

MARKOV FIELDS OF HIGHER SPIN

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LIM SWEE CHENG

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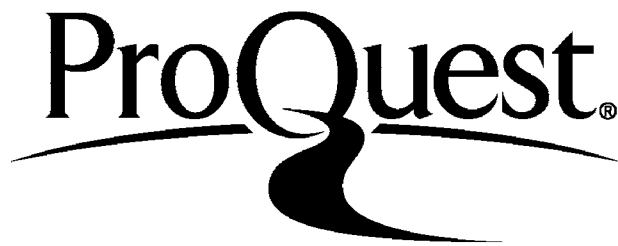
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## ABSTRACT

Various formulations of free Euclidean Markov spin one fields with  $m > 0$  and  $m = 0$  are studied. Attempts to construct Euclidean Markov spin two tensor fields with  $m > 0$  and  $m = 0$  are only partially successful.

ERRATA

- Page 15 Delete last line of equation (20) (i.e.  $\langle \Omega, \varphi(x) \varphi(y) \Omega \rangle$ ) and " and  $\Omega$  the vacuum state " that follows.
- Page 16 Lines 5 & 6 from the bottom of the page should read "Define  $\{ \Gamma_n(\mathcal{M}_r) \}$  as the orthogonal complement of  $\Gamma_{n \leq 1}(\mathcal{M}_r)$  in  $\Gamma_{\leq n}(\mathcal{M}_r)$  ....."
- Page 18 All the  $\mathcal{M}$  and  $\mathcal{M}'$  in Theorem (Nelson Ne : 3) should be replaced by  $\mathcal{Q}$  and  $\mathcal{Q}'$  respectively. (e.g.  $L^p(\mathcal{M})$  should be replaced by  $L^p(\mathcal{Q})$  , etc.)
- Page 49 Delete the first paragraph (line 2 to line 8)
- Page 55 Add the following remark after the proof :  
 "The anti-symmetric spin - 1 tensor field is equivalent to Proca field if  $\epsilon_{\mu\nu\gamma\delta} A^{\mu\nu\rho\sigma} = 0$  , otherwise  $A^{\mu\nu}$  and  $\phi^M$  are not the same".
- Page 57 Add the following parenthesis after line 11 :  
 "(For  $m \neq 0$  the Lagrangian is non-local in "Landau gauge" which implies non-Markovicity)".
- Page 76 Equation (31) should read :  
 " 
$$U(a, \Lambda) F^{\mu\nu}(f_{\mu\nu}) U(a, \Lambda)^{-1} = F^{\mu\nu}(f_{\mu\nu}^{\{a, \Lambda\}}) \quad (31)$$
 where  $f_{\mu\nu}^{\{a, \Lambda\}}(x) = \Lambda_{\mu}^{\rho} \Lambda_{\nu}^{\sigma} f_{\rho\sigma}(\Lambda^{-1}(x-a))$  "
- Page 83 The sentence in line 8 of last paragraph should read :  
 "Furthermore the proof of TCP theorem does not hold since the assumption of local commutativity may not be valid in non-local field theory ....."

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## CHAPTER ONE

## INTRODUCTION

I. 1. Summary Of Results

In the first chapter we give a brief historical account of the progress in Euclidean field theory, some essential mathematical tools and a brief account of Nelson's work. We generalize some of the basic concepts such as Euclidean covariance, reflection property, Markov property, etc. so that they are defined for Euclidean random tensor field of arbitrary rank.

We introduce a general Lagrangian formulation for massive particles with higher integer spin due to Umezawa and Takahashi in Section II. 1. This formulation has the advantage in the sense that if the corresponding Euclidean region field exists, then Nelson's proof of Markovicity can be applied directly. The reason for this is that the Euclidean propagator in this formulation always has a local inverse. The work of L. Gross and T.H. Yao is then considered in the light of this approach. In Section II. 3. we construct a anti-symmetric rank-two Euclidean tensor field which is Markovian. This Euclidean tensor field can be shown to be equivalent to the Euclidean Proca field. We conclude our study of Euclidean massive spin - 1 field by constructing a Euclidean vector field with a one parameter family of covariant gauges. This model is interesting for it enables one to take the  $m \rightarrow 0$  limit. Although it is Markovian, but it does not satisfy the reflection property. The corresponding Minkowski region field is renormalizable but require



the use of indefinite metric Hilbert space because of the presence of unphysical states (ghosts).

Chapter three is devoted to the study of Euclidean electromagnetic potential and field. We first discuss Euclidean electromagnetic potential with one parameter family of covariant gauges, which is just the  $m \rightarrow 0$  limit of the Euclidean massive vector field discussed in Section II. 4. Again, such field though is Markovian, it violates the reflection property, this may explain why it does not lead to a Wightman field. Next we show that the Euclidean electromagnetic field in terms of anti-symmetric rank-two field intensity tensor is also Markovian, but the proof for this is more involved. In this case the reflection property is satisfied and it does give rise to a Wightman field in Minkowski space-time.

Finally we study Euclidean massive and massless spin - 2 tensor field in chapter four. We start with Umezawa-Takehashi formulation of massive spin - 2 tensor field. However the corresponding Schwinger two-point function is not positive-semidefinite so we cannot construct a Euclidean field in the usual manner. In order to make the Schwinger function positive-definite, we impose traceless condition on the tensor-valued test functions, but now we do not have the Markov property. In another attempt, we impose, in addition to the traceless condition, a differential condition (some kind of divergenceless condition) on the test functions; however this again fails to give a Euclidean Markov tensor field. For the massless case, we do get a Euclidean Markov tensor field in some covariant gauges. This Markov field does not satisfy the reflection property and the Minkowski region field does not give a Wightman theory.

## I. 2. Historical Account

The idea of using Euclidean space in quantum field theory has a long history. It goes as far back as 1949 in the work of Dyson (Dy 1) on renormalization theory. In order to get rid of the mass-shell singularities of the Feynman propagator  $(p_0^2 - \underline{p}^2 + m^2 - i\epsilon)^{-1}$  he replaced it by the well-behaved Euclidean propagator  $-(p_0^2 + \underline{p}^2 + m^2)^{-1}$ . Less than a decade later, Wightman and Hall (Wi 1, W - H1) considered the analytic continuation of vacuum expectation values of field products to a region including the Euclidean region, however, no emphasis has been placed on the importance of the Euclidean region vacuum expectation values.

Later on, Schwinger (Sc 1, 2) and Nakano (Na 1) independently studied in some details the vacuum expectation values of time-ordered field products taken at purely imaginary time and real space, called the Euclidean Green's functions or Schwinger functions. The former noted that such functions are invariant under Euclidean group, and he also stressed the importance of such an approach.

In the past few years there have been much progress in the Euclidean approach to quantum field theory, in particular the systematic use of the ideas and mathematical methods of statistical and probability theory. The first step towards this direction was carried out by Symanzik (Sy 1, 2, 3) who realized that it might be easier to construct Euclidean Green functions than the direct construction of the Wightman functions from a given Lagrangian density. He also developed many of the ideas special to Euclidean field theory, and established a useful connection between Euclidean field theory and classical statistical mechanics for certain class of interactions. It was found that the existence of Euclidean fields depend crucially

on the requirement that the Schwinger functions need to be the expectation values of products of fields with respect to a positive measure, i.e.

$$S(x_1, \dots, x_n) = \int \Phi(x_1) \cdots \Phi(x_n) d\mu(\Phi) \quad (1)$$

where the  $\Phi(x)$  are commuting fields defined for  $X$  in Euclidean space-time. Since the functional integrations involved in the theory are respectable probability theory, the hyperbolic field equations are replaced by elliptic equations, the complications of Lorentz invariance are replaced by simple Euclidean covariance, and the Euclidean fields commute at all space-time points, all these ought to make the Euclidean field theory much easier to handle than ordinary quantum field theory.

The importance of Euclidean method in quantum field theory was not realised by many constructive field theorists before the publication of the decisive papers by Nelson (Ne 1, 2, 3). He gave a mathematically rigorous formulation of Euclidean massive scalar boson field, and also solved the question left open by Symanzik's work, namely, the determination of quantum field theory in Minkowski region given a Euclidean field theory. The main emphasis in Nelson's work was on the probabilistic method, in particular, he isolated a crucial property of Euclidean scalar boson field - the Markov property. Actually this property was first discovered by Symanzik (Sz 4) but he did not explore it further. Nelson was able to show that given a Euclidean region field which satisfies certain suitable axioms, one could construct an associated Minkowski region field obeying the Wightman axioms. Thus, it would be suffice for one

to establish the proof of the existence of solutions in model theories in the Euclidean version of the theory; the existence of solutions in the Minkowski space version then follows from Nelson's general theory.

Another important contribution to Euclidean method in constructive quantum field theory came from the work of Osterwalder and Schrader (O - S 1, 2, 3). They showed that the Euclidean formulation of relativistic quantum field can be carried out in terms of Schwinger functions alone, without using additional assumptions provided by the existence of Euclidean region fields. The Schwinger functions are required to satisfy a set of axioms (O - S axioms) analogous to the Wightman axioms for Wightman functions. In general, O - S axioms do not guarantee the existence of Euclidean region fields. In this respect Nelson's axioms are strictly stronger than the O - S axioms (Si - 1).

The results of Osterwalder and Schrader have been successfully extended to field with arbitrary spin by Ozkaynak (Oz 1, 2). However the generalization of Nelson's probabilistic approach to arbitrary spin field has not been carried out except for the case of spin - 1 Proca field which has been studied by L. Gross (Gr 1) and T.H. Yao (Ya 1) independently. The latter showed that the Euclidean Proca field is Markovian. Although the massless spin - 1 Euclidean vector field (or Euclidean electromagnetic potential) in Lorentz gauge has also been considered by Gross, however his conclusion that such a field is non-Markovian is incorrect.

In this work we shall study Euclidean spin - 1 and spin - 2 fields in the spirit of Nelson and Yao. Various formulations of free massive and massless spin - 1 Euclidean fields will be studied and they are shown to be Markovian. We have found that those Euclidean

Markov fields which do not lead to Wightman fields in Minkowski space-time violate the reflection property. Attempts will be made to formulate Euclidean spin - 2 fields using the same approach. For Euclidean spin - 2 massless tensor field we have managed to show that it is Markovian in certain covariant gauges; whereas for the Euclidean massive spin - 2 tensor field there is a strong indication that it is non-Markovian. Throughout our analysis we rely heavily on two points, namely, the positive-definiteness of the Schwinger functions and the existence of a local inverse for the covariance functionals.

### I. 3. Some Basic Notions In Probability Theory

We shall give a brief introduction to the basic concepts and theorems in the theory of probability which are useful in the formulation of Euclidean field theory. We begin with some basic definitions.

A probability space is a triple  $(Q, \Sigma, \mu)$ , where  $Q$  is a set,  $\Sigma$  is a  $\sigma$ -algebra of subsets of  $Q$ , and  $\mu$  is a positive measure defined on  $\Sigma$  with  $\mu(Q) = 1$ . A real random variable is a measurable real-valued function  $f$  from  $Q$  to  $\mathbb{R}$  (i.e. such that  $f^{-1}(B)$  is in  $\Sigma$  for each Borel set  $B$  in  $\mathbb{R}$ ). If  $f$  is a positive or integrable random variable, its mean or expectation  $E[f]$  is given by  $\int_Q f d\mu$ . Given a random variable  $f$ , the measure induced on  $\mathbb{R}$  by

$$\mu_f[B] = \mu[f^{-1}(B)] = \mu[\{\omega \in Q \mid f(\omega) \in B\}] \quad (2)$$

is called the probability distribution for  $f$ , and its Fourier transform

$$C_f(t) = \int_{\mathbb{R}} e^{itx} d\mu_f(x) = \int_{\mathcal{Q}} e^{itf(\omega)} d\mu(\omega) \quad (3)$$

is called the characteristic function of  $f$ . A knowledge of  $\mu_f$  is equivalent to that of  $C_f$ , which can be taken as the expectation of the random variable  $e^{itf}$ , i.e.  $C_f(t) = E[e^{itf}]$

A nice property of the characteristic function is its connection to the moments of  $f$ . Recall that the  $n^{\text{th}}$  moment of  $f$  is  $\langle f^n \rangle$ , the expectation of the  $n^{\text{th}}$  power of  $f$ .  $f$  has moments of all orders if and only if  $C_f$  is  $C^\infty$  and in that case,

$$\begin{aligned} \langle f^n \rangle &= E[f^n] = \int_{\mathcal{Q}} f^n(\omega) d\mu(\omega) \\ &= \int x^n d\mu_f(x) = (-i)^n \left. \frac{dC_f}{dt} \right|_{t=0} \end{aligned} \quad (4)$$

An important criterion for a function  $C$  to be a characteristic function is given by :

Bochner's Theorem (Bo 1)

A necessary and sufficient condition for a function  $C$  (from  $\mathbb{R}$  to  $\mathbb{C}$ ) to be the characteristic function of a random variable is that  $C(\cdot)$  obey :

(a)  $C(0) = 1$

(b)  $t \mapsto C(t)$  is continuous

(c) For any  $\{t_i\}_{i=1}^n$  in  $\mathbb{R}$  and  $\{z_i\}_{i=1}^n$  in  $\mathbb{C}$

$$\sum_{i,j=1}^n z_i \bar{z}_j C(t_i - t_j) \geq 0$$

Amongst the functions of positive type on the reals the Gaussians play a particularly important role. A random variable  $f$  is called a Gaussian random variable of mean zero whenever its characteristic function has the form

$$C_f(t) = e^{-\frac{1}{2}at^2}, \quad a \geq 0 \quad (5)$$

By Fourier inversion, (5) is equivalent to

$$\begin{aligned} d\mu_f(x) &= \delta(x) dx \quad \text{if } a=0 \\ &= (2\pi a)^{-\frac{1}{2}} e^{-\frac{x^2}{2a}} dx \quad \text{if } a>0 \end{aligned} \quad (5')$$

It can be checked that the moments of a Gaussian random variable are all finite. (4) and (5) imply that

$$\begin{aligned} E[f^{2n+1}] &= 0 \\ E[f^{2n}] &= \frac{(2n)!}{2^n n!} a^n = 1 \cdot 3 \cdots (2n-1) a^n \end{aligned} \quad (6)$$

In particular,

$$a = E[f^2] = \langle f^2 \rangle \quad (7)$$

Let  $f_i, i=1, \dots, n$  be random variables, the matrix

$$(\Gamma_{ij}) = \langle (f_i - \langle f_i \rangle)(f_j - \langle f_j \rangle) \rangle \quad (8)$$

is called the covariance matrix of the random variables

Since  $\sum_{i,j=1}^n \alpha_i \alpha_j \Gamma_{ij} = \langle |\sum \alpha_i (f_i - \langle f_i \rangle)|^2 \rangle \geq 0$ , so the covariance matrix is positive-definite, real and symmetric. If  $f_i$  and  $f_j$  have zero mean, then  $\Gamma_{ij} = \langle f_i, f_j \rangle_{L^2(Q, d\mu)}$ .

The joint probability distribution  $d\mu_{f_1, \dots, f_n}$  on  $\mathbb{R}^n$  of the random variables  $f_i, i=1, \dots, n$ , all defined on  $(Q, \Sigma, \mu)$ , is given by

$$d\mu_{f_1, \dots, f_n}(B) = d\mu((f_1 \otimes \dots \otimes f_n)^{-1}(B)) \quad (9)$$

with  $(f_1 \otimes \cdots \otimes f_n)(\omega) = (f_1(\omega), \dots, f_n(\omega))$  for each Borel set  $B$  in  $\mathbb{R}^n$ . The joint characteristic function is

$$C_{f_1, \dots, f_n}(t_1, \dots, t_n) = \int e^{i \sum_{j=1}^n t_j f_j} d\mu = \langle e^{i \sum_{j=1}^n t_j f_j} \rangle \quad (10)$$

A finite set  $f_i, i=1, \dots, n$  of random variables all defined on  $(Q, \Sigma, \mu)$  is called jointly Gaussian if and only if their joint characteristic function has the form

$$C_{f_1, \dots, f_n}(t_1, \dots, t_n) = e^{-\frac{1}{2} \sum_{i,j=1}^n \Gamma_{ij} t_i t_j} \quad (11)$$

where  $(\Gamma_{ij})$  is the covariance matrix. For  $f_1, \dots, f_n$  to be jointly Gaussian it is necessary and sufficient that all real linear combinations  $f = \sum_{i=1}^n t_i f_i$  of the  $f_i$  be Gaussian random variables. The Fourier inverse shows that if the covariance matrix  $(\Gamma_{ij})$  is invertible with inverse  $(Q_{ij})$ , then

$$d\mu_{f_1, \dots, f_n}(x_1, \dots, x_n) = (2\pi)^{-\frac{n}{2}} (\det Q_{ij})^{\frac{1}{2}} e^{-\frac{1}{2} \sum_{i,j} Q_{ij} x_i x_j} dx_1 \cdots dx_n \quad (12)$$

Thus it is the inverse of covariance matrix that enters in

Wick's Theorem (Si 1, page 9)

Let  $f_1, \dots, f_{2n}$  be jointly Gaussian random variables (not necessarily distinct), then

$$\int f_1 \cdots f_{2n} d\mu = \sum_{\text{pairings}} \langle f_{i_1} f_{j_1} \rangle \cdots \langle f_{i_n} f_{j_n} \rangle$$

where  $\langle f_i f_j \rangle = \int f_i f_j d\mu$  and  $\sum_{\text{pairings}}$  means the sum over all  $(2n)! / (2^n n!)$  ways of writing  $1, \dots, 2n$  as  $n$



distinct (unordered) pairs  $(i_1, j_1), \dots, (i_n, j_n)$ .

Suppose  $f$  is a random variable with finite moments, then the  $n^{\text{th}}$  Wick power of  $f$  denoted by  $:f^n:$ ,  $n=0, 1, \dots$ , is defined recursively by :

$$:f^0: = 1 \quad (13a)$$

$$\frac{\partial}{\partial f} :f^n: = n :f^{n-1}: \quad ; \quad n=1, 2, \dots \quad (13b)$$

$$\langle :f^n: \rangle = 0 \quad ; \quad n=1, 2, \dots \quad (13c)$$

For a Gaussian random variable with zero mean,

$$:f^n: = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{m!(n-2m)!} f^{n-2m} \left(-\frac{1}{2} \langle f^2 \rangle\right)^m \quad (14)$$

where  $\lfloor \frac{n}{2} \rfloor$  is the largest integer  $\leq \frac{n}{2}$ . If  $f$  and  $g$  are both Gaussian random variables, then

$$\langle :f^n: :g^m: \rangle = \delta_{mn} n! \langle fg \rangle^n \quad (15)$$

This expression can be generalized to the case where  $f_1, \dots, f_k$  are Gaussian random variables with  $\langle f_i f_j \rangle = \delta_{ij}$ , now we have

$$\langle :f_1^{n_1} \dots f_k^{n_k} : : f_1^{m_1} \dots f_k^{m_k} : \rangle = \delta_{n_1 m_1} \dots \delta_{n_k m_k} n_1! \dots n_k! \quad (16)$$

Furthermore, if  $f_1, \dots, f_n$  and  $g_1, \dots, g_m$  are Gaussian and  $n \neq m$ , then

$$\langle : f_1 \cdots f_n : : g_1 \cdots g_m : \rangle = 0 \quad (17)$$

Remark

One can also consider a random variable as an equivalence class of measurable functions, the equivalence relation being equality almost everywhere with respect to  $\mu$ . The set of classes of bounded random variables on the probability space  $(Q, \Sigma, \mu)$  is in one-one correspondence with  $L^\infty(Q, d\mu)$ , which is known to be a von Neumann algebra. The unbounded random variables can be viewed as (normal) unbounded operators affiliated with  $L^\infty(Q, d\mu)$ .

Conversely if  $\mathcal{O}$  is a von Neumann algebra with a faithful normal state, then  $\mathcal{O}$  can be realized as  $L^\infty(Q, d\mu)$  acting by pointwise multiplication on  $L^2(Q, d\mu)$  for some (non-unique) probability space  $(Q, \Sigma, \mu)$ ; whereby the unbounded operators affiliated with  $\mathcal{O}$  become unbounded measurable functions on  $Q$ . Note that in this context the polynomials, exponentials, and more generally the Borel functions of random variables go over into the same polynomials, exponentials and Borel functions of the corresponding operators.

I. 4. Generalized Random Field

A stochastic or random process indexed by a set  $\Lambda$  is a function from  $\Lambda$  to the set of random variables on some underlying probability space  $(Q, \Sigma, \mu)$ . Let  $\mathcal{K}$  denotes either  $\mathcal{S}_r(\mathbb{R}^n)$ , the real Schwartz space of test functions; or  $\mathcal{D}_r(\mathbb{R}^n)$ , the space of real-valued  $C^\infty$ -functions with compact support. A generalized random field  $\Phi$  over  $\mathcal{K}$  is defined as the stochastic

process indexed by  $\mathcal{K}$  such that if  $\{f_\alpha\} \subset \mathcal{K}$ ,  $f_\alpha \rightarrow f$  in  $\mathcal{K}$ , then  $\Phi(f_\alpha) \rightarrow \Phi(f)$  in measure.

We need to generalize Bochner's theorem so that nuclear spaces like  $\mathcal{S}'(\mathbb{R}^n)$  and  $\mathcal{D}'(\mathbb{R}^n)$  can be included in the underlying measure spaces. To do this some new notions are required. We consider the theory on the space  $\mathcal{S}'$ , but the same theory work on  $\mathcal{D}'$  as well. A cylinder set in  $\mathcal{S}'$  is the set of distributions  $F$  so that  $(F(f_1), \dots, F(f_n)) \in B$  where  $f_1, \dots, f_n$  are  $n$  fixed elements in  $\mathcal{S}$  and  $B$  is a fixed Borel set in  $\mathbb{R}^n$  which indexes the cylinder set. A cylinder set measure is a measure

$\mu$  on the  $\sigma$ -algebra generated by the cylinder sets with  $\mu(\mathcal{S}') = 1$ . By definition there corresponds to each  $f \in \mathcal{S}$  a measurable function  $\Phi(f)$  on  $\mathcal{S}'$  by  $\Phi(f)(F) = F(f)$ . If  $f_n \rightarrow f$  weakly, then  $\Phi(f_n) \rightarrow \Phi(f)$  pointwise.

Now we can state the generalization of Bochner's theorem:

Minlos' Theorem (Si 1, page 21)

Let  $C$  be a function on  $\mathcal{S}(\mathbb{R}^n)$ . A necessary and sufficient condition for there to be a cylinder set measure,  $d\mu$ , on  $\mathcal{S}'(\mathbb{R}^n)$  so that

$$C(f) = \int e^{i\Phi(f)} d\mu \quad (18)$$

is that

- (a)  $C(0) = 1$ ,
- (b)  $f \mapsto C(f)$  be continuous in the strong topology,
- (c) for any  $f_1, \dots, f_n \in \mathcal{S}$  and  $z_1, \dots, z_n \in \mathbb{C}$ ,
 
$$\sum_{i,j=1}^n z_i \bar{z}_j C(f_i - f_j) \geq 0$$

(18) sets up a one-one correspondence between cylinder set measures

and functions  $C(\cdot)$  obeying (a) - (c).

Thus if  $B(f, g)$  is a positive semi-definite quadratic form on  $\mathcal{F}$  which is weakly continuous and let  $C(f) = e^{-\frac{1}{2}B(f, f)}$ , then by Minlos' theorem one can construct a measure  $d\mu$  on  $\mathcal{F}'$ . The generalized stochastic process  $\bar{\Phi}$  with such a characteristic functional  $C(f)$  is a generalized Gaussian random field with mean zero and covariance functional  $B(f, g)$ . It can be shown that the probability distribution of a generalized Gaussian random field  $\bar{\Phi}$  is uniquely defined by the covariance functional  $B(f, g)$  and the mean  $E(f)$  of  $\bar{\Phi}$  (for proof see for example G - V 1, page 252). Note that for Gaussian random field the map  $f, g \mapsto \langle \bar{\Phi}(f) \bar{\Phi}(g) \rangle$  defines a semi-definite inner product on  $\mathcal{K}$ . We can then form a Hilbert space by completing the inner product space quotient  $\{ f \mid \langle \bar{\Phi}(f) \rangle^2 = 0 \}$ . If  $\mathcal{K}$  is itself already a Hilbert space, then we can always form a generalized Gaussian field indexed by  $\mathcal{K}$  with mean zero and covariance functional given by the inner product.

We end this brief introduction to random field with two important notions, the conditional expectation and Markov property.

#### Theorem

Let  $(Q, \Sigma, \mu)$  be a probability space and let  $\Sigma'$  be a sub- $\sigma$ -algebra of  $\Sigma$ . Let  $f \in L_r^1(Q, d\mu)$ . Then there exists a unique function  $E[(f | \Sigma')]$  so that

- (i)  $E[(f | \Sigma')]$  is  $\Sigma'$ -measurable;
- (ii)  $\int g E[(f | \Sigma')] d\mu = \int f g d\mu$  for all  $g$  which are  $\Sigma'$ -measurable and in  $L_r^\infty(Q, d\mu)$ .

(For proof we refer to Si 1, page 91. We remark that

the existence of such a function  $E[(f|\Sigma')]$

follows from the Radon-Nikodym Theorem.)

