SOME PROPERTIES OF HAUSDORFF MEASURE THEORY

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SUMMARY

CHAPTER I

The definition of all the measure functions used in the thesis.

CHAPTER II

The condition for a measure function to be a Hausdorff diametral dimension function in p-dimensional real Euclidean space is first established. Then the fact that an analytical set of infinite Hausdorff diametral measure is then proved and the necessary and sufficient conditions for a subset of a set with Hausdorff diametral dimension function h(x) to have dimension function g(x) are established.

CHAPTER III

Conditions on the dimension function of the cartesian product of two one-dimensional sets whose dimension functions are known, are established.

CHAPTER IV

The proof of the existence of a plane set S with Hausdorff diametral dimension function $x^{2\alpha}$ $\alpha = \frac{\log 2}{\log 3}$, such that if S is translated through any distance in the plane then the intersection of S with itself translated has zero Hausdorff diametral measure with dimension function $x^{2\alpha}$.

CHAPTER V

The two area measures are considered in two dimensional real Euclidean space only. The necessary and sufficient condition for a measure function to be a non-metric-area

dimension function is established and the metric area measure of sets which are the cartesian products of intervals with linear sets is found. These are used to deduce that non-metric-area measure is in fact non-metric. The condition for x^{α} to be a metric area measure is also established.

CHAPTER VI

This deals with sets on the frontier of the unit circle. First the connection between the area measures and the generalized affine length is established. Then the triangle of minimum area covering a given total arc length is found and finally the necessary and sufficient condition for a measure function to be a Hausdorff diametral dimension function for such sets is found.

CHAPTER I

SECTION I, 1.

NOTATION

The following is the general scheme of notation used throughout the thesis:

The letter S represents a set and the small letter s represents a point belonging to S.

Other letters used to represent sets are F, G, E, D and H.

J is used to represent a cube and $\mathcal{J}$ represents a class of cubes J.

V is used to represent an open set and $\mathcal{V}$ represents a sequence of sets V.

The following letters are used to denote coverings of a given set:

$\mathcal{U}$ denotes a covering of convex sets U.

$\mathcal{I}$ denotes a covering of p-dimensional intervals I.

$\mathcal{R}$ denotes a covering of rectangles R.

$\mathcal{P}$ denotes a covering of parallelograms P.

$\mathcal{T}$ denotes a covering of tangent triangles T (this is limited to sets on the frontier of a circle: a tangent triangle is one consisting of two tangents and the line joining their points of contact).

The diameter of a set U is denoted by d(U) and the area of the greatest triangle that can be taken with vertices in U is denoted by $\Delta(U)$. Then

$\mathcal{A}_\delta$ denotes the class of all coverings $\mathcal{U}$ consisting of convex sets U with $d(U) < \delta$ for all $U \in \mathcal{U}$.

$\mathcal{B}_\delta$ denotes the class of all coverings $\mathcal{U}$ consisting of convex sets U with $\Delta(U) < \delta$ for all $U \in \mathcal{U}$.

All other notations are given in the following sections or explained in the theorem in which they occur.

SECTION I, 2.

DEFINITION - MEASURE FUNCTION

A function $h(x)$ is said to be a measure function if it has the following properties:

(a) $h(x) > 0$ for all $x > 0$.

(b) $h(x) \to 0+$ as $x \to 0+$.

(c) $h(x)$ increases as x increases.

SECTION I, 3.

DEFINITION - HAUSDORFF DIAMETRAL MEASURE of a set S with measure function $h(x)$.

Let δ be any positive number. Let $\mathcal{A}_\delta$ be the class of all coverings $\mathcal{U}$ of the set S, $\mathcal{U}$ consisting of convex sets U with $d(U) < \delta$ for all $U \in \mathcal{U}$. Then the Hausdorff diametral measure of S with measure function $h(x)$ is defined as

$$\lim_{\delta \to 0} \inf_{\mathcal{U} \in \mathcal{A}_\delta} \sum_{U \in \mathcal{U}} h(d(U)).$$

This will be denoted by $m(S,h)$. The function $\inf_{\mathcal{U} \in \mathcal{A}_\delta} \sum_{U \in \mathcal{U}} h(d(U))$ will be denoted by $m_\delta(S,h)$.

SECTION I, 4.

DEFINITION - DIMENSION FUNCTIONS

A function $h(x)$ is said to be a dimension function if there exists a set S having finite non-zero Hausdorff diametral measure with measure function $h(x)$.

A set S is said to have dimension function $h(x)$ if

$$0 < m(S,h) < \infty$$

SECTION I, 5.

DEFINITION - METRIC AREA MEASURE of a set S with measure function h(x).

Let δ be any positive number and let $\mathfrak{A}_\delta$ be the class of all coverings $\mathfrak{U}$ of the set S, $\mathfrak{U}$ consisting of convex sets U with $d(U) < \delta$ for all $U \in \mathfrak{U}$. Then the metric area measure of the set S with measure function h(x) is defined as

$$\lim_{\delta \to 0} \inf_{\mathfrak{U} \in \mathfrak{A}_\delta} \sum_{U \in \mathfrak{U}} h(\Delta(U))$$

This will be denoted by A(S,h). The function $\inf_{\mathfrak{U} \in \mathfrak{A}_\delta} \sum_{U \in \mathfrak{U}} h(\Delta(U))$ will be denoted by $A_\delta(S,h)$.

A measure function h(x) is said to be a M.A. dimension function if there exists a set S such that

$$0 < A(S,h) < \infty$$

SECTION I, 6.

DEFINITION - NON-METRIC AREA MEASURE of a set S with measure function h(x).

Let δ be any positive number and let $\mathfrak{B}_\delta$ be the class of all coverings $\mathfrak{U}$ of the set S, $\mathfrak{U}$ consisting of convex sets U with $\Delta(U) < \delta$ for all $U \in \mathfrak{U}$. Then the non-metric area measure of the set S with measure function h(x) is defined as

$$\lim_{\delta \to 0} \inf_{\mathfrak{U} \in \mathfrak{B}_\delta} \sum_{U \in \mathfrak{U}} h(\Delta(U))$$

This will be denoted by B(S,h). The function $\inf_{\mathfrak{U} \in \mathfrak{B}_\delta} \sum_{U \in \mathfrak{U}} h(\Delta(U))$ will be denoted by $B_\delta(S,h)$.

A measure function h(x) is said to be a N.M.A. dimension function if there exists a set S such that

$$0 < B(S,h) < \infty$$

SECTION I, 7.

DEFINITION - AFFINE LENGTH and the GENERALIZED AFFINE LENGTH of a set S on the frontier of the unit circle.

Let δ be any positive number and $\mathcal{A}_\delta$ be the class of all coverings τ of the set S, τ consisting of tangent triangles T with $d(T) < \delta$ for all $T \in \tau$. Then the affine length of the set S is defined as

$$\lim_{\delta \to 0} \inf_{\tau \in \mathcal{A}_\delta} \sum_{T \in \tau} \Delta(T)^{1/3}$$

The generalized affine length of the set S with measure function h(x) is defined as

$$\lim_{\delta \to 0} \inf_{\tau \in \mathcal{A}_\delta} \sum_{T \in \tau} h(\Delta(T))$$

This is denoted by F(S,h).

SECTION I, 8.

DEFINITION - UPPER and LOWER DENSITIES of a set S with respect to Hausdorff diametral measure with measure function h(x).

Let C(a,r) be an open sphere centre the point a and radius r. Then the upper h-density at the point $a \in S$ is defined as

$$\lim_{r \to 0} \sup_{r > 0} \frac{m(S.C(a,r),h)}{h(2r)}$$

The lower h-density at the point $a \in S$ is defined as

$$\lim_{r \to 0} \inf_{r > 0} \frac{m(S.C(a,r),h)}{h(2r)}$$

The upper h-density is denoted by $\overline{D}(a,h)$ and the lower h-density at the point a is denoted by $\underline{D}(a,h)$

CHAPTER II

SECTION II, 1.

THEOREM

In p-dimensional real Euclidean space the necessary and sufficient condition for a measure function h(x) to be a dimension function is that

$$\lim_{x\to 0} \inf \frac{h(x)}{x^p} > 0$$

Necessity

Assume that $\lim_{x\to 0} \inf \frac{h(x)}{x^p} = 0$ and let S be any bounded set in the p-dimensional space. Then S can be completely contained in a p-dimensional cube of p-dimensional volume v. Let $\mathcal{A}_x$ be the class of all coverings $\mathcal{U}$ of the set S, $\mathcal{U}$ being a class of convex sets U with $d(U) \leqslant x$.

Then since a cube of volume v can be covered by at most $\left(\frac{v^{1/p}\sqrt{p+1}}{x}\right)^p$ cubes of diameter x it follows that

$$\inf_{\mathcal{U}\in\mathcal{A}_x} \sum_{U\in\mathcal{U}} h(d(U)) \leqslant \inf\left(\frac{v}{x}^{1/p}\sqrt{p+1}\right)^p h(x)$$

i.e. $$m_x(S,h) \leqslant \inf\left[\left(v^{1/p}\sqrt{p+x}\right)^p \frac{h(x)}{x^p}\right] \to 0 \text{ as } x \to 0.$$

and hence $m(S,h) = 0$. Since this is true for any bounded set S it is true for any set S.

Sufficiency

(i) Let $\lim_{x\to 0} \inf \frac{h(x)}{x^p} = \alpha \qquad 0 < \alpha < \infty$

Let J be a p-dimensional cube of volume v. Then by the same argument as above

$$m(J,h) \leqslant (\sqrt{p})^p v\alpha.$$

Given $\epsilon > 0$ there exists δ such that

$$\frac{h(x)}{x^p} > \alpha - \epsilon \quad \text{for all } x < \delta$$

Let $\mathcal{U}$ be a covering of the cube J consisting of convex sets U with $d(U) < \delta$.

Then
$$\sum_{U \in \mathcal{U}} h(d(U)) > (\alpha - \epsilon) \sum_{U \in \mathcal{U}} [d(U)]^p$$
$$\geqslant v(\alpha - \epsilon)$$

because U is contained in a cube of side d(U)

This holds for all such coverings $\mathcal{U}$, and hence
$$m_\delta(J,h) \geqslant v(\alpha - \epsilon)$$
i.e.
$$m(J,h) \geqslant v(\alpha - \epsilon)$$

This is true for all $\epsilon > 0$ and thus
$$m(J,h) \geqslant v\alpha.$$

(ii) Let
$$\liminf_{x \to 0} \frac{h(x)}{x^p} = \infty \tag{1}$$

This implies that there exist arbitrarily small positive numbers x such that
$$\frac{h(x)}{x^p} \leqslant 2 \inf_{0 < t \leqslant x} \frac{h(t)}{t^p} \tag{2}$$

Since if (2) does not hold there exists a sequence of positive numbers t_n, $t_n \to 0$ as $n \to \infty$ and such that
$$\frac{h(t_n)}{t_n^p} > \frac{h(t_{n+1})}{t_{n+1}^p}$$

which contradicts (1).

A special sequence of numbers x_n is now constructed in the following way: Let $\{A_n\}$ be any positive sequence of numbers such that $A_n \geqslant 2$ for all n and $\Sigma\, 1/A_n$ is convergent.

Any positive number satisfying (2), is chosen to be x_0, and x_n is then chosen so that

(a) (2) holds for all $x = x_n$, $n = 1, 2, \ldots$

(b) $h(x_{n-1}) = C_n^p h(x_n)$ $\quad C_n > A_n$.

(c) $2C_n x_n < x_{n-1}$

These three conditions can be satisfied simultaneously since $h(x) \to 0$ as $x \to 0+$, and (2) holds for arbitrarily small x. Clearly $x_n \to 0$ as $n \to \infty$.

Denote the integral part of C_n by K_n and define y_n by the relation

$$x_{n-1} = K_n x_n + (K_n - 1) y_n,$$

for every n.

A set S is now constructed as follows: S_0 is the set of points of a p-dimensional cube side $x_0/\sqrt{p}$ and diameter x_0. On each side of S_0 construct $(K_1 - 1)$ open intervals of length $y_1/\sqrt{p}$ alternating with K_1 closed intervals length $x_1/\sqrt{p}$. From these intervals a network of K_1^p cubes sides $x_1/\sqrt{p}$, $(K_1 - 1)^p$ cubes sides $y_1/\sqrt{p}$ and cuboids of sides $x_1/\sqrt{p}$ and $y_1/\sqrt{p}$ is constructed. All but the K_1^p cubes sides $x_1/\sqrt{p}$ are deleted from S_0 and the remaining set is denoted by S_1. The construction is then repeated in each of the cubes of S_1, replacing x_0 by x_1, x_1 by x_2, K_1 by K_2 and y_1 by y_2 obtaining $K_1^p K_2^p$ cubes of side length $x_2/\sqrt{p}$. This set is denoted by S_2. The process is repeated to form the set S_3 and so on.

The set S is now defined as

$$S = \bigcap_{n=0}^{\infty} S_n$$

(A) To show that $m(S,h) < \infty$

Given any number $\rho > 0$ there exists m such that $x_m < \rho$, since $x_n \to 0$ as $n \to \infty$. For any $n \geq m$ S_n is a covering of S consisting of cubes with diameter $x_n < \rho$.

Hence
$$\begin{aligned} m_\rho(S,h) &= \inf_{\mathcal{U} \in A_\rho} \sum_{U \in \mathcal{U}} h(d(U)) \\ &\leq K_1^p K_2^p \cdots\cdots\cdots K_n^p \, h(x_n) \\ &\leq K_1^p K_2^p \cdots\cdots\cdots K_{n-1}^p \, h(x_{n-1}) \\ &\leq h(x_0) \end{aligned}$$

from condition (b) noting that $K_n \leq C_n$.

Thus $m_\rho(S,h) \leq h(x_0)$ for all $\rho > 0$ and
$m(S,h) \leq h(x_0) < \infty$.

(B) To show that $m(S,h) > 0$.

$$K_1^p K_2^p \ldots\ldots K_n^p\, h(x_n) > (C_1 - 1)^p (C_2 - 1)^p \ldots\ldots (C_n - 1)^p\, h(x_n)$$

$$= (C_1 - 1)^p (C_2 - 1)^p \ldots\ldots \frac{(C_n - 1)^p}{C_n^p}\, h(x_{n-1})$$

$$= \prod_{\nu=1}^{n} \frac{(C_\nu - 1)^p}{C_\nu^p}\, h(x_0)$$

$$= \left[\prod_{\nu=1}^{n} (1 - 1/C_\nu)\right]^p h(x_0)$$

Since $C_n > A_n$ and Σ^1/A_n was a convergent series the infinite product $\prod_{\nu=1}^{\infty} (1 - 1/C_\nu)$ is also convergent. Thus for any set S_n

$$\sum_{S_n} h(x_n) \geqslant \beta h(x_0)$$

where β is a constant.

It will now be established that S_n is a sufficiently good covering of S, i.e. given any $\rho > 0$

$$\inf \sum_{S_n} h(x_n) \leqslant \gamma[\inf_{\mathcal{U} \in A_\delta} \sum_{U \in \mathcal{U}} h(d(U))]$$

using the same notation as before, γ being a constant.

Let $\mathcal{U}$ be a class of open convex sets U covering S. Every $U \in \mathcal{U}$ can be contained in a cube of side d(U) which can be orientated so that it lies parallel to S_0. This cube in turn can be covered by at most $([\sqrt{p}] + 1)^p$ cubes each of diameter d(U), where $[\sqrt{p}]$ denotes the integral part of $\sqrt{p}$. The latter will be denoted by L and the class of all cubes L corresponding to the sets U belonging to $\mathcal{U}$ will be denoted by $\mathcal{L}$.

Then $$\sum_{L \in \mathcal{L}} h(d(L)) \leqslant ([\sqrt{p}] + 1)^p \sum_{U \in \mathcal{U}} h(d(U))$$

Any L that does not contain a point of S is omitted from the sum and the remaining cubes shrunk so that at least one pair of faces contain points of S. Denote the cubes thus obtained by I and the class of all these cubes by $\mathcal{I}$.

Then $$\sum_{L \in \mathcal{L}} h(d(L)) \geqslant \sum_{I \in \mathcal{I}} h(d(I))$$

Denote the cubes which form the set S_n by s_n and consider any one cube I. There exists an m such that $s_{m-1} \supset I$ whilst I is not contained in any one cube s_m. Let I have points in common with $r > 1$ cubes s_m. Then $x_{m-1} \geqslant d(I)$ and by (2)

$$h(d(I)) \geqslant \tfrac{1}{2}\frac{(d(I))^p}{(x_{m-1})^p} h(x_{m-1}) \qquad (3)$$

<u>Lemma</u>

(α) If $I \subset s_{m-1}$ and meets t^p $(t \geqslant 2)$ or more cubes s_m then $d(I) \geqslant (t-1)y_m$.

(β) If $I \subset s_{m-1}$ and meets r cubes s_m when $1 < r < 2^p$ then $d(I) \geqslant y_m$.

<u>Proof</u>.

If I meets t^p cubes s_m any line contained in I parallel to one of the sides of S_0 which meets one cube s_m meets t such cubes; for if it meets only $(t-1)$, $[(t-1)x_m + ty_m] >$ side length of I, and no other line parallel to a side can meet t cubes s_m. But this implies that I does not meet t^p cubes s_m which contradicts the initial assumption. A hyperplane perpendicular to such a line and meeting one cube s_m meets $t^{(p-1)}$ such cubes and is cut in a $(p-1)$ dimensional cube by I. Assume (α) is true in $(p-1)$ dimensions. Then

$$\frac{\sqrt{p-1}}{\sqrt{p}} d(I) \geqslant (t-1)\frac{\sqrt{p-1}}{\sqrt{p}} y_m$$

in the hyperplane and hence

$$d(I) \geqslant (t-1)y_m.$$

The result is obvious in 1-dimension since

$$d(I) \geqslant (t-2)x_m + (t-1)y_m$$

and by induction the result is true in p-dimensions.

If I meets 2 cubes s_m then $d(I) \geqslant y_m$ because the minimum distance between 2 cubes s_m is $y_m/\sqrt{p}$ in a direction parallel to the sides of S_0, and the side length of I, $d(I)/\sqrt{p}$ satisfies

$$\frac{d(I)}{\sqrt{p}} \geqslant \frac{y_m}{\sqrt{p}}$$

This gives the second part of the lemma.

To show that if I meets r cubes s_m where $r \geqslant t^p$, $t \geqslant 2$ then

$$\frac{d(I)}{x_{m-1}} > \frac{(t-1)}{2K_m-1}$$

In constructing S the relation

$$x_{m-1} = K_m x_m + (K_m - 1)y_m \qquad (4)$$

was used. By the inequality (c) on page 6 and the fact that K_m is the integral part of C_m (4) becomes

$$2K_m x_m < K_m x_m + (K_m - 1)y_m$$

and

$$x_m < y_m$$

Substituting for x_m in (4) we get

$$(2K_m - 1)y_m > x_{m-1}$$

and combining this with the inequality of the lemma gives

$$\frac{d(I)}{x_{m-1}} > \frac{(t-1)}{2K_m-1} \qquad t \geqslant 2 \qquad (5)$$

Let the cube I meet r cubes s_m.

(i) $t^p \leqslant r < (t+1)^p$, $t \geqslant 2$.

