GAIFMAN OPERATIONS, MINIMAL MODELS AND THE NUMBER OF COUNTABLE MODELS.

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Abstract.

We deal with questions and problems in first order countable model theory.

Chapter 1 examines countable first order Gaifman operations, which are theories whose models are determined, up to isomorphism, by their relativised reducts. We first prove some reduction and preservation results. Then we prove that the class of relativised reducts of a Gaifman operation is generalised elementary. Finally, we examine the degree of 1-cardinality of such theories.

Chapter 2 is basically concerned with trying to get lots of pairwise elementarily equivalent countable models, or to begin with, at least four models, to which my friend Salim Salem would say, "It's hard enough to get one." We first show that a minimal prime model is "fairly" algebraic. Then, under various conditions on the algebraicity of the countable models of a theory, we prove results concerning the number of its countable models.
The main result is that a countable complete theory which has a model with an infinite definable subset all of whose elements are algebraic of degree at most two, has at least four countable models, up to isomorphism.

Chapters 1 and 2 are formally independent and self-contained. However there are certain common themes. The notion of a minimal model is important in both chapters. More generally, both chapters are concerned with a question at the centre of model theory - the number of models of a theory. In Chapter 1, it is the number of models over a

predicate, in particular the case where the number is one. In Chapter 2 it is the number of countable models.

Contents

Chapter 0.

Notation and preliminaries.

I shall be concerned in this thesis with first order model theory.

A general reference for the basic definitions and results is Chang and Keisler [3] . I assume familiarity with the basic notions of model, language, theory, consistency, satisfaction and semantic and syntactic implication. For the fundamental properties of first order logic, in particular compactness, the consistency theorem (every consistent set of sentences has a model), and the Lowenheim-Skolem theorem, see [3] .

I shall denote models by A, B, M, M',... , and theories by T, T',... . If X is a set, then $|X|$ will denote the cardinality of X. If A is a model, then $|A|$ will denote the universe of A. So $\|A\|$ denotes the cardinality of the universe of A.

If A is a model for a language L, then the language of A, L(A), is just L. If T is a theory, we say that T is countable if the number of non-logical symbols appearing in T is countable. We identify a theory with its deductive closure.

Variables are denoted by x , y , z , x_1 , x_2, .. . Constants are denoted by a , b , c , a_1, ... We often use the same symbol to denote a constant and the element which it names in a model. If A is a model, and X is a subset of $|A|$, then

$(A,a)_{a\in X}$ is the model got from A by adding names for the elements of X.

$\vec{x},\vec{y},..$ and $\vec{a},\vec{b},...$ denote finite sequences of variables and constants respectively. If $\vec{a} = (a_1, \quad a_n)$ and A is a model, then we often write $\vec{a} \in A$ to mean that $a_i \in |A|$ for i = 1,... n. Formulae are denoted by ϕ, ψ, θ, ... When we write a formula as $\phi(x_1, \dots x_n)$, we mean that the free variables of ϕ are among $x_1, \dots x_n$. If $\vec{a} \in A$, then $A \models \phi(\vec{a})$ means that $\phi(\vec{x})$ is satisfied in A by $\vec{a}$. If $\phi(\vec{x})$ is a formula, and $\vec{a}$ is a sequence of constants, then we also denote by $\phi(\vec{a})$ the result of replacing each x_i in ϕ by a_i.

We often write $\forall \vec{x}$ to mean $\forall x_1 \dots \forall x_n$.

Similarly with $\exists \vec{x}$.

If θ is an n-ary formula (i.e. θ is $\theta(x_1, \dots x_n)$), then

$\forall \vec{x} \in \theta(\dots.)$ means $\forall \vec{x}(\theta(\vec{x}) \rightarrow \dots)$, and

$\exists \vec{x} \in \theta(\dots)$ means $\exists \vec{x}(\theta(x) \wedge \dots)$

$\exists^k \vec{x}(\theta(\vec{x}))$ 'means' there are exactly k distinct n-tuples $\vec{x}$ such that $\theta(\vec{x})$.

$\exists^{\leq k} \vec{x}(\theta(\vec{x}))$ 'means' there are at most k distinct n-tuples $\vec{x}$ such that $\theta(\vec{x})$.

If $\phi(x)$ is a 1-ary formula of a language L, and A is a model for L, then $\phi^A = \{a \in A : A \models \phi(a)\}$

Let L be a language, P be a unary predicate of L, and $L_0 \subseteq L - \{P\}$. If P^A is closed under the functions of A which are in L_0, then we get an L_0-structure whose universe is P^A, and whose relations and functions are just the restrictions of the relevant ones of A. We call this model $A^P | L_0$. And in this situation we say that $A^P | L_0$ is defined.

Let A_1, A_2 be models, and $A_0 = A_1^P L = A_2^P L$.

We then say that A_1 is isomorphic to A_2 over A_0, in symbols

$A_1 \simeq_{A_0} A_2$, if there is an isomorphism of A_1 onto A_2 which is the identity on A_0.

$A \equiv B$ and $A \preccurlyeq B$ as usual mean that A is elementarily equivalent to B, and that A is an elementary substructure of B, respectively. We write $f: A \preccurlyeq B$ to mean that f is an elementary embedding of A into B. $f: A \simeq B$ means that f is an isomorphism of A onto B.

I assume familiarity with the notions of ultrafilter, ultraproduct and ultrapower. For details, and for the important Los' Theorem, see [3].

If A is a model for the language L, then Th(A) is the set of sentences of L which are true in A.

T is a complete theory means that for any sentence σ in the language of T, $T \vdash \sigma$ or $T \vdash \neg\sigma$.
If ϕ is a formula, then $T \vdash \phi$ means that $T \vdash \forall\vec{x}\,\phi$ where $\vec{x}$ is a sequence which contains the free variables of ϕ.
If K is a class of models for L, then Th(K) is the set of sentences of L which are true in every model A in K. The class K is said to be elementary, if there is a sentence σ such that

$$A \in K \quad \text{if and only if} \quad A \models \sigma .$$

K is said to be generalised elementary if there is a set of sentences Σ such that $A \in K$ if and only if $A \models \Sigma$
Let T be a theory and n a natural number. Then an n-type of T is a set of formulae, each of whose free variables is among say x_1, x_n, which is consistent with T. A type of T is just an n-type of T for some n.

A complete n-type of T is an n-type Σ of T such that for each n-formula ϕ, $\phi \in \Sigma$ or $\neg\phi \in \Sigma$.

In Chapter 2, whenever we talk about types we shall mean <u>complete</u> types, unless we say otherwise.

If Σ is an n-type and $\vec{a}$ is an n-tuple of a model A, then we say that $\vec{a}$ realises Σ , if $A \models \phi(\vec{a})$ for all $\phi \in \Sigma$.

The type of a tuple $\vec{a}$ in a model A is the set of formulae ϕ such that $A \models \phi(\vec{a})$.

We say that Σ is a principal n- type of T, if Σ is an n-type of T and there is an n-formula $\phi(x_1, \ldots x_n)$ consistent with T such that

$$T \vdash \phi \longrightarrow \psi \qquad \text{for all } \psi(x_1, \ldots x_n) \in \Sigma \quad .$$

Let A be a model and $\vec{b} \in A$. When we say that $\vec{b}$ realises a principal type in A, we shall mean that the type of $\vec{b}$ in A is a principal type of Th(A).

A model A is said to omit a type, if no tuple in A realises the type.

Let T be a complete theory. An n-formula $\phi(x_1, \ldots x_n)$ is said to be complete for T, if for every $\psi(x_1, \ldots x_n)$

$$T \vdash \phi \longrightarrow \psi \quad \text{or} \quad T \vdash \phi \longrightarrow \neg\psi .$$

A model A is <u>atomic</u>, if every finite sequence of elements of A satisfies a complete formula of Th(A) (or equivalently, realises a principal type of Th(A)).

A is a prime model of T, if for all $B \models T$ there is $f: A \preccurlyeq B$. A is said to be <u>prime</u>, if A is a prime model of Th(A). A complete theory T is <u>atomic</u>, if for every n-formula ϕ there is a complete n-formula ψ of T such that $T \vdash \psi \longrightarrow \phi$, for all n.

We state the following classical results.

(A) (Grzegorizyk et al[10]) <u>The Omitting Types Theorem</u>

Let T be a countable theory, and $\{\Sigma_n : n < \omega\}$ a collection of non-principal types of T. Then T has a model which omits Σ_n for all n.

(B) (Vaught [25]) Let A be a model for a countable language. Then A is prime if and only if A is countable and atomic.

(C) (Vaught [25]) Let T be a complete countable theory. Then T is atomic if and only if T has a countable atomic model.

Let κ be a cardinal. A theory T is κ-categorical, if all models of T of cardinality κ are isomorphic to one another.

A model A is κ-saturated, if for all $X \subseteq |A|$ such that $|X| < \kappa$, $(A,a)_{a \in X}$ realises all types of $Th((A,a)_{a \in X})$. A is said to be saturated if A is $\|A\|$-saturated.

Let A,B be models, I be a set, and a_i, b_i, be elements of A,B respectively, for all $i \in I$.

Then $(A,a_i)_{i \in I} \equiv (B,b_i)_{i \in I}$ means that $Th((A,a_i)_{i \in I}) = Th((B,b_i)_{i \in I})$, where we represent a_i and b_i by the same constant for each i.

So if $\vec{a}$ and $\vec{b}$ are finite sequences of the same length, $(A,\vec{a}) \equiv (B,\vec{b})$ if and only if $\vec{a}$ realises the same type in A as $\vec{b}$ realises in B.

Similarly, we write $(A,a_i)_{i \in I} \simeq (B,b_i)_{i \in I}$ to mean that there is an isomorphism $f: A \simeq B$ such that $f(a_i) = b_i$ for all i.

A is homogeneous, if whenever $|I| < \|A\|$, and $(A,a_i)_{i \in I} \equiv (A,b_i)_{i \in I}$ then $(A,a_i)_{i \in I} \simeq (A,b_i)_{i \in I}$.

A is universal, if $B \equiv A$, and $\|B\| \leq \|A\|$ implies that there is $f: B \preccurlyeq A$.

A is full, means that A realises all types of Th(A).

The following fact is easy to establish.

(D) If A is a countable model which is homogeneous and full, then A is saturated.

We also have the following :

(E) (Vaught [25]) Let T be a countable complete theory with only countably many complete types. Then T has a prime model and a countable saturated model.

(F) (Ryll-Nardzewski [19]) Let T be a complete countable theory. Then T is $\aleph_0$-categorical iff T has finitely many complete n-types for all $n < \omega$ iff all complete types of T are principal.

A simple extension of a theory T, is a theory T' such that $T \subseteq T'$, and such that the language of T' is got from the language of T by adding at most finitely many new constants.

Although we do not really work with stability notions, stability is referred to now and again. So we give the definitions. Let κ be a cardinal. Then we say that a theory T is κ-stable if whenever $A \models T$, $X \subseteq |A|$, and $|X| \leq \kappa$, then $Th((A,a)_{a\in X})$ has at most κ complete 1-types.

T is stable if it is κ-stable for some infinite cardinal κ.

T is superstable if it is κ-stable for all sufficiently large κ.

The notion of stability has been a useful and important tool in the study of the number of <u>uncountable</u> models of a countable theory. For example, the following have been proved :

If T is unstable, then T has 2^{κ} models of cardinality κ, for all uncountable κ. (Shelah [23])

T is categorical in all uncountable powers if and only if T is $\aleph_0$-stable and T does not satisfy the hypothesis of Vaught's two-cardinal theorem. (Baldwin and Lachlan [1])

However I do not think that stability is such a sharp tool when it comes to analysing the difference between theories with finitely many, and theories with infinitely many, countable models.

We work in general only with countable languages and countable theories. In Chapter 1 we sometimes get an uncountable language,by adding names for elements of an uncountable model. However, in Chapter 2 everything is countable. Also for Chapter 2 we make the general assumption that all the complete theories we talk about, have only infinite models. And of course, whenever we talk about the number of models of a theory, we mean up to isomorphism.

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Chapter 1

Gaifman operations

1.0 Introduction

Gaifman [8] originally defined his 'single valued operations', as a means of placing in a model-theoretic setting, or of elucidating the model-theoretic content of, certain standard algebraic constructions and operations. The kind of operations which we have in mind are exemplified by the following - forming the field of fractions of an integral domain; forming the ultrapower from a model, a set and an ultrafilter on that set; given a field, forming the n-dimensional vector space over that field. Then Gaifman's idea is the following. Let us suppose that the operation under consideration takes certain models for a language L_1 to models for a language L_2. (L_1 and L_2 may possibly be many-sorted). Then there is a theory T such that a value of the operation for A is the model B, if and only if there is a set of relations and functions r, such that $(A, B, r) \models T$. Also, if $(A, B_i, r_i) \models T$, $i = 1,2$, then (A, B_1, r_1) is isomorphic to (A, B_2, r_2) over A, in symbols $(A, B_1, r_1) \simeq_A (A, B_2, r_2)$

The set of relations and functions r, will serve to connect A and B, or possibly define B from A. We may stipulate that T be a first order theory, or a theory in

$L_{\omega_1, \omega}$ or whatever. We may also add extra conditions concerning the degree of rigidity of (A, B, r) over A, for (A, B, r) $\models$ T.

Although the above many-sorted definition is more natural in the algebraic context, we can express everything in a one sorted logic, as in Gaifman [9]. This will be the definition which we shall actually work with. We shall only be concerned with the situation where all languages are countable, and the theory T is first order.

Definition 1.

Let L_0 and L be countable languages, and P a unary predicate in L and not in L_0, such that $L_0 \subseteq L$. Let T be a first order theory in L such that whenever $A \models T$, then $A^P | L_0$ is defined. We then say that (T, P, L_0) is a countable first order Gaifman operation if and only if $A_i \models T$, $i = 1,2$ and $A_1^P | L_0 = A_2^P | L_0 = A_0$ implies that A_1 is isomorphic to A_2 over A_0.

As we will only consider the countable first order situation, and as P and L_0 will usually be clear from the context, we shall just use the phrase "T is a Gaifman operation".

Taken apart from the algebraic motivation, the study of Gaifman operations essentially boils down to the study of categority over a predicate. Also Gaifman [9] notes that the property of T of being a Gaifman operation is a generalisation of that implicit definability which is characterised by Beth's theorem. In the situation of Beth's theorem we have languages L_0 and L with $L_0 \subseteq L$, and a theory T in L such that any model A for L_0 has at most one expansion to a model of T. Beth's theorem then says that T explicitly defines each new relation of L in terms of formulae in L_0. In the case of Gaifman operations however, new elements as well as new relations and functions are added to the model. So the immediate question is whether there is an analogous explicit definability result in this Gaifman situation. The obvious interpretation of explicit definability (which is the same as Hodges's word constructions [11], and Ershov's method of elementary definability [5]) is that for A a model of T, every element of A can be interpreted as an n-tuple of elements from $A^D | L_c$, and the holding of a predicate of L for a sequence of elements of A depends uniformly on the holding of some formula for the corresponding sequence of n-tuples in $A^D | L_0$. If such an explicit definability result held for Gaifman operations, then it would validate Gaifman's definition of his single valued operations as a standard characterisation of algebraic constructions. However, Hodges [11] has given an example of a Gaifman operation T, for which no such explicit definability holds. Strengthenings and variations of the above, do hold however, for Gaifman operations which in addition satisfy certain conditions on the degree of rigidity of A over

$A^p|L_0$. But for plain Gaifman operations, the question of what we can say in the way of explicit definability is still open.

