INTERACTIONS BETWEEN STARS AND INTERSTELLAR MATERIAL

T H E S I S

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by

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P R E F A C E

The contribution of the work described in this thesis to the advancement of the subject lies primarily in strengthening the foundations of the accretion theory by the examination of various aspects of the mechanism. The work also indicates that the accretion process may play an important part in the evolution of binary stars.

With the exception of the account of the Bondi-Hoyle mechanism, the work in Chapter I of this thesis is original. However, the idea of considering the effect of density gradient arose from the work of Gething and some of the discussion in this chapter is due to Professor McCrea. The account of the Bondi accretion process in Chapter IV is not original, but the present writer differs slightly from Bondi on the interpretation of the results and reasons are given for this. The remaining work of the thesis is original, but with the exception of Chapters II and III, the problems were suggested by Professor McCrea.

The work in Chapter II, Chapter VII, the second part of Chapter IV and the original part of Chapter I has already been published as (8)(15)(13)(9) respectively.

ABSTRACT

In Chapters I to IV of this thesis, the mechanism, originally put forward by Hoyle, Lyttleton and Bondi, by which stars can capture large amounts of interstellar material is examined and extended. The rate of accretion of interstellar material is determined on various assumptions about the nature of this material. The resistive force is also evaluated under various conditions. The effect of temperature and variation of material density is considered.

In Chapter V, a theory of binary star formation is investigated in which the resistive force is considered to remove part of the gravitational energy of a pair of stars and so leave them gravitationally bound together. It is found that under suitable circumstances such binary star formations can occur, but no estimate has been obtained of the probability of such a formation.

In the remaining two chapters, the modifications are considered which have to be applied to the accretion theory when the star is a binary. The dynamical effects of accretion on the binary are considered and estimates are made of the time required for the size of the binary orbit to be appreciably reduced.

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Introduction

There is direct astronomical evidence of the existence of diffuse clouds of matter in interstellar space. These clouds are concentrated fairly close to the plane of the galaxy and almost certainly exist mainly along the spiral arms. The clouds consist mainly of atomic hydrogen with probably about ten per cent (by numbers of atoms) of helium and traces of heavier elements. Mixed with the gas is about one per cent of dust particles whose sizes are of order 10^{-5} cm. The composition of the particles is uncertain. On the whole, the evidence indicates that the gas and dust are well mixed although clouds which appear to contain large proportions of dust can be observed. The kinetic temperature of the gas in normal regions of space (i.e. not too close to hot stars) is between 50 and 100^{o}A, the hydrogen being largely neutral and not ionised. But the internal temperature of the dust particles is only a few degrees absolute. The presence of the dust in the gas causes heat to be radiated and so enables the gas temperature to be fairly uniform over large regions of space. In HII regions near O and B type stars, the kinetic temperature is of the order of $10,000^{o}$A and there is probably no dust.

The average density of the gas over the entire region which it occupies may be described as of the order of one hydrogen atom per c.c. However, the density is by no means uniform and regions of much higher density exist. Strömgren [2] estimates that in typical regions of density maxima, there are 100 hydrogen atoms per c.c. But even denser regions exist, such as the neighbourhood of the Orion nebula where there may be 1000 hydrogen atoms per c.c. Some photographs of the denser regions show a filamentary structure so that the density may be even higher within the filaments.

The fact that the density of the gas is so variable enables the dense regions to be referred to as gas "clouds". These clouds move with the general rotation of the galaxy but they also have random velocities relative to each other. The stars in the dense regions are of course immersed in the clouds and are moving relative to them owing to their own peculiar motions.

In the phase of evolution of the galaxy before the formation of stars, these gaseous regions probably existed in similar conditions to those we observe today but the density may have been considerably greater. The stars are thought to have been formed by condensing from the gaseous material.

The existence of the clouds of interstellar material raises a number of important questions in theoretical astronomy. There is the question of the gain or loss of material by the star and the influence of this on the star's evolution. There is also the question of the gain or loss of momentum by the star and the consequent effect on the star's motion. This may be caused by the gravitational attractions of the clouds as a whole and also by the effects which occur in the immediate neighbourhood of the star due to its motion relative to the cloud in which it is immersed. The possibility and conditions for condensations to form is also an important problem. All these questions require to be answered in relation to the past evolution of the stars but it is also necessary to know if the processes are still active in present conditions.

There is observational evidence that interactions between stars and interstellar material must have occurred and in comparatively recent times. Recent observation has revealed an association between regions of interstellar material and population I stars. Detailed observation has shown an association of particular categories of very luminous stars with the regions of highest density and this strongly suggests continuing intake of interstellar material by the stars. Some of the brightest stars must have been formed in comparatively recent times and so were presumably formed from interstellar material by fresh condensations or else by absorption of interstellar material by already existing stars.

Although all this evidence indicates a gain of material by the stars, there are certain classes of stars in particular, novae which eject material into space. Novae may be responsible for the distribution of the heavier elements in the universe.

In this thesis an attempt has not been made to examine all the problems arising from the existence of the interstellar clouds but merely to examine some possible mechanisms of the interactions between stars and interstellar material and to consider one or two of their implications.

It has been realised for some time that when a star moves through a region of space containing interstellar material, the star may pick up or "accrete" some of this material. However, it appeared from Eddington's work[3] that the star would only pick up material lying within a distance from its path of the order of the radius of the star. Considering the low density of the interstellar material, this "accretion" would be too small to produce any important astrophysical effects.

In 1939, Hoyle and Lyttleton[4] suggested a mechanism by which substantially larger amounts of material could be captured by a star and they indicated the importance of such a process to stellar evolution. It may be mentioned that some aspects of the mechanism had already been realised by Nölke[5] as far back as 1910 and by Moreux[6] in 1922.

Between 1939 and 1944 a series of papers by Hoyle and Lyttleton developed the "Accretion Theory" and its applications. But it was not until 1944 that the mathematical theory of the accretion process was considered in detail in a paper by Bondi and Hoyle[7]. The mechanism which they described involved an "accretion column" which extended in the wake of the star. In this paper, the resistive force exerted on the star by the material was also considered and a value for it was obtained by considering the behaviour of the accretion column at a great distance from the star. The unsteady behaviour of the accretion column was later examined numerically in a paper by Dodd[8]. In 1952, a paper by Dodd and McCrea[9] showed that the value of the resistive force does not depend on the formation of an accretion column provided the interstellar material has the properties assumed in the Bondi-Hoyle mechanism. This paper gave consideration to the effects of non-uniformity in the density of the interstellar material (although this aspect had previously been considered by Gething[10]). The importance of the resistive force was also stressed. A year later, McCrea[11] considered the effects of density and velocity variations in the interstellar material on the Lyttleton theory of comet formation which is one of the applications of the accretion theory. In all the work up to this stage, the undisturbed interstellar material had been treated as consisting of inelastic particles with negligible relative motion. These properties are possessed by the particles of a gas at very low absolute temperatures provided that any heat developed in the gas can be radiated away rapidly. In 1952, Bondi[12] considered the rate of accretion by a star at rest in a cloud of gas at temperatures other than zero absolute. In 1953, Dodd[13] examined numerically the rate of accretion by a star moving through a gas treating the latter as a hydrodynamical medium, but results were obtained only in the case of isothermal flow. Up to the present, no estimate has been obtained of the resistive force experienced by a star in moving through a gas at temperatures other than zero absolute (where the random particle motions do not exist). Recently a paper by McCrea[14] has indicated the possible importance of a star being "trapped" by a cloud of material. It was shown that if a star enters a cloud with a suitably small velocity, the resistive force will bring the star to rest after which the star will undergo a symmetrical accretion of material, as for example in the Bondi process.

In Chapter I of this thesis, the Bondi-Hoyle mechanism of accretion is described and expressions for the rate of accretion and for the resistive force are obtained. The resistive force is then determined on the assumption that the particles of the cloud do not experience collisions between themselves. The agreement between the dominant terms of the expressions for the forces is then noted and the implications of this are discussed. A discussion follows of the "cut-off effect" which limits the magnitude of the force. The effects of a non-uniform cloud density are then examined and it is shown that in certain circumstances a circulatory current of material can be established about the star. The chapter ends with a mention of the temperatures under which the above consideration can be expected to be valid.

In Chapter II, an account is given of a numerical investigation of the unsteady behaviour of the Bondi-Hoyle accretion mechanism. It is found, as already suggested by Bondi and Hoyle that large perturbations tend to reduce the rate of accretion and their value for the minimum rate of accretion is approximately confirmed.

In Chapter III, an account is given of an elaboration of the Bondi-Hoyle mechanism in which the accretion column is treated as consisting of a gas at a temperature removed from 0° A. This treatment indicates that there is a definite limit to the temperature which can be allowed in the accretion column. The resistive force is examined and an upper limit to its value is found. This treatment is midway between the inelastic particle mechanism of Bondi and Hoyle and the hydrodynamical treatment of the next chapter.

In Chapter IV, a brief discussion is given of the Bondi symmetrical accretion process for a gas at temperatures removed from 0°A. This is followed by an account of a numerical investigation to determine the accretion rate by a star moving through an isothermal gas at temperatures removed from 0°A. The results are in good agreement with a formula conjectured by Bondi on the basis of results for the simpler mathematical models of the situation.

In Chapter V, a theory of binary star formation is investigated in which the resistance of the cloud is considered to remove part of the gravitational energy of a pair of stars and so leave them gravitationally bound together. It is found that under suitable circumstances such binary star formations can occur but no estimate has been obtained of the probability of such a formation. It is shown that the encounter between the stars must occur early in the lives of the stars if it is to result in a binary star and it is shown that during the encounter, the stars have their masses greatly increased by accretion. The chapter ends with an interesting little piece of mathematics concerned with the statistics of multiple system formation but it is not really relevant to the main work of the chapter.

In Chapter VI, the modifications are considered which have to be applied to the accretion theory when the star is a binary. It is found that the accretion process depends on the ratio of the separation of the binary components to the local mean interstellar distance. The dynamical effects of accretion on the binary are considered and estimates are made of the time required for the size of the binary orbit to be appreciably reduced.

Chapter VII is an account of a numerical investigation to check one of the results obtained in the last chpater. The work is also of interest as it involves the restricted three-body problem. The result is in satisfactory agreement with the work of the last chapter.

Chapter I: The Mechanism of Accretion.

Motion of a Particle Under an Inverse Square Central Force.

In order to give an account of the mechanism of accretion we have to use the results of the elementary theory of the motion of a particle under an inverse square central force so it will be useful to start by considering this theory. The particle whose motion is examined will later be considered as a particle of interstellar material and the centre of force will be a star.

We suppose a particle to experience a force μ/r^2 per unit mass of the particle directed towards the centre O which is at rest in a Newtonian frame of reference. μ is a constant and r is the distance of the particle from O. The particle moves in a plane through O so let its position be indicated by polar coordinates (r , θ) with respect to some axis through O. Then we have the equations

$$r^2\dot{\theta} = h \qquad \text{(conservation of angular momentum)}, \qquad (1.1)$$

$$\tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{r} = E \qquad \text{(conservation of energy)}, \qquad (1.2)$$

where h and E are constants; h being the angular momentum of unit mass of the particle about O and E being the energy of unit mass of the particle. Eliminating $\dot{\theta}$ from (1.1) and (1.2) gives

$$\dot{r}^2 + \frac{h^2}{r^2} = \frac{2\mu}{r} + 2E. \qquad (1.3)$$

Writing $r' = dr/d\theta$ we have $\dot{r}^2 = r'^2\dot{\theta}^2$

so from (1.1) $$\dot{r}^2 = r'^2 \frac{h^2}{r^4} \tag{1.4}$$

and $$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{r'}{r^2}. \tag{1.5}$$

From (1.3) and (1.4), $\frac{r'^2 h^2}{r^4} + \frac{h^2}{r^2} = \frac{2\mu}{r} + 2E$

hence from (1.5), $\frac{d}{d\theta}\left(\frac{1}{r}\right) = \pm\sqrt{\frac{2E}{h^2} + \frac{2\mu}{h^2 r} - \frac{1}{r^2}}.$

Taking the plus sign,

$$\frac{d(1/r)}{\sqrt{\left(\frac{2E}{h^2} + \frac{\mu^2}{h^4}\right) - \left(\frac{1}{r} - \frac{\mu}{h^2}\right)^2}} = d\theta$$

$$\therefore \cos^{-1}\left\{\frac{\frac{1}{r} - \frac{\mu}{h^2}}{\sqrt{\frac{2E}{h^2} + \frac{\mu^2}{h^4}}}\right\} = \theta - C$$

where C is a constant. This may be written in the form

$$\frac{l}{r} = 1 + e\cos(\theta - C) \tag{1.6}$$

which is a conic section, where

$$l = h^2/\mu \tag{1.7}$$

$$e = \sqrt{\frac{2Eh^2}{\mu^2} + 1} \tag{1.8}$$

and $\theta = C$ is the major axis of the conic section.

We wish to consider in more detail the motion when the particle is projected from infinity with speed v parallel to a fixed direction Ox and at a perpendicular distance σ

from O , as in Fig. 1.1. It is evident from (1.2) that the particle will eventually recede to infinity with speed v after being deflected through an angle β .

We have

$$h = v\sigma \quad , \quad E = \tfrac{1}{2} v^2.$$

We also note that by putting $r \to \infty$, $\theta \to \pi$, (1.6) gives

$$\cos C = 1/e. \tag{1.9}$$

To obtain the distance x from O at which the particle hits Ox we put $r = x$, $\theta = 0$ in (1.6) which gives

$$x = \frac{l}{2} = \frac{h^2}{2\mu} = \frac{v^2\sigma^2}{2\mu}. \tag{1.10}$$

From (1.1), the velocity component of the particle perpendicular to Ox when it hits Ox is

$$r\dot{\theta} = \frac{h}{x} = \frac{v\sigma}{x} ,$$

from (1.10)

$$= \sqrt{\frac{2\mu}{x}} . \tag{1.11}$$

Hence at this instant, from (1.2),

$$\frac{1}{2}\left(\dot{r}^2 + \frac{2\mu}{x}\right) - \frac{\mu}{x} = E = \frac{1}{2} v^2$$

$$\therefore \dot{r} = v. \tag{1.12}$$

The Bondi-Hoyle Accretion Mechanism.

We are now in a position to consider the accretion mechanism due to Bondi and Hoyle [7]. This mechanism

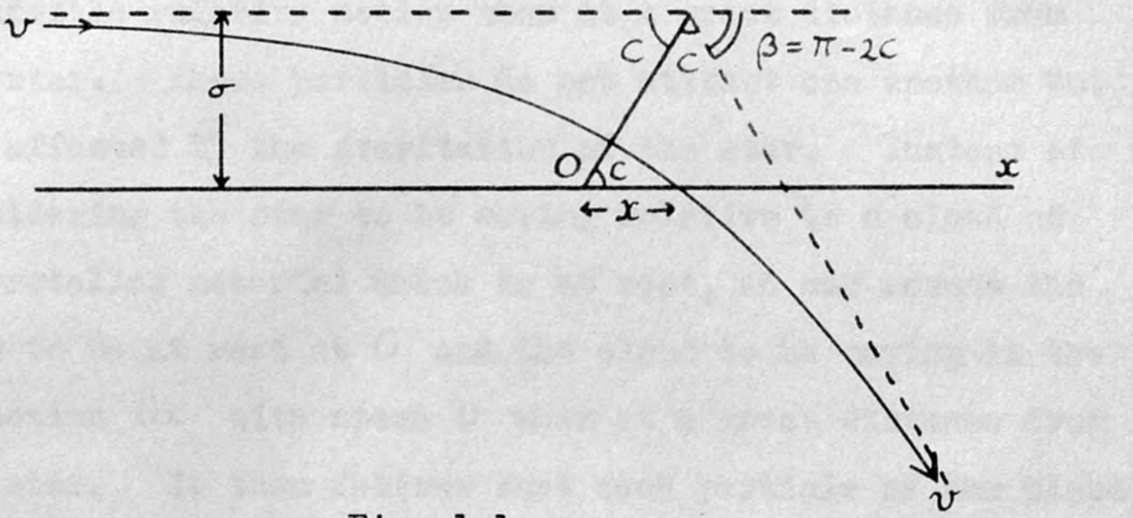


Fig. 1.1.

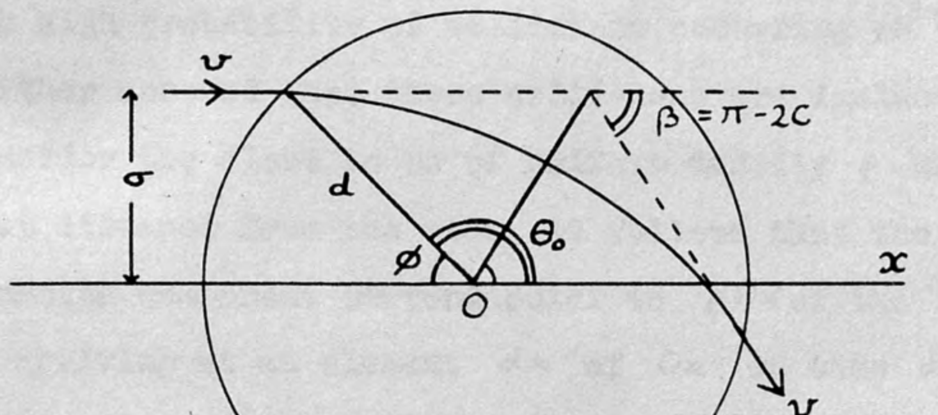


Fig. 1.2.

is a development of that originally put forward by Hoyle and Lyttleton [4]. Bondi and Hoyle considered the interstellar material to consist of particles which have negligible relative motion when at a great distance from the star. These particles do not attract one another but are affected by the gravitation of the star. Instead of considering the star to be moving relative to a cloud of interstellar material which is at rest, we may assume the star to be at rest at O and the cloud to be moving in the direction Ox with speed v when at a great distance from the star. It then follows that each particle of the cloud will describe a hyperbolic orbit, as considered above. Since, however, all particles hit Ox, it follows that there will be a high probability of collisions occurring on Ox. It is further assumed that these collisions are inelastic. If we consider the cloud to be of uniform density ρ when at a great distance from the star, it follows that the total momentum component perpendicular to Ox of the material arriving at an element dx of Ox in time dt will be zero. Consequently, from (1.12), this material will commence to move along Ox with velocity v and will become involved in further collisions with incoming particles. When the process begins, a collision on Ox may only be a chance occurrence but once material has begun to accumulate along Ox, collisions rapidly become inevitable. Thus a system is set up in which there is an accumulation of material along Ox and this is continually supplied with further material by incoming particles.

The assumption of inelastic collisions on Ox would appear to be justified if the cloud consists of dust particles. If the cloud consists of a gas, the initial assumption that the particles have negligible relative motion is equivalent to supposing that the gas is near zero absolute temperature. The assumption of inelastic collisions is justified if it can be supposed that the heat generated in the collisions is radiated away quickly so that the gas does not gain any appreciable thermal energy. This supposition will be justified if the gas contains limited amounts of dust, as this facilitates radiation.