Then
$$rh(x_m) < (t+1)^p h(x_m)$$
$$\leqslant \left(\frac{t+1}{K_m}\right)^p h(x_{m-1})$$

(from the equation (b) on page 6 and the fact that $K_m < C_m$)

$$\leqslant 2\left(\frac{t+1}{K_m}\right)^p \left(\frac{x_{m-1}}{d(I)}\right)^p h(d(I))$$

(from the inequality (3))

$$< 2\left[\frac{(2K_m-1)(t+1)}{K_m(t-1)}\right]^p h(d(I))$$

(from the inequality (5))

$$< 2.6^p\, h(d(I)) \qquad (7)$$

(ii) $1 < r < 2^p$

It has been established that

$$x_{m-1} < (2K_m - 1)y_m$$

and from (β) in the lemma $d(I) \geqslant y_m$

Thus in this case $$\frac{d(I)}{x_{m-1}} > \frac{1}{2K_m - 1} \qquad (6)$$

and
$$rh(x_m) \leqslant \frac{r}{K_m{}^p} h(x_{m-1})$$

$$\leqslant 2 \frac{r}{K_m{}^p} \left(\frac{x_{m-1}}{d(I)}\right)^p h(d(I))$$

(as in (i))

$$< 2r \left(\frac{2K_m - 1}{K_m}\right)^p h(d(I))$$

(from the inequality (6))

$$< 2.4^p h(d(I)) < 2.6^p h(d(I)) \qquad (7)$$

If $r = 1$ $d(I) < x_m$ since the cubes I were chosen so that at least one pair of faces contained points of S. In this case I is contained in one cube s_m and the procedure is repeated using x_m instead of x_{m-1} and considering the cubes s_{m+1} contained in s_m.

To each cube I there corresponds a number m such that one of the cubes $s_{m-1} \supset I$ and I has points in common with $r > 1$ cubes s_m, and if I is replaced by these r cubes the inequality (7) holds. Let $n = \max$ (m corresponding to $I \in \mathcal{J}$). By (b) on page 6 and the fact that $K_n < C_n$ and inequality (7)

$$r(K_n^p \ldots K_{m-1}^p) h(x_n) < 2.6^p h(d(I))$$

Thus if each cube $I \in \mathcal{J}$ is replaced by the cubes s_n with which it has points in common with all the cubes $s_n \in S_n$ will be used at least once and hence

$$\sum_{S_n} h(x_n) \leqslant 2.6^p \sum_{I \in \mathcal{J}} h(d(I))$$

For any covering $\mathcal{U}$ of the set S consisting of sets U the corresponding class $\mathcal{I}$ satisfies the relation

$$\sum_{U \in \mathcal{U}} h(d(U)) \geqslant \left(\frac{1}{[\sqrt{p}]+1}\right)^p \sum_{I \in \mathcal{I}} h(d(I))$$

$$\geqslant \tfrac{1}{2}\left(\frac{1}{6([\sqrt{p}]+1)}\right)^p \sum_{S_n} h(x_n)$$

for some value of n,

$$\geqslant \gamma h(x_0)$$

where γ is a constant.

Since this holds for all coverings $\mathcal{U} \in A_\rho$

$$m_\rho(S,h) \geqslant \gamma h(x_0) \qquad \text{for all } \rho > 0$$

and hence $m(S,h) > 0$.

It has now been established that if $h(x)$ is any measure function satisfying

$$\liminf_{x \to 0} \frac{h(x)}{x^p} > 0$$

there exists a set S having finite non-zero Hausdorff diametral measure with this measure function, i.e. $h(x)$ is a dimension function.

SECTION II,2.

THEOREM I

In p-dimensional real Euclidean space a closed set of infinite Hausdorff diametral measure has a subset of finite non-zero Hausdorff diametral measure.

Proof Let S be the given set. Take rectangular cartesian axes and over the whole space construct a network of closed cubes J_n each of side length 2^{-n} and lying parallel to the axes, no two cubes having common interior points. Denote by $\mathcal{J}_n$ any class of such cubes which covers S and consists of cubes J_ν $\nu \geqslant n$, and let Λ_n be the class of all such coverings $\mathcal{J}_n$.

Define $L_n(S,h) = \inf_{\mathcal{J}_n \in \Lambda_n} \sum_{J_\nu \in \mathcal{J}_n} h(d(J_\nu))$

and $L(S,h) = \lim_{n\to\infty} L_n(S,h)$

(1) If $\rho \geqslant \frac{(\sqrt{p})}{2^n}$

then $m_\rho(S,h) \leqslant L_n(S,h)$

and hence $m(S,h) \leqslant L(S,h)$

(2) Given any $\rho > 0$ let $\mathcal{U}$ be an arbitrary covering of convex sets U of the set S with $d(U) < \rho$. Consider any $U \in \mathcal{U}$ Then there exists an n such that

$$\frac{1}{2^{n-1}} > \frac{\bar{d}(U)}{\sqrt{p}} \geqslant \frac{1}{2^n}$$

As in Section II,1 any convex set U can be replaced by at most $([\sqrt{p}]+1)^p$ cubes I of diameter d(U) and if $\mathcal{I}$ is the class of all cubes I which cover S

$$\sum_{I\in\mathcal{I}} h(d(I)) \leqslant ([\sqrt{p}]+1)^p \sum_{U\in\mathcal{U}} h(d(U)) \qquad (a)$$

the cubes I being parallel to the axes. Consider the intervals cut by a cube I and the cubes J_ν on a line parallel to one of the axes.

Since $\frac{1}{2^{n-1}} > \frac{d(I)}{\sqrt{p}} \geqslant \frac{1}{2^n}$

the interval cut by I which is of length $d(I)/\sqrt{p}$ has points in common with at most three of the intervals cut by the cubes J_n, and hence I has points in common with at most 3^p cubes J_n. Since h(x) is a measure function it is monotonically increasing and

$$3^p\, h(d(I)) \geqslant 3^p\, h(\sqrt{p}/2^n) \qquad (b)$$

Let m be the integer such that $\frac{1}{2^{m-1}} > \frac{\rho}{\sqrt{p}} \geqslant \frac{1}{2^m}$

(b) holds for all cubes $I \in \mathcal{I}$ and hence there exists a covering $\mathcal{J}_m$ of S such that if $\mathcal{I}$ is replaced by this $\mathcal{J}_m$ then

$$3^p \sum_{I \in \mathcal{I}} h(d(I)) \geqslant \sum_{J_n \in \mathcal{J}_m} h(\sqrt{p}/2^n)$$

$$\geqslant L_m(S,h)$$

This holds for all coverings $\mathcal{I}$ and thus combining this result with (a) gives

$$[3([\sqrt{p}]+1)]^p \, m_\rho(S,h) \geqslant L_m(S,h)$$

Henry $(3[\sqrt{p}]+1)^p m(S,h) \geqslant L(S,h)$

(3) LEMMA I

If S_n is the intersection of S with a cube J_n and

$$L_{n+1}(S_n,h) > h(\sqrt{p}/2^n)$$

there exists a subset S_n' of S_n such that

$$L_{n+1}(S_n',h) = h(\sqrt{p}/2^n)$$

Proof

Divide the cube J_n into a network of λ_1 closed cubes e_{1j} $(j = 1 \ldots \lambda_1)$ which are parallel to the axes and such that if s^n_{1j} denotes the part of S_n contained in e_{1j} then

$$L_{n+1}(s^n_{1j},h) < h(\sqrt{p}/2^n)$$

for all j.

Since $\bigcup_{j=1}^{\lambda_1} s^n_{1j} \supset S_n$

$$\sum_{j=1}^{\lambda_1} L_{n+1}(s^n_{1j},h) \geqslant L_{n+1}(S_n,h)$$

$$> h(\sqrt{p}/2^n)$$

and there exists an integer m_1 such that

$$\sum_{j=1}^{m_1-1} L_{n+1}(s^n_{1j},h) < h(\sqrt{p}/2^n)$$

and $$\sum_{j=1}^{m_1} L_{n+1}(s^n_{1j},h) \geqslant h(\sqrt{p}/2^n)$$

Consider this cube e_{1m_1} and as before cover it with a network of λ_2 cubes e_{2j}. Then there exists an integer m_2 $1 \leq m_2 \leq \lambda_2$ such that

$$\sum_{j=1}^{m_1-1} L_{n+1}(s^n_{1j},h) + \sum_{j=1}^{m_2-1} L_{n+1}(s^n_{2j},h) < h(\sqrt{p}/2^n)$$

and

$$\sum_{j=1}^{m_1-1} L_{n+1}(s^n_{1j},h) + \sum_{j=1}^{m_2} L_{n+1}(s^n_{2j},h) \geq h(\sqrt{p}/2^n)$$

Repeat this process using e_{2m_2} and so on to obtain a decreasing sequence of cubes. Let E denote the set

$$\bigcup_{j=1}^{m_1-1} e_{1j} + \bigcup_{j=1}^{m_2-1} e_{2j} + \dots\dots\dots\dots \bigcup_{j=1}^{m_t-1} e_{tj} + \dots\dots\dots + \bigcap_{i=1}^{\infty} e_{im_i}$$

then $$L_{n+1}(S_n \wedge E,h) = h(\sqrt{p}/2^n)$$

E is a closed set since the limit point of any sequence of points belonging to E will either belong to one of the sets $\bigcup_{j=1}^{m_t-1} e_{tj}$ since all these sets are adjoining or will be the point $\bigcap_{i=1}^{\infty} e_{im_i}$

Thus $S_n \wedge E$ is the required subset S'_n.

(4) If $$L_{n+1}(S_n,h) \leq h(\sqrt{p}/2^n)$$

$$L_{n+1}(S_n,h) = L_n(S_n,h)$$

On the other hand if

$$L_{n+1}(S_n,h) > h(\sqrt{p}/2^n)$$

then from the lemma there exists a subset S'_n of S_n such that

$$L_{n+1}(S'_n,h) = h(\sqrt{p}/2^n)$$

and also

$$L_n(S'_n,h) = h(\sqrt{p}/2^n)$$

PROOF OF THE THEOREM I

Itis sufficient to prove that S has a subset of finite Hausdorff diametral measure greater than a given number b.

Since $m(S,h) = \infty$ it follows from paragraph (1) that $L(S,h) = \infty$, and given $b > 0$ there exists an integer m such that

$$L_m(S,h) > (3\{[\sqrt{p}]+1\})^p b.$$

Write $S = S^m$. Define a subset S^{m+1} of S^m in the following way. In every cube J_m in which $L_{m+1}(S_m,h) \leqslant h(\sqrt{p}/2^m)$ $S^{m+1}_m = S^m_m$, and in those cubes J_m in which $L_{m+1}(S_m,h) > h(\sqrt{p}/2^m)$ S^{m+1}_m is the subset S'_m such that $L_{m+1}(S'_m,h) = h(\sqrt{p}/2^m)$.

By paragraph (4)

$$L_{m+1}(S^{m+1},h) = L_m(S^{m+1},h)$$
$$= L_m(S^m,h)$$

A subset S^{m+2} of S^{m+1} is defined in the same way as S^{m+1} was defined from S^m, and so on. A decreasing sequence S^n is thus obtained taking $n = m+1, m+2, \ldots$ such that for any $n > m$

$$L_n(S^n,h) = L_{n-1}(S^n,h) = \ldots..$$
$$= L_m(S^n,h) = L_m(S^m,h)$$

Write $\lim S^n = R$. R is a closed subset of S. Given any $\eta > 0$ there exists an n_0 such that for any $n > n_0$ any point of S^n is within η of R. Let $\mathcal{U}(R)$ be any finite open covering of R such that for any $U \in \mathcal{U}(R)$ $d(U) < \rho$.
As $\mathcal{U}(R) \supset R$, $\mathcal{U}(R)$ contains all the points that are within

a certain $\eta > 0$ from R and consequently all S^n for n greater than a certain n_0. Thus $\mathfrak{U}(R)$ is also a covering of S^n.

Hence

$$\sum_{U\in\mathfrak{U}(R)} h(d(U)) \geq m_\rho(S^n,h)$$
$$\geq \frac{1}{(3\{[\sqrt{p}]+1\})^p} L_q(S^n,h)$$

where q is the integer such that $\frac{1}{2q-1} > \frac{\rho}{\sqrt{p}} \geq \frac{1}{2q}$

from paragraph (2)

$$= \frac{1}{(3([\sqrt{p}]+1))^p} L_m(S^m,h)$$

from which it follows that

$$m(R,h) \geq \frac{1}{(3([\sqrt{p}]+1))^p} L_m(S^m,h) > b$$

On the other hand

$$m_\rho(R,h) \leq m_\rho(S^n,h) \leq L_n(S^n,h)$$
$$\leq L_m(S^m,h)$$

Hence $m(R,h) \leq L_m(S^m,h)$ and the theorem is proved.

Corollary

If $\{F_n\}$ is any decreasing sequence of bounded closed sets and if $F = \prod_{n=1}^{\infty} F_n$ then for any integer q

$$m_{2^{-(q-1)}}(F,h) \geq \frac{1}{\{3([\sqrt{p}]+1)\}} \lim_{n\to\infty} L_q(F_n,h)$$

---

(5) Remark Denote the p-dimensional cube with sides on each axis given by the interval (0,1) by C. Let F be a set contained in C and m a positive integer.

Then
$$L_m(F,h) = \sum_{k=0}^{2^{p(m-1)}} L_m(F \wedge J_{mk},h)$$

where J_{mk} is a cube cutting intervals $(r_i\ 2^{-m}, (r_i+1)\ 2^{-m})$ on the i^{th} axis and $k = ri$.

For any cube J_m with diameter ℓ we have either

$$\text{(i)} \quad L_m(F_\wedge J_m,h) = h(\ell)$$

if $$L_{m+1}(F_\wedge J_m,h) > h(\ell)$$

or $$\text{(ii)} \quad L_m(F_\wedge J_m,h) = L_{m+1}(F_\wedge J_m,h)$$

Denoting by Q_1 the sum of the cubes for which (i) holds

$$L_m(F,h) = \sum_{Q_1} h(\ell) + L_{m+1}(F_\wedge(C-Q_1),h)$$

Similarly denoting by Q_2 the sum of cubes $J_{m+1} \subset C - Q_1$ for which

$$L_{m+2}(F_\wedge J_{m+1},h) > h(\ell)$$

we get

$$L_m(F,h) = \sum_{Q_1+Q_2} h(\ell) + L_{m+2}(F_\wedge(C-Q_1-Q_2),h)$$

Generally after the sets $Q_1, Q_2, \ldots, Q_{n-1}$ have been defined we denote by Q_n the sum of cubes $J_{m+n+1} \subset C - \sum_{q=1}^{n-1} Q_q$ on which

$$L_{m+n}(F_\wedge J_{m+n+1},h) > h(\ell)$$

and we get

$$L_m(F,h) = \sum_{Q_1+Q_2+\ldots Q_n} h(\ell) + L_{m+n}(F_\wedge(C-\sum_1^n Q_q),h) \quad (3)$$

Write $Q = Q_1 + Q_2 + \ldots\ldots$, $D = C - Q$.

(6) <u>LEMMA 2</u>

$$L_m(F,h) = \sum_Q h(\ell) + L(F_\wedge D,h)$$

<u>Proof</u> If $\mathfrak{J}_n$ is a covering of $F_\wedge D$ consisting of cubes J_i $i \geq m+n$, then $\mathfrak{J}_n + Q_{n+1} + Q_{n+2} + \ldots$ is a covering of $F_\wedge(C - Q_1 - Q_2 - \ldots\ldots - Q_n)$.

Thus $$\sum_{\mathfrak{J}_n} h(\ell) + \sum_{Q_{n+1}+Q_{n+2}+\ldots\ldots} h(\ell)$$

$$\geq L_{m+n}(F_\wedge(C-Q_1-\ldots\ldots-Q_n),h)$$

Hence

$$L_{m+n}(F_\wedge D,h) \geqslant L_{m+n}(F_\wedge(C-Q_1-Q_2-\dots-Q_n),h) - \sum_{Q_{n+1}+\dots} h(\ell) \qquad (4)$$

On the other hand

$$L_{m+n}(F_\wedge D,h) \leqslant L_{m+n}(F_\wedge(C-Q_1-Q_2-\dots-Q_n),h) \qquad (5)$$

By (3) the sum $\sum_{Q_1+Q_2+\dots+Q_n+\dots} h(\ell)$ is finite

since $\quad L_m(F,h) \leqslant 2^{pm}\, h(\sqrt{p}/2^m)$

Hence $\epsilon_n = \sum_{Q_{1+n}+Q_{2+n}+\dots} h(\ell) \to 0$ as $n\to\infty$

and by (4) and (5)

$$L_{m+n}(F_\wedge D,h) = L_{m+n}(F_\wedge(C-Q_1-Q_2-\dots-Q_n),h) - \theta\epsilon_n \qquad (6)$$

where $0 \leqslant \theta \leqslant 1$.

By (3) and (6)

$$L_m(F,h) = \sum_{Q_1+\dots Q_n} h(\ell) + L_{m+n}(F_\wedge D,h) + \theta\epsilon_n$$

and letting $n\to\infty$

$$L_m(F,h) = \sum_{Q} h(\ell) + L(F_\wedge D,h)$$

(7) <u>LEMMA</u> 3

If $\{G_n\}$ is any increasing sequence of sets and $G = \cup G_n$ is bounded then for any integer m

$$L_m(G,h) = \lim_{n\to\infty} L_m(G_n,h)$$

<u>Proof</u> Let Q^n and D^n be the sets Q and D of lemma 2 corresponding to the set G_n. Then we have

$$Q^n + D^n = C, \qquad Q^n \subset Q^{n+1}, \qquad D^n \supset D^{n+1}$$

and writing $\lim_{n\to\infty} Q^n = Q$ and $\lim_{n\to\infty} D^n = D$

we have $\lim_{n\to\infty} \sum_{Q^n} h(\ell) = \sum_{Q} h(\ell) \qquad C - Q = D$

and further

$$L_m(G_n,h) = \sum_{Q^n} h(\ell) + L(G_n \wedge D_n^n, h)$$

$$\geq \sum_{Q^n} h(\ell) + L(G_n \wedge D, h)$$

Hence

$$\lim_{n\to\infty} L_m(G_n,h) \geq \sum_{Q} h(\ell) + L(G \wedge D, h) \qquad (7)$$

On the other hand

$$\lim_{n\to\infty} L_m(G_n,h) \leq L_m(G,h) \leq L_m(G \wedge Q,h) + L_m(G \wedge D,h) \qquad (8)$$

$$\leq \sum_{Q} h(\ell) + L(G \wedge D,h)$$

$$\leq \lim_{n\to\infty} L_m(G_n,h).$$

Hence $\lim_{n\to\infty} L_m(G_n,h) = L_m(G,h)$

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(8) THEOREM II .

Any analytic set which is not the sum of a countable sequence of sets of finite Hausdorff diametral measure with dimension $h(x)$ contains a closed subset of infinite measure.

Proof

Let F be an analytic set which we may suppose bounded, which is not a countable sum of sets of finite Hausdorff diametral measure with dimension $h(x)$. For this set F choose a determining system $\{I_{i_1 \dots i_n}\}$ of closed p-dimensional cubes so that

$$I_{i_1 \dots\dots i_n i_{n+1}} \subset I_{i_1 \dots\dots i_n} \qquad (n=1,2,3,\dots)$$

and (a) $F = \sum_{i_1 i_2 \dots} I_{i_1} \wedge I_{i_1 i_2} \wedge \dots\dots \wedge I_{i_1 i_2 \dots i_n} \wedge \dots\dots$

the summation being extended over all infinite sequences of positive integers $i_1 i_2 \dots i_n \dots$

Let m_1 be an integer for which

$$L_{m_1}(F,h) > 1$$

Let N_r denote the sum (a) extended over all sequences $i_1 i_2 \ldots\ldots$ for which $1 \leq i_1 \leq r$. Then $\{N_r\}$ is an increasing sequence of sets and $F = \Sigma N_r$.

Choose an integer r_1 so large that

(i) $L_m(N_{r_1}, h) > 1$ this is possible by lemma 3 and

(ii) N_{r_1} is not a countable sum of sets of finite measure.