We can quite easily prove results which for a Gaifman operation T, uniformly reduce certain properties of models A of T to properties of $A^p|L_0$. We do this in 1.1. Some of these results are known or have been stated in the literature. We show also that if T is a Gaifman operation, then if $A_i \models T$, $i = 1,2$, $f : A_1^p|L_0 \preccurlyeq A_2^p|L_0$ and $\|A_1^p|L_0\| \leq \aleph_0$, then there is $g : A_1 \preccurlyeq A_2$ which extends f.

Gaifman [9] states that Shelah has proved this without the cardinality restriction on $A_1^p|L_0$. We use this latter result of Shelah as well as our earlier results to answer affirmatively a conjecture of Gaifman [8], that the domain of a Gaifman operation is EC_Δ. We also rely heavily on Shelah's results to give an interesting characterisation of Gaifman operations.

In 1.2 we connect the 1-cardinality of Gaifman operations with a certain characterisation of minimality by Deissler [4]. Finally we give a counterexample to a conjecture of Wilfred Hodges characterising Gaifman operations as being 1-cardinal and satisfying some further reduction property.

The study of categoricity over a predicate, is part of a wider study of general model theory over a predicate. This looks to be quite difficult. Shelah [22] has generalised stability concepts to this area. Our results however, tend to be centered around consequences of the definitions, for countable models.

1.1 Preservation, reduction and related results

Usually, the following preservation theorem (Theorem 2) is deduced from the uniform reduction theorem (Theorem 3) which in turn follows from Fefeman's many-sorted interpolation theorem [6]. Here, however we prove the preservation result independently, using Shelah's ultrapower theorem.

Theorem 2 Let T be a Gaifman operation. Let M_1, M_2 be models of T, $\vec{a}_1 \in P^{M_1}$, $\vec{a}_2 \in P^{M_2}$ be n-tuples, and

$$(M_1^P | L_0, \vec{a}_1) \equiv M_2^P | L_0, \vec{a}_2)$$

Then $(M_1, \vec{a}_1) \equiv (M_2, \vec{a}_2)$

Proof

Let M_1, M_2, $\vec{a}_1$, $\vec{a}_2$ be as in the assumptions of the theorem. By Shelah [20], there is a set I and an ultrafilter U on I such that

$$(M_1^P | L_0, \vec{a}_1)^I/U \simeq (M_2^P | L_0, \vec{a}_2)^I/U$$

Consider now $(M_i, \vec{a}_i)^I/U$ for $i = 1,2$.

It is easily seen that $((M_i, \vec{a}_i)^I/U)^P | L_n = (M_i^P L_o, \vec{a}_i)^I/U$ for $i = 1,2$.

So $((M_1, \vec{a}_1)^I/U)^P | L_o \simeq ((M_2, \vec{a}_2)^I/U)^P | L_o$ (*)

By Los' theorem, $M_i \equiv M_i^I/U$, $i = 1,2$.

So $M^I/U \models T$ for $i = 1,2$.

Thus as T is a Gaifman operation it follows from (*) that

$$(M_1, \vec{a}_1)^I/U \simeq (M_2, \vec{a}_2)^I/U$$

So in particular, $(M_1, \vec{a}_1)^I/U \cdot \equiv (M_2, \vec{a}_2)^I/U$

Again, by Los it follows that

$(M_1, \vec{a}_1) \equiv (M_2, \vec{a}_2)$, proving the theorem.

In particular, for $n = 0$, we have that, $M_i \models T$, $i = 1,2$ and $M_1^P | L_o \equiv M_2^P | L_o$, implies that $M_1 \equiv M_2$.

We can now prove the uniform reduction theorem.

<u>Theorem 3</u>. Let T be a Gaifman operation, $n < \omega$.

Then for every formula $\phi(x_1, \ldots x_n)$ of L, there is a formula $\psi(x_1, \ldots x_n)$ of L_o, such that, for every $M \models T$ and $\vec{a} = (a_1, \ldots a_n) \in P^M$, $M \models \phi(\vec{a})$ if and only if $M^P | L_n \models \psi(\bar{a})$.

<u>Proof</u>.

We first define "P-L_n formulae". A P-L_o formula ψ is an L-formula, which is in prenex normal form, whose matrix contains symbols only from L_o, and whose quantifiers are all relativised to P.

Then, given a P-L_0 n-formula $\psi(\vec{x})$ there is an n-formula $\psi'(\vec{x})$ in L_0, such that :

$$\left.\begin{array}{l}\text{for any L-structure } M \text{ and } \vec{a} = (a_1, \ldots a_n) \in P^M, \\ M \models \psi(\vec{a}) \text{ if and only if } M^P | L_0 \models \psi'(\vec{a}).\end{array}\right\} (*)$$

And conversely, given any n-formula $\psi'(\vec{x})$ in L_0 there is a P-L_0 formula $\psi(\vec{x})$ such that (*) holds.

Now let $\phi(x_1, \ldots x_n)$ be an L-formula. Let $c_1, \ldots c_n$ be new constants. We write $P\vec{c}$ for $\bigwedge_{i=1\ldots n} Pc_i$.

Put $\Gamma = \Gamma(\vec{x}) = \{ \psi(\vec{x}) : \psi$ is P-L_0 formula such that $T, P\vec{c} \vdash \phi(\vec{c}) \rightarrow \psi(\vec{c}) \}$

We will show that $T, P\vec{c}, \Gamma(\vec{c}) \vdash \phi(\vec{c})$.

So let $(M, \vec{a}) \models T \cup \{P\vec{c}\} \cup \Gamma(\vec{c})$

(Here the elements a_i are interpretations of the constants c_i)

Let $\Gamma'(\vec{x}) = \{ \psi(\vec{x}) : \psi$ a P-L_0 formula, and $M \models \psi(\vec{a}) \}$

Then we assert that $T \cup \Gamma'(\vec{c}) \cup \{ P\vec{c}, \phi(\vec{c}) \}$ is consistent.

or if not, then

$T, P\vec{c} \vdash \phi(\vec{c}) \rightarrow \neg\psi(\vec{c})$, for some $\psi(\vec{x})$ in Γ'. But then $\neg\psi(\vec{x})$ is in Γ, whereby $M \models \neg\psi(\vec{a})$. This is a contradiction.

So let $(N, \vec{a}') \models T \cup \Gamma'(\vec{c}) \cup \{ P\vec{c}, \phi(\vec{c}) \}$

As $(N, \vec{a}') \models \Gamma'(\vec{a})$, we can see that

$$(M^P | L_0, \vec{a}) \equiv (N^P | L_0, \vec{a}')$$

By Theorem 2, $(M, \vec{a}) \equiv (N, \vec{a}')$

Thus $M \models \phi(\vec{a})$.

As $(M, \vec{a})$ was an arbitrary model of $T \cup \{P\vec{c}\} \cup \Gamma(\vec{c})$, we have established that $T, P\vec{c}, \Gamma(\vec{c}) \vdash \phi(\vec{c})$.

By compactness there is $\psi(\vec{c})$ in $\Gamma(\vec{c})$

such that $T, P\vec{c} \vdash \psi(\vec{c}) \rightarrow \phi(\vec{c})$

So $T, P\vec{c} \vdash \psi(\vec{c}) \leftrightarrow \phi(\vec{c})$

so $T \vdash \forall \vec{x}(P\vec{x} \rightarrow (\psi(\vec{x}) \leftrightarrow \phi(\vec{x})))$.

Let $\psi'(\vec{x})$ be the L_0 formula which corresponds to the P-L_0 formula $\psi(\vec{x})$. Then for any $M \models T$ and $\vec{a} \in P^M$

$$M \models \phi(\vec{a}) \text{ iff } M \models \psi(\vec{a}) \text{ iff } M^P | L_0 \models \psi'(\vec{a})$$

Lemma 4 Let T be a Gaifman operation, M be an infinite model of T and $\vec{b}$ an n-tuple of M. Then $\vec{b}$ realises in $(M, a)_{a \in P^M}$ a principal n-type of $Th(M, a)_{a \in P^M}$.

Proof

We first prove the lemma for the case in which M is countable. So suppose that M is countable. Let $\Gamma(\vec{x})$ be the type which $\vec{b}$ realises in $(M, a)_{a \in P^M}$. Suppose that $\Gamma(\vec{x})$ were a nonprincipal type of $Th(M, a)_{a \in P^M}$. Notice that P^M must be countably infinite, for otherwise we could characterise $M^P | L_0$ up to isomorphism by a set of sentenses in $L(M, a)_{a \in P^M}$, and then by the Lowenheim-Skolem theorem, we could find models N of T of any cardinality such that $N^P | L_0 = M^P | L_0$. Thus $|P^M| = \aleph_0$

So $\Lambda(y) = \{ y \neq a : a \in P^M \} \cup \{Py\}$ is a consistent type of $Th(M, a)_{a \in P^M}$. Moreover $\Lambda(y)$ is omitted in

$(M, a)_{a \in P^M}$. So $\Lambda(y)$ must be nonprincipal. So by the omitting types theorem there is a model of $Th(M, a)_{a \in P^M}$, which omits the types $\Gamma(\vec{x})$ and $\Lambda(y)$. Let this model be $(N, a)_{a \in P^M}$. Then $N^P | L_0 = M^P | L_0$. However, $(M, a)_{a \in P^M}$ realises $\Gamma(\vec{x})$ whereas $(N, a)_{a \in P^M}$ omits $\Gamma(\vec{x})$, so the two models cannot be isomorphic. This contradicts T being a Gaifman operation. So $\Gamma(\vec{x})$ must be principal.

Now let $M \models T$ be of arbitrary infinite cardinality. Let $\vec{b} \in M$. Let $N \preccurlyeq M$, $\vec{b} \in N$ and $\|N\| = \aleph_0$. Then from above $\vec{b}$ realises a principal complete type of $Th(N, a)_{a \in P^N}$. Let this type of $\vec{b}$ be generated by the formula $\phi(\vec{x}; \vec{a})$ where $\vec{a} \in P^N$ and $\phi(\vec{x}; \vec{y})$ is an L(T) formula. We assert that $\phi(\vec{x}; \vec{a})$ also generates the type of $\vec{b}$ in $(M, a)_{a \in P^M}$. For if not, there is a formula $\psi(\vec{x}; \vec{c})$, $\vec{c} \in P^M$ and

$$M \models (\exists \vec{x})(\phi(\vec{x}; \vec{a}) \wedge \psi(\vec{x}; \vec{c})) \wedge (\exists \vec{x})(\phi(\vec{x}; \vec{a}) \wedge \neg\psi(\vec{x}; \vec{c}) \wedge P\vec{c}$$

As $N \preccurlyeq M$, and $\vec{a} \in N$ there is $\vec{c}' \in P^N$ such that

$$N \models (\exists \vec{x})(\phi(\vec{x}; \vec{a}) \wedge \psi(\vec{x}; \vec{c}')) \wedge (\exists \vec{x})(\phi(\vec{x}; \vec{a}) \wedge \neg\psi(\vec{x}; \vec{c}')$$

But this contradicts the fact that $\phi(\vec{x}; \vec{a})$ generates the type of $\vec{b}$ in $(N, a)_{a \in P^M}$.
So the lemma is proved.

Lemma 4 now enables us to prove that elementary embeddings of "ground models" can be extended, provided that the embedded model is countable.

<u>Theorem 5</u> Let T be a Gaifman operation.

Let M, N be models of T, $\|M\| \leqslant \aleph_0$ and $f : M^P|L_0 \preccurlyeq N^P|L_0$. Then there is $g : M \preccurlyeq N$ which extends f.

<u>Proof</u>

Add to the language L of T, new names for the elements of P^M to get a countable language L'.
Similarly, expand L_0 to L_0'.
Let $T' = T \cup \{$ P-L_0' sentences $\psi(\vec{a})$ such that $M \models \psi(\vec{a})\}$
Then by Theorem 3, T is a complete theory in L'.
So $T' = Th(M, a)_{a \in P^M}$.
Let us assume, with no loss of generality, that f is an elementary inclusion.
So $(N, a)_{a \in P^M} \models Th(M, a)_{a \in P^M}$.
We may also assume that M is infinite. By lemma 4, every n-tuple of $(M, a)_{a \in P^M}$ realises a principal type of T'. i.e. $(M, a)_{a \in P^M}$ is atomic. Thus, as it is also countable, $(M, a)_{a \in P^M}$ is a prime model of T'. So there is an elementary embedding g of $(M, a)_{a \in P^M}$ into $(N, a)_{a \in P^M}$. So $g : M \preccurlyeq N$ extends f, proving the theorem.

Gaifman [9] quotes Shelah as having extended our result above to the case where M is of any cardinality.
So -

<u>Theorem 6</u> (Shelah) Let T be a Gaifman operation.

Then for any M, N which are models of T and $f : M^P|L_0 \preccurlyeq N^P|L_0$ there is $g : M \preccurlyeq N$ which extends f.

Gaifman [8] gives a slightly weaker version of the following theorem, without proof.

<u>Theorem 7</u> Let T be a Gaifman operation, and $n < \omega$.

Then for any $\phi(x_0, \dots x_{n-1}; \vec{y})$ in L, there is $m < \omega$ and $\psi(z_0, \dots z_{m-1}; \vec{y})$ in L_0, such that for every $M \models T$, there is $f : |M|^n \longrightarrow (P^M)^m$ such that for any $\vec{b} = (b_0, \dots b_{n-1})$ in M and for all $\vec{a} \in P^M$
$M \models \phi(\vec{b}; \vec{a})$ if and only if $M^P|L_0 \models \psi(f(\vec{b}); \vec{a})$.

<u>Proof</u>

Let $\phi(x_0, \dots x_{n-1}; \vec{y})$ be a formula in L. Let $M \models T$ and $\vec{b} = (b_0, \dots b_{n-1}) \in M$. Then, by lemma 4, there is a formula $\theta(\vec{x}; \vec{a})$ which generates the type of $\vec{b}$ in $(M, a)_{a \in P^M}$.

Therefore $M \models (\forall \vec{y} \in P)(\phi(\vec{b}; \vec{y}) \leftrightarrow (\forall \vec{x})(\theta(\vec{x}; \vec{a}) \rightarrow \phi(\vec{x}; \vec{y})))$

Putting $\psi_{\vec{b}}(\vec{z}; \vec{y})$ for $(\forall \vec{x})(\theta(\vec{x}; \vec{z}) \rightarrow \phi(\vec{x}; \vec{y}))$

we have $M \models (\forall \vec{y} \in P)(\phi(\vec{b}; \vec{y}) \leftrightarrow \psi_{\vec{b}}(\vec{a}; \vec{y}))$

so for each $\vec{b} \in M$ there is $\psi_{\vec{b}}(\vec{z}; \vec{y})$ such that

$M \models (\exists \vec{z} \in P)(\forall \vec{y} \in P)(\phi(\vec{b}; \vec{y}) \leftrightarrow \psi_{\vec{b}}(\vec{z}; \vec{y}))$

So $T \vdash \forall \vec{x} \bigvee_{\text{all } \psi_i} (\exists \vec{z}_i \in P)(\forall \vec{y} \in P)(\phi(\vec{x}; \vec{y}) \leftrightarrow \psi_i(\vec{z}_i; \vec{y}))$

By compactness there is $r < \omega$ such that

$$T \vdash \forall \vec{x} \bigvee_{i=1,..r} (\exists \vec{z}_i \in P)(\forall \vec{y} \in P)(\phi(\vec{x};\vec{y}) \leftrightarrow \psi_i(\vec{z}_i;\vec{y}))$$

Define $\psi'(\vec{z}_1, \ldots, \vec{z}_r, z, z_1, \ldots, z_r, \vec{y})$

to be $\bigwedge_{i=1,..r} (z = z_i \rightarrow \psi_i(\vec{z}_i;\vec{y})$

Put $\psi'(\vec{z};\vec{y})$ to be $\psi'(\vec{z}_1, \ldots, \vec{z}_r, z, z_1, \ldots, z_r, \vec{y}))$

Then we easily have

$$T \vdash (\forall \vec{x})(\exists \vec{z} \in P)(\forall \vec{y} \in P)(\phi(\vec{x};\vec{y}) \leftrightarrow \psi'(\vec{z};\vec{y}))$$

The uniform reduction theorem now gives us an L_0 formula $\psi(\vec{z};\vec{y})$ for the formula $\psi'(\vec{z};\vec{y})$. We can easily see that, given $M \models T$ and $\vec{b} \in M$, there if $\vec{c} \in P^M$ such that for all $\vec{a} \in P^M$ $M \models \phi(\vec{b};\vec{a})$ if and only if $M^P | L_0 \models \psi(\vec{c};\vec{a})$.