We shall next establish the equations governing the unsteady motion of the material on Ox. Let $v(x,t)$ be the velocity of every particle at x in the direction Ox at time t and let $m(x,t)$ be the mass of material per unit length of Ox at x at time t. First, consider the conservation of mass on the element of the axis Ox between x and $x+dx$. The increase of mass in this element in time dt is

$$d\left(m(x,t)\,dx\right) = \frac{\partial m}{\partial t}\,dx\,dt + O\left([dt]^2\right).$$

In this time, a mass $\mathcal{M}(x,t)\mathcal{V}(x,t)\,dt$ enters the element from the left and a mass $\mathcal{M}(x+dx,t)\mathcal{V}(x+dx,t)dt$ leaves from the right. We also have to consider the material entering the element from the surrounding space. From (1.10) we have

$$dx = \frac{v^2}{\mu}\sigma\, d\sigma. \tag{1.13}$$

Consequently the material hitting the element dx will have originally come from a tubular region with Ox as axis of thickness $d\sigma$ and of radius σ. The mass of material passing any section of this region in time dt is

$$2\pi\sigma\, d\sigma . \rho v dt$$

from (1.13) $$= \frac{2\pi\mu\rho}{v}\, dx\, dt.$$

Consequently the mass of material arriving at Ox per unit length per unit time is

$$m = 2\pi\mu\rho/v \tag{1.14}$$

and so the mass arriving at our element dx is $m\,dx\,dt$. The conservation of mass then gives

$$\frac{\partial\mathcal{M}}{\partial t}dx\,dt + O\left([dt]^2\right) = m\,dx\,dt + \mathcal{M}(x,t)\mathcal{V}(x,t)dt - \mathcal{M}(x+dx,t)\mathcal{V}(x+dx,t)dt$$

$$= m\,dx\,dt - \frac{\partial}{\partial x}(\mathcal{M}\mathcal{V})\,dx\,dt + O\left([dx]^2\right).$$

Letting dx and $dt \to 0$ we have

$$\frac{\partial \mathcal{M}}{\partial t} = m - \frac{\partial(\mathcal{M}\mathcal{V})}{\partial x}. \tag{1.15}$$

Consider the material on the axis between x and $x+dx$ at time t. Its mass is $\mathcal{M}(x,t)\,dx$. As t increases, this material will move along the axis and will also be mixed with material which has recently hit the axis. The original material will gain momentum from the material added. This is equivalent to a force on the original material equal to the loss of momentum by the incoming material in the direction Ox per unit time. The mass of material per unit time hitting the axis between x and $x+dx$ is $m\,dx$. This has its velocity reduced by $v - V(x,t)$ so the force acting on the original material in the direction Ox is $m(v-V)\,dx$. It is also acted on by a force $-\mu \mathcal{M}\,dx/x^2$ in the direction Ox due to the gravitation of the star. (Here μ = mass of star × constant of gravitation). Hence the equation of motion for the original material is

$$\mathcal{M}\,dx\frac{dV}{dt} = m(v-V)\,dx - \mu\mathcal{M}\frac{dx}{x^2}. \qquad (1.16)$$

Since the x and t in $V(x,t)$ are connected by

$$\frac{dx}{dt} = V$$

we have
$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x}$$

so (1.16) becomes

$$\mathcal{M}V\frac{\partial V}{\partial x} + \mathcal{M}\frac{\partial V}{\partial t} = m(v-V) - \frac{\mu\mathcal{M}}{x^2}. \qquad (1.17)$$

(1.15) and (1.17) are the equations determining $\mathcal{M}$ and V.

In forming equation (1.17), it was assumed that a particle at x is not influenced by the particles on the axis at greater or smaller distances from the star. This is reasonable provided that two separated particles on the axis do not come together. This is the case if

$$\frac{\partial v}{\partial x} > 0. \tag{1.18}$$

Further consideration will be given to the unsteady problem in Chapter II. Here we shall consider the possibility of a "steady state," in which m and v are independent of t. In this case, (1.15) and (1.17) become

$$0 = m - \frac{d}{dx}(mv) \tag{1.19}$$

$$mv\frac{dv}{dx} = m(v - v) - \frac{\mu m}{x^2}. \tag{1.20}$$

m is independent of x, so (1.19) integrates to

$$mv = m(x - \alpha) \tag{1.21}$$

where α is a constant. For $x > \alpha$, $mv > 0$ and as $m \geqslant 0$, v must be positive. For $x < \alpha$, v must be negative. By considering the dynamics of the material on the axis near $x = \alpha$, we see that at $x = \alpha$, we must have $v = 0$. Putting $v = 0$, $x = \alpha$ in (1.20), we obtain

$$m(\alpha) = mv\alpha^2/\mu$$

which is useful as an order of magnitude estimate of m along the axis.

It is clear from this, that all the material hitting Ox at $x < \alpha$ must eventually fall into the star, while all the material hitting Ox at $x > \alpha$ will eventually escape to infinity. However, equations (1.19) and (1.20) do not determine α and consequently its value in any particular case must depend on the unsteady conditions existing before the steady state was established. But limits can be put on the value of α. In order that (1.18) should be satisfied, we require $\alpha = \mu/v^2$. Also Bondi and Hoyle claim that $\alpha < 2\mu/v^2$ in order that the system shall be stable, though their derivation of this is not immediately obvious. Even when α is given, the equations (1.19) and (1.20) do not determine a unique solution but so far as the star is concerned, the value of α gives all that is required, for the rate at which the star collects material (i.e. "the rate of accretion") is

$$m\alpha = \frac{2\pi\mu\rho}{v}\alpha \qquad (1.22)$$

Apart from the accretion, the mechanism also causes the star to experience a resistive force. To determine this force, consider a point x ($\gg \alpha$) on the axis. In the steady state, the amount of material passing along the axis past this point in unit time is the same as the amount of material hitting the axis in unit time between the point x and $x = \alpha$. Thus the mass of material per unit time passing x is

$$m(x-\alpha) \simeq mx.$$

Eliminating m from (1.20), (1.21) we obtain

$$\frac{dV}{dx} = \frac{v-V}{x-\alpha} - \frac{\mu}{x^2 V}$$

and putting $V = v(1+u)$ in this gives approximately, for large x,

$$\frac{du}{dx} + \frac{u}{x} = -\frac{\mu}{x^2 v^2}$$

the solution of which is

$$u = -\frac{\mu}{v^2 x}(\ln x + A)$$

where A is a constant which depends on the particular solution of (1.20) and (1.21). Therefore

$$V \sim v\left(1 - \frac{\mu}{v^2 x}\ln x\right) \tag{1.23}$$

Hence the momentum of the material passing x in unit time is

$$mxv\left(1 - \frac{\mu}{v^2 x}\ln x\right)$$

but the original momentum of this material was mxv so the loss of momentum is

$$\frac{m\mu}{v}\ln x = \frac{2\pi\rho\mu^2}{v^2}\ln x \tag{1.24}$$

which is an expression for the resistive force on the star. This expression tends to infinity with x but in reality other stars must produce a "cut-off effect" which enables us to limit the value of x. This will be dealt with in more detail below.

In the Bondi-Hoyle mechanism, it is interesting to note that, once a steady state has been established and provided a small region of the accretion column near $x = \alpha$ remains unchanged, the rest of the material on the axis may undergo disturbances without the steady state being permanently destroyed. For new stems of material will "grow" in both directions from the neutral point $x = \alpha$ to replace the disturbed material. (This result follows from the assumption that there is no pressure in the material on the axis.) The disturbances may be due to the gravitational interaction of material on the axis, as taken into account in the Lyttleton theory of comets, or in the outward moving stream they may be due to a drift of material off the axis for reasons which will be given later.

The Resistive Force in the Absence of Collisions.

If we entirely exclude the possibility of collisions between the particles of the cloud, even in the vicinity of Ox, then every particle will describe a hyperbolic orbit about O. We shall evaluate the resistive force experienced by the star in this case. As is usual in such work, we neglect the effect of particles colliding directly with the surface of the star because such particles will constitute a negligible portion of all the particles involved.

In order to obtain a general result we relax the condition that the interstellar material is of uniform density and assume only that the density, when at a great distance from the star, is independent of x. At O let us introduce right handed rectangular axes $Oxyz$ and in the plane Oyz let us introduce polar coordinates (σ, φ) with O as pole and Oy as axis. Then at great distances from the star, the density of the interstellar material is $\rho \equiv \rho(\sigma, \varphi)$. Consider the element of the plane Oyz which is bounded by the lines φ and $\varphi + d\varphi$ and the arcs σ and $\sigma + d\sigma$. The mass of material moving past the star in unit time, which was originally directed towards points in this element, is $d\sigma . \sigma d\varphi . v\rho(\sigma, \varphi)$. Now the components of the velocity gained by this material in passing the star, in the directions Ox, Oy, Oz are

$$-v + v\cos\beta, \quad -v\sin\beta\cos\varphi, \quad -v\sin\beta\sin\varphi$$

where $\beta = \pi - 2C$ (see Fig. 1.1). Since C is given by (1.9) and using (1.8), these components become

$$\frac{-2v}{\frac{v^4\sigma^2}{\mu^2}+1}, \quad \frac{-2\frac{v^3\sigma}{\mu}\cos\varphi}{\frac{v^4\sigma^2}{\mu^2}+1}, \quad \frac{-2\frac{v^3\sigma}{\mu}\sin\varphi}{\frac{v^4\sigma^2}{\mu^2}+1}$$

Now the force experienced by the star is equal and opposite to the momentum gained by all the material passing the star in unit time. Thus, the components of the force are

$$
\left.\begin{aligned}
X &= 2v^2\int_0^{\Sigma}\int_0^{2\pi}\sigma\rho\left(1+\frac{v^4\sigma^2}{\mu^2}\right)^{-1}d\sigma\, d\varphi \\
Y &= \frac{2v^4}{\mu}\int_0^{\Sigma}\int_0^{2\pi}\sigma^2\rho\cos\varphi\left(1+\frac{v^4\sigma^2}{\mu^2}\right)^{-1}d\sigma\, d\varphi \\
Z &= \frac{2v^4}{\mu}\int_0^{\Sigma}\int_0^{2\pi}\sigma^2\rho\sin\varphi\left(1+\frac{v^4\sigma^2}{\mu^2}\right)^{-1}d\sigma\, d\varphi
\end{aligned}\right\} \qquad (1.25)
$$

where Σ is the limit of integration for σ and will be discussed later.

We shall first apply (1.25) to the case of $\rho =$ constant. In this case we obtain

$$
\begin{aligned}
X &= \frac{2\pi\rho\mu^2}{v^2}\ln\left(1+\frac{v^4\Sigma^2}{\mu^2}\right) \qquad (1.26) \\
Y &= Z = 0
\end{aligned}
$$

If we let x be the distance at which the limiting particles (i.e. those originally at a distance Σ from Ox) cut Ox, we can use (1.10) to write (1.26) in the form

$$
\begin{aligned}
X &= \frac{2\pi\rho\mu^2}{v^2}\ln\left(1+2\frac{v^2}{\mu}x\right) \\
&\sim \frac{2\pi\rho\mu^2}{v^2}\ln x \qquad (1.27)
\end{aligned}
$$

for large x. It will be noted that this expression is identical with that for the force in the Bondi-Hoyle mechanism, (1.24). Consequently the value of the resistance does not depend upon the realization of the special conditions required to produce streaming along the axis. This statement applies only to the dominant term of the resistance. Since, however, the problem is in any event somewhat idealized, only this term need be considered.

This result is of some importance for reasons which will now be considered. The resistance (1.24) appears to depend upon the existence of the outward-flowing stream along the accretion axis. This depends in the first place upon the density of the cloud ahead of the star being sufficiently uniform. For, if the density varies appreciably in a direction transverse to the motion of the star, the stream will not form. In this case, even if collisions between cloud particles along the axis have their maximum possible effectiveness, it will be shown later that the material will drift to one side of the axis instead of accumulating along the axis.

In the second place, even when the density is uniform, the existence of the stream depends upon the attainment of a steady state. This in turn depends upon the particle-orbits passing through the accretion axis. But perturbations by other stars, though insufficient to change the general character of the orbits below the cut-off value of x , may nevertheless cause them to miss the axis by a small amount and so prevent the formation of the stream. Such effects will apply more to particles associated with the larger values of σ concerned (which are those contributing the main part of the force, as shown by the non-convergence of the expression for the force) than to particles associated with the smaller values of σ. Thus the latter particles may still produce an inward-flowing stream on the axis near the star in the manner of the Bondi-Hoyle mechanism, even where the outward-moving stream is not established, or only partially established.

The importance of the agreement between (1.24) and (1.27) lies in the fact that the force remains the same in spite of the above difficulties. At any rate, the force is the same in the two extreme cases where no particles collide and where all particles collide and we may therefore expect it to be the same for intermediate cases. We note further that the derivation of (1.27) does not depend physically upon the assumption of a steady state. The result will continue to hold for a non-steady state (including the case where the density depends on x) provided ρ is taken to be a suitable mean value of the density ahead of the star. It will also continue to hold if the path of the star is not rectilinear.

Conservation of Energy in the Absence of Collisions.

In the case of a fixed star with material moving past it, it is clear that the speed of a given particle tends eventually to the original value v. So there is no energy change. Since the star does not move, the force $F\ (= X\ )$ performs no work and this is in accordance with the conservation of energy. It is interesting to check that the energy is also conserved in the case where the star is moving with velocity v and the material is at rest. In the case of the star being at rest, let a particle originally at a distance σ from the axis be deflected through an angle β. Then the force on the star is

$$F = \int_0^\Sigma (2\pi\sigma\, d\sigma\, \rho v) \times v(1-\cos\beta)$$

which reduces to (1.26). If now we superpose a velocity v on the system, so that the star is moving through a medium at rest, the force F is unchanged, but it does work at a rate Fv. The ultimate velocity of the particle originally at a distance σ from the axis is now V where

$$V^2 = v^2 + v^2 - 2vv\cos\beta$$
$$= 2v^2(1-\cos\beta)$$

The energy gained by the particle is

$$\tfrac{1}{2} \times \text{mass} \times 2v^2(1-\cos\beta)$$

Hence the total energy gained by the medium

$$= \int_0^\Sigma \tfrac{1}{2} \times (2\pi\sigma\, d\sigma . \rho v) \times 2v^2(1-\cos\beta) = vF,$$

so the energy of the system is still conserved.

The Cut-off Distance.

Since (1.26) tends to infinity with Σ, there must in reality be some limitation on the value of Σ. This limitation is determined by the presence of neighbouring stars which prevent the orbits of the particles from being affected by the star under consideration. It seems, therefore, that the upper limit Σ , or "cut-off distance," should be taken as about half of the local mean interstellar distance. We have seen that (1.26) may be written

$$F = \frac{2\pi\varrho\mu^2}{v^2}\ln\left(1+2\frac{v^2}{\mu}x\right). \qquad (1.28)$$

Bondi and Hoyle considered that this formula should be used where the cut-off distance is substituted for x . This gives somewhat different results and so it is important to look more closely into the matter. We can do this by determining the value of F when the star is surrounded by a "sphere of cut-off." In reality, the cut-off surface will not be a sphere but will depend on the positions and masses of neighbouring stars. In Fig. 1.2, the star is considered to be at rest at the centre O of the cut-off sphere of radius d . Material particles enter the sphere from the left with velocity in a direction Ox and leave it with the same speed after being deflected through various angles β but without suffering collision. Consider a particle entering the sphere at a distance σ from the axis Ox . Let the coordinates of the point of entry with respect to O be (d, θ_0) and let the axis of symmetry of the particle's orbit make an angle C with Ox. Then from the geometry $\beta = \pi - 2C$. The force on the star is therefore

$$\int_0^d 2\pi\sigma d\sigma . v\rho . v\left(1-\cos\overline{\pi-2C}\right)$$

$$= 2\pi\rho v^2 \int_0^d (1+\cos 2C)\sigma d\sigma \qquad (1.29)$$

The equation of the orbit of the particle originally at a distance σ from Ox is given by (1.6),(1.7) and (1.8) where $h=\sigma v$, $E=\frac{1}{2}v^2-\frac{\mu}{d}$. Putting (d, θ_0) in (1.6) gives $C=\theta_0-\psi$ where $\cos\psi=\frac{1}{e}\left(\frac{\ell}{d}-1\right)$.

But $\theta_0=\pi-\phi$, where $\sin\phi=\sigma/d$.

$$\therefore C=\pi-\phi-\psi$$

$$\therefore \cos 2C=\cos(2\pi-2\phi-2\psi)=\cos(2\phi+2\psi).$$

Now $\cos 2\phi = 1-2\sin^2\phi = 1-2\frac{\sigma^2}{d^2}$

$$\sin 2\phi = 2\sin\phi\cos\phi = 2\frac{\sigma}{d}\cdot\frac{\sqrt{d^2-\sigma^2}}{d}$$

$$\cos 2\psi = 2\cos^2\psi - 1$$

$$= 2\frac{1}{e^2}\left(\frac{\ell}{d}-1\right)^2-1 = \frac{1}{e^2}\left(1-2\frac{\sigma^2v^2}{d\mu}+2\frac{\sigma^4v^4}{d^2\mu^2}-\frac{v^4\sigma^2}{\mu^2}\right)$$

$$\sin 2\psi = 2\sin\psi\cos\psi = 2\sqrt{1-\frac{1}{e^2}\left(\frac{\ell}{d}-1\right)^2}\cdot\frac{1}{e}\left(\frac{\ell}{d}-1\right)$$

$$= \frac{2}{e^2}\cdot\frac{\sigma v^2}{d\mu}\left(\frac{\sigma^2v^2}{d\mu}-1\right)\sqrt{d^2-\sigma^2}.$$

So (1.29) is

$$2\pi\rho v^2\int\left(1+\cos 2\phi\cos 2\psi - \sin 2\phi\sin 2\psi\right)\sigma d\sigma$$

which reduces to

$$\frac{4\pi\rho v^2}{d^2}\int\frac{(d^2-\sigma^2)\,\sigma\,d\sigma}{1+2\frac{v^2E}{\mu^2}\sigma^2}.$$

Integrating from $\sigma=0$ to Σ we obtain

$$\frac{\pi\rho\mu^2}{E}\left[\left(1+\frac{\mu^2}{2d^2v^2E}\right)\ln\left(1+\frac{2v^2E}{\mu^2}\Sigma^2\right)-\frac{\Sigma^2}{d^2}\right]. \qquad (1.30)$$

We note that if we keep Σ fixed and let $d\to\infty$, (1.30) becomes the usual formula (1.26). To get the full force we put $\Sigma=d$ in (1.30), i.e.,

$$\frac{\pi\rho\mu^2}{E}\left[\left(1+\frac{\mu^2}{2d^2v^2E}\right)\ln\left(1+\frac{2v^2Ed^2}{\mu^2}\right)-1\right]. \qquad (1.31)$$

We can now compare (1.31) with (1.26) and (1.28) putting $\Sigma=d$ and $x=d$ respectively. For $v\to 0$, (1.26) and (1.31) both behave like $2\pi\rho v^2d^2$ which $\to 0$ with v, whereas (1.28) behaves like $4\pi\rho\mu d$ which is constant. For large v, (1.26) and (1.31) both behave like

$$\frac{2\pi\rho\mu^2}{v^2}\ln\left(\frac{v^4d^2}{\mu^2}\right)$$

whereas (1.28) behaves like

$$\frac{2\pi\rho\mu^2}{v^2}\ln\left(\frac{v^2d}{\mu}\right)$$

which is less by a factor of 2. When $E=0$, i.e. $\frac{1}{2}v^2=\mu/d$, (1.31) has value

$$\pi\rho\mu d\times 2$$

while (1.26) and (1.28) both have value

$$\pi\rho\mu d\times 1{\cdot}6.$$

In view of the closer agreement between (1.31) and (1.26) than between (1.31) and (1.28), it appears that (1.26) with Σ equal to the cut-off distance is a better formula for the force than that suggested by Bondi and Hoyle.