$E[(f|\Sigma')]$  is called the conditional expectation of  $f$  given  $\Sigma'$ . Suppose  $\Sigma$  is the smallest  $\sigma$ -algebra for which all  $\bar{\Phi}(f)$  are measurable functions and  $\Sigma_\Lambda \subset \Sigma$  is the smallest sub- $\sigma$ -algebra for which the functions  $\{\bar{\Phi}(f) \mid \text{supp } f \subset \Lambda, \Lambda \subset \mathbb{R}^n \text{ a closed set}\}$  are measurable, then  $E[f|\Sigma_\Lambda]$  is the conditional expectation of  $f$  given  $\Sigma_\Lambda$ .

We can now give a succinct statement of the multidimensional Markov property in terms of conditional expectations. Let  $\bar{\Phi}$  be a generalized random field over  $\mathcal{K}$ . Let  $\mathcal{U} \subset \mathbb{R}^n$  and  $\Sigma_{\mathcal{U}}$  be the sub- $\sigma$ -algebra generated by the  $\bar{\Phi}(f)$  with  $f \in \mathcal{K}$  and  $\text{supp } f \subset \mathcal{U}$ . Then  $\bar{\Phi}$  is a Markov field if

$$E[u|\Sigma_{\mathcal{O}'}] = E[u|\Sigma_{\partial\mathcal{O}}] \quad (19)$$

where  $\mathcal{O} \subset \mathbb{R}^n$  is any open set,  $\mathcal{O}'$  its complement and  $\partial\mathcal{O}$  its boundary, and  $u$  is a positive random variable. Express in words, (19) states that there is no more information to be gained inside  $\mathcal{O}$  from knowing the random field everywhere outside than in knowing it on the boundary.

### I. 5. Relativistic One Particle Hilbert Space And Fock Space

To describe the free relativistic scalar field theory we first construct a suitable one particle space and then second quantize. A system of one relativistic scalar particle of mass  $m$  is represented by the positive frequency solutions of the Klein-Gordon equation. Therefore the Hilbert space of states or the one particle Hilbert space  $\mathcal{M}$

is obtained by completing the inner product space whose elements are equivalence classes of elements of  $\mathcal{S}(\mathbb{R}^4)$ , with equivalence defined with respect to the norm given by the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ :

$$\langle f, g \rangle_{\mathcal{M}} = \frac{1}{2} \int dx dy \Delta_+(x-y, m^2) f(x) g(y) \quad (20)$$

$$\begin{aligned} \text{where } \Delta_+(x-y, m^2) &= \int d^4 p e^{ip(x-y)} \theta(p_0) \delta(p^2 - m^2) \\ &= \int \frac{d^3 p}{2\omega(p)} e^{i\omega(p)(x-y) - \mathbf{p}(x-y)} \\ &= \langle \Omega, \varphi(x) \varphi(y) \Omega \rangle \end{aligned}$$

with  $\omega(p) = (p^2 + m^2)^{1/2}$  and  $\Omega$  the vacuum state. If we let  $\hat{f}$  be the Fourier transform of  $f$  defined by  $\hat{f}(p) = \int dx e^{ip \cdot x} f(x)$ , then

$$\langle f, g \rangle_{\mathcal{M}} = \frac{1}{2} \int \frac{\overline{\hat{f}(p)} g(p) d^4 p}{2\omega(p)} \quad (21)$$

A unitary irreducible representation of the Poincare group can be defined on  $\mathcal{M}$  in the usual way.

The boson Fock space is given by  $\mathcal{F}(\mathcal{M}) = \sum_{n=0}^{\infty} \oplus \mathcal{M}_n$  where  $\mathcal{M}_n = \mathcal{M} \otimes_s \cdots \otimes_s \mathcal{M}$  is the  $n$ -fold symmetric tensor product of  $\mathcal{M}$  with itself, and  $\mathcal{M}_0 \equiv \mathbb{C}$ . The inner product in  $\mathcal{M}_n$  is given by

$$\langle \text{Sym } f_1 \otimes \cdots \otimes f_n, \text{Sym } g_1 \otimes \cdots \otimes g_n \rangle = \sum_{\pi} \langle f_{\pi(1)}, g_1 \rangle \cdots \langle f_{\pi(n)}, g_n \rangle \quad (22)$$

where  $\text{sym.}$  is the symmetrization operator

$$\text{Sym } f_1 \otimes \cdots \otimes f_n = \frac{1}{n!} \sum_{\pi} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}$$

Given  $f \in \mathcal{M}$  and  $\psi_n \in \mathcal{M}_n$  we define the creation operator  $a^*(f): \mathcal{M}_n \rightarrow \mathcal{M}_{n+1}$  is defined as  $a^*(f)\psi_n = (n+1)^{\frac{1}{2}} f \otimes_s \psi_n$ . The annihilation operator  $a(f)$  is the adjoint of  $a^*(f)$ . The field  $\varphi(f)$  is defined by  $\varphi(f) = \frac{1}{\sqrt{2}} [a^*(f) + a(f)]$ . Let  $\mathcal{F}_\lambda$  be the incomplete direct sum (finite particle vectors). Since the vectors in  $\mathcal{F}_\lambda$  are analytic vectors of  $\varphi(f)$ , so  $\varphi(f)$  are essentially selfadjoint on  $\mathcal{F}_\lambda$ . They commute in the strong sense of Nelson and thus generate an abelian von Neumann algebra  $\mathcal{O}$ .

In order to see how the probabilistic method comes into play in quantum field theory we consider the abstract formalism of second quantization for Boson field due to I. Segal (Se 1). Let  $\varphi$  be the Gaussian generalized stochastic process indexed by real  $\mathcal{M}$  ( $\mathcal{M}_r$ ) with mean zero and covariance functional given by

$$E[\varphi(f)\varphi(g)] = \langle f, g \rangle_{\mathcal{M}_r}$$

If  $(Q, \Sigma, \mu)$  is the underlying probability space of this process, then  $\varphi$  maps each  $f \in \mathcal{M}_r$  into a Gaussian random variable on  $(Q, \Sigma, \mu)$ .

Let  $L^2(Q, d\mu)$  be the  $L^2$ -space over  $Q$  with respect to the measure  $d\mu$ , we shall denote it by  $\Gamma(\mathcal{M}_r)$ . Let  $\Gamma_{\leq n}(\mathcal{M}_r)$  be the closed subspace spanned by  $\{\varphi(f_1), \dots, \varphi(f_m)\}$ ,  $m \leq n$ . Then  $\bigcup_n \Gamma_{\leq n}(\mathcal{M}_r)$  is dense in  $L^2$  and  $\Gamma_{\leq n-1}(\mathcal{M}_r) \subset \Gamma_{\leq n}(\mathcal{M}_r)$ . Introduce the Hermite polynomial  $\{\Gamma_n(\mathcal{M}_r)\}$  of degree  $n$  as the orthogonal complement of  $\Gamma_{\leq n-1}(\mathcal{M}_r)$  in  $\Gamma_{\leq n}(\mathcal{M}_r)$ , i.e.  $\Gamma_{\leq n}(\mathcal{M}_r) = \Gamma_{\leq n-1}(\mathcal{M}_r) \oplus \Gamma_n(\mathcal{M}_r)$ . Since  $\bigcup_n \Gamma_{\leq n}(\mathcal{M}_r)$  is dense in  $L^2$ , there exists a natural isomorphism between  $L^2(Q, d\mu)$  and  $\Gamma(\mathcal{M}_r) = \sum_{n=0}^{\infty} \Gamma_n(\mathcal{M}_r)$  with  $\Gamma_0(\mathcal{M}_r) = \mathbb{C}$ .

Following Segal we define Wick ordered monomial  $:\varphi(f_1)\cdots\varphi(f_n):$  as the orthogonal projection of  $\varphi(f_1)\cdots\varphi(f_n)$  on  $\Gamma_n(\mathcal{M}_r)$ .

We then have

$$\langle :\varphi(f_1)\cdots\varphi(f_n):, :\varphi(g_1)\cdots\varphi(g_n): \rangle = \sum_{\pi} \langle f_{\pi(1)}, g_1 \rangle \cdots \langle f_{\pi(n)}, g_n \rangle \quad (23)$$

It is clear from (22) and (23) that

$$:\varphi(f_1)\cdots\varphi(f_n): \longmapsto \text{Sym } f_1 \otimes \cdots \otimes f_n$$

extends uniquely to be unitary from  $\Gamma_n(\mathcal{M})$  onto  $\mathcal{M}_n$ . We shall use this unitary mapping to identify  $\Gamma_n(\mathcal{M})$  and  $\mathcal{M}_n$  the  $n$ -particle space, and hence  $\Gamma(\mathcal{M}) \equiv \mathcal{F}(\mathcal{M})$  is a Fock space. We have the following theorem:

Theorem (Segal : Se 1)

The boson Fock space  $\mathcal{F}(\mathcal{M})$  is unitarily equivalent to  $\Gamma(\mathcal{M})$  under a unitary map  $V$  so that

- (a)  $V\Omega = 1$
- (b)  $V\mathcal{M}_n = \Gamma_n(\mathcal{M})$
- (c)  $V\varphi(f)V^{-1} = \varphi(f)$

Suppose  $A$  is a contraction on  $\mathcal{M}$  (i.e. a linear mapping of norm  $\leq 1$ ). Then there exists a unique contraction  $\Gamma(A)$  on  $\Gamma(\mathcal{M})$  such that

$$\Gamma(A) : \varphi(f_1)\cdots\varphi(f_n) : = : \varphi(Af_1)\cdots\varphi(Af_n) :$$

In Fock space notation,

$$\Gamma(A) \upharpoonright \mathcal{F}_n(\mathcal{M}) = A \otimes \cdots \otimes A \quad (n \text{ times})$$

Note that  $\Gamma(A)\Gamma(B) = \Gamma(A)\Gamma(B)$  and  $\Gamma(1) = 1$ .



We shall end this section with a remarkable theorem of Nelson which characterizes completely  $\Gamma(A)$  in all cases of interest.

Theorem (Nelson : Ne 3)

Let  $A : \mathcal{M} \rightarrow \mathcal{M}'$  be a contraction from one real Hilbert spaces to another. Then  $\Gamma(A)$  is a contraction from  $L^q(\mathcal{M})$  to  $L^p(\mathcal{M}')$  for  $1 \leq q \leq p \leq \infty$ , provided that

$$\|A\| \leq \left( \frac{q-1}{p-1} \right)^{\frac{1}{2}} \quad (24)$$

If (24) does not hold, then  $\Gamma(A)$  is not a bounded operator from  $L^q(\mathcal{M})$  to  $L^p(\mathcal{M}')$ .

#### I. 6. Free Euclidean Markov Field

To the physical single particle system living in Minkowski space we associate a mathematical image, living in Euclidean space, from which all properties of the physical system can be easily derived. The starting point is given by the two point Schwinger function

$$S(x-y) = (2\pi)^{-4} \int e^{ip \cdot (x-y)} (p^2 + m^2)^{-1} d^4p \quad (25)$$

which is just the analytic continuation of the two point Wightman function to pure imaginary time and real space. It defines the Green's function for  $(-\Delta + m^2)$  where  $\Delta = \sum_{\mu=1}^4 \partial_{\mu}^2$  so that

$$(-\Delta_x + m^2) S(x-y) = \delta(x-y) \quad (26)$$

$S(x-y)$  is positive-definite and analytic for  $x \neq y$ , it

decreases like  $e^{-m|x-y|}$  as  $|x-y| \rightarrow \infty$ .

We now introduce Sobolev space  $\mathcal{H}^\alpha(\mathbb{R}^n)$ , which is the space of all distributions  $f$  on  $\mathbb{R}^n$  with finite norm

$$\|f\|_{\mathcal{H}^\alpha}^2 = \frac{1}{2} \int |\hat{f}(p)|^2 (p^2 + m^2)^\alpha d^n p \quad (27)$$

Recall that the relativistic one particle space  $\mathcal{M}$  for a free boson with mass  $m > 0$  in 3 space dimensions is the space  $\mathcal{H}^{-\frac{1}{2}}(\mathbb{R}^3)$ .

The one particle space  $\mathcal{N}$  for the corresponding Euclidean field is taken to be  $\mathcal{H}^{-1}(\mathbb{R}^4)$  with finite norm

$$\|g\|_{\mathcal{N}}^2 = \int |\hat{g}(p)|^2 (p^2 + m^2)^{-1} d^4 p \quad (28)$$

The choice of inner product in  $\mathcal{N}$  is dictated by the fact that the two point Schwinger function is formally the kernel of the operator  $(-\Delta + m^2)^{-1}$ , so that

$$\langle f, g \rangle_{\mathcal{N}} = \iint \overline{f(x)} S(x-y) g(y) d^4 x d^4 y = \langle f, (-\Delta + m^2)^{-1} g \rangle_{L^2} \quad (29)$$

A generalized Gaussian random field  $\Phi$  indexed by  $\mathcal{N}$  can then be defined as a generalized stochastic process with mean zero and covariance functional given by the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{N}}$ . The representation of the Euclidean group  $IO(4)$  on the underlying probability space  $(\Omega, \Sigma, \mu)$  of  $\Phi$  is given by the measure-preserving automorphism  $T_\beta$  of the measure algebra  $\Sigma$ . Given  $u, v$  in  $L^\infty$ ,  $\beta \mapsto \int u(v \cdot T_\beta) d\mu$  is a measurable function on  $IO(4)$ . Euclidean covariance of the field is given by

$$T_{\beta} \bar{\Phi}(f) = \bar{\Phi}(f \circ \beta^{-1}) \quad (30)$$

If  $\beta$  is the reflection in the hyperplane  $x_4 = S$ , then the Gaussian random field is said to satisfy the reflection property if

$$T_{\beta} u = u \quad (31)$$

where  $u$  is any random variable which is measurable with respect to the sub- $\sigma$ -algebra  $\Sigma_{x_4=S}$ . If the reflection property holds for one hyperplane, it holds for all hyperplane.

In order to find the connection between the Minkowski and Euclidean one particle spaces, we introduce the following mapping for  $f \in \mathcal{M}$ ,

$$j_s : f(x, t) \longrightarrow f(x) \delta(t-s) \quad (32)$$

We have the following theorem:

Theorem

- (i)  $j_s$  is an isometry from  $\mathcal{M} \equiv \mathcal{H}^{-\frac{1}{2}}(\mathbb{R}^3)$  to  $\mathcal{N} \equiv \mathcal{H}^{-1}(\mathbb{R}^4)$ .
- (ii) The range  $\mathcal{M}_{(S)}$  of  $j_s$  consists precisely of those elements of  $\mathcal{N}$  with support in the hyperplane  $x_4 = S$ .
- (iii)  $j_r^* j_s = e^{-|r-s|\mu}$  where  $\mu = (-\Delta + m^2)^{\frac{1}{2}}$  and  $j_r^*$  is the adjoint of  $j_r$ .

Proof

(i) and (iii) follow immediately from the identities

$$\int_{-\infty}^{+\infty} \frac{e^{ips}}{p^2 + m^2} dp = \frac{\pi e^{-m|s|}}{m} \quad (33)$$

and  $(\widehat{j_s f})(p, p_4) = (2\pi)^{-\frac{1}{2}} \widehat{f}(p) e^{iSp_4}$  . As for (ii) we need only to show that any element of  $\mathcal{N}$  with support in the hyperplane  $x_4 = S$  has the form  $f(x) \delta(S)$  . But any distribution supported on the hyperplane has the form  $\sum_{r=0}^n f_{(r)}(x) \delta_{(r)}(x_4 - S)$  and from (28) we see that this distribution belongs to  $\mathcal{N}$  if and only if  $n = 0$  .

Q.E.D.

We can thus naturally imbed  $\mathcal{M}$  onto the subspace  $\mathcal{N}$  of distribution with support on the hyperplane  $x_4 = S$  . Since the most general element of  $\mathcal{N}_0$  is of the form  $f \otimes \delta_0 = f(x) \delta(x_4)$  we have  $\|f \otimes \delta_0\|_{\mathcal{N}} = \|f\|_{\mathcal{M}}$  . Therefore there is an isomorphic and isometric identification of two Hilbert spaces  $\mathcal{M}$  and  $\mathcal{N}$  . In other words, the Euclidean scalar boson field  $\bar{\Phi}(x)$  agrees at time zero with the relativistic boson field, i.e.  $\bar{\Phi}(0, x) = \varphi(x)$  . Similar relation holds between the Fock spaces  $\Gamma(\mathcal{N})$  and  $\Gamma(\mathcal{M})$  . The second quantized version of the above theorem can be formulated as follows. Let  $\mathcal{N} = \Gamma(\mathcal{N})$ ,  $\mathcal{F} = \Gamma(\mathcal{M})$ ,  $\mathcal{J}_s = \Gamma(j_s)$ , then we have

Theorem (Nelson)

- (i)  $\mathcal{J}_s$  is an isometric imbedding of  $\mathcal{F}$  into  $\mathcal{N}$  . The range of  $\mathcal{J}_s$  is the subspace  $\mathcal{F}_{(s)}$  of  $\mathcal{N}$  concentrated at  $x_4 = S$  .
- (ii)  $\mathcal{J}_t^* \mathcal{J}_s = e^{-|t-s|H_0}$  , where  $H_0 = \Gamma(\mu)$  is the free Hamiltonian on  $\mathcal{F}$  .

Let  $e_s = j_s j_s^*$  be the projection in  $\mathcal{N}$  onto  $\mathcal{M}_{(s)}$  . Suppose  $u(t)$  denotes translation along the  $x_4$ -direction and  $\mathcal{V}_0$

reflection in the hyperplane  $x_4 = 0$ , then from the theorem above we have

$$e_0 u(s) e_0 = j_0 e^{-|s|\mu} j_0^* \quad (34)$$

$$\text{and } r_0 j_0 = j_0 ; \quad j_0^* r_0 = j_0^* \quad (35)$$

(35) is an immediate consequence of the fact that  $\delta(x) = \delta(-x)$ , and it tells us that  $r_0$  leaves  $\mathcal{N}$  pointwise invariant. If we identify  $\mathcal{M}_{(0)}$  and  $\mathcal{M}$  by  $\mathcal{M} \xrightleftharpoons[j_0^*]{j_0} \mathcal{M}_{(0)}$ , then (34) becomes

$$e_0 u(s) e_0 = e^{-|s|\mu} \quad (36)$$

Let  $E_s = \Gamma(e_s)$ ,  $U(t) = \Gamma(u(t))$  and  $R_t = \Gamma(r_t)$ .

We have

$$E_0 U(t) E_0 = J_0 e^{-|t|H_0} J_0^* \quad (37)$$

with  $J_s J_s^* = E_s$ ;  $J_s = U(s) J_0$

$R_t$  leaves  $\mathcal{F}_{(0)}$  pointwise invariant. So  $\mathcal{F}$  is naturally realized as a subspace of  $\mathcal{N}$  such that

$$e^{-|t|H_0} = E U(t) E \upharpoonright \mathcal{F} \quad (38)$$

where  $E$  is the projection onto  $\mathcal{F}$  as imbedded in  $\mathcal{N}$ .

We remark that there exists a connection between Nelson's theory and the theory of unitary dilation of Foias-Sz-Nagy (F - S 1). It is clear that the group  $U(t)$  is a dilation of the semigroup  $e^{-tH_0}$ .

on  $\mathcal{F}$ . However it is not the minimal dilation. Rather  $U(t)$  is the minimal dilation of  $e^{-t\mu}$ , so that  $U(t)$  is the second quantization of the minimal dilation.

We now introduce a local structure on  $\mathcal{N}$  by considering a close set  $\Lambda \subset \mathbb{R}^4$ , then the subspace  $\mathcal{N}_\Lambda$  of  $\mathcal{N}$  consists of distributions  $\{f \mid f \in \mathcal{N}, \text{supp } f \subset \Lambda\}$ . Denote by  $e_\Lambda$  the orthogonal projection on  $\mathcal{N}_\Lambda$ . It is obvious that if  $A \subseteq B$  then  $\mathcal{N}_A \subseteq \mathcal{N}_B$ , and  $e_A e_B = e_B e_A = e_A$ . Let  $\Gamma(e_\Lambda) = E_\Lambda$  be the second quantized operator. We can interpret  $E_\Lambda$  as the conditional expectation with respect to the sub- $\sigma$ -algebra  $\Sigma_\Lambda$  of  $\Sigma$  generated by the fields  $\tilde{\Phi}(f)$  with  $f \in \mathcal{N}$ ,  $\text{supp } f \subset \Lambda$ . We can now consider the Markov property for one particle system in terms of the projection operator  $e_\Lambda$ .

Theorem (Pre-Markov Property)

Let  $A$  and  $B$  be closed subsets of  $\mathbb{R}^4$  with  $\overset{\circ}{A} \cap B = \emptyset$ , where  $\overset{\circ}{A}$  is the interior of  $A$ .

Then

- (i)  $e_A e_{\partial A} e_B = e_A e_B$ ,  $\partial A$  denotes the boundary of  $A$ .
- (ii) If  $f \in \mathcal{N}_B$ , then  $e_A f$  lies in  $\mathcal{N}_{\partial A}$ .

Proof

Since  $e_A e_{\partial A} = e_{\partial A}$ , so (i) is equivalent to  $e_{\partial A} e_B = e_A e_B$  which is the same as (ii). To prove (ii) we must show that  $e_A f$  has support in  $\partial A$  as a distribution. Since  $A$  is closed,  $e_A f$  has support in  $A$ , so we need only to prove that  $\int (e_A f)(x) g(x) = 0$ , if  $g$  is in  $C_0^\infty$ , with support in  $\overset{\circ}{A}$ .

$$\begin{aligned}
\int (e_A f)(x) g(x) &= \langle e_A f, (-\Delta + m^2) g \rangle_{\mathcal{N}} \\
&= \langle f, e_A (-\Delta + m^2) g \rangle_{\mathcal{N}} \\
&= \langle f, (-\Delta + m^2) g \rangle_{\mathcal{N}} \\
&= \langle f, g \rangle_{L^2} = 0
\end{aligned}$$

The second equality depends on the fact that  $e_A$  is an  $\mathcal{N}$ -orthogonal projection, and the next on the fact that  $(-\Delta + m^2)g$  also has support in  $\overset{\circ}{A}$  since  $(-\Delta + m^2)$  is local.

Q.E.D.

Theorem (The Markov Property)

Let  $A$  and  $B$  be closed subsets of  $\mathbb{R}^4$  with

$$A^{\text{int}} \cap B = \emptyset. \quad \text{Then}$$

- (i)  $E_A E_{\partial A} E_B = E_A E_B$
- (ii) If  $f$  is any  $\Sigma_B$ -measurable function, then the conditional expectation  $E_A(f)$  is  $\Sigma_{\partial A}$ -measurable.
- (iii) If  $f$  is any  $\Sigma_B$ -measurable function, then  $E_A(f) = E_{\partial A}(f)$ .

Proof

The proof for (i) follows from second quantization of (i) in the previous theorem. Now since  $E_A E_{\partial A} = E_{\partial A}$ , (ii) and (iii) follow.

Q.E.D.

Remarks

- (i) In one dimension (i.e.  $A = (-\infty, 0]$ ,  $B = [0, \infty)$ ) this property reduces to the familiar Markov relation that for questions about the future ( $u$   $\Sigma_B$ -measurable) knowledge of the present ( $E_{\partial A} u$ ) is as good as knowledge of the entire past ( $E_A u$ ).
- (ii) We have taken  $A$  and  $B$  closed merely for the sake

of convenience, the result holds for arbitrary measurable sets.

- (iii) The proof of Markovicity depends critically on the fact that  $\mathcal{N}$  has a non-local inner product such that the kernel is the inverse of a local operator.

Before we can state the reconstruction theorem of Nelson two more assumptions are necessary.

(I) Regularity Assumption

There exist  $k$  and  $\ell$  such that for each  $f \in \mathcal{S}(\mathbb{R}^3)$   
 $(1+H_0)^{-k} \bar{\Phi}(f \otimes \delta_0) (1+H_0)^{-\ell}$  is bounded (where  
 $\bar{\Phi}(f \otimes \delta_0) = \varphi(f)$ .)

(II) Ergodicity Assumption

The translation subgroup of  $IO(4)$  acts ergodically, i.e.

the only translation invariant measurable functions are constant.

Theorem (Nelson's Reconstruction Theorem)

Given Euclidean covariant Markov field theory which satisfies the regularity and ergodicity assumptions, then it is associated with an essentially unique Wightman theory.

Proof : See (Ne 1, 2, 3)

I. 7. Euclidean Tensor Field

In this section we state the Wightman axioms and Osterwalder-Schrader axioms in terms of the Wightman distributions and Schwinger distributions respectively, for massive tensor field with integer spin  $S$ .