Let T be a Gaifman operation, and let us define the class of models K to be $\{ A :$ there is $M \models T, M^P | L_0 = A\}$. Looking for a moment at the situation described by Beth's theorem, where we only add new relations and functions to the model, Beth's theorem implies that the class of models which can be expanded to models of the theory in question is a generalised elementary class. This follows by just replacing each new symbol by its defining formula in the smaller language.

Gaifman [8] asks whether an analogous result holds for Gaifman operations. Namely, is K a generalised elementary class. Below, we answer this question affirmatively. What we do is to first prove that $K_{\aleph_0} = \{ A \in K : \|A\| = \aleph_0 \}$ is generalised elementary, in the sense that it is the class of countable models of some theory. Then we use Shelah's Theorem 6 to extend this to models of higher cardinality.

Theorem 8 Let T be a Gaifman operation. Then the class K as defined above is generalised elementary.

Proof

We show that K is the class of models of Th(K). To prove this, it is enough to show that K is closed under elementary equivalence. For let $A \models Th(K)$. If no member of K is elementarily equivalent to A, then for each $B \in K$ there is sentence σ_B such that $B \models \sigma_B$ and $A \models \neg \sigma_B$. So $K \models \bigvee_{B \in K} \sigma_B$. Let σ'_B be the P-L_0 sentence corresponding to σ_B. Then $T \vdash \bigvee_{B \in K} \sigma'_B$

So by compactness there are $B_1, \ldots B_r \in K$, $r < \omega$ such that $T \vdash \bigvee_{i=1..r} \sigma'_{B_i}$

but then $K \models \bigvee_{i=1..r} \sigma_{B_i}$, so $\bigvee_{i=1..r} \sigma_{B_i} \in Th(K)$

But this contradicts the fact that $A \models \neg \sigma_{B_i}$, $i = 1, \ldots r$

and $A \models Th(K)$. So there is $B \in K$ such that $A \equiv B$.
We will prove that K is closed under elementary equivalence. First, some terminology.
Let A be a structure for L_0. Let L_0^A be the language L_0 together with names for the elements of A. We define $Th_P(\hat{A})$ to be the set of those P-L_0^A sentences $\psi(\vec{a})$ which correspond to L_0^A sentences $\psi'(\vec{a})$ for which $A \models \psi'(\vec{a})$.
So for any L-structure M,

$(M,a)_{a\in A} \models Th_P(\hat{A})$ if and only if $(M^P | L_0, a)_{a\in A} \models Th(A,a)_{a\in A}$

Now suppose that $A \equiv B$, and $B \in K$.
Then it is quite easy to see that $T \cup Th_P(\hat{A})$ is consistent.
So to prove that K is closed under elementary equivalence, it suffices to show that -:
whenever $T \cup Th_P(\hat{A})$ is consistent, $A \in K$ (*)

We prove (*) by induction on the infinite cardinality of A. So let A be countable and $T \cup Th_P(\hat{A})$ consistent. By Theorem 3, $T \cup Th_P(\hat{A})$ is complete.
We show that $T \cup Th_P(\hat{A})$ has a model which omits the type $\Sigma(x) = \{Px\} \cup \{x \neq a : a \in A\}$. If not, then by the omitting types theorem, $\Sigma(x)$ is a principal type of $T \cup Th_P(\hat{A})$.
Namely, there is a formula $\psi(x;\vec{a})$ of $L(T \cup Th_P(\hat{A}))$ (where we exhibit all the names of elements of A), such that $T \cup Th_P(\hat{A}) \vdash \psi(x,\vec{a}) \rightarrow \Sigma(x)$.
Now let $(B,a)_{a\in A} \models T \cup Th_P(\hat{A})$. So $(B,a)_{a\in A} \models (\exists x \in P)\psi(x;\vec{a})$.

$(B,a)_{a\in A} \models \psi(b;\vec{a})$ for example, where $b \in P^B$.

Let $\psi'(x;\vec{y})$ be the L_0 formula which corresponds to $\psi(x;\vec{y})$ by the uniform reduction theorem.

Then $(B^P|L_0,a)_{a\in A} \models \psi'(b,\vec{a})$.

But $(A,a)_{a\in A} \preceq (B^P|L_0,a)_{a\in A}$

So $(B^P|L_c,a)_{a\in A} \models \psi'(a,\vec{a})$ for some $a\in A$.

So again $(B,a)_{a\in A} \models \psi(a,\vec{a})$.

But this contradicts the fact that

$$(B,a)_{a\in A} \models \psi(x;\vec{a}) \rightarrow x \neq a \quad \text{for each } a\in A .$$

So $T \cup Th_p(\hat{A})$ has a model M which omits $\Sigma(x)$.

Then $M \models T$, and $M^P|L_0 = A$ whereby $A\in K$.

Now suppose that we have proved (*) for A of cardinality $< \kappa$ Now let A be an L_0-structure, $\|A\| = \kappa$ and $T \cup Th_p(\hat{A})$ be consistent.

There are λ and models A_α for $\alpha < \lambda$ such that $\|A_\alpha\| < \kappa$ for all $\alpha < \lambda$, $\alpha < \beta < \lambda$ implies $A_\alpha \preceq A_\beta$, $A_\alpha \preceq A$ for all $\alpha < \lambda$ and $A = \bigcup_{\alpha<\lambda} A_\alpha$.

It is easy to show that $T \cup Th_p(\hat{A}_\alpha)$ is consistent for all $\alpha < \lambda$. So by the induction hypothesis, for each $\alpha < \lambda$, there is $B_\alpha \models T$ with $A_\alpha = B_\alpha{}^P|L_0$.

By Theorem 6, for each $\alpha < \lambda$, we can easily elementarily embed B_α in $B_{\alpha+1}$, over the elementary inclusion $A_\alpha \preceq A_{\alpha+1}$.

So we can assume that $B_\alpha \leqslant B_{\alpha+1}$, for all $\alpha < \lambda$.
By the condition of T being a Gaifman operation, we can also assume that for δ a limit ordinal,

$$\bigcup_{\alpha<\delta} B_\alpha = B_\delta \ .$$

Let $B = \bigcup_{\alpha<\lambda} B_\alpha$. Then $B \models T$, as $B_\alpha \leqslant B$ for all α.
Also $B^P | L_0 = \bigcup_{\alpha<\lambda} (B_\alpha{}^P | L_0) = \bigcup_{\alpha<\lambda} A_\alpha = A$.

So $A \in K$.

Thus the induction step is completed, and so the theorem is proved.

1.2 Gaifman operations and 1-cardinality.

Viewed abstractly, the way in which the model M is implicitly defined from the model $M^P | L_0$ by the Gaifman operation T, can be regarded from two aspects. One the one hand, the language L_0 is expanded to the language L, and on the other hand new elements are added, and the original model is assigned the unary predicate P. The uniform reduction theorem essentially solves the problems relating to the first aspect (the expansion of the language). So, as expected, the main difficulty arises in trying to work out the relationship between the model M and its P part.

One aspect of this is the question of cardinality.

Definition 9. Let T be a theory, and P a unary predicate in the language of T. Then we say that

(T,P) is 1-cardinal if and only if whenever M is an infinite model of T, then $\|M\| = |P^M|$.

When P is clear from the context, we shall just say that "T is 1-cardinal", to mean the obvious thing.

Wilfrid Hodges has conjectured that T is a Gaifman operation if and only if T satisfies the conclusion of the uniform reduction theorem and T is 1-cardinal. In Example 21 below, we disprove this conjecture.

A very strong tool in the study of 1-cardinality is the following result of Vaught[16].

Theorem 10. (T,P) is 1-cardinal if and only if it is not the case that there are models M and N of T such that $M \preccurlyeq N$, $M \neq N$ and $P^M = P^N$.

Proposition 11. Let T be a Gaifman operation.
Then T is 1-cardinal.

Proof.

If T were not 1-cardinal, then there would be $M \models T$, such that $\|M\| > |P^M| \geq \aleph_0$. (From things we have mentioned before, the case where $|P^M| < \aleph_0$ cannot arise.)

Then, by the Lowenheim-Skolem theorem, there is $N \preccurlyeq M$, with $P^M \subseteq |N|$, and $\|N\| = |P^M|$. But then $P^N = P^M$, whereby $M^P|L_0 = N^P|L_0$. However, there can be no isomorphism between M and N , as $\|N\| < \|M\|$. But this contradicts T being a Gaifman operation.

So the proposition is proved.

We say that a model (not necessarily in a countable language) is minimal, if it has no proper elementary substructure. We say that M is minimal over $M^P|L_0$, if there is no N such that

$$N \preccurlyeq M \ , \ N \neq M \ , \quad \text{and} \quad N^P|L_0 = M^P|L_0 \quad .$$

Then obviously M is minimal over $M^P|L_0$ if and only if the model $(M,a)_{a\in P^M}$ is minimal.

Proposition 12. Let T be a Gaifman operation, and let M be a model of T.

Then M is minimal over $M^P|L_0$.

Proof.

Note first that this follows immediately from Theorem 10 and Proposition 11. However, we can use the strong property of T being a Gaifman operation to do the work of Theorem 10 directly.

For suppose that we had

$$N \preccurlyeq M \ , N \neq M, \quad \text{and} \quad N^P|L_0 = M^P|L_0 \ .$$

We may suppose that $\|M\| = \|N\| = |P^M| = \lambda$,say.

As T is a Gaifman operation, N is isomorphic to M over $M^P|L_0$. We can thus build a strictly increasing, continuous, elementary chain of models

$$\{ M_\alpha : \alpha < \lambda^+ \} \text{ , such that}$$

$$M_\alpha{}^P|L_0 = M^P|L_0 \quad \text{for all} \quad \alpha < \lambda^+ \text{ , and}$$

$$\|M_\alpha\| = \lambda \quad \text{for all} \quad \alpha < \lambda^+.$$

The fact that T is a Gaifman operation allows us to carry on the construction at the limit stage.

$$\text{Let} \quad M' = \bigcup \{ M_\alpha : \alpha < \lambda^+ \} \ .$$

Then $M' \models T$, $M'^P|L_0 = M^P|L_0$, and $\|M'\| = \lambda^+$. But $|P^{M'}| = \lambda$, and so this contradicts the 1-cardinality of T. So M must be minimal over $M^P|L_0$.

We can now put together the above proposition and Theorem 6, to help us characterise Gaifman operations.

<u>Theorem 13</u>. T is a Gaifman operation if and only if whenever $M_i \models T$, $i = 1,2$, and $f: M_1{}^P|L_0 \preccurlyeq M_2{}^P|L_0$, then there is $g: M_1 \preccurlyeq M_2$ which extends f.

<u>Proof</u>.

The direction from left to right is just Theorem 6. For the converse, let us suppose that the condition on extending elementary embeddings holds. Firstly, this implies that T is 1-cardinal. For, if not, then

there would be M,N models of T, such that

$$\|M\| > \|N\| \text{ , and } M^P|L_0 = N^P|L_0 \text{ .}$$

But then we would be unable to elementarily embed M in N. So now let M and N be models of T such that

$$M^P|L_0 = N^P|L_0 \text{ .}$$

Then there is $f: M \preccurlyeq N$ such that f is the identity on $M^P|L_0$. But then by 1-cardinality and Theorem 10, f must be onto, whereby f is an isomorphism of M and N.

Thus T is a Gaifman operation.

Actually the proof of the above proposition tells us something more. Let T be any theory with a unary predicate P. Then -:

Proposition 14. Suppose that for every $M \models T$, $(M,a)_{a \in P^M}$ is a prime model. Then for every $M \models T$, $(M,a)_{a \in P^M}$ is the unique prime model of $Th((M,a)_{a \in P^M})$.

In the next few definitions and results, T will be just a (countable) theory, whose language contains (among other things) a unary predicate P.

If M is a model for the language of T, we will denote by $\hat{M}$ the expanded model $(M,a)_{a \in P^M}$. (So the language of $\hat{M}$ may be uncountable.) So then Theorem 10 just says that T is 1-cardinal if and only if $\hat{M}$ is minimal for all $M \models T$. Proposition 12 and Theorem 13 imply that, if $\hat{M}$ is prime for every model M of T, then $\hat{M}$ is minimal for every model M of T.

This differs from the situation for "fixed" models, where we may have models which are prime, but not minimal. Example 21 below will be, among other things, an example of a theory T such that for all models M of T, $\hat{M}$ is minimal, but for which there are models M, with $\hat{M}$ not prime.

We will first, however, look further into the relationship between the 1-cardinality of a theory T, and the minimality of its expanded models.

Deissler[4] has defined a notion of rank for elements of a model, which enables him to characterise countable minimal models.

His definition is as follows :

<u>Definition 15</u>. Let M be a model (in a language of any cardinality).

The rank in M of an element $a \in M$ over a subset X of M, rk(a,X,M), is defined by induction :
rk(a,X,M) = 0 if there is a formula $\phi(x,\vec{y})$ in L(M), and $\vec{c} \in X$, such that

$$M \models \phi(a,\vec{c}) \wedge \exists^1 x \phi(x,\vec{c}) .$$

For ζ an ordinal larger than 0
rk(a,X,M) = ζ if not rk(a,X,M) = η for $\eta < \zeta$, and if there is $\phi(x,\vec{y})$ and $\vec{c} \in X$ such that

$$M \models \exists x \phi(x,\vec{c}),$$

and such that for all $b \in M$ with $M \models \phi(b,\vec{c})$

$$rk(a,X \cup \{b\},M) < \zeta .$$

We say that rk(a,X,M) = ∞ if there is no ordinal ζ with rk(a,X,M) = ζ. (By convention $\zeta < \infty$ for all ordinals ζ.)

We define rk(a,M) to be rk(a,0,M) , and rk(M) to be $\sup\{ rk(a,M)+1 : a \in M \}$.

<u>Lemma 16</u>. a) Let M be a model (in a language of any cardinality). Then rk(M) $< \infty$ implies that M is minimal.
b) If M is a countable model in a countable language, then rk(M) $< \infty$ if and only if M is minimal .

<u>Proof</u>. Quite straightforward, as for example in Flum[7].

Let A be a model with a unary predicate P. Then we say that A is a 2-cardinal model if

$$\|A\| > |P^A| \geq \aleph_0 .$$

We say that a set of sentences Σ, <u>almost axiomatises</u> a class of structures K, if for any model A, $A \models \Sigma$ if and only if there is $B \equiv A$ such that $B \in K$.

Keisler[12] has given a set of sentences which almost axiomatises the class of 2-cardinal models :

<u>Theorem 17</u>. (Keisler) Let L be a countable language which contains a unary predicate P. Then the following set of sentences Σ almost axiomatises the class of 2-cardinal models for L.

$$\Sigma = \{ \exists v_0 \forall x_0 w_0 \in P \exists y_0 z_0 \ldots\ldots \forall x_n w_n \in P \exists y_n z_n [\bigwedge_{i=0}^{n} v_0 \neq y_i \wedge \bigwedge_{j=0}^{n} \phi_j (x_0, \ldots x_n, z_0, \ldots z_n) \longleftrightarrow \phi_j (y_0, \ldots y_n, w_0, \ldots w_n)] :$$

$$n < \omega,\ m < \omega,\ \phi_0, \ldots \phi_m,\ 2n+1\text{-ary formulae of } L \}$$

It follows that if T is a theory in L, and (T,P) is 1-cardinal, then T ∪ Σ is inconsistent.