It may be noted that a formula similar to (1.31) was obtained by Moiseyev in 1932 [16].

<u>Value of the Froce when Accretion is present.</u>

In reality, we expect that the material passing close to the star will collide on the axis Ox and form an accretion column as described by Bondi and Hoyle. According to their theory, the particles hitting the axis up to a maximum distance $x = 2\mu/v^2$ from the star are captured by it. Thus, these particles cannot be considered to contribute to the resistive force. Now a particle originally at a distance σ from Ox hits Ox at $x = v^2\sigma^2/2\mu$, from (1.10). Thus a particle cutting Ox at $x = 2\mu/v^2$ must originally have been at a distance σ from Ox given by

$$\frac{2\mu}{v^2} = \frac{v^2\sigma^2}{2\mu}$$

$$\therefore \sigma = 2\mu/v^2 \qquad (1.32)$$

Thus when accretion is present, the resistive force is

$$F = \frac{2\pi\varrho\mu^2}{v^2}\left\{\ln\left(1+\frac{v^4\Sigma^2}{\mu^2}\right) - \ln\left(1+\frac{v^4\sigma^2}{\mu^2}\right)\right\}$$

$$= \frac{2\pi\varrho\mu^2}{v^2}\left\{\ln\left(1+\frac{v^4\Sigma^2}{\mu^2}\right) - \ln\left(1+\frac{v^4}{\mu^2}\left(\frac{2\mu}{v^2}\right)^2\right)\right\}$$

$$= \frac{2\pi\varrho\mu^2}{v^2}\ln\frac{1}{5}\left(1+\frac{v^4\Sigma^2}{\mu^2}\right). \tag{1.33}$$

In considering the rectilinear motion of the star while undergoing accretion we use the equation of motion

$$\frac{d(mv)}{dt} = -F$$

where m is the mass of the star and F is given by (1.33). Thus

$$v\frac{dm}{dt} + m\frac{dv}{dt} = -F.$$

But dm/dt is the rate of accretion which we may denote by M. Hence

$$m\frac{dv}{dt} = -\left(F + Mv\right).$$

The total effective "resistance" is therefore

$$F + Mv. \tag{1.34}$$

It will be seen that for large values of v, the formulae (1.26) and (1.33) give similar results. At small velocities, however, the difference is more marked. In particular, (1.33) vanishes at a non-zero velocity, namely when

$$\frac{v^4\Sigma^2}{\mu^2} = 4$$

i.e.
$$v^2 = 2\mu/\Sigma. \tag{1.35}$$

For this value of v, we notice from (1.32) that $\sigma = \Sigma$ so that, at this velocity, the star is accreting all the material which comes within its cut-off sphere. For higher velocities, the accretion rate is

$$\frac{4\pi\mu^2\rho}{v^3} \tag{1.36}$$

as in the Bondi-Hoyle mechanism (maximum value), but for lower velocities, the accretion rate cannot exceed

$$\pi\Sigma^2\rho v, \tag{1.37}$$

this expression being the mass of all the material which comes within the cut-off sphere in unit time.

<u>Comparison of the Accretion and Resistive Force at different Velocities.</u>

For the purpose of numerical examples, it is found convenient in this work to use units of measurement which make ρ and G (the constant of gravitation) unity. We also take the unit of distance to be about one parsec. Taking

$$G = 6{\cdot}67 \times 10^{-8} \text{ c. g. s.} \quad [M^{-1}L^3T^{-2}]$$

$$\rho = 10^{-22} \text{ gm./c.c.} \quad [ML^{-3}]$$

and the unit of distance $= 3 \times 10^{18}$ cm. $[L]$

we obtain

unit of mass $= 2.7 \times 10^{33}$ gm.,

unit of time $= 3.87 \times 10^{14}$ sec. $= 1.23 \times 10^{7}$ years,

unit of velocity $= 0.0775$ km./sec.

It may be noted that if we take the unit of density to be $f \times 10^{-22}$ gm./c.c. , then the units of mass, time and velocity must be multiplied by f, $f^{-\frac{1}{2}}$ and $f^{\frac{1}{2}}$ respectively.

For the purpose of a comparison of the accretion and the force at different velocities, the graphs in Fig. 1.3 have been drawn for a star of unit mass. Using the above units and taking the cut-off distance to be half a parsec, the force (1.33) becomes

$$F = \frac{2\pi}{v^2} \ln \frac{1}{5} \left(1 + \frac{v^4}{4}\right).$$

This is the curve DEB in Fig. 1.3. The accretion formula (1.36) becomes

$$M = \frac{4\pi}{v^3}$$

which is the curve CB. The accretion formula (1.37) for low velocities becomes

$$M = \frac{\pi}{4} v$$

which is the curve OC. The accretion rate multiplied by the velocity gives the Mv term of (1.34) and this is the curve OAGB. When the force (1.33) is added to this, we obtain the total effective resistance to the motion of the

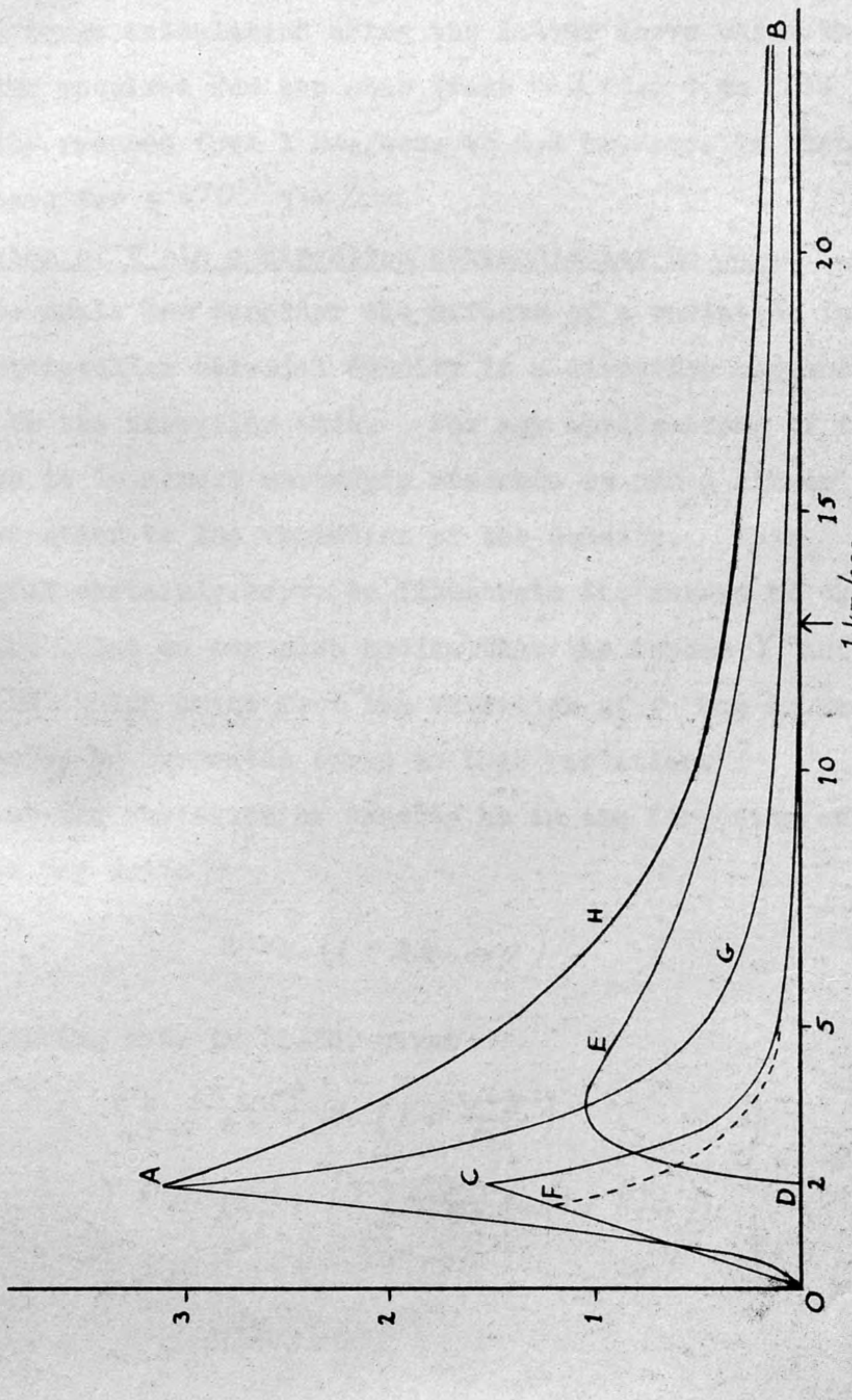


Fig. 1.3.

star (1.34) which is the curve OAHB.

A rough calculation using the latter curve shows that the time required for the star (mass $\simeq 1 M_\odot$) to have its velocity reduced from 1 km./sec. to 0.1 km./sec. is about 10 years for $\rho = 10^{-22}$ gm./c.c.

<u>Variation of ρ in a direction perpendicular to Ox</u>.

We shall now consider the effects of a variation in the interstellar material density in a direction perpendicular to the accretion axis. For any applications of the results it is almost certainly adequate to use a linear approximation to the variation of the density. This case will certainly serve to illustrate the nature of the results. But we may also notice that the forces Y and Z in (1.25) which arise from the variation of ρ are in fact unaffected by quadratic terms in this variation.

Let the variation of density be in the direction of Oy. Then we may write

$$\rho = \rho_0 (1 + b\sigma\cos\phi) \tag{1.38}$$

Substituting this in (1.25) gives

$$\left.\begin{aligned} X &= \frac{2\pi\rho_0\mu^2}{v^2}\ln\left(1+\frac{v^4\Sigma^2}{\mu^2}\right) \\ Y &= \pi\rho_0 b\mu\left\{\Sigma^2 - \frac{\mu^2}{v^4}\ln\left(1+\frac{v^4\Sigma^2}{\mu^2}\right)\right\} \\ Z &= 0 \end{aligned}\right\} \tag{1.39}$$

which are the components of the force experienced by the star when there are no collisions between the particles of the interstellar material. Thus if the value of b is appreciable, the star will tend to be deflected towards the denser regions of the cloud by the force Y.

We shall now obtain approximate values for the force components in the event of the particles colliding on the axis Ox. Consider the element of Ox between x and $x + dx$. We suppose all the particles arriving at this element in time dt to suffer inelastic collisions and thus coalesce. It should be remarked that the term "coalesce" is not intended to have any special physical connotations. It is convenient merely as a general description of the state of affairs treated by the mathematics. All that is implied is that subsequent to the collision, the particles move with a common velocity. This velocity will have an initial component v along Ox as we have seen, but it will also have a component U perpendicular to Ox since there will be a greater mass of material arriving from some directions than from others. We must now obtain this component U. The material hitting the axis in time dt which was originally directed towards points in the element of the plane Oyz bounded by the lines φ and $\varphi + d\varphi$ and the arcs σ and $\sigma + d\sigma$, has mass

$$d\sigma . \sigma d\varphi . v \rho dt \tag{1.40}$$

and on hitting the axis has a velocity component parallel to Oy of

$$-\sqrt{\frac{2\mu}{x}}\cos\varphi \tag{1.41}$$

from (1.11). The component of momentum of this material parallel to Oy is therefore

$$d\sigma.\,\sigma d\varphi.\,\varrho v dt\sqrt{\frac{2\mu}{x}}\cos\varphi.$$

Thus the resultant momentum of all the material originally at distances between σ and $\sigma+d\sigma$ from Ox is

$$\int_{\varphi=0}^{2\pi} d\sigma.\,\sigma d\varphi.\,\varrho v dt\sqrt{\frac{2\mu}{x}}\cos\varphi$$

$$=\sqrt{\frac{2\mu}{x}}\,\sigma d\sigma.\,v dt\int_0^{2\pi}\varrho\cos\varphi d\varphi. \tag{1.42}$$

From (1.40), the mass of such material is

$$\int_{\varphi=0}^{2\pi} d\sigma.\,\sigma d\varphi.\,\varrho v dt$$

$$=\sigma d\sigma.\,v dt\int_0^{2\pi}\varrho d\varphi. \tag{1.43}$$

Hence the velocity component of this material parallel to yO , after coalescing on the axis is obtained by dividing (1.42) by (1.43):

$$U=\sqrt{\frac{2\mu}{x}}\frac{\int_0^{2\pi}\varrho\cos\varphi d\varphi}{\int_0^{2\pi}\varrho d\varphi}. \tag{1.44}$$

Substituting for ρ from (1.38), this gives

$$U = b\mu/v \tag{1.45}$$

which, we note, is independent of x.

The next step is to determine the ultimate velocity components (v_∞, U_∞) along and perpendicular to Ox of the material colliding on the element of axis dx. The path of this material is given by (1.6), (1.7) and (1.8) with

$$\ell = xU \quad \text{and} \quad E = \tfrac{1}{2}(U^2+v^2) - \mu/x.$$

Putting $r = x$, $\theta = 0$ in (1.6) gives

$$\cos C = \frac{1}{e}\left(\frac{\ell}{x} - 1\right).$$

Putting $r = \infty$, $\theta = \theta_\infty$ in (1.6) gives

$$\cos(\theta_\infty - C) = \frac{-1}{e}$$

where θ_∞ is the direction of the other asymptote of the orbit from that which is approached by the material. It follows that the direction of the asymptote which is approached by the material is given by

$$\theta_\infty' = -(\theta_\infty - 2C)$$

The ultimate velocity $\omega = \sqrt{v_\infty^2 + U_\infty^2}$ of the material is given by the energy equation to be

$$\tfrac{1}{2}\omega^2 = E$$

$$\therefore \omega^2 = U^2 + v^2 - 2\mu/x.$$

Hence $\quad v_\infty = \omega\cos\theta_\infty', \quad U_\infty = \omega\sin\theta_\infty'$

which reduce to

$$v_\infty = \left(\mu^2 - \mu x U^2 + x^2 v U^2 \omega\right)\omega \big/ \left(\mu^2 + x^2 U^2 \omega^2\right) \tag{1.46}$$

$$U_\infty = \left(\mu x v U - \mu x U \omega + x^2 U^3 \omega\right)\omega \big/ \left(\mu^2 + x^2 U^2 \omega^2\right) \tag{1.47}$$

The mass of the material pursuing this orbit per unit time is, from (1.43),

$$2\pi\rho_0\mu\,dx/v.$$

The components of the momentum lost by this mass are the components of the force on the star, i.e.,

$$\left.\begin{aligned} X &= \int \frac{2\pi\rho_0\mu\,dx}{v}(v - v_\infty) \\ Y &= \int \frac{2\pi\rho_0\mu\,dx}{v} U_\infty \end{aligned}\right\} \tag{1.48}$$

Retaining only the dominant term for large x and $U \ll v$, we have

$$v - v_\infty \sim \frac{\mu}{xv}, \quad U_\infty \sim U.$$

Substituting these values in (1.48) gives

$$X = \frac{2\pi\rho_0\mu^2}{v^2}\ln x$$

$$Y = \pi\rho_0 v U \Sigma^2 = \pi\rho_0 b\mu\Sigma^2.$$

These expressions are the same as the dominant terms of (1.39). So again the force is the same whether collisions do or do not take place on Ox. We may expect that the force will be the same in the intermediate case where some particles collide and some do not.

In the case of collisions on Ox, not all the resulting material has sufficient energy to escape to infinity. This applies to material for which $E<0$

i.e.,
$$\tfrac{1}{2}(U^2+v^2)-\mu/x<0$$

i.e.,
$$x<2\mu/(U^2+v^2). \tag{1.49}$$

Thus all the material colliding on the axis at points x satisfying (1.49) remains gravitationally bound to the star. Such material therefore begins to move in elliptic orbits about the star. It will be shown later that if U is independent of x, which is the case for a linear variation of density, these elliptic orbits do not intersect, so that a species of circulatory flow is established. The total mass of material which is thus captured by the star per unit time is

$$\frac{4\pi\varrho_0\mu^2}{v(U^2+v^2)}. \tag{1.50}$$

Neglecting U^2, this rate is the same as the maximum rate in the Bondi-Hoyle mechanism for uniform density ϱ_0.

Strictly speaking, the condition (1.49) is not exact, since to escape from the star it is only necessary for the material to reach the cut-off distance d from the star. A particle just reaching this point would have energy $-\mu/d$. The condition should therefore be $E < -\mu/d$

i.e.
$$x < 2\mu/(U^2 + v^2 + 2\mu/d)$$

so the capture rate is

$$\frac{4\pi\rho_0\mu^2}{v\left(U^2 + v^2 + \frac{2\mu}{d}\right)}.$$

This rate differs little from (1.50) and is shown in Fig. 1.3 (taking $d = \frac{1}{2}$ parsec) by the curve FB, neglecting U^2. This compares with CB which is for (1.50), neglecting U^2.

We shall now prove that the elliptic orbits do not intersect when U is independent of x. We shall first find the condition that the equation

$$A + B\cos\theta + C\sin\theta = 0$$

has no real solution in θ. This may be written

$$A + \sqrt{B^2 + C^2}\cos(\theta + \gamma) = 0, \quad \gamma = \text{constant},$$

and the left hand side of this is never zero if

$$A^2 > B^2 + C^2$$

which is the required condition.

Let the particles, resulting from collisions on Ox at a distance x from O, move in the ellipse

$$\frac{l}{r} = 1 + e\cos(\theta+\varepsilon).$$

Then $l = x^2U^2/\mu$, $e = \sqrt{1+\frac{x^2U^2}{\mu^2}\left(V^2-\frac{2\mu}{x}\right)}$ where $V^2 = v^2 + U^2$.