Notations

$$[\mu^i]_s = \mu_1^i \cdots \mu_s^i, \quad i = 1, \dots, n.$$

$$[\mu]_s^n = [\mu^1 \cdots \mu^n]_s; \quad [\mu]_s^0 = 0; \quad [\mu]_s^{n*} = [\mu^n \cdots \mu^1]_s.$$

$$[k]^n = k^1 \cdots k^n; \quad [k]_s^0 = 0; \quad [k]_s^{n*} = -k_n \cdots -k_1.$$

$$\mathbb{R}^{4n} \ni \underline{x} = x_1 \cdots x_n, \quad x_j \in \mathbb{R}^4$$

$$\mathcal{S}(\mathbb{R}^{4n}) \ni f^{[\mu]_s^n} = f_1^{[\mu^1]_s} \otimes \cdots \otimes f_n^{[\mu^n]_s}; \quad f_i^{[\mu^i]_s} \in \mathcal{S}(\mathbb{R}^4)$$

Let  $\varphi_{[\mu]_s}$  be a massive tensor field obeying all the Wightman axioms (W-S 1). Then its vacuum expectation values or Wightman distribution

$$\begin{aligned} & \langle \Omega, \varphi_{[\mu^1]_s}^{k^1}(f_1^{[\mu^1]_s}) \cdots \varphi_{[\mu^n]_s}^{k^n}(f_n^{[\mu^n]_s}) \Omega \rangle \\ &= W_{[\mu^1 \cdots \mu^n]_s}^{k^1 \cdots k^n}(f_1^{[\mu^1]_s} \otimes \cdots \otimes f_n^{[\mu^n]_s}) = W_{[\mu]_s^n}^{[k]^n}(f^{[\mu]_s^n}) \end{aligned} \quad (39)$$

have the following properties :

(W0). Distribution Property

For each  $n$ ,  $W_{[\mu]_s^n}^{[k]^n}$  is a tempered distribution belonging to  $\mathcal{S}'(\mathbb{R}^{4n})$  with  $W_0^0 = 1$ .

(W1). Relativistic Covariance

For each  $n$ ,  $W_{[\mu]_s^n}^{[k]^n}$  is Poincare invariant.

$$W_{[\mu]_s^n}^{[k]^n}(\underline{x}) = \prod_{j=1}^n (\Lambda^{-1})_{\mu_j^i}^{\nu_j^i} \cdots (\Lambda^{-1})_{\mu_j^n}^{\nu_j^n} W_{[\nu]_s^n}^{[k]^n}(\Lambda \underline{x} + a) \quad (40)$$

for all  $(a, \Lambda) \in \mathcal{P}_+^\uparrow$ , where  $\Lambda \underline{x} + a = (\Lambda x_1 + a, \dots, \Lambda x_n + a)$ .

(W2). Positivity

For all finite sequences  $f_0^{[\mu]_s^0}, f_1^{[\mu^1]_s}, \dots, f_n^{[\mu^n]_s}$  of test functions,  $f_0^{[\mu]_s^0} \in \mathbb{C}$ ,  $f_n^{[\mu^n]_s} \in \mathcal{S}(\mathbb{R}^{4n})$ ,  $n = 1, \dots, N$ ,

$$\sum_{\substack{n,m \\ \mu, \nu \\ k, l}} W_{[\mu]_S^n [\nu]_S^m}^{[k]^{n-1} [l]^m} \left( (f^*)_{n, [k]^{n-1}}^{[\mu]_S^n} \times f_{m, [l]^m}^{[\nu]_S^m} \right) \geq 0 \quad (41)$$

where  $(f^*)_{n, [k]^{n-1}}^{[\mu]_S^n}(\underline{x}) = \overline{f}_{n, [k]^{n-1}}^{[\mu]_S^n} (x_n, \dots, x_1)$ .

(W3). Locality

For any  $n$  and all permutation  $\pi$  of  $1, \dots, n$ ,

$$W_{[\mu]_S^n}^{[k]^{n-1}}(x_1, \dots, x_n) = W_{[\hat{\mu}]_S^n}^{[\hat{k}]^{n-1}}(x_{\pi(1)}, \dots, x_{\pi(n)}) \quad (42)$$

where  $[\hat{k}]^{n-1} = k^{\pi(1)} \dots k^{\pi(n)}$ ,  $[\hat{\mu}]_S^n = [\mu^{\pi(1)} \dots \mu^{\pi(n)}]_S$ .

(W4). Cluster Property

For any spacelike  $a$ ,

$$\lim_{\lambda \rightarrow \infty} \sum_{\substack{n,m \\ \mu, \nu \\ k, l}} \left\{ W_{[\mu]_S^n [\nu]_S^m}^{[k]^{n-1} [l]^m} \left( (f^*)_{n, [k]^{n-1}}^{[\mu]_S^n} \times g_{m, [l]^m}^{[\nu]_S^m}(\lambda a) \right) - W_{[\mu]_S^n}^{[k]^{n-1}} \left( (f^*)_{n, [k]^{n-1}}^{[\mu]_S^n} \right) W_{[\nu]_S^m}^{[l]^m} \left( g_{m, [l]^m}^{[\nu]_S^m} \right) \right\} = 0 \quad (43)$$

where  $g_{m, [l]^m}^{[\nu]_S^m}(\lambda a)(\underline{x}) = g_{m, [l]^m}^{[\nu]_S^m}(\underline{x} - \lambda a)$ .

(W5). Spectral Condition

Translation invariance of the  $W_{[\mu]_S^n}^{[k]^{n-1}}$  implies there exist distributions  $W_{[\mu]_S^{n-1}}^{[k]^{n-1}} \in \mathcal{S}'(\mathbb{R}^{4(n-1)})$ , such that  $W_{[\mu]_S^n}^{[k]^{n-1}}(\underline{x}) = W_{[\mu]_S^{n-1}}^{[k]^{n-1}}(\underline{\xi})$  where  $\underline{\xi} = (\xi_1, \dots, \xi_{n-1})$  and  $\xi_j = x_{j+1} - x_j$ .

Then

$$\text{Supp } \hat{W}_{[\mu]_S^{n-1}}^{[k]^{n-1}} \subset \overline{V}_+^{n-1} = \{ \underline{q} \mid q_j \in \overline{V}_+, j=1, \dots, n-1 \} \quad (44)$$

where  $\hat{W}_{[\mu]_S^{n-1}}^{[k]^{n-1}}(\underline{q}) = (2\pi)^{-4(n-1)} \int e^{i \underline{q} \cdot \underline{\xi}} W_{[\mu]_S^{n-1}}^{[k]^{n-1}}(\underline{\xi}) d^{4(n-1)} \underline{\xi}$  is the Fourier transform of  $W_{[\mu]_S^{n-1}}^{[k]^{n-1}}$ ,  $\overline{V}_+$  is the closed forward light cone, and  $q_j \xi_j = q_j^0 \xi_j^0 - \underline{q}_j \cdot \underline{\xi}_j$ .

One can recover the field structure by the following theorem:

Theorem (Wightman's Reconstruction Theorem)

Given a series  $\left\{ W_{[\mu]_S}^{[k]_n} \right\}_{n=0}^{\infty}$  obeying axioms (W 1 - 5), there is an essentially unique field theory of Garding-Wightman type for which  $\left\{ W_{[\mu]_S}^{[k]_n} \right\}$  are the Wightman functions.

Proof : Refer to (Ar 1)

The analytic continuation of the Wightman function  $W_{[\mu]_S}^{[k]_n}$  to Euclidean region with pure imaginary time and real space gives the  $n$ -point Euclidean Green's function or Schwinger function.

$$S_{[\mu]_S}^{[k]_n}(\underline{x}) = S_{[\mu]_S}^{[k]_n}(x_1, \dots, x_n) = W_{[\mu]_S}^{[k]_n}((ix_1^0, \underline{x}_1), \dots, (ix_n^0, \underline{x}_n)) \quad (45)$$

for  $\underline{x} \in E^n = \{ \underline{x} \mid x_i \neq x_j ; \text{ for all } 1 \leq i < j \leq n \}$ . We also set  $S_0 = W_0 = 1$ .

Let  $\mathbb{R}_+^{4n} = \{ \underline{x} \in \mathbb{R}^{4n} \mid x_j^0 > 0 \text{ for all } j=1, \dots, n \}$ , and  $\mathcal{S}(\mathbb{R}_+^{4n})$  be the space of test functions with support of  $f$  in  $\mathbb{R}_+^{4n}$ , given the induced topology. The Euclidean Green's functions satisfy the following axioms of Osterwalder and Schrader.

(E0). Distribution Property

There is a Schwartz norm  $|\cdot|_{\mathcal{S}}$  on  $\mathcal{S}(\mathbb{R}_+^{4n})$  and some  $L > 0$  such that for all  $n$  and for all  $f_j^{[\mu]_S} \in \mathcal{S}(\mathbb{R}_+^4)$ ,  $j=1, \dots, n$ ,

$$\left| S_{[\mu]_S}^{[k]_n} \left( f_{[\mu]_S}^{[\mu]_n} \right) \right| \leq (n!)^L \prod_{i=1}^n |f_{\mu_i}^{[\mu]_S}|_{\mathcal{S}} \quad (46)$$

(E1). Euclidean Covariance

$$S_{[\mu]_S}^{[k]_n}(\underline{x}) = \prod_{i=1}^n (R^{-1})_{\mu_i}^{\nu_i} \cdots (R^{-1})_{\mu_n}^{\nu_n} S_{[\mu]_S}^{[k]_n}(R\underline{x}+a) \quad (47)$$

where  $(a, R) \in IO(4)$ .

(E2). Positivity

$$\sum_{\substack{n,m \\ \mu,\nu \\ k,\ell}} S_{[\mu]_S^n [\nu]_S^m}^{[k]_S^n [\ell]_S^m} \left( \Theta(f^*)_{n,[k]_S^n}^{[\mu]_S^n} \times f_{m,[\ell]_S^m}^{[\nu]_S^m} \right) \geq 0 \quad (48)$$

where  $(\Theta f)_n(x_1, \dots, x_n) = f_n(\mathfrak{I}x_1, \dots, \mathfrak{I}x_n)$ ,  $\mathfrak{I}x = (-x^0, \underline{x})$ .

(E3). Symmetry

$$S_{[\mu]_S^n}^{[k]_S^n}(x_1, \dots, x_n) = S_{[\hat{\mu}]_S^n}^{[\hat{k}]_S^n}(x_{\pi(1)}, \dots, x_{\pi(n)}) \quad (49)$$

(E4). Cluster Property

$$\lim_{\lambda \rightarrow \infty} \sum_{\substack{n,m \\ \mu,\nu \\ k,\ell}} \left\{ S_{[\mu]_S^n [\nu]_S^m}^{[k]_S^n [\ell]_S^m} \left( \Theta(f^*)_{n,[k]_S^n}^{[\mu]_S^n} \times g_{m,[\ell]_S^m}^{[\nu]_S^m}(\lambda a) \right) - S_{[\mu]_S^n}^{[k]_S^n} \left( \Theta(f^*)_{n,[k]_S^n}^{[\mu]_S^n} \right) S_{[\nu]_S^m}^{[\ell]_S^m} \left( g_{m,[\ell]_S^m}^{[\nu]_S^m} \right) \right\} = 0 \quad (50)$$

The main result of Osterwalder and Schrader is as follows:

Theorem (Osterwalder-Schrader Reconstruction Theorem)

A sequence of distributions  $\left\{ S_{[\mu]_S^n}^{[k]_S^n} \right\}_{n=0}^{\infty}$  satisfying axioms (E 0 - 4) is the sequence of Euclidean Green's functions of a uniquely determined Wightman quantum field theory.

Proof : See (0 - S 3)

Remark

In general the 0 - S axioms do not imply a field structure. The 0 - S positivity condition is distinct from the one needed for fields (Symanzik-Nelson positivity). However, for the free scalar field, the field positivity condition follows from that for  $S_2$ . Thus in this case, 0 - S positivity does imply the possibility of fields. We shall see in chapter four that for higher integer spin  $S > 1$ , 0 - S positivity does not imply field structure.

Now we give a brief introduction to Euclidean tensor random field following Nelson's approach. A Euclidean covariant rank- $S$  tensor random field over  $\mathcal{S}(\mathbb{R}^4)$  is a collection of random variables  $\bar{\Phi}_{[\mu]_S}(f^{[\mu]_S})$  on a probability space  $(\Omega, \Sigma, \mu)$  indexed by  $S$  indices  $\mu_i = 1, 2, 3, 4, i=1, \dots, S$ , such that  $\bar{\Phi}_{[\mu]_S}(f^{[\mu]_S})$  is linear and  $\bar{\Phi}_{[\mu]_S}(f^{[\mu]_S}_\alpha) \rightarrow \bar{\Phi}_{[\mu]_S}(f^{[\mu]_S})$  in measure if  $f^{[\mu]_S}_\alpha \rightarrow f^{[\mu]_S}$  in the usual topology of  $\mathcal{S}(\mathbb{R}^4)$ , and there is a representation  $T$  of the full Euclidean group (including reflection) on the underlying probability space such that

$$T_{(a,R)} \bar{\Phi}_{[\mu]_S}(f^{[\mu]_S}) = \sum_{\nu_1, \dots, \nu_S=1}^4 R_{\mu_1 \nu_1}^{-1} \cdots R_{\mu_S \nu_S}^{-1} \bar{\Phi}_{[\nu]_S}(f^{[\nu]_S}_{(a,R)}) \quad (51)$$

where  $T_{(a,R)}$  is the representative of the element  $(a, R) \in IO(4)$  in the representation  $T$ , with  $a \in \mathbb{R}^4$  a translation and  $R$  a rotation, and  $f^{[\mu]_S}_{(a,R)}(x) = f^{[\mu]_S}(R^{-1}(x-a))$ .

Let  $\rho = (0, R_\rho)$  denote reflection on hyperplane with

$$R_\rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \text{ then}$$

$$(T_\rho \bar{\Phi}_{[\mu]_S})(f^{[\mu]_S}) = (-1)^{\sum_{j=1}^S \delta_{\mu_j, 4}} \bar{\Phi}_{[\mu]_S}(f^{[\mu]_S}_\rho) \quad (52)$$

where  $f^{[\mu]_S}_\rho(x) = f^{[\mu]_S}(x, -x_4)$ . A Euclidean covariant tensor field is said to satisfy the reflection property if  $T_\rho u = u$  for all  $u \in L^2(\Omega, \Sigma, \mu)$  localized at the hyperplane  $x_4 = 0$ .

Let  $\mathcal{O} \subset \mathbb{R}^4$  be an open set and  $\Sigma_{\mathcal{O}}$  the  $\sigma$ -algebra generated by the random variables  $\bar{\Phi}_{[\mu]_S}(f^{[\mu]_S})$  with  $\text{supp } f^{[\mu]_S} \subset \mathcal{O}$ .

If  $\mathcal{U}$  is any arbitrary subset of  $\mathbb{R}^4$ , then  $\Sigma_{\mathcal{U}} = \bigcap_{v \in \mathcal{U}} \Sigma_v$ .  
 Then the Euclidean tensor random field  $\Phi_{(\mu)_s}$  is  
 Markovian if

$$E[u | \Sigma_{\mathcal{O}}] = E[u | \Sigma_{\partial \mathcal{O}}] \quad (53)$$

for all positive random variables  $u$  measurable with respect to  $\Sigma_{\mathcal{O}}$ .  
 Examples of Euclidean Markov tensor fields will be given in the next  
 three chapters.

Finally we remark that Nelson's axioms are more restrictive  
 than that of Osterwalder and Schrader, and thus lead to a richer  
 structure. However they seem to be harder to work with in constructive  
 field theory, and none of the non-trivial models constructed so far  
 has the Markov property of Symanzik and Nelson. The Osterwalder and  
 Schrader axiom scheme provides a convenient route to the Wightman  
 axioms, especially in cases where one can control Schwinger functions  
 rather than Markov field measures.

## CHAPTER TWO

## EUCLIDEAN MASSIVE SPIN ONE FIELD

II. 1. Takahashi-Umezawa Formulation

The Euclidean formulation of massive spin - 1 vector field was first studied by L. Gross (Gr 1) and T.H. Yao (Ya 1) independently. Yao has correctly shown that the Euclidean Proca field is Markovian, contrary to the implied conjecture of Gross (Remark 2.3 in Gr 1). A crucial point in Yao's proof of Markovicity for the Euclidean Proca field is that the matrix inverse of the propagator exists and is a local differential operator, just like the case in scalar field. Therefore, Nelson's proof of Markovicity can be imitated.

Actually the fact that the propagator has a local inverse is closely related to a familiar problem in quantum field theory, which has been clearly expounded by Umezawa and Takahashi (U - T 1; Um 1; Ta 1, 2). The main idea is as follows. In the theory of higher spin field first studied by Dirac (Di 1), Fierz and Pauli (Fi 1; F - P 1) and later developed by Rarita and Schwinger (R - S 1) and others (Ch 1; S - H 1), massive particles with integral spin  $S \gg 1$  are generally described by a tensor field of rank  $S$ , Since a tensor index carries spin - 1 as well as spin - 0, the tensor field  $\varphi_{\mu_1 \dots \mu_S}(x)$  will therefore include spin -  $S$  as well as lower spins associated with it. Some of these lower spins enter with negative metric in the Wightman function. In order to obtain a field describing definite spin  $S$ , subsidiary conditions need to be imposed

on the field to eliminate all the lower spins. Explicitly, the tensor field  $\phi_{\mu_1 \dots \mu_s}(x)$  describing free massive spin - S particles satisfies in addition to the Klein-Gordon equation\*

$$(\square + m^2) \phi_{\mu_1 \dots \mu_s}(x) = 0 \quad (1)$$

the following subsidiary conditions

$$\phi_{\mu_1 \dots \mu_2 \dots \mu_n \dots \mu_s}(x) = \phi_{\mu_1 \dots \mu_n \dots \mu_2 \dots \mu_s}(x) \quad (2)$$

$$\phi_{\mu \mu_2 \dots \mu_s}^{\mu}(x) = 0 \quad (3)$$

$$\partial^{\mu} \phi_{\mu \mu_2 \dots \mu_s}(x) = 0 \quad (4)$$

For example, the real massive vector field  $\phi_{\mu}(x)$  will describe a unique spin - 1 field if it satisfies not only the Klein-Gordon equation  $(\square + m^2) \phi_{\mu}(x) = 0$ , but also the subsidiary condition  $\partial^{\mu} \phi_{\mu}(x) = 0$ . This condition removes the spin - 0 part of the field which would have had a negative metric.

In the above formulation, it is difficult to construct a free Lagrangian from equations (1), (2), (3) and (4), because to do this one requires a compact, single matrix equation. Furthermore, the introduction of interaction is difficult since the interaction frequently contradicted the subsidiary conditions. To overcome these problems, Umezawa and Takahashi proposed a new formulation. Their method is to express the wave equations in the form of single matrix

\* We use the following convention for Minkowski metric :  $g_{\mu\nu} = (+1, -1, -1, -1)$



local differential equation

$$\Lambda(\partial) \phi(x) = 0 \quad (5)$$

such that it can be reduced to Klein-Gordon equation and all the subsidiary conditions by a finite number of differentiations and algebraic operations. In other words, there exists an operator  $d(\partial)$ , called Klein-Gordon divisor, such that

$$d(\partial) \Lambda(\partial) = \Lambda(\partial) d(\partial) = \square + m^2 \quad (6)$$

Furthermore, there exists a non-singular matrix  $\eta$  satisfying

$$[\eta \Lambda(\partial)]^* = \eta \Lambda(-\partial)$$

so that the equation of motion (5) can be derived from the local Lagrangian defined by

$$\mathcal{L} = -\phi^*(x) \eta \Lambda(\partial) \phi(x) \quad (7)$$

where  $*$  denotes hermitian-conjugate. It is interesting to note that the Klein-Gordon divisor  $d(\partial)$  is closely related to the spin-projection operators. To be more specific,  $d(\partial)$  acts as a spin-projection operator for energy-momentum on the mass-shell.

From the above discussion it is clear that the Green's function is given by  $d(\partial) (\square + m^2)^{-1}$  whose inverse is the local operator  $\Lambda(\partial)$ . The locality of  $\Lambda(\partial)$  ensures the Markov property for the corresponding Euclidean theory. The Markov property is thus related to the possibility of finding

a local Lagrangian. We remark that the above discussion on Markovicity holds only if the Euclidean Green's function is positive-definite (or positive-semidefinite). Otherwise, even if  $\Lambda(\partial)$  exists as a local operator, one cannot define a Euclidean random field, hence there is no Markov property to talk about.

## II. 2. Euclidean Proca Field

In this section we shall like to apply the Umezawa-Takahashi formulation to Proca field ( $S = 1, m > 0$ ), so as to make the Markov property of the corresponding Euclidean field more transparent. The two equations describing the spin - 1 massive vector field

$$(\square + m^2) \phi^\mu(x) = 0 \quad (8)$$

and

$$\partial_\mu \phi^\mu(x) = 0 \quad (9)$$

can be combined into one, namely the Proca equation

$$\Lambda^\mu_\nu(\partial) \phi^\nu(x) = 0$$

where

$$\Lambda^\mu_\nu(\partial) = -(\square + m^2) g^\mu_\nu + \partial^\mu \partial_\nu \quad (10)$$

This equation can be derived from the Lagrangian

$$\mathcal{L} = -\phi_\mu^*(x) g^\mu_\lambda \Lambda^\lambda_\nu(\partial) \phi^\nu(x) \quad (11)$$

Here we have let  $\eta$  in equation (7) be  $g^\mu_\lambda$ . The Klein-Gordon divisor is then given by

$$d^\mu_\nu(\partial) = -(g^\mu_\nu + m^{-2} \partial^\mu \partial_\nu) \quad (12)$$

as can be easily verified that

$$d^\mu_\lambda(\partial) \Lambda^\lambda_\nu(\partial) = \Lambda^\mu_\lambda(\partial) d^\lambda_\nu(\partial) = (\square + m^2) \delta^\mu_\nu \quad (13)$$

The Fourier-transform of the Green's function is

$$G_{\mu\nu}(p) = \frac{-g_{\mu\nu} + m^{-2} p_\mu p_\nu}{m^2 - p^2 - i\varepsilon} \quad (14)$$

The two-point Wightman function can be expressed in the following form

$$\begin{aligned} W_{\mu\nu}(x-y) &= \langle \Omega, \phi_\mu(x) \phi_\nu(y) \Omega \rangle \\ &= (2\pi)^{-3} \int \frac{d^4 p}{2\omega(p)} e^{ip(x-y)} \left( -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \right) \end{aligned} \quad (15)$$

where  $\omega(p) = (p^2 + m^2)^{1/2}$ .

We now proceed to construct the relativistic one particle Hilbert space  $\mathcal{M}$ . Let  $\mathcal{V}$  be the space of complex vector functions  $\hat{f}_\mu(p)$  defined on the positive mass hyperboloid

$$V_m^+ = \{ p \in \mathbb{R}^4 \mid p_0^2 = \underline{p}^2 + m^2, p_0 > 0 \},$$

with Lorentz-invariant inner product

$$\langle f, g \rangle = \int \frac{d\underline{p}}{\omega(\underline{p})} \overline{\hat{f}_\mu(\underline{p})} \left( -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right) \hat{g}_\nu(\underline{p})$$

This inner product does not lead to a positive-definite norm, however it does give rise to a positive-semidefinite norm. In order to obtain a Hilbert space one needs to consider the set of equivalence classes modulo the subspace of zero norm. These non-zero vectors with vanishing norm are of the form  $\hat{f}_\mu(\underline{p}) = \alpha(\underline{p}) P_\mu$ . If we denote by  $\mathcal{N}_0$  the subspace spanned by these vectors, then the relativistic one particle Hilbert space  $\mathcal{M}$  is defined as the quotient space  $\mathcal{V}/\mathcal{N}_0$ . For the time zero vector field, one can choose a suitable co-ordinate system such that  $f_0 = 0$  and thus leaves only three independent spatial components. Now the relevant test functions at time zero are real vector functions  $f(\underline{x})$  of the space variable  $\underline{x}$ . We have the following reality condition in the momentum space :

$$\hat{f}(\underline{p}) = \overline{\hat{f}(-\underline{p})}$$

In this representation the relativistic one particle space can be taken as the completion of the test function space  $[\mathcal{S}(\mathbb{R}^3)]^3 \equiv \mathcal{S}(\mathbb{R}^3) \times \mathcal{S}(\mathbb{R}^3) \times \mathcal{S}(\mathbb{R}^3)$  in the topology given by the inner product

$$\langle f, g \rangle_M = \sum_{i,j=1}^3 \int \frac{dP}{\omega(P)} \widehat{f}_i(P) (\delta_{ij} + \frac{P_i P_j}{m^2}) \widehat{g}_j(P)$$

such that

$$\|f\|_M^2 = \|f\|_{\frac{1}{2}}^2 + m^{-2} \|\operatorname{div} f\|_{-\frac{1}{2}}^2 < \infty, \quad (16),$$

for  $f \in [\mathcal{S}(\mathbb{R}^3)]^3$ , where

$$\|f\|_{\frac{1}{2}}^2 = \langle \widehat{f}, (P^2 + m^2)^{\frac{1}{2}} \widehat{f} \rangle_{L^2(\mathbb{R}^3)}$$

and

$$\operatorname{div} f(x) = \sum_{i=1}^3 \partial_i f_i(x)$$

The transition from relativistic Green's function to Euclidean Green's function is not as direct as that in scalar case. The main difference lies in the fact that the Minkowski metric  $g_{\mu\nu}$  appears in the relativistic Green's function and no amount of analytic continuation is going to change the indefinite  $g_{\mu\nu}$  into the definite  $\delta_{\mu\nu}$  needed for a probabilistic interpretation. To overcome this difficulty, we shall use an idea suggested by Streater that one should re-introduce the "old-fashioned" four-vector  $\phi_\mu$ ,  $\mu = 1, 2, 3, 4$ , with  $\phi_4(x) = i\phi_0(x)$ . We shall call  $\phi_\mu(x)$  with  $\mu = 1, 2, 3, 4$ , a Minkowski four-vector to distinguish it from the Lorentz four-vector  $\phi_\mu(x)$ ,  $\mu = 0, 1, 2, 3$ . It is the Schwinger functions of Minkowski four-vector field that are covariant under the

real Euclidean group. This fact is a bit obscure in (O-S 1, 2, 3) where for the most part, covariance relative to the complex Lorentz group is considered.