Thus there are $\sigma_1, \ldots \sigma_r$ in Σ such that

$$T \vdash \bigvee_{i=1}^{r} \neg \sigma_r .$$

But if σ is in Σ, then ¬σ is a sentence of the form

$$\forall v_0 \exists x_0 w_0 \in P \forall y_0 z_0 \ldots\ldots \exists x_n w_n \in P \forall y_n z_n$$

$$[(\bigwedge_{j=0}^{m} \phi_j (x_0, \ldots x_n, z_0, \ldots z_n) \longleftrightarrow \phi_j (y_0, \ldots y_n, w_0, \ldots w_n)) \longrightarrow \bigvee_{i=0}^{n} v_0 = y_i] ,\ \text{for some } n, m < \omega .$$

Let M be a model for the language of T , and suppose that

$$M \models \neg\sigma .$$

If we look closely at $\neg\sigma$, we can see that this implies that for every $a \in M$ $rk(a,\mathfrak{M}) < n$.

So $rk(\mathfrak{M}) < n+1$.

Now for every model M of T, there is $i \leq r$ such that

$$M \models \neg\sigma_i .$$

Thus there is some $n_1 < \omega$ such that

$rk(\mathfrak{M}) < n_1$ for all $M \models T$. So we have proved :

Proposition 18. Let T be a (countable) theory which has a unary predicate P.

Then (T,P) is 1-cardinal if and only if there is $n \leq \omega$ such that $rk(\mathfrak{M}) < n$, for all $M \models T$.

Let us now return to the context of Gaifman operations. Namely T is a theory in a countable language L, P is a unary predicate in L, and $L_0 \subseteq L - \{P\}$.

Definition 19. a) T has the uniform reduction property if for any $\phi(\vec{x})$ in L there is $\psi(\vec{x})$ in L_0, such that

for all $M \models T$ and for all $\vec{a} \in P^M$,

$M \models \phi(\vec{a})$ if and only if $M^P|L_0 \models \psi(\vec{a})$.

b) T is 1-cardinal of rank n if n is the least natural number such that $rk(\mathfrak{M}) < n$ for all $M \models T$.

As mentioned before, Wilfrid Hodges conjectured that T is a Gaifman operation if and only if T is 1-cardinal and has the uniform reduction proprty. The direction left to right is given by Proposition 11 and Theorem 3. The opposite direction does not hold. However, it does hold if we stipulate that T is 1-cardinal of rank 1 .

<u>Proposition 20</u>. Suppose T is 1-cardinal of rank 1, and has the uniform reduction property.

Then T is a Gaifman operation.

<u>Proof.</u>

Let M_1, M_2 be models of T, such that

$$M_1^P | L_0 = M_2^P | L_0 = M_0 .$$

As T has the uniform reduction property, it follows that

$$(M_1, a)_{a \in M_0} \equiv (M_2, a)_{a \in M_0}.$$

As the model $(M_1, a)_{a \in M_0}$ has rank 1 , then for every element b of M_1 there is a formula $\psi(x)$ of $L((M_1, a)_{a \in M_0})$, such that $M_1 \models \psi(b) \wedge \exists^1 x \psi(x)$.

Similarly for $(M_2, a)_{a \in M_0}$.

So let $b \in M_2$, and $\psi(x)$ define b as above.

Then $M_2 \models \exists^1 x \psi(x)$. Suppose that $M_2 \models \psi(c)$.

Then we put $f(b) = c$.

It is easily seen that the map $f: M_1 \rightarrow M_2$, thus defined is an isomorphism, and that $f(a) = a$ for all a in M_0. Thus T is a Gaifman operation.

However in the general case, we have a counter-example.

Our example is actually based on an idea of Shelah[21] . Shelah gives an example of a countable non-prime minimal model, which has $2^{\aleph_0}$ models which are elementarily equivalent to it and minimal.

In what follows ${}^{\omega}2$ will denote the set of functions from ω to 2. ${}^{\omega>}2$ will denote the set of functions from n to 2, for $n < \omega$.

Example 21. A theory T in a countable language L, with a unary predicate P and a sublanguage $L_0 \subseteq L - \{P\}$, such that T is 1-cardinal, T has the uniform reduction property, but T is not a Gaifman operation.

We will define a model M, and T will be Th(M). L, the language of M, will have as its non-logical symbols , a unary predicate letter P, a unary predicate letter Q_ν for each $\nu \in {}^{\omega>}2$, and a binary operation letter + . L is then a countable language.

Let us fix $\eta_0 \in {}^{\omega}2$.

Then we will put $P^M = \{ \sigma \in {}^{\omega}2 : (\exists k<\omega)(\forall n>k)\sigma(n)=0 \}$

And $(\neg P)^M = \{ \eta \in {}^{\omega}2 : (\exists k<\omega)(\forall n>k)\eta(n)=\eta_0(n) \}$

So intuitively, the elements of the P part of M are the sequences of 0's and 1's of length ω which are eventually 0. And the rest of M consists of sequences which are eventually the same as η_0.

If ν_1, ν_2 are in ${}^{\omega}2 \cup {}^{\omega>}2$, then we will write $\nu_1 \lessdot \nu_2$ to mean that ν_1 is an initial segment of ν_2. Then for all $\sigma \in M$, we stipulate that

$M \models Q_\nu \sigma$ if and only if $\nu \lessdot \sigma$, for all $\nu \in {}^{\omega>}2$.

And for η_1, η_2, η_3 in M

$M \models \eta_1 + \eta_2 = \eta_3$ if and only if

$\eta_1(n) + \eta_2(n) = \eta_3(n) \pmod 2$, for all $n < \omega$.

In this example the sublanguage L_0 will just consist of the language L without the predicate P. This is all right as in the model M, P^M is closed under the operation + . So the same will be true in any model of T = Th(M) .

Also, note that for any $\eta \in (\neg P)^M$

$(\neg P)^M = \{ \sigma + \eta : \sigma \in P^M \}$

Thus $M \models \forall x \forall y \exists z \in P(x + z = y)$.

So $T \vdash \forall x \forall y \exists z \in P(x + z = y)$

So any model N of T is generated by one element over P^N. So <u>T is 1-cardinal</u>. Actually we can see that T is 1-cardinal of rank 2.

To prove the rest, we need a set of axioms for T. So we propose the following :

1) $(\exists x)(Px \wedge Q_\nu x)$, $(\exists x)(\neg Px \wedge Q_\nu x)$ — for each $\nu \in {}^{\omega>}2$.

2) $(\forall x)(Q_{\nu_1} x \rightarrow Q_{\nu_2} x)$ whenever $\nu_1 \lessdot \nu_2$.

3) $(\forall x)(Q_\nu x \rightarrow (Q_{\nu\frown\langle 0\rangle} x \vee Q_{\nu\frown\langle 1\rangle} x))$ for all $\nu \in {}^{\omega>}2$.

4) $(\forall x)(\neg(Q_{\nu\frown\langle 0\rangle} x \wedge Q_{\nu\frown\langle 1\rangle} x))$ for all $\nu \in {}^{\omega>}2$.

5) $(\forall xyz)(x + y = z \wedge Px \wedge Py \rightarrow Pz)$

6) $(\forall xyz)(x + y = z \wedge Px \wedge \neg Py \rightarrow \neg Pz)$

7) $(\forall xyz)(x + y = z \wedge \neg Px \wedge \neg Py \rightarrow Pz)$

8) $(\forall xy)(x + y = y + x)$

9) $(\forall xyz)(x + (y + z) = (x + y) + z)$

10) $(\forall xyz)(x + y = z \rightarrow x = y + z)$

11) $(\forall xyz)(x + y = z \wedge Q_{\nu_1} x \wedge Q_{\nu_2} y \rightarrow Q_{\nu_3} z)$

where ν_1, ν_2 and ν_3 are in ${}^{\omega>}2$, length(ν_3) = min(length(ν_1),length(ν_2)) ,

and $\nu_3(i) = \nu_1(i) + \nu_2(i) \pmod 2$ for all $i <$ length(ν_3).

Let Σ be the set of sentences 1) to 10).

It is obvious that $\Sigma \subseteq T$.

It is routine to show that Σ has elimination of quantifiers, i.e. that for any formula $\phi(\vec{x})$, there is a quantifier free formula $\psi(\vec{x})$, such that

$$T \vdash \phi \longleftrightarrow \psi.$$

The main point to note in doing the quantifier elimination is that a formula of the sort

$(\exists x)(Q_{\nu_1} x \wedge Q_{\nu_2}(x+y))$ is equivalent under Σ to the quantifier free formula $Q_{\nu_3} y$, where length(ν_3) = min(length(ν_1),length(ν_2)) and $\nu_3(i) = \nu_1(i) + \nu_2(i) \pmod 2$ for all $i <$ length(ν_3).

It fellws that Σ must be complete, whereby Σ does axiomatise T. So T also has elimination of quantifiers.

We now show that T has the uniform reduction property.

So let $\phi(\vec{x})$ be a formula of L such that

$$T \cup \{(\exists \vec{x} \in P)\phi(\vec{x})\} \text{ is consistent.}$$

By elimination of quantifiers there is a quantifier free formula of L, $\psi(\vec{x})$, such that

$$T \vdash \forall \vec{x}(\phi(\vec{x}) \longleftrightarrow \psi(\vec{x})).$$

We can then easily get a quantifier free formula $\psi'(\vec{x})$ of $L - \{P\}$, such that

$$T \vdash \forall \vec{x} \in P(\phi(\vec{x}) \longleftrightarrow \psi'(x)).$$

But then, for any model N of T, and $\vec{a} \in P^N$,

$N \models \phi(\vec{a})$ iff $N \models \psi'(\vec{a})$ iff $N^P | L_0 \models \psi'(\vec{a})$.

(as ψ is quantifier free and $L_0 = L - \{P\}$)

So T has the uniform reduction property.

It just remains to show that T is not a Gaifman operation.

Remember that to define the model M, we began with a fixed $\eta_0 \in {}^{\omega}2$. Let us now choose $\eta_1 \in {}^{\omega}2$ such that for all $k<\omega$ there is $i>k$ such that

$$\eta_1(i) \neq \eta_0(i) \ .$$

Now define a model M′ from η_1, exactly as we defined M from η_0. It is easily checked that M′ satisfies the axioms Σ, whereby M′ is a model of T. It is also clear that

$(M')^P = M^P$, as the P part of M was defined independently of η_0.

However each element of $(\neg P)^M$ is different at arbitrarily large points from each element of $(\neg P)^{M'}$, and thus $M \not\cong M'$. In fact there are $2^{\aleph_0}$ pairwise non-isomorphic models N with $N^P = M^P$, and $N \models T$.

Chapter 2

Minimal models and the number of countable models

2.0 Introduction

The original motivation behind this chapter is the attempt to determine the possible number of countable models up to isomorphism of a complete countable theory which has a minimal model. The conjecture is that such a theory has infinitely many countable models. Such a result would strengthen the Baldwin-Lachlan theorem, which says that an $\aleph_1$-categorical non $\aleph_0$-categorical theory has $\aleph_0$ countable models. What we end up proving, however, are some comparatively weak results on the number of countable models of a theory with a certain kind of "very algebraic" prime model. We show that such a theory has at least four countable models. Now it is known that $n(T)$(= number of countable models of a countable theory T) can never be equal to two. So if $n(T)>1$ then $n(T) \geq 3$. Thus to show that a theory has at least four countable models is the weakest possible nontrivial result.

Essentially the only known example of a theory T with $n(T)=3$, is the "Ehrenfeucht example". And the examples of T with $n(T)$ finite are modifications of this example. We show that any theory T for which n(T)=3 is quite a bit like the Ehrenfeucht example.

It has been thought that one could obtain a nice characterisation of those theories T for which n(T) is finite, analogous to the Ryll-Nardjewski characterisation of $\aleph_0$-categorical theories. I think that such a neat characterisation is unlikely to be found, partly because theories with more than one, but only finitely many countable models are such an anomaly. Any characterisation will probably be of a rather complicated structural nature. However, if we look at $\aleph_0$-categorical theories, we can, rather crudely, divide them into -

a) those theories which are $\aleph_0$-categorical because of lack of structure (e.g. theory of equality, theory of infinite abelian groups of order p), and

b) those theories which are $\aleph_0$-categorical due to the presence of structure (e.g. theory of dense linear orderings, theory of atomless Boolean algebras).

In case a) there is nothing to distinguish countable models of the theory. Whereas in case b) there is enough going on in the models to enable us to construct isomorphisms. The feeling is then that theories T with n(T) greater than one, but finite, arise from modifications of $\aleph_0$-categorical theories of type b), as for example Ehrenfeucht's example comes from adding a sequence of constants to a dense linear ordering.

I present a general framework for obtaining or constructing non-isomorphic countable models. This essentially centres around the presence in our theory of certain exchange properties, which allow us to get models of arbitrary finite "dimension". As stated above, under certain strong assumptions on the degree of algebraicity of the prime model, we have as yet only been able to obtain at least four models. However I also prove a quite general exchange result, which under quite strong conditions on the algebraicity of the <u>theory</u> (namely that every model prime over a finite set is actually algebraic over that set),enables us to get infinitely many countable models. This latter result, whereby one obtains infinitely many models has been proved directly by Lascar[14], but I feel that the above-mentioned exchange result is fairly interesting for it's own sake.

As for minimal models, we view minimality(of a model) as a generalisation of algebraicity. In the case of a model which is algebraic, one can see directly what is responsible for it's minmality, so we would like to connect the two notions. I actually show that a minimal prime model has a large part which is algebraic over a finite set. This also connects our original conjecture to the later results on the number of countable models, although there are obviously many gaps to be filled in order to prove the conjecture.

I will first state a few preliminary definitions and observations. In this chapter all models and theories that we talk about, will be in a countable language. Models will be infinite unless otherwise stated.

Definition 22. We say that a model A is minimal if there is no B such that $B \preccurlyeq A$ and $B \neq A$.

It follows that a minimal model is countable.

Definition 23. (i) Let A be a model and $a \in A$. We say that a is algebraic in A, if there is a fomula $\theta(x)$ of $L(A)$, and $n < \omega$ such that $A \models \exists^n x \theta(x) \wedge \theta(a)$.

(ii) A is algebraic if for all $a \in A$, a is algebraic in A.

(iii) Let T be a complete theory. Then T is algebraic if T has an algebraic model.

It is easy to see that if a model is algebraic then it is prime and minimal.

Let T be a theory. Then as mentioned before we denote by $n(T)$ the number of countable models of T up to isomorphism. It would be worthwhile to state and prove the following classical result of Vaught[25].

Theorem 24. Let T be a complete theory. Then $n(T) \neq 2$.

Proof. Let us assume that $n(T) > 1$, and $n(T) \leq \aleph_0$. We will show that $n(T) \geq 3$. First of all then, T can only have $\aleph_0$ types, for otherwise we would get too many models. Thus T has a prime model and a countable saturated model. Also as T is not $\aleph_0$-categorical, there must be a non-principal n-type $p(\vec{x})$ for some $n < \omega$, whereby the prime and saturated models cannot be isomorphic. Let $T' = T \cup p(\vec{c})$, where $\vec{c}$ is a sequence of n new constants. Then T′ has again only $\aleph_0$ types, and thus has a prime model $(A,\vec{c})$. A is a countable model of T. As A realises $p(\vec{x})$, A is not prime. Now T′ must have some non-principal n-type $q(\vec{c},\vec{x})$ (because T, and thus T′ has infinitely many n-types). $(A,\vec{c})$ omits this type, and thus A cannot be saturated. Thus we have at least three countable models of T.