Thus $L(N_{r_1}, h) = \infty$ and we can choose m_2 so that $m_2 \geq m_1$ and
$$L_{m_2}(N_{r_1}, h) > 2.$$

Let $N_{r_1 r}$ denote the sum (a) extended over all sequences $i_1 i_2 \ldots$ for which $1 \leq i_1 \leq r_1$ and $1 \leq i_2 \leq r$ Then $\{N_{r_1 r}\}$ is an increasing sequence of sets and $N_{r_1} = \Sigma N_{r_1 r}$

Choose an integer r_2 so large that

(i) $L_{m_1}(N_{r_1 r_2}, h) > 1$ and (ii) $L_{m_2}(N_{r_1 r_2}, h) > 2$

this is possible by lemma 3 and

(iii) $N_{r_1 r_2}$ is not a countable sum of sets of finite measure.

Continuing in this way we obtain a sequence of integers $m_1 \leq m_2 \leq \ldots\ldots$ and $r_1, r_2 \ldots$ such that for each n

(b) $L_{m_\nu}(N_{r_1 \ldots r_n}, h) > \nu$ $(\nu = 1 \ldots n)$

and $N_{r_1 \ldots r_n}$ is not a countable sum of sets of finite measure.

Write $H_n = \sum_{1 \leq i_\nu \leq r_\nu} I_{i_1} \wedge I_{i_1 i_2} \wedge \ldots I_{i_1 i_2 \ldots i_n}$

and $H = \pi H_n$.

Each set H_n is the sum of a finite number of closed cubes and is thus closed, also $N_{r_1 r_2 \ldots r_n} \subset H_n$.

By (b)

(c) $$L_{m_\nu}(H_n,h) > \nu \quad \text{for all } \nu \text{ and } n.$$

Now the sets H_n form a decreasing sequence of bounded closed sets and so by (c) and the corollary to theorem I

$$m_{2^{-(m_\nu+1)}}(H,h) \geq \frac{1}{(3[\sqrt{p}]+1))^p} \nu \text{ for all } \nu,$$

and therefore $m(H,h) = \infty$. Now it has been shown that $H \subset F$. Since H is closed the theorem is proved.

Corollary

Any analytic set which is not the sum of a countable sequence of sets of finite Hausdorff measure with measure function h(x) contains a subset of finite non-zero Hausdorff diametral measure with dimension h(x).

SECTION II,3

To deduce that in p-dimensional space if

$$\liminf_{x\to 0} \frac{h(x)}{x^p} = \infty$$

there exists a set of finite non-zero Hausdorff diametral measure with dimension function h(x)

Let J be the unit closed cube in p-dimensional space. Given $\alpha > 0$ there exists δ such that

$$\frac{h(x)}{x^p} > \alpha \qquad \text{for all } x < \delta$$

Let $\mathcal{U}$ be a covering of the cube J consisting of convex sets U with $d(U) < \delta$. Then

$$\sum_{U\in\mathcal{U}} h(d(U)) > \alpha \sum_{U\in\mathcal{U}} (d(U))^p$$
$$> \alpha (\sqrt{p})^p$$

Hence $\quad m_\delta(J,h) > \alpha\ (\sqrt{p})^p$

Since this holds for all $\alpha > 0 \quad m(J,h) = \infty$.

Thus the cube J fulfils the conditions for the given set in theorem I section II,2, and hence there exists a subset S of J having finite non-zero Hausdorff diametral measure with dimension function h(x).

SECTION II,4

THEOREM

Let S be a closed set with dimension function h(x). Then

(a) If $\liminf_{x\to 0} \frac{g(x)}{h(x)} > 0$ there exists a subset F of S having dimension function g(x).

(b) If F is any subset of S with dimension function g(x) then g(x) satisfies $\limsup_{x\to 0} \frac{g(x)}{h(x)} > 0$.

Proof of (a)

(1) If $\liminf_{x\to 0} \frac{g(x)}{h(x)} = \infty$ then given any $\alpha > 0$ there exists δ such that

$$\frac{g(x)}{h(x)} > \alpha \qquad \text{for all } x < \delta$$

For any covering $\mathcal{U}$ of the set S, $\mathcal{U}$ consisting of convex sets U such that $d(U) < \delta$ for all $U \in \mathcal{U}$

$$\sum_{U\in\mathcal{U}} g(x) > \sum_{U\in\mathcal{U}} \alpha\, h(x)$$

where $d(U) = x$. If A_δ is the class of all such coverings $\mathcal{U}$

$$\inf_{\mathcal{U}\in A_\delta} \sum_{U\in\mathcal{U}} g(x) \geqslant \alpha \inf_{\mathcal{U}\in A_\delta} \sum_{U\in\mathcal{U}} h(x)$$

Since this holds for all $x < \delta$

$$m(S,g) \geqslant \alpha\, m(S,h)$$

But $m(S,h)$ is finite and non-zero and the inequality holds for all $\alpha > 0$, thus $m(S,g) = \infty$.

S is closed and thus the conditions of theorem I in Section II,2, are satisfied and there exists a subset F of S such that $0 < m(F,g) < \infty$.

(2) Let $\liminf_{x\to 0} \dfrac{g(x)}{h(x)} = \beta \qquad 0 < \beta < \infty$

Given $\epsilon > 0$ there exists δ such that

$$\frac{g(x)}{h(x)} > (\beta - \epsilon) \qquad \text{all } x < \delta$$

For any covering $\mathcal{U}$ of the set S consisting of convex sets U with $d(U) < \delta$ writing $x = d(U)$,

$$(\beta-\epsilon) \sum_{U\in\mathcal{U}} h(x) < \sum_{U\in\mathcal{U}} g(x)$$

Let A_δ be the class of all such coverings $\mathcal{U}$. Then

$$(\beta-\epsilon) \inf_{\mathcal{U}\in A_\delta} \sum_{U\in\mathcal{U}} h(x) \leqslant \inf_{\mathcal{U}\in A_\delta} \sum_{U\in\mathcal{U}} g(x)$$

and since this holds for all $x < \delta$ and all $\epsilon > 0$

$$\beta\, m(S,h) \leqslant m(S,g)$$

Since $m(S,h) > 0$, $m(S,g) > 0$.

If $m(S,g) = \infty$ then as in the first case there exists a subset F of S such that $0 < m(F,g) < \infty$

If $m(S,g)$ is finite then S itself may be taken as the required subset.

<u>Proof of (b)</u>

Since F is a subset of S

$$m(S,g) \geq m(F,g) > 0.$$

Hence given $\delta > 0$ there exists ∂_1 such that for any covering of the set S consisting of convex sets U with $d(U) < \delta$ for all $U \in \mathcal{U}$, writing $d(U) = x$,

$$\sum_{U \in \mathcal{U}} g(x) > \partial_1$$

This holds for all $\mathcal{U}$ such that $d(U) < \delta$ all $U \in \mathcal{U}$.

Since $m(S,h)$ is finite and non-zero there exists ∂_2 such that

$$\sum_{U \in \mathcal{U}} h(x) \leq \partial_2$$

for infinitely many coverings $\mathcal{U}$ of S with $d(U) < \delta$ for all $U \in \mathcal{U}$
Thus for at least one covering $\mathcal{U}$

$$\frac{\sum_{U \in \mathcal{U}} g(x)}{\sum_{U \in \mathcal{U}} h(x)} \geq \frac{\partial_1}{\partial_2} \qquad (1)$$

The set of values ∂_1/∂_2 which depend on δ, is bounded below by some fixed positive number.

If $\lim_{x \to 0} \sup \frac{g(x)}{h(x)} = 0$ then there exists a constant ∂_3 such that $\frac{g(x)}{h(x)} < \partial_3$ for all $x < \delta$ and hence for all coverings $\mathcal{U}$ defined above

$$\frac{\sum_{U \in \mathcal{U}} g(x)}{\sum_{U \in \mathcal{U}} h(x)} < \partial_3$$

The number ∂_3 depends on δ and $\partial_3 \to 0$ as $\delta \to 0$.

But this is a contradiction of (1) and hence

$$\lim_{x\to 0} \sup \frac{g(x)}{h(x)} > 0$$

This completes the proof of the theorem.

In the statement of the theorem it is not possible to replace "lim sup" in the inequality (b) by "lim inf" as the following example will establish. Thus the theorem gives the best possible result.

EXAMPLE

To construct a set S having finite non-zero Hausdorff diametral measure with dimension function h(x) and such that there exists a measure function g(x) satisfying

(1) $\lim_{x\to 0} \inf \frac{g(x)}{h(x)} = 0$

(2) $\lim_{x\to 0} \sup \frac{g(x)}{h(x)} > 0$

(3) $0 < m(S,g) < \infty$

Consider the function h(x) given by

$$h(x) = \frac{1}{2^{n^2/2}} \quad \text{when } \frac{1}{2^{n^2}} \geq x > \frac{1}{2^{(n+1)^2}} \qquad (n=1,2,\ldots)$$

This function is monotonically increasing and $h(x) \to 0$ as $x \to 0$

Also $\lim_{x\to 0} \inf \frac{h(x)}{x} = \lim_{n\to\infty} \frac{2^{n^2}}{2^{n^2/2}} = \infty$

The function $\frac{h(x)}{x}$ decreases strictly in the interval

$\frac{1}{2^{n^2}} \geq x > \frac{1}{2^{(n+1)^2}}$ and the set of values of $\frac{h(x)}{x}$ at the points $x = \frac{1}{2^{n^2}}$ decreases strictly as x increases. Thus every point of this set satisfies

$$\frac{h(x)}{x} = \min_{0<t\leqslant x} \frac{h(t)}{t} \leqslant 2 \inf_{0<t\leqslant x} \frac{h(t)}{t}$$

Hence the method of constructing a set in one dimension established in Section II,1 on page 6 can be used if the decreasing sequence of points in this construction which were denoted by $\{x_n\}$ are limited to be of the form $x_i = \frac{1}{2^{j^2}}$

The set S obtained by this construction satisfies

$$0 < m(S,h) < \infty$$

Denote by $I(S,h)$ the measure of S obtained by considering only the coverings of intervals with end points given by $x_i = \frac{1}{2^{j^2}}$. Then it was proved in Section II,1, that

$$\alpha_1 \, I(S,h) \leqslant m(S,h) \leqslant I(S,h)$$

where α_1 is a constant.

Consider the function given by $g(x) = x^{\frac{1}{2}}$

Then $g(x) = h(x)$ whenever $x = \frac{1}{2^{j^2}}$

and since $\frac{g(x)}{x}$ is strictly increasing as x increases the set S will also satisfy

$$\alpha_2 \, I(S,g) \leqslant m(S,g) \leqslant I(S,g)$$

where α_2 is a constant.

But $I(S,h) = I(S,g)$ and hence since $0 < m(S,h) < \infty$ $m(S,g)$ is finite and non-zero.

Condition (3) is thus satisfied by this set S and the two functions $h(x)$ and $g(x)$. Conditions (1) and (2) are also satisfied as follows.

Given any $\epsilon > 0$ there exists m such that $\frac{1}{2^{(m+\frac{1}{2})}} < \frac{\epsilon}{2}$

and $\frac{g(x)}{h(x)} < \frac{2^{n^2/2}}{2^{(n+1)^2/2}} + \frac{\epsilon}{2}$

for infinitely many $n >$ some n_0.

Thus $\frac{g(x)}{h(x)} < \frac{1}{2^{(n+\frac{1}{2})}} + \frac{\epsilon}{2} < \epsilon$

for infinitely many $n > \max(n_0, m)$

Also $\frac{g(x)}{h(x)} > 0$ for all $x > 0$ and hence

$$\lim_{x\to 0} \inf \frac{g(x)}{h(x)} = 0$$

On the other hand $\frac{g(x)}{h(x)} \leqslant 1$ whenever $x \leqslant 1$.

and $\frac{g(x)}{h(x)} = 1$ whenever $x = 1/2^{j^2}$

and thus $\lim_{x\to 0} \sup \frac{g(x)}{h(x)} = 1$.

CHAPTER III

SECTION III,1.

THEOREM 1

Given a closed bounded set S in p-dimensional real Euclidean space a sufficient condition for $m(S,h) > 0$ is that there exists an additive set function $\phi(R)$ defined over half-open figures (a half-open figure is a set expressible as a finite sum of half-open, i.e. open on the right, p-dimensional intervals) R such that

(1) for any figure R $\phi(R) \geq 0$

(2) if $R \supset S$ $\phi(R) \geq b > 0$ where b is some fixed constant,

(3) there is a finite non-zero constant K such that

$$\phi(R) \leq K.h(d(R))$$

Then in fact $m(S,h) \geq b/K$.

Proof

By the Heine-Borel theorem we can take any covering of S by open sets to be finite. Let $\mathcal{U}$ be a class of open convex sets U covering S. Each set $U \in \mathcal{U}$ can be enclosed in a half-open figure R of diameter d(R) so near d(U) that

$$h(d(R)) < (1 + \epsilon)\, h(d(U))$$

where ϵ is a given small positive number.

Thus we have for each figure R

$$h(d(R)) \geq \frac{1}{K}\, \phi(R)$$

and so $$\sum_{U \in \mathcal{U}} h(d(U)) \geq \frac{1}{K(1+\epsilon)} \sum \phi(R)$$

$$\geq \frac{1}{K(1+\epsilon)} \phi(\cup R)$$

because $\sum \phi(R) \geq \phi(\cup R)$ since the figures R may be overlapping, and $\phi(R)$ is additive.

Moreover $\cup R$ contains S and so

$$\phi(\cup R) \geq b$$

therefore $$\sum_{U \in \mathcal{U}} h(d(U)) \geq \frac{b}{K(1+\epsilon)}$$

Since this holds for all coverings $\mathcal{U}$ and for all $\epsilon > 0$

$$m(S,h) \geq b/K.$$

THEOREM 2

If S is a set in p-dimensional real Euclidean space which is measurable with regard to Hausdorff diametral measure with measure function $h(x)$, then $\overline{D}(x,h) \leq 1$ at all points $x \in S$ except possibly for a set of measure zero.

LEMMA

Given a measurable set F for which $m(F,h) < \infty$ and any positive number ϵ, there exists a number δ depending on F and ϵ such that for any sequence $\mathcal{V}$ of open sets V with $d(V) < \delta$ all $V \in \mathcal{V}$

$$m(F_\wedge \mathcal{V},h) \leq \sum_{V \in \mathcal{V}} h(d(V)) + \epsilon$$

Proof of the lemma

By the definition of Hausdorff diametral measure there exists a number δ depending on F and ϵ such that for any covering $\mathcal{U}$ of open sets U of the set F, which is such that

$d(U) < \delta$ all $U \in \mathcal{U}$ then

(1) $\sum_{U \in \mathcal{U}} h(d(U)) > m(F,h) - \frac{\epsilon}{2}$

where ϵ is any prescribed positive number.

Let $\mathcal{V}$ be any sequence of open sets V with $d(V) < \delta$ all $V \in \mathcal{V}$ Then since any $\mathcal{V}$ is an open set $F_\wedge \mathcal{V}$ is measurable and

(2) $m(F,h) = m(F_\wedge \mathcal{V},h) + m(F - \overline{F_\wedge \mathcal{V}},h)$

and from (1) and (2) we conclude that

(3) $\sum_{U \in \mathcal{U}} h(d(U)) > m(F_\wedge \mathcal{V},h) + m(F - \overline{F_\wedge \mathcal{V}},h) - \frac{\epsilon}{2}$

Now given any $\mathcal{V}$ we can find a set $\mathcal{U}_1$ of open sets U_1 with $d(U_1) < \delta$ all $U_1 \in \mathcal{U}_1$ such that $\mathcal{U}_1$ covers the set $F - \overline{F_\wedge \mathcal{V}}$ and also

(4) $\sum_{U_1 \in \mathcal{U}_1} h(d(U_1)) \leqslant m(F - \overline{F_\wedge \mathcal{V}},h) + \frac{\epsilon}{2}$

Let $\mathcal{U}$ be the class of sets $\mathcal{V} + \mathcal{U}_1$. Then

(5) $\sum_{U \in \mathcal{U}} h(d(U)) \leqslant \sum_{V \in \mathcal{V}} h(d(V)) + \sum_{U_1 \in \mathcal{U}_1} h(d(U_1))$

and $\mathcal{U}$ is a covering of F with $d(U) < \delta$ all $U \in \mathcal{U}$. Thus this class $\mathcal{U}$ satisfies the inequality (3).

From (4) and (5)

$$\sum_{V \in \mathcal{V}} h(d(V)) + m(F - \overline{F_\wedge \mathcal{V}},h) + \frac{\epsilon}{2}$$

$$\geqslant \sum_{V \in \mathcal{V}} h(d(V)) + \sum_{U_1 \in \mathcal{U}_1} h(d(U_1))$$

$$\geqslant \sum_{U \in \mathcal{U}} h(d(U))$$

and from (3) $\geqslant m(F_\wedge \mathcal{V},h) + m(F - \overline{F_\wedge \mathcal{V}},h) - \frac{\epsilon}{2}$

Hence $\sum_{V \in \mathcal{V}} h(d(V)) \geqslant m(F_\wedge \mathcal{V},h) - \epsilon$

which proves the lemma.

Proof of the theorem

Given any number $u > 0$ the set of points x such that $\overline{D}(x,h) > u$ is measurable. Let F' be the set where $\overline{D}(x,h) > 1$ and suppose that $m(F',h) > 0$. Then there exists a positive number b such that $m(F'',h) > 0$ where F'' is the set at which $\overline{D}(x,h) > 1 + b$.

Write $\epsilon = \min(\frac{1}{2}.m(F'',h), \ \frac{b}{72} m(F'',h))$

In the case of a set in $p > 2$ dimensions the factor 1/72 is replaced by $1/2(3[\sqrt{p}]+1))^p$

Let ρ be a positive number such that the inequality of the lemma is satisfied for the given set S, the number ϵ and $\rho = \delta$.

Write

$$F'' = F_1 + F_2 + \ldots\ldots + F_j + \ldots\ldots$$

where F_j is a set of points $x \in F''$ about which it is possible to draw an open circle $C(x,r)$ of centre x and radius r where

$$\frac{\rho}{j} \leqslant 2r < \frac{\rho}{j+1}$$

and such that

$$\frac{m(S.C(x,r),h)}{h(2r)} > 1 + b$$

F_j is measurable and $C(x,r)$ is called a density circle of class j.

About any point of F_1 we draw a circle of radius r and class 1 and a concentric circle of radius 3r. Then about any point of F_1 outside these two circles we describe two concentric circles in a similar way. We continue the process at each stage taking a point outside the circles already drawn and so that a density circle of the lowest possible class can be drawn together with a circle three times its radius. We thus obtain a finite or enumerably infinite ($m(S,h)$ being finite only a finite number of non-overlapping density circles can be drawn) set C of non-overlapping density circles and a set C' of concentric circles such that C' covers all the points of F''.

Each circle radius $3r$ can be completely covered by at most 36 circles radius r. Thus if the class C' is replaced by a class C'' which consists of circles radius r and which covers F'', C'' satisfies the inequality

$$\sum_{C} h(2r) \geqslant \frac{1}{36} \sum_{C''} h(2r)$$

In p-dimensional space the factor $\frac{1}{36}$ is replaced by $\frac{1}{(3([\sqrt{p}]+1))^{p}}$

The radius of any circle of C'' is $r < \rho$ and so

$$\sum_{C''} h(2r) \geqslant m(F'',h) - \epsilon \geqslant \tfrac{1}{2}m(F'',h)$$

and hence $\sum_{C} h(2r) \geqslant \frac{1}{72}\, m(F'',h)$

Now the circles of C do not overlap and so

$$m(F''_{\wedge}C,h) > (1+b) \sum_{C} h(2r)$$

$$\geqslant \sum_{C} h(2r) + \frac{b}{36}\, m(F'',h)$$

$$(7) \qquad > \sum_{C} h(2r) + \epsilon$$

But taking C as the set γ of the lemma (which is possible since $2r < \rho$) we have

$$(8) \qquad m(F''_{\wedge}C,h) \leqslant \sum_{C} h(2r) + \epsilon$$

(7) and (8) being contradictory we conclude that $m(F'',h) = 0$. This proves the theorem.

SECTION III,2.