Now the Schwinger two-point function of the Minkowski four-vector field can be found by noting that

$$\begin{aligned}\langle \phi_4(x) \phi_4(y) \rangle_E &= \langle i \phi_0(\underline{x}, ix^0) i \phi_0(\underline{y}, iy^0) \rangle \\ &= i^2 (-g_{00} - m^2 \partial_0 \partial_0) \Delta_+(x-y, i(x^0 - y^0)) \\ &= (1 - m^2 \partial_4^2) S(x-y)\end{aligned}$$

$$\begin{aligned}\langle \phi_4(x) \phi_j(y) \rangle_E &= \langle i \phi_0(\underline{x}, ix^0) \phi_j(\underline{y}, iy^0) \rangle \\ &= -i m^2 \partial_0 \partial_j S(x-y) \\ &= -m^2 \partial_4 \partial_j S(x-y)\end{aligned}$$

where  $\langle \quad \rangle_E$  denotes vacuum expectation values taken at Schwinger points, and  $S(x-y)$  is the Schwinger two-point function for scalar boson field. Combining these results we obtain for the two-point Schwinger function for Minkowski vector field as

$$\begin{aligned}S_{\mu\nu}(x-y) &= \langle \phi_\mu(x) \phi_\nu(y) \rangle_E \\ &= (\delta_{\mu\nu} - m^2 \partial_\mu \partial_\nu) S(x-y)\end{aligned}$$

whose Fourier-transform is  $(\delta_{\mu\nu} + m^2 P_\mu P_\nu) (p^2 + m^2)^{-1}$ .

By direct computation one can verify that the two-point Schwinger function  $S_{\mu\nu}(x-y)$  is positive-definite. Thus one can define Euclidean one particle Hilbert space  $\mathcal{K}$  as the completion of the vector-valued test-function space

$[\mathcal{S}(\mathbb{R}^4)]^4 \equiv \mathcal{S}(\mathbb{R}^4) \times \mathcal{S}(\mathbb{R}^4) \times \mathcal{S}(\mathbb{R}^4) \times \mathcal{S}(\mathbb{R}^4)$  with respect to the norm

$$\begin{aligned} \|f\|_{\mathcal{K}}^2 &= \sum_{\mu, \nu=1}^4 \langle f_{\mu}, S_{\mu\nu} f_{\nu} \rangle_{L^2(\mathbb{R}^4)} \\ &= \sum_{\mu, \nu=1}^4 \int \frac{\hat{f}_{\mu}(p) (\delta_{\mu\nu} + m^{-2} p_{\mu} p_{\nu}) \hat{f}_{\nu}(p) d^4 p}{p^2 + m^2} \\ &= \|\hat{f}\|_{-1}^2 + m^{-2} \|p \cdot \hat{f}\|_{-1}^2 \\ &= \|f\|_{-1}^2 + m^{-2} \|\operatorname{div} f\|_{-1}^2 < \infty \end{aligned}$$

where

$$\operatorname{div} f = \sum_{\mu=1}^4 \partial_{\mu} f_{\mu}(x).$$

The relation between the relativistic one particle space  $\mathcal{M}$  and the Euclidean one particle space  $\mathcal{K}$  is given by the following theorem due to Gross (Gr 1) :

Theorem II. 1.

Let  $\mathcal{K}_0$  be the time zero subspace of  $\mathcal{K}$  consisting of those distributions with support in the hyperplane  $x_4 = 0$ . Then  $\mathcal{K}_0$  can be naturally identified with  $\mathcal{M}$ .

L. Gross has also shown that the space  $\mathcal{K}$  can be obtained by "dilating" the semigroup  $e^{-tH_0}$ , where  $H_0$  is the Hamiltonian of a single free particle of mass  $m$  and spin = 1. Our remarks show that this dilation is connected to the analytic continuation, just as it is in Nelson's theory of scalar boson. By a straight forward generalization one can obtain a similar relation for the corresponding Fock spaces  $\Gamma(\mathcal{M})$  and  $\Gamma(\mathcal{K})$ . For real Proca fields the one particle spaces will be real  $\mathcal{M}$  and  $\mathcal{K}$  which we shall denote by  $\mathcal{M}_r$  and  $\mathcal{K}_r$  respectively.

We shall now consider a generalized Gaussian random vector field  $\bar{\Phi}$  over  $\mathcal{K}_r$  with mean zero and covariance given by

$$E[\bar{\Phi}(f)\bar{\Phi}(g)] = \langle f, g \rangle_{\mathcal{K}} \quad , \quad f, g \in \mathcal{K}_r$$

so that  $\bar{\Phi}$  maps  $f \in \mathcal{K}_r$  into a measurable function (a Gaussian random variable) on a probability space  $(\Omega, \Sigma, \mu)$ . We have  $\bar{\Phi}(f) = \bar{\Phi}_1(f_1) + \bar{\Phi}_2(f_2) + \bar{\Phi}_3(f_3) + \bar{\Phi}_4(f_4) = \sum_{\mu=1}^4 \bar{\Phi}_{\mu}(f_{\mu})$  and if  $F = \{F_1, F_2, \dots\}$  is a fundamental sequence with  $F_i \in [\mathcal{S}_r(\mathbb{R}^4)]^4$ , then  $\bar{\Phi}(F) = \{\bar{\Phi}(F_1), \bar{\Phi}(F_2), \dots\} \in L^2(\Omega, \Sigma, \mu)$ , the space of square-integrable functions on the sample space of  $\bar{\Phi}$  which are measurable with respect to the  $\sigma$ -algebra  $\Sigma$  generated by  $\{\bar{\Phi}(f), f \in \mathcal{K}_r\}$ .

To see that  $\bar{\Phi}$  is Euclidean covariance we let  $J_{(a,R)}$  be a transformation in  $\mathcal{K}_r$  such that

$$J_{(a,R)} f = R f_{(a,R)}$$

where  $f_{(a,R)} = f(R^{-1}(x-a))$ ,  $f \in \mathcal{K}_r$ .

Then we have

$$\langle J_{(a,R)} f, J_{(a,R)} f \rangle_{\mathcal{K}} = \langle f, f \rangle_{\mathcal{K}}$$

Hence  $J_{(a,R)}$  is an orthogonal transformation in  $\mathcal{K}_r$  then there exists an automorphism  $T_{(a,R)}$  of the measure algebra of  $(\Omega, \Sigma, \mu)$  such that

$$T_{(a,R)} \bar{\Phi}(f) = \bar{\Phi}(J_{(a,R)} f)$$



Thus,

$$T_{(a,R)} \left[ \sum_{\mu=1}^4 \bar{\Phi}_{\mu}(f_{\mu}) \right] = \sum_{\mu,\nu=1}^4 R_{\mu\nu}^{-1} \bar{\Phi}_{\nu}(f_{\mu(a,R)})$$

where  $f_{\mu(a,R)}(x) = f_{\mu}(R^{-1}(x-a))$ .

Therefore we have

$$T_{(a,R)} \bar{\Phi}_{\mu}(f) = \sum_{\nu=1}^4 R_{\mu\nu}^{-1} \bar{\Phi}_{\nu}(f_{(a,R)})$$

for  $f \in \mathcal{S}_r(\mathbb{R}^4)$ . This relation holds also for  $f \in \mathcal{S}(\mathbb{R}^4)$ . Hence  $T$  is a representation of the full Euclidean group on  $(\mathcal{Q}, \Sigma, \mu)$ . This completes the verification of Euclidean covariance for the Gaussian random field  $\bar{\Phi}$ , which we can now call Euclidean Proca field.

It was first suggested by Yao (Ya - 2) that the Euclidean Proca field  $\bar{\Phi}$  satisfies a reflection property similar to that given by Nelson. We shall prove this property in the following theorem.

#### Theorem 2

Let  $\Sigma_0$  be the  $\sigma$ -algebra generated by  $\{\bar{\Phi}(f), f \in \mathcal{K}_{0,r}\}$  where  $\mathcal{K}_{0,r}$  is the real time zero subspace of  $\mathcal{K}$ . If  $\rho$  is the reflection in the hyperplane  $X_4 = 0$  then

$$\tau_{\rho} u = u, \quad \forall u \in \Sigma_0$$

#### Proof :

Let  $T$  be the time-reversal transformation on  $\mathcal{K}_{0,r}$  defined as follows:

$$(\mathcal{T}f)^\mu = (-1)^{\delta_{\mu 4}} f_\rho^\mu$$

where

$$f_\rho^\mu(x) = f^\mu(\underline{x}, -x_4)$$

By Euclidean covariance we have

$$\begin{aligned} \bar{\Phi}(\mathcal{T}f) &= \sum_{\mu=1}^4 \bar{\Phi}_\mu((\mathcal{T}f)^\mu) \\ &= \sum_{\mu=1}^4 (-1)^{\delta_{\mu 4}} \bar{\Phi}_\mu(f_\rho^\mu) \\ &= \tau \left[ \sum_{\mu=1}^4 \bar{\Phi}_\mu(f^\mu) \right] \\ &= \tau \bar{\Phi}(f) \end{aligned}$$

Since the elements of  $\mathcal{K}_{or}$  are distributions of the form  $f = \{ \underline{f}_1 \otimes \delta, \underline{f}_2 \otimes \delta, \underline{f}_3 \otimes \delta, 0 \}$  where  $\underline{f}_j \in \mathcal{S}_r(\mathbb{R}^3)$ ,  $\mathcal{T}$  leaves  $\mathcal{K}_{or}$  pointwise invariant. Therefore we have

$$\tau_\rho \bar{\Phi}(f) = \bar{\Phi}(f)$$

By Segal isomorphism the Fock space over  $\mathcal{K}_{or}$ ,  $\Gamma(\mathcal{K}_{or})$ , is isomorphic to  $L^2(Q, \Sigma, d\mu)$  and  $\Gamma(\mathcal{T}) = \tau$  is the operator on  $L^2(Q, \Sigma, d\mu)$  corresponding to  $\mathcal{T}$  on  $\mathcal{K}_{or}$ . Consequently we have

$$\tau_\rho u = u \quad \forall u \in L^2(Q, \Sigma, d\mu)$$

Q.E.D.

The following result on Markovicity first proved by Yao (Ya 1) can be easily seen in Umezawa-Takahashi formulation.

Theorem 3

The generalized Gaussian vector field  $\bar{\Phi}$  over  $\mathcal{K}_r$  is Markovian.

Proof :

The proof is similar to that for scalar boson given by Nelson (Ne 1). We remark that the Markovicity for the Euclidean Proca field  $\bar{\Phi}$  is obvious if one consider Umezawa-Takahashi formulation. The existence of the local inverse  $\Lambda_{\mu\nu}(\partial)$  for the Green's function  $G_{\mu\nu}(\partial)$  guarantees the Markovicity. In the Euclidean region the Euclidean Green's function  $(S) = S_{\mu\nu}$  has a local inverse,

$$(S_{\mu\nu})^{-1} = (-\Delta + m^2) \delta_{\mu\nu} + \partial_\mu \partial_\nu$$

The rest of the proof then follows from Nelson's argument.

Q.E.D.

Remark

In L. Gross' work (Gr 1) he considered  $\langle (m^2 - \Delta)g, h \rangle_{\mathcal{K}_r}$  instead of  $\langle Sg, h \rangle_{\mathcal{K}_r}$ . Since  $\langle (m^2 - \Delta)g, h \rangle_{\mathcal{K}_r}$  is irrelevant to the question (of Markovicity), therefore it is not surprising that it lead to a wrong conclusion that  $\bar{\Phi}$  is non-Markovian.

We shall conclude this section with a remark on the subsidiary condition  $\partial_\mu \phi^\mu = 0$  in Euclidean region. To see what happens to this divergenceless condition in this model, we introduce the following definition:

Definition

A Euclidean field is said to be ultra-local if all its cumulants  $E_T [\bar{\Phi}(x_1) \cdots \bar{\Phi}(x_n)]$ , i.e. all its truncated expectation values, are zero unless all  $x_1, \dots, x_n$  are equal.

If we assume that the first moments  $E[\bar{\Phi}(f)]$  are vanish, then the Wightman field obtained from an ultra-local Euclidean field is zero. This is because, by definition, its Wightman functions are obtained by analytic continuation of the Euclidean Schwinger functions at unequal points at which points they vanish. Quantum fields with this property are related to infinitely divisible group representation (Str 1, 2).

The Euclidean Proca field  $\bar{\Phi}_\mu$  does not satisfy  $\partial_\mu \bar{\Phi}_\mu^{(x)} = 0$  even though its Wightman field  $\phi_\mu$  satisfies  $\partial_\mu \phi^\mu(x) = 0$ . However  $\partial_\mu \bar{\Phi}_\mu(x)$  is ultra-local. Indeed, in momentum space,

$$\begin{aligned} \hat{E} [\partial_\mu \bar{\Phi}_\mu \partial_\nu \bar{\Phi}_\nu] &= \frac{p^\mu p^\nu}{p^2 + m^2} \left[ \delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \right] \\ &= \frac{p^2}{m^2} \end{aligned}$$

Then, the covariance function in  $x$ -space is  $-\frac{\Delta}{m^2} \delta^{(4)}(x)$  leading to an ultra-local field since the field is Gaussian.

Thus there exists some kind of ambiguity in the Green's functions of the Euclidean fields, they are unique up to some ultra-local terms. The addition or subtraction of such ultra-local terms

will still give the same Wightman theory.

### II. 3. Euclidean Massive Anti-symmetric Tensor Field

In the previous section we have discussed the usual four-vector representation  $\mathbb{D}^{(\frac{1}{2}, \frac{1}{2})}$  of the massive spin - 1 field, i.e. the Proca field. From the group-theoretical point of view, the most natural representation for the spin - 1 massive field is  $\mathbb{D}^{(1,0)} \oplus \mathbb{D}^{(0,1)}$  which contains entirely the spin - 1 components. This representation is described by a pair of symmetric spinors of rank-two  $\psi_{\alpha\beta}$  and  $\psi_{\dot{\alpha}\dot{\beta}}$ , which define uniquely an anti-symmetric rank-two tensor  $A_{\mu\nu}$  as follows:

$$A_{\mu\nu} = \left(\frac{1}{8}\right)^{\frac{1}{2}} \left[ \sigma_{\mu\nu; \alpha\beta} \psi^{\alpha\beta} + \sigma_{\mu\nu; \dot{\alpha}\dot{\beta}}^* \psi^{\dot{\alpha}\dot{\beta}} \right]$$

where  $\sigma_{\mu\nu; \alpha\beta} = \frac{1}{2} (\sigma_{\mu; \alpha}^{\dot{\gamma}} \sigma_{\nu; \dot{\gamma}\beta} - \sigma_{\nu; \alpha}^{\dot{\gamma}} \sigma_{\mu; \dot{\gamma}\beta})$

and  $\sigma_{\mu} = (\sigma_i, I)$  is the  $2 \times 2$  Pauli matrices.

In this way the massive spin - 1 particles can be described by a rank-two tensor  $A^{\mu\nu}$  satisfying the following equations:

$$(\square + m^2) A^{\mu\nu}(x) = 0 \quad (17)$$

$$A^{\mu\nu}(x) + A^{\nu\mu}(x) = 0 \quad (18)$$

$$\partial_{\mu} A^{\mu\nu}(x) = 0 \quad (19)$$

Condition (18) leaves  $A^{\mu\nu}$  with 6 independent components, and

the divergenceless condition (19) eliminates another 3 components hence leaves 3 independent components for  $A^{\mu\nu}$  which correctly describe the spin - 1 meson.

Now we shall see that these 3 equations can be derived from a single equation as in Umezawa-Takahashi formulation. is not assume a priori an anti-symmetric tensor. Using a method due to Aurilla and Umezawa (A - U 1, 2), which was further developed and simplified by Macfarlane and Tait (M - T 1) we have found that the required single compact matrix equation is given by

$$\Lambda_{\rho\sigma}^{\mu\nu}(\partial) A^{\rho\sigma}(x) = 0$$

with

$$\begin{aligned} \Lambda_{\rho\sigma}^{\mu\nu}(\partial) = & \frac{1}{2}(\square + m^2)(g_\rho^\mu g_\sigma^\nu - g_\sigma^\mu g_\rho^\nu) \\ & - \frac{1}{2}(g_\rho^\mu \partial^\nu \partial_\sigma - g_\sigma^\mu \partial^\nu \partial_\rho + g_\sigma^\nu \partial^\mu \partial_\rho - g_\rho^\nu \partial^\mu \partial_\sigma) \\ & - \frac{\lambda}{2} m^2 (g_\rho^\mu g_\sigma^\nu + g_\sigma^\mu g_\rho^\nu) \end{aligned} \quad (20)$$

where  $\lambda$  is any real number  $\neq 0$ . Equation (20) is the Euler-Lagrangian equation derived from the Lagrangian

$$\begin{aligned} \mathcal{L} &= [A^*(x) \eta \Lambda(\partial) A(x)] \\ &= A_{\mu\nu}^*(x) g_\kappa^\mu g_\tau^\nu \Lambda_{\rho\sigma}^{\kappa\tau}(\partial) A^{\rho\sigma}(x) \end{aligned}$$

To see that equation (20) indeed can be reduced to the Klein-Gordon equation and the two subsidiary conditions (18) and (19), we

write out equation (20) in full.

$$\begin{aligned}
 & (\square + m^2) (A^{\mu\nu}(x) - A^{\nu\mu}(x)) + (\partial^\mu \partial_\tau A^{\nu\tau}(x) - \partial^\nu \partial_\tau A^{\mu\tau}(x) \\
 & + \partial^\nu \partial_\tau A^{\tau\mu}(x) - \partial^\mu \partial_\tau A^{\tau\nu}(x)) - \lambda m^2 (A^{\mu\nu}(x) + A^{\nu\mu}(x)) = 0
 \end{aligned} \tag{20'}$$

Interchanging  $\mu$  and  $\nu$  indices and add the resulting equation to (20') we get

$$\lambda m^2 (A^{\mu\nu}(x) + A^{\nu\mu}(x)) = 0$$

Since  $\lambda \neq 0$  and  $m^2 \neq 0$ , this implies  $A^{\mu\nu}(x) + A^{\nu\mu}(x) = 0$  which is just the subsidiary condition (17). Applying this condition in (19') gives

$$(\square + m^2) A^{\mu\nu}(x) - \partial^\mu \partial_\tau A^{\tau\nu}(x) + \partial^\nu \partial_\tau A^{\tau\mu}(x) = 0 \tag{21}$$

Multiplying (21) by  $\partial_\mu$  from the left we obtain

$$(\square + m^2) \partial_\mu A^{\mu\nu}(x) - \square \partial_\tau A^{\tau\nu}(x) + \partial_\mu \partial^\nu \partial_\tau A^{\tau\mu}(x) = 0$$

Since  $A^{\tau\mu}(x)$  is anti-symmetric,  $\partial_\mu \partial^\nu \partial_\tau A^{\tau\mu}(x)$  vanishes identically. Therefore we are left with

$$-m^2 \partial_\mu A^{\mu\nu}(x) = 0$$

which implies  $\partial_\mu A^{\mu\nu}(x) = 0$  which is just the

divergenceless condition (18). This completes the proof.

Unlike the Proca field, equation (17), (18) and (19) are actually invariant under gauge transformation of second<sup>kind</sup> and they are equivalent to the Proca equation when the gauge is fixed. This can be verified by using a similar method as in electromagnetic field, noting that the relation between  $A^{\mu\nu}(x)$  and Proca field

$\phi^\mu(x)$  is analogous to that between electromagnetic field  $F^{\mu\nu}(x)$  and electromagnetic potential  $A^\mu(x)$ .

The existence of a Klein-Gordon divisor  $d(\partial)$  is guaranteed by the ability to reduce equation (19) to the Klein-Gordon equation with a finite number of differentiation and algebraic operations. By direct computation,  $d(\partial)$  is found to be

$$\begin{aligned} d_{\rho\sigma}^{\mu\nu}(\partial) &= \frac{1}{2} (g_\rho^\mu g_\sigma^\nu - g_\sigma^\mu g_\rho^\nu) \\ &+ \frac{1}{2m^2} (g_\sigma^\nu \partial^\mu \partial_\rho - g_\rho^\nu \partial^\mu \partial_\sigma + g_\sigma^\mu \partial^\nu \partial_\rho - g_\rho^\mu \partial^\nu \partial_\sigma) \\ &- \frac{(\square + m^2)}{2\lambda m^2} (g_\rho^\mu g_\sigma^\nu + g_\sigma^\mu g_\rho^\nu) \end{aligned} \quad (22)$$

It satisfies

$$\Lambda_{\rho\sigma}^{\mu\nu}(\partial) d_{\mu'\nu'}^{\rho\sigma}(\partial) = d_{\rho\sigma}^{\mu\nu}(\partial) \Lambda_{\mu'\nu'}^{\rho\sigma} = (\square + m^2) \delta_{\mu'}^\mu \delta_{\nu'}^\nu \quad (23)$$

The Fourier transform of Green's function is given by

$$\hat{d}_{\rho\sigma}^{\mu\nu}(p) (-p^2 + m^2 - i\varepsilon)^{-1}$$

The theorem can be simplified a little by assuming  $A^{\mu\nu}(x)$  a priori anti-symmetric. The necessary changes in  $\Lambda_{\rho\sigma}^{\mu\nu}(\partial)$  and  $d_{\rho\sigma}^{\mu\nu}(\partial)$  are just to drop the symmetric terms proportional to



$(g_\rho^\mu g_\sigma^\nu + g_\sigma^\mu g_\rho^\nu)$  . Such alternations do not affect the Wightman theory because the term dropped in the Green's function, i.e.  $\frac{(\square + m^2)}{2\lambda m^2} (g_\rho^\mu g_\sigma^\nu + g_\sigma^\mu g_\rho^\nu)$  , is an ultra-local term which does not contribute to the Wightman function. The new Green's function is

$$G_{\rho\sigma}^{\mu\nu}(p) = \frac{\frac{1}{2}(g_\rho^\mu g_\sigma^\nu - g_\sigma^\mu g_\rho^\nu) + \frac{1}{2m^2}(g_\rho^\nu p_\sigma^\mu - g_\sigma^\nu p_\rho^\mu + g_\rho^\mu p_\sigma^\nu - g_\sigma^\mu p_\rho^\nu)}{-p^2 + m^2 - i\varepsilon} \quad (24)$$

Let us see what happens if we try to generalize this Green's function to a one-parameter family

$$\frac{\frac{1}{2}(g_\rho^\mu g_\sigma^\nu - g_\sigma^\mu g_\rho^\nu) + \frac{\alpha}{2m^2}(g_\rho^\nu p_\sigma^\mu - g_\sigma^\nu p_\rho^\mu + g_\rho^\mu p_\sigma^\nu - g_\sigma^\mu p_\rho^\nu)}{-p^2 + m^2 - i\varepsilon}$$

where  $\alpha$  is any real number  $\neq 0$  . It has an inverse given by

$$\frac{1}{2}(\square + m^2)(g_\rho^\mu g_\sigma^\nu - g_\sigma^\mu g_\rho^\nu) + \left(1 - (\alpha - 1)\frac{m^2}{-(\alpha p^2 + m^2)}\right)(g_\rho^\mu p_\sigma^\nu - g_\sigma^\mu p_\rho^\nu + g_\rho^\nu p_\sigma^\mu - g_\sigma^\nu p_\rho^\mu)$$

which is non-local except for  $\alpha = 1$  . This implies that for  $\alpha \neq 1$  we obtain a non-local Lagrangian field theory which as we have remarked before, does not lead to a Euclidean Markov field theory. Therefore we shall restrict ourselves to the case  $\alpha = 1$  .

The relativistic one particle Hilbert space  $\mathcal{M}$  is defined as the completion of the space of anti-symmetric tensor-valued test functions  $[\mathcal{S}(\mathbb{R}^3)]_A^{16}$  with respect to the inner product

$$\langle f, g \rangle_{\mathcal{M}} = \sum_{\mu, \nu=1}^3 \langle f^{\mu\nu}, G_{\mu\nu, \rho\sigma} g^{\rho\sigma} \rangle_{L^2(\mathbb{R}^3)}$$

with  $f, g \in [\mathcal{S}(\mathbb{R}^3)]_A^{16}$  such that  
 $f^{\mu\nu} = -f^{\nu\mu}$  and  $\|f\|_{\mathcal{M}} < \infty$

After some simplifications we find that

$$\|f\|_{\mathcal{M}}^2 = 2 \left( \|f\|_{-\frac{1}{2}}^2 + 2m^{-2} \|\text{div. } f\|_{-\frac{1}{2}}^2 \right) < \infty \quad (25)$$

where  $\text{div } f = \sum_{\mu=1}^3 \partial_{\mu} f^{\mu\nu}(x)$ .

We note the close similarity between this norm and that for the one particle space for Proca field (compare equations (16) and (25)).

Therefore it is of no surprise that both Hilbert spaces describe the same system.