<u>Observation 25</u>. Let T be a complete theory with no prime model. Then $n(T) = 2^{\aleph_0}$.

<u>Proof</u>.

T must be non-atomic , and thus there is some n-formula $\theta(\vec{x})$ which is not implied by any complete n-formula of T over T. By a standard tree method we can get $2^{\aleph_0}$ n-types of T, and so T must have at least $2^{\aleph_0}$ countable models , to realise all these types. But $n(T) \leq 2^{\aleph_0}$, so the result follows.

<u>Observation 26.</u> Let T be a complete theory with a prime model. Suppose that $A \models T$, and A is minimal. Then A is prime.

Proof.

Let B be the prime model of T. Then $B \leqslant A$, and so $B = A$ by minimality of A.

Note from this that if T has a prime model, then T has at most one minimal model up to isomorphism.

Also, in so far as we are interested in the number of countable models of a theory with a minimal model, we can by Observations 25 and 26, assume that the minimal model of the theory is prime. Thus in the section following, we restrict our attention to prime minimal models.

2.1 Prime Minimal Models.

Proposition 27. Let T be a complete atomic theory, and A be a countable model of T. Then A is minimal if and only if A is atomic and has no atomic proper elementary extension.

Proof.

Let A be minimal. Then as T has a prime model, A is prime and thus atomic. Suppose that we had $B \succcurlyeq A$, $B \neq A$, and B atomic. We may take B to be countable, for if not take a countable elementary substructure. But then $A \simeq B$, so we could find $C \preccurlyeq A$, $C \neq A$, contradicting the minimality of A.

Conversely, suppose A were atomic and not minimal, There would be $B \preccurlyeq A$, $B \neq A$. But then obviously B would also be atomic, and thus $B \simeq A$, so we could find C, with $A \preccurlyeq C$, $A \neq C$, and C atomic.

The above proposition says that if A is a prime model, then A is minimal if and only if, whenever $A \preccurlyeq B$, and $A \neq B$, there is $n < \omega$ and an n- tuple $\vec{b}$ from $|B|^n - |A|^n$ which realises a non-principal n-type in B.

Compare this with the situation for algebraic models.

Observation 28. Let A be a prime model. Then A is algebraic if and only if whenever $A \preccurlyeq B$ and $A \neq B$, every element of B − A realises a non-principal type.

<u>Proof</u>.

Note that if A is algebraic, then for every complete 1-formula $\theta(x)$ of $Th(A)$, there is $n < \omega$ such that $A \models \exists^n x\theta(x)$. So if $A \preccurlyeq B$, all the realisations of θ must be in A. So if $b \in B-A$ then b cannot realise a principal type.

Conversely, suppose A were not algebraic. Then there is a complete formula $\theta(x)$ (i.e. complete for $Th(A)$) which is satisfied by infinitely many elements of A. Consider the following set of sentences in the language got by adding names for elements of A and a new constant c

$\Sigma = Th(A,a)_{a \in A} \cup \theta(c) \cup \{ c \neq a : a \in A \}$.

By compactness Σ has a model $(B,a,c)_{a \in A}$. Then $A \preccurlyeq B$, $c \in B-A$, and c realises the principal type of $Th(A)$ determined by $\theta(x)$.

<u>Proposition 29</u>. Let T be a complete atomic theory. Then T has a minimal model if and only if all atomic models of T are countable.

<u>Proof</u>.

Suppose that A is a minimal model of T. Then A is prime, and if there were an uncountable atomic model of T, we would have $A \preccurlyeq B$, $A \neq B$, contradicting Proposition 27.

Conversely, suppose that T had no minimal model. Let A be the prime model of T. Then by Proposition 27 A would have an atomic proper elementary extension A_1. If A_1 is uncountable, there is nothing more to prove. If A_1 is countable, then $A \simeq A_1$, and we can therefore build a

strictly increasing continuous elementary chain $\{A_i : i<\aleph_1\}$ of countable atomic models of T. (We can continue at the limit stage, because for δ a limit ordinal, $\delta<\aleph_1$, $A_\delta = \bigcup\{A_i : i<\delta\}$ is a countable atomic model of T, and is thus isomorphic to A_0.)
Put $A' = \bigcup_{i<\aleph_1} A_i$. Then A' is an uncountable model of T. Also, any finite tuple from A' is in A_i for some $i<\aleph_1$, and so realises a principal type in A_i and so also in A'.
Thus A' is atomic.

Proposition 30. Let T be a complete theory. Then
a) if T has a minimal model, then T is not $\aleph_0$-categorical.
b) if T is $\aleph_1$-categorical and not $\aleph_0$-categorical, then T has a minimal model.

Proof.

a) Suppose T were $\aleph_0$-categorical. Then T would be atomic, and moreover by Ryll-Nardjewski, all types of T would be principal. Thus all models of T would be atomic. So by the previous proposition, T could not have a minimal model.

b) Let T be $\aleph_1$-categorical and not $\aleph_0$-categorical. By $\aleph_1$-categoricity, T is atomic. By non-$\aleph_0$-categoricity, T has a non-principal type p. This type p will be realised in some, and thus in all, models of T of cardinality $\aleph_1$. Thus it is easy to see that T has no uncountable atomic model. So by the previous proposition, T has a minimal model.

Proposition 31. Let T be a complete theory, and p a not necessarily complete type, such that all models of T which omit p are isomorphic. Then if a model A of T omits p, A is prime and minimal.

Proof.

Let A be a model of T which omits p. If A is not prime, then A realises some non-principal type q. But then, by the Omitting Types Theorem, T has a model B which omits both p and q. But then B cannot be isomorphic to A. Contradiction. So A must be prime. T could not have an uncountable atomic model, for such a model would omit p, but would be non-isomorphic to A. Thus by Proposition 29 A is minimal.

We now come towards the main result of this section. We first need a few more definitions.

Definition 32. (i) Let A be a model. We say that A′ is a principal expansion of A, if $A' = (A, \vec{a})$, where $\vec{a}$ is a finite tuple from A which realises a principal type.

(ii) Let T be a complete theory. We say that T′ is a principal extension of T, if $T' = Th((A, \vec{a}))$, where $(A, \vec{a})$ is a principal expansion of some model A of T.

Note that if A is a prime model, then every expansion of A got by adding finitely many names, is a principal expansion.

Definition 33. Let A be a model, and $X \subset |A|$.
Then $cl(X,A) = \{ a\in A$: there is a formula $\phi(x;\bar{y})$ of $L(A)$, $\bar{b}\in X$, and $k<\omega$ such that $A \models \exists^{k} x\phi(x;\bar{b})$ and $A \models \phi(a;\bar{b}) \}$.
If $a\in cl(X,A)$, we say that a is algebraic over X in A.
If $X = \{b\}$, we say that a is algebraic over b in A, and if $X = 0$, we just say that a is algebraic in A.

Note that to say that A is algebraic, is just to say that $cl(0,A) = A$.
We can now relate minimality to algebraicity.

Theorem 34. Let A be a prime minimal model. Then A has a principal expansion A′, such that in $L(A')$ there is a formula $\phi(x)$, such that $\phi^{A'}$ is infinite and $\phi^{A'} \subseteq cl(0,A')$.
Proof.

By Proposition 27, A has no atomic proper elementary extension. Let L be L(A). Let us add names for all the elements of A and a new constant c, so as to expand L to L″. Consider the following theory in L″ :
$T'' = Th((A,a)_{a\in A}) \cup \{c\neq a : a\in A\}$. Then the L-reduct of any model of T″ is a proper elementary extension of A.
Thus no model of T″ can be atomic, when viewed as a model of Th(A). For each $n<\omega$, put

$\Sigma_n(x_1,..x_n) = \{\neg\psi(x_1,..x_n) : \psi$ a complete n-formula of Th(A) $\}$.

Then it is easy to see that the L-reduct of a model of T″ is atomic if and only if the model omits $\Sigma_n(x_1,..x_n)$ for all $n<\omega$. So by the Omitting Types Theorem, there is $n<\omega$ such that $\Sigma_n(\bar{x})$ is principal over T″.

So, there are a finite tuple $\vec{a}\in A$, and an L-formula $\phi(\vec{x},y,\vec{z})$ such that :

(i) $T'' \cup \{\phi(\vec{x},c,\vec{a})\}$ is consistent, and

(ii) $T'' \vdash \phi(\vec{x},c,\vec{a}) \rightarrow \neg\psi(\vec{x})$, for each $\neg\psi(\vec{x}) \in \Sigma_n(\vec{x})$.

By (i), $\{ y\in A : A \models \exists\vec{x}\phi(\vec{x},y,\vec{a}) \}$ is an infinite set.

By (ii), for each $\neg\psi(x)\in\Sigma_n$, there is by compactness $b_1,\dots b_r$ in A such that

$Th((A,a)_{a\in A}) \cup \bigwedge_{i=1\dots r} c\neq b_i \vdash \phi(\vec{x},c,\vec{a}) \rightarrow \neg\psi(\vec{x})$.

So $Th((A,a)_{a\in A}) \vdash \phi(\vec{x},c,\vec{a}) \wedge \psi(\vec{x}) \rightarrow \bigvee_{i=1\dots r} c=b_i$.

So by syntax or semantics, there is k<r such that

$Th((A,\vec{a})) \vdash \exists^k y \exists\vec{x}(\phi(\vec{x},y,\vec{a}) \wedge \psi(\vec{x}))$.

Now we take $(A,\vec{a})$ to be the principal expansion A' of A that we wanted to find.

Then the formula $\exists\vec{x}\phi(\vec{x},y,\vec{a})$ is a formula of $L(A')$ and is satisfied by infinitely many elements of A'.

It remains to show that every element of A' satisfying $\exists\vec{x}\phi(\vec{x},y,\vec{a})$ is algebraic in A'.

So let $A' \models \exists\vec{x}\phi(\vec{x},b,\vec{a})$. But every n-tuple of A' satisfies a principal n-type of Th(A), so there is some complete n-formula $\psi(\vec{x})$ of Th(A) such that

$A \models \exists\vec{x}(\phi(\vec{x},b,\vec{a}) \wedge \psi(\vec{x}))$.

But from above, there is $k<\omega$ such that

$A \models \exists^k y \exists\vec{x}(\phi(\vec{x},y,\vec{a}) \wedge \psi(\vec{x}))$, whereby $b\in cl(0,A')$.

This proves the theorem.

Proposition 35. Let A be a homogeneous model with a principal expansion $(A,\vec{a})$ which is minimal. Then A is minimal.

Proof.

Let A be as in the hypothesis. Suppose that $B \preccurlyeq A$. As $\vec{a}$ realises a principal type, there is $\vec{b} \in B$ such that $(B,\vec{b}) \equiv (A,\vec{a})$. Thus $(A,\vec{b}) \equiv (A,\vec{a})$, and so by homogeneity of A, $(A,\vec{b}) \simeq (A,\vec{a})$. Thus $(A,\vec{b})$ is minimal. But $(B,\vec{b}) \preccurlyeq (A,\vec{b})$, and so B = A. So A is minimal.

Corollary 36. Let A be a model with a principal expansion $(A,\vec{a})$ which is minimal and prime. then A is minimal and prime.

Proof.

Firstly it is clear that, as $(A,\vec{a})$ is prime and $\vec{a}$ realises a principal type in A, then A must also be prime. Thus A is also homogeneous. The result now follows from Proposition 35.

Corollary 37. Let A have a principal expansion $(A,\vec{a})$ which is algebraic. Then A is minimal and prime.

Proof.

Note that $(A,\vec{a})$ is minimal and prime. Now use Corollary 36.

We are interested in the extent to which the implication in Corollary 37 can be reversed. Theorem 34 gives a partial result in this direction, by showing that a prime minimal model has a principal expansion with a large definable algebraic bit. In a special case we can get a stronger result.

First of all, we make some more definitions.

Definition 38. Let A be a model, and $\phi(x)$ a formula in L(A). Then we say that $\phi(x)$ is minimal in $(A,a)_{a\in A}$ if

(i) $\{ x\in A : A\models\phi(x) \}$ is infinite, and

(ii) for each formula $\psi(x;\vec{y})$ of L(A) and $\vec{a}\in A$, either

$\{ x\in A : A\models\phi(x)\wedge\psi(x;\vec{a}) \}$ is finite, or

$\{ x\in A : A\models\phi(x)\wedge\neg\psi(x;\vec{a}) \}$ is finite.

Then as in the literature, a complete theory T is said to be strongly minimal if for every model A of T, the formula 'x=x' is minimal in $(A,a)_{a\in A}$.

Proposition 39. Let A be a model such that 'x=x' is minimal in $(A,a)_{a\in A}$. Then A is minimal and prime if and only if A has a principal expansion which is algebraic.

Proof.

One direction is given by Corollary 37.

For the other direction, let A be minimal and prime. Theorem 34 then gives us a tuple $\vec{a}$ in A, and a formula $\phi(x)$ of $L((A,\vec{a}))$, such that $\phi^{(A,\vec{a})}$ is infinite and is a subset of $cl(\vec{a},A)$. As 'x=x' is minimal in $(A,a)_{a\in A}$, it must be the case that $\neg\phi^{(A,\vec{a})}$ is finite.

But then $A = \phi^{(A,\vec{a})} \cup \neg\phi^{(A,\vec{a})} \subseteq cl(\vec{a},A)$.

Thus $(A,\vec{a})$ is algebraic.

Corollary 40. Let T be a strongly minimal theory. Then T has a minimal model if and only if T has a principal expansion T′ with an algebraic model.

It is interesting to note that the above Corollary can also be deduced from the Marsh-Baldwin-Lachlan framework in the following way :

Assuming T to be strongly minimal, let A be a minimal model of T. A will be prime. If the(Baldwin-Lachlan) dimension of the universe in A is infinite, then every countable model of T will have infinite dimension,whereby T will be $\aleph_0$-categorical. But this contradicts the fact that T has a minimal model. So A must have finite dimension. But this just means that there is a finite tuple $\vec{a} \in A$, with $(A, \vec{a})$ algebraic.

It is easy to find examples which show that the conclusion of Proposition 39 does not in general hold. We can just put together a lot of minimal models. For example, let our model consist of ω disjoint copies of $(\mathbb{Z}, <)$, each copy distinguished by a unary predicate. Then the model is minimal, but it cannot be algebraic over any finite set.

2.2 The number of countable models.

There are very few examples known of theories with more than one, but only finitely many countable models. Such a theory would thus seem to be a pathological case. Moreover all the examples are more or less modifications of the original Ehrenfeucht example, which gives a theory T with $n(T) = 3$. We now give this theory.

<u>Example</u>

Let $A = (A, <, a_i)_{i<\omega}$ be a countable model, where $<$ is a dense linear ordering without endpoints, $A \models a_i < a_j$ iff $i<j$, for all $i,j < \omega$, and the a_i are unbounded above in A. We put $T = Th(A)$. Then T has just three countable models.

A is the prime model.

The 'middle model' A_1 is such that $\{ x \in A_1 : A_1 \models a_i < x$ for all $i \}$ is non-empty and has a first element c. A_1 is actually prime over c.

The third model A_2 is saturated, and $\{ x \in A_2 : A_2 \models a_i < x$ for all $i \}$ is non-empty, but has no first element.

We take the opportunity to observe that in the model A_1, if $d > c$, then d realises a principal type over c, but c does not realise a principal type over d. This is, in a sense, what is <u>responsible</u> for the fact that $n(T) = 3$.