THEOREM

In two dimensional real Euclidean space take rectangular cartesian axes OX and OY. If D is a measurable set on the OX axis with dimension function $h(x)$ i.e. $0 < m(D,h) < \infty$ and E is any measurable linear set on the OY axis where $0 < m(E,x) < \infty$, then

$$\alpha\, m(D,h).m(E,x) \geqslant m(D_{\times} E,xh) \geqslant \beta\, m(D,h).m(E,x)$$

where $D_{\times}E$ is the cartesian product of D and E, and α and β are constants. Constant w.r.t. what?

Proof of the left hand inequality

Let $\mathcal{J}$ be an enumerable class of intervals I_i on the OY axis covering E and $d(I_i) < \delta$ where δ is a small positive number and such that

$$\sum_{I_i \in \mathcal{J}} d(I_i) < m(E,x) + \epsilon$$

ϵ being a small positive number.

For any particular interval I_j consider the set $D_{\times}I_j$ Let n be a large integer and cover D by intervals I_i' where $d(I_i') < \dfrac{d(I_j)}{n}$ and such that if $\mathcal{J}'$ is the class of all intervals I_i'

$$\sum_{I_i' \in \mathcal{J}'} h(d(I_i')) < m(D,h) + \epsilon$$

The product of these intervals with I_j consists of a set of rectangles. Each of these rectangles can be covered by not more than $4\left\{\left[\frac{d(I_j)}{d(I_i')}\right] + 1\right\}$ squares of diagonal $d(I_i')$ where $\left[\frac{d(I_j)}{d(I_i')}\right]$ is the integral part of $\frac{d(I_j)}{d(I_i')}$. Thus the set $D_x I_j$ has been covered by squares of diagonal a_i such that

$$\sum_i a_i h(a_i) \leqslant \sum_i 4\left\{\left[\frac{d(I_j)}{d(I_i')}\right] + 1\right\} d(I_i')\, h(d(I_i'))$$

thus

$$\sum_i a_i h(a_i) \leqslant 4\left\{\sum_i d(I_j)\, h(d(I_i')) + \sum_i d(I_i')\, h(d(I_i'))\right\}$$

$$\leqslant 4\, d(I_j) \left\{\{m(D,h) + \epsilon\}\{1 + \tfrac{1}{n}\}\right\}$$

Carrying out the same process for all the intervals $I_j \in \mathcal{J}$ we obtain a covering of squares such that

$$\sum_i a_i h(a_i) \leqslant 4\,\{1 + \tfrac{1}{n}\}\{m(D,h) + \epsilon\}\{m(E,x) + \epsilon\}$$

Since δ, ϵ and $1/n$ are arbitrarily small we have

$$m(D_x E, xh) \leqslant 4\, m(D,h)\, m(E,x)$$

Proof of the right hand inequality

We first establish that it is sufficient to take D and E and therefore $D_x E$ to be closed sets. To do this we need the following lemma.

LEMMA

If F is a measurable set with dimension function $h(x)$, given any $\epsilon > 0$ there exists a closed set F' such that

$$m(F - F', h) + m(F' - F, h) < 3\epsilon$$

Proof

Given $\epsilon > 0$ let $\mathcal{U}_1$ be an enumerable class of open convex sets U_{1i} covering F with $d(U_{1i}) < \rho$ where ρ is a number > 0 depending on ϵ, such that

$$\sum_{U_{1i} \in \mathcal{U}_1} h(d(U_{1i})) < m(F,h) + \epsilon$$

Choose n_1 so that

$$m(F \wedge (\bigcup_{n_1+1}^{\infty} U_{1i}), h) < \frac{\epsilon}{2}$$

This can be done by using the lemma proved in theorem 2 in Section III,1, and choosing n_1 so that U_{1i} $(i=n_1+1 \ldots .)$ have $d(U_{1i}) < \delta$ where δ is a number such that the inequality of the lemma holds with $\epsilon/2$.

Let F_1' be the closure of $\bigcup_{i=1}^{n_1} U_{1i}$ and write $E_1 = F_1' \wedge F$. Now suppose that E_1 is covered by a class $\mathcal{U}_2$ of open convex sets U_{2i} such that

$$\sum_{U_{2i} \in \mathcal{U}_2} h(d(U_{2i})) < m(F,h) + \epsilon$$

and also such that $d(U_{2i}) < \frac{\rho}{2}$ all $U_{2i} \in \mathcal{U}_2$. Now choose n_2 so that

$$m(F \wedge (\bigcup_{n_2+1}^{\infty} U_{2i}), h) < \frac{\epsilon}{2^2}$$

Let F' be the closure of $\bigcup_{i=1}^{n_2} U_{2i}$ and $E_2 = F_2' \wedge F$. Continue the process using a covering $\mathcal{U}_3$ of E_2 and so on.

Let $F' = \prod_{1}^{\infty} F'_j$. Then F' exists and is closed. Moreover since it is contained in each of the finite sequences $\{U_{ji}\}$ $(i=1\ldots n_j)$ in which the convex sets U_{ji} have $d(U_{ji}) < \frac{\rho}{2^{j-1}}$ we have

$$m(F',h) \leqslant m(F,h) + \epsilon$$

and

$$m(F - F',h) \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2^2} + \frac{\epsilon}{2^3} + \ldots\ldots \leqslant \epsilon$$

Also
$$m(F_\wedge F',h) \geqslant m(F,h) - \epsilon$$

so
$$m(F' - F,h) \leqslant m(F,h) + \epsilon - m(F,h) + \epsilon$$
$$\leqslant 2\epsilon$$

This completes the proof of the lemma.

We now establish that it is sufficient to prove the theorem when D and E are closed. For suppose that they are not closed. Then there exists a closed set D' satisfying the inequality of the lemma with the set D. And

$$\begin{aligned} m(D'_\times E,xh) &= m(\{D + \overline{D' - D}\}_\times E,xh) \\ &= m(D_\times E,xh) + m(\overline{D' - D}_\times E,xh) \\ &\geqslant m(D_\times E,xh) + 3\epsilon\beta\, m(E,xh) \end{aligned}$$

Wrong D' mixed up with D.

where ϵ can be taken as small as we please. Hence it is sufficient to prove the result for D' and E, and we can therefore take D to be closed. By a similar argument we can take E to be closed also.

From theorem 2, Section III,1, we have $\overline{D}(x,h) \leqslant 1$ for all points $x \in D$ except possibly for a set of measure zero. Let ϵ be small and take D_η as the set of points belonging to D for which

$$\frac{m(D_\wedge I,h)}{h(d(I))} \leqslant 1+\epsilon$$

whenever $d(I) < \eta$, I being an open interval centred on the point x. By decreasing η the value $m(D_\eta, h)$ can be made as near that of $m(D,h)$ as we please.

Let η_1 be the number such that

(1) $$m(D_{\eta_1}, h) \leqslant m(D,h)\,[1+\epsilon]$$

Using a similar argument for E we obtain a number η_2 such that

(2) $$m(E_{\eta_2}, x) \leqslant m(E,x)\,[1+\epsilon]$$

Let δ_0 be the smaller of η_1 and η_2. Then given any half open rectangle R let I_x and I_y be the projections of R on the X and Y axes respectively. Define

$$\phi(R) = [m(D_{\eta^\wedge} I_x, h)][m(E_{\eta^\wedge} I_y, x)]$$

and for any half open figure $\mathcal{R}$ consisting of rectangles R we define

$$\phi(\mathcal{R}) = \sum_{R\in\mathcal{R}} \phi(R)$$

If $d(\mathcal{R}) \leqslant \delta_0$ we shall show that

$$\phi(\mathcal{R}) \leqslant 2(1+\epsilon)^2\,[h(d(\mathcal{R}))]\,d(\mathcal{R})$$

Consider a rectangle R with projections I_x and I_y on the two axes. From (1) we have

$$\frac{m(D_{\eta^\wedge} I_x, h)}{h(d(I_x))} \leqslant 1+\epsilon$$

and from (2)

$$\frac{m(E_{\eta^\wedge} I_y, x)}{d(I_y)} \leqslant 1+\epsilon$$

Thus $$\phi(R) \leqslant [1+\epsilon]^2\ d(I_y)\ h(d(I_x))$$

$$\leqslant [1+\epsilon]^2\ d(R)\ h(d(R))$$

All $R \in \mathcal{R}$ can be completely covered by a square J side $d(\mathcal{R})$ with diameter $\sqrt{2}\, d(\mathcal{R})$. J in turn can be contained in two rectangles R' with diameters $d(\mathcal{R})$ and since these two rectangles contain $\mathcal{R}$

$$\phi(\mathcal{R}) \leqslant 2\,\phi(R') \leqslant 2\,(1+\epsilon)^2\, d(R')\, h(d(R'))$$

i.e.

$$\phi(\mathcal{R}) \leqslant 2\,(1+\epsilon)^2\, d(\mathcal{R})\, h(d(\mathcal{R}))$$

The conditions of theorem 1 Section III,1, are thus satisfied and hence

$$m(D_\times E, xh) \geqslant \frac{m(D_\eta, h)\, m(E_\eta, x)}{2}$$

and thus

$$m(D_\times E, xh) \geqslant \frac{m(D,h)\, m(E,x)}{2}$$

since $m(D_\eta, h)$ can be made as near to $m(D,h)$ and $m(E_\eta, x)$ as near to $m(E,x)$ as we please.

This completes the proof of the theorem.

SECTION III,3.

THEOREM 1.

If D is a set of dimension $h(x)$ on the X axis and E is a set of dimension $g(x)$ on the Y axis then the dimension function $f(x)$ of the cartesian product $D_\times E$ of D and E satisfies

(1) $$\limsup_{x\to 0} \frac{f(x)}{x\,h(x)} > 0$$

(2) $$\limsup_{x\to 0} \frac{f(x)}{x\,g(x)} > 0$$

Proof

This follows from the theorems in Sections II,4, and III,2, since the set $D \times E$ is a subset of the cartesian product of the set D with a linear set on the Y axis and also a subset of the cartesian product of the set E with a linear set on the X axis.

CHAPTER IV

SECTION IV,1.

THEOREM

and finite L measure.

If E is a closed set on the x axis having dimension function x^α and such that the lower x^α-density at every point a belonging to E is greater than some positive constant μ then the dimension function of the cartesian product of E and a similar set E' on the y axis is $x^{2\alpha}$.

Proof

It is a known result that $m(E_x E', x^{2\alpha}) > 0$.

Let δ be any small positive number. Then we are given that there exists ϵ such that

Not quite. This assumes a uniformity condition that should be established by taking a subset of E

$$\frac{m(E_\wedge I, x^\alpha)}{d(I)^\alpha} > \mu + \epsilon$$

for all $d(I) < \delta$ and where I is an interval centred on a point $a \in E$. Take any point $a_1 \in E$ and cover it by an interval I_1 length $2r < \delta$ which is centred on the point a_1. Take any other point $a_2 \in E$ which does not belong to I_1 and enclose this point in an open interval I_2 length 2r centred on a_2. Now take a point $a_3 \in E$ which does not belong to I_1 or I_2 and an open interval I_3 length 2r surrounding this point. Continuing in this way a finite sequence of open sets I is obtained which covers E and which is such that no point of E belongs to more than two intervals I. Denote this sequence by $\mathcal{J}$. Each interval I has diameter 2r and satisfies

$$m(E \wedge I, x^{\alpha}) > (\mu + \epsilon)\, d(I)^{\alpha}$$

Thus $\sum_{I \in \mathcal{J}} m(E \wedge I, x^{\alpha}) > (\mu + \epsilon) \sum_{I \in \mathcal{J}} d(I)^{\alpha}$

Since no point of E can belong to more than two intervals I

$$2m(E, x^{\alpha}) > \sum_{I \in \mathcal{J}} m(E \wedge I, x^{\alpha})$$

and hence $2m(E, x^{\alpha}) > (\mu + \epsilon) \sum_{I \in \mathcal{J}} d(I)^{\alpha}$

Cover the set E' on the y axis by a similar sequence $\mathcal{J}'$ of open sets I'. Then $\mathcal{J} \times \mathcal{J}'$ that is the class of squares $I \times I'$ covers the set $E \times E'$ and

$$\sum_{I \times I' \in \mathcal{J} \times \mathcal{J}'} d(I \times I')^{2\alpha} = 2^{\alpha} \left\{ \sum_{I \in \mathcal{J}} d(I)^{\alpha} \right\}^{2}$$

$$< 2^{\alpha} \left\{ \frac{2\, m(E, x^{\alpha})}{(\mu + \epsilon)} \right\}^{2}$$

from the above inequality.

Hence we have established that

$$m_{\delta}(E \times E', x^{2\alpha}) < \infty$$

and since the above result holds for all $\delta > 0$ and the number μ is non-zero we have

$$m(E \times E', x^{2\alpha}) < \infty$$

This completes the proof of the theorem.

SECTION IV, 2.

THEOREM

In two-dimensional real Euclidean space there exists a set S having finite non-zero Hausdorff diametral measure with dimension function $x^{2\mu}$ where $\mu = \log 2/\log 3$ such that if

S(a,b) is the translation of S through a distance a parallel to the x axis and b parallel to the y axis then $S_{\wedge}S(a,b)$ is a set having zero Hausdorff diametral measure with measure function $x^{2\mu}$ for all values of a and b such that either a or b is non-zero.

Proof

Take rectangular cartesian axes in four-dimensional real Euclidean space. Let D be a plane set in the x,y plane and E a plane set in the z,t plane; then the plane $x = z-a$, $y = t-b$ intersects the set $D_{\times}E$ in a set whose projection in the x,y plane is $D_{\wedge}E(a,b)$.

Take four different axes O'X', Y',Z',T'. On the O'X' axis construct the Cantor ternary set in the interval $0 \leqslant x' \leqslant 1$. On the O'Y' axis construct the Cantor ternary set in the interval $0 \leqslant y' \leqslant 1$ and let F' denote the cartesian product of these two sets. Denote by I'_{ni} $(i=1 \ldots . 2^{2n})$ the closed squares belonging to F' at the n^{th} stage. Construct a similar set in the O'Z'T' plane.

Map these two sets onto the OXY, OZT planes by the relations $x = x' + dx'^2$ $y = y' + dy'^2$ $z = z' + dz'^2$ $t = t' + dt'^2$ where $d = 10^{-6}$ (say). Call the resulting sets F_1 and F_2 respectively and the rectangles corresponding to I'_{ni} $I^{(1)}_{ni}$ $I^{(2)}_{ni}$ respectively.

Consider the set on the OX axis. Denote it by Q. Then

$Q = \bigcap_{n=0}^{\infty} Q_n$ where Q_n is the union of 2^n intervals lengths ℓ_i

$$\ell_i = x'_i - x'_{i-1} + d(x'^2_i - x'^2_{i-1})$$

and $\quad x'_i - x'_{i-1} = \frac{1}{3^n} \qquad 0 < x'_i \leqslant 1.$

Hence $\quad \frac{1}{3^n} \leqslant \ell_i \leqslant \frac{(1+2d)}{3^n} \qquad$ for all i.

Thus if C is the Cantor ternary set in the interval [0,1]

$$m(C,x^{\mu}) \leqslant m(Q,x^{\mu}) \leqslant (1+2d)^{\mu} m(C,x^{\mu})$$

i.e. $\qquad 0 < m(Q,x^{\mu}) < \infty$

From the theorem in Section IV,1, on page 41 the dimension function of both F_1 and F_2 is $x^{2\mu}$.

It is sufficient to prove that $F_1 \times F_2$ is intersected by the plane $x = z-a \quad y = t-b$ in a set of zero Hausdorff diametral measure with measure function $x^{2\mu}$ for all values of a and b such that either a or b is non-zero.

Let $D_1 \ldots . D_{2n}$ be the parts of F_1 covered by the rectangles $I^{(1)}_{ni}$ and $E_1 \ldots .. E_{2n}$ parts of F_2 for $I^{(2)}_{nj}$. Then it is required to prove that $x = z-a$, $y = t-b$ intercepts the set $D_i \times E_j$ in a set of zero measure whenever $i \neq j$. If the plane intercepts the set $D_i \times E_i$ then by increasing n we can ensure that it ceases to do so since only the plane $x = z$, $y = t$ meets the set $D_i \times E_i$ for all n. Let $x = z-a$, $y = t-b$ intercept the set $D_i \times E_j$ in G. Then G lies in the cuboid $I^{(1)}_{ni} \times I^{(2)}_{nj}$ which has the point (x_0, y_0, z_0, t_0) as the point nearest the origin. For any $m > 0$ consider the cuboids $I^{(1)}_{m+n,p} \times I^{(2)}_{m+n,q} \in I^{(1)}_{ni} \times I^{(2)}_{nj}$. These are intersected by the plane $x = z-a$, $y = t-b$ in a set of closed convex sets $J_1 \dots J_K$ covering G. Then we have to prove that

$$\lim_{m\to\infty} \sum_{1}^{k} d(J_i)^{2\mu} = 0$$

It is sufficient to prove that

$$\sum_{i=1}^{k} d(J_i)^{2\mu} \leqslant K\, d(J_0)^{2\mu}$$

where J_0 is the rectangle in which $x = z-a$, $y = t-b$ intercepts the four-dimensional cuboid $I^{(1)}_{ni} \times I^{(2)}_{nj}$ and K is a constant less than one which is independent of (x_0, y_0, z_0, t_0). For the argument can then be repeated in each of the cuboids $I^{(1)}_{n+m,p} \times I^{(2)}_{n+m,q}$ to get the desired result.

To prove this we need the following lemma.

LEMMA

Let $d = 10^{-6}$ and let $0 \leqslant A_0 \leqslant A \leqslant A_1 < B_0 \leqslant B \leqslant B_1 \leqslant 1$ $A_0 < A_1$ and $B_0 < B_1$ where A_0, A_1, B_0 and B_1 are given constants and A and B are any numbers satisfying the above inequality. Let c be a number such that $0 \leqslant c \leqslant d(B_0 - A_1)$. Take A_0', A', A_1' B_0', B', B_1' and c' satisfying similar inequalities.

Consider the four dimensional cuboid which has the three dimensional planes

$$x = 0 \qquad y = 0$$
$$x = 1 + 2dA + dc \qquad y = 1 + 2dA' + dc'$$
$$z = 0 \qquad t = 0$$
$$z = 1 + 2dB + dc \qquad z = 1 + 2dB' + dc'$$

as its faces. This is approximately the unit four dimensional cube: denote it by $\mathcal{J}_0$. Divide this cuboid into 81 cuboids by the three dimensional planes

$$x = \frac{1}{3}(1 + 2dA) + \frac{1}{9}\,dc \qquad y = \frac{1}{3}(1 + 2dA') + \frac{1}{9}\,dc'$$

$$x = \frac{2}{3}(1 + 2dA) + \frac{4}{9}\,dc \qquad y = \frac{2}{3}(1 + 2dA') + \frac{4}{9}\,dc'$$

$$z = \frac{1}{3}(1 + 2dB) + \frac{1}{9}\,dc \qquad t = \frac{1}{3}(1 + 2dB') + \frac{1}{9}\,dc'$$

$$z = \frac{2}{3}(1 + 2dB) + \frac{4}{9}\,dc \qquad t = \frac{2}{3}(1 + 2dB') + \frac{4}{9}\,dc'$$

Consider the sixteen cuboids which have one vertex in common with $\mathcal{J}_0$ and let the plane $x = z + a_1 \quad y = t + b_1$ intersect them in sets J_i the original cuboid being intersected in a set J_0. Then there exists $K < 1$ such that

$$\Sigma\, d(J_i)^{2\mu} \leqslant K\, d(J_0)^{2\mu}$$

the summation being extended over all the smaller cuboids intersected by the plane and the inequality holding uniformly for A, A', B, B', c and c' satisfying the inequalities given above and a_1 and b_1 satisfying

$$\tfrac{1}{2} \geqslant a_1 \geqslant -\left[\frac{2}{3}(1 + 2dB) + \frac{4}{9}\,dc\right] \quad \text{and}$$

$$1 + 2dA' + dc' \geqslant b_1 \geqslant -(1 + 2dB' + dc')$$

or similar inequalities with a_1 and b_1 reversed.