Since $(x, 0)$ lies on this ellipse, we have

$$\cos\varepsilon = \frac{1}{e}\left(\frac{l}{x}-1\right) = \frac{1}{e}\left(\frac{U^2}{\mu}x-1\right).$$

So

$$\sin\varepsilon = \sqrt{1-\frac{1}{e^2}\left(\frac{U^2}{\mu}x-1\right)^2}$$

$$= \frac{1}{e}\sqrt{1+\frac{x^2U^2V^2}{\mu^2}-\frac{2U^2x}{\mu}-\frac{U^4x^2}{\mu^2}-1+\frac{2U^2x}{\mu}}$$

$$= \frac{1}{e}\sqrt{\frac{x^2U^2v^2}{\mu^2}} = \frac{xUv}{e\mu}.$$

Now let two of the ellipses be

$$\frac{l_1}{r} = 1+e_1\cos(\theta+\varepsilon_1),$$

$$\frac{l_2}{r} = 1+e_2\cos(\theta+\varepsilon_2).$$

Eliminating r, $\dfrac{l_1}{l_2} = \dfrac{1+e_1\cos(\theta+\varepsilon_1)}{1+e_2\cos(\theta+\varepsilon_2)}$

So $l_1 + l_1e_2\cos(\theta+\varepsilon_2) = l_2 + l_2e_1\cos(\theta+\varepsilon_1)$

$$\therefore l_1+l_1e_2\left[\cos\theta\cos\varepsilon_2-\sin\theta\sin\varepsilon_2\right] = l_2+l_2e_1\left[\cos\theta\cos\varepsilon_1-\sin\theta\sin\varepsilon_1\right]$$

$$\therefore (l_1-l_2)+\cos\theta\left(l_1e_2\cos\varepsilon_2-l_2e_1\cos\varepsilon_1\right)+\sin\theta\left(l_2e_1\sin\varepsilon_1-l_1e_2\sin\varepsilon_2\right)=0.$$

For no real solution we require

$$l_1^2e_2^2\cos^2\varepsilon_2 + l_2^2e_1^2\cos^2\varepsilon_1 - 2l_1l_2e_1e_2\cos\varepsilon_1\cos\varepsilon_2$$
$$+ l_2^2e_1^2\sin^2\varepsilon_1 + l_1^2e_2^2\sin^2\varepsilon_2 - 2l_1l_2e_1e_2\sin\varepsilon_1\sin\varepsilon_2 < l_1^2+l_2^2-2l_1l_2$$

i.e. $0 < \ell_1^2(1-e_2^2) + \ell_2^2(1-e_1^2) - 2\ell_1\ell_2\left(1 - e_1 e_2[\cos\varepsilon_1\cos\varepsilon_2 + \sin\varepsilon_1\sin\varepsilon_2]\right)$

i.e. $0 < \frac{U^4}{\mu^2}x_1^4\left(2\frac{U^2x_2}{\mu} - \frac{U^2V^2}{\mu^2}x_2^2\right) + \frac{U^4}{\mu^2}x_2^4\left(2\frac{U^2x_1}{\mu} - \frac{U^2V^2}{\mu^2}x_1^2\right)$

$$-2\frac{U^4}{\mu^2}x_1^2x_2^2\left(1 - \left[\left(\frac{U^2}{\mu}x_1 - 1\right)\left(\frac{U^2}{\mu}x_2 - 1\right) + \frac{U^2v^2}{\mu^2}x_1x_2\right]\right)$$

i.e. $0 < x_1^4\left(2x_2 - \frac{V^2}{\mu}x_2^2\right) + x_2^4\left(2x_1 - \frac{V^2}{\mu}x_1^2\right) + 2x_1^2x_2^2\left(\frac{V^2}{\mu}x_1x_2 - x_1 - x_2\right)$

i.e. $0 < 2x_1^4x_2 + 2x_1x_2^4 - 2x_1^3x_2^2 - 2x_1^2x_2^3 - \frac{V^2}{\mu}\left(x_1^4x_2^2 + x_2^4x_1^2 - 2x_1^3x_2^3\right)$

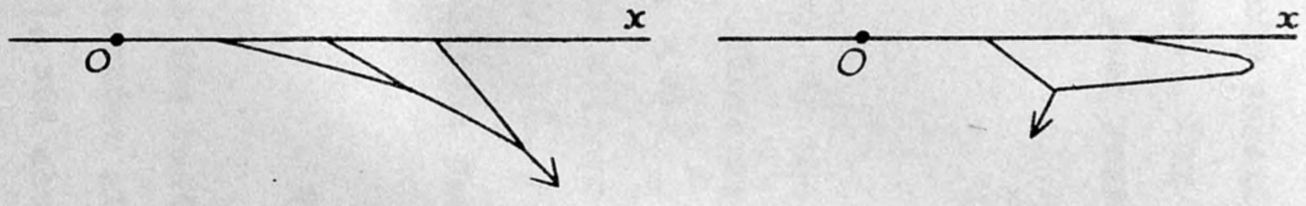
i.e. $0 < 2(x_1 - x_2)^2(x_1 + x_2) - \frac{V^2}{\mu}(x_1 - x_2)^2 x_1 x_2$

i.e. $\frac{V^2}{2} < \frac{\mu}{x_1} + \frac{\mu}{x_2}$

which is true since from the energy equation,

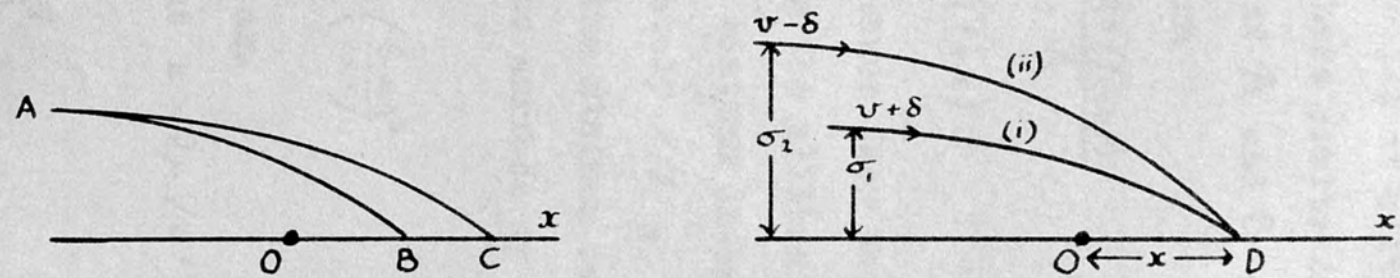
$$\tfrac{1}{2}V^2 - \mu/x_1 < 0 \quad \text{and} \quad \tfrac{1}{2}V^2 - \mu/x_2 < 0.$$

Hence the ellipses do not intersect. This result has been proved only for the case where U is independent of x. If dU/dx is positive and of sufficient magnitude, material in different orbits will suffer further collisions, as illustrated in Fig. 1.4(a) and the resulting motion will be highly complicated. Similarly if dU/dx is negative and of sufficient magnitude, collisions will occur as illustrated in Fig. 1.4(b). Even in the case of a linear variation of density, the circulating streams may be destroyed and the material absorbed by the star if the density gradient varies from place to place along the path of the star.



(a) (b)

Fig. 1.4.



(a) (b)

Fig. 1.5.

Returning to the case of a linear variation of density, we have seen that the material which remains gravitationally bound to the star describes a series of elliptic orbits. We shall now consider whether these orbits cut the surface of the star. For an ellipse, if A and B are the major and minor axes respectively, then

$$l = \frac{B^2}{A} = \frac{A^2(1-e^2)}{A}$$

$$\therefore A(1-e) = l/(1+e)$$

But $A(1-e)$ is the minimum distance from the star to the ellipse. Since we are assuming the ellipse to be highly eccentric, $e \simeq 1$. Hence the minimum distance from the star to the ellipse is approximately $l/2$. But $l = h^2/\mu$ and $h = xU = xb\mu/v$. Hence the minimum value of b such that the ellipse does not cut the surface of the star is given by

$$R = l/2 = \frac{\mu}{2}\left(\frac{bx}{v}\right)^2,$$

where R is the radius of the star.
For the outermost ellipse, we put $x = 2\mu/v^2$ approximately, from (1.49), and obtain

$$\frac{1}{b} = \sqrt{\frac{2}{R}\left(\frac{\mu}{v^2}\right)^3}.$$

As an example, consider a star like the sun with $R = 7 \times 10^{10}$ cm. and mass 2×10^{33} gm. We obtain

$$\frac{1}{b} = \frac{8 \cdot 23 \times 10^{19}}{v^3}$$

where v is here in km./sec. For $v = 5$ km./sec., we need $1/b = 6 \cdot 58 \times 10^{17}$ cm., i.e., a 100 per cent variation of density in less than a parsec. For $v = 1$ km./sec., on the other hand, we only need $1/b = 8 \cdot 23 \times 10^{19}$ cm., i.e., a 100 per cent variation of density in about 28 parsecs. Thus, while at high velocities, all captured material is likely to fall on to the star; at lower velocities, circulatory streams of material may be formed without any undue variation of density.

The differences between the accretion process in the Bondi-Hoyle mechanism and the capture process here described may now be discussed. In the first place we have to recall that when an accretion column has been established in accordance with the Bondi-Hoyle theory, the incoming cloud-particles collide with material already in the column and not directly with each other. Now the consequence of a transverse density-gradient in the cloud is to give the incoming material a resultant transverse momentum. In that case there cannot therefore be an accretion column lying symmetrically along the accretion axis. If the resultant transverse momentum is sufficiently large, there can in fact be no accumulation of material along the axis for the incoming particles to collide with;

the only collisions to be considered would be collisions along the axis between the incoming particles themselves. If also the cloud-density were sufficiently small, such collisions would be rare and there would be no accretion.

However, for any given cloud-density ρ_0, there must be a range of values of the density-gradient as measured by b which leads to a situation intermediate between that of the Bondi-Hoyle theory ($b = 0$) and the extreme case of no accretion. Such an intermediate case must have the general character of that treated above.

In this case all the material arriving at the axis Ox in $0 \leqslant x \leqslant X$ where X is given by (1.49), is captured by the star. Thus the amount of material "captured" is approximately the same as that "accreted" in the Bondi-Hoyle theory. In the present case, however, the captured material may not all fall on to the star but some may go to form a distribution revolving in the vicinity of the star.

We notice that, whereas the parameter X is indeterminate on steady-state considerations in the case $b = 0$, it becomes determinate in the case of $b \neq 0$ here considered.

It is important to appreciate how this case differs from that considered by Bondi and Hoyle. In the present case the incoming particles are assumed to "coalesce" on the accretion axis. This means merely that, owing to collisions in the vicinity of the axis, the particles arriving there at any instant acquire a common velocity. Owing to the assumption that b is not zero (though it may be very small) this velocity at any point of the axis has a non-zero component perpendicular to the axis. It is part of the assumption that on account of this lateral drift the material concerned is not involved in further collisions with subsequently arriving incoming particles. In the case considered by Bondi and Hoyle, on the other hand, the material proceeds to move along the axis where it does encounter further incoming particles. Thus, although the present case is intermediate between that of Bondi and Hoyle and that of no collisions, it does not yield either of these cases as limits. In particular, it is not possible to infer that X must be determinate in Bondi and Hoyle's case by attempting to treat it as a limit of this case.

The more interesting point is that the angular momentum about O per unit mass of the captured material is, on the average

$$\frac{1}{X}\int_0^X U\,dx = b\frac{\mu^2}{v^3} \tag{1.51}$$

taking $X = 2\mu/v^2$. If we take μ to correspond to the mass of the Sun and $v = 5$ km./sec. as a typical value, this gives about $1 \cdot 4 \times 10^{35} b$ cm.^{2}sec.$^{-1}$

We may compare this with the angular momentum per unit mass of the Sun itself. According to values given by Chapman[17], this is about $5 \times 10^{14} \, cm^2 sec^{-1}$. Thus the value given by (1.51) is greater than or equal to this if b^{-1} is less than about 100 parsecs. Now b^{-1} is the distance over which the cloud-density changes by 100 per cent, and 100 parsecs would be a large value for the radius of a cloud[18].

It is then tempting to conclude that any significant accretion of mass is likely to be accompanied by a significant change in the star's angular momentum. Nevertheless it has to be noted that, although the density-gradient required to produce this result is so small, its component transverse to the star's path would have to be in the same direction over a great length of this path. Without much further investigation it is therefore impossible to say whether the effect has any general significance in regard to the phenomenon of stellar rotation. It may serve merely to endow separate portions of accreted material with varying amounts of angular momentum about the star in a manner suggested by Gething in relation to Lyttleton's theory of comets.

In connection with the above discussion, it should be noted that a variation of the undisturbed material velocity in a direction transverse to the direction of the star's motion can also produce angular momentum in the accreted material and this has been investigated by McCrea[11].

The Effect of Temperature on the Resistive Force.

We have assumed so far, that the particles of the undisturbed interstellar material have no relative motion. If the particles are the molecules of a gas this is only true if the gas is at zero absolute temperature. At other temperatures each particle has a random component of velocity superposed on the common velocity v relative to the star. Provided this random component is not too large, we may expect the mechanism producing the resistive force still to hold approximately. It is difficult to make a quantitative estimate of the temperature such that the force is not appreciably changed, but this will now be attempted in the case where there are no collisions between particles.

In the case where there is no random velocity, the resistive force is $\rho F(v)$ where

$$F(v) = \frac{\pi\mu^2}{E}\left[\left(1+\frac{\mu^2}{2d^2v^2E}\right)\ln\left(1+\frac{2v^2Ed^2}{\mu^2}\right)-1\right] \qquad (1.52)$$

from (1.31).

Now suppose each particle has superposed on its initial velocity v, a velocity c. Let c have the same magnitude and direction for every particle and let the direction be that of a given line through O making an angle Θ with Ox (Fig. 1.2). Then the resultant initial velocity of each particle will be V given by

$$V^2 = v^2 + c^2 + 2vc\cos\Theta \tag{1.53}$$

and this will make an angle Φ with Ox, given by

$$\sin\Phi = \frac{c}{V}\sin\Theta \tag{1.54}$$

It follows that the resulting force will be $\rho F(V)$ making an angle Φ with Ox. The component of this along Ox is $\rho F(V)\cos\Phi$.

Next suppose the particles have superposed on their initial velocity v a random velocity component. This component is to be considered as of constant magnitude but uniformly distributed in direction. The resulting force will now be along Ox and of magnitude

$$\frac{\rho}{4\pi}\int_{\alpha=0}^{2\pi}\int_{\Theta=0}^{\pi} F(V)\cos\Phi\sin\Theta\, d\Theta\, d\alpha$$

$$= \frac{\rho}{2}\int_0^{\pi} F(V)\cos\Phi\sin\Theta\, d\Theta. \tag{1.55}$$

We shall evaluate this integral on the assumption that terms of order $(c/v)^3$ and higher terms may be neglected. Let us write $V = v + \delta v$ and $F(V) = A_0 + A_1\delta v + A_2(\delta v)^2 + \ldots$ where $A_0, A_1, A_2, \ldots$ are independent of δv and Θ.

We have

$$\delta v = v\left(\frac{1}{2}\frac{c^2 + 2vc\cos\Theta}{v^2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left[\frac{c^2+2vc\cos\Theta}{v^2}\right]^2 + \ldots\right)$$

$$= v\left(\frac{c}{v}\cos\Theta + \frac{1}{2}\frac{c^2}{v^2}\sin^2\Theta\right)$$

and $$\cos\Phi = \sqrt{1-\frac{c^2}{V^2}\sin^2\Theta}$$

$$= \sqrt{1-\frac{c^2}{v^2}\sin^2\Theta\left(1-\frac{2\delta v}{v}+\dots\right)}$$

$$= 1-\frac{1}{2}\frac{c^2}{v^2}\sin^2\Theta .$$

Thus (1.55) becomes

$$\frac{\rho}{2}\int_0^\pi\left[A_0+A_1 v\left(\frac{c}{v}\cos\Theta+\frac{1}{2}\frac{c^2}{v^2}\sin^2\Theta\right)+A_2 v^2\frac{c^2}{v^2}\cos^2\Theta\right]$$
$$\times\left(1-\frac{1}{2}\frac{c^2}{v^2}\sin^2\Theta\right)\sin\Theta\, d\Theta .$$

$$= \rho A_0 + \rho\frac{c^2}{v^2}\left[-\frac{1}{3}A_0+\frac{1}{3}A_1 v+\frac{1}{2}A_2 v^2\right] \qquad (1.56)$$

so the fractional change in the force is

$$\left(\frac{c^2}{v^2}\right)\left[-\frac{1}{3}+\frac{1}{3}\frac{A_1 v}{A_0}+\frac{1}{2}\frac{A_2 v^2}{A_0}\right].$$

In this expression, the sum of the terms in the square bracket is of order unity. This can be indicated by considering the limiting forms of $F(v)$ when $v\to 0$ and when $v\to\infty$, i.e.

$$F(v)\sim 2\pi d^2 v^2 \quad \text{as} \quad v\to 0$$

in which case the sum of these terms is 5/6, and

$$F(v)\sim 8\pi\mu^2\frac{\ln v}{v^2} \quad \text{as} \quad v\to\infty$$

in which case the sum of these terms is ½.

Thus the fractional change in the force is of order c^2/v^2.

This formula will also hold approximately in the case where the velocity magnitude is also random if we take c to represent the R.M.S. velocity. For the change not to be more than about 25 per cent we need $c \leq \frac{1}{2}v$. Now for gases we have the relation

$$pv = RT = \frac{1}{3}Mc^2$$

$$\therefore T = Mc^2/3R$$

where T is the absolute temperature, R is the gas constant, c is the R.M.S. velocity of the gas molecules and M is the molecular weight of the gas. For atomic hydrogen, $M=1$ and putting $c^2 = \frac{1}{4}v^2$ we have

$$T = v^2/12R$$

$$\therefore T = 10v^2 \,°A \qquad (1.57)$$

if v is measured in km./sec. (1.57) gives the value of T above which the force will be altered by more than about 25 per cent from its value at $T = 0°A$. (1.57) applies only to the case of no collisions between the particles.

It is extremely difficult to assess the effect of temperature when collisions occur and an accretion column forms. It may be worthwhile to mention here an extremely simplified case which has been considered, not to determine the effect of temperature, but to examine if under certain circumstances the accretion rate can be increased. Consider the star to be at rest at O and the particles which issue from A (at a great distance) in Fig. 1.5(a) to be of two kinds: (i) those with velocity $v+\delta$ and (ii) those with velocity $v-\delta$; there being equal numbers of each type of particle so that the average velocity is the usual v. The first type of particle will hit the axis at C and the second type at B, say. As will now be shown, this model gives an increased rate of accretion by a factor of $(1-\delta^2/v^2)^{-3}$ over the rate in the case where all the particles have velocity v. The effect is small, e.g. for δ/v as large as $0{\cdot}45$, the rate of accretion is only doubled.