In order to go over to the Euclidean region, we need to introduce Minkowski tensor field as follows:

$$A^{4j} = i A^{0j}, \quad j = 1, 2, 3$$

This is necessary to change  $g_{\mu\nu}$  to  $\delta_{\mu\nu}$ . The two-point Schwinger function of the Minkowski tensor field is obtained by replacing  $g_{\mu\nu}$  by  $\delta_{\mu\nu}$  and  $\square$  by  $-\Delta$  in  $G_{\mu\nu, \rho\sigma}(x-y)$ . We get

$$S_{\mu\nu, \rho\sigma}(x-y) = \frac{(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) - \frac{1}{2m^2}(\delta_{\mu\sigma}\partial_{\nu}\partial_{\rho} - \delta_{\mu\rho}\partial_{\nu}\partial_{\sigma} + \delta_{\nu\sigma}\partial_{\mu}\partial_{\rho} - \delta_{\nu\rho}\partial_{\mu}\partial_{\sigma})}{-\Delta + m^2} \quad (26)$$

This is positive-definite as can be verified by direct computation that the determinants of all its major minors are positive in the

anti-symmetric subspace. We can now construct the Euclidean one particle Hilbert space  $\mathcal{K}$  by completing the anti-symmetric tensor valued test function space  $[\mathcal{S}(\mathbb{R}^4)]_A^{16}$  with respect to the inner product

$$\langle f, g \rangle_{\mathcal{K}} = \sum_{\mu, \nu=1}^4 \langle f_{\mu\nu}, S_{\mu\nu\rho\sigma} g_{\rho\sigma} \rangle_{L^2(\mathbb{R}^4)}$$

such that  $\|f\|_{\mathcal{K}}^2 = 2(\|f\|_{-1}^2 + 2m^{-2}\|\operatorname{div}f\|_{-1}^2) < \infty$   
 where  $\operatorname{div}f(x) = \sum_{\mu=1}^4 \partial_{\mu} f_{\mu\nu}(x)$ .

This norm is clearly translation-invariant. If  $R$  is an orthogonal transformation in  $\mathbb{R}^4$ , one has

$$\operatorname{div}[Rg(R^{-1}x)] = \operatorname{div}(g)(R^{-1}x)$$

This, together with the fact that the Sobolev norm is invariant under orthogonal transformation, enables us to conclude that the induced action on  $\mathcal{K}$  by Euclidean group is Unitary. For a unitary representation  $U$  of the Euclidean group,

$$U(a, R) f^{\mu\nu}(x) U^{-1}(0, R) = R_{\mu\alpha} R_{\nu\beta} f^{\alpha\beta}(R^{-1}(x-a))$$

where  $a \in \mathbb{R}^4$ .

Similar to the case in Proca field, if we denote by the subspace of  $\mathcal{K}_0$ , consisting of those distributions with support in the hyperplane  $X_4 = 0$ , we have the following theorem:

Theorem 4

The time zero subspace  $\mathcal{K}_0$  of  $\mathcal{K}$  is naturally identical to  $\mathcal{M}$ .

Proof:

Every element of  $\mathcal{K}_0$  can be expressed in the form

$$f(x) = \underline{f}(\underline{x}) \otimes \delta(x_4)$$

whose components are

$$f_{\mu\nu}(x) = \underline{f}_{\mu\nu}(\underline{x}) \otimes \delta(x_4)$$

where  $\underline{f} \in [\mathcal{F}(\mathbb{R}^3)]_A^{16}$  and  $\underline{f}_{\mu\nu} \in \mathcal{F}(\mathbb{R}^3)$

since

therefore  $\|\operatorname{div} f\|_{-1} < \infty$  implies all  $f_{4\nu} = 0$

for  $\nu = 1, 2, 3, 4$  . This requires all  $f_{4\nu} = 0$

for  $\nu = 1, 2, 3, 4$  . Now using the identity

$$\int_{-\infty}^{+\infty} \frac{dP_4}{P_4^2 + (P^2 + m^2)} = \frac{\pi}{(P^2 + m^2)^{1/2}}$$

it immediately follows that

$$\|f \otimes \delta\|_{\mathcal{K}} = \|\underline{f}\|_{\mathcal{M}}$$

which implies that there exists a unitary map from  $\mathcal{M}$  onto

$\mathcal{K}_0$ ,  $f \mapsto \sqrt{2} \underline{f} \otimes \delta(x_4)$  . Therefore we can identify

$\mathcal{M}$  and  $\mathcal{K}_0$  in a natural way.

Q.E.D.

We can now define the Euclidean tensor field  $\ominus$  as the generalized Gaussian tensor field over real  $\mathcal{K}$  (i.e.  $\mathcal{K}_r$ )

with mean zero and covariance given by

$$E[\theta(f)\theta(g)] = \langle f, g \rangle_{\mathcal{K}_r}$$

Theorem 5

- (i) The Euclidean tensor field  $\theta$  on  $\mathcal{K}_r$  satisfies the reflection property.
- (ii)  $\theta$  is Markovian.

Proof:

- (i) The proof will be the same as that given for the Euclidean Proca field because the covariance functions for these two fields have the same kind of singularities.
- (ii) First of all we note that relation (23) still holds even after we have dropped the ultra-local terms in  $\Lambda(\partial)$  and  $d(\partial)$ . This is because the symmetric ultra-local terms vanish in the anti-symmetric subspace. In the Euclidean region a relation similar to (23) holds

$$S_{\mu\nu,\rho\sigma}(p) \Pi_{\rho\sigma,\mu'\nu'}(p) = \delta_{\mu\mu'} \delta_{\nu\nu'}$$

where  $\Pi_{\mu\nu,\rho\sigma}(p)$  is the matrix inverse of the two-point Schwinger function  $S_{\mu\nu,\rho\sigma}(p)$ , and is given by

$$\begin{aligned} \Pi_{\mu\nu,\rho\sigma}(p) &= \frac{1}{2} (p^2 + m^2) (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) \\ &+ \frac{1}{2} (\delta_{\nu\sigma} p_\mu p_\rho - \delta_{\nu\rho} p_\mu p_\sigma + \delta_{\mu\rho} p_\nu p_\sigma - \delta_{\mu\sigma} p_\nu p_\rho) \end{aligned}$$

This is a local operator, therefore the rest of the proof of Markovicity follows from Nelson's argument.

Q.E.D.

#### II. 4. Euclidean Massive Vector Field With "Covariant Gauges"

Now we shall like to consider a theory of vector mesons which differs considerably from those given in the previous sections. This is a model of free vector mesons with "one parameter family of covariant gauges". The notion of gauge invariance plays no intrinsic role in the theory of Proca field, this can be seen from the field equations. However, we study this model for the following reasons. It is a well-known fact that Proca field does not lead to a renormalizable theory because the free propagator  $(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2})(-p^2 + m^2 - i\varepsilon)^{-1}$  does not tend to zero at high momentum. In order to obtain a renormalizable Lagrangian field theory, one introduces an extra ghost particle with mass  $m_g$  such that the free propagator of vector meson becomes

$$\left[ \frac{-g^{\mu\nu}}{-p^2 + m^2 - i\varepsilon} + \frac{p^\mu p^\nu}{m^2} \left( \frac{1}{-p^2 + m^2 - i\varepsilon} - \frac{1}{-p^2 + m_g^2 - i\varepsilon} \right) \right] \quad (27)$$

which falls off like  $p^2$  as in the scalar theory. We have achieved renormalizability by the method of regularization, however, at the expense of introducing an indefinite metric Hilbert space. Furthermore, unlike the Proca field which does not have  $m \rightarrow 0$  limit, in this formulation the  $m \rightarrow 0$  limit is well-defined and thus enables one to treat the massive and massless vector fields in a unified way. The most important reason, looking from our point of view, is that this model of vector meson provides us an example of

Euclidean Markov vector field satisfying Nelson's axioms except the reflection property and it does not lead to a Wightman theory.

For a detailed discussion of the Lorentz-region field using indefinite metric Hilbert space we refer to (Yo 1) where the unphysical states are considered as dipole ghosts. A review of the recent developments in the renormalization theory of vector meson is given in (F - T 1) which includes discussion on spontaneous symmetry breaking of gauge group and other recent advances in gauge theory. We shall only remark that Gupta-Bleuler formulation (Gu 1, Bl 1) needs to be used and in the physical Hilbert space the ghost field has zero expectation value. The observables of the theory are those quantities which commute with the divergence of the vector field, and they are independent of the mass of ghost particle.

In order to make our discussion more transparent, from now on we shall let  $m_g = \sqrt{\alpha} m$  in (27) where  $m$  is the usual mass of vector meson and  $\alpha$  is a positive real number. The free propagator now becomes

$$\begin{aligned}
 G^{\mu\nu}(p) &= \frac{-g^{\mu\nu} + m^{-2} p^\mu p^\nu}{-p^2 + m^2 - i\epsilon} - \frac{m^{-2} p^\mu p^\nu}{-p^2 + \alpha m^2 - i\epsilon} \\
 &= \frac{-g^{\mu\nu}}{-p^2 + m^2 - i\epsilon} + \frac{(\alpha - 1) p^\mu p^\nu}{(-p^2 + m^2 - i\epsilon)(-p^2 + \alpha m^2 - i\epsilon)} \quad (28)
 \end{aligned}$$

This propagator contains a gauge invariant term representing the propagator of the Proca field of mass  $m$ , and a gauge-dependent term arising from the ghost particle of mass  $\sqrt{\alpha} m$ . Thus  $\alpha$  can be viewed as a real parameter which characterized the covariant gauges. We also note that the  $m \rightarrow 0$  limit is well-

defined in (28). Indeed this massless limit can be taken as a special case of our more general formulation of electromagnetic potential with covariant gauges in next chapter.

There are two limiting cases worth mentioning, namely  $\alpha = 0$  and  $\alpha = \infty$ . The latter corresponds to the usual Proca formulation of vector meson. As for the case  $\alpha = 0$ , we get a massive vector field theory in a gauge which corresponds to the Landau gauge in the electromagnetic potential theory. In this case the propagator becomes  $(-g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2})(-p^2 + m^2 - i\varepsilon)^{-1}$ . We shall see later on that the Euclidean massive vector field in Landau gauge is non-Markovian.

Euclidean Green's function is obtained by replacing  $p^2$  by  $-p^2$  and  $g^{\mu\nu}$  by  $-\delta^{\mu\nu}$  (again we have to use Minkowski vector field with  $\phi^4 = i\phi^0$  for the same reason as before), and we get

$$\begin{aligned} S^{\mu\nu}(p) &= \frac{\delta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}}{p^2 + m^2} - \frac{\frac{p^\mu p^\nu}{m^2}}{p^2 + \alpha m^2} \\ &= \frac{1}{p^2 + m^2} \left[ \delta^{\mu\nu} + \frac{(\alpha - 1) p^\mu p^\nu}{p^2 + \alpha m^2} \right] \quad (29) \end{aligned}$$

This two-point Schwinger function is positive-definite if  $\alpha > 0$ . The Euclidean one particle space Hilbert space  $\mathcal{K}$  can be defined as the completion of the inner product space of vector-valued functions  $f, g \in [\mathcal{S}(\mathbb{R}^4)]^4$  with respect to the inner product



$$\begin{aligned}
\langle f, g \rangle_{\mathcal{K}} &= \langle f, S g \rangle_{L^2(\mathbb{R}^4)} \\
&= \sum_{\mu, \nu=1}^4 \frac{1}{\hat{f}_{\mu}(p)} \frac{\delta^{\mu\nu} + (\alpha-1) \frac{p^{\mu} p^{\nu}}{p^2 + \alpha m^2}}{p^2 + m^2} \hat{g}_{\nu}(p) d^4 p \\
&= \sum_{\mu, \nu=1}^4 \langle \hat{f}_{\mu}, \left( \delta^{\mu\nu} + \frac{(\alpha-1) p^{\mu} p^{\nu}}{p^2 + \alpha m^2} \right) \hat{g}_{\nu} \rangle_{-1}
\end{aligned}$$

with  $\|f\|_{\mathcal{K}}^2 = \left( \|f\|_{-1}^2 + (\alpha-1) \|(-\Delta + \alpha m^2)^{-\frac{1}{2}} \operatorname{div} f\|_{-1}^2 \right) < \infty$

The natural action of the four-dimensional Euclidean group in  $\mathcal{K}$  is unitary. This follows from the Euclidean invariance of the  $\|\cdot\|_{-1}$  norm and that  $\operatorname{div}(R f(R^{-1}x)) = (\operatorname{div} f)(R^{-1}x)$  where  $R$  is an orthogonal rotational transformation in  $\mathbb{R}^4$ ; and  $(-\Delta + \alpha m^2)$  is an Euclidean-covariance quantity.

As in the previous section, we define the Euclidean vector field  $\Theta$  as a generalized Gaussian vector field indexed by  $\mathcal{K}_r$  with mean zero and covariance given by

$$E[\Theta(f)\Theta(g)] = \langle f, g \rangle_{\mathcal{K}}$$

The following theorem holds for the Euclidean field  $\Theta$  :

Theorem 6

- (i) The Euclidean vector field  $\Theta$  on  $\mathcal{K}_r$  is Markovian.
- (ii) The Euclidean vector field  $\Theta$  does not satisfy the reflection property.

Proof:

- (i) The Markovicity of  $\Theta$  is proved by noting that the matrix kernel of the inner product in  $\mathcal{K}_r$ ,  $S^{\mu\nu}(p)$ , has a local matrix inverse

$$(S^{\mu\nu})^{-1} = (p^2 + m^2) \delta^{\mu\nu} + \left(\frac{1}{\alpha} - 1\right) p^\mu p^\nu$$

Now the rest of the proof follows from Nelson's argument.

- (ii) To prove that  $\Theta$  violates reflection property, we shall show that reflection property does not hold for certain class of test functions. The 4-4 component of the Schwinger two-point function,  $S^{44}(p)$  contains the term  $p^{4^2} [(p^2 + m^2)(p^2 + \alpha m^2)]^{-1}$  which allows test functions localized at the hyperplane  $x_4 = 0$ , of the form

$$f_\mu(x) = \underline{f}_\mu(\underline{x}) \otimes \delta(x_4)$$

with  $f_4 \neq 0$  and  $\underline{f}_\mu \in \mathcal{S}_r(\mathbb{R}^3)$ . For such a test function we have

$$\begin{aligned} T_p f_4(x) &= (-1)^{\delta_{44}} \underline{f}_{4p}(x) \\ &= -\underline{f}_4(\underline{x}) \otimes \delta(-x_4) \\ &= -\underline{f}_4(\underline{x}) \otimes \delta(x_4) \\ &= -f_4(x) \end{aligned}$$

Therefore  $T_p \Theta(f) \neq \Theta(f)$ .

Q.E.D.

We note that in the present case the Schwinger two-point function is less singular than that of the Proca field which contains 4-4 term  $\frac{(p^4)^2}{m^2} (p^2 + m^2)^{-1}$ , thus ruling out test functions localized at the hyperplane  $x_4 = 0$ , of the form  $\underline{f}_4 \otimes \delta(x_4)$  thereby preserves the reflection property. The effect of the reflection property may be considered as to prevent the theory from being too regular in its ultraviolet behaviour. As can be seen in Nelson's theory, reflection property excludes scalar boson fields with "good" covariance function of the form  $(-\Delta + m^2)^{-n}$ ,  $n > 1$ . Actually such fields give rise to indefinite metric Hilbert space with ghost states (or nonlocal theory without ghost states), hence do not form a Wightman theory. The fact that the Euclidean Markov vector field constructed above does not lead to a Wightman theory may be explained by its failure to satisfy the reflection property.

Finally we shall like to prove a remark made earlier in this section, namely the Euclidean vector field in Landau gauge (corresponding to  $\alpha = 0$ ) is non-Markovian. We observe two points, first the covariance function in Landau gauge  $(\delta^{\mu\nu} - \frac{p_\mu p_\nu}{p^2}) (p^2 + m^2)^{-1}$  is only positive-semidefinite; second point is that it is a singular matrix. Nelson's proof of Markovicity does not apply since the covariance matrix does not have an inverse. However, if we restrict the physical space to the subspace  $\mathcal{K}_0 \subset \mathcal{K}_r$  of distribution satisfying  $\sum_\mu \partial_\mu f_\mu = 0$ , we may hope to obtain Markov property since now  $S^{\mu\nu}$  has a local inverse in  $\mathcal{K}_0$ . This is not possible as can be seen in the following argument.

In  $\mathcal{K}_0$ , we have

$$\langle f, g \rangle_{\mathcal{K}_0} = \sum_{\mu, \nu=1}^4 \left\langle f_\mu, \frac{\delta^{\mu\nu}}{-\Delta + m^2} g_\nu \right\rangle_{L^2}$$

Now let  $f \in \mathcal{K}_0$ , with  $\text{supp } f \subset \mathcal{O}$ , where  $\mathcal{O}$  is an open set in  $\mathbb{R}^4$ . Let  $g \in \mathcal{D}(\mathcal{O}')$ ,  $\mathcal{O}'$  is the interior of the complement of  $\mathcal{O}$ . Denote the projection onto  $\mathcal{K}_0(\mathcal{O}')$  by  $e_{\mathcal{O}'}$ , then we have

$$\begin{aligned}
 \langle e_{\mathcal{O}'} f, g \rangle_{L^2} &= \sum_{\mu, \nu=1}^4 \langle e_{\mathcal{O}'} \hat{f}_\mu, \delta^{\mu\nu} \hat{g}_\nu \rangle_{L^2} \\
 &= \sum_{\mu, \nu=1}^4 \langle e_{\mathcal{O}'} \hat{f}_\mu, (\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}) \hat{g}_\nu \rangle_{L^2} \\
 (\because p^\mu \hat{f}_\mu &= 0) \\
 &= \sum_{\mu, \nu=1}^4 \langle e_{\mathcal{O}'} \hat{f}_\mu, \frac{p^2 + m^2}{p^2 + m^2} (\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}) \hat{g}_\nu \rangle_{L^2} \\
 &= \sum_{\mu, \nu=1}^4 \langle e_{\mathcal{O}'} f, (p^2 + m^2) (\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}) \hat{g}_\nu \rangle_{\mathcal{K}_0} \\
 &[\because (p^2 + m^2) (\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}) \hat{g}_\nu \in \mathcal{K}_0(\mathcal{O}')]
 \end{aligned}$$

Now the crucial step of the proof of Markovicity cannot be carried out because this requires that  $(p^2 + m^2) (\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2})$  be a local operator, which is not true. Therefore we cannot conclude that the support of  $e_{\mathcal{O}'} (p^2 + m^2) (\delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}) \hat{g}_\nu$  is not contained in  $\mathcal{O}' - \mathcal{O}' = \partial\mathcal{O}$ . This strongly suggests that the Euclidean vector field in Landau gauge is non-Markovian. This differs from Euclidean electromagnetic potential in Landau gauge which is Markovian as we shall see in next chapter.

CHAPTER THREE  
EUCLIDEAN MASSLESS SPIN-ONE FIELD

Free Euclidean electromagnetic potential in Lorentz gauge has been studied by L. Gross (Gr 1). However his conclusion that such a field is non-Markovian is incorrect. In section two of this chapter we shall consider free Euclidean electromagnetic potential in a general class of covariant gauges which includes Gross' field as a particular case. Such Euclidean field is Markovian. The failure of this Euclidean theory to lead to a Wightman theory when analytic continue back to Minkowski region may be explained by its failure to obey the reflection principle. In the second part of this chapter we shall construct a free Euclidean electromagnetic field in terms of electromagnetic field tensor  $F_{\mu\nu}$  only. This field is also Markovian but the proof for Markovicity differs from the previous proof. The Euclidean electromagnetic field  $\Theta_{\mu\nu}$  satisfies the reflection principle and it leads to a Wightman theory in the Minkowski region.

Before we consider the Euclidean formulation, we shall like to discuss briefly the main difficulties present in the quantization of the free electromagnetic theory.

### III. 1. Relativistic Quantum Electromagnetic Field Theory

The theory of massless spin - 1 particles or photons differs considerably from that of massive spin - 1 particles. The main difficulties in the quantized theory of electromagnetic field can be seen

in the following discussion.

Let  $F^{\mu\nu}$  be an anti-symmetric electromagnetic field tensor with  $F^{0k} = E^k$  as the electric field vector and  $F^{ij} = \epsilon^{ijk} H_k$  as the magnetic field vector. Then the gauge invariant Lagrangian for the relativistic classical electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) = -\frac{1}{2} (\underline{E}^2 - \underline{H}^2) \quad (1)$$

which gives rise to Maxwell equations

$$\partial_\nu F^{\mu\nu}(x) = 0 \quad (2)$$

If we introduce electromagnetic potential vector  $A^\mu = (\phi, \underline{A})$  such that

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \quad (3)$$

equation (2) then becomes

$$\square A^\mu(x) - \partial^\mu \partial_\nu A^\nu(x) = 0 \quad (4)$$

From the point of view of Takehashi-Umezawa formulation the difficulty exists in the massless higher spin theory can be considered to stem from the non-existence of Klein-Gordon divisor  $d(\partial)$ . To illustrate this point explicitly for the massless spin - 1 field, we write equation (3) in the form

$$\Lambda^\mu_\nu(\partial) A^\nu(x) = 0 \quad (5)$$

where

$$\Lambda^\mu_\nu(\partial) = \square g^\nu_\mu - \partial^\mu \partial_\nu$$

Now the Klein-Gordon divisor  $d(\partial)$  cannot be obtained from the equation

$$\Lambda^\mu_\nu(\partial) d^\nu_{\mu'}(\partial) = \square \cdot \delta^\mu_{\mu'} \quad (6)$$

since the determinant of l. h. S. of equation (6) is zero whereas the determinant of r. h. S. is simply  $\square$ . This situation corresponds to the Strocchi's difficulty in electromagnetic field which we shall discuss later on.

Due to the gauge invariant nature of the theory, the original equation (4) can be split into two if we choose the Lorentz gauge

$$\chi = \partial_\nu A^\nu(x) = 0 \quad (7)$$

we then obtain

$$\square A^\nu = 0 \quad (8)$$

$$\partial_\nu A^\nu = 0 \quad (9)$$

There exists a difficulty in the above Lagrangian approach since the momentum conjugate to  $A^\mu$ ,

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial^\circ A^\mu(x))} = -F^{\circ\mu}(x) \quad (10)$$

has its  $\mu = 0$  - component vanishes identically. Thus canonical quantization cannot be applied to this component.

To overcome this difficulty, one can choose the Fermi-gauge Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left( \frac{\partial A^\mu}{\partial x^\mu} \right)^2 \quad (11)$$

which is not gauge-invariant due to the presence of the term  $\left( \frac{\partial A^\mu}{\partial x^\mu} \right)^2$  however it is relativistic invariant. The corresponding equation of motion is

$$\square A^\mu(x) = 0 \quad (12)$$

This is equivalent to Maxwell's equations if we impose the following subsidiary conditions

$$\begin{aligned} \chi &= 0 \quad \text{at} \quad t = 0 \\ \frac{\partial \chi}{\partial t} &= 0 \quad \text{at} \quad t = 0 \end{aligned} \quad (13)$$

Then  $\chi = 0$  for all times is implied by the equation of motion

$\square \chi = 0$ . The Fermi Lagrangian can also be expressed in the following form

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu A_\nu(x)) (\partial^\mu A^\nu(x))$$

Using this we get

$$\pi^{\mu(x)} = \frac{\partial \mathcal{L}}{\partial (\partial^\circ A^\mu(x))} = -\partial^\circ A^\mu \quad (14)$$



so that the Hamiltonian is given by

$$H = -\frac{1}{2} \int [(\partial_i A_\mu)(\partial^i A^\mu) - (\partial_0 A_\mu)(\partial^0 A^\mu)] d^3x \quad (15)$$

Now we are facing with a new problem, namely the energy is not positive-definite because the  $\mu=0$  - component contributes a negative-definite quantities to  $H$ . The reason is that we have as yet not made use of the subsidiary condition, so that the theory considered so far does not correspond to the Maxwell theory. In the classical theory, the Lorentz condition  $\partial_\mu A^\mu(x) = 0$  guarantees that the field equation  $\square A^\mu(x) = 0$  corresponds to Maxwell's equations and thus ensure the positive-definiteness of the total energy. In the quantized theory, the Lorentz condition does not hold as an operator identity, otherwise it would lead to a contradiction because

$$\left[ \frac{\partial A^\mu(x)}{\partial x^\mu}, A^\nu(y) \right] = i \frac{\partial D(x-y)}{\partial x^\nu} \neq 0 \quad (16)$$

So far we have only discussed the difficulties present in the Lagrangian formulation of electromagnetic field, the situation is no better in the axiomatic formulation. Garding and Wightman (G - W 1) have shown that in free quantum electrodynamics that the weak locality and relativistic covariance of electromagnetic potential leads inevitably to indefinite metric Hilbert space. The analysis of Strocchi (St 1, 2) has further indicated that if equations (2) and (3) hold as operator equations, where  $A^\mu$  transforms as a four-vector, then one gets a trivial theory with  $F^{\mu\nu} |\Omega\rangle = 0$  ( $|\Omega\rangle$  is a vacuum state). These results were obtained without

the use of spectral condition, the positive-definiteness of the Hilbert space or the locality condition. If one further requires that  $F^{\mu\nu}$  should be local and the underlying Hilbert space of states should have a positive-definite metric, then one obtains  $F^{\mu\nu} = 0$ . In addition, Strocchi has also shown that if  $A^\mu$  is quasi-local and satisfies (2) and (3) then the theory of the field  $F^{\mu\nu}$  is again trivial. All these results do not depend on whether the metric of the Hilbert space is positive-definite or indefinite. They also rule out the procedure proposed by Fermi.

