$$

with $n=0,1,2,\ldots$. Similarly the normalized odd eigenfunctions are

$$\psi_n^-=3^{-1/3}\sqrt{\frac{1}{2}\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{5}{3}\right)}}|x|^{2/3}L_n^{2/3}\left(\frac{|x|}{3}\right)\exp\left(-\frac{|x|}{6}\right) \tag{19}$$

for $x\geq 0$ and $-\psi_n^-$ for $x<0$ with $n=0,1,2,\ldots$, and $x=z^3$.

The solutions of the Fokker-Planck equation (2) can be written in terms of the propagator for the equation which in turn can be written exactly in terms of the eigenvalues and eigenfunctions above. Equilibrium correlation coefficients for the ensemble average of an observable $O(z)$ can then be written as

$$\langle O(z_0)O(z_{t'})\rangle=\sum_{n\sigma}|\langle\psi_0^+|\hat{O}|\psi_n^\sigma\rangle|^2\exp(\lambda_n^\sigma t'). \tag{20}$$

For example, it is shown in Arvedson *et al.* [1] that in order to evaluate $\langle x^2(t)\rangle$ one needs to calculate the correlation coefficient

$$\langle z_{t_2'}z_{t_1'}\rangle=\sum_{n,m}\frac{\psi_m^+(0)}{\psi_0^+(0)}\langle\psi_0^+|\hat{z}|\psi_n^-\rangle\langle\psi_n^-|\hat{z}|\psi_m^+\rangle\exp[\lambda_n^-(t_2'-t_1')+\lambda_m^+t_1']. \tag{21}$$

All of these expressions for the correlation coefficients require the matrix elements of an operator for the given set of eigenfunctions. It is very straightforward to evaluate these expressions using the eigenfunctions when they are represented explicitly in terms of associated Laguerre polynomials due to their well known properties such as recurrence relations, etc. The matrix element $\langle\psi_m^+|\hat{z}|\psi_n^-\rangle$ is evaluated below to illustrate this point.

We note that the expression $\frac{\psi_n^+(0)}{\psi_0^+(0)}$ in Eq. (21) can be read off immediately from Eq. (17) since

$$\frac{\psi_n^+(0)}{\psi_0^+(0)} = \frac{N_n^+}{N_0^+} = \sqrt{\frac{\Gamma\left(n+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma(n+1)}} \tag{22}$$

which is identical to that obtained in Ref. [1] up to an arbitrary phase factor. [Note that $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})=\frac{2\pi}{\sqrt{3}}$.]

The matrix element $Z_{mn}=\langle\psi_m^+|\hat{z}|\psi_n^-\rangle$ can be written using Eqs. (18) and (19) as follows:

$$\begin{aligned} Z_{mn} &= \int_{-\infty}^{\infty}\psi_m^+ z\psi_n^- dz = \frac{2}{3}\int_0^{\infty}\psi_m^+ x^{-1/3}\psi_n^- dx \\ &= \frac{1}{3}\sqrt{\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{5}{3}\right)}}\sqrt{\frac{\Gamma(m+1)}{\Gamma\left(m+\frac{1}{3}\right)}} \\ &\quad\times\int_0^{\infty} e^{-x/3}x^{1/3}L_m^{-2/3}\left(\frac{x}{3}\right)L_n^{2/3}\left(\frac{x}{3}\right)dx. \end{aligned} \tag{23}$$

The integral in the equation can be evaluated using well known properties of the Laguerre polynomials [5]. First, letting $u=x/3$, the integral becomes

$$\begin{aligned} I_{mn} &= \int_0^{\infty} e^{-x/3}x^{1/3}L_m^{-2/3}\left(\frac{x}{3}\right)L_n^{2/3}\left(\frac{x}{3}\right)dx \\ &= 3^{4/3}\int_0^{\infty} e^{-u}u^{-2/3}L_m^{-2/3}(u)uL_n^{2/3}(u)du \end{aligned} \tag{24}$$

and using the recurrence relation for the Laguerre polynomials [5], $uL_n^{2/3}=\left(n+\frac{2}{3}\right)L_n^{-1/3}-(n+1)L_{n+1}^{-1/3}$, we get

$$I_{mn} = 3^{4/3}\int_0^{\infty} e^{-u}u^{-2/3}L_m^{-2/3}\left[\left(n+\frac{2}{3}\right)L_n^{-1/3}-(n+1)L_{n+1}^{-1/3}\right]du. \tag{25}$$