In Fig. 1.5(b), consider the point D on the axis at a distance x from the star. Two streams of particles will collide at D. These are type (i) particles originally at a distance σ_1 from the axis and type (ii) particles originally at a distance σ_2 from the axis. From (1.10),

$$x = \frac{\sigma_1^2(v+\delta)^2}{2\mu} = \frac{\sigma_2^2(v-\delta)^2}{2\mu}$$

$$\therefore dx = 2\sigma_1 d\sigma_1 \frac{(v+\delta)^2}{2\mu} = 2\sigma_2 d\sigma_2 \frac{(v-\delta)^2}{2\mu}. \qquad (1.58)$$

The mass of type (i) particles hitting Ox between x and $x+dx$ in time dt is

$$2\pi\sigma_1 d\sigma_1 (v+\delta)dt(\rho/2).$$

(The $(\rho/2)$ is because half of the density is due to type (i) and half to type (ii) particles). This is, from (1.58)

$$m_1 = \pi\rho\, dt \frac{\mu\, dx}{(v+\delta)}.$$

Similarly, the mass of type (ii) particles is

$$m_2 = \pi\rho\, dt \frac{\mu\, dx}{(v-\delta)}.$$

Hence the total mass hitting dx in time dt is

$$m_1 + m_2 = \pi\rho\, dt\, \mu\, dx \frac{2v}{v^2-\delta^2}. \qquad (1.59)$$

The velocity V of this along the axis is given by the momentum equation (assuming all the material coalesces and then moves freely under the gravitation of the star),

$$(m_1+m_2)V = m_1(v+\delta) + m_2(v-\delta)$$

$$\therefore V = (v^2-\delta^2)/v.$$

The maximum value of x such that material hitting the axis between x and $x+dx$ does not escape to infinity is

$$X = \frac{2\mu}{V^2} = \frac{2\mu v^2}{(v^2-\delta^2)^2}.$$

From (1.59), the accretion rate is therefore

$$(m_1 + m_2)X = \pi \rho \mu \cdot \frac{2v}{v^2 - \delta^2} \cdot \frac{2\mu v^2}{(v^2 - \delta^2)^2}$$

$$= \frac{4\pi \mu^2 \rho}{v^3}\left(1 - \frac{\delta^2}{v^2}\right)^{-3}.$$

Chapter II: The Unsteady Accretion Problem.

In the Bondi-Hoyle mechanism of accretion, we saw in Chapter I that a consideration of the "steady state", in which a star is moving through a cloud of interstellar material of uniform density, does not give a unique solution for the motion. Bondi and Hoyle therefore concluded that the amount of material captured by a star must depend on the perturbations previously suffered by the star and its column of interstellar material. They then obtained an approximate solution in the case of a star moving from empty space into a cloud of uniform density, the density of the interstellar material being discontinuous across the surface of the cloud.

In the present chapter a method is suggested which may be used when the density is not discontinuous at the edge of the cloud. The method may also be used when the density within the cloud varies with time. A case similar to that considered by Bondi and Hoyle and also one other case have been examined using this method. Considerable computation was required and this was performed on the Manchester University Electronic Computer.

The Manchester University Electronic Computer has been constructed by Ferranti Ltd. and is the only engineered electronic machine in operation in this country.

It is a development of the experimental machine built in the Department of Electrical Engineering at Manchester University by T. Kilburn and F.C. Williams. Also the Manchester University Electronic Computer is the only machine in the country with an auxiliary magnetic storage system, which is essential for large problems.

The Equations.

Using a slightly different notation, the equations governing the unsteady state, (1.15) and (1.17), may be written

$$\frac{\partial m}{\partial \tau} = a - \frac{\partial (mv)}{\partial \xi} \tag{2.1}$$

$$mv\frac{\partial v}{\partial \xi} + m\frac{\partial v}{\partial \tau} = a(c - v) - \frac{\mu m}{\xi^2} \tag{2.2}$$

where τ is the time, ξ is the distance along the accretion axis, c is the speed of the star relative to the cloud, a is the mass of material hitting unit length of the axis per unit time and the other variables are as in Chapter I. In the new notation, we let the neutral point ($x = \alpha$) be $\xi = \beta$.

It may be noted here that a steady state (defined as a state in which m, v and a are all independent of τ) cannot exist unless ρ is independent of τ . But if ρ becomes independent of τ , it does not follow that a steady state will occur immediately. However, the system will immediately start tending towards such a steady state.

The Proposed Method.

Let us concentrate attention on a particle of material in the column. Its position at any time will be $\xi = X(\tau)$, say. Let the values of $\mathcal{V}$, a and m in the neighbourhood of the particle be V, A and M respectively. Now $\mathcal{V}(\xi, \tau)$ is the velocity of a particle at point ξ at time τ. So

$$\mathcal{V}(X(\tau), \tau) = V(\tau) = \frac{dX}{d\tau}, \tag{2.3}$$

and

$$\frac{dV}{d\tau} = \frac{\partial \mathcal{V}}{\partial \xi}\frac{dX}{d\tau} + \frac{\partial \mathcal{V}}{\partial \tau};$$

which from (2.3)

$$= \frac{\partial \mathcal{V}}{\partial \xi}\mathcal{V} + \frac{\partial \mathcal{V}}{\partial \tau}. \tag{2.4}$$

Substituting (2.4) in (2.2) we obtain

$$\frac{dV}{d\tau} = \frac{A}{M}(c - V) - \frac{\mu}{X^2}. \tag{2.5}$$

We also have

$$M(\tau) = m(X(\tau), \tau),$$

so

$$\frac{dM}{d\tau} = \frac{\partial m}{\partial \xi}\frac{dX}{d\tau} + \frac{\partial m}{\partial \tau}$$

$$= \frac{\partial m}{\partial \xi} v + \frac{\partial m}{\partial \tau}. \qquad (2.6)$$

Writing (2.1) as

$$\frac{\partial m}{\partial \tau} + \frac{\partial m}{\partial \xi} v = a - m\frac{\partial v}{\partial \xi},$$

and substituting for the left-hand side from (2.6) we have

$$\frac{dM}{d\tau} = A - M\left(\frac{\partial v}{\partial \xi}\right)_{\xi = X}. \qquad (2.7)$$

Let us suppose equations (2.3), (2.5) and (2.7) to be solved, giving X, V and M as functions of τ starting from the boundary condition $\xi = \xi_1$ when $\tau = \tau_0$. Then the function X would represent the path of the particle as shown by $\xi_1 MBK$ in Fig. 2.1. In this case (as shown in the diagram) the particle starts to move away from the star but then returns and falls into it. If a somewhat larger value of ξ had been taken in the boundary condition, say $\xi = \xi_2$ then the path would have been as $\xi_2 AI$, indicating that the particle escapes from the star. If the density ρ is constant so that a also is constant, then the system must eventually become steady and there is a value ξ_3 of ξ at $\tau = \tau_0$ such that the particle neither falls into the star nor escapes from it, but follows a

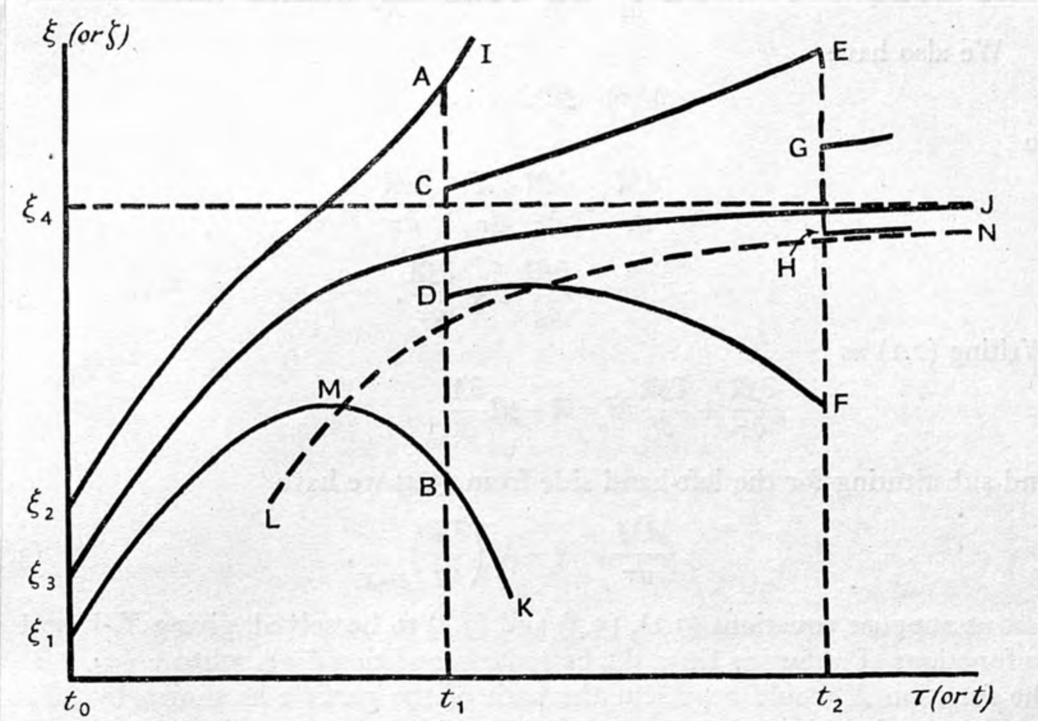

ξ (or ζ)
A
I
E
G
C
ξ4
J
N
H
D
M
F
B
L
ξ2
ξ3
K
ξ1
t0
t1
t2
τ (or t)

Fig. 2.1.

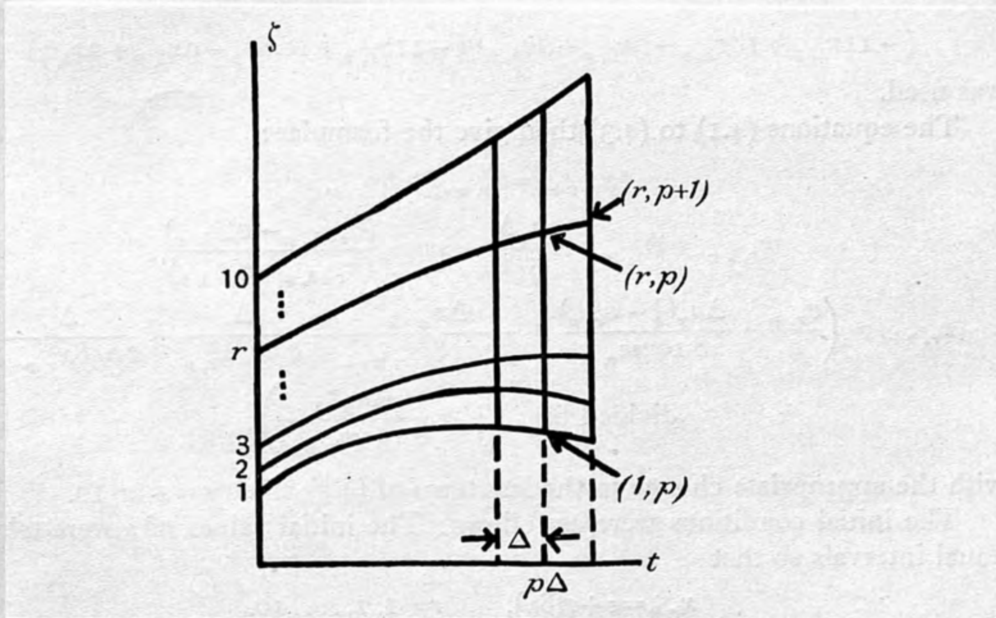

ζ
(r, p+1)
(r, p)
10
r
3
2
1
(1, p)
Δ
pΔ
t

Fig. 2.2.

path $\xi_3 J$ and comes to rest at $\xi = \xi_4$. This value ξ_4 is what we denoted by β above.

In general, the method may be looked upon as a way of determining the alteration in β caused by a change from one constant value of ρ to another. It is not necessary that the change be instantaneous; the value of ρ between the two constant values may be any function of time.

The Numerical Solution.

The object of the numerical investigation was to obtain the value of β in the case of a star entering a cloud with a discontinuous change of density at its edge, and also to examine the effect on β of a slower change in ρ .

On account of the gradient on the right-hand side of (2.7), it is not possible to solve (2.3), (2.5) and (2.7) for a single particle. It is necessary to solve for a number of paths simultaneously. It is a property of the equations (parabolic equations) that disturbances are propagated along the paths $\xi_1 BK$, etc. and not from one path to another.

The star is supposed to enter a cloud from empty space at a time $\tau = 0$. The value of a is supposed to increase either instantaneously, or else linearly with time from zero to its final value which is reached at $\tau = \mu t_c / c_3$, after which a remains constant. (The factor μ/c^3 is to make t_c a dimensionless constant.) The surface of the cloud is taken to be of such a shape that a is a function of τ only. It is at this point that the problem here considered differs from that of Bondi and Hoyle. They considered the cloud to have a plane boundary when at a great distance from the star. ~~As the star approaches the cloud, the boundary becomes distorted from the star.~~ As the star approaches the cloud, the boundary becomes distorted and they take this into account. In this problem, on the other hand, the surface of the cloud (and in general the surfaces of constant density), when at a great distance from the star, is supposed to be such that as the star approaches, the surface will become distorted and will fall on to the axis $O\xi$ in such a way that all points on it hit the axis at the same instant. This makes a a function of τ only and independent of ξ. The surfaces of constant density in the undisturbed cloud will be surfaces of revolution with the path of the star as axis, and the surfaces will present a concave side towards the star. In view of the difference between this problem and that of Bondi and Hoyle, it is not justifiable to expect that the values of β will be the same in the two cases.

The general method given in this paper could be applied to a cloud with a plane boundary, but this would make a a function of τ as well as of ξ. The extra information that would have to have been stored in the electronic machine would have considerably reduced the rate of computation. For this reason it was decided to use the simpler case for the purpose of obtaining a result.

The equations may first be made dimensionless by the transformation

$$X = 8(\mu/c^2)x,\ \xi = 8(\mu/c^2)\zeta,\ \tau = (\mu/c^3)t,\ M = 1024(\mu\bar{A}/c^3)m,$$

$$U = 4c\bar{v},\ V = 4cv,\ A = \bar{A}a$$

where $\bar{A}$ here indicates the final value attained by A. The equations (2.3), (2.5) and (2.7) become

$$\frac{dx}{dt} = \frac{v}{2}, \tag{2.8}$$

$$\frac{dv}{dt} = \frac{1}{4}\left(\frac{a(1-4v)}{1024\,m} - \frac{1}{64x^2}\right), \tag{2.9}$$

$$\frac{dm}{dt} = \frac{a}{1024} - \frac{m}{2}\left(\frac{\partial\bar{v}}{\partial\zeta}\right)_{\zeta=x}. \tag{2.10}$$

The numerical factors in the transformation were necessitated by the decision to use an electronic calculating machine. In any machine there are limits between which numbers must lie. In the machine used the limits were

$$-\tfrac{1}{2} \leq N < \tfrac{1}{2}.$$

Scale factors must therefore be inserted to ensure that the variables all lie in this range. The equations involve a single parameter a. In the equations, the derivatives with respect to t indicate ratios of small changes which, in the distance-time diagram (Fig. 2.1), are taken <u>in the direction of the path</u>.

The solution of the problem was obtained by building up a network formed by ten particle paths and the lines $t =$ constant as in Fig. 2.2. Denoting the paths by the integers 1 to 10 and taking equal intervals Δ in t, we may attach a number pair (r, p) to each point of the network, where r is the number of the path on which the point lies and p indicates that the point is on $t = p\Delta$. The values of the variables at any point of the network will be denoted by the appropriate symbol with the number pair of the point as a suffix.

Since only about three significant figures were required in β, it was not thought worth while to use more than second-order approximations for the derivatives in equations (2.8) to (2.10). Thus in (2.8), $(dx/dt)_{r,p}$ was represented by

$$(x_{r,p+1} - x_{r,p-1})/2\Delta,$$

and similarly for $(dm/dt)_{r,p}$ in (2.10). As a formula of this sort for $(dv/dt)_{r,p}$ would have caused oscillation in the solutions, it was decided to use

$$v_{r,p+1} - v_{r,p} = \frac{\Delta}{2}\left\{\left(\frac{dv}{dt}\right)_{r,p+1} + \left(\frac{dv}{dt}\right)_{r,p}\right\}.$$

For $(d\bar{v}/d\xi)_{r,p}$ in (2.10) the approximation

$$(v_{r+1,p} - v_{r-1,p})/(x_{r+1,p} - x_{r-1,p})$$

was used except at the edges of the network (r = 1 and 10) where the third-order approximation of the type

$$(-11v_{1,p} + 18v_{2,p} - 9v_{3,p} + 2v_{4,p})/(-11x_{1,p} + 18x_{2,p} - 9x_{3,p} + 2x_{4,p})$$

was used.

The equations (2.8) to (2.10) then give the formulae:

$$x_{r,p+1} = x_{r,p-1} + \Delta v_{r,p}, \tag{2.11}$$

$$m_{r,p+1} = m_{r,p-1} + \frac{a_p\Delta}{512} - \Delta m_{r,p}\frac{(v_{r+1,p} - v_{r-1,p})}{(x_{r+1,p} - x_{r-1,p})}, \tag{2.12}$$

$$v_{r,p+1} = \left(\frac{v_{r,p}}{4} + \frac{\Delta a_p(\frac{1}{4} - v_{r,p})}{8192\, m_{r,p}} + \frac{\Delta a_{p+1}}{32768\, m_{r,p+1}} - \frac{\Delta}{2048\, x^2_{r,p}} - \frac{\Delta}{2048\, x^2_{r,p+1}}\right)$$

$$\text{divided by } \left(\frac{1}{4} + \frac{\Delta a_{p+1}}{8192\, m_{r,p+1}}\right), \tag{2.13}$$

with the appropriate change in the last term of (2.12) when r = 1 or 10.

The initial conditions were as follows: The initial values of x were taken at equal intervals so that

$$x_{r,0} = \epsilon + r(\delta x), \qquad r = 1, 2, \ldots, 10.$$

ϵ and δx being suitably chosen constants. There is no material in the column initially, so

$$m_{r,0} = 0$$

and the initial velocity is c, so

$$v_{r,0} = \frac{1}{4} \quad \text{and} \quad a_0 = 0.$$

The first-order approximations for the variables at $t = \Delta$ were taken:

$$x_{r,1} = \epsilon + r(\delta x) + \tfrac{1}{8}\Delta,$$

$$m_{r,1} = \Delta^2 / 2048 t_c,$$

$$v_{r,1} = \frac{1}{4} - \frac{\Delta}{1024 x_{r,1}^2},$$

$$a_1 = \Delta / t_c.$$

These formulae are for the case where a increases linearly from zero at $t = 0$ to unity at $t = t_c$ and then remains constant. In the case of an instantaneous increase in the value of a from zero to unity, the last three formulae are changed to

$$m_{r,1} = \Delta / 1024,$$

$$v_{r,1} = \frac{1}{4} - \frac{\Delta}{768 x_{r,1}^2}$$

$$a_1 = 1.$$

By using the above formulae the network can be built up. The best way to determine the value of β is to find, for various values of t, the value of x (which we shall refer to as η) at which $v = 0$. This can be determined by an interpolation between the points of the network on the lines $t =$ constant. If the results are plotted, a curve is obtained like LMN in Fig. 2.1. This curve passes through all the maxima of the paths and so tends to $\zeta = \beta/8(\mu/c^2)$.