To circumvent these difficulties it has been usual to follow one of the following two routes. One can either abandon the requirement of relativistic covariance and quasi-locality for  $A^\mu$  or accept the point of view that the Maxwell equations are not satisfied as operator equalities. The first method is known as Coulomb (or radiation) gauge formalism, in which the gauge condition is  $\nabla \cdot \underline{A} = 0$ . Now the underlying Hilbert space has a positive-definite metric. However the theory is no longer manifestly covariant and local, so it is necessary to supply under a Lorentz transformation with a gauge term in order to obtain covariance of the Coulomb condition. Since we are only interested in covariant theory, we shall not pursue this approach any further in this work.

The second method is known as Gupta-Bleuler formulation (Gu 1, Bl 1), which allows local and covariant potential  $A^\mu$ . This can only be done at the expense of the positive-definiteness of the underlying Hilbert space, and now Maxwell equations can no longer be satisfied in the whole Hilbert space. By imposing the nonlocal condition

$$\partial_\mu A^{\mu(+)}(x) |\varphi\rangle = 0 \quad (17)$$

where  $\partial_\mu A^{\mu(+)}$  is the positive frequency part of the operator  $\partial_\mu A^\mu$ , one can define the Hilbert space of physical states  $|\varphi\rangle$ , which has a positive-definite metric. However this subspace spanned by the physical states is not dense in the original indefinite metric Hilbert space, so the Maxwell equations are only satisfied in the sense that they hold when one takes the matrix elements of these equations between physical states.

From the above brief discussion we conclude that one cannot have a local and covariant theory of electromagnetic potential without introducing indefinite metric Hilbert space and unphysical states.

### III. 2. Free Euclidean Electromagnetic Potential

In quantum electrodynamics, different form of free photon propagator determines different type of gauge. The free propagator for electromagnetic potential  $A^\mu$  in Gupta-Bleuler formulation is

$$\langle A^\mu A^\nu \rangle = \frac{g^{\mu\nu}}{p^2} \quad (18)$$

This propagator determines the Lorentz gauge for  $A^\mu$ . The corresponding two-point Schwinger function is  $\delta^{\mu\nu} p^{-2}$  which is positive-definite. We can construct a Euclidean vector potential  $A^\mu$  with

$$\langle A^\mu A^\nu \rangle = \frac{\delta^{\mu\nu}}{p^2} \quad (19)$$

In order to obtain a theory of Euclidean electromagnetic potential which includes a wide class of covariant gauges, we introduce the following transformation:

$$A^\mu \longrightarrow A^\mu_{\#} = A^\mu + \partial^\mu \Lambda \quad (20)$$

where  $\Lambda(x)$  is a real scalar random field independent of  $A^\mu(x)$ . The propagator for the transformed field  $A^\mu_{\#}(x)$  has the general form (in the Euclidean region)

$$\left( \delta^{\mu\nu} - F(p^2) \frac{p^\mu p^\nu}{p^2} \right) \frac{1}{p^2}, \quad F \leq 1 \quad (21)$$

where we have imposed a condition on  $F$  which guarantees that the Euclidean field has a positive metric, but we have ignored the condition that the Lorentz-region field should have positive metric. This propagator has an inverse of the form

$$p^2 \delta^{\mu\nu} + G(p^2) p^\mu p^\nu \quad \text{with} \quad F = \frac{G}{1+G} \quad (22)$$

A limiting case of this is the Landau gauge for which  $G = \infty$ .

This, together with the condition  $F \leq 1$  give the inequalities

$$-1 \leq G \leq \infty.$$

For cases of physical interest it is sufficient to consider a parameter family of covariant gauges. This can be done by noting that the massive spin - 1 propagator  $\left( \delta^{\mu\nu} + \frac{(\alpha-1)p^\mu p^\nu}{p^2 + \alpha m^2} \right) \frac{1}{p^2 + m^2}$  given in equation (29), section II. 4 of previous chapter, has  $m \rightarrow 0$  limit. This limit becomes one parameter family of photon propagator

$$S^{\mu\nu}(p) = \left[ \delta^{\mu\nu} + \frac{(\alpha-1) p^\mu p^\nu}{p^2} \right] \frac{1}{p^2} \quad (23)$$

This is just a special case of (21).  $S^{\mu\nu}(p)$  is positive-definite if  $\alpha > 0$ , so we can define a Euclidean vector potential in one-parameter family of covariant gauges characterized by  $\alpha$ . The Euclidean one particle Hilbert space  $\mathcal{K}$  is defined as the completion of the inner product space  $[\mathcal{S}(\mathbb{R}^4)]^4$ , with the inner product given by

$$\langle f, g \rangle_{\mathcal{K}} = \sum_{\mu, \nu=1}^4 \overline{f^\mu(p)} \left( \frac{\delta^{\mu\nu} + (\alpha-1) \frac{p^\mu p^\nu}{p^2}}{p^2} \right) g^\nu(p) d^4 p \quad (24)$$

and  $\|f\|_{\mathcal{K}} < \infty$  ( $\alpha > 0$ ).

Then the Euclidean vector potential  $A^\mu$  is the generalized random vector field over  $\mathcal{K}_r$  with mean zero and covariance  $E[A^\mu(f)A^\nu(g)] = \langle f, g \rangle_{\mathcal{K}}$ . Because  $S^{\mu\nu}(p)$  has a local inverse

$$(S^{\mu\nu}(p))^{-1} = p^2 \delta^{\mu\nu} + \left(\frac{1}{\alpha} - 1\right) p^\mu p^\nu \quad (25)$$

therefore the Euclidean electromagnetic potential  $A^\mu$  over  $\mathcal{K}_r$  is Markovian for all covariant gauges characterized by  $\alpha > 0$ .

We shall introduce the following definition:

Definition

The covariant gauges characterized by  $\alpha > 0$  are called Markov gauges of the Euclidean electromagnetic potential over  $\mathcal{K}_r$ .

Some examples of Markov gauges are  $\alpha = 1$ , the Lorentz (or Fermi) gauge; and  $\alpha = 3$ , the Fried-Yennie gauge. For

the limiting case  $\alpha = 0$  which corresponds to Landau gauge, the Markov property is not obvious because now  $S^{\mu\nu}$  is singular so its inverse does not exist on the space of four-vector test functions. Furthermore  $S^{\mu\nu}$  is only positive-semidefinite to obtain a Hilbert space we need to take the quotient space  $\mathcal{K}_r / \text{Kernel } \|\cdot\|$ .

### Theorem 1

The Euclidean electromagnetic potential over  $\mathcal{K}_r$  with covariant gauges  $\alpha > 0$  is Markovian.

### Proof

For  $\alpha > 0$ , equation (25) implies that the Euclidean propagator has a local inverse, so Nelson's proof applies.

For  $\alpha = 0$  (i.e. Landau gauge), we shall restrict to test functions satisfying  $\sum_{\mu} \partial_{\mu} f^{\mu}(x) = 0$  so that the inverse of  $S^{\mu\nu}$  exists in this subspace, and Nelson's argument can be used.

Let  $\mathcal{O} \subset \mathbb{R}^4$  be an open set and let  $\mathcal{K}_{\mathcal{O}}$  be the set of distribution vector fields  $f^{\mu}$  such that  $\sum_{\mu} \partial_{\mu} f^{\mu}(x) = 0$  and  $\int \hat{f}^{\mu}(p) p^{-2} \hat{f}_{\mu}(p) d^4 p < \infty$ . Let  $\Sigma_{\mathcal{O}}$  be the Borel  $\sigma$ -ring generated by the Gaussian field  $A^{\mu}$  over  $\mathcal{K}_{\mathcal{O}}$  and  $\Sigma_{\mathcal{O}^c}$  the  $\sigma$ -ring generated by  $A(f)$ ,  $\text{supp } f \subset \mathcal{O}$ ,  $f \in \mathcal{K}_{\mathcal{O}}$ . Let  $f \in \Sigma_{\mathcal{O}^c}$  and let  $E_{\mathcal{O}^c}$  be the projection onto  $\mathcal{K}_{\mathcal{O}^c}(\mathcal{O}^c)$ , the subset of  $\mathcal{K}_{\mathcal{O}^c}$  of distributions with support in  $\mathcal{O}^c$ , the complement of  $\mathcal{O}$ . Then if  $h \in \mathcal{D}(\mathcal{O}^c)$ , where  $\mathcal{O}^c$  is the interior of  $\mathcal{O}^c$ , we have

$$\langle e_{\mathcal{O}^c} f, h \rangle_{L^2} = \langle (e_{\mathcal{O}^c} \hat{f})^{\mu} \frac{1}{p^2} p^2 (\delta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2}) \hat{h}^{\nu} \rangle_{L^2}$$

since  $p_{\mu} (e_{\mathcal{O}^c} \hat{f})^{\mu} = 0$ ,  $\quad = \langle (e_{\mathcal{O}^c} \hat{f})^{\mu} \frac{1}{p^2} (p^2 \delta_{\mu\nu} - p_{\mu} p_{\nu}) \hat{h}^{\nu} \rangle_{L^2}$

$$\begin{aligned}
\text{Now } (p^2 \delta_{\mu\nu} - p_\mu p_\nu) h^\nu &\in K_{\text{or}}(\hat{0})_{\text{so}} \\
&= \langle e_0, \hat{f}^\mu, (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \hat{h}^\nu \rangle_K \\
&= \langle \hat{f}^\mu, (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \hat{h}^\nu \rangle_K \\
&= \langle \hat{f}^\mu, \frac{(p^2 \delta_{\mu\nu} - p_\mu p_\nu) \hat{h}^\nu}{p^2} \rangle_{L^2} \\
&= \langle \hat{f}, \hat{h} \rangle_{L^2} = 0.
\end{aligned}$$

since  $p_\mu \hat{f}^\mu = 0$

Hence  $\text{Supp } e_0, f \subset \partial U$  and the field in Landau gauge is Markovian.

Q.E.D.

What we have just proved is the Markov property for a Euclidean electromagnetic potential satisfying the Lorentz condition. We note that only in Landau gauge does the electromagnetic potential satisfy Lorentz condition as an operator identity. Thus we obtain the same theory as in (Gr 1) where test functions are subjected to  $\sum_\mu \partial_\mu f^\mu = 0$  conditions and the covariance is  $\delta^{\mu\nu}/p^2$ . However we remark that in no gauge do we get a Wightman theory. This follows from our discussion in section III. 1. For example  $\alpha = 0$  leads to a non-local theory and  $\alpha = 1$  leads to an indefinite metric. It is interesting to see that the Euclidean electromagnetic potential in covariant gauges does not satisfy the reflection property, just like the case of Euclidean vector meson in covariant gauges. Therefore we have another example of Euclidean Markov field which does not give rise to Wightman theory in the Minkowski region. This clearly indicates that the Markov property alone is not enough to ensure that

a Euclidean field will lead to a Wightman field, reflection property is also needed to be satisfied.

We might ask for a property which, while more general than the reflection property (so as to include theories like electrodynamics and gravitation), yet still excludes very nonlocal theories like one with propagator of the form  $(-\Delta + m^2)^{-n}$ ,  $n > 1$ . Such a property is known as the classical Markov property which can be formulated as follows. Let  $\mathcal{K}$  be the completion of  $\mathcal{F}(\mathbb{R}^4)$  in the norm defined by the covariance function of the random field in question. For any open set  $\mathcal{O} \subset \mathbb{R}^4$ , let  $\mathcal{K}(\mathcal{O})$  be the (closed) subspace generated by  $\{f \in \mathcal{K}, \text{supp } f \subset \mathcal{O}\}$  and let  $\Sigma_{\mathcal{O}}$  be the  $\sigma$ -algebra generated by the  $\{\Phi(f) \mid f \in \mathcal{K}(\mathcal{O})\}$ . Let  $\mathcal{K}_0(\mathcal{O})$  be the subspace of  $\mathcal{K}(\mathcal{O})$ , consisting of measures, and let  $\Sigma_{0,\mathcal{O}}$  denote the Borel  $\sigma$ -ring generated by  $\{\Phi(f) \mid f \in \mathcal{K}_0(\mathcal{O})\}$ . For any subset  $\mathcal{U} \subset \mathbb{R}^4$ , denote by  $\Sigma_{0,\mathcal{U}}$  the intersection  $\bigcap \{\Sigma_{0,\mathcal{O}} \mid \mathcal{O} \supset \mathcal{U}\}$ ,  $\mathcal{O}$  open. Then we say that a field  $\bar{\Phi}$  satisfies the classical Markov property if, for every function  $F: \mathcal{Q} \rightarrow \mathbb{R}$  which is measurable, and every open set  $\mathcal{O} \subset \mathbb{R}^4$ ,

$$E[F \mid \Sigma_{\mathcal{O}'}] = E[F \mid \Sigma_{\partial\mathcal{O}}]$$

holds, where  $\mathcal{O}'$  is the complement of  $\mathcal{O}$  in  $\mathbb{R}^4$ , and  $\partial\mathcal{O}$  is the boundary of  $\mathcal{O}$ .

The random fields defined by the Euclidean electromagnetic potential in various Markov gauges are such that  $\Sigma_{0,\partial\mathcal{O}}$  coincides with



$$\Sigma_{\partial\mathcal{V}} = \bigcap_{\mathcal{O}_i} \{ \Sigma_{\mathcal{O}_i} \mid \mathcal{O}_i \subset \partial\mathcal{V} \}$$

Hence the classical and Nelson's definitions of Markovicity coincide.

These fields therefore also satisfy the classical Markov property.

### Remarks

- (i) In the Minkowski-region fields, except for the case  $\alpha = 1$  (which corresponds to Lorentz or Fermi gauge), one needs to introduce dipole ghost field in Gupta-Bleuler formalism of quantum electrodynamics. Supplementary condition is required to remove such ghost states.
- (ii) Different values of  $\alpha$  are related by the  $g$ -number gauge transformation:

$$A_{\alpha}^{\mu} \longrightarrow A_{\alpha'}^{\mu} = A_{\alpha}^{\mu} + \beta \partial^{\mu} B, \quad \beta \text{ real.}$$

where  $\langle B B \rangle = \frac{1}{p^4}$

and  $\langle A_{\alpha}^{\mu} A_{\alpha}^{\nu} \rangle = \left[ \delta^{\mu\nu} + (\alpha - 1) \frac{p^{\mu} p^{\nu}}{p^2} \right] \frac{1}{p^2}$

such that  $\langle A_{\alpha}^{\mu} A_{\alpha'}^{\nu} \rangle = \left[ \delta^{\mu\nu} + (\alpha + \beta^2) \frac{p^{\mu} p^{\nu}}{p^2} \right] \frac{1}{p^2}$

This gauge transformation changes a given value of gauge parameter  $\alpha$  into  $\alpha + \beta^2$ . Therefore by choosing the value of  $\beta$  appropriately, we can connect any two covariant gauges by this  $g$ -number gauge transformation.

- (iii) The propagator in Landau gauge is a singular matrix

$$G_{\nu}^{\mu} = \left( g_{\nu}^{\mu} - \frac{p^{\mu} p_{\nu}}{p^2} \right) \frac{1}{p^2} \quad . \quad \text{This}$$

means that there is no direct relation between the quantization procedure in this gauge and the operator form

of the classical field equation  $\Lambda^\mu_\nu(\partial) A^\nu = 0$ ,  
 as given by  $G^\mu_\lambda(p) \Lambda^\lambda_\nu(p) = -p^2 g^\mu_\nu$ . However,  
 one hopes to find such a relation in the spin - 1  
 subspace. If we replace the Minkowski metric  $g^\mu_\nu$  by  
 the metric tensor in the spin - 1 subspace

$$\begin{aligned} d^\mu_\nu &= g^\mu_\nu - \frac{p^\mu p_\nu}{p^2} & \text{we get } G^\mu_\lambda(p) \Lambda^\lambda_\nu(p) \\ &= -d^\mu_\nu p^2 & \text{with } \Lambda^\lambda_\nu(p) = p^2 g^\lambda_\nu - p^\lambda p_\nu \end{aligned}$$

- (iv) The gauge ambiguity of the propagator can be used to simplify certain calculations in quantum electrodynamics. For example, Landau gauge is particularly useful in the consideration of ultraviolet divergences; on the other hand, Fried and Yennie gauge is used in the study of infrared divergences. These two gauges are just special cases of our general covariant gauge.

### III. 3. Free Euclidean Electromagnetic Field

We have seen in the beginning of this chapter that it is impossible to formulate a manifestly covariant and ~~for~~ local theory of quantized electromagnetic potential without using Gupta-Bleuler formalism. However there is a completely consistent covariant formulation of the free quantum electrodynamics in terms of the electromagnetic field intensity  $F^{\mu\nu}$  only, without introducing the vector potential at all. In this formulation the indefinite metric does not arise and there is no unphysical state. This is a gauge-covariant formulation which does not suffer from the ambiguities connected with the gauge problem.

The anti-symmetric electromagnetic field strength tensor

satisfies following equations:

$$\square F^{\mu\nu}(x) = 0 \quad (26)$$

and 
$$\partial_\mu F^{\mu\nu}(x) = 0 \quad (27)$$

We obtain a Wightman theory with the following two-point Wightman function:

$$\begin{aligned} W^{\mu\nu,\rho\sigma}(x-y) &= \langle F^{\mu\nu}(x) F^{\rho\sigma}(y) \rangle \\ &= i (g^{\nu\sigma} \partial^\mu \partial^\rho - g^{\nu\rho} \partial^\mu \partial^\sigma + g^{\mu\rho} \partial^\nu \partial^\sigma - g^{\mu\sigma} \partial^\nu \partial^\rho) D_+(x-y) \end{aligned} \quad (28)$$

where  $\square D_+(x) = 0$ . The relativistic one particle Hilbert space  $\mathcal{M}$  can then be defined as the completion of the test function space  $[\mathcal{S}(\mathbb{R}^3)]_A^{16}$  with respect to the inner product

$$\langle f, g \rangle_{\mathcal{M}} = \sum_{\substack{\mu, \nu=1 \\ \rho, \sigma=1}}^3 \langle f^{\mu\nu}, W^{\mu\nu,\rho\sigma} g^{\rho\sigma} \rangle_{L^2(\mathbb{R}^3)} \quad (29)$$

with  $f^{\mu\nu} = -f^{\nu\mu}$  and the norm  $\|f\|_{\mathcal{M}}^2 = 4 \|\operatorname{div} f\|_{-\frac{1}{2}}^2 < \infty$ , where  $\operatorname{div} f = \sum_{\mu=1}^3 \partial_\mu f^{\mu\nu}(x)$  and  $\|f\|_{-\frac{1}{2}}^2 = \langle \hat{f}, \frac{1}{p} \hat{f} \rangle_{L^2}$ .

If  $U(a, \Lambda)$  is the irreducible unitary representation of inhomogeneous Lorentz group, then

$$U(a, \Lambda) F^{\mu\nu}(x) U(a, \Lambda)^{-1} = (\Lambda^{-1})^\mu_\rho (\Lambda^{\nu-1})^\sigma F^{\rho\sigma}(\Lambda x + a) \quad (30)$$

For smeared field  $F^{\mu\nu}(f) = \int F^{\mu\nu}(x) f_{\mu\nu}(x) d^3x$  we have

$$U(a, \Lambda) F^{\mu\nu}(f) U(a, \Lambda)^{-1} = F^{\mu\nu}(f_{\{a, \Lambda\}}) \quad (31)$$

where  $f_{\{a, \Lambda\}}(x) = f(\Lambda^{-1}(x-a))$

Now we define Minkowski tensor field as  $F^{4j} = i F^{0j}$

Analytic continuation to Schwinger points of the two-point Wightman function of this Minkowski tensor field gives two-point Schwinger function in momentum space as

$$S^{\mu\nu, \rho\sigma}(p) = [P^\mu P^\rho \delta^{\nu\sigma} - P^\mu P^\sigma \delta^{\nu\rho} + P^\nu P^\sigma \delta^{\mu\rho} - P^\nu P^\rho \delta^{\mu\sigma}] \frac{1}{p^2} \quad (32)$$

This is positive-definite in the subspace of anti-symmetric tensor-valued test function space  $[\mathcal{S}(\mathbb{R}^4)]_A^{16}$ . The Euclidean one particle Hilbert space is the completion of this space with respect to the inner product

$$\langle f, g \rangle_{\mathcal{K}} = \sum_{\substack{\mu, \nu=1 \\ \rho, \sigma=1}}^4 \langle f^{\mu\nu}, S^{\mu\nu, \rho\sigma} g^{\rho\sigma} \rangle_{L^2(\mathbb{R}^4)} \quad (33)$$

with  $\|f\|_{\mathcal{K}}^2 = 4 \|\operatorname{div} f\|_{-1}^2 < \infty$

$$\|f\|_{-1}^2 = \langle \hat{f}, \frac{1}{p^2} \hat{f} \rangle_{L^2(\mathbb{R}^4)} \quad \text{and} \quad \operatorname{div} f = \sum_{\mu=1}^4 \partial_\mu f_{\mu\nu} \quad \text{where}$$

The Euclidean electromagnetic field  $\Theta$  is defined as the generalized tensor-valued Gaussian random field over  $\mathcal{K}_r$  with mean zero and covariance given by the inner product  $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ .

In the probability space,  $(\Omega, \Sigma, \mu)$  on which the fields  $\Theta^{\mu\nu}(f)$  are random variables is furnished with a realization of the Euclidean group  $\mathrm{IO}(4)$ , by measure-preserving transformations

$T_\beta : \Omega \rightarrow \Omega$ ,  $\beta \in \mathrm{IO}(4)$ . This induces a unitary

representation  $U$  of  $IO(4)$  on  $L^2(Q, d\mu)$  by

$$(U(\beta)u)(\omega) = u(T_\beta^{-1}\omega), \quad u \in L^2(Q, d\mu).$$

The tensor field  $\theta^{\mu\nu}$  is covariant under  $IO(4)$  such that for  $(a, R) \in IO(4)$ ,

$$U(a, R) \theta^{\mu\nu}(f) U(a, R)^{-1} = \sum_{\mu', \nu'=1}^4 R^{\mu\mu'} R^{\nu\nu'} \theta^{\mu'\nu'}(f_{(a,R)}) \quad (34)$$

where  $f_{(a,R)}(x) = f(R^{-1}(x-a))$ .

### Theorem 2

- (i) The time zero Euclidean one particle space is naturally identical to the relativistic one particle space.
- (ii) The Euclidean electromagnetic field satisfies the reflection property.

### Proof

The proof of this theorem is similar to the corresponding theorem for the anti-symmetric tensor field of massive spin - 1 mesons (Theorem 6 of chapter two). The main point to be observed is that the two-point functions in both cases have similar kind of singularities.

### Theorem 3

The Euclidean electromagnetic field over  $\mathcal{K}_r$  is Markovian.

### Proof

Since the two-point Schwinger function is a singular matrix, Markov property is not so obvious as we cannot apply Nelson's proof directly. However for the case of two dimensional space-time there exists a very simple proof. In this case, there are only one independent components for  $f$  namely  $f^{12} = -f^{21}$  with  $f^{11} = f^{22} = 0$ . The inner product in  $\mathcal{K}_r$  now becomes

$$\begin{aligned}
\langle f, g \rangle_K &= 4 \sum_{\lambda, \mu, \nu=1}^2 \langle f_{\mu\lambda}, \frac{P_\mu P_\lambda}{P^2} g_{\mu\lambda} \rangle_{L^2(\mathbb{R}^2)} \\
&= 4 \int (P_1 \hat{f}_{12})(P_1 \hat{g}_{12}) + (P_2 \hat{f}_{21})(P_2 \hat{g}_{21}) \frac{d^2 p}{P^2} \\
&= 4 \int \frac{P_1^2 + P_2^2}{P^2} \hat{f}_{12}(p) \hat{g}_{12}(p) d^2 p \\
&= 4 \int \hat{f}_{12}(p) \hat{g}_{12}(p) d^2 p
\end{aligned}$$

Markovicity then follows from Nelson's argument.