Proof

Define α_i, β_i (i=0,1,2,3) by

$$\alpha_0 = 0, \quad \alpha_1 = \frac{1}{3}(1 + 2dA) + \frac{1}{9}\,dc \quad \alpha_2 = \frac{2}{3}(1 + 2dA) + \frac{4}{9}\,dc$$

$$\alpha_3 = 1 + 2dA + dc,$$

$\beta_0 = 0, \quad \beta_1 = \frac{1}{3}(1 + 2dB) + \frac{1}{9}\,dc \quad \beta_2 = \frac{2}{3}(1 + 2dB) + \frac{4}{9}\,dc$

$\beta_3 = 1 + 2dB + dc,$

and α_i', β_i' $(i=0,1,2,3)$ similarly.

Denote the cuboid defined by

$\alpha_i \leqslant x \leqslant \alpha_{i+1} \quad \alpha_j' \leqslant y \leqslant \alpha_{j+1}' \quad \beta_k \leqslant z \leqslant \beta_{k+1}$ and

$\beta_\ell' \leqslant t \leqslant \beta_{\ell+1}'$ by $\mathcal{J}(i,j,k,\ell)$ where i, j, k, ℓ, independently take the values 0,1, and 2.

Consider the plane P $x = z + a_1$, $y = t + b_1$. This meets $\mathcal{J}(i,j,k,\ell)$ if and only if the inequalities

$$\alpha_i - a_1 \leqslant z \leqslant \alpha_{i+1} - a_1 \quad \beta_k \leqslant z \leqslant \beta_{k+1}$$

and

$$\alpha_j' - b_1 \leqslant t \leqslant \alpha_{j+1}' - b_1 \quad \beta_\ell' \leqslant t \leqslant \beta_{\ell+1}'$$

are not inconsistent i.e. if and only if

$$\alpha_{i+1} - a_1 \geqslant \beta_k \quad \alpha_i - a_1 \leqslant \beta_{k+1}$$

and

$$\alpha_{j+1}' - b_1 \geqslant \beta_\ell' \quad \alpha_j' - b_1 \leqslant \beta_{\ell+1}'$$

i.e. if and only if

$$\alpha_i - \beta_{k+1} \leqslant a_1 \leqslant \alpha_{i+1} - \beta_k$$

and

$$\alpha_j' - \beta_{\ell+1}' \leqslant b_1 \leqslant \alpha_{j+1}' - \beta_\ell'$$

Of the 81 cuboids $\mathcal{J}(i,j,k,\ell)$ 16 have vertices in common with a vertex of $\mathcal{J}_0$. They are the cuboids $\mathcal{J}(i,j,k,\ell)$ for which each argument i,j,k,ℓ is either 0 or 2. We wish to establish which of these are intersected by the plane P for the various values of a_1 and b_1.

By direct calculation, since $0 \leqslant A < B \leqslant 1$ and $c \leqslant d(B - A)$

$\alpha_2 - \beta_3 < \alpha_1 - \beta_2 < -\beta_1$ and $\alpha_3 - \beta_2 < \alpha_2 - \beta_1 < \alpha_1$ and it

$\alpha_2 - \alpha_1 = \frac{1}{3}(1 + 2dA + dc) < \frac{1}{3}(1 + 2dB) + \frac{1}{9}dc$

$A + c < B.$

$2d(A - B) < 2dc$

$A < B - c \quad c < d(B - A)$

follows that P meets the following cuboids $J(i, j, k, \ell)$ with i, j, k, ℓ all 0 or 2.

values of b_1 \ values of a_1	$-\beta_3$ to $\alpha_2-\beta_3$	$\alpha_2-\beta_3$ to $\alpha_1-\beta_2$	$\alpha_1-\beta_2$ to $-\beta_1$	$-\beta_1$ to $\alpha_3-\beta_2$	$\alpha_3-\beta_2$ to $\alpha_2-\beta_1$	$\alpha_2-\beta_1$ to α_1	α_1 to α_3
α_1' to α_3'	(0220)	{(0220), (2220)}	(2220)	(2220) (0200)	(0200)	{(0200), (2200)}	(2200)
$\alpha_2'-\beta_1'$ to α_1'	(0220) (0020)	{(0220), (0020), (2220), (2020)}	(2220) (2020)	(2220) (0200) (2020) (0000)	(0200) (0000)	{(0200), (0000), (2200), (2000)}	(2200) (2000)
$\alpha_3'-\beta_2'$ to $\alpha_2'-\beta_1'$	(0020)	{(0020), (2020)}	(2020)	(2020) (0000)	(0000)	{(0000), (2000)}	(2000)
$-\beta_1'$ to $\alpha_3'-\beta_2'$	(0020) (0222)	{(0222), (0020), (2222), (2020)}	(2222) (2020)	(2222) (0202) (2020) (0000)	(0202) (0000)	{(0202), (0000), (2202), (2000)}	(2202) (2000)
$\alpha_1'-\beta_2'$ to $-\beta_1'$	(0222)	{(0222), (2222)}	(2222)	(2222) (0202)	(0202)	{(0202), (2202)}	(2202)
$\alpha_2'-\beta_3'$ to $\alpha_1'-\beta_2'$	(0222) (0022)	{(0222), (0022), (2222), (2022)}	(2222) (2022)	(2222) (0202) (2022) (0002)	(0202) (0002)	{(0202), (0002), (2202), (2002)}	(2202) (2002)
$-\beta_3'$ to $\alpha_2'-\beta_3'$	(0022)	{(0022), (2022)}	(2022)	(2022) (0002)	(0002)	{(0002), (2002)}	(2002)

Transform the coordinates by

$$x = x'' + z'' \qquad y = y'' + t''$$

$$z = z'' - x'' - a_1 \qquad t = t'' - y'' - b_1$$

Then the plane P becomes the plane $x'' = 0$, $y'' = 0$. The sets $J(i, j, k, \ell)$ all have sides parallel to the z'' and t'' axes and are thus rectangles.

Consider the lines cut by the cubes $\mathcal{J}_{(i,j,k,\ell)}$ $(i,j,k,\ell = 0 \text{ or } 2)$ parallel to the z'' axis.

For the cuboid $\mathcal{J}_0$ these are given by

$$\begin{array}{lll} z'' = 0 & z'' = a_1 + \beta_3 & -\beta_3 \leqslant a_1 \leqslant \alpha_3 - \beta_3 \\ z'' = 0 & z'' = \alpha_3 & \alpha_3 - \beta_3 \leqslant a_1 \leqslant 0 \\ z'' = a_1 & z'' = \alpha_3 & 0 \leqslant a_1 \leqslant \alpha_3 \end{array}$$

For $\mathcal{J}(0220)$, $\mathcal{J}(0020)$, $\mathcal{J}(0222)$ and $\mathcal{J}(0022)$

$$\begin{array}{lll} z'' = 0 & z'' = a_1 + \beta_3 & -\beta_3 \leqslant a_1 \leqslant \alpha_1 - \beta_3 \\ z'' = 0 & z'' = \alpha_1 & \alpha_1 - \beta_3 \leqslant a_1 \leqslant -\beta_2 \\ z'' = a_1 + \beta_2 & z'' = \alpha_1 & -\beta_2 \leqslant a_1 \leqslant \alpha_1 - \beta_2 \end{array}$$

For $\mathcal{J}(2220)$, $\mathcal{J}(2020)$, $\mathcal{J}(2222)$ and $\mathcal{J}(2022)$

$$\begin{array}{lll} z'' = \alpha_2 & z'' = a_1 + \beta_3 & \alpha_2 - \beta_3 \leqslant a_1 \leqslant \alpha_3 - \beta_3 \\ z'' = \alpha_2 & z'' = \alpha_3 & \alpha_3 - \beta_3 \leqslant a_1 \leqslant \alpha_2 - \beta_2 \\ z'' = a_1 + \beta_2 & z'' = \alpha_3 & \alpha_2 - \beta_2 \leqslant a_1 \leqslant \alpha_3 - \beta_2 \end{array}$$

For $\mathcal{J}(0200)$, $\mathcal{J}(0000)$, $\mathcal{J}(0202)$ and $\mathcal{J}(0002)$

$$\begin{array}{lll} z'' = 0 & z'' = a_1 + \beta_1 & -\beta_1 \leqslant a_1 \leqslant \alpha_1 - \beta_1 \\ z'' = 0 & z'' = \alpha_1 & \alpha_1 - \beta_1 \leqslant a_1 \leqslant 0 \\ z'' = a_1 & z'' = \alpha_1 & 0 \leqslant a_1 \leqslant \alpha_1 \end{array}$$

For $\mathcal{J}(2200)$, $\mathcal{J}(2000)$, $\mathcal{J}(2202)$ and $\mathcal{J}(2002)$

$$\begin{array}{lll} z'' = \alpha_2 & z'' = a_1 + \beta_1 & \alpha_2 - \beta_1 \leqslant a_1 \leqslant \alpha_3 - \beta_1 \\ z'' = \alpha_2 & z'' = \alpha_3 & \alpha_3 - \beta_1 \leqslant a_1 \leqslant \alpha_2 \\ z'' = a_1 & z'' = \alpha_3 & \alpha_2 \leqslant a_1 \leqslant \alpha_3 \end{array}$$

In each case the rectangle $J_{(i,j,k,\ell)}$ increases in size when the first two equations hold, remains constant when the second pair holds and decreases when the third pair holds.

The lines parallel to the t'' axis are simiilar.

We note that $4\,(1/3)^{2\mu} = 1$. Let $f(a_1,b_1)$ denote $d(J_0)^{-2\mu}\ \Sigma\ d(J_{(i,j,k,\ell)})^{2\mu}$ where the summation is taken over all the cuboids intersected by the plane P for these values of a_1 and b_1. Then if P intersects 4 cuboids $\mathcal{J}(i,j,k,\ell)$ and $d(J_{(i,j,k,\ell)}) \leqslant \partial_1\ d(J_0)$ where $\partial_1 < 1/3$ $f(a_1,b_1) \leqslant K_1 < 1$. Also if P intersects 2 cuboids $\mathcal{J}(i,j,k,\ell)$ and if $d(J_{(i,j,k,\ell)}) \leqslant \partial_2\ d(J_0)$ where $\partial_2 < 1/\sqrt{3}$ for both the cuboids intersected then $f(a_1,b_1) \leqslant K_2 < 1$. If P intersects only one cuboid $\mathcal{J}(i,j,k,\ell)$ and $d(J(i,j,k,\ell)) \leqslant \partial_3\ d(J_0)$ where $\partial_3 < 1$ then $f(a_1,b_1) \leqslant K_3 < 1$. Clearly one of these three cases holds for the values of a_1 and b_1 satisfying

$$\alpha_1 \geqslant a_1 \geqslant \alpha_3 - \beta_2 \quad \text{or} \quad -\beta_1 \geqslant a_1 \geqslant \alpha_2 - \beta_3$$

$$\text{or} \quad \alpha'_1 \geqslant b_1 \geqslant \alpha'_3 - \beta'_2 \quad \text{or} \quad -\beta'_1 \geqslant b_1 \geqslant \alpha'_2 - \beta'_3$$

Consider the range $-\beta_1 \leqslant a_1 \leqslant \alpha_3 - \beta_2$

and $-\beta'_3 \leqslant b_1 \leqslant \alpha'_2 - \beta'_3$

The cuboids $\mathcal{J}_{(2022)}$ and $\mathcal{J}_{(0002)}$ are met. The maximum value of $f(a_1,b_1)$ occurs when $b_1 = \alpha'_1 - \beta'_3$ and a_1 lies somewhere in the interval $[(\alpha_3 - \beta_3),(\alpha_1 - \beta_1)]$. Thus

$$f(a_1,b_1) \leqslant \frac{[\alpha_1'^2 + (\alpha_3 - \alpha_2)^2]^\mu + [\alpha_1'^2 - \alpha_1^2]^\mu}{[\alpha_3^2 + \alpha_1'^2]^\mu}$$

$$\leqslant K_2 < 1$$

by the above argument. A similar result holds for

$-\beta_1 \leqslant a_1 \leqslant \alpha_3 - \beta_2$ and $+\alpha_3' \geqslant b_1 \geqslant \alpha_1'$

and for $-\beta_1' \leqslant b_1 \leqslant \alpha_3' - \beta_2'$ and either

$-\beta_3 \leqslant a_1 \leqslant \alpha_2 - \beta_3$ or $\alpha_1 \leqslant a_1 \leqslant \alpha_3$.

Now consider the range

$$-\beta_1 \leqslant a_1 \leqslant \alpha_3 - \beta_2 \text{ and } -\beta_1' \leqslant b_1 \leqslant \alpha_3' - \beta_2'.$$

The four cuboids met are $\mathcal{J}_{(2222)}$, $\mathcal{J}_{(2020)}$, $\mathcal{J}_{(0202)}$ and $\mathcal{J}_{(0000)}$.
The maximum value of $f(a_1,b_1)$ occurs at some point in the intervals $\alpha_2 - \beta_2 \leqslant a_1 \leqslant \alpha_1 - \beta_1$ and $\alpha_2' - \beta_2' \leqslant b_1 \leqslant \alpha_1' - \beta_1'$ since J_0 is constant throughout these intervals, and all four rectangles increase when a_1 and b_1 are less than the values belonging to these intervals and all four rectangles decrease when a_1 and b_1 are greater than these intervals, a_1 and b_1 increasing.

Consider the intervals $\alpha_2 - \beta_2 \leqslant a_1 \leqslant (\alpha_3 - \beta_3)/2$

and $\alpha_2' - \beta_2' \leqslant b_1 \leqslant (\alpha_3 - \beta_3)/2$

Then in this double interval

$$f(a_1,b_1) \leqslant \frac{\begin{aligned}&[(\alpha_3-\alpha_2)^2+(\alpha_3'-\alpha_2')^2]^\mu + [(\alpha_3-\alpha_2)^2+(\tfrac{\alpha_3'-\beta_3'+\beta_1}{2})^2]^\mu \\ &+ [(\tfrac{\alpha_3-\beta_3+\beta_1}{2})^2+(\alpha_3'-\alpha_2')^2]^\mu + [(\tfrac{\alpha_3-\beta_3+\beta_1}{2})^2-(\tfrac{\alpha_3'-\beta_3'+\beta_1'}{2})^2]^\mu\end{aligned}}{[\alpha_3{}^2+\alpha_3'^2]^\mu}$$

$$\begin{aligned}= &[\tfrac{1}{3}(1+2dA+\tfrac{5}{3}dc)^2 + \tfrac{1}{3}(1+2dA'+\tfrac{5}{3}dc')^2]^\mu + \\ &[\tfrac{1}{3}(1+2dA+\tfrac{5}{3}dc)^2 + \tfrac{1}{3}(1+d(3A'-B')+\tfrac{1}{3}dc')^2]^\mu + \\ &[\tfrac{1}{3}(1+d(3A-B)+\tfrac{1}{3}dc)^2 + \tfrac{1}{3}(1+2dA'+\tfrac{5}{3}dc')^2]^\mu +\end{aligned}$$

$$\frac{[\frac{1}{3} + d(3A-B) + \frac{1}{3}dc)^2 + \frac{1}{3}(1 + d(3A'-B') + \frac{1}{3}dc')^2]^{\mu}}{[(1 + 2dA + dc)^2 + (1 + 2dA' + dc')^2]^{\mu}}$$

$= K < 1$ since $c \leqslant d(B_0 - A_1) \leqslant d(B - A)$

and hence $\frac{5}{3}dc$ is very much less than $d(B - A)$ and also $\frac{5}{3}dc'$ is very much less than $d(B' - A')$. A similar result holds for the double intervals $\frac{\alpha_3 - \beta_3}{2} \leqslant a_1 \leqslant \alpha_1 - \beta_1$

and $\alpha_2' - \beta_2' \leqslant b_1 \leqslant \frac{\alpha_3' - \beta_3'}{2}$ or $\frac{\alpha_3' - \beta_3'}{2} \leqslant b_1 \leqslant \alpha_1' - \beta_1'$

and the double interval

$\alpha_2 - \beta_2 \leqslant a_1 \leqslant \frac{\alpha_3 - \beta_3}{2}$ and $\frac{\alpha_3' - \beta_3'}{2} \leqslant b_1 \leqslant \alpha_1' - \beta_1'$

Now consider the range $-\beta_3 \leqslant a_1 \leqslant \alpha_2 - \beta_3$

$-\beta_3' \leqslant b_1 \leqslant \alpha_2' - \beta_3'$

For $-\beta_3 \leqslant a_1 \leqslant \alpha_1 - \beta_3$ and $-\beta_3' \leqslant b_1 \leqslant \alpha_1' - \beta_3'$

$f(a_1, b_1) = 1$. As either a_1 or b_1 increases beyond these intervals $f(a_1, b_1)$ decreases. We take either $a_1 \geqslant -\beta_2$ or $b_1 \geqslant -\beta_2'$.

Then $f(a_1, b_1) \leqslant \frac{[\alpha_1{}^2 + \alpha_1'{}^2]^{\mu}}{[\alpha_1{}^2 + (\beta_3' - \beta_2')^2]^{\mu}}$ or $\frac{[\alpha_1{}^2 - \alpha_1'{}^2]^{\mu}}{[\alpha_1'{}^2 + (\beta_3 - \beta_2)^2]^{\mu}}$

$= K < 1$.

A similar result holds for the double intervals

$-\beta_3 \leqslant a_1 \leqslant \alpha_2 - \beta_3$ and $\alpha_3' \geqslant b_1 \geqslant \alpha_1'$

and $\alpha_3 \geqslant a_1 \geqslant \alpha_1$ and $\alpha_3' \geqslant b_1 \geqslant \alpha_1'$ or $-\beta_3' \leqslant b_1 \leqslant \alpha_2' - \beta_3'$

This completes the proof of the lemma.

Proof of the theorem

Consider the cuboid $I^{(1)}_{ni} \times I^{(2)}_{nj}$ which has the point (x_0, y_0, z_0, t_0) as its vertex nearest the origin. Assume that $x_0 < z_0$ and $y_0 < t_0$. Then if x', y', z', t' are the coordinates used in the construction of F_1 and F_2

$$x - x_0 = (x' - x'_0)(1 + 2dx'_0) + d(x' - x'_0)^2$$
$$y - y_0 = (y' - y'_0)(1 + 2dy'_0) + d(y' - y'_0)^2$$
$$z - z_0 = (z' - z'_0)(1 + 2dz'_0) + d(z' - z'_0)^2$$
$$t - t_0 = (t' - t'_0)(1 + 2dt'_0) + d(t' - t'_0)^2$$

Changing the origins to the points (x_0, y_0, z_0, t_0) and (x'_0, y'_0, z'_0, t'_0) respectively and the scale by 3^{-n} we get

$$x = x'(1 + 2dx'_0) + 3^{-n}dx'^2$$
$$y = y'(1 + 2dy'_0) + 3^{-n}dy'^2$$
$$z = z'(1 + 2dz'_0) + 3^{-n}dz'^2$$
$$t = t'(1 + 2dt'_0) + 3^{-n}dt'^2$$

Replacing x'_0 by A, y'_0 by A', z'_0 by B and t'_0 by B' the conditions of the lemma are satisfied if n is large enough. Moreover for any cuboid lying inside $I^{(1)}_{ni} \times I^{(2)}_{nj}$ the conditions are satisfied for values of A, A', B and B' lying in the ranges $x'_0 \leq A \leq x'_0 + 3^{-n} < z'_0 \leq B \leq z'_0 + 3^{-n}$ and $y'_0 \leq A' \leq y'_0 + 3^{-n} < t'_0 \leq B' \leq t'_0 + 3^{-n}$, provided n is chosen so that

$$x'_0 + 3^{-n} < z'_0 \quad \text{and} \quad y'_0 + 3^{-n} < t'_0.$$

Let the plane $x = z - a$, $y = t - b$ be transformed into the plane $x = z + a_1$, $y = t + b_1$ in the new coordinates. If either

$$\tfrac{1}{2} \geq a_1 \geq -\tfrac{2}{3}(1 + 2dB + \tfrac{2}{3}dc)$$

or
$$\tfrac{1}{2} \geq b_1 \geq -\tfrac{2}{3}(1 + 2dB' + \tfrac{2}{3}dc')$$

B, B', c and c' being those determined above we can replace the cuboid $I^{(1)}_{ni} \times I^{(2)}_{nj}$ which corresponds to the cuboid $\mathcal{J}_0$ of the lemma by the cuboids $I^{(1)}_{n+1,p} \times I^{(2)}_{n+1,q}$ which are intersected by the given plane since these are the cuboids $\mathcal{J}(i,j,k,\ell)$ $(i,j,k,\ell = 0 \text{ or } 2)$ of the lemma. In this case

$$\Sigma\, d(J_i)^{2\mu} < K\, d(J_0)^{2\mu} \quad \text{where} \quad K < 1.$$

If $a_1 < -\tfrac{2}{3}(1 + 2dB + \tfrac{2}{3}dc)$ and $b_1 < -\tfrac{2}{3}(1 + 2dB' + \tfrac{2}{3}dc')$

then we apply the whole argument to the cube $\mathcal{J}_{(0022)}$, choosing the suitable values of A, B, A', B', c and c'. If a_1 and b_1 still do neither satisfy the given relations we continue subdividing. Similar arguments hold for the other extreme values of a_1 and b_1. The process of subdivision must come to an end after a finite number of steps unless $x = z+a_1$, $y = t+b_1$ passes only through one of the corner points. It then forms an isolated point of intersection and can be ignored.