One can modify the above example to get a larger finite number of countable models, by adding for any n say, a set of n unary predicates P_i, $i<n$, which partition the model A, and each of which is dense in A. Then we get n 'middle models', like A_1, but distinguished from each other by which of the P_i holds for $c = \lim_{n<\omega} a_n$. Altogether therefore we have n+2 countable models.

Lachlan has modified the example in a slightly different way to obtain a theory T with n(T) = 6. What he does is to add to the dense linear ordering two sequences of constants, one going up, and the other going down, and all the members of the first sequence less than all the members of the second. The countable models of the theory are then determined by whether the interval between the two sets of constants is empty, open, half-open, etc. Peretyat'kin[17] has given an example of a theory T with n(T) = 3, by adding a sequence of constants to a certain kind of dense tree. Woodrow[24] has shown that if T is a countable complete theory in the same language as the Ehrenfeucht example, and with elimination of quantifiers, then n(T) = 3 implies that T is very much like the Ehrenfeucht example. I show below that any theory T such that n(T) = 3, is 'similar to' the Ehrenfeucht example.

Some other studies have been made of theories with more than one but finitely many countable models. Rosenstein[18] showed that any such theory has a countable model which is not saturated, but realises all types of the theory. Benda[2] has shown that, if, not only T but also every complete extension of T by finitely many constants, has only finitely many countable models, and T is not $\aleph_0$-categorical then T has a countable universal model which is not saturated. This method of placing conditions on all simple extensions of a theory, is rather artificial, but enables one to prove results by iterating certain constructions. We examine later on, what happens when every complete simple extension of a theory is algebraic.

There have been only a few non-trivial results telling us when a theory has infinitely many countable models. Baldwin and Lachlan[1] proved that if T is $\aleph_1$-categorical and not $\aleph_0$-categorical, then $n(T) = \aleph_0$. Lachlan[13] strengthened this by proving that if T is superstable and not $\aleph_0$-categorical, then $n(T) \geq \aleph_0$. Both proofs rely very heavily on the stability of the theories in question, and the proof of the former result relies a lot on the existence of a strongly minimal formula in a principal extension. We would like to prove results without any stability assumptions. Lascar[14] proves <u>essentially</u> that <u>if every</u> complete simple extension of a theory T is algebraic, then $n(T) \geq \aleph_0$. This follows from some lemmas that he proves on Cantor-Bendixon ranks of types of the theory. I will rework some of the Lascar material in a more model-theoretic way, proving an interesting exchange result while doing so.

However if we place algebraicity conditions <u>only</u> on the prime model of the theory, then it looks to be much more difficult to <u>prove</u> that there are many countable models. I get some comparatively weak results below.

And from the proofs of these results, it seems that any attempt to push the results further will involve one in many combinatorial problems.

However, I first present a general schema for obtaining non-isomorphic countable models.

2.2 I A general framework for getting models.

Let us first note that if all the countable models of a complete theory are homogeneous, then the theory must have infinitely many countable models. This follows at once from Rosenstein's result mentioned above, for if a countable full model is homogeneous, then it must be saturated. However, this criterion is not all that helpful, for there are an abundance of theories with infinitely many countable models, not all of which are homogeneous. Look, for example at the theory $T = Th((Z,<))$. We get lots of countable models of T by adding extra copies of $\mathbb{Z}$. However, the model $\mathbb{Z} + \mathbb{Z}$ is not homogeneous. For any element in the first copy realises the same type as any element in the second copy, but there can be no automorphism of the model taking the one element to the other. A more helpful observation which is concerned rather with relative homogeneity, is the following—

Lemma 41. Let T be a complete theory which is not $\aleph_0$-categorical. Suppose that if $A \models T$, $\bar{a}$ is a finite sequence from A, and $(A,\vec{a})$ is prime, then for all $\vec{b}$ in A of the same length as $\vec{a}$, $(A,\vec{a}) \equiv (A,\vec{b})$ implies that $(A,\vec{a}) \cong (A,\vec{b})$. Then $n(T) \geq \aleph_0$.

Proof.

First we may assume that T has $\leq \aleph_0$ n-types, for all $n < \omega$. For otherwise we will have to have more than countably many countable models, to fit in all these types. It also follows that any complete simple extension of T has only $\aleph_0$ types. Thus, every complete simple extension of T has a prime model.

Now, as T is not $\aleph_0$-categorical, T has infinitely many n-types for some $n < \omega$. W.n.l.o.g. assume n to be 1. So let $p_1(x_1)$ be a non-principal 1-type of T. Let c_1 be a new constant. Then again $T_1 = T \cup p_1(c_1)$ has infinitely many 1-types, so we can find $p_2(c_1,x)$ a non-principal 1-type of T_1. Proceeding inductively, we can thus find n-types $p_n(x_1,\dots x_n)$ of T, for $1 \leq n < \omega$, and corresponding theories $T_n = T \cup p_n(c_1,\dots c_n)$, such that, for all n $p_n(x_1,\dots x_n) \subseteq p_{n+1}(x_1,\dots x_{n+1})$, and $p_{n+1}(c_1,\dots c_n,x)$ is a non-principal 1-type of T_n.

Now, for each n let A_n' be a prime model of T_n, and let A_n be the L(T)-reduct of A_n'. Each A_n is then a countable model of T, and we assert that $m \neq n$ implies that A_m is not isomorphic to A_n.

Suppose, by way of contradiction, that for some n $A_n \cong A_{n+1}$. As A_{n+1}' is a model of T_{n+1}, then there are $c_1,\dots c_{n+1}$ in A_n such that $(A_n,c_1,\dots c_{n+1}) \models T_{n+1}$. But there are $b_1,\dots b_n$ in A_n such that $(A_n,b_1,\dots b_n)$ is a prime model of T_n. By construction of the T_i,

$(A_n, b_1, \ldots b_n) \equiv (A_n, c_1, \ldots c_n)$,whereby $(A_n, b_1, \ldots b_n) \cong (A_n, c_1, \ldots c_n)$,by the conditions of the lemma. So $(A_n, c_1, \ldots c_n)$ is prime. But this contradicts the fact that c_{n+1} realises a non-principal type over $(c_1, \ldots c_n)$ in A_n.

Thus it is easy to see that $\{ A_m : 1 \leq m < \omega \}$ is a set of pairwise non-isomorphic countable models of T.

Along similar lines we have :

Lemma 42. Let T be a complete theory which has a model A such that : 1) A is not prime
2) there is a finite tuple $\vec{a}$ in A such that $(A, \vec{a})$ is prime,
3) for any $\vec{b}$ in A such that $(A, \vec{a}) \equiv (A, \vec{b})$, it is the case that $(A, \vec{a}) \cong (A, \vec{b})$.

Then $n(T) \geq 4$.

Proof.

We may assume that T has a prime model and a countable saturated model. These cannot be isomorphic, as the conditions of the lemma imply that T has a non-principal type. Let $p(\vec{x})$ be the type of $\vec{a}$ in A. Then $p(\vec{x})$ is non-principal. Let $q(\vec{x}, \vec{y})$ be a type of T which is non-principal over $p(\vec{x})$ and extends $p(\vec{x})$ (i.e. $p(\vec{x}) \subseteq q(\vec{x}, \vec{y})$ and $q(\vec{b}, \vec{y})$ is a non-principal type of $T \cup p(\vec{b})$). Let $(B, \vec{b}, \vec{c})$ be a prime model of $T \cup q(\vec{b}, \vec{c})$. Then, as in the proof of Lemma 41, A and B are non-isomorphic countable models of T. Also, as in the proof of Theorem 24, neither A nor B can be prime or saturated. Thus T has at least four countable models.

<u>Definition 43</u>. Let A be a model, and $\vec{a}$ and $\vec{b}$ finite tuples from A. We say that $\vec{b}$ is principal over $\vec{a}$ in A, if $\vec{b}$ realises a principal type in $(A,\vec{a})$.

The following lemma is widely known.(e.g. Benda[2])

<u>Lemma 44</u>. Let $(A,\vec{a})$ be prime. Suppose that $\vec{b}\in A$ and $\vec{a}$ is principal over $\vec{b}$ in A. Then $(A,\vec{b})$ is prime.

<u>Proof</u>.

Let $\vec{c}\in A$. We show that $\vec{c}$ realises a principal type in $(A,\vec{b})$. Firstly, there is a formula $\phi(\vec{z},\vec{x},\vec{y})$ which generates the type of $(\vec{b},\vec{c})$ in $(A,\vec{a})$. Let $\psi(\vec{b},\vec{z})$ generate the type of $\vec{a}$ in $(A,\vec{b})$. Then it is quite easy to see that the formula $(\exists\vec{z})(\psi(\vec{b},\vec{z}) \wedge \phi(\vec{z},\vec{b},\vec{y}))$ generates the type of $\vec{c}$ in $(A,\vec{b})$. So $\vec{c}$ realises a principal type in $(A,\vec{b})$. As $\vec{c}$ was an arbitrary finite tuple from A, it follows that $(A,\vec{b})$ is prime.

<u>Observation 45.</u> It follows that if $(A,\vec{a})$ is prime, $(A,\vec{a}) \equiv (A,\vec{b})$, and $\vec{a}$ is principal over $\vec{b}$, then $(A,\vec{a}) \simeq (A,\vec{b})$ (as both these models will be prime models of the same complete theory).

Note also that if $(A,\vec{a})$ is prime, then any $\vec{b}$ in A is already principal over $\vec{a}$ in A. So we can see already that the problem of getting non-isomorphic countable models has been reduced to the problem of proving exchange results of the following sort : if $\vec{a}$ and $\vec{b}$ are finite tuples from a model A, and $(A,\vec{a}) \equiv (A,\vec{b})$, then $\vec{a}$ principal over $\vec{b}$ implies that $\vec{b}$ is principal over $\vec{a}$.

2.2 II Getting infinitely many models.

We will first define the 'Cantor-Bendixson' rank on the types of a theory. So let us fix a complete theory T. Let $S_n(T)$ denote the set of (complete) n-types of T.

Definition 46. (i) We define for each ordinal α, a subset $S_n^\alpha(T)$ of $S_n(T)$ by :

1) $S_n^0(T) = S_n(T)$.

2) For δ a limit ordinal, $S_n^\delta(T) = \bigcap S_n^\alpha(T)$.

3) $S_n^{\alpha+1}(T) = \{ p : p \in S_n^\alpha(T)$ and for all $\phi \in p$ there is $q \in S_n^\alpha(T)$ such that $q \neq p$ and $\phi \in q \}$

(ii) If $p \in S_n(T)$, then we define

$\mathrm{Rank}_n p$ = the least α such that $p \in S_n^\alpha(T)$, if there is such an α. Otherwise $\mathrm{Rank}_n p = \infty$.

(iii) We also define with no confusion ranks and degrees of formulae, with respect to T. So let ϕ be an n-formula consistent with T. Then

$\mathrm{Rank}_n \phi = \sup\{ \alpha :$ there is $p \in S_n^\alpha(T)$ with $\phi \in p \}$, if such a sup exists. Otherwise $\mathrm{Rank}_n \phi = \infty$.

If $\mathrm{Rank}_n \phi = \alpha$, then we define $\mathrm{Deg}_n \phi = |\{p \in S_n^\alpha(T) : \phi \in p\}|$.

The following facts are then easy to prove.

Lemma 47. (i) Suppose that ϕ is an n-formula, $p \in S_n(T)$ and $\phi \in p$. Then $\mathrm{Rank}_n p \leqslant \mathrm{Rank}_n \phi$.

(ii) Suppose that $\mathrm{Rank}_n p = \alpha < \infty$. Then there is an n-formula ϕ such that $\mathrm{Rank}_n \phi = \alpha$, $\mathrm{Deg}_n \phi = 1$, and p is generated over T by $\{\phi\} \cup \{ \neg \psi : \psi$ is an n-formula, $T \vdash \psi \rightarrow \phi$, and $\mathrm{Rank}_n \psi < \alpha \}$.

(iii) Suppose that $p \in S_n(T)$ and that p is generated over T by $\{\phi\} \cup \{\neg\psi_i : i \in I\}$, where $\mathrm{Rank}_n \psi_i < \alpha$ for all $i \in I$. Then $\mathrm{Rank}_n p \leq \alpha$.

We also note the following.

Observation 48. (i) Let $p \in S_n(T)$. Then $\mathrm{Rank}_n p = 0$ if and only if p is a principal n-type of T.

(ii) Let ϕ be an n-formula. Then $\mathrm{Rank}_n \phi = 1$ and $\mathrm{Deg}_n \phi = 1$ if and only if ϕ is minimal, where by ϕ being minimal we mean that

1) there are infinitely many complete n-formula ψ such that $T \vdash \psi \rightarrow \phi$, and

2) if ϕ' is any n-formula, then either there are only finitely many complete n-formula ψ such that

$$T \vdash \psi \rightarrow \phi \wedge \phi'$$

or there are only finitely many complete n-formula ψ such that $T \vdash \psi \rightarrow \phi \wedge \neg\phi'$

Baldwin and Lachlan[1] prove an exchange result for strongly minimal formulae, one case of which is :

if $\phi(x)$ is strongly minimal in a theory T, $A \models T$, $a, b \in A$, $A \models \phi(a)$, $a \notin cl(0,A)$, $b \notin cl(0,A)$, then $b \in cl(\{a\},A)$ implies that $a \in cl(\{b\},A)$.

This, however, does not hold for minimal formulae as defined above, even when all elements of the prime model of the theory are named. Look at the following example for instance :

Let $A = (A,R)$ be a countable model, where R is a binary relation on A. $R(x,y)$ "says" that y is an immediate successor of x. Under the induced ordering A is a tree with a first element, such that every element at the nth level has exactly $n+2$ immediate successors. Every element is at level n for some $n < \omega$. All elements of A are named by constants. Let $T = Th(A)$. Then '$x = x$' is minimal for T. Let $B \models T$, and $B \neq A$. $A \preccurlyeq B$ in the obvious way. Let $b \in B-A$. Then b has a unique immediate predecessor a, whereby a is algebraic over b. But a is in $B-A$, and so has infinitely many immediate successors. So b is not algebraic over a.

We <u>can</u> prove a weaker exchange result, which however, holds between <u>any</u> two tuples whose types are of the same Cantor-Bendixson rank less than infinity.

For the next few results, let M be an $\aleph_0$-saturated model of a complete theory T. Any tuples we talk about will be in M, and for such an n-tuple $\vec{x}$, $tp(\vec{x})$ will denote the n-type realised by $\vec{x}$ in M. Ranks and degrees of types and formulae will be obviously relative to T. Any countable model of T will be isomorphic to an elementary substructure of M. So Lemmas 49 and 50 following, are valid if we are working inside any countable model of T. Before we can prove the exchange result, we need the following lemma :

<u>Lemma 49</u>. Let $\vec{a}$ and $\vec{b}$ be n and m-tuples respectively, such that $Rank_n tp(\vec{a}) = \alpha < \infty$, and $\vec{b}$ is algebraic over a.

Then $Rank_m tp(\vec{b}) \leq \alpha$.

Proof.

We prove the lemma by induction on α.

It is clearly true for $\alpha = 0$.

Now suppose the lemma is true for all $\beta < \alpha$. Let $\vec{a},\vec{b}$ be tuples as in the hypothesis. Let k be the least natural number such that there is a formula $\psi(\vec{x},\vec{y})$ and

$$M \models \exists^k \vec{y}\psi(\vec{a},\vec{y}) \quad \text{and} \quad M \models \psi(\vec{a},\vec{b}) .$$

Then $\psi(\vec{a},\vec{y})$ generates the type of $\vec{b}$ over $\vec{a}$.

Now $tp(\vec{a})$ is generated over T by

$$\{\phi(\vec{x})\} \cup \{\neg\phi_i(\vec{x}) : i<\omega\} ,$$

where $Rank_n\phi = \alpha$, $Deg_n\phi = 1$, and $Rank_n\phi_i < \alpha$ for all i.