It was found that as soon as the maximum of a path had been passed, the path rapidly fell away towards the t axis. In machine computation this meant that the velocity grew (negatively) so rapidly that there was a danger of it going out of the number range. To avoid this, arrangements were made at certain intervals of t to interpolate between the paths and to continue the integration with ten new paths. Thus, in Fig. 2.1, suppose that the integration had been performed up to $t = t_1$, so that the network was $\xi_1 \xi_2$ AB. At this point a new set of paths would be formed by selecting five of the original paths and forming five new ones by mid-point interpolation between the old paths. These ten new paths would intersect AB between C and D, say. The next part of the network would be like CDFE. At $t = t_2$ a further ten paths would be formed and the integration would continue from GH.

Considering the points on AB, the changes in v and x from one point to another are small until the velocity goes negative. After this, the changes become large. These large changes reduce the accuracy of the interpolation between paths. For this reason it was found wise to choose the disposition of CD with respect to AB so that at D, v was just slightly positive.

The instructions to cause the machine to carry out the calculations had to be punched in a certain code on to paper tape. The tape was about 24 feet long. These instructions were put into the machine by means of a tape reader. The value of t_c was specified on the tape and so each different value of t_c involved a slight alteration of the tape. The values of ϵ, δx (used in forming $x_{r,0}$) and Δ were specified by the setting of switches on the machine. When these switches had been set, the machine was caused to form the initial conditions and to integrate up to $t = 1$ and then stop. The values of the variables at the points on $t = 1$ could then be printed out on paper if desired or they could be examined by means of a matrix of dots which appeared on the screen of a cathode ray tube. It was then necessary to tell the machine, by the manipulation of switches, whether to carry on the integration or whether first to form ten new paths. In the latter case, the disposition of the new paths with respect to the old ones had to be specified again by switches. The machine would then integrate up to $t = 2$, when the decisions would be taken again as at $t = 1$, after an examination of the variables. To avoid the risk of errors due to a unit of the machine not functioning properly, it was arranged for each part of the calculation to be repeated until two consecutive results agreed. This could be tested by the machine itself. Only when agreement was reached, would the next part of the calculation be attempted. Integrations were tried with $\Delta = \frac{1}{8}$ and $\Delta = \frac{1}{16}$ and as these agreed to the desired limits of accuracy, it was decided to use $\Delta = \frac{1}{8}$. A value of about 0.001 was found to be satisfactory for δx. It may be thought that the accuracy would be improved by making δx as small as possible but this did not appear to be the case because when δx was reduced much below 0.001, oscillations rapidly appeared in the values of the variables. The reason for this is not clear. It may ~~possible~~ possibly have been due to some number range trouble.

When the machine was in good working order it could integrate between two integral values of t with $\Delta = \frac{1}{8}$ in five minutes. This involved 80 applications of each of the formulae (2.11), (2.12) and (2.13).

Results and Conclusion.

Most of the work consisted of preliminary integrations to determine the best values to take for parameters such as Δ , ϵ and δx. The final results are set out in the accompanying table. This shows the value of $\delta\eta$ (i.e. the values of δx at which $v=0$) for various values of t . The column headed $t_c = 0$ is for an instantaneous change in density when the star enters the cloud. The figures in this column converge to a value near 1.13. This may be compared with the case considered by Bondi and Hoyle who obtained 1.25. These values are the estimates of $\beta c^2/\mu$. The figures for $t_c = 1$ could not be carried far enough to get an accurate value for $\beta c^2/\mu$ because beyond $t = 7$ the accumulation of truncation errors caused the solutions to become inaccurate. They show, however, that the value of β is changed substantially from the case $t_c = 0$.

It is interesting to notice that although the curve of zero velocity (i.e. $\zeta = \eta(t)$) tends to a higher value in the case of $t_c = 1$ than in the case of $t_c = 0$, it is initially below the $t_c = 0$ curve. The first few values of the $t_c = 2$ curve show that it starts below the $t_c = 1$ curve although it is expected to go above it eventually.

The general conclusion from the numerical investigation is that the more violent the change of density, the smaller is the value of β. This confirms the findings of Bondi and Hoyle.

Table of 8η

t	$t_c = 0$	$t_c = 1$	$t_c = 2$
1.125	0.65	0.53	0.52
2.125	0.84	0.73	0.67
3.125	0.98	0.93	0.85
4.125	1.09	1.03	
5.125	1.116	1.11	
6.125	1.12	1.16	
7.125		1.21	

Chapter III: An Elaboration of the Bondi-Hoyle Mechanism.

Bondi and Hoyle[7] considered the flow of material in the accretion column which, according to their theory, would be formed by particle collisions occurring in the wake of a star. They supposed the properties of the material entering the accretion column to be such that it rapidly loses heat by radiation and so the accretion column could be treated mathematically as a linear distribution of material. In this chapter we formulate the equations governing a more elaborate model of the accretion column.

We suppose the mechanism by which material is "focussed" on to the axis Ox, is the same as described by Bondi and Hoyle. We assume however, that the accretion column consists of a stream of gas around the accretion axis. This stream is kept in position by the steady shower of particles which arrive from the surrounding space. Although the boundary of this stream will not be clearly defined in reality, we introduce for the purpose of a mathematical treatment a definite boundary with circular section of radius $r(x)$ where x is the distance from the star O. We suppose that within this boundary, conditions are uniform over any cross-section. Let $m(x)$ be the mass of gas per unit length of the stream, $v(x)$ the velocity and $p(x)$ the pressure of the gas.

It should be mentioned here that the width of the stream has been briefly discussed on page 89 of [19].

We now obtain the equations determining the conditions in the stream of gas in a steady state. By considering the conservation of mass on an element of the axis, we obtain

$$mV = m(x-\alpha) \tag{3.1}$$

as in the Bondi-Hoyle mechanism, (1.21). As in the Bondi-Hoyle mechanism we can also obtain an equation of motion like (1.16) but there will now be an additional term on the right hand side representing the force on the element of material between x and $x+dx$ due to the pressure in the gas. The force due to pressure on the plane end of the element at x is

$$\pi r^2(x)\,p(x)$$

and the force due to pressure on the plane end at $x+dx$ is

$$\pi r^2(x+dx)\,p(x+dx),$$

the difference between these being

$$dx\frac{d}{dx}(\pi r^2 p).$$

This is the force due to pressure on the element and it is in the direction xO. So setting up the equation of motion we have, for the steady state,

$$mv\frac{dV}{dx} = m(v-V) - \frac{\mu M}{x^2} - \frac{d}{dx}(\pi r^2 p). \tag{3.2}$$

The stream has its shape maintained by its pressure being balanced by the force due to the momentum component, perpendicular to Ox, of the particles which hit unit length of the axis in unit time. From (1.11) this is

$$m\sqrt{\frac{2\mu}{x}}.$$

The outward force per unit length of axis due to pressure is

$$2\pi r(x)p(x).$$

Equating these forces,

$$2\pi r p = m\sqrt{\frac{2\mu}{x}}. \qquad (3.3)$$

In deriving (3.3) we assume that

$$\frac{dr}{dx} \ll 1,$$

otherwise both (3.2) and (3.3) would be affected, since the force due to the outward pressure of the gas in the stream would not quite be perpendicular to Ox.

If we consider the gas in the stream to be perfect we have

$$pV = \frac{1}{3}nMc^2 \qquad (3.4)$$

where p is the pressure of a volume V of gas containing n particles each of mass M and having R.M.S. velocity c. That is

$$nc^2 = \sum_j c_j^2,$$

c_j being the velocity of the j th particle at any time. Now c is constant for a given temperature so we can take c^2 to be a measure of the temperature. For our element of the tube, we have

$$V = \pi r^2 dx$$
$$nM = \mathcal{M} dx$$

so (3.4) becomes

$$p.\pi r^2 dx = \tfrac{1}{3}\mathcal{M} dx\, c^2$$

$$\therefore \pi r^2 p = \tfrac{1}{3}\mathcal{M} c^2 \qquad (3.5)$$

taking c^2 to be a function of x and the same over any section of the stream perpendicular to Ox.

We obtain a further equation by considering the conservation of energy within the element between x and $x+dx$. The energy consists of kinetic energy of translational motion, heat energy and gravitational potential energy. Thus:

energy entering per unit time energy leaving per unit time.

$$\left[\mathcal{M}V\left(\tfrac{1}{2}V^2+\tfrac{1}{2}c^2-\tfrac{\mu}{x}\right)\right]_x + \tfrac{1}{2}mv^2dx = \left[\mathcal{M}V\left(\tfrac{1}{2}V^2+\tfrac{1}{2}c^2-\tfrac{\mu}{x}\right)\right]_{x+dx}$$

+ radiation loss per unit time.

So

$$mv^2 = \frac{d}{dx}\left[\mathcal{M}V\left(V^2 + c^2 - \frac{2\mu}{x}\right)\right] + 2(\text{radiation loss from}$$

$$\text{stream per unit length per unit time}). \quad (3.6)$$

Equations (3.1),(3.2),(3.3),(3.5) and (3.6) are five equations for the five quantities $\mathcal{M}, V, p, r, c^2$.

<u>The Radius of the Stream.</u>

It is not proposed to discuss these equations in general but we shall use them to get an estimate of r near the neutral point $x = \alpha$, which exists, as in the Bondi-Hoyle mechanism by virtue of equation (3.1). As a suitable function for the radiation loss is not known, we shall assume a value for c and so avoid the use of (3.6). Remembering that $V = 0$ at the neutral point $x = \alpha$, we obtain from (3.2), neglecting the last term,

$$\mathcal{M}(\alpha) = \frac{mv\alpha^2}{\mu}. \quad (3.7)$$

Dividing (3.5) by (3.3) gives

$$\frac{r}{2} = \frac{\frac{1}{3}\mathcal{M}c^2}{m\sqrt{\frac{2\mu}{\alpha}}}$$

For definiteness let us take $\alpha = 2\mu/v^2$ so that

$$r = \frac{8\mu RT}{v^4}$$

where R is the gas constant and T is the temperature in the stream ($RT = \frac{1}{3}\overline{M}c^2$ where $\overline{M} = 1$ for atomic hydrogen). The values of r and $2\mu/v^2$ are compared in the accompanying table for a few values of v.

It will be seen that only in the lower two cases is $r \ll 2\mu/v^2$, which is a necessary condition for the Bondi-Hoyle mechanism to operate. For $v = 1$ Km./sec., T must not exceed a few degrees absolute in order that this condition should be satisfied, although at 5 Km./sec. it can go up to about $100°$A. If $v = 0.1$ Km./sec., it is evident that T must be a fraction of a degree absolute. In general $4RT/v^2$ must be small compared with unity since

$$r = \frac{4RT}{v^2} \cdot \frac{2\mu}{v^2}.$$

Knowing r, the density ρ_0 in the stream can be obtained from $\mathfrak{M} = \pi r^2 \rho_0$

$$\therefore \rho_0 = \frac{8\mu^2}{r^2 v^4} \rho$$

where ρ is the density in the undisturbed interstellar material. Two values of ρ_0/ρ are given in the accompanying table.

By the above calculations, estimates have been obtained of the thickness of the axial stream and of its density in the region of the neutral point. We have seen that if the temperature of the material in this stream goes above a certain value, determined by the

Table of r and e_0/e for $T^\circ A$

and a star of solar mass.

v (km./sec.)	r (parsec)	$2\mu/v^2$ (parsec)	e_0/e
0.1	$2.9 \times T$	0.89	-
1.0	$3.0 \times 10^{-4} \times T$	8.9×10^{-3}	$1.81 \times 10^{3}/T^2$
5.0	$4.8 \times 10^{-7} \times T$	3.6×10^{-4}	$1.13 \times 10^{6}/T^2$

velocity of the star through the cloud, the value of r becomes so large that our equations can no longer be expected to hold. But it follows that at such temperatures, the Bondi-Hoyle mechanism cannot hold either. This does not imply that there is no accretion. It merely means that the Bondi-Hoyle mechanism is no longer an adequate representation of the accretion process. It seems that there are three stages of the accretion process depending on the temperature in the accretion column. For temperatures very near 0°A., the mechanism will be that of Bondi and Hoyle. For temperatures somewhat greater, the mechanism will be that described in the present chapter and for still greater temperatures the mechanism will be a completely aerodynamic one. The temperature at which the second stage gives way to the third is not clearly defined. It depends on the velocity of the star relative to the cloud and will be about the temperature at which r becomes of the order of $2\mu/v^2$. In the next chapter an account is given of the attempts which have been made to estimate the accretion rate when the problem is treated aerodynamically.

<u>The Resistive Force</u>.

It is possible to obtain an expression for the resistive force on the basis of the model considered here in the same way as in the Bondi-Hoyle mechanism. This will now be done on the assumption that there is no radiation loss. We shall see that the resistive force is increased by a factor of 3. This gives an upper limit to the value of the force when there is a radiation loss.

For large x, (3.1) becomes

$$M\mathcal{V} \simeq mx. \tag{3.8}$$

Substituting (3.5) in (3.2), we obtain

$$M\mathcal{V}\frac{d\mathcal{V}}{dx} = m(v-\mathcal{V}) - \frac{\mu M}{x^2} - \frac{1}{3}\frac{d}{dx}(Mc^2)$$

which becomes, using (3.8),

$$\frac{d\mathcal{V}}{dx} = \frac{v-\mathcal{V}}{x} - \frac{\mu}{\mathcal{V}x^2} - \frac{1}{3mx}\frac{d}{dx}(Mc^2). \tag{3.9}$$

If the radiation loss is zero, (3.6) integrates to

$$mv^2x + A = M\mathcal{V}\left(c^2 + \mathcal{V}^2 - \frac{2\mu}{x}\right)$$

where A is a constant of integration. Using (3.8) we get

$$c^2 = v^2 - \mathcal{V}^2 + \frac{A}{mx} + \frac{2\mu}{x}. \tag{3.10}$$

We then substitute this in (3.9). We put $\mathcal{V} = v(1+u)$ and expand the various terms, retaining only first powers of u. We obtain approximately,

$$\frac{du}{dx} + \frac{u}{x} = -\frac{3\mu}{v^2x^2}$$

of which the solution is

$$u = -\frac{3\mu}{v^2}\frac{\ln x}{x} + \frac{B}{x}$$

So

$$V \sim v\left(1 - \frac{3\mu}{v^2}\frac{\ln x}{x}\right)$$

from which the factor of 3 in the force arises. Substituting this expression for V in (3.10) gives

$$c^2 \sim 6\mu \ln x / x$$

so $c^2 \to 0$ as $x \to \infty$. Dividing (3.5) by (3.3) gives

$$\frac{r}{2} = \frac{\frac{1}{3}Mc^2}{m\sqrt{\frac{2\mu}{x}}}$$

$$\therefore r \sim \frac{\sqrt{2\mu x}\,\ln x}{v},$$

so $r \to \infty$ with x . Thus, the thermal energy is used up in expanding the stream of gas. Although $r \to \infty$,

$$\frac{dr}{dx} \propto \frac{1}{2}\frac{1}{\sqrt{x}}\cdot \ln x + \sqrt{x}\cdot\frac{1}{x} \to 0 \qquad \text{as } x \to \infty$$

so our assumption that $dr/dx \ll 1$ is satisfied for large x . Substituting the above expression for r in (3.3) gives

$$p = \frac{m\sqrt{\frac{2\mu}{x}}}{2\pi\cdot\frac{\sqrt{2\mu x}\,\ln x}{v}} \to 0 \qquad \text{as } x \to \infty.$$

From this consideration of the asymptotic behaviour of the stream of gas, we see that the velocity U and hence the resistive force is not very different when pressure is taken into account from the case of inelastic collisions considered by Bondi and Hoyle. When the radiation loss is taken into account, the difference will be even less. Since this treatment is midway between that of Bondi and Hoyle and the aerodynamical case, we have an indication that the force will be little different in the aerodynamical case. This is useful because no estimate of the force has been made in the latter case.

Chapter IV: Gaseous Accretion.

The Bondi Process.

The problem of the interaction of a star with interstellar material, when the latter is a gas at a temperature other than zero absolute, was first treated by Bondi[12] in the special case of a star at rest relative to the material. Bondi considered the problem for various values of the ratio of specific heats of the gas, γ, but for the purpose of illustrating the results, we shall here examine only the isothermal case where $\gamma = 1$.

The star is considered to be at rest in an infinite cloud of gas which at infinity is also at rest and of uniform density ρ_∞ and pressure p_∞. The motion of the gas is spherically symmetrical and steady, the increase in mass of the star being ignored so that the field of force is unchanging. The pressure p and density ρ are related everywhere by

$$p = A\rho, \tag{4.1}$$

A being a constant. If we take r to be the radial coordinate and v the inward velocity of the gas, the equation of continuity is

$$4\pi r^2 \rho v = \text{constant} = B \text{ (say)}, \tag{4.2}$$

where B is the accretion rate. Bernoulli's equation is

$$\frac{1}{2} v^2 + \int \frac{dp}{\rho} - \frac{\mu}{r} = \text{constant}$$

i.e.

$$\frac{1}{2} v^2 + A \ln\left(\frac{\rho}{\rho_\infty}\right) - \frac{\mu}{r} = 0. \tag{4.3}$$

Let c denote the sound speed, then

$$c^2 = \frac{dp}{d\rho} = A. \tag{4.4}$$

Let us introduce non-dimensional variables x, y, z as follows:

$$r = x\frac{\mu}{c^2},$$

$$v = y c,$$

$$\rho = z \rho_\infty.$$

Then (4.2) and (4.3) take the forms

$$x^2 y z = \lambda \tag{4.5}$$

$$\frac{1}{2} y^2 + \ln z = \frac{1}{x} \tag{4.6}$$

where λ is given by

$$B = 4\pi \lambda \mu^2 \rho_\infty / c^3. \tag{4.7}$$

Eliminating z from (4.6), using (4.5), we obtain

$$\left(\frac{1}{2} y^2 - \ln y\right) = -\ln\lambda + \left(\frac{1}{x} + 2\ln x\right) \tag{4.8}$$

The two quantities in brackets are plotted in Fig. 4.1. Our problem is to find y as a function of x. Suppose for the present that $\ln\lambda = 0$. Then for any value G of x, we obtain two values H and I for y. By letting G take all possible values of x, we obtain a set of points H and a set I as possible solutions of (4.8). The set H is excluded however as it does not satisfy $y \to 0$ as $x \to \infty$. The solution I is therefore the only one possible. It will be noted that this solution $\to 0$ as $x \to \infty$ and as $x \to 0$. If $\ln\lambda \neq 0$, the same procedure is possible except that the curve DEF is displaced downwards a distance $\ln\lambda$. However no solution is possible if E is displaced below B, i.e. if

$$2 - 2\ln 2 - \ln\lambda < \tfrac{1}{2},$$

and there is a critical case when E and B are at equal heights. Let the value of λ in this case be denoted by λ_c, then

$$2 - 2\ln 2 - \ln\lambda_c = \tfrac{1}{2}$$

$$\therefore \lambda_c = \tfrac{1}{4} e^{3/2} = 1{\cdot}12.$$

In the critical case, the solution I has two branches, one $\to 0$ and the other $\to \infty$ as $x \to 0$. Bondi refers to the two branches as Type I and II respectively.