Thus a set of rectangles covering the set G can be replaced by a larger number whose sum of diagonals raised to the power 2^{μ} is less than K times the sum of the diagonals of the original set raised to the power 2^{μ}, and hence

$$m(D, x^{2\mu}) = 0 \quad \text{since } K < 1$$

i.e.
$$m(F_1 \times F_2, x^{2\mu}) = 0$$

We assumed that the vertex of the cuboid $I^{(1)}_{ni} \times I^{(2)}_{nj}$ which contains G, that is the point (x_0, y_0, z_0, t_0) satisfies $x_0 < z_0$ and $y_0 < t_0$. This implies that a and b (the values before the origin of coordinates was changed) are positive. But $S \wedge S(a,b) = S(-a,-b) \wedge S$ and hence the theorem has been proved for either a and b both positive or a and b both negative. If a is positive and b negative then a different proof is required. If in the lemma B′ is taken to be smaller than A′ i.e. the A′s and the B′s are interchanged then the lemma will still hold although the limits on the value of b_1 will be changed. The following proof of the theorem can be then applied taking $y_0 > t_0$. This can be done since in proving the lemma the value of a_1 taken to find the maximum value of $f(a_1, b_1)$ was independent of the value of b_1 taken. Thus the theorem can be proved for a positive and b negative or a negative and b positive.

This completes the proof of the theorem.

CHAPTER V

THEOREM V,1

In real Euclidean space of two dimensions the necessary and sufficient condition for a measure function h(x) to be a N.M.A.-dimension function is that

$$\liminf_{x\to 0} \frac{h(x)}{x} > 0$$

Proof

Necessity Let S be any finite set in the plane and h(x) a measure function such that $\liminf_{x\to 0} \frac{h(x)}{x} = 0$. Then S can be completely contained in a convex set of area α. Let $\mathcal{B}_\delta$ be the class of all coverings $\mathcal{U}$ of the set S such that $\mathcal{U}$ is a class of convex sets U with $\Delta(U) < \delta$.

$$\text{Then} \quad \inf_{\mathcal{U}\in\mathcal{B}_\delta} \sum_{U\in\mathcal{U}} h(\Delta(U)) \leqslant \inf \frac{\alpha}{\delta} h(\delta)$$

i.e.

$$B_\delta(S,h) \leqslant \alpha \inf \frac{h(\delta)}{\delta}$$

and hence

$$B(S,h) = 0$$

Sufficiency (1) Assume that $\liminf_{x\to 0} \frac{h(x)}{x} = \beta \quad 0 < \beta < \infty$

Let J be the unit square in the real Euclidean plane. Then by the same argument as the above

$$B(J,h) \leqslant \beta$$

Given $\epsilon > 0$ there exists δ such that

$$\frac{h(x)}{x} > (\beta - \epsilon) \qquad \text{all } x < \delta$$

Let $\mathcal{B}_\delta$ be the class of all coverings $\mathcal{U}$ of the set J such that $\mathcal{U}$ is a class of convex sets U with $\Delta(U) < \delta$

Then $$\sum_{U \in \mathcal{U}} h(\Delta(U)) > (\beta - \epsilon) \sum_{U \in \mathcal{U}} \Delta(U)$$

This holds for all $\mathcal{U} \in \mathcal{B}_\delta$ and hence

$$B_\delta(J,h) \geq (\beta - \epsilon)$$

i.e. $$B(J,h) \geq (\beta - \epsilon)$$

This is true for all $\epsilon > 0$ and hence

$$B(J,h) = \beta$$

(2) $\lim\limits_{x \to 0} \inf \dfrac{h(x)}{x} = \infty$

As in theorem in section II,1, on page 5 this implies that there exists arbitrarily small positive numbers x such that

$$\frac{h(x)}{x} \leq 2 \inf_{0<t\leq x} \frac{h(t)}{t} \qquad \text{(i)}$$

Let $\{A_n\}$ be any positive sequence of increasing numbers such that $\Sigma \frac{1}{A_n}$ is convergent. Define a sequence of numbers in the following way.

(a) Let x_0 be any number satisfying (i)

(b) x_n satisfies (i) for all n.

(c) $h(x_{n-1}) = C_n\, h(x_n) \qquad C_n \geq A_n$

(d) $2.C_n.x_n < x_{n-1}$

All three conditions can be satisfied simultaneously and $x_n \to 0$ as $n \to \infty$.

Construct a set S in the following way. In the 2-dimensional coordinate plane take a rectangle S_0 which is the cartesian product of a closed interval length x_0 on the x axis and a closed interval length 2 on the y axis, the interval on the y axis being denoted by J.

Let K_n denote the integral part of C_n and define z_n by

$$K_n x_n + (K_n - 1) z_n = x_{n-1} \qquad \text{for all } n.$$

On the base of S_0 take K_1 closed intervals length x_1 so that between any two such intervals lies an interval length z_1. Take the cartesian product of these K_1 intervals length x_1 with the interval J and denote this set by S_1. On the base of each rectangle of S_1 take K_2 closed intervals length x_2 two such intervals being separated by an open interval length z_2. Take the cartesian product of these $K_1 K_2$ closed intervals with J and denote this set by S_2. Continuing in this way we obtain a sequence of sets $S_1 \ldots . S_n \ldots$ such that each S_n is closed and $S_n \supset S_{n+1}$.

Let $S = \prod_1^\infty S_n$. Then S is also closed.

(α) To prove that $B(S,h) < \infty$

Since $x_n \to 0$ as $n \to \infty$ given any number $\delta > 0$ there exists a number n_0 such that x_{n_0} is less than δ. Any set S_n is a covering of S and if s_n denotes a rectangle of S_n $\Delta(s_n) = x_n$ Thus for any $n > n_0$ S_n is a covering of S with $\Delta(s_n) < \delta$ for all $s_n \in S_n$.

Then $$K_1 \ldots . K_n \, h(\Delta(s_n)) \leq K_1 \ldots . K_{n-1} \, h(\Delta(s_{n-1}))$$

$$\leq h(x_0)$$ by (c), and the fact that $K_n \leq C_n$.

Thus $$B_\delta(S,h) \leq h(x_0) \qquad \text{for all } \delta > 0$$

and hence $$B(S,h) \leq h(x_0)$$

(β) To prove that $B(S,h) > 0$

Let $\mathfrak{U}$ be any covering of open convex sets U, $\mathfrak{U}$ covering the set S. Then any set $U \in \mathcal{U}$ can be completely contained in a triangle area $4\Delta(U)$ or 4 triangles area $\Delta(U)$. Any such triangle in turn can be completely contained in a parallelogram P with one pair of sides lying parallel to the y axis. Let $\mathcal{P}$ be the class of all such parallelograms P corresponding to the sets $U \in \mathcal{U}$.

Then

$$8 \sum_{U \in \mathcal{U}} h(\Delta(U)) \geq \sum_{P \in \mathcal{P}} h(\Delta(P))$$

LEMMA

Given any finite covering $\mathcal{P}$ of the set S, $\mathcal{P}$ consisting of open parallelograms with one pair of sides parallel to the y axis, there exists a covering $\mathcal{R}$ of rectangles R which have sides parallel to the axes, and their bases lying on the x axis and heights 2. ~~Then~~ $\mathcal{R}$ is such that

$$4 \sum_{P \in \mathcal{P}} h(\Delta(P)) \geq \sum_{R \in \mathcal{R}} h(\Delta(R))$$

Proof of the lemma

Since S is a closed set and the limit of a decreasing sequence of sets $\{S_n\}$, given any $\eta > 0$ there exists an integer n such that every point of S_n $n \geq n_0$ is within a distance η of S. Any finite covering by open convex sets of the set S will also cover all points within a certain η of S and in particular the covering of parallelograms $\mathcal{P}$ covers all sets S_n for $n \geq$ some n_0.

Take $n \geq n_0$ and consider the set $S_n \wedge P$ $P \in \mathcal{P}$. This set consists of parallelograms of finite area. Denote the total area of these parallelograms i.e. the total area of S_n contained in P by $\Lambda_n(P)$. $\Lambda_n(P)$ is a number and not a set and

$$\sum_{P \in \mathcal{P}} \Lambda_n(P) \geqslant 2K_n K_{n-1} \ldots K_1 \; x_n$$
$$= \Lambda_n(P)$$

Consider any $P \in \mathcal{P}$. Let the length of the sides of P parallel to the y axis be β and the perpendicular distance between them be α. Then $\Delta(P) = \frac{1}{2}\alpha\beta$. If P is replaced by a rectangle W with sides parallel to the axes of length α and β, the side length α lying parallel to the x axis, and W is placed so that the sides parallel to the y axis have the same x coordinate as the corresponding sides of P then

$$\Delta(W) = \tfrac{1}{2}\alpha\beta = \Delta(P)$$

and $$\Lambda_n(P) = \Lambda_n(W)$$

Now consider the variation in $\Lambda_n(W)$ as the position of W is changed its size and shape remaining constant, and the sides remaining parallel to the axes.

(a) Moving the rectangle W in a direction parallel to the y axis. This leaves the value of $\Lambda_n(W)$ unaltered as long as W is contained in S_0 throughout.

(b) Moving W in a direction parallel to the x axis.

Let t be the integer such that $x_{t-1} >$ base of W $\geqslant x_t$. If W is not originally contained completely in a rectangle $s_{t-1} \in S_{t-1}$ then moving it until it is just contained in such a rectangle can only increase the value of $\Lambda_n(W)$. If W is completely contained in a rectangle s_{t-1} and is moved so that in its new position its right hand side coincides with the right hand side of a rectangle $s_t \in S_t$ whilst still being contained in S_{t-1} then $\Lambda_n(W)$ either remains constant or increases by at most a factor $2x_t$.

Thus if W and W' are two similar rectangles parallel to the axes, W placed anywhere in S_0 and W' placed so that it is completely contained in a rectangle s_{t-1} and its right hand side coinciding with the right hand side of one of the rectangles s_t and its base lying on the x axis then

$$\Lambda_n(W') \geqslant \Lambda_n(W)$$

This position of W' will then be referred to as the best position for a rectangle. For the purpose of this definition we have stated that the right hand sides must be coincident. It is clearly immaterial whether we choose the right hand sides or the left hand sides to be coincident as long as the rectangle is contained in S_{t-1}.

Consider a rectangle W placed in the best position. Let W have sides length α and β, α being placed on the x axis and β the height. Let $\beta \leqslant 1$. Divide W into two sets W_1 and W_2 by a perpendicular line bisecting the base. Then only one of the sets W_1 and W_2 is necessarily in the best position say W_1.

Thus $$\Lambda_n(W_1) \geqslant \Lambda_n(W_2)$$

If W_2 is placed above W_1 to form a rectangle W' sides $\frac{\alpha}{2}$ and 2β

then $$\Lambda_n(W') = \Lambda_n(W_1) + \Lambda_n(W_1)$$

since moving a rectangle W_1 parallel to the y axis leaves $\Lambda_n(W_1)$ unaltered,

$$\geqslant \Lambda_n(W_1) + \Lambda_n(W_2)$$
$$= \Lambda_n(W)$$

If $2\beta \leqslant 1$ also then a rectangle W'' is obtained sides $\frac{\alpha}{4}$ and 4β

such that $$\Lambda_n(W'') \geqslant \Lambda_n(W') \geqslant \Lambda_n(W)$$

Proceeding in this way a rectangle Q can be obtained such that

(1) Q is in the best position

(2) The height of Q is greater than 1

(3) $\Lambda_n(Q) \geqslant \Lambda_n(W)$

(4) $\Delta(Q) = \Delta(W)$

Thus it has been established that given any parallelogram P there exists a rectangle Q such that if Q is placed in the best position for itself,

(1) Q has sides parallel to the axes

(2) height of $Q > 1$

(3) $\Delta(Q) = \Delta(P)$

(4) $\Lambda_n(Q) \geqslant \Lambda_n(P)$

Corresponding to every $P \in \mathcal{P}$ take one such rectangle Q and denote the class of all the rectangles Q by $\mathcal{Q}$. Then $\mathcal{Q}$ is finite. Enumerate the rectangles of $\mathcal{Q}$ so that if a_i is the base length of Q_i

$$a_i \leqslant a_j \quad \text{if} \quad i > j.$$

The class $\mathcal{Q}$ is now placed on the set S_n so that every $Q \in \mathcal{Q}$ is in the best position and they are all disjoint. In order that the set S_n will be covered we have to place 3 additional sets Q around the first one and these extra sets may overlap. Thus a class 4 $\mathcal{Q}$ is actually made to cover S_n

Let q_i be the integer such that

$$x_{q_i - 1} > a_i \geqslant x_{q_i}$$

Denote each rectangle of S_m by s_m^j and enumerate them from the right.

Place Q_1 in the right hand bottom corner of $s^1_{q_1-1}$. This is clearly the best position. Place a similar rectangle above this one and two more alongside these so that in all they cover a rectangle height $\geqslant 2$ and base $2a_1$. We consider next the different cases that can occur.

<u>Case (1)</u>

Suppose that $a_1 < x_{q_1-1}/2$ and $q_1 = q_2 = \dots = q_{j+1}$

Let Q_1 alone meet r_1 rectangles s_{q_1}. Then the 4 rectangles Q_1 will completely cover these first r_1 rectangles s_{q_1} but will not cover the rectangle s_{q_1-1}. Place Q_2 so that its right hand bottom corner coincides with the right hand bottom corner of $s_{q_1}^{r_1+1}$ As with Q_1 place 3 more rectangles similar to Q_2 one above and two alongside the first Q_2 to the right so that a rectangle height 2 and base $2a_2$ is covered. Then if the first rectangle Q_2 met r_2 rectangles s_{q_1} (q_2 was assumed to be equal to q_1) the 4 rectangles Q_2 will completely cover these r_2 rectangles. Thus all the rectangles $s^1_{q_1}$ $s^2_{q_1}$ ..... $s_{q_1}^{(r_1+r_2)}$ have been covered. Place Q_3 so that its right hand bottom corner coincides with the right hand bottom corner of the rectangle $s_{q_1}^{(r_1+r_2+1)}$. Repeat the process with Q_3 and so on. An integer t is then obtained such that when the first rectangle Q_t is placed in position as described above it is completely contained in $s^1_{q_1-1}$ and either

(i) when the three additional sets Q_t are placed in their prescribed positions alongside and above the first one the covering of all the rectangles s_{q_1} contained in $s^1_{q_1-1}$ is completed or

(ii) the three extra rectangles Q_t do not complete the covering of all the rectangles $s_{q_1} \in s^1_{q_1-1}$ but the set Q_{t+1} when placed in its prescribed position that is with its right hand bottom corner coinciding with the right hand bottom corner of the rectangle $s_{q_1}^{(r_1+r_2+\ldots r_t+1)}$, is not completely contained in $s^1_{q_1-1}$.

In the case (i) the set Q_{t+1} is placed in the right hand bottom corner of $s^2_{q_1-1}$ and the process repeated over this rectangle.

In the case (ii) let w be the distance between the left hand bottom corner of $s^1_{q_1-1}$ and the left hand bottom corner of the first rectangle Q_t. Then since the three additional sets Q_t did not complete the covering of $s^1_{q_1-1}$ $w > a_t \geqslant a_{t+1}$. Also since Q_{t+1} when placed in position was not contained in $s^1_{q_1-1}$ $a_{t+1} > w - x_{q_1} - z_{q_1}$. Thus if Q_{t+1} is placed so that its left hand bottom corner coincides with the left hand bottom corner of $s^1_{q_1-1}$ it will not overlap the first rectangle Q_t and when the extra rectangles Q_t and Q_{t+1} are placed in position the covering of all $s_{q_1} \in s^1_{q_1-1}$ is completed. The rectangle Q_{t+2} is then placed in the right hand bottom corner of $s^2_{q_1-1}$ and the process repeated over this rectangle.

Case (2)

Suppose that $a_1 < x_{q_1-1}/2$ but for some $j < t$ (t as defined in the previous case) $q_j < q_{j-1}$. In this case when Q_j is placed so that its right hand bottom corner coincides with the right hand bottom corner of the rectangle $s_{q_1}^{(r_1+r_2+\ldots r_{j-1}+1)}$ it will be completely contained in this rectangle and the process can be repeated using this smaller rectangle $s_{q_{j-1}}$ instead of $s^1_{q_1-1}$.

Case (3)

Suppose that $a_1 \geq x_{q_1-1}/2$. When the 4 rectangles Q_1 are placed in position they completely cover the rectangle $s^1_{q_1-1}$ and Q_2 can then be placed so that its right hand bottom corner coincides with the right hand bottom corner of $s^2_{q_1-1}$ and the process is then repeated over this rectangle.

In each case the first rectangle Q was placed so that it was in the best position and also so that it did not overlap any other first rectangle.

The method described above can be used as long as $q_i > n$. But if for some i, $q_i \leq n$ a slightly different approach is needed. Let j be the first integer such that $q_j \leqslant n$. Then by the above method Q_j is placed in the right hand bottom corner of a rectangle s_n. If $a_j \geqslant x_n/2$ the 4 rectangles Q_j when placed in their prescribed positions will completely cover the rectangle s_n and there is no difficulty. If $a_j < x_n/2$ since a_i is a decreasing sequence of numbers there exists an integer k such that

$$x_n/2 \leqslant a_j + a_{j+1} + \ldots\ldots + a_k < x_n.$$

These $(k-j+1)$ rectangles are then placed in the right hand bottom corner of the rectangle s_n, each one with a similar rectangle above it. Then these two sets of rectangles cover a rectangle contained in s_n of height 2 and base $a_j + \ldots + a_k \geqslant x_n/2$. If two more sets of rectangles $Q_j \ldots Q_k$ are now placed alongside these the covering of s_n is completed. In this case it is trivial that if P_i is the parallelogram corresponding to Q_i then for the first Q_i used

$$\Lambda_n(Q_i) \geqslant \Lambda_n(P_i) \qquad\qquad i \geqslant j.$$

Continue in this way until the covering of S_n is completed or all the sets $Q \in \mathcal{Q}$ have been used, without completing the covering of S_n. The second alternative is impossible since considering only the first rectangle Q_i used each time all the Q_i are disjoint and hence if the covering has not been completed

$$\sum_{Q_i \in \mathcal{Q}} \Lambda_n(Q_i) < \Lambda_n(S_0)$$

But

$$\sum_{Q_i \in \mathcal{Q}} \Lambda_n(Q_i) \geqslant \sum_{P_i \in \mathcal{P}} \Lambda_n(P_i)$$

$$\geqslant \Lambda_n(S_0)$$

which gives a contradiction.