So $T \cup \{\phi(\vec{x})\} \cup \{\neg\phi_i(\vec{x}) : i<\omega\} \vdash \exists^k\vec{y}\psi(\vec{x},\vec{y})$.

By compactness there is $r<\omega$ such that

$$T \cup \{\phi(\vec{x})\} \cup \{\neg\phi_i(\vec{x}) : i=0,..r\} \vdash \exists^k\vec{y}\psi(\vec{x},\vec{y}) \qquad (*)$$

Consider the following set of formulae :

$$\Sigma(\vec{y}) = \{(\exists\vec{x})(\phi(\vec{x})\wedge\bigwedge_{i=0..r}\neg\phi_i(\vec{x})\wedge\psi(\vec{x},\vec{y}))\} \cup$$
$$\{\neg(\exists\vec{x})(\phi_j(\vec{x})\wedge\bigwedge_{i=0..r}\neg\phi_i(\vec{x})\wedge\psi(\vec{x},\vec{y})) : r<j<\omega\}$$

Now suppose that $\vec{b}'$ realised $\Sigma(\vec{y})$. Then, from looking at Σ we can see that there must be an n-tuple $\vec{a}'$ with

$$tp(\vec{a}') = tp(\vec{a}) \quad \text{and} \quad M \models \psi(\vec{a}',\vec{b}') .$$

But then, as $\psi(\vec{a},\vec{y})$ generates the type of $\vec{b}$ over $\vec{a}$, it is clear that $tp(\vec{b}') = tp(\vec{b})$.

Thus $tp(\vec{b})$ is determined by $\Sigma(\vec{y})$ over T.

Look now at one of the formulae

$$\sigma_j(\vec{y}) = (\exists\vec{x})(\phi_j(\vec{x})\wedge\bigwedge_{i=0..r}\neg\phi_i(\vec{x})\wedge\psi(\vec{x},\vec{y})) \quad \text{where } j>r .$$

We may suppose $\sigma_j(\vec{y})$ to be consistent, so it is satisfied by some $\vec{d}$.

But then there is $\vec{c}$ such that

$$M \models \phi_j(\vec{c}) \wedge \bigwedge_{i=0,\ldots,r} \neg\phi_i(\vec{c}) \wedge \psi(\vec{c},\vec{d}) .$$

From (*), it follows that

$$M \models \exists^k y \psi(\vec{c},\vec{y}) \text{ , whereby } \vec{d} \text{ is algebraic over } \vec{c}.$$

Let p be the type of $\vec{c}$. Then $\phi_j \in p$.

But $\mathrm{Rank}_n \phi_j < \alpha$, so by Lemma 47, $\mathrm{Rank}_n p < \alpha$.

But then, by the induction hypothesis $\mathrm{Rank}_m tp(\vec{d}) < \alpha$.

So, for all $\vec{d}$ satisfying $\sigma_j(\vec{y})$, $\mathrm{Rank}_m tp(\vec{d}) < \alpha$.

Therefore $\mathrm{Rank}_m \sigma_j < \alpha$.

So the type of $\vec{b}$ is generated over T by one formula, and a set of negations of formulae of rank less than α.

So, from Lemma 47, $\mathrm{Rank}_m tp(\vec{b}) \leq \alpha$.

Thus the lemma is proved.

We can now prove the exchange result that we have been aiming for. This result actually follows from some lemmas on ranks in Lascar[14]. But it is not clear whether he noticed it in this form. Anyway, our proof here will be rather more longwinded, as an introduction to techniques used in the next section.

<u>Lemma 50</u>. Let $\vec{a}$ and $\vec{b}$ be m and n-tuples respectively, such that $\mathrm{Rank}_m tp(\vec{a}) = \mathrm{Rank}_n tp(\vec{b}) = \alpha < \infty$.

Then $\vec{b}$ algebraic over $\vec{a}$ implies that $\vec{a}$ is principal over $\vec{b}$.

<u>Proof</u>.

Let $p(\vec{x})$ be the type of $\vec{a}$, and let $\psi(\vec{a},\vec{y})$ generate the type of $\vec{b}$ over $\vec{a}$.

So $M \models \psi(\vec{a},\vec{b})$ and $T \cup p(\vec{x}) \vdash \exists^k \vec{y} \psi(\vec{x},\vec{y})$ for some $k<\omega$.

First of all, let us note that if there were a formula $\sigma(\vec{x},\vec{b})$ such that $M \models \sigma(\vec{a},\vec{b})$, and such that

$M \models \psi(\vec{x},\vec{b}) \wedge \sigma(\vec{x},\vec{b}) \rightarrow \theta(\vec{x})$ for all $\theta(\vec{x})$ in $p(\vec{x})$,

then $\vec{a}$ would realise a principal type over $\vec{b}$, generated by

$$\psi(\vec{x},\vec{b}) \wedge \sigma(\vec{x},\vec{b}) .$$

For let $\psi'(\vec{x},\vec{b})$ be a formula. By the completeness of the formula $\psi(\vec{a},\vec{y})$, we have that either

(i) $M \models (\forall \vec{y})(\psi(\vec{a},\vec{y}) \rightarrow \psi'(\vec{a},\vec{y}))$ or

(ii) $M \models (\forall \vec{y})(\psi(\vec{a},\vec{y}) \rightarrow \neg\psi'(\vec{a},\vec{y}))$.

Suppose (i) to be the case.

Now suppose that $M \models \psi(\vec{a}',\vec{b}) \wedge \sigma(\vec{a}',\vec{b})$.

Then $\vec{a}'$ realises $p(\vec{x})$ and so has the same type as $\vec{a}$.

So by (i) we have that $M \models \psi'(\vec{a}',\vec{b})$.

Thus $M \models (\forall \vec{y})(\psi(\vec{x},\vec{b}) \wedge \sigma(\vec{x},\vec{b}) \rightarrow \psi'(\vec{x},\vec{b}))$.

Similarly, if (ii) is true, then

$M \models (\forall \vec{y})(\psi(\vec{x},\vec{b}) \wedge \sigma(\vec{x},\vec{b}) \rightarrow \neg\psi'(\vec{x},\vec{b}))$.

So the above note is established.

Now $p(\vec{x})$ is generated over T by

$\{\phi(\vec{x})\} \cup \{\neg \phi_i(\vec{x}) : i < \omega\}$, where

$\mathrm{Rank}_m \phi = \alpha$, $\mathrm{Deg}_m \phi = 1$, and $\mathrm{Rank}_m \phi_i < \alpha$ for all $i < \omega$.

By compactness, there is $n < \omega$ such that

$T \cup \{\phi(\vec{x})\} \cup \{\neg\phi_i(\vec{x}) : i < n\} \vdash \exists^{K} y \psi(\vec{x},\vec{y})$ (*)

Now suppose that for all $j > n$

$M \models (\psi(\vec{x},\vec{b}) \wedge \phi(\vec{x}) \wedge \bigwedge_{i<n} \neg\phi_i(\vec{x})) \rightarrow \neg\phi_j(\vec{x})$.

Then, the formula $\psi(\vec{x},\vec{b}) \wedge \phi(\vec{x}) \wedge \bigwedge_{i<n} \neg\phi_i(\vec{x})$ would determine the type of $\vec{x}$ as being $p(\vec{x})$, whereby from what we noted above, $\vec{a}$ would be principal over $\vec{b}$, and the lemma would be proved. So let us assume that for some $j > n$

$$M \models (\exists\vec{x})(\psi(\vec{x},\vec{b}) \wedge \phi(\vec{x}) \wedge \bigwedge_{i<n} \neg\phi_i(\vec{x}) \wedge \phi_j(\vec{x})) \ .$$

So for some $\vec{c}$, $M \models \psi(\vec{c},\vec{b}) \wedge \phi(\vec{c}) \wedge \bigwedge_{i<n} \neg\phi_i(\vec{c}) \wedge \phi_j(\vec{c})$.

But then, by (*), $\vec{b}$ is algebraic over $\vec{c}$.

Also, as $M \models \phi_j(\vec{c})$ and $\mathrm{Rank}_m \phi_j < \alpha$, we have that

$$\mathrm{Rank}_m \, tp(\vec{c}) < \alpha \ .$$

But then by Lemma 49, $\mathrm{Rank}_n \, tp(\vec{b}) < \alpha$.

This is a contradiction, and thus the theorem is proved.

We can now apply this lemma to the results of the preceding section to prove the following theorem, which is essentially due to Lascar[14] .

Theorem 51. Let T be a complete theory, such that every complete extension of T by finitely many constants is algebraic. Then $n(T) \geq \aleph_0$.

Proof.

Firstly, we may as usual assume that T has not more than $\aleph_0$ n-types, for all $n < \omega$. It follows easily from this that $\mathrm{Rank}_n p < \infty$, for every n-type p of T.

Now let $A \models T$, and $\vec{a}$ a finite tuple from A such that $(A,\vec{a})$ is prime.

Then, by the conditions of the theorem,

$$(A,\vec{a}) \text{ is algebraic.}$$

Suppose that $\vec{b} \in A$ and $(A,\vec{a}) \equiv (A,\vec{b})$.

Then $\vec{b}$ is algebraic over $\vec{a}$, and the types of the two tuples being the same, must have the same rank.

Thus from Lemma 50, $\vec{a}$ is principal over $\vec{b}$, whereby

$$(A,\vec{a}) \cong (A,\vec{b}) \ .$$

The theorem now follows from Lemma 41.

The following proposition is implicit in the proof of the above theorem. It is, however, an elegant expression of the exchange result developed in this section, and provides an almost unqualified generalisation of the notion of dimension which is found in, for example, algebraically closed fields.

Proposition 52. Let A be a model which is algebraic over a finite tuple $\vec{a}$, where the type of $\vec{a}$ in A has rank less than infinity. Then A is algebraic over any other $\vec{b}$ in A which realises the same type as $\vec{a}$.

Certain important classes of theories can be extended to complete theories which satisfy the hypothesis of Theorem 51. Thus, for example :

Corollary 53. Let T be a countable theory with Skolem functions. Then $n(T) \geq \aleph_0$.

In fact, as Lascar notes, it is enough that a theory T have a simple extension satisfying the conditions of Theorem 51. For then, in building our non-isomorphic countable models of T, we just ensure that all these models realise the type which defines the simple extension. Using this fact, and through the mediating property of the strong elementary intersection property, Lascar proves :

Theorem 54. Let T be a countable theory which is convex and model-complete. Then $n(T) \geq \aleph_0$.

2.2 III Getting at least four models.

We now examine what happens when we place conditions only on the prime model of a theory. In this case, results are more difficult to come by. We use the same techniques as in the previous section, namely proving exchange results, but the proofs are not as immediate.

We first look at the situation in which $n(T) = 3$. We know that in this case, the countable models consist of a prime, a saturated, and a 'middle' model. Recall that in the Ehrenfeucht example, the middle model A_1 is prime over an element c, where $c = \lim_{n<\omega} a_n$. I now show that for any theory T for which $n(T) = 3$, a similar situation holds.

Theorem 55. Let T be a complete theory such that $n(T) = 3$. Suppose that T has infinitely many 1-types. Then there is a formula $\phi(x)$, formulae $\phi_i(x)$ for $i<\omega$, and a formula $\psi(x,y)$ such that

1) If A is the prime model of T, then $\phi_i^A \subseteq \phi^A$ for all $i<\omega$, $\phi_i^A \cap \phi_j^A = 0$ for all $i \neq j$, and the relation "$\phi_i^A < \phi_j^A$" which we define to hold if and only if $A \models \exists x \exists y(\phi_i(x) \wedge \phi_j(y) \wedge \psi(x,y))$, is a total ordering such that $\phi_i^A < \phi_j^A$ iff $i < j$.

2) If B is the middle model of T, then B is prime over an element $c \in B$, where

a) $B \models \phi(c)$, and

b) $c = \lim_{n<\omega} \phi_n^B$ in ϕ^B , in the sense that

$B \models \exists x \epsilon \phi_n(\psi(x,c))$ for all $n<\omega$, and if for some $d \in B$, $B \models \phi(d)$ and $B \models (\exists x \in \phi_n)(\psi(x,d))$ for all $n<\omega$, then $B \models \neg\psi(d,c)$.

Proof.

(Let us first note that if $n(T) = 3$, then T must have infinitely many n-types for some $n<\omega$. We have taken n to be 1. In the general case the proof below will give the same conclusion, but for n-formulae and n-tuples, rather than for 1-formulae and single elements.)

So let T be as in the hypothesis of the theorem. Then as usual, for any type $p(\vec{x})$ of T, $T \cup p(\vec{c})$ has a prime model. Also T must have a minimal 1-formula (where by minimal we mean the same as in Observation 48). For, if not, we can by a tree argument, get $2^{\aleph_0}$ 1-types, which would give us too many countable models.

So let this minimal formula be $\phi(x)$. Let $\{\psi_i(x) : i<\omega\}$ be the set of complete 1-formulae of T such that $T \vdash \psi_i \rightarrow \phi$. Then $i \neq j$ implies that

$$T \vdash \neg(\exists x)(\psi_i(x) \wedge \psi_j(x)).$$

Also, $\{\phi(x)\} \cup \{\neg\psi_i(x) : i<\omega\}$ determine a complete 1-type of T, whereby $T'_c = T \cup \{\phi(c)\} \cup \{\neg\psi_i(c) : i<\omega\}$ is a complete theory.

Let A be the prime model of T, and let (B,c) be the prime model of T'_c. Then $A \models (\forall x)(\phi(x) \leftrightarrow \bigvee_{i<\omega} \psi_i(x))$.
B must be the middle model of T, as a non-principal type is realised in B, and B is not saturated.

Also, by Lemma 42, there must be an element d in B, such that $(B,c) \equiv (B,d)$ but not $(B,c) \cong (B,d)$. But then d is principal over c, whereas c is not principal over d. As d is principal over c, there is a formula $\psi(c,x)$ which generates a principal 1-type of T'_c and such that $B \models \psi(c,d)$.

Put $T'_d = Th((B,d))$. Then T'_d is the same as T'_c but with c replaced by d.

Now suppose that there were a formula $\psi'(y,d)$, consistent with T'_d such that

$$T'_d \vdash \psi'(y,d) \rightarrow \psi(y,d) \wedge \phi(y) \wedge \bigwedge_{i<\omega} \neg\psi_i(y) .$$

Then as in the proof of Theorem 50, $\psi'(y,d)$ would be a complete 1-formula of T'_d satisfied in B by c, whereby c would be principal over d. Thus there is no such formula $\psi'(y,d)$. So, in particular

(i) $X = \{ i<\omega : B \models \exists x(\psi_i(x) \wedge \psi(x,d)) \}$ is infinite, for if not then $\psi(y,d) \wedge \bigwedge_{i \in X} \neg\psi_i(y)$ would do the job of $\psi'(y,d)$. And also

(ii) By the Omitting types theorem, T'_d has a model omitting the set

$$\Sigma(y,d) = \{\psi(y,d)\} \cup \{\phi(y)\} \cup \{\neg\psi_i(y) : i<\omega\} .$$

Now as T'_c is just the same as T'_d, and as (B,c) is a prime model of T'_c, then the set X is also equal to

$$\{ i<\omega : B \models \exists x(\psi_i(x) \wedge \psi(x,c)) \} ,$$

and (B,c) omits the set of formulae $\Sigma(y,c)$.