We have thus shown that a solution to the problem is

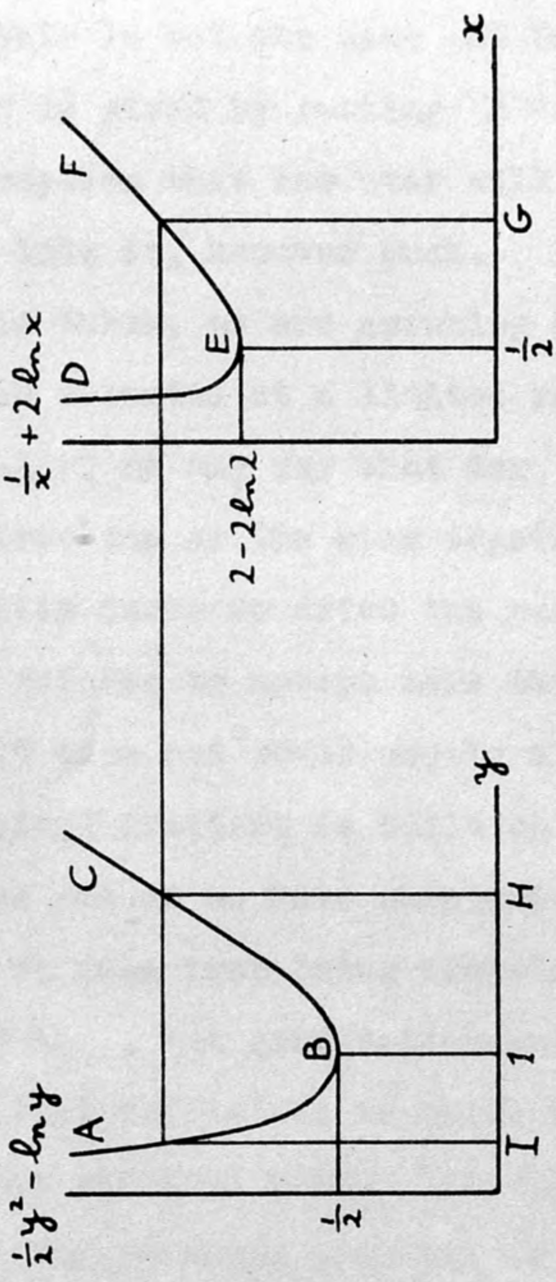


Fig. 4.1.

possible if $0 \leqslant \lambda \leqslant \lambda_c$. Bondi concluded that the rate of accretion is not determinate without introducing other considerations such as stability. It is the opinion of the present writer that this is not the case and that in fact the rate of accretion is given by putting $\lambda = \lambda_c$. This follows from the assumption that the star will swallow up any material falling into it, however much. If any smaller value of λ is taken, we are assuming at the start that the star swallows material at a limited rate. Looking at the matter physically, we may say that for $\lambda < \lambda_c$, the gravitational attraction of the star together with the gas pressure at infinity serve to drive the gas towards the star. But the star refuses to accept more than a certain amount of gas per unit time and consequently a pressure (and therefore a density) gradient is built up near the star which retards the gas as we have seen and so prevents the specified accretion rate from being exceeded.

However when $\lambda = \lambda_c$, the gravitation and gas pressure at infinity are only just sufficient to drive the gas towards the star at the required rate. In this case, there is no retardation and the pressure gradient is less than for $\lambda < \lambda_c$. In the opinion of the present writer, the Type I motion has no physical significance.

The assumption actually made, namely that the star is capable of swallowing up an infinitely large amount of gas is equivalent to putting $\lambda = \infty$. A solution in this case is not possible, however, since the inertia of the gas prevents it reaching the star at any rate greater than that given by $\lambda = \lambda_c$. Consequently this is the maximum possible accretion rate and is the rate which will occur in reality.

Taking $\lambda = \lambda_c$, the accretion rate is given by (4.7) to be

$$4\pi (1.12) \mu^2 \rho_\infty / c^3.$$

This compares with the maximum rate in the Bondi-Hoyle mechanism which is

$$\frac{4\pi \mu^2 \rho_\infty}{V^3},$$

V being the velocity of the star. In the light of these two results, Bondi conjectured that the formula

$$\frac{4\pi \mu^2 \rho_\infty}{(c^2 + V^2)^{3/2}} \tag{4.9}$$

should give approximately the rate of capture in the case of a star moving through the gas. Actually, Bondi's own 'general purposes' estimate is one-half this rate, but his arguments, when applied to an isothermal gas, would lead to (4.9) as given.

<u>Accretion by the Star at Supersonic Velocities.</u>

We now give the results of a study of the isothermal flow of a gas near a star which is moving with a supersonic velocity relative to the gas. The reason for considering isothermal flow is that it introduces a certain mathematical simplification. However, two physical arguments can be given to justify this. In the first place, the temperature of the gas will probably be affected by the heat radiated from the star, and so will, to some extent, be a function of the distance from the star. In this case, the simplified problem, in which this radiation is ignored, is unlikely to represent the situation accurately whether isothermal or adiabatic flow is considered. In the second place, it is believed that, when dust is mixed with a gas, this enables the gas to radiate heat and thus maintain a uniform temperature. The assumption of isothermal flow is not, therefore, as artificial as may at first be thought. The restriction to supersonic velocities of the star is necessitated by the method used to study the flow; this will be shown below. It may be explained, however, that in the supersonic case the rate of accretion by the star is not affected by any boundary condition at the surface of the star, whereas in the subsonic case it is necessary to specify the rate of accretion before it is possible to determine the flow.

We may consider the star to be at rest at the origin of coordinates and the gas to be moving with a velocity V when at a great distance from the star. Let the z-axis of cylindrical polar coordinates (r, α, z) be taken in the direction of V. Then the steady flow of gas past the star will be axially symmetric. Thus we may write down the equations of continuity,

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho v r) + \frac{\partial}{\partial z}(\rho u) = 0, \tag{4.10}$$

the equations of motion,

$$v\frac{\partial v}{\partial r} + u\frac{\partial v}{\partial z} = R - \frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{4.11}$$

$$v\frac{\partial u}{\partial r} + u\frac{\partial u}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{4.12}$$

and the equation of state for the gas,

$$p = A\rho \tag{4.13}$$

In these equations, u, v are the velocity components of the gas parallel to the directions of z and r respectively, ρ is the density of the gas, p is the pressure of the gas, R, Z are the components of the gravitational force of the star, i.e.

$$R = -\frac{\mu r}{(r^2+z^2)^{3/2}}, \quad Z = -\frac{\mu z}{(r^2+z^2)^{3/2}}, \tag{4.14}$$

and A is a constant depending on the temperature of the gas. The speed of sound c in the gas is given by

$$c^2 = \frac{dp}{d\rho} = A, \tag{4.15}$$

and so is constant throughout the field.

The method used to study the flow was the 'method of characteristics'. The detailed theory of the method cannot be given here, but the derivation of the formulae used can be outlined.

We assume the existence of a velocity potential ϕ such that

$$u = \frac{\partial \phi}{\partial z}, \quad v = \frac{\partial \phi}{\partial r}.$$

From (4.11) and (4.15) we obtain

$$\frac{\partial \rho}{\partial r} = \frac{\rho}{c^2}\left[R - v\frac{\partial v}{\partial r} - u\frac{\partial v}{\partial z}\right], \qquad (4.16)$$

and from (4.12) and (4.15) we obtain an expression for $\partial \rho/\partial z$. Substituting this and (4.16) in (4.10) we obtain

$$R(u^2 - c^2) + 2uvS + T(v^2 - c^2) = vR + uZ + vc^2/r, \qquad (4.17)$$

where $R = \dfrac{\partial^2 \phi}{\partial z^2}$, $\quad S = \dfrac{\partial^2 \phi}{\partial r \partial z}$, $\quad T = \dfrac{\partial^2 \phi}{\partial r^2}$.

Now $\quad du = R\,dz + S\,dr, \; dv = S\,dz + T\,dr,$

so $\quad R = (du - S\,dr)/dz, \; T = (dv - S\,dz)/dr. \qquad (4.18)$

Substituting (4.18) in (4.17) we obtain

$$S = \frac{(u^2 - c^2)\,du\,dr + (v^2 - c^2)\,dv\,dz - vR\,dz\,dr - uZ\,dz\,dr - vc^2\,dz\,dr/r}{(u^2 - c^2)(dr)^2 - 2uv\,dz\,dr + (v^2 - c^2)(dz)^2} \qquad (4.19)$$

as an expression for S at any point in the field. For a solution to the problem, S must be finite at all points in the field. Hence, when the denominator is zero, the numerator must be also. The denominator is zero if

$$\frac{dr}{dz} = \frac{uv \pm c(u^2+v^2-c^2)^{\frac{1}{2}}}{u^2-c^2}. \tag{4.20}$$

This equation represents two directions through any point, and hence it also represents two families of curves called 'characteristics', a member of each family passing through each point. It will be noticed that these curves are real only if the flow is supersonic, i.e. if $u^2+v^2>c^2$. If we introduce the new variables θ and m, where θ is the inclination of the streamline to the z-axis, i.e.

$$\tan\theta = v/u, \tag{4.21}$$

and m, the Mach angle, is defined by

$$c = (u^2+v^2)^{\frac{1}{2}} \sin m, \tag{4.22}$$

then (4.20) becomes

$$\frac{dr}{dz} = \tan(\theta \pm m). \tag{4.23}$$

From (4.23) it follows that at any point the streamline bisects the angle between the characteristics. To obtain relations holding along the characteristics, we equate to zero the numerator of (4.19) and use the substitution (4.22). We also write $U^2 = u^2+v^2$. After some reduction we obtain

$$\frac{dU}{U} \mp \tan m\, d\theta - \frac{dz}{\cos(\theta \pm m)\cos m}\left[\frac{\sin\theta \sin^2 m}{r} + \frac{\sin\theta}{U^2}R + \frac{\cos\theta}{U^2}Z\right] = 0 \qquad (4.24)$$

for these relations. For isothermal flow c is a constant, and so from (4.22)

$$dU/U = -\cot m\, dm.$$

Substituting this in (4.24) and using (4.14), we find

$$dm \pm \tan^2 m\, d\theta + \frac{\tan^2 m \sin m\, dz}{\cos(\theta \pm m)}\left[\frac{\sin\theta}{r} - \frac{\mu}{c^2}\frac{r\sin\theta + z\cos\theta}{(r^2+z^2)^{3/2}}\right] = 0, \qquad (4.25)$$

or, say,

$$dm + \Theta_{\pm}\, d\theta + Z_{\pm}\, dz = 0. \qquad (4.26)$$

The characteristic relations require further consideration on the z-axis, since it is necessary to determine the value of $\sin\theta/r$. On the z-axis we have $r=0,\ v=0,\ \theta=0$; hence $S = \partial v/\partial z = 0$. Distinguishing values of variables on the z-axis by the suffix 0, we see that

$$\lim_{r\to 0}\frac{\sin\theta}{r} = \frac{1}{U_0}\lim_{r\to 0}\frac{U\sin\theta}{r} = \frac{1}{U_0}\lim_{r\to 0}\frac{v}{r} = \frac{1}{u_0}\left(\frac{\partial v}{\partial r}\right)_0 = \frac{T_0}{u_0}. \qquad (4.27)$$

On the z-axis, (4.17) takes the form

$$R_0(u_0^2 - c^2) - T_0 c^2 = T_0 c^2 + u_0\mu/z^2. \qquad (4.28)$$

Now $R_0 = \dfrac{\partial u_0}{\partial z} = \dfrac{\partial}{\partial z}\dfrac{c}{\sin m_0} = -\dfrac{c\cot m_0}{\sin m_0}\dfrac{\partial m_0}{\partial z}.$

Substituting this in (4.28), we have

$$\frac{T_0}{u_0} = \frac{1}{2}\left[-\cot^3 m_0 \frac{\partial m_0}{\partial z} - \frac{\mu}{c^2 z_0^2}\right]. \qquad (4.29)$$

It is convenient to measure distances in terms of μ/c^2. Let us refer to the families of characteristic curves as the (−) family and the (+) family, according to the sign in (4.23). The method consists of taking an arbitrary curve AD at a sufficient distance upstream from the star (at the origin), so that on it the flow may be considered to be almost undisturbed by the gravitational effect of the star (Fig. 4.2). A number of points A, B, C, D are taken on the curve. Consider the points C and D. Suppose through D a straight line is drawn in the (−) characteristic direction and through C a straight line is drawn in the (+) characteristic direction, the directions being obtained at D and C on the assumption that the flow is uniform. Then these will intersect at a point $E_1(r_1, z_1)$. A pair of relations like (4.26) can then be used to obtain values of m and θ at E_1 (m_1, θ_1, say), where the differentials in (4.26) are replaced by finite differences and the functions $\Theta_\pm$ and $Z_\pm$ are evaluated at C and D. Knowing r_1, z_1, m_1 and θ_1, a new point $E_2(r_2, z_2, m_2, \theta_2)$ can be found from D and C using gradients $\theta \pm m$ and functions $\Theta_\pm$ and $Z_\pm$ which are means of their values at D and E_1

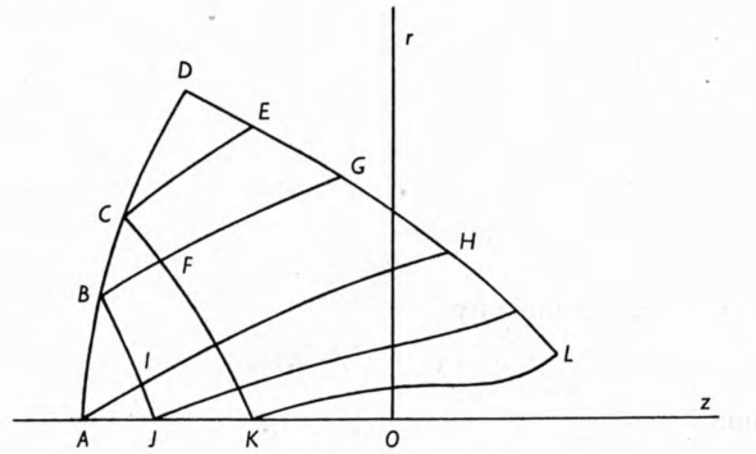


Fig. 4.2.

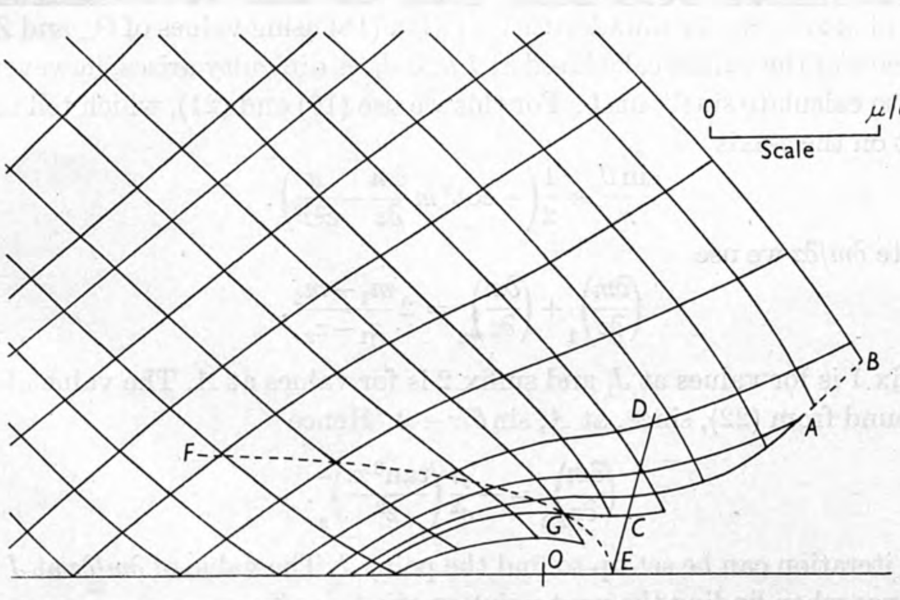


Fig. 4.3.

and at C and E_1. Thus an iteration can be set up leading to a point E. This point is considered to be the intersection of the characteristics through C and D. In a similar manner, F is found from C and B, then G from F and E, and so on, until a region ADH has been filled in.

In the undisturbed flow, $\theta = 0$ and U = constant = V, hence m = constant. Thus in the undisturbed flow, each family of characteristics is a set of parallel straight lines. If the curve AD is taken sufficiently upstream, the characteristics in the region ADH are almost straight. It can be seen from this discussion that the curve AD is not completely arbitrary as it must intersect every characteristic of either family not more than once.

The point I (being the mesh point adjacent to A on the (+) characteristic through A) requires special treatment because it is necessary to evaluate $\sin\theta/r$ at A. Since we assume uniform flow on AD, it follows that $\sin\theta/r \to 0$ at A.

The calculation in the region on the other side of AH is similar except that a different calculation is required to obtain points like J on the z-axis. We know that on the z-axis, $r = \theta = 0$. Using the (-) characteristic direction at I, we can find a point J_1 on the z-axis. The value of m at this point can be found by a relation

(4.26) using Θ_- and Z_- evaluated at I. Using the values at J_1, we find a point J_2 such that IJ_2 has a gradient which is the mean between the values of the gradient evaluated at I and at J_1. The value of m at J_2 can be found with a relation (4.26) using values of Θ_- and Z_- which are the means of the values calculated at I and J_1. A difficulty arises, however, as it is necessary to calculate $\sin\theta/r$ at J_1. For this we use (4.27) and (4.29), which tell us that at each point on the z-axis

$$\frac{\sin\theta}{r} \to \frac{1}{2}\left(-\cot^3 m \frac{\partial m}{\partial z} - \frac{\mu}{c^2 z^2}\right). \qquad (4.30)$$

To evaluate $\partial m/\partial z$ we use

$$\left(\frac{\partial m}{\partial z}\right)_1 + \left(\frac{\partial m}{\partial z}\right)_2 = 2\frac{m_1 - m_2}{z_1 - z_2},$$

where suffix 1 is for values at J_1 and suffix 2 is for values at A. The value of $(\partial m/\partial z)_2$ is easily found from (4.30), since, at A, $\sin\theta/r \to 0$. Hence

$$\left(\frac{\partial m}{\partial z}\right)_2 = -\frac{\mu}{c^2}\left(\frac{\tan^3 m}{z^2}\right)_2.$$

Hence, an iteration can be set up to find the point J. The value of $\partial m/\partial z$ at J must be stored for use when finding the next point on the axis, K.