As for 4-dimensional space-time, the proof is a bit more involved. We shall follow the proof due to Yao (Ya 2). First we note that the 6 independent components of  $f \in K_V$  can be indexed by  $I = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$  so that the matrix  $\hat{S}_{\mu\nu\rho\sigma}(p)$  is reduced to

$$\begin{array}{l}
\begin{array}{l}
(\mu, \nu) \rightarrow \\
(p, \sigma) \\
\downarrow \\
(1,2) \\
(1,3) \\
(1,4) \\
(2,3) \\
(2,4) \\
(3,4)
\end{array}
\begin{array}{cccccc}
(1,2) & (1,3) & (1,4) & (2,3) & (2,4) & (3,4) \\
\left[ \tilde{Q}_{\mu\nu\rho\sigma}(p) \right] = & \left( \begin{array}{cccccc}
P_1^2 + P_2^2 & P_2 P_3 & P_2 P_4 & -P_1 P_3 & -P_1 P_4 & 0 \\
P_2 P_3 & P_1^2 + P_3^2 & P_3 P_4 & P_1 P_2 & 0 & -P_1 P_4 \\
P_2 P_4 & P_3 P_4 & P_1^2 + P_4^2 & 0 & P_1 P_2 & P_1 P_3 \\
-P_1 P_3 & P_1 P_2 & 0 & P_2^2 + P_3^2 & P_3 P_4 & -P_2 P_4 \\
-P_1 P_4 & 0 & P_1 P_2 & P_3 P_4 & P_2^2 + P_4^2 & P_2 P_3 \\
0 & -P_1 P_4 & P_1 P_3 & -P_2 P_3 & P_2 P_3 & P_3^2 + P_4^2
\end{array} \right)
\end{array}
\end{array}$$

With this reduced matrix we have

$$\langle f, g \rangle_{\mathcal{K}} = \sum_{\substack{(\mu, \nu) \in I \\ (p, \sigma) \in I}} \int d^4 p \hat{f}_{\mu\nu}(p) \hat{Q}_{\mu\nu, \rho\sigma}(p) \hat{g}_{\rho\sigma}(p)$$

Now let  $\mathcal{O}$  be an open set in  $\mathbb{R}^4$  and  $\mathcal{O}'$  be its complement and  $\partial\mathcal{O}$  its boundary. Define

$$\mathcal{K}_{\mathcal{O}} = \{ f \in \mathcal{K}_r \mid \text{supp } f \subset \mathcal{O} \}$$

$$\mathcal{K}_{\mathcal{O}'} = \{ f \in \mathcal{K}_r \mid \text{supp } f \subset \mathcal{O}' \}$$

$$\mathcal{K}_{\partial\mathcal{O}} = \{ f \in \mathcal{K}_r \mid \text{supp } f \subset \partial\mathcal{O} \}$$

Let  $e_{\mathcal{O}'}$  and  $e_{\partial\mathcal{O}}$  be the projection operator onto  $\mathcal{K}_{\mathcal{O}'}$  and  $\mathcal{K}_{\partial\mathcal{O}}$  respectively. If  $f \in \mathcal{K}_{\mathcal{O}}$  is such that  $f_{\mu\nu} = 0$  except for  $(\mu, \nu) = (1, 2)$  and  $(2, 1)$ , then  $e_{\mathcal{O}'} f = h$  has only non-vanishing components  $h_{12}$  and  $h_{21}$  with  $h_{12} = -h_{21}$ . Similarly let  $g \in \mathcal{K}_r$  with non-vanishing components  $g_{12}$  and  $g_{21}$  belonging to  $\mathcal{D}_r(\mathcal{O}')$ , the space of infinitely differentiable functions with compact support contained in the interior of  $\mathcal{O}'$ , such that all  $g_{\mu\nu} = 0$  except for components  $g_{12}$  and  $g_{21}$ . Then we have

$$\langle g, h \rangle_{\mathcal{K}_r} = \langle g, f \rangle_{\mathcal{K}_r}$$

Written in full,

$$\begin{aligned} & \int dx g_{12}(x) \frac{(\partial_1^2 + \partial_2^2)}{-\Delta} h_{12}(x) \\ &= \int dx g_{12}(x) \frac{(\partial_1^2 + \partial_2^2)}{-\Delta} f_{12}(x) \end{aligned}$$

Since  $\partial_3^2 + \partial_4^2$  is a local differential operator the following step is permissible

$$\begin{aligned} & \int dx g(x) (\partial_1^2 + \partial_2^2) (\partial_3^2 + \partial_4^2) h_{12}(x) \\ &= \int dx g(x) (\partial_1^2 + \partial_2^2) (\partial_3^2 + \partial_4^2) f_{12}(x) \end{aligned}$$

for any  $g \in \mathcal{D}_r(\mathcal{O}')$ .

In a similar way we can have  $f' \in \mathcal{K}_0$  such that  $f'_{\mu\nu} = 0$  except for  $f'_{34} = -f'_{43}$ . Then  $e_{\mathcal{O}}, f' = h'$  has only non-vanishing components  $h'_{34} = -h'_{43}$ . Again let  $g' \in \mathcal{K}_r$  with the only non-vanishing components  $g'_{34}$  and  $g'_{43}$  belonging to  $\mathcal{D}_r(\mathcal{O}')$ . Using a similar reasoning as given above, we get for any  $g \in \mathcal{D}_r(\mathcal{O}')$ ,

$$\begin{aligned} & \int dx g(x) (\partial_1^2 + \partial_2^2) (\partial_3^2 + \partial_4^2) h'_{34} \\ &= \int dx g(x) (\partial_1^2 + \partial_2^2) (\partial_3^2 + \partial_4^2) f'_{34}(x) \end{aligned}$$

If we choose  $f'_{34} = f_{12}$  we obtain

$$\begin{aligned} & \int dx g(x) (\partial_1^2 + \partial_2^2) (\partial_3^2 + \partial_4^2) h_{12}(x) \\ &= \int dx g(x) (\partial_1^2 + \partial_2^2) (\partial_3^2 + \partial_4^2) h'_{34}(x) \end{aligned}$$



∴ We get  $(\partial_1^2 + \partial_2^2)(\partial_3^2 + \partial_4^2) h_{12}(x)$   
 $= (\partial_1^2 + \partial_2^2)(\partial_3^2 + \partial_4^2) h'_{34}(x)$

as distributions on  $\mathcal{O}'$ . This implies

$$h_{12} = h'_{34} + \gamma$$

where  $\gamma$  is a distributional solution to

$$(\partial_1^2 + \partial_2^2)(\partial_3^2 + \partial_4^2)\gamma(x) = 0 \quad \text{in } \mathcal{O}'.$$

If we let  $\gamma^{(1,2)}(x_1, x_2)$  and  $\gamma^{(3,4)}(x_3, x_4)$  be the distributional solutions to the Laplace equations

$$(\partial_1^2 + \partial_2^2)\gamma^{(1,2)}(x_1, x_2) = 0$$

$$(\partial_3^2 + \partial_4^2)\gamma^{(3,4)}(x_3, x_4) = 0$$

respectively, then  $\gamma(x)$  is given by

$$\gamma(x) = \sum_{j=1}^{\infty} \gamma^{(1,2)}(x_1, x_2) \gamma_j^{(3,4)}(x_3, x_4)$$

in some sufficiently small neighbourhood of any point in  $\mathcal{O}'$ .

If we choose  $g_{12} = g'_{34} = g$  for some  $g \in \mathcal{D}_r(\mathcal{O}')$ ,

then

$$\int dx g(x) \frac{(\partial_1^2 + \partial_2^2)}{-\Delta} h_{12}(x) = \int dx g(x) \frac{(\partial_1^2 + \partial_2^2)}{-\Delta} f_{12}(x)$$

and

$$\begin{aligned} \int dx g(x) \frac{(\partial_3^2 + \partial_4^2)}{-\Delta} h_{12}(x) &= \int dx g(x) \frac{(\partial_3^2 + \partial_4^2)}{-\Delta} (h'_{34} + \gamma(x)) \\ &= \int dx g(x) \frac{(\partial_3^2 + \partial_4^2)}{-\Delta} h'_{34}(x) \\ &= \int dx g(x) \frac{(\partial_3^2 + \partial_4^2)}{-\Delta} f'_{12}(x) \end{aligned}$$

Hence  $\int dx g(x) h_{12}(x) = \int dx g(x) f_{12}(x) = 0$

For any  $g \in \mathcal{D}_r(\mathcal{O}')$  which implies  $h \in \mathcal{K}_{20}$ .

Using the similar arguments we can show that for any

$f \in \mathcal{K}_0$ ,  $e_0 f \in \mathcal{K}_{20}$ . Therefore we can conclude,

by Nelson's proof, that the Euclidean electromagnetic field over  $\mathcal{K}_r$  is indeed Markovian.

Q.E.D.

Before we conclude this section, we would like to point out some short comings in the above formulation. The main limitation is its restriction to the interaction-free case which is, in many respect a rather trivial one. One does not know how to formulate a local interacting theory since we are not able to write down a local-interaction Lagrangian in terms of  $F_{\mu\nu}$  only. Furthermore the proof of T C P theorem does not hold as it is based on local Lagrangian field theory. In addition, the assumption of local commutativity may not hold for non-local Lagrangian field theory. Therefore the above for formulation with positive metric quantization of electromagnetic field is ~~only~~ for academic interest only.

## CHAPTER FOUR

## EUCLIDEAN SPIN TWO MASSIVE AND MASSLESS FIELDS

In this chapter we shall attempt to construct Euclidean spin - 2 massive and massless tensor fields using the method similar to that used in the previous chapters. The fact that spin - 2 fields differ considerably from spin - 1 and spin - 0 fields suggests that one should not expect this method to work smoothly for the spin - 2 case. This is indeed the case, for there exist serious difficulties in the formulation of spin - 2 Euclidean tensor field. The main difficulty is that the Schwinger functions obtained from analytic continuation to Schwinger points of the Wightman functions are not positive-semidefinite. In an attempt to overcome this difficulty, supplementary conditions have been imposed on the test functions. However new problem arises since now the two-point Schwinger function does not have a local inverse, which therefore prevents us from getting a Markov tensor field via the usual method. For the massive spin - 2 field, our results strongly indicate that the Euclidean field is non-Markovian. The massless spin - 2 tensor field does give rise to a Euclidean Markov tensor field in certain covariant gauges, although it does not lead to a Wightman theory. This, like the case of electromagnetic potential with covariant gauges, may be explained by the fact that the reflection property is violated.

IV. 1. Relativistic Massive Spin Two Tensor Field

A real massive spin - 2 free field can be described by a rank - 2

tensor  $\phi^{\mu\nu}$  which satisfies the following equations:

$$(\square + m^2) \phi^{\mu\nu}(x) = 0 \quad (1)$$

$$\phi^{\mu\nu}(x) = \phi^{\nu\mu}(x) \quad (2)$$

$$\phi_{\mu}^{\mu}(x) = 0 \quad (3)$$

$$\partial_{\mu} \phi^{\mu\nu}(x) = 0 \quad (4)$$

The subsidiary conditions (2), (3) and (4) impose six, one and four conditions on  $\phi^{\mu\nu}(x)$  respectively, so that only five out of the sixteen components of  $\phi^{\mu\nu}(x)$  are independent (i.e. exactly  $(2S+1)$  components correspond to  $S=2$ ). In terms of group-theoretic language the field  $\phi^{\mu\nu}(x)$  transforms according to the representation  $\mathbb{D} = \mathbb{D} \otimes \mathbb{D}$  where  $\mathbb{D} = \mathbb{D}(1) \oplus \mathbb{D}(0)$  has the transformation properties of a vector,  $\mathbb{D}(S)$  being the irreducible representation corresponding to spin value  $S$ . Since each tensor index carries spin - 1 and spin - 0, we get

$$\begin{array}{l} \mathbb{D}(0) \otimes \mathbb{D}(0) = \mathbb{D}(0) \quad : \text{ scalar} \\ \mathbb{D}(0) \otimes \mathbb{D}(1) = \mathbb{D}(1) \quad : \text{ vector} \\ \mathbb{D}(1) \otimes \mathbb{D}(0) = \mathbb{D}(1) \quad : \text{ vector} \\ \mathbb{D}(1) \otimes \mathbb{D}(1) = \mathbb{D}(0) \oplus \mathbb{D}(1) : \text{ scalar \& vector} \end{array}$$

These are eliminated  
by the subsidiary  
conditions (2), (3)  
and (4)

$\oplus \mathbb{D}(2) : \text{ tensor } \rangle$  This is what remained

Therefore

Now we shall like to see that equations (1), (2), (3) and (4) can be combined into one compact wave equation as given by Umezawa-Takahashi formulation (M - T 1) . By a suitable choice of Lagrangian, we are able to derive a single matrix equation from which the Klein-Gordon equation and all the subsidiary conditions can be derived. This free Lagrangian can be expressed in the following form:

$$\mathcal{L} = \phi_{\mu\nu}^*(x) \eta^{\mu\nu} \Lambda_{\rho\sigma}^{(\partial)} \phi^{\rho\sigma}(x) \quad (5)$$

where

$$\begin{aligned} \Lambda_{\rho\sigma}^{\mu\nu}(\partial) = & \frac{1}{2} (\square + m^2) (g_{\rho}^{\mu} g_{\sigma}^{\nu} + g_{\sigma}^{\mu} g_{\rho}^{\nu}) \\ & - \frac{1}{2} (g_{\rho}^{\mu} \partial^{\nu} \partial_{\sigma} + g_{\sigma}^{\mu} \partial^{\nu} \partial_{\rho} + g_{\rho}^{\nu} \partial^{\mu} \partial_{\sigma} + g_{\sigma}^{\nu} \partial^{\mu} \partial_{\rho}) \\ & - \alpha (g^{\mu\nu} \partial_{\rho} \partial_{\sigma} + \partial_{\rho}^{\mu} \partial_{\sigma}^{\nu} g_{\rho\sigma}) - \beta \square g^{\mu\nu} g_{\rho\sigma} \quad (6) \\ & - \gamma m^2 g^{\mu\nu} g_{\rho\sigma} - \delta m^2 (g_{\rho}^{\mu} g_{\sigma}^{\nu} - g_{\sigma}^{\mu} g_{\rho}^{\nu}) \end{aligned}$$

with  $\beta = \frac{1}{2} (3\alpha^2 + 2\alpha + 1)$ ,  $\gamma = \alpha + 2\beta$  and  $\eta_{\rho\sigma}^{\mu\nu} = g_{\rho}^{\mu} g_{\sigma}^{\nu}$ .

$\mathcal{L}$  is not unique because it contains two real parameters  $\alpha$  and  $\delta (\neq 0)$  . From this Lagrangian we can derive the following field equation:

$$\Lambda_{\rho\sigma}^{\mu\nu}(\partial) \phi^{\rho\sigma}(x) = 0 \quad (7)$$

It can be shown that equations (1), (2), (3) and (4) can be deduced from equation (7) by a finite number of differentiations and algebraic operations. For the sake of simplicity, we shall do this for the

case  $\alpha = -1$  (then  $\beta = 1, \gamma = 1$ ). Equation (7) when written out in full is

$$\begin{aligned} & \frac{1}{2}(\square + m^2)(\phi^{\mu\nu} + \phi^{\nu\mu}) - \frac{1}{2}(\partial^\mu \partial_\rho \phi^{\nu\rho} + \partial^\nu \partial_\sigma \phi^{\mu\sigma} \\ & + \partial^\nu \partial_\rho \phi^{\mu\rho} + \partial^\nu \partial_\sigma \phi^{\mu\sigma}) + (g^{\mu\nu} \partial_\rho \partial_\sigma \phi^{\rho\sigma} + \partial^\mu \partial^\nu \phi) \\ & - g^{\mu\nu}(\square + m^2)\phi + m^2 \delta(\phi^{\mu\nu} - \phi^{\nu\mu}) = 0 \end{aligned} \quad (7')$$

Here we have put  $\phi = \phi^\mu_\mu = \phi^{00} - \sum_{j=1}^3 \phi^{jj}$ . If we interchange the indices  $\mu$  and  $\nu$  in (7') and subtracting the resulted equation from (7') we get

$$m^2 \delta(\phi^{\mu\nu} - \phi^{\nu\mu}) = 0 \quad (8)$$

Since  $m^2 \neq 0$  and  $\delta \neq 0$ , this implies  $\phi^{\mu\nu} = \phi^{\nu\mu}$  which is just equation (2). Using this symmetric property in (7') we obtain

$$\begin{aligned} & (\square + m^2)\phi^{\mu\nu} - (\partial^\mu \partial_\rho \phi^{\rho\nu} + \partial^\nu \partial_\rho \phi^{\rho\mu}) \\ & + g^{\mu\nu} \partial_\rho \partial_\sigma \phi^{\rho\sigma} + \partial^\mu \partial^\nu \phi - g^{\mu\nu}(\square + m^2)\phi = 0 \end{aligned} \quad (9)$$

Multiplying (9) by  $\partial_\mu$  from the left and summing over  $\mu$  given

$$m^2(\partial_\mu \phi^{\mu\nu} - \partial^\nu \phi) = 0 \quad \text{or} \quad \partial_\mu \phi^{\mu\nu} - \partial^\nu \phi = 0 \quad (10)$$

Let  $\mu = \nu$  in (9) and sum over  $\mu$  we get

$$4(\partial_\mu \partial_\rho \phi^{\rho\nu} - \square \phi) + 3m^2 \phi = 0 \quad (11)$$

Multiplying (10) by  $\partial_\nu$  from the left and sum over  $\nu$  gives

$$\partial_\mu \partial_\nu \phi^{\mu\nu} - \square \phi = 0 \quad \text{or} \quad 4(\partial_\mu \partial_\nu \phi^{\mu\nu} - \square \phi) = 0 \quad (12)$$

Subtracting (12) from (11) we obtain  $3 m^2 \phi = 0$ . Since  $m^2 \neq 0$ , this implies  $\phi = \phi^\mu_\mu = 0$ , which is the traceless condition (3). Substituting this condition into (9) gives

$$\partial_\mu \phi^{\mu\nu} = 0 \quad \text{which is just the divergenceless condition}$$

(4). By taking into account these three subsidiary conditions the field equation (7') reduces to the Klein-Gordon equation (1).

We have just verified that we can reduce (7) into Klein-Gordon equation plus all the three subsidiary conditions by a finite number of differentiations and algebraic manipulations. This is equivalent to the existence of a Klein-Gordon divisor  $d(\partial)$  such that

$$\Lambda^{\mu\nu}_{\rho\sigma}(\partial) d^{\rho\sigma}_{\mu\nu}(\partial) = d^{\mu\nu}_{\rho\sigma}(\partial) \Lambda^{\rho\sigma}_{\mu\nu} = (\square + m^2) \delta^\mu_\rho \delta^\nu_\sigma \quad (13)$$

$d(\partial)$  is given in (M - T 1) as

$$\begin{aligned} d^{\mu\nu}_{\rho\sigma}(\partial) = & \left[ \frac{1}{2} (g^\mu_\rho g^\nu_\sigma + g^\mu_\sigma g^\nu_\rho) - \frac{1}{3} g^{\mu\nu} g_{\rho\sigma} - \frac{1}{3m^2} (\partial^\mu \partial^\nu g_{\rho\sigma} + g^{\mu\nu} \partial_\rho \partial_\sigma) \right. \\ & \left. + \frac{1}{2m^2} (g^\mu_\rho \partial^\nu \partial_\sigma + g^\mu_\sigma \partial^\nu \partial_\rho + g^\nu_\rho \partial^\mu \partial_\sigma + g^\nu_\sigma \partial^\mu \partial_\rho) + \frac{2}{3m^2} \partial^\mu \partial^\nu \partial_\rho \partial_\sigma \right] \\ & - \frac{\square + m^2}{m^2} \left[ -\frac{1}{3} \frac{\alpha(\alpha+1)}{(2\alpha+1)^2} g^{\mu\nu} g_{\rho\sigma} - \frac{(\alpha+1)^2}{6(2\alpha+1)^2} \frac{\square}{m^2} g^{\mu\nu} g_{\rho\sigma} \right. \\ & \left. + \frac{1}{3} \frac{(\alpha+1)}{(2\alpha+1)} \frac{\partial^\mu \partial^\nu g_{\rho\sigma} + g^{\mu\nu} \partial_\rho \partial_\sigma}{m^2} + \frac{1}{\delta} (g^\mu_\rho g^\nu_\sigma + g^\mu_\sigma g^\nu_\rho) \right] \end{aligned} \quad (14)$$

We note that the Klein-Gordon divisor can be grouped into two terms : the first term  $d'(\partial)$  is parameter-free and it is just the spin - 2 projection operator restricted to the mass-shell ( $p^2 = m^2$ ) ; the second term  $d''(\partial) (\square + m^2)$  is a contact term which is parameter-dependent and is ultra-local. This contact term can be made to vanish by a specific choice of parameters  $\alpha = -1$  and  $\delta = 0$  . The free propagator is given by  $d^{\mu\nu\rho\sigma}(\partial) (\square + m^2)^{-1}$  . Since  $(\square + m^2) \Delta_+(x-y) = 0$  , only  $d'(\partial)$  term contributes to the commutation relation. From now on we shall simplify our discussion by taking  $\alpha = -1$  . This choice of parameter removes some of the ultra-local terms which do not affect the Wightman theory. Moreover we can drop the anti-symmetric terms by putting  $\delta = 0$  , this requires us to assume a priori that the tensor field  $\phi^{\mu\nu}(x)$  is symmetric. The two-point Wightman function is independent of the parameters, since only  $d'(\partial)$  contributes. We have

$$W^{\mu\nu\rho\sigma}(x-y) = d^{\mu\nu\rho\sigma}(\partial) W(x-y) \quad (15)$$

where  $W(x-y)$  is the two-point Wightman function for the massive scalar bosons.  $W^{\mu\nu\rho\sigma}$  is positive-semidefinite, so we can construct a relativistic one particle Hilbert space in the usual manner.

#### IV. 2. Euclidean Massive Spin Two Tensor Field

We shall first of all introduce the Minkowski tensor field defined by  $\phi^{4j} = i \phi^{0j}$  and  $\phi^{44} = (i)^2 \phi^{00} = -\phi^{00}$ . The Schwinger two-point is then the analytic continuation of the



Minkowski tensor field to Schwinger points and is given by

$$\begin{aligned}
 S^{\mu\nu,\rho\sigma}(x-y) &= \left[ \frac{1}{2} (\delta^{\mu\sigma}\delta^{\nu\rho} + \delta^{\mu\rho}\delta^{\nu\sigma}) - \frac{1}{3} \delta^{\mu\nu}\delta^{\rho\sigma} \right] \\
 &\quad - \frac{1}{2m^2} (\delta^{\mu\rho}\partial^\nu\partial^\sigma + \delta^{\mu\sigma}\partial^\nu\partial^\rho + \delta^{\nu\rho}\partial^\mu\partial^\sigma + \delta^{\nu\sigma}\partial^\mu\partial^\rho) \\
 &\quad + \frac{1}{3m^2} (\delta^{\rho\sigma}\partial^\mu\partial^\nu + \delta^{\mu\nu}\partial^\rho\partial^\sigma) + \frac{2}{3m^4}\partial^\mu\partial^\nu\partial^\rho\partial^\sigma \Big] \frac{1}{-\Delta+m^2} \\
 (\text{Let } d^{\mu\nu} &= \delta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{m^2}) \\
 &= \frac{1}{2} (d^{\mu\rho}d^{\nu\sigma} + d^{\mu\sigma}d^{\nu\rho} - \frac{2}{3}d^{\mu\nu}d^{\rho\sigma}) \frac{1}{-\Delta+m^2}
 \end{aligned} \tag{16}$$

If we write the first line of the l.h.s. of equation (16) in the form  $\frac{1}{2} (\delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho} - \frac{1}{2}\delta^{\mu\nu}\delta^{\rho\sigma}) - \frac{1}{12}\delta^{\mu\nu}\delta^{\rho\sigma}$ , then we see that the term  $-\frac{1}{12}\delta^{\mu\nu}\delta^{\rho\sigma}$  contributes a term with negative norm associated with the trace. This term is not ultralocal, and thus cannot be cancelled by a term in equation (14) by another choice of real  $\alpha$ . Hence we cannot use this  $S^{\mu\nu,\rho\sigma}(x-y)$  to define a Euclidean random Gaussian field.

Because of equation (13) we know that  $S^{\mu\nu,\rho\sigma}$  has a local matrix inverse. We might get a Markov field if we can make  $S^{\mu\nu,\rho\sigma}$  positive-definite by imposing some supplementary conditions on the test functions. The simplest choice is to restrict the space of test functions to the subspace of traceless tensors, but this is insufficient since  $S^{\mu\nu,\rho\sigma}$  does not map this subspace to itself. Instead, we consider a slightly different operator given by

$$\hat{d} \otimes \hat{d} = \left( \delta^{\mu\mu'} + \frac{p^\mu p^{\mu'}}{m^2} \right) \left( \delta^{\nu\nu'} + \frac{p^\nu p^{\nu'}}{m^2} \right) \tag{17}$$

This operator is the Klein-Gordon divisor of

$$\frac{\hat{b} \otimes \hat{b}}{p^2 + m^2} = \left[ \frac{((p^2 + m^2) \delta^{\mu\mu'} - p^\mu p^{\mu'}) ((p^2 + m^2) \delta^{\nu\nu'} - p^\nu p^{\nu'})}{p^2 + m^2} \right] \quad (17')$$

since  $\hat{b} \hat{d} = \hat{d} \hat{b} = (p^2 + m^2) \delta^{\mu\nu}$ . We shall now take the Euclidean Green's function to be  $(\hat{d} \otimes \hat{d})(p^2 + m^2)^{-1}$  which is positive-semidefinite on the space of tensor-valued test function space  $[\mathcal{S}(\mathbb{R}^4)]^{16}$ . Since the inverse of  $(\hat{d} \otimes \hat{d})(p^2 + m^2)^{-1}$  is not local, the random field with  $(\hat{d} \otimes \hat{d})(p^2 + m^2)^{-1}$  as covariance is not obviously Markovian for an argument analogous to that of Nelson cannot be given. This is in spite of the fact that the analytic continuation of this Schwinger function leads to a free Wightman field theory (of  $m > 0$ , spins 2 and 0).