Note that, by compactness, for any formula $\theta(x)$ of $L(T)$, $T'_c \vdash \theta(c)$ if and only if $\{j<\omega : A \models \psi_j(y) \rightarrow \theta(y) \}$ is a cofinite set of natural numbers. (1)

Also note that, as $B \models \psi(c,d)$, $\psi(c,x)$ is a complete 1-formula of T'_c, and $(B,d) \equiv (B,c)$, then

$$T'_c \vdash \psi(c,x) \rightarrow \phi(x) \wedge \neg\psi_i(x) \quad \text{for all } i<\omega.$$

So, by compactness, for each finite $Z_1 \subseteq \omega$, there is a

finite $Z_2 \subseteq \omega$ such that

$$T \cup \{\phi(c)\} \cup \{\neg\psi_i(c): i \in Z_2\} \vdash \psi(c,x) \longrightarrow \phi(x) \wedge \bigwedge_{j \in Z_1} \neg\psi_j(x). \quad (2)$$

And finally note that from (1) and the definition of X, it follows that, if $i \in X$ then

$$X_i = \{j<\omega : A \models \psi_j(y) \longrightarrow (\exists x)(\psi_i(x) \wedge \psi(x,y))\} \text{ is a cofinite set.} \quad (3)$$

We now define inductively $i_n \in X$, for $n<\omega$, such that

$\underline{X_{i_{n+1}} \cup \{i_{n+1}\} \subseteq X_{i_n} - \{i_n\}}$ for all $n<\omega$.

Suppose that i_n has been defined. As $i_n \in X$, we have by (3) that X_{i_n} is cofinite.

By (2) there is a finite $Z \subseteq \omega$ such that if

$T \cup \{\phi(c)\} \cup \{\neg\psi_j(c): j \in Z\} \vdash (\exists x)(\psi(c,x) \wedge \psi_i(x))$

then $i \in X_{i_n} - \{i_n\}$.

Then $Y = X_{i_n} \cap (\omega - Z) \cap (X - i_n)$ is an infinite set.

We choose $i_{n+1} \in Y$.

Note that, by the completeness of the 1-formulae ψ_i, we have that for all $i,j<\omega$, $i \neq j$,

$T \vdash (\forall x \in \psi_i)(\exists y \in \psi_j)(\psi(x,y))$ if and only if

$T \vdash (\forall y \in \psi_j)(\exists x \in \psi_i)(\psi(x,y))$ if and only if

$T \vdash (\exists x \in \psi_i)(\exists y \in \psi_j)(\psi(x,y))$.

It is now easily seen that i_{n+1} satisfies the induction condition. Thus the definition of the i_n can be carried out.

We now put ϕ_k to be ψ_{i_k} for all $k<\omega$.

Let $k < m < \omega$. Then $X_{i_m} \cup \{i_m\} \subseteq X_{i_k} - \{i_k\}$.

As $i_m \in X_{i_k}$ we have $A \models (\exists x \in \phi_k)(\exists y \in \phi_m)(\psi(x,y))$.

As $i_k \notin X_{i_m}$ we have $A \models (\exists x \in \phi_k)(\forall y \in \phi_m)(\neg\psi(y,x))$,

but then by the completeness of ϕ_k,

$A \models \neg(\exists x \in \phi_k)(\exists y \in \phi_m)(\psi(y,x))$.

So part 1) of the theorem is proved.

For part 2) we have to show that there is no element b in B such that $B \models \psi(b,c)$, $B \models \phi(b)$, and

$$B \models (\exists x)(\phi_i(x) \wedge \psi(x,b)) \quad \text{for all } i<\omega.$$

But this follows immediately from the fact that (B,c) omits $\Sigma(y,c)$, and from the fact that there can be no ψ_j for which $\{ i<\omega : A \models (\exists x)(\exists y)(\psi_i(x) \wedge \psi_j(y) \wedge \psi(x,y)) \}$ is infinite for then we would have $T'_c \vdash (\exists y)(\psi_j(y) \wedge \psi(c,y))$, which is impossible.

This completes the proof of the theorem.

We now come to the main result of this chapter. We would like to be able to prove that if a theory T has a prime model A with an infinite definable subset X such that $X \subseteq cl(0,A)$ (i.e. all elements of X are algebraic), then $n(T) \geq 4$. However, we have as yet only been able to prove this in the special case that every element of X is algebraic 'of degree at most two'.

Theorem 56. Let T be a complete theory, with a model A and a formula $\phi(x)$ such that, ϕ^A is infinite, and for every $a \in \phi^A$ there is a formula $\psi(x)$ such that

$$A \models \psi(a) \quad \text{and} \quad A \models \exists^{\leq 2} x\psi(x).$$

Then $n(T) \geq 4$.

Proof.

We may assume that T has a prime model, and that A is this prime model. Also we may assume that T has a minimal 1-formula $\psi(x)$ such that $T \vdash \psi(x) \rightarrow \phi(x)$.

We will assume, for ease of notation, that $\psi(x)$ is 'x=x'. Let $\{\phi_i(x):i<\omega\}$ be the set of complete 1-formulae of T. Then it follows that for each $i<\omega$, $A \models \exists^{\leq 2} x \phi_i(x)$,and for each $a \in A$ there is $i<\omega$ with $A \models \phi_i(a)$.

By minimality of 'x=x', $T'_c = T \cup \{\neg\phi_i(c):i<\omega\}$ is a complete theory. Let (B,c) be a prime model of T'_c. So c realises a non-principal type of T in B, whereby B is not a prime model of T. Therefore, by Lemma 42, to show that $n(T) \geq 4$, it is enough to show that if $d \in B$ and $(B,d) \equiv (B,c)$, then c is principal over d in B.

So let $d \in B$, and $(B,d) \equiv (B,c)$. As d realises a principal 1-type in (B,c), there is a complete 1-formula $\psi(c,x)$ of T'_c such that $B \models \psi(c,d)$.

Let $X = \{i<\omega : B \models \exists x(\phi_i(x) \wedge \psi(x,d))\}$. It is clear that $\phi_i^B \cap \phi_j^B = 0$ for $i \neq j$.

If X is finite, then as we noted in the proof of Theorem 55, the formula $\psi(y,d) \wedge \bigwedge_{i \in X} \neg\phi_i(y)$ is a complete 1-formula of T'_d satisfied by c in B, and we are done (where T'_d is again the same as T'_c but with d replacing c). So we assume that X is infinite, and aim for a contradiction.

Firstly, we may assume that

1) $T'_c \vdash (\forall x)(\psi(c,x) \rightarrow \neg\psi(x,c))$,

for if not, then by the completeness of $\psi(c,x)$

$T'_c \vdash (\forall x)(\psi(c,x) \rightarrow \psi(x,c))$, in which case c will be obviously principal over d.

Note that as $(B,c) \equiv (B,d)$, X is also equal to

$\{i<\omega : B \models (\exists x)(\phi_i(x) \wedge \psi(x,c))\}$.

I assert that

2) $T'_c \vdash (\forall x)(\psi(c,x) \rightarrow (\forall y_1 y_2)(\psi(x,y_1) \wedge \psi(y_1,y_2) \rightarrow \psi(c,y_2)))$

(i.e. that ψ is 2-transitive) .

For suppose not. Then by the completeness of $\psi(c,x)$,

$$T'_c \vdash \psi(c,x) \rightarrow (\exists y_1 y_2)(\psi(x,y_1) \wedge \psi(y_1,y_2) \wedge \neg\psi(c,y_2)) .$$

By compactness, there is $n<\omega$ such that

$$T \cup \{\neg\phi_i(a) : i<n\} \vdash \psi(a,x) \longrightarrow (\exists y_1 y_2)(\psi(x,y_1) \wedge \psi(y_1,y_2) \wedge \neg\psi(a,y_2))$$

(where a is a new constant) .

Now as X is infinite, we may choose $j \in X$ such that $j>n$. Then by the definition of X, there is $a \in \phi_j^B$ such that

$$B \models \psi(a,c) .$$

And so, from above there are b_1, b_2 in B such that

$$B \models \psi(c,b_1) \wedge \psi(b_1,b_2) \wedge \neg\psi(a,b_2) .$$

Now $B \models \psi(c,d)$, $(B,c) \equiv (B,d)$ and $\psi(c,x)$ is a complete 1-formula of T'_c. So it follows that

$$(B,b_1) \models T'_{b_1} \quad \text{and} \quad (B,b_2) \models T'_{b_2} .$$

We know that $|\phi_j^B| \leq 2$.

If $|\phi_j^B| = 1$, then the formula ϕ_j defines a, so as $B \models (\exists x \in \phi_j)(\psi(x,b_2))$, we would have that $B \models \psi(a,b_2)$.

So $|\phi_j^B| = 2$. Let a' be the other element in ϕ_j^B.

Now $B \models (\exists x \in \phi_j)(\psi(x,b_2))$, so we must have that $B \models \psi(a',b_2)$ and also $B \models (\exists^1 x)(\phi_j(x) \wedge \psi(x,b_2))$.

So also, $T'_c \vdash (\exists^1 x)(\phi_j(x) \wedge \psi(x,c))$.

Now either $B \models \psi(a,b_1)$ or $B \models \psi(a',b_1)$.

If $B \models \psi(a,b_1)$, then by the completeness of $\psi(c,y)$

$$B \models \psi(c,y) \rightarrow (\forall x \in \phi_j)(\psi(x,c) \rightarrow \psi(x,y)) .$$

But then $B \models \psi(b_1,y) \rightarrow (\forall x \in \phi_j)(\psi(x,b_1) \rightarrow \psi(x,y))$,

whereby $B \models \psi(a,b_2)$. Contradiction .

So it must be the case that $B \models \psi(a', b_1)$, but then again, as $B \models \psi(a', b_2)$, we must have that

$$B \models \psi(b_1, y) \rightarrow (\forall x \in \phi_j)(\psi(x, b_1) \rightarrow \psi(x, y)) .$$

Replacing b_1 by c in the line above, we conclude that

$$B \models \psi(a, b_1) .$$

But this was impossible. This contradiction proves assertion 2)

Now suppose that

$$B \models (\exists y_1 y_2)(\psi(c, y_1) \wedge \psi(y_1, y_2) \wedge \psi(y_2, c')) .$$

Then from 2), $B \models \psi(c, c')$.

Thus by completeness of $\psi(c,x)$, it follows that

3) $T'_c \vdash (\forall x)(\psi(c,x) \rightarrow (\exists y_1 y_2)(\psi(c, y_1) \wedge \psi(y_1, y_2) \wedge \psi(y_2, x)))$

(i.e. ψ is 2-dense)

Put $\theta(c,x)$ to be the formula

$$[\neg \psi(x,c) \wedge (\forall y_1 y_2)(\psi(x, y_1) \wedge \psi(y_1, y_2) \rightarrow \psi(c, y_2)) \wedge (\exists y_1 y_2)(\psi(c, y_1) \wedge \psi(y_1, y_2) \wedge \psi(y_2, x))]$$

Then by 1), 2), 3) and compactness, there is $m_1 < \omega$ such that

4) $T \cup \{\neg \phi_i(c) : i < m_1\} \vdash (\forall x)(\psi(c,x) \rightarrow \theta(c,x))$.

Also, as $T'_c \vdash \psi(c,x) \rightarrow \neg \phi_i(x)$ for all $i < \omega$,

there is $m_2 < \omega$, $m_2 > m_1$ such that

5) $T \cup \{\neg \phi_i(c) : i < m_2\} \vdash (\forall x)(\psi(c,x) \rightarrow \bigwedge_{i<m_1} \neg \phi_i(x))$.

Now choose $a \in A$, $a \notin \phi_i^A$ for $i < m_2$, such that there is $b \in A$ with $A \models \psi(a,b)$.

Then, by 4) and "2-denseness", there are a_1, a_2 in A with

$$A \models \psi(a, a_2) \wedge \psi(a_2, a_1) \wedge \psi(a_1, b) .$$

Once again, there are a_3, a_4 in A with

$$A \models \psi(a, a_4) \wedge \psi(a_4, a_3) \wedge \psi(a_3, a_2) .$$

Continuing in this way, we can find a set $\{ a_n : n < \omega \}$ of elements of A, such that for each $n \geq 1$,

$A \models \psi(a,a_{2n}) \wedge \psi(a_{2n},a_{2n-1}) \wedge \ldots\ldots \wedge \psi(a_2,a_1) \wedge \psi(a_1,b)$.

By 4)(2-transitivity) and 5), for each m,n, with $m>n$,

$$A \models (\exists x)(\psi(a_{2m},x) \wedge \psi(x,a_{2n})) \ .$$

So by 4)(asymmetry) and 5), $a_{2m} \neq a_{2n}$.

Thus $\{ a_{2n} : n<\omega \}$ is an infinite set.

Again, by 4)(2-transitivity) and 5), for each n,

$$A \models (\exists y)(\psi(a_{2n},y) \wedge \psi(y,b)) \ .$$

So $\{ x \in A : A \models (\exists y)(\psi(x,y) \wedge \psi(y,b)) \}$ is infinite.

Now $A \models \phi_r(b)$ for some $r<\omega$. Then

$X_1 = \{ x \in A: A \models (\exists z)(\exists y)(\psi(x,y) \wedge \psi(y,z) \wedge \phi_r(z)) \}$ is also infinite. As each of the complete 1-formulae ϕ_i of T is satisfied by at most two elements, it follows that $\{ i<\omega : \phi_i^A \subseteq X_1 \}$ is infinite, and thus by minimality of 'x=x', cofinite. But then

$$T'_c \vdash (\exists z)(\exists y)(\psi(c,y) \wedge \psi(y,z) \wedge \phi_r(z)) \ .$$

So there is $c_1 \in B$ such that

$$B \models \psi(c,c_1) \wedge (\exists z)(\psi(c_1,z) \wedge \phi_r(z)) \ .$$

But $(B,c_1) \models T'_{c_1}$ and $T'_{c_1} \vdash \psi(c_1,x) \longrightarrow \neg\phi_i(x)$ for all $i<\omega$. So we have a contradiction, and the theorem is proved.

Corollary 57. Let A be a countable model. Then A has at least three countable proper elementary extensions, up to isomorphism over itself.

Proof.

Put $T = Th(A,a)_{a \in A}$. Then T satisfies the conditions of Theorem 56. So $n(T) \geq 4$. One of the countable models of T will be $(A,a)_{a \in A}$. The $L(A)$-reducts of the other three will be countable proper elementary extensions of A, pairwise non-isomorphic over A.

There are obviously many gaps to be filled in order to get from the results above to anywhere near proving the original conjecture that a theory with a minimal model has infinitely many countable models. But I think that the above work has at least pointed out a possible approach.

I view facts about the minimality and algebraicity of the prime model of a theory, as tools for obtaining lots of countable models, but by no means as a characterisation of those theories with infinitely many countable models. Or putting it another way, the converse to the conjecture is not true.

However, we can now, after having been through the proofs in this chapter, view our original intuitions in a slightly more educated light. Firstly, what is no doubt responsible for n(T) being finite, in the known examples, is the "denseness" of the orderings or relations in the models in the theories concerned. This also makes sense, when we note that the canonical methods for getting lots of countable models involve getting models of different finite "dimensions". And the notion of dimension involves the notion of nearness and thus of discreteness. For, a model is intuitively of dimension one, for example, if all its elements are near each other. Then, using compactness one can get a model of larger dimension, by adding elements that are far away. Denseness, however, implies that one cannot distinguish elements as being near to, or far away from, each other. Looked at more technically, if the models of our theory are "discrete" in some sense, then whenever $\vec{b}$ is principal over $\vec{a}$ in some model, the formula $\psi(x,y)$ which makes $\vec{b}$ principal over $\vec{a}$ will in some sense "say" that $\vec{b}$ is "near to" $\vec{a}$. Then, by using the compactness methods of Lemma 49, we can, as in Lemma 49, prove nice rank properties by induction, which will enable us to get lots of countable models. The situation where $\vec{b}$ is <u>algebraic</u> over $\vec{a}$, as in section 2.2 II, is just a very transparent case of nearness.

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