To complete the proof of the lemma each set of 4 rectangles Q_i are replaced by 4 rectangles R_i $\Delta(R_i) = \Delta(Q_i)$ where R has height 2 and base on the x axis. The first rectangle R is placed so that its right hand side (or left hand side in the special case of a rectangle Q_i placed as described in case 1 (ii)) coincides with the right hand side (or left hand side) of the first rectangle Q_i and the three remaining rectangles R_i are placed adjacent to this one. Then if $\mathcal{R}$ is the class of all such rectangles R, $\mathcal{R}$ covers $4\mathcal{Q}$ and

$$\sum_{R \in \mathcal{R}} h(\Delta(R)) \leqslant \sum_{Q \in 4\mathcal{Q}} h(\Delta(Q))$$

Since $\Delta(Q_i) = \Delta(P_i)$ for all i

$$\sum_{Q_i \in 4\mathcal{Q}} h(\Delta(Q_i)) \leqslant 4 \sum_{P_i \in \mathcal{P}} h(\Delta(P_i))$$

hence
$$\sum_{R \in \mathcal{R}} h(\Delta(R)) \leqslant 4 \sum_{P \in \mathcal{P}} h(\Delta(P))$$

which completes the proof of the lemma.

Denote by R(S,h) the measure obtained by limiting the class $\mathcal{B}_\delta$ of coverings $\mathcal{U}$ of S to coverings $\mathcal{R}$ of rectangles R as defined in the lemma.

Then it has been proved that

$$\sum_{U \in \mathcal{U}} h(\Delta(U)) \geqslant \frac{1}{16} \sum_{R \in \mathcal{R}} h(\Delta(R))$$

This holds for all $\mathcal{U} \in \mathcal{B}_\delta$ and hence

$$B(S,h) \geqslant \frac{1}{16} R(S,h)$$

Consider any rectangle R with base length a. Then $\Delta(R) = a$. Also for any $s_n \in S_n$ $\Delta(s_n) = x_n$. The problem now reduces to the one dimensional case of the proof of the sufficiency in the theorem of Section II,1, on page 5. For if S' is the projection of S on the x-axis and $\mathcal{I}$ the class of intervals projected by $\mathcal{R}$ on the x-axis

$$d(I) = \Delta(R) \qquad I \in \mathcal{I} \quad \text{and } R \in \mathcal{R}$$

$$\sum_{I \in \mathcal{I}} h(d(I)) = \sum_{R \in \mathcal{R}} h(\Delta(R))$$

and hence

$$m(S',h) = R(S,h)$$

It was established in the proof referred to above that

$$m(S',h) > 0$$

Thus $\quad R(S,h) > 0$

and so $\quad B(S,h) > 0$

This completes the proof of the theorem.

SECTION V, 2.

THEOREM

The metric area measure of a plane set formed by taking the cartesian product of any set on the x-axis and an interval on the y-axis is zero or infinite for any measure function h(x) such that

$$\lim_{x \to 0} \frac{h(\lambda x)}{h(x)} = \alpha \quad \text{where } \lambda \text{ is any}$$

positive integer and $\lambda \neq \alpha$.

Proof

Denote the plane set by S and let $\mathfrak{A}_\delta$ be the class of all coverings $\mathfrak{U}$ of the set S, $\mathfrak{U}$ consisting of convex sets U with $d(U) < \delta$. Take cartesian coordinates so that S is the cartesian product of a set D on the x-axis and an interval J on the y-axis. Multiply the y coordinate of every point of S by λ. Denote the set obtained thus by S_λ. Then S_λ may be divided into λ sets similar to S by drawing $(\lambda-1)$ lines parallel to the x-axis at a mutual distance d(J) apart.

Since the measure is metric

$$\lambda A(S,h) = A(S_\lambda,h) \tag{1}$$

Multiply the y coordinate of every point $u \in U \in \mathfrak{U}$ by λ Then the resulting class $\mathfrak{U}_\lambda$ of convex sets U_λ covers the set S_λ and $\Delta(U_\lambda) = \lambda\,\Delta(U)$ for all $U \in \mathfrak{U}$

Hence

$$A_{\lambda\delta}(S_\lambda,h) \geq \inf_{\mathfrak{U} \in \mathfrak{A}_\delta} \sum_{U \in \mathfrak{U}} h(\lambda\Delta(U))$$

Now consider any covering $\mathfrak{U}'$ of S_λ. Divide the y coordinate of every point belonging to a member of this class by λ and the resulting class is then a covering of S.

Hence

$$A_{\lambda\delta}(S_\lambda,h) \leq \inf_{\mathfrak{U} \in \mathfrak{A}_\delta} \sum_{U \in \mathfrak{U}} h(\lambda\Delta(U))$$

i.e.

$$A_{\lambda\delta}(S_\lambda,h) = \inf_{\mathfrak{U} \in \mathfrak{A}_\delta} \sum_{U \in \mathfrak{U}} h(\lambda\Delta(U))$$

If $\dfrac{h(\lambda x)}{h(x)} \to \alpha$ as $x \to 0$

then

$$A(S_\lambda,h) = \alpha\, A(S,h) \tag{2}$$

But if $\lambda \neq \alpha$ (1) and (2) are contradictory unless A(S,h) is zero or infinite.

This proves the theorem.

Remark This theorem applies to $h(x) = x^r$ $0 < r < 1$.

COROLLARY

Non-metric area measure is in fact non-metric. If non-metric area measure was metric the above proof could be used to establish that B(S,h) is zero or infinite for any set S consisting of the cartesian product of a set on the x axis and a closed interval J on the y axis whenever $\frac{h(\lambda x)}{x} \to \alpha$ as $x \to 0$ $\lambda \neq \alpha$. But this is a contradiction of the theorem established in Section V,1, and hence this measure is non-metric.

SECTION V,3,

THEOREM

If $0 < \alpha < 1$ there exists a set S having finite non-zero metric area measure with dimension function x^α, in 2-dimensional real Euclidean space, α rational.

Proof

Construct the set S as follows. Let $\{K_n\}$ be a rapidly increasing sequence of positive integers. Now choose a sequence of numbers x_n such that $x_0 = 1$ and

$$K_n \, x_n{}^\alpha = x_{n-1}{}^\alpha \tag{1}$$

Then $x_n \to 0$ as $n \to \infty$. Let

$$z_n = \frac{1}{(K_n-1)} [x_{n-1} - K_n x_n] \qquad (2)$$

Let S_0 denote the unit square in two dimensional real Euclidean space with sides parallel to rectangular cartesian axes. Along each side of S_0 take K_1 closed intervals of length x_1 interspaced by (K_1-1) open intervals of length z_1 and from these intervals construct a network of K_1^2 closed squares side x_1, $(K_1-1)^2$ open squares side z_1 and rectangles sides x_1 and z_1. Delete all but the K_1^2 closed squares from S_0 and denote the set so obtained by S_1 and any square belonging to S_1 by s_1. Along each side of each $s_1 \in S_1$ construct K_2 closed intervals length x_2 interspaced by (K_2-1) open intervals length z_2 and as before construct the network to obtain the set S_2 consisting of $K_1^2 K_2^2$ closed squares s_2 side x_2. Continuing in this way a decreasing sequence of sets S_n is obtained. Let $S = \prod_1^\infty S_n$. Then S is the required set.

(i) To prove that $A(S,x^\alpha) < \infty$

Since $x_n \to 0$ as $n \to \infty$ given any $\delta > 0$ there exists n such that $\sqrt{2}\, x_n < \delta$. Thus for such an integer n S_n is a covering of S which consists of squares s_n with $d(s_n) < \delta$ and also $\Delta(s_n) = \frac{x_n^2}{2}$

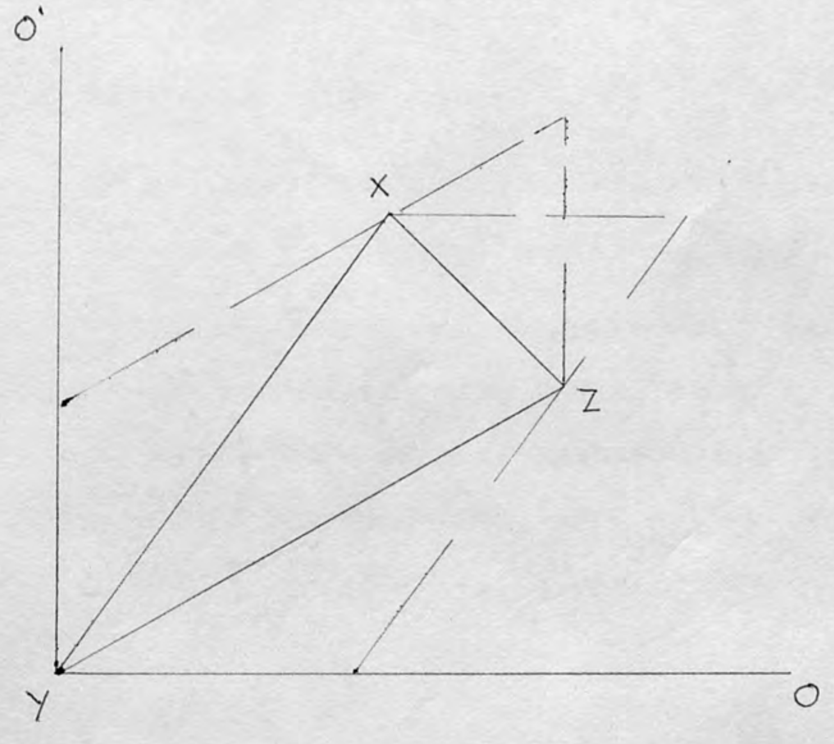
Then $$(K_1 \ldots K_n)^2 \left(\frac{x_n^2}{2}\right)^\alpha = (K_1 \ldots K_{n-1})^2 \left(\frac{x_{n-1}^2}{2}\right)^\alpha$$

$$= \frac{1}{2^\alpha} < \infty$$

Hence $A_\delta(S,x^\alpha) < \infty$. Since this holds for all $\delta > 0$ $A(S,x^\alpha) < \infty$.

(ii) To prove that $A(S,x^\alpha) > 0$.

From the proof of the theorem in Section V,1, it is sufficient to prove this result considering only coverings of parallelograms with one pair of sides parallel to the axes. Since the sides can be chosen to be parallel to either of the axes the parallelogram can be chosen so that its acute interior angle is greater than $\pi/4$. For if U is any set belonging to a covering $\mathfrak{U}$ of S and in the diagram below the triangle XYZ contains U and is such that the area of XYZ = $4\Delta(U)$, the parallelogram P is then taken so that $\Delta(P) = \Delta(XYZ)$. Let XZ be the shortest side. Then P must have Y as a vertex and either YZ or YX as one side.



Let YO and YO' be the lines parallel to the axes.

Then the interior acute angle of P is either $\widehat{XYO}$ or $\widehat{ZYO'}$.

$$\widehat{XYO} + \widehat{ZYO'} - \widehat{XYZ} = \pi/2$$

and hence either $\widehat{XYO}$ or $\widehat{ZYO'}$ is greater than $\pi/4$

Let P be such a parallelogram and let the acute interior angle be β. Let the length of the pair of sides parallel to the axis be a and the perpendicular distance between these sides be b. Then $\Delta(P) = \frac{1}{2}ab$.

Let m be the integer such that $x_m \leqslant b < x_{m-1}$.
The set S_m consists of $(K_1 \ldots K_m)^2$ squares s_m which are arranged in $(K_1 \ldots . K_m)$ columns parallel to the sides of P length a. Let P meet r of these columns.

Then $$b > (r-1)z_m + (r-2)x_m$$

and $$(\tfrac{1}{2}ab)^\alpha > (\tfrac{1}{2}a)^\alpha [(r-1)z_m + (r-2)x_m]^\alpha$$

From (1) and (2) $$z_m = \frac{x_m}{K_m - 1} [K_m^{1/\alpha} - K_m]$$

therefore
$$(\tfrac{1}{2}ab)^\alpha > (\tfrac{1}{2}ax_m)^\alpha \left[\frac{(r-1)}{K_m - 1} \{K_m^{1/\alpha} - K_m\} + (r-2) \right]^\alpha$$

Let $\mathcal{P}$ be the class of parallelograms P which covers S. Then if for every $P \in \mathcal{P}$ $(\frac{1}{2}ab)^\alpha > \frac{r}{2} (\frac{1}{2}ax_m)^\alpha$ each parallelogram P can be replaced by r parallelograms which have one pair of parallel sides coinciding with the sides of a column of squares s_m and if such parallelograms are denoted by Q and the class of all Q needed to cover S by $\mathcal{Q}$ then

$$\sum_{P \in \mathcal{P}} (\Delta(P))^\alpha > \tfrac{1}{2}\{ \sum_{Q \in \mathcal{Q}} (\Delta(Q))^\alpha \}$$

Hence it is sufficient to consider only coverings of parallelograms Q if we can establish that

$$\left\{ \frac{(r-1)}{K_m - 1} [K_m^{1/\alpha} - K_m] + (r-2) \right\} > \frac{r}{2}$$

or
$$(r-1)K_m^{1/\alpha} - K_m - (r-2) > \left(\frac{r}{2}\right)^{1/\alpha}(K_m - 1)$$

i.e.
$$\left(\frac{r}{2}\right)^{1/\alpha}(K_m-1) - r(K_m^{1/\alpha} - 1) + K_m^{1/\alpha} + K_m - 2$$
$$= f(r) < 0.$$

Consider the function f(r) for $r \geqslant 2$. Differentiating

$$\frac{df}{dr} = \frac{(K_m - 1)}{\alpha\, 2^{1/\alpha}} r^{1/\alpha - 1} - (K_m^{1/\alpha} - 1)$$

and thus there is only one real positive value of r for which $\frac{df}{dr} = 0$.

When $r = 0$ $\quad f(r) = K_m^{1/\alpha} + K_m - 2 > 0.$

When $r = 2$ $\quad f(r) = -K_m^{1/\alpha} + 2K_m - 1 < 0$

since K_m is large.

When $r = K_m$ $\quad f(r) = -\{1 - (\frac{1}{2})^{1/\alpha}\}K_m^{1/\alpha + 1} + \{1 - (\frac{1}{2})^{1/\alpha}\}K_m^{1/\alpha}$

$$+ 2K_m - 2 < 0$$

since K_m is large.

As $r \to +\infty$ $f(r) \to +\infty$.

Since f(r) has only one turning point $f(r) < 0$ throughout the range $2 \leqslant r \leqslant K_m$. Since the integer m was chosen so that $x_{m-1} > b$ the parallelogram P will meet at most K_m columns of squares s_m and so we have established the desired result.

A parallelogram Q with sides coinciding with the sides of a column of squares s_m can be replaced by K_{m+1} parallelograms Q with sides coinciding with those of columns of squares s_{m+1} without affecting the sum $\sum_{Q \in \mathcal{Q}} (\Delta(Q))^{\alpha}$

Now consider any parallelogram P with sides parallel to the axes length a perpendicular distance between them b and the acute interior angle $\beta > \pi/4$. Let q be an integer such that $x_{q-1} > a \geq x_q$. From the above we can take $b \leq x_q$. For if $b > x_q$ wecan replace P by parallelograms Q with $b = x_q$. Let P meet t rows of squares s_q, the rows being perpendicular to the sides of P length a.

Then $$a + b \cot\beta > (t-2)x_q + (t-1)z_q$$

and since $b \cot\beta \leq x_q$,

$$a > (t-3)x_q + (t-1)z_q$$

By the same argument as the above

$$(\tfrac{1}{2}ab)^{\alpha} \geq \frac{t}{2} (\tfrac{1}{2}x_q b)^{\alpha}$$

Combining these results if n is an integer such that $a \geq x_n$ and $b \geq x_n$ and P meets μ squares s_n then

$$(\tfrac{1}{2}ab)^{\alpha} \geq \frac{\mu}{4} (\tfrac{1}{2}x_n{}^2)^{\alpha}$$

Thus $$4 \sum_{P \in \mathcal{P}} (\Delta(P))^{\alpha} \geq \sum_{s_n \in S_n} (\Delta(s_n))^{\alpha}$$

and since this holds for all coverings of parallelograms P it is sufficient to prove the result for coverings S_n only. But we have already established that

$$\sum_{s_n \in S_n} (\Delta(s_n))^{\alpha} = (\tfrac{1}{2})^{\alpha} > 0 \qquad \text{for all } n$$

and hence $A(S,x^{\alpha}) > 0$.

This completes the proof of the theorem.

CHAPTER VI

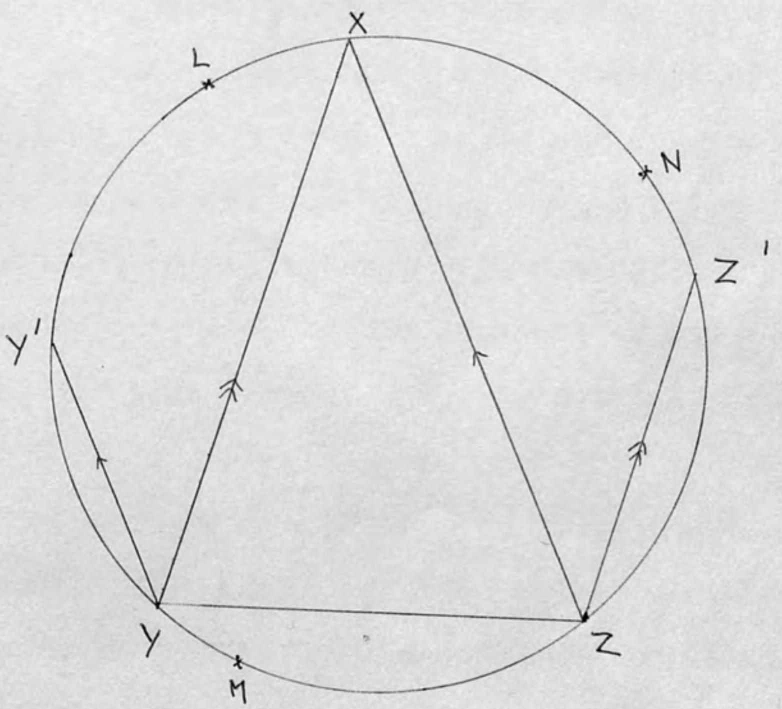
SECTION VI,1.

THEOREM

Let $\mathfrak{U}$ be the class of all convex sets U covering a given total arc length δ on the circumference of a unit circle. Then the least value of $\Delta(U)$, $U \in \mathfrak{U}$ is the area of the greatest triangle that can be drawn in a segment of the circle covering an arc length δ.

LEMMA

Let S be a closed set of linear measure $\lambda \leqslant 4\pi/3$ on the frontier of the unit circle. Then there exists three points belonging to S such that the least length of arc between any two of them is $\lambda/2$.



Consider the class of all triangles having vertices in S. Let XYZ be one such triangle having the greatest value of least side length and let YZ be this shortest side. Draw YY' parallel to XZ and ZZ' parallel to XY cutting the circle again in Y' and Z' respectively. Y' lies at Y or on the side of XY opposite to Z and Z' lies at Z or on the side of XZ opposite to Y. For if, for example Y' lies on the same side of XY as Z then XY = Y'Z < YZ which contradicts the hypothesis.

If XY = YZ = ZX each vertex is at an arc length $2\pi/3$ from the other two and since $\lambda \leqslant 4\pi/3$ the lemma follows.