We might again attempt to seek a subspace of test functions for which the Markov property does hold. A reasonable choice is as follows. Let  $\mathcal{K}$  be the completion of  $[\mathcal{S}_r(\mathbb{R}^4)]^{16}$  in the inner product

$$\langle f, g \rangle_{\mathcal{K}} = \sum_{\substack{\mu, \nu=1 \\ \rho, \sigma=1}}^4 \int d^4 p \hat{f}^{\mu\nu}(p) \frac{\hat{d}^{\mu\mu'} \hat{d}^{\nu\nu'}}{p^2 + m^2} \hat{g}^{\mu'\nu'}(p) \quad (18)$$

and let  $\mathcal{K}_0$  be the subset of traceless symmetric tensors  $f$  such that  $f \in \mathcal{K}_0$ ,  $\partial_\mu \partial_\nu f^{\mu\nu}(x) = 0$ . Restricting to this subspace will serve to remove the spin - 0 component from the corresponding Wightman theory, leaving a field of pure spin - 2. We note that the symmetric spin - 1 component of the field is ultra-local since

$$\begin{aligned} \hat{E} [\partial_\mu \Phi^{\mu\nu} \partial_{\mu'} \Phi^{\mu'\nu'}] &= \frac{p_\mu p_{\mu'}}{p^2 + m^2} \left( \delta^{\mu\mu'} + \frac{p^\mu p^{\mu'}}{m^2} \right) \left( \delta^{\nu\nu'} + \frac{p^\nu p^{\nu'}}{m^2} \right) \\ &= \frac{p^2}{m^2} \left( \delta^{\nu\nu'} + \frac{p^\nu p^{\nu'}}{m^2} \right) \end{aligned}$$

is a local differential operator.

Now let  $\bar{\Phi}$  be the generalized Gaussian random tensor field over  $\mathcal{K}_0$ , with mean zero and covariance given by the inner product

$\langle \cdot, \cdot \rangle_{\mathcal{K}_0}$ . To see whether  $\bar{\Phi}$  is Markovian or not, we proceed as follows. First of all we need to show that  $\hat{d} \otimes \hat{d}$  and  $\hat{b} \otimes \hat{b}$  map  $\mathcal{K}_0$  to itself. Let  $f \in \mathcal{K}_0$ ,  $g = (\hat{d} \otimes \hat{d})f$ , then

$$\begin{aligned} g^{\mu\mu} &= \left( \delta_{\mu'}^{\mu} + \frac{P^{\mu} P^{\mu'}}{m^2} \right) \left( \delta_{\nu'}^{\mu} + \frac{P^{\mu} P^{\nu'}}{m^2} \right) f^{\mu'\nu'} \\ &= f^{\mu\mu} + 2 \frac{P^{\mu'} P^{\nu'}}{m^2} f^{\mu'\nu'} + \frac{P^2}{m^4} P^{\mu'} P^{\nu'} f^{\mu'\nu'} = 0 \end{aligned}$$

Also  $P^{\mu} P^{\nu} g^{\mu\nu} = P^{\mu} P^{\nu} \left( \delta_{\mu'}^{\mu} + \frac{P^{\mu} P^{\mu'}}{m^2} \right) \left( \delta_{\nu'}^{\nu} + \frac{P^{\nu} P^{\nu'}}{m^2} \right) f^{\mu'\nu'}$

$$= P^{\mu'} P^{\nu'} f^{\mu'\nu'} + \frac{P^2}{m^2} P^{\mu'} P^{\nu'} f^{\mu'\nu'} = 0$$

Further, defining  $h^{\mu\nu} = (\hat{b} \otimes \hat{b})f$ ,  $f \in \mathcal{K}_0$ , then

$$\begin{aligned} h^{\mu\mu} &= \left[ \left( (P^2 + m^2) \delta_{\mu'}^{\mu} - P^{\mu} P^{\mu'} \right) \left( (P^2 + m^2) \delta_{\nu'}^{\mu} + P^{\mu} P^{\nu'} \right) \right] f^{\mu'\nu'} \\ &= (P^2 + m^2) f^{\mu\mu} - 2(P^2 + m^2) P^{\mu'} P^{\nu'} f^{\mu'\nu'} + P^2 P^{\mu'} P^{\nu'} f^{\mu'\nu'} = 0 \end{aligned}$$

It is also elementary to show that  $P^{\mu} P^{\nu} h^{\mu\nu} = 0$ . Thus it follows that on  $\mathcal{K}_0$ ,  $\left( \frac{\hat{d} \otimes \hat{d}}{P^2 + m^2} \right)^{-1} = \frac{\hat{b} \otimes \hat{b}}{P^2 + m^2}$ .

Now we need to show that the inverse of the Green's function is a local operator in  $\mathcal{K}_0$ . Clearly,  $\frac{\hat{b} \otimes \hat{b}}{P^2 + m^2}$  is local except for the term  $\left( \frac{P^{\mu} P^{\nu} P^{\mu'} P^{\nu'}}{m^4} \right) (P^2 + m^2)^{-1}$ . But this vanishes on  $\mathcal{K}_0$ . It appears as if we can now apply Nelson's proof of Markovicity

to our case. However this cannot be done. The reason is simply that  $(\hat{d} \otimes \hat{d})(p^2 + m^2)^{-1}$  and  $(\hat{b} \otimes \hat{b})(p^2 + m^2)^{-1}$  do not map an arbitrary

$g \in \mathcal{D}_r(\mathcal{O})$  to an element of  $\mathcal{K}_0(\mathcal{O})$ . Therefore we cannot conclude that for  $f \in \mathcal{K}_0$ ,  $\text{supp } f \subset \mathcal{O}$  and for all

$$g \in C^\infty(\mathcal{O}')$$

$$\int (e_{\mathcal{O}}, f)(x) g(x) dx = 0$$

We only have  $\int (e_{\mathcal{O}}, f)(x) g_0(x) dx = 0$  for all  $g_0 \in \mathcal{K}_0(\mathcal{O}')$

which is not sufficient for us to conclude that the Gaussian random field  $\bar{\Phi}$  is Markovian.

Thus we are only able to construct a generalized Gaussian random field  $\bar{\Phi}$  with covariance function  $E[\bar{\Phi}(f)\bar{\Phi}(g)]$  defined on  $\mathcal{K}_0 \times \mathcal{K}_0$ , having an extension to a distribution which is the Schwinger function of the usual free field with mass  $m > 0$  and spin 2. It is possible to do this in many way, even if we require the extension to be the Schwinger function of some Wightman theory. This choice also happen to define a Euclidean field too, it does not describe a unique spin however.

#### Remarks

(i) If  $\alpha$  is taken to be  $\infty$ , then the negative trace term

$$-\frac{1}{12} \delta^{\mu\nu} \delta^{\rho\sigma}$$

in (16) is cancelled out by a

similar term of opposite sign in the contact terms.

However in this case we do not have a inverse for

and also the field equation ceases to exist.

(ii) It is interesting to note that the "soft propagator"

given by

$$S^{\mu\nu, \rho\sigma}(p) = \frac{1}{2} \frac{(\delta^{\mu\rho} \delta^{\nu\sigma} + \delta^{\mu\sigma} \delta^{\nu\rho} - \frac{2}{3} \delta^{\mu\nu} \delta^{\rho\sigma})}{p^2 + m^2}$$

does give rise to a Markov tensor field if we restrict the space of test functions to a traceless, symmetric subspace. Since  $S^{\mu\nu,\rho\sigma}$  is positive-definite in this subspace, therefore we can define a Hilbert space  $\mathcal{K}_0$  in the usual manner, with inner product

$$\langle f, g \rangle_{\mathcal{K}_0} = \sum_{\substack{\mu, \nu, \rho, \sigma=1 \\ \mu, \nu, \rho, \sigma=1}}^4 \left\langle \hat{f}^{\mu\nu}, \frac{\delta^{\mu\rho} \delta^{\nu\sigma}}{p^2 + m^2} \hat{g}^{\rho\sigma} \right\rangle_{L^2}$$

noting that  $\sum_{\mu, \nu, \rho, \sigma} \langle f^{\mu\nu}, \frac{\delta^{\mu\nu} \delta^{\rho\sigma}}{p^2 + m^2} g^{\rho\sigma} \rangle_{L^2}$  vanishes for  $f, g \in \mathcal{K}_0$ . Now  $S^{\mu\nu,\rho\sigma}$  maps  $\mathcal{K}_0$  to itself. Let  $h \in \mathcal{K}_0$ ,

$$\begin{aligned} \text{Tr}(S^{\mu\nu,\rho\sigma} h^{\rho\sigma}) &= (\delta^{\mu\rho} \delta^{\nu\sigma} - \frac{1}{3} \delta^{\mu\nu} \delta^{\rho\sigma}) h^{\rho\sigma} \\ &= h^{\mu\mu} - \frac{1}{3} \delta^{\mu\mu} h^{\rho\rho} = 0 \end{aligned}$$

Since tracelessness and symmetry are just algebraic conditions, they do not affect the support properties of the test functions. Therefore Nelson's argument can be used to show the Markovicity.

#### IV. 3. Relativistic Massless Spin Two Tensor Field

In the classical theory, massless spin - 2 particles can be described by a rank-two tensor field  $\psi^{\mu\nu}$  satisfying the following equations:

$$\square \psi^{\mu\nu}(x) = 0 \quad (19)$$

$$\psi^{\mu\nu}(x) = \psi^{\nu\mu}(x) \quad (20)$$

$$\psi^{\mu}_{\mu}(x) = 0 \quad (21)$$

$$\partial_{\mu} \psi^{\mu\nu}(x) = 0 \quad (22)$$

However, in contrast to the non-zero mass case, the subsidiary conditions, in particular equation (22), cannot be derived from a variational principle. This means that Umezawa-Takahashi formulation fails to work for massless spin - 2 field, just like in the photon field. Again, if we attempt to quantize the field  $\psi^{\mu\nu}(x)$  in the usual manner, then there arise inconsistencies similar to those in electromagnetic potential. Furthermore, the analysis carried out by Strocchi and Bracci (St 3, S - B 1, 2, 3) has shown that gauge problems in spin - 2 massless field theory give rise to difficulties analogous to those appear in quantum electrodynamics. Their results indicate that a local and covariant description of massless spin - 2 particles by means of a symmetric rank-two tensor field  $\psi^{\mu\nu}(x)$  is possible only in a Hilbert space with indefinite metric. In other words, one needs to introduce Gupta-Bleuler formalism with unphysical states.

The quantized theory in Gupta formalism (Gu 2, 3) requires the subsidiary conditions (21) and (22) to hold only on physical states  $|\varphi\rangle$  in the form

$$\psi^{\mu}_{\mu}{}^{(+)}(x) |\varphi\rangle = 0 \quad (23)$$

$$\partial_{\mu} \psi^{\mu\nu}{}^{(+)}(x) |\varphi\rangle = 0 \quad (24)$$

where  $\Psi^{\mu\nu(+)}(x)$  denotes the positive frequency part of  $\Psi^{\mu\nu}(x)$ .

The free propagator is then given by

$$\langle \hat{\Psi}^{\mu\nu} \hat{\Psi}^{\rho\sigma} \rangle = \frac{-1}{2} (g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \frac{1}{p^2} \quad (25)$$

This gives us a theory of massless spin - 2 field in Lorentz gauge (or Gupta gauge). In order to generalize the theory so as to include a class of covariant gauges, we introduce the following general gauge transformation:

$$\Psi^{\mu\nu} \longrightarrow \Psi_{\#}^{\mu\nu} = \Psi^{\mu\nu} + \lambda (\partial^{\mu} B^{\nu} + \partial^{\nu} B^{\mu}) \quad (26)$$

where  $\lambda$  is an arbitrary real parameter,  $B^{\mu}(x)$  is a vector ghost field with the following two point function:

$$\langle \hat{B}^{\mu} \hat{B}^{\nu} \rangle = - \left( \frac{\alpha g^{\mu\nu}}{p^2} - \frac{\beta p^{\mu} p^{\nu}}{p^4} \right) \frac{1}{p^2} \quad (27)$$

and  $\langle \hat{B}^{\mu} \hat{\Psi}^{\mu\nu} \rangle = 0 \quad (28)$

Using these we get

$$\langle \hat{\Psi}_{\#}^{\mu\nu} \hat{\Psi}_{\#}^{\rho\sigma} \rangle = \tilde{G}^{\mu\nu\rho\sigma}(p) = \left[ (g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) - \frac{\lambda^2 \alpha}{p^2} (p^{\mu} p^{\rho} g^{\nu\sigma} + p^{\mu} p^{\sigma} g^{\nu\rho} + p^{\nu} p^{\rho} g^{\mu\sigma} + p^{\nu} p^{\sigma} g^{\mu\rho}) + \frac{4\lambda^2 \beta}{p^4} p^{\mu} p^{\nu} p^{\rho} p^{\sigma} \right] \frac{1}{(-p^2)} \quad (29)$$

We note that this propagator contains three real parameters  $\alpha$ ,  $\beta$  and  $\lambda$  whereas there is only one real parameter in electromagnetic potential. This is because the gauge transformation in

spin - 2 potential  $\psi^{\mu\nu}(x)$  is specified by four-vector  $B^\mu(x)$  whereas in spin - 1 potential it is specified by single gauge function  $\Lambda(x)$ . This clearly indicates that the spin - 2 propagator will be of much larger variety than its spin - 1 counterparts.

If the transformed field  $\psi_{\#}^{\mu\nu}(x)$  is to describe the same particles then one needs to eliminate the effective ghost states due to  $B^\mu(x)$ . This can be done by imposing the following subsidiary condition on the physical state vector  $|\varphi\rangle$  :

$$B^{\mu(+)}(x) |\varphi\rangle = 0 \quad (30)$$

Hence we have obtained a theory of massless spin - 2 field in a general class of covariant gauges. We note that for  $\lambda^2\alpha = +1$  and  $2\lambda\beta = +1$  we get

$$G^{\mu\nu\rho\sigma}(\partial) = \left[ (g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) - \frac{1}{\square} (g^{\mu\rho}\partial^\nu\partial^\sigma + g^{\mu\sigma}\partial^\nu\partial^\rho + g^{\nu\rho}\partial^\mu\partial^\sigma + g^{\nu\sigma}\partial^\mu\partial^\rho) + 2\frac{\partial^\mu\partial^\nu\partial^\rho\partial^\sigma}{\square^2} \right] \frac{1}{\square} \quad (31)$$

This propagator satisfies  $\partial_\mu G^{\mu\nu\rho\sigma}(\partial) = 0$ , thus corresponds to the Landau gauge. We shall see later on that in this gauge the corresponding Euclidean field is Markovian.

#### IV. 4. Euclidean Massless Spin Two Tensor Field

The Euclidean version of the above theory can be achieved as follows. By analytic continuing the two-point function of the Minkowski tensor field to the Schwinger points we obtain the two-point Schwinger function



$$S^{\mu\nu,\rho\sigma}(p) = \left[ (\delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}) - \frac{\lambda^2\alpha}{p^2} (P^\mu P^\rho \delta^{\nu\sigma} + P^\mu P^\sigma \delta^{\nu\rho} + P^\mu P^\sigma \delta^{\nu\rho} + P^\nu P^\rho \delta^{\mu\sigma}) + \frac{4\lambda^2\beta}{p^4} P^\mu P^\nu P^\rho P^\sigma \right] \frac{1}{p^2} \quad (32)$$

For  $S^{\mu\nu,\rho\sigma}$  to be positive-definite (or semidefinite) we require  $\lambda^2\alpha \leq 1$  and  $4\lambda^2\beta \geq 0$ . We can now use  $S^{\mu\nu,\rho\sigma}$  to define a Euclidean one particle Hilbert space  $\mathcal{K}$  as before, with inner product

$$\langle f, g \rangle_{\mathcal{K}} = \sum_{\substack{\mu,\nu=1 \\ \rho,\sigma=1}}^4 \langle \hat{f}^{\mu\nu}, S^{\mu\nu,\rho\sigma}(p) \hat{g}^{\rho\sigma} \rangle_{L^2} \quad (33)$$

The Euclidean spin - 2 mass tensor field  $\bar{\Psi}$  over  $\mathcal{K}_r$  is defined as the generalized Gaussian random tensor field with mean zero and covariance given by

#### Theorem 1

The Euclidean massless spin - 2 tensor field  $\bar{\Psi}$  over  $\mathcal{K}_r$  violates the reflection property.

#### Proof

We shall omit the proof since it is similar to that for Euclidean electromagnetic potential.

#### Theorem 2

The Euclidean massless spin - 2 tensor field  $\bar{\Psi}$  is Markovian in

- (i) Lorentz gauge ( $\lambda = \alpha = \beta = 0$ ).
- (ii) Landau gauge ( $\lambda^2\alpha = +1, 2\lambda\beta = +1$ ).

#### Proof

- (i) Denote by  $\mathcal{K}_0$  the one particle Hilbert space which

consists of symmetric, traceless tensor  $f^{\mu\nu}$ ,  
with inner product

$$\langle f, g \rangle_{\mathcal{K}_0} = \sum_{\substack{\mu, \nu=1 \\ \mu, \nu=1}}^4 \langle \hat{f}^{\mu\nu}, \frac{\delta^{\mu\rho} \delta^{\nu\sigma}}{p^2} \hat{g}^{\rho\sigma} \rangle_{L^2}$$

Tracelessness is preserved by  $\delta^{\mu\rho} \delta^{\nu\sigma} p^{-2}$ ,  
since for any  $h \in \mathcal{K}_0$ ,

$$\begin{aligned} \text{Tr} (\delta^{\mu\rho} \delta^{\nu\sigma} p^{-2}) h^{\rho\sigma} &= \sum_{\mu} p^{-2} \delta^{\mu\rho} \delta^{\mu\sigma} h^{\rho\sigma} \\ &= \sum_{\mu} p^{-2} h^{\mu\mu} = 0 \end{aligned}$$

Since this is just an algebraic condition and the covariance matrix  $p^{-2} \delta^{\mu\rho} \delta^{\nu\sigma}$  is the inverse of a local matrix operator, therefore Nelson's proof of Markovicity carries through.

(ii) In Landau gauge, the covariance matrix is

$$\begin{aligned} S^{\mu\nu\rho\sigma}(p) &= \left[ (\delta^{\mu\rho} \delta^{\nu\sigma} + \delta^{\mu\sigma} \delta^{\nu\rho}) - \frac{1}{p^2} (p^\mu p^\rho \delta^{\nu\sigma} + p^\mu p^\sigma \delta^{\nu\rho} \right. \\ &\quad \left. + p^\nu p^\rho \delta^{\mu\sigma} + p^\nu p^\sigma \delta^{\mu\rho}) + \frac{2 p^\mu p^\nu p^\rho p^\sigma}{p^4} \right] \frac{1}{p^2} \end{aligned}$$

In the space of symmetric test functions we can write

$$\begin{aligned} S^{\mu\nu\rho\sigma}(p) &= \left[ \delta^{\mu\rho} \delta^{\nu\sigma} - \frac{1}{p^2} (\delta^{\mu\rho} p^\nu p^\sigma + \delta^{\nu\sigma} p^\mu p^\rho) \right. \\ &\quad \left. + \frac{p^\mu p^\nu p^\rho p^\sigma}{p^4} \right] \frac{1}{p^2} \\ &= \left( \delta^{\mu\rho} - \frac{p^\mu p^\rho}{p^2} \right) \left( \delta^{\nu\sigma} - \frac{p^\nu p^\sigma}{p^2} \right) \frac{1}{p^2} \end{aligned}$$

Now consider the subset of test functions satisfying the following conditions:

$$\text{symmetry} \quad : \quad f^{\mu\nu} = f^{\nu\mu}$$

$$\text{tracelessness} \quad : \quad \sum_{\mu} f^{\mu\mu} = 0$$

$$\text{divergenceless} \quad : \quad \sum_{\mu} \partial_{\mu} f^{\mu\nu} = 0$$

The inner product  $\langle f, g \rangle_{K_0} = \sum_{\substack{\mu, \nu \\ \rho, \sigma}} \langle f^{\mu\nu}, S^{\mu\nu, \rho\sigma} g^{\rho\sigma} \rangle_{L^2}$  reduces to  $\sum_{\substack{\mu, \nu \\ \rho, \sigma}} \langle f^{\mu\nu}, \frac{\delta^{\mu\rho} \delta^{\nu\sigma}}{p^2} g^{\rho\sigma} \rangle_{L^2}$  for such test functions.

Furthermore,  $S^{\mu\nu, \rho\sigma}$  maps  $K_0$  to  $K_0$  since for any  $h \in K_0$ ,

$$\begin{aligned} & \sum_{\mu} p^{\mu} (\delta^{\mu\rho} - p^{-2} p^{\mu} p^{\rho}) (\delta^{\nu\sigma} - p^{-2} p^{\nu} p^{\sigma}) h^{\rho\sigma} \\ &= (p^{\rho} - p^{\rho}) (\delta^{\nu\sigma} - p^{-2} p^{\nu} p^{\sigma}) h^{\rho\sigma} = 0 \end{aligned}$$

$$\begin{aligned} & \sum_{\nu} p^{\nu} (\delta^{\mu\rho} - p^{-2} p^{\mu} p^{\rho}) (\delta^{\nu\sigma} - p^{-2} p^{\nu} p^{\sigma}) h^{\rho\sigma} \\ &= (\delta^{\mu\rho} - p^{-2} p^{\mu} p^{\rho}) (p^{\sigma} - p^{\sigma}) h^{\rho\sigma} = 0 \end{aligned}$$

$$\begin{aligned} & \text{Tr} [ (\delta^{\mu\rho} - p^{-2} p^{\mu} p^{\rho}) (\delta^{\nu\sigma} - p^{-2} p^{\nu} p^{\sigma}) h^{\rho\sigma} ] \\ &= \sum_{\mu} (\delta^{\mu\rho} - p^{-2} p^{\mu} p^{\rho}) (\delta^{\mu\sigma} - p^{-2} p^{\mu} p^{\sigma}) h^{\rho\sigma} \\ &= (\delta^{\rho\sigma} - p^{-2} p^{\rho} p^{\sigma}) h^{\rho\sigma} = 0 \end{aligned}$$

Let  $\mathcal{O} \subset \mathbb{R}^4$  be an open set and  $f \in K_0$  with

$\text{Supp } f \subset \mathcal{U}$ . If  $e_{\mathcal{U}'}$  is the projection onto

$\mathcal{K}_0(\mathcal{U}')$ , then one needs to show that

$$\langle e_{\mathcal{U}'} f, h \rangle_{L^2} = 0 \quad \text{for all } h \in C^\infty(\mathcal{U}')$$

where  $\mathcal{U}'$  is the interior of the complement of  $\mathcal{U}$ .

The proof of this depends crucially on two points.

Firstly,  $(p^2 \delta^{\mu\rho} - p^\mu p^\rho)$  or  $(p^2 \delta^{\nu\sigma} - p^\nu p^\sigma)$  maps any  $h^{\rho\sigma} \in C^\infty(\mathcal{U}')$  to element satisfying

$$p_\rho h^{\rho\sigma} = 0 \quad \text{i.e. to element belonging to}$$

$\mathcal{K}_0(\mathcal{U}')$ . Second point is that the inverse of

$$p^{-2} \delta^{\mu\rho} \delta^{\nu\sigma} \quad \text{is a local differential operator.}$$

Thus we have

$$\langle e_{\mathcal{U}'} f, h \rangle_{L^2} = \langle (e_{\mathcal{U}'} f)^{\mu\nu}, \frac{p^2}{p^2} (\delta_{\mu\rho} - \frac{p_\mu p_\rho}{p^2}) (\delta_{\nu\sigma} - \frac{p_\nu p_\sigma}{p^2}) h^{\rho\sigma} \rangle_{L^2}$$

$$\text{since } p_\mu f^{\mu\nu} = p_\nu f^{\mu\nu} = 0, \quad = \langle (e_{\mathcal{U}'} f)^{\mu\nu}, \frac{1}{p^2} (\delta_{\mu\rho} - \frac{p_\mu p_\rho}{p^2}) (p^2 \delta_{\nu\sigma} - p_\nu p_\sigma) h^{\rho\sigma} \rangle_{L^2}$$

$$= \langle (e_{\mathcal{U}'} f)^{\mu\nu}, \frac{1}{p^2} (\delta_{\mu\rho} - \frac{p_\mu p_\rho}{p^2}) g^{\rho\nu} \rangle_{L^2}$$

(where  $g^{\rho\nu} = (p^2 \delta_{\nu\sigma} - p_\nu p_\sigma) h^{\rho\sigma} \in \mathcal{K}_0(\mathcal{U}')$  since  $p_\rho g^{\rho\nu} = 0$ )

$$\text{since } p_\rho g^{\rho\nu} = 0, \quad = \langle e_{\mathcal{U}'} f^{\mu\nu}, \frac{\delta_{\mu\rho}}{p^2} g^{\rho\nu} \rangle_{L^2}$$

$$= \langle e_{\mathcal{U}'} f^{\mu\nu}, \delta_{\mu\rho} g^{\rho\nu} \rangle_{\mathcal{K}_0}$$

$$= \langle f^{\mu\nu}, \frac{\delta_{\mu\rho}}{p^2} (p^2 \delta_{\nu\sigma} - p_\nu p_\sigma) h^{\rho\sigma} \rangle_{L^2}$$

$$\text{since } p_\nu f^{\mu\nu} = 0, \quad = \langle f^{\mu\nu}, h_{\mu\nu} \rangle_{L^2} = 0$$

Hence  $\text{supp } e_0, f \subset \partial U$ , and the tensor field in Landau gauge is Markovian.

Q.E.D.

Remark

We have only considered two important covariant gauges, namely the analogues of Lorentz and Landau gauges, and we are able to show that the Euclidean massless tensor field in these gauges is Markovian. The generalization to include other covariant gauges faces similar kind of difficulty as in Euclidean massive tensor field in I. 2. Furthermore, no attempt has been made to discuss gravitation field theory which, though is very interesting, is outside the scope of this work.

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