Using the relationship between Laguerre polynomials of different order [5]

$$L_n^{\alpha} = \sum_{k=0}^{n}\frac{\Gamma(\alpha-\beta+k)}{\Gamma(\alpha-\beta)\Gamma(k+1)}L_{n-k}^{\beta}, \tag{26}$$

we get

$$\begin{aligned} I_{mn} = 3^{4/3}\Bigg[&\left(n+\frac{2}{3}\right)\sum_{k=0}^{n}\frac{\Gamma\left(k+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma(k+1)} \\ &\times\int_0^{\infty}e^{-u}u^{-2/3}L_m^{-2/3}L_{n-k}^{-2/3}du-(n+1) \\ &\times\sum_{k=0}^{n+1}\frac{\Gamma\left(k+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma(k+1)}\int_0^{\infty}e^{-u}u^{-2/3}L_m^{-2/3}L_{n+1-k}^{-2/3}du\Bigg] \end{aligned} \tag{27}$$

which from the orthonormality relation (16) gives

$$\begin{aligned} I_{mn} = 3^{4/3}\Bigg[&\left(n+\frac{2}{3}\right)\sum_{k=0}^{n}\frac{\Gamma\left(k+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma(k+1)}\left(\frac{\Gamma\left(m+\frac{1}{3}\right)}{\Gamma(m+1)}\delta_{mn-k}\right) \\ &-(n+1)\sum_{k=0}^{n+1}\frac{\Gamma\left(k+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma(k+1)}\left(\frac{\Gamma\left(m+\frac{1}{3}\right)}{\Gamma(m+1)}\delta_{mn+1-k}\right)\Bigg]. \end{aligned} \tag{28}$$

We get immediately the selection rule that $I_{mn}=0$ and hence $Z_{mn}=0$ unless $m\leq n+1$. The sums have just one term and we get

$$\begin{aligned} I_{mn} = 3^{4/3}\Bigg\{&\left(n+\frac{2}{3}\right)\frac{\Gamma\left(n-m+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma(n-m+1)}\frac{\Gamma\left(m+\frac{1}{3}\right)}{\Gamma(m+1)} \\ &-(n+1)\frac{\Gamma\left(n-m+\frac{4}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma(n-m+2)}\frac{\Gamma\left(m+\frac{1}{3}\right)}{\Gamma(m+1)}\Bigg\} \end{aligned} \tag{29}$$

or

$$I_{mn} = 3^{1/3}\frac{(n+m+1)}{\Gamma\left(\frac{1}{3}\right)}\frac{\Gamma\left(m+\frac{1}{3}\right)}{\Gamma(m+1)}\frac{\Gamma\left(n-m+\frac{1}{3}\right)}{\Gamma(n-m+2)}. \tag{30}$$

Finally substituting for I_{mn} in Eq. (23) the matrix element Z_{mn} is

$$\begin{aligned} Z_{mn} = 3^{-2/3}&\frac{(n+m+1)}{\Gamma\left(\frac{1}{3}\right)} \\ &\times\sqrt{\frac{\Gamma(n+1)}{\Gamma(m+1)}\frac{\Gamma\left(m+\frac{1}{3}\right)}{\Gamma\left(n+\frac{5}{3}\right)}\frac{\Gamma\left(n-m+\frac{1}{3}\right)}{\Gamma(n-m+2)}} \end{aligned} \tag{31}$$

for $m\leq n+1$ and zero otherwise.

The above can be generalized in a straightforward way to the case when the momentum diffusion constant in Eq. (2) $D(p)$ has a more general form $D(p)\sim|p|^{-\alpha}$, where $\alpha>0$ [3]. After transforming the Fokker-Planck equation to its Hermitian form, the eigenvalue problem in the general case becomes

$$\frac{d}{dz}\frac{1}{|z|^{\alpha}}\frac{d\psi}{dz}+\left(\frac{1}{2}-\frac{|z|^{2+\alpha}}{4}\right)\psi=\lambda\psi. \tag{32}$$

The solution proceeds in a similar way to the above and the results are summarized below. Taking $x=z^{2+\alpha}$, for $x>0$ the equation becomes

$$(2+\alpha)^2 x\frac{d^2\psi}{dx^2}+(2+\alpha)\frac{d\psi}{dx}-\left(\frac{x}{4}+\lambda-\frac{1}{2}\right)\psi=0. \quad (33)$$

Letting $\psi(x)=g(x)\exp\left(-\frac{x}{2(2+\alpha)}\right)$ the equation for $g(x)$ is

$$x\frac{d^2g}{dx^2}+\left(\frac{1}{(2+\alpha)}-\frac{1}{(2+\alpha)x}\right)\frac{dg}{dx}-\frac{\lambda}{(2+\alpha)^2}g=0. \quad (34)$$