Otherwise there is no loss in generality in assuming that XY > YZ, which implies that Y' and Y do not coincide. Take L on the arc XY', M on the arc YZ and N on the arc Z'X such

$$\text{arc } YM = \text{arc } XL - 2\delta$$

$$\text{arc } Z'N = \text{arc } XL - \delta$$

where δ is small and positive. Then

$$\text{arc } LN = \text{arc } XZ' + \delta$$

$$\text{arc } LM = \text{arc } XY - 2\delta$$

$$\text{arc } MN = \text{arc } YZ' + \delta$$

Choose δ so small that arc XY $- 2\delta >$ arc XY'. Then since arc XY' = arc XZ' = arc YZ it follows by the extremal property of the triangle XYZ that at least one of the points L, M and N does not belong to S. If L is allowed to vary in such a way that $L \in XY'$ $M \in YZ$ and $N \in XZ'$ then of the three points at most two belong to S, and hence the measure of S in the arcs YZ, Z'X and XY' is at most $2(3\text{arc } YZ + 6\delta)/3$. This is true for all $\delta > 0$ and hence the measure of S in the arcs YZ, Z'X and XY' is at most twice the arc YZ. But since no point of S can lie in the arcs YY' and ZZ' by the extremal property of the triangle XYZ the measure of S in the arcs YZ, Z'X and XY' must be λ and thus the arc YZ $\geqslant \lambda/2$. This completes the proof of the lemma.

Proof of the theorem

Let U_0 be one set belonging to the given class. Then U_0 cuts the frontier of the circle in a set of linear measure λ say. By the lemma if $\lambda \leqslant 4\pi/3$, there exists three points X, Y and Z belonging to U_0 and such that the least length of arc between any two of them is $\lambda/2$. Let arc YZ = α, arc ZX = β. Then

$$\text{area of triangle XYZ} = \tfrac{1}{2}(\sin\alpha + \sin\beta - \sin(\alpha+\beta))$$
$$= f(\alpha,\beta)$$

$f(\alpha,\beta)$ takes minimum value when α and β take their minimum values i.e. when $\alpha = \beta = \lambda/2$. In this case the points Z and Z' of the lemma coincide, arc XZY = λ and the triangle XYZ is the greatest triangle that can be drawn in a segment of the circle covering an arc of length λ.

When $\lambda > 4\pi/3$ then by the lemma applied to a subset of linear measure $4\pi/3$ there exists three points each at an arc length $2\pi/3$ from the other two and this is the greatest triangle that can be drawn in a segment of the circle covering an arc length $\lambda > 4\pi/3$. This completes the proof of the t theorem.

REMARK

Consider a covering $\mathfrak{U}$ of convex sets U of a set S on the frontier of the unit circle. Let U' be the segment of the circle covering the same total length of arc as U and let $\mathfrak{U}'$ be the class of all such sets U'. Then by the theorem

$$\Delta(U) \geqslant \Delta(U')$$

and for any measure function h(x)

$$h(\Delta(U)) \geqslant h(\Delta(U'))$$

This is true for all $U \in \mathfrak{U}$ and the corresponding $U' \in \mathfrak{U}'$ and hence

$$\sum_{U \in \mathfrak{U}} h(\Delta(U)) \geqslant \sum_{U' \in \mathfrak{U}'} h(\Delta(U')).$$

This inequality holds for any covering $\mathfrak{U}$. Given any $\delta > 0$ it is possible to find coverings of the form $\mathfrak{U}'$ such that both $\Delta(U') < \delta$ and $d(U') < \delta$ for all $U' \in \mathfrak{U}'$. Hence in calculating the metric area measure or the non-metric area measure of a set S on the frontier of the unit circle it is sufficient to consider only coverings $\mathfrak{U}'$ which consist of sets U' which are segments of the circle. Clearly for any such set S the metric area measure is equal to the non-metric area measure.

If λ is the arc length covered by a set U'

$$\Delta(U') = \tfrac{1}{2}(2 \sin \lambda/2 - \sin\lambda)$$
$$= \sin(\lambda/2)(1 - \cos \lambda/2)$$

SECTION VI, 2.

Relation between the metric or non-metric area measure of a set on the frontier of the unit circle and its generalized affine length.

The definition of the generalized affine length of a set S on the frontier of the unit circle with measure function h(x) was given in chapter I, Section I, 7 on page 4 To calculate it we consider only coverings τ of tangent triangles T that is triangles formed by two tangents to the circle and the line joining their points of contact. Then if such a triangle T covers an arc length λ

$$\Delta(T) = \sin(\lambda/2)(\sec\lambda/2 - \cos\lambda/2)$$

Consider a covering $\mathcal{U}'$ of the given set S, $\mathcal{U}'$ consisting of sets U' which are limited to be segments of the circle as explained in the preceding section. Then any arc length covered by a set U' can certainly be covered by two sets T where $\Delta(U') = \Delta(T)$. Hence there exists a covering τ of tangent triangles T such that

$$\sum_{T \in \tau} h(\Delta(T)) \leq 2 \inf_{\mathcal{U}' \in A_\delta} \sum_{U' \in \mathcal{U}'} h(\Delta(U'))$$

where A_δ is the class of all coverings $\mathcal{U}$ of the set S where $\mathcal{U}$ is such that $d(U) < \delta$ for all $U \in \mathcal{U}$, and U is any convex set.

Therefore $$\inf_{\tau \in A_\delta} \sum_{T \in \tau} h(\Delta(T)) \leq 2 \inf_{\mathcal{U}' \in A_\delta} \sum_{U' \in \mathcal{U}'} h(\Delta(U'))$$

This holds for all $\delta > 0$ and thus

$$F(S,h) \leq 2\, A(S,h)$$

Now consider any covering τ of triangles T. Then any arc length covered by a set T can certainly be covered by a set U' where $\Delta(T) = \Delta(U')$, and by a similar argument

$$F(S,h) \geqslant A(S,h)$$

Thus the generalized affine length and metric area measure satisfy the inequality

$$A(S,h) \leqslant F(S,h) \leqslant 2A(S,h)$$

Also the generalized affine length and non-metric area measure of a set S satisfy

$$B(S,h) \leqslant F(S,h) \leqslant 2B(S,h)$$

SECTION VI,3.

THEOREM

The triangle of minimum area which covers arcs of the unit circle of given total length is one such that two of its sides are tangents which touch the circle at their midpoints.

LEMMA

In the class of all triangles which cover arcs of the unit circle of given total length, there exists one which has the minimum area.

Proof

The class of triangles is bounded since every such triangle must contain at least the segment of the circle determined by a third of the given arc length. From this class it is possible to select an infinite sequence of triangles decreasing in area. By the Blaschke selection theorem such a sequence contains an infinite subsequence which converges and the limit of such a subsequence will be the required minimum triangle.

Proof of the theorem

Let XYZ be the triangle of minimum area given by the lemma, and let the total arc length covered by XYZ be 2λ. Then since XYZ is a triangle of minimum area each side of XYZ must either touch or cut the circle.

Case I

XYZ has three vertices outside the circle and only one side YZ cutting the circle.

Let XY and XZ touch the circle at L and M respectively. A small rotation ϕ of the point L around the circle leaves the area of XYZ unchanged to the first order in ϕ only if L is the midpoint of XY. Such a rotation does not alter the arc length covered and since XYZ is the triangle of minimum area L must be the midpoint of XY. Also by a similar argument M must be the midpoint of XZ.

Case II

XYZ has two sides XY and XZ cutting the circle and all three vertices outside the circle.

Let XY cut the circle in L and M and let N be the midpoint of LM. Then a small rotation of XY through an angle ψ about N leaves the total arc length covered by XYZ unaltered. Since XYZ is the triangle of minimum area such a rotation must also leave the area of XYZ unchanged to the first order in ψ i.e. N must also be the midpoint of XY. The same result holds for XZ and also the point of contact of YZ is the midpoint of YZ.

Let XZ cut the circle in L' and M'. Let α = angle between the tangent to the circle at L and the line XY, β = angle between the tangent to the circle at L' and XZ, a = length of XY and b = length of XZ. Move XY a distance η parallel to itself so as to increase the arc length covered and XZ a distance μ parallel to itself so as to decrease the arc length covered.

Then increase in area = $a\eta - b\mu$

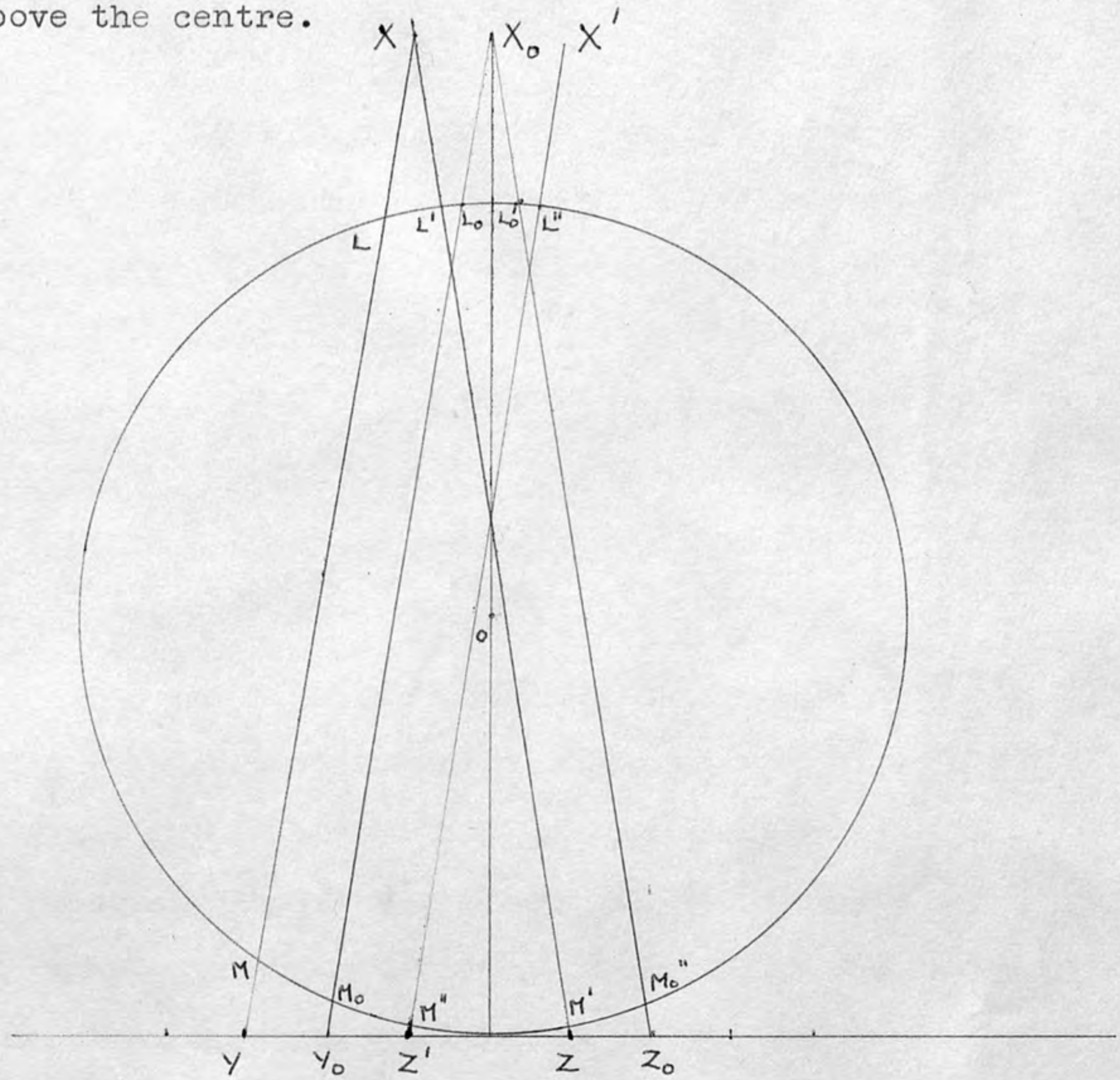
and the increase in arc length covered $= \dfrac{2\eta}{\sin\alpha} - \dfrac{2\mu}{\sin\beta}$

to the first order in η and μ. Since the triangle XYZ is the triangle of minimum area both these must be zero simultaneously

i.e. $$a \sin\alpha = b \sin\beta$$

But if a > b $\sin\alpha > \sin\beta$ (since we have already established that X, Y and Z must be equidistant from the centre of the circle) and hence $a = b$ and $\alpha = \beta$. Thus if XYZ is given by this case it must be an isosceles triangle symmetrically placed about the centre.

However it is easy to see that such a triangle is in fact the triangle of maximum and not minimum area. For consider a large displacement of the triangle in a direction parallel to YZ and such that the side XY still cuts the circle and the side XZ cuts the diameter perpendicular to YZ in a point above the centre.



Draw the line X'Z' which is the reflection of XZ in the diameter perpendicular to YZ and let X'Z' cut the circle in L" and M". Denoting the original positions of X,Y,Z... by $X_0, Y_0, Z_0 \ldots$ the increase in arc length covered is $2(L\,L_0' - L_0 L'')$ > 0 since XY, X_0Y_0 and X'Z' are parallel equidistant lines, all on the same side of the parallel diameter and XY is furthest from and X'Z' nearest to this diameter.

Thus XYZ, the triangle of minimum area is not given by this case.

Case III

XYZ has all three vertices outside the circle and all three sides cutting the circle.

By thesame argument as in Case II it can be shown that XYZ would have to be an equilateral triangle placed symmetrically about the circle and also that such a triangle is that of maximum and not minimum area. Thus XYZ the triangle of minimum area cannot be given by this case.

Case IV

XYZ is a triangle having one vertex outside the circle and two vertices inside the circle.

Let L and M be the endpoints of the arc covered. Then since XYZ is the triangle of minimum area Y and Z, the two vertices inside the circle must coincide with L and M respectively and XY and XZ must be the tangents at L and M. But such a triangle has one vertex outside and two on the circle and may be considered as an extreme case of case I. Thus the triangle of minimum area will not be of this form either.

Case V

XYZ is a triangle with one vertex inside the circle and the opposite side not cutting the circle.

Let X be the vertex inside the circle. As we proved in case II if XY and XZ cut the circle in L and M respectively L and M must be the midpoints of XY and XZ. Denote this triangle by Q and the extreme triangle obtained in case I by W, where W also covers the arc LM.

Let b be the height of the line joining the points of contact of the tangents forming W above the centre. Then

$$\Delta(W) = \frac{4}{b}(1-b^2)^{3/2}$$

$$\Delta(Q) = 4(1-(2b-1/b)^2)^{\frac{1}{2}}(1-(2b-1/b))$$

and $\Delta(Q) - \Delta(W) = \frac{4(1-b)^{3/2}(1+b)^{\frac{1}{2}}}{b^2}\left[(1+2b)^{3/2}(2b-1)^{\frac{1}{2}} - b(1+b)\right]$

and $\Delta(Q) - \Delta(W) = 0$ when $b = 1, -1$ or

$$(2b+1)^3(2b-1) - b^2(1+b)^2 = 0$$

i.e. $$15b^4 + 14b^3 - b^2 - 4b - 1 = f(b) = 0$$

By Descartes rule of signs this has at most one positive root.

When $b = 1$ $\qquad f(b) = 23$

$b = \frac{1}{\sqrt{2}}$ $\qquad f(b) = \frac{9+6\sqrt{2}}{4}$

$b = 0$ $\qquad f(b) = -1,$

and thus the root occurs in the interval $0 < b < \frac{1}{\sqrt{2}}$

But the least value of b giving a triangle of the form described in case V is $b = 1/\sqrt{2}$ since for this value of b the vertex X lies on the circumference of the circle. Thus for all possible triangles Q and W $\Delta(Q) > \Delta(W)$ and thus the minimum triangle cannot be one of the form Q i.e. one belonging to case V.

Case VI

XYZ is a triangle having one vertex X inside the circle and the opposite side cutting the circle.

Let XY, XZ cut the circle in L and M respectively and let YZ cut the circle in L' and M'. Then as established in Case II L is the midpoint of XY and M the midpoint of XZ and also YL' = M'Z. Thus XYZ is an isosceles triangle. Denote this triangle by Q' and as in the previous case let W denote the triangle of minimum area obtained in Case I which covers the same arc length as Q'. If N is the midpoint of XY and O the centre of the circle let $N\hat{O}L' = \alpha$.

Then $$\Delta(Q') = 4\sin(\lambda+\alpha)(\cos\lambda - \cos(\lambda+\alpha))$$

$$\frac{d\,\Delta(Q')}{d\alpha} = 4\,(\cos(2\alpha+\lambda) - \cos(2\alpha+2\lambda))$$

and $$\frac{d\,\Delta(Q')}{d\alpha} = 0 \quad \text{when} \quad = \frac{\pi}{2} - \frac{3\lambda}{4}$$

giving the maximum value of the area. The minimum value of $\Delta(Q')$ occurs either when $\alpha = 0$ in which case the triangle Q' becomes one of the kind dealt with in Case V or when the vertex X, which is the one inside the circle lies on the circumference. This may be considered as an extreme case of Case III. But neither case V nor case III gives the triangle of minimum area and hence this case does not either.

If XYZ is such that one side cuts the circle its position is given by one of these six cases. It has been established that when $\lambda < \pi$ the only case giving a triangle of minimum area is case I. This is the triangle described in the statement of the theorem.

If $\lambda = \pi$ no side of XYZ can cut the circle and the triangle of minimum area is equilateral with each side touching the circle at its midpoint. But this is the extreme case of the triangle described in the statement of the theorem and thus we have completed the proof of the theorem.

SECTION VI,4.

THEOREM

The necessary and sifficient condition for the existence of a set S on the frontier of the unit circle having finite non-zero metric or non-metric area measure with measure function h(x) is that

$$\liminf_{x\to 0} \frac{h(x)}{x^{1/3}} > 0$$

The result is proved for the generalized affine length F(S,h) and since in Section VI,2, the relations

$$A(S,h) \leqslant F(S,h) \leqslant 2A(S,h)$$

and

$$B(S,h) \leqslant F(S,h) \leqslant 2B(S,h)$$

were established the result holds for the two area measures also.

Proof

The area of a tangent triangle covering an arc length 2λ is

$$\frac{\sin^3\lambda}{\cos\lambda} \approx \lambda^3 \qquad \text{for small } \lambda$$

Consider any set S on the frontier of the unit circle and a covering τ of triangles T as defined in Section VI,2.

Then

$$F(S,h) = \lim_{\delta\to 0} \inf_{\tau\in_\delta} \sum_{T\in\tau} h(\Delta(T))$$

$$= \lim_{\delta\to 0} \inf_{\tau\in_\delta} \sum_{T\in\tau} h \frac{\sin^3\lambda}{\cos\lambda}$$

$$\geqslant m(S',g)$$

where $g(x) = h\left(\frac{\sin^3 x}{\cos x}\right)$ and $m(S',g)$ is the Hausdorff diametral measure of a set S' which is the set S considered as a set on an interval length 2π on the real Euclidean line with measure function $g(x)$.

Consider a covering $\mathcal{J}$ of intervals I of the set S'. Then this is the same as a covering $\mathcal{J}'$ of arcs I' of the set S on the frontier of the unit circle.

Then
$$m(S',g) = \lim_{\delta\to 0} \inf_{\mathcal{J}\in A_\delta} \sum_{I\in\mathcal{J}} \frac{g(d(I))}{2}$$
$$\geq F(S,h)$$

Hence
$$m(S',g) = F(S,h)$$

But it was proved in Section II,1, on page 5 that the necessary and sufficient condition for a measure function $g(x)$ to be the dimension function of a set on the real line is that

$$\lim_{x\to 0} \inf \frac{g(x)}{x} > 0$$

and this gives
$$\lim_{x\to 0} \inf \frac{h\,\frac{\sin^3 x}{\cos x}}{x} > 0$$

But for small values of x, $\frac{\sin^3 x}{\cos x} \approx x^3$ and hence the condition becomes

$$\lim_{\Delta(T)\to 0} \inf \frac{h(\Delta(T))}{\Delta(T)^{1/3}} > 0$$

and this is the required condition.

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