The general solution for $x>0$ can be written as

$$g(x)=A\,{}_1F_1\left(\frac{\lambda}{2+\alpha},\frac{1}{2+\alpha},\frac{x}{2+\alpha}\right)+Bx^{1-1/(2+\alpha)}\,{}_1F_1\left(\frac{\lambda-1}{2+\alpha}+1,2-\frac{1}{2+\alpha},\frac{x}{2+\alpha}\right), \quad (35)$$

where A and B are arbitrary constants. The even and odd solutions can thus be written down directly as

$$\psi^+(x)=N^+\,{}_1F_1\left(\frac{\lambda}{2+\alpha},\frac{1}{2+\alpha},\frac{|x|}{2+\alpha}\right)\exp\left(-\frac{|x|}{2(2+\alpha)}\right), \quad (36)$$

$$\psi_n^-(x)=N^-|x|^{1-1/(2+\alpha)}\,{}_1F_1\left(\frac{\lambda-1}{2+\alpha}-1,2-\frac{1}{2+\alpha},\frac{|x|}{2+\alpha}\right)\times\exp\left(-\frac{|x|}{2(2+\alpha)}\right) \quad (37)$$

for $x\geq 0$ and $-\psi_n^-$ for $x<0$ with $n=0,1,2,\ldots$, and $x=z^{2+\alpha}$.

The even eigenvalues are given by

$$\lambda_n^+=-(2+\alpha)n, \quad n=0,1,2,\ldots \quad (38)$$

and the odd eigenvalues by

$$\lambda_n^-=-(2+\alpha)n-\alpha-1, \quad n=0,1,2,\ldots\,. \quad (39)$$

The eigenfunctions can be written in terms of associated Laguerre polynomials. For example the even eigenfunctions are

$$\psi_n^+=N_n^+\,{}_1F_1\left(-n,\frac{1}{2+\alpha},\frac{x}{2+\alpha}\right)\exp\left(-\frac{|x|}{2(2+\alpha)}\right)=N_n^+\frac{\Gamma\left(\frac{1}{2+\alpha}\right)\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2+\alpha}\right)}L_n^{1/(2+\alpha)-1}\left(\frac{|x|}{2+\alpha}\right)\times\exp\left(-\frac{|x|}{2(2+\alpha)}\right). \quad (40)$$

Normalizing gives

$$N_n^+=\frac{(2+\alpha)^{(\alpha+1)/(2\alpha+4)}}{\Gamma\left(\frac{1}{2+\alpha}\right)}\sqrt{\frac{1}{2}\frac{\Gamma\left(n+\frac{1}{2+\alpha}\right)}{\Gamma(n+1)}}, \quad (41)$$

and the normalized even eigenfunctions are

$$\psi_n^+=(2+\alpha)^{(\alpha+1)/(2\alpha+4)}\times\sqrt{\frac{1}{2}\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2+\alpha}\right)}}L_n^{1/(2+\alpha)-1}\left(\frac{|x|}{2+\alpha}\right)\times\exp\left(-\frac{|x|}{2(2+\alpha)}\right), \quad (42)$$

$n=0,1,2,\ldots$. Similarly the normalized odd eigenfunctions are

$$\psi_n^-=(2+\alpha)^{-(\alpha+1)/(2\alpha+4)}\times\sqrt{\frac{1}{2}\frac{\Gamma(n+1)}{\Gamma\left(n+2-\frac{1}{2+\alpha}\right)}}|x|^{1-1/(2+\alpha)}L_n^{1-1/(2+\alpha)}\left(\frac{|x|}{2+\alpha}\right)\times\exp\left(-\frac{|x|}{2(2+\alpha)}\right) \quad (43)$$

for $x\geq 0$ and $-\psi_n^-$ for $x<0$. All the required matrix elements, such as Z_{mn}, can be obtained from the properties of the associated Laguerre polynomials.

In summary we have presented an alternate solution to an eigenvalue problem which arises from the Fokker-Planck equation in the analysis of generalized Ornstein-Uhlenbeck processes. The solution allows one to write down explicitly the normalized eigenfunctions in terms of associated Laguerre polynomials. This greatly facilitates the calculation of matrix elements required in the computation of ensemble averages and correlation coefficients. The eigenvalues and the normalized eigenfunctions are obtained for the generic case where the momentum diffusion constant $D(p)\sim|p|^{-1}$ and the general case $D(p)\sim|p|^{-\alpha}$ where $\alpha>0$.

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