Towards the end of the calculation, at L, the (−)

characteristics become steeply inclined to the z-axis and if carried far enough would turn right over. For this reason it is better to use $dr/\sin(\theta - m)$ instead of $dz/\cos(\theta - m)$ in the (−) characteristic relation (4.25).

In the calculations which were performed, the curve AD was taken to be an arc of the circle $r^2 + z^2 = 4\mu/c^2$, with modifications in some cases to suit the geometry of the characteristics. This boundary appeared to be sufficiently removed from the star, because the characteristics in the region ADH were almost straight. Sixteen points were taken on this boundary, and the flow was investigated for four values of the initial Mach angle (i.e. four values of V/c).

Fig. 4.3 shows the characteristic pattern in the case where $m = 45°$ in the undisturbed flow (i.e. $V/c = \sqrt{2}$). Only the characteristic pattern in the region of the star is shown. The initial boundary is not shown, as it is off the left-hand edge of the figure. Some of the (+) characteristics in the top left corner of the figure have been omitted. This accounts for their irregular spacing in this region. It will be seen that the pattern has not been continued downwards sufficiently to touch the positive z-axis. To have done so would have required a considerably smaller mesh size, owing to the rate at which the gravitational force changes near the star, and so a greatly increased time for the calculation. The pattern has, however, been continued sufficiently for an estimate to be made of the rate of accretion by the star. If material is captured by the star, then there must be a stagnation point S on the positive z-axis. Since $u = v = 0$ at a stagnation point, there must be a surface enclosing a small volume round S upon which the gas velocity is equal to the sound speed c. On this surface it follows from (4.22) that $m = 90°$. Since the stream line through the stagnation point will cut the z-axis at right angles, it follows that the (+) characteristic will be parallel to the z-axis near S. (Incidentally, the (−) characteristics will also be parallel to the z-axis on the small surface enclosing S.) Thus the dip in the (+) characteristic near C (Fig. 4.3) gives an indication of the position of S.

The following consideration also gives an indication of the position of S. On any line parallel to the z-axis, θ is negative and $\theta \to 0$ as $|z| \to \infty$. For lines fairly near the z-axis, θ will clearly have its minimum value in the region of the stagnation point. But this does not apply for lines very close to the z-axis, and consequently this method of determining S is less reliable and is only used as a check on the position of S determined by the above method. In Fig. 4.3, the (+) characteristics in the region of C are approximately parallel to the z-axis.

The line DC is the locus of the points of minimum θ on these characteristics. By extrapolating this line, the point E is obtained. Since, in Fig. 4.3, the dip in the characteristics is near this point, we may suppose that E is close to the stagnation point S. Taking E as the estimated position of S, the rate of accretion by the star can be estimated by finding the stream-line FGE which passes through E. The part GE of the stream-line has to be guessed, with the help of the fact that the stream-line at E cuts the z-axis at right angles, and with the knowledge of θ along the innermost (+) characteristic. Having obtained the point G, the part FG of the stream-line can be found, since the values of θ are known at all points of the mesh. The limiting distance a of F from the z-axis is the radius of the 'tunnel' swept out of the gas by the star. The rate of accretion of gas by the star is then $\pi a^2 V \rho_\infty$.

The values of the radius a in terms of μ/c^2 are given for the four initial values of m in the accompanying table under the heading 'Found'. The values under '(4.9)' are those those calculated from the formula (4.9). Owing to the crudeness of the method by which the points E, G and F have been determined, the conclusion to be drawn from this table is merely that the calculations indicate a rate of accretion of the same order as would be given by formula

Table of ac^2/μ

m	Found	(4.9)
20^0	0.19	0.23
30^0	0.45	0.42
45^0	0.70	0.74
60^0	0.90	0.99

(4.9). No importance should be attached to precise values in the column 'Found', since the method of determining E and G might well produce errors up to 50% in the value of a.

It is hoped that these calculations will help to bridge the gap between the cases of zero gas temperature and of zero star velocity, which have been previously studied, and so will strengthen the foundation of the accretion theory. It will be noted that in the case of gaseous accretion, there is no accretion column, and so the possibility of the inflowing material missing this axis cannot arise. It is also of mathematical interest that in the supersonic gaseous case the rate of accretion is determined by the steady-state equations, whereas in the mechanism of Bondi and Hoyle it is necessary to consider unsteady conditions before the precise accretion rate can be obtained.

From the hydrodynamical point of view, the calculation may also be of interest. In Fig. 4.3, two of the (+) characteristics run together at A. This is an indication of the formation of a shock wave which extends in the direction AB.

The computation of the characteristic mesh was performed on the Manchester University Electronic Computer. It took about four hours to compute the mesh shown in Fig. 4.3. A similar time was required for each of the other three cases.

Chapter V: A Theory of Binary Star Formation.

The number of binary stars that have been observed is very considerable and in the neighbourhood of the sun, binary stars are about half as numerous as single stars. The problem of the origin of binary systems is therefore one of some importance to astronomy. There appear to be only three logically possible theories: the binaries may have been formed by the disintegration of originally single stars; they may have been born or created as double stars; or they may have been formed by the coming together of two originally single stars.

We shall consider briefly the three types of theory. The first or fission theory was for some time the accepted one of binary formation. However, later examination[20][21] indicated that it is inadequate to explain the formation of binaries. The second theory which says that binary stars were formed by the condensation of two stars within each other's gravitational influence is now the theory receiving most attention. In the third theory, two originally independent stars are considered to come together and remain together. If we look upon the stars as particles, it can be shown that if two stars approach from a great distance, they must necessarily separate again to a great distance in the absence of forces other than their gravitational attraction. Although the fact that stars are finite bodies may alter this statement in certain circumstances, the problem has never been seriously considered and all theories of this type assume that forces are brought into play in some other way. The most obvious way is by the introduction of a third star. Thus, if three stars approach from a great distance, it is possible for one to recede to infinity leaving the other two gravitationally bound together.

The theory we wish to consider here is a combination of the second and third types, but the place of the third star is taken by the interstellar material. In other words, two stars are considered to approach from a great distance apart in a region of interstellar material. During the approach and subsequent encounter, each star will experience a resistance to its motion and so will lose energy. If this energy loss is sufficient, the stars will not again recede to a great distance apart but will remain gravitationally bound together. Once the binary has formed in this way, its orbital elements may be subsequently affected by the interstellar material as will be explained in the next chapter.

If we examine more closely the second theory of binary formation, we see that some similar process must occur because after the stars have condensed, their subsequent motions must for some time be influenced by the star-forming medium. It is during this early stage that we consider the theory here put forward to have been of most importance.

<u>Mathematical Investigation</u>.

In considering the dynamics of a binary forming encounter between two stars, as described above, we must imagine the stars to approach from a separation of the order of the local mean interstellar distance since at greater separations, each star is more likely to be influenced by other neighbouring stars. The maximum initial velocities of the stars necessary to produce a capture will depend on the initial directions of motion and on the masses of the stars. A full investigation of the binary forming encounter would be quite difficult in view of the numerous parameters involved. However, a simple example has been investigated in order to find the order of magnitude of the initial velocities. The case considered is that of two stars of equal mass approaching with equal speeds in almost opposite directions. Starting with an initial separation d the initial speeds are found so that the stars never again separate to a distance of more than $\frac{1}{2}d$. The centre of mass of the stars is taken to be at rest in the medium which has density ρ.

Since the velocities of the stars are necessarily small, the accretion mechanism must be considered to be that in which all the material coming within the cut-off distance of either star is captured by it so that the accretion rate is given by (1.37) and there is no resistive force of the form (1.33). However, owing to the presence of the other star, we would expect the cut-off distance for either star to be reduced to about half the separation of the two stars at any instant. With this modification we can write down the equations of motion for either of the stars. Let the frame of reference be such that the origin O is at the mass centre of the stars and let Ox pass through the starting points of the stars. Then the motion of either star will be almost rectilinear and coincident with Ox. Let x be the position, v the velocity along Ox and m the mass of one of the stars at time t, then

$$\frac{dx}{dt} = v \tag{5.1}$$

The equation of motion is

$$\frac{d(mv)}{dt} = \frac{Gm^2}{4x^2} \tag{5.2}$$

when the stars are approaching, where G is the constant of gravitation. The modified rate of accretion is, from (1.37)

$$\frac{dm}{dt} = \pi x^2 \rho |v|. \tag{5.3}$$

Consider the star which is initially at $x = -\frac{1}{2}d$. Then the velocity of this star will be positive so dividing (5.3) by (5.1) gives

$$\frac{dm}{dx} = \pi \rho x^2$$

whence $$m = \tfrac{1}{3}\pi \rho x^3 + M \tag{5.4}$$

where M is the mass of the star when it reaches the origin. From (5.2),

$$m\frac{dv}{dt} + v\frac{dm}{dt} = \frac{Gm^2}{4x^2}$$

and using (5.1), (5.3) and (5.4), this becomes

$$\frac{d(v^2)}{dx} + \frac{2\pi\rho x^2}{M + \frac{1}{3}\pi\rho x^3}v^2 = \frac{G(M+\frac{1}{3}\pi\rho x^3)}{2x^2}$$

of which the solution is

$$v^2 = \frac{1}{(M+\frac{1}{3}\pi\rho x^3)^2}\left[G\left(-\frac{M^3}{2x} + \frac{\pi\rho}{4}M^2x^2 + \frac{\pi^2\rho^2}{30}Mx^5 + \frac{\pi^3\rho^3}{432}x^8\right) + A\right] \tag{5.5}$$

A being a constant of integration. Now (5.4) implies that as the stars approach each other, they accrete the material lying within the cone whose vertex is at O and whose semi-angle is $\pi/4$. It is supposed that the stars narrowly miss a collision at O and then proceed and come to rest at $x = \pm\frac{1}{4}d$. Since after passing O each star will be entering a region which has just been cleared of material by the other star, we assume that the stars do not accrete between O and their first position of rest. Thus, in this

Thus there is a limit above which x must not go. This limit is obtained by equating to zero the cubic expression in (5.7).

part of the motion, we have from elementary considerations (or else by putting $\rho = 0$ and changing the sign of G in (5.5)),

$$v^2 = \frac{GM}{2x} + B, \tag{5.6}$$

B being a constant which is determined by the fact that $v = 0$ at $x = \frac{1}{4}d$. So $B = -\frac{2GM}{d}$.

The condition at the origin is that at $x = \delta$ (where δ is a small positive quantity), v^2 given by (5.6) must be the same as v^2 at $x = -\delta$ given by (5.5). From this it follows that $A = -\frac{2GM^3}{d}$.

To obtain the maximum initial velocity of a star we put $x = -\frac{1}{2}d$ in (5.5). We obtain

$$\left.\begin{aligned} v^2 &= \frac{G\rho d^2}{\left(\alpha - \frac{\pi}{24}\right)^2}\left[-\alpha^3 + \frac{\pi}{16}\alpha^2 - \frac{\pi^2}{960}\alpha + \frac{\pi^3}{110592}\right] \\ \alpha &= M/\rho d^3. \end{aligned}\right\} \tag{5.7}$$

which gives v for various values of M. It will be noted that when α is large enough, v^2 will be negative owing to the negative coefficient of α^3 in (5.7). SEE OPPOSITE The solution is approximately $\alpha = 0.135$. The physical significance is that stars with $\alpha > 0.135$ cannot be brought to rest at a separation of $\frac{1}{2}d$ even if their initial velocities (at separation d) are zero.

Discussion of Results.

The accompanying tables give the maximum initial velocities and resulting masses (M) of the stars undergoing such a binary forming encounter, for given initial masses and interstellar material densities. The initial masses are such that the diagonal figures in the tables correspond to $\alpha = 0{\cdot}135$. For these tables, the value of d was taken to be half a parsec. This value was taken because although the present mean interstellar distance in the neighbourhood of the sun is about one parsec, if the binary stars were formed by the method suggested then the mean interstellar distance of the original single stars must have been rather less.

The initial velocities given in the table are unfortunately too large for the assumption about the accretion rate to be satisfied. The maximum velocity for this assumption to hold is given by the condition (1.35). Applying this condition at the point $x = -\frac{1}{2}d$ we obtain the velocities given in brackets in the table. We must therefore look upon these as representing the order of magnitude of the maximum velocities. It may be thought that by using the appropriate accretion conditions for higher velocities, a higher maximum could be obtained. It is not likely that much can be gained in this direction however because as Fig. 1.3 shows, the accretion rate and effective resistance to the star fall off rapidly after the point D which represents the velocity given by condition (1.35).

Table of masses of stars resulting from the binary forming encounter of stars of initial separation ½ parsec and of given initial masses in a medium of given density.

ρ gm./c.c.	Initial Masses (in Solar Masses)			
	0.00069	0.0069	0.069	0.69
10^{-22}	0.023	(0.029)	(0.091)	(0.712)
10^{-21}	0.222	0.230	(0.290)	(0.91)
10^{-20}	2.21	2.22	2.30	(2.90)
10^{-19}	22.1	22.1	22.2	23.0

Table of maximum initial velocities (km./sec.) of stars in the binary forming encounter.

ρ gm./c.c.	Initial Masses (in Solar Masses)			
	0.00069	0.0069	0.069	0.69
10^{-22}	0.0299 (0.0035)	-	-	-
10^{-21}	2.13 (0.0035)	0.0945 (0.011)	-	-
10^{-20}	70.1 (0.0035)	6.75 (0.011)	0.299 (0.035)	-
10^{-19}	2240 (0.0035)	222 (0.011)	21.3 (0.035)	0.945 (0.11)

The bracketed velocities are very small and consequently this binary forming encounter process can only be considered to apply to stars which are almost at rest initially. But this is all that is necessary for the application to the second theory. (This point, incidentally, shows that the above rectilinear treatment of the problem is all that is necessary.) The next point to note is that the resulting masses of the stars are almost independent of the initial masses. The resulting masses are therefore only dependent on the density ρ and also on d, being proportional to d^3. To obtain a binary consisting of two components of solar mass we would require a density of between 10^{-21} and 10^{-20} gm./c.c. (with $d = \frac{1}{2}$ parsec). This is high compared with densities of clouds observed at present (10^{-22} gm./c.c.) but it must be remembered that when the binaries were formed, considerably different conditions of the interstellar material must have prevailed. Of course, for a given value of ρ, any desired resulting mass can be obtained by a suitable choice of d. The importance of the present calculations is that the masses of observable binaries can be obtained with quite reasonable values of ρ and d.

In the table of resulting masses, the bracketed figures are included for the cases which do not result in captures. It is seen, of course, that in all cases the masses of the stars are increased. It follows that if a star does not form a binary in its first encounter with another star, then it can never again do so in a medium of the same density except possibly by a three body encounter. It also follows that if a binary forms, it cannot subsequently pick up a third star to form a tertiary system, under the influence of the medium alone. It is however possible for three or more initial condensations to become mutually attracted and form a multiple system. To investigate the case of three condensations, we may consider three equal condensations at the corners of an equilateral triangle and examine their motions as they approach their common centre of mass through the medium. If we do this, we obtain an expression like (5.7) except that G is increased by a certain factor. Thus, the initial velocities of approach v, as given by (5.7), are no smaller than for a binary forming encounter. Similarly for four condensations originally at the corners of a tetrahedron.

When a binary forming encounter of the type discussed above occurs, the mass of material accreted by the two stars is about a half of that originally contained within a sphere of diameter d. Consequently, there will still be sufficient interstellar material left to form a resistive medium for further accretion effects to occur. The formation of multiple systems leaves somewhat less unaccreted material.

In the above treatment of the binary forming encounter, the stars subsequently form a binary star whose components have highly elongated orbits. If the stars originally have slight lateral velocities off Ox , then the stars will follow paths which are curved but none-the-less close to Ox , so that the above treatment may still be expected to hold approximately. However, the orbits of the resulting binaries will be elliptic, or in the extreme case, circular. It is important to consider the maximum separation in such cases. Consider a binary with an elongated orbit and one with a circular orbit. Let the stars be of equal mass m and in the elongated orbit, let the maximum separation be $2R$. Then the energy of the binary is $-Gm^2/2R$. In the circular orbit, the energy is

$$\frac{1}{2}mv^2 + \frac{1}{2}mv^2 - \frac{Gm^2}{2r}$$

where v is the velocity of each star and r is the radius of the orbit. But $v = \sqrt{Gm/4r}$ (see equation (6.1)) so the energy is $-Gm/4r$. Equating these energies we have $R = 2r$ so that in a binary forming encounter, if the stars separate to $\frac{1}{2}d$ in the elongated case, they would separate to about $\frac{1}{4}d$ in the circular case and to intermediate distances in the elliptic case.

Effect of Unequal Initial Masses.

If the masses of the condensations are not equal initially, let the masses at any instant be m_1, m_2, the positions x_1, x_2 and the inward directed velocities v_1 , v_2 respectively. Then (5.2) becomes

$$\frac{d(m_1 v_1)}{dt} = \frac{G m_1 m_2}{(x_1 - x_2)^2} = \frac{d(m_2 v_2)}{dt}$$

from which it follows that at any instant,

$$m_1 v_1 = m_2 v_2 + \text{constant.} \tag{5.8}$$

If, in this case, we consider the masses to tunnel out regions of radii equal to their distances from the neutral point of the gravitational field (i.e. the point where the gravitational attraction of m_1 and m_2 are equal and opposite) then the accretion rate for m_1 is given by

$$\frac{dm_1}{dt} = \pi \rho v_1 \left[\frac{\sqrt{m_1}\,|x_1 - x_2|}{\sqrt{m_1} + \sqrt{m_2}} \right]^2. \tag{5.9}$$

If the condensations start from rest, the constant in (5.8) is zero and so (5.9) and the corresponding expression for dm_2/dt are equal. Thus during the encounter, both masses accrete equally and since the total mass of the accreted material greatly exceeds the masses of the initial condensations, it follows that the masses of the stars after the encounter are practically equal. If, however, the initial velocities are not zero, the accretion rates will be unequal and so some difference may be expected in the resulting masses. It is not possible to estimate the extent of this difference in general but a rough integration of the equations on a hand machine for unequal initial masses with velocities of the required order has given resulting stars of mass-ratio 1:2. It is difficult without further investigation to say whether mass-ratios as large as 1:10 could be obtained, but this process can at any rate account for some differences between the masses of the components of the resulting binaries.

Time of Encounter.

Returning to the case of equal initial masses, we shall estimate the time required for the condensations to move from their initial positions to their positions of minimum separation. For definiteness we shall consider the case where the initial velocities are zero and $\alpha = 0{\cdot}135$. Then the time is

$$\int_{-d/2}^{0} \frac{dx}{v} = I_1 + I_2$$

where

$$I_1 = \int_{-0.5d}^{-0.375d} \frac{dx}{v} \;, \quad I_2 = \int_{-0.375d}^{0} \frac{dx}{v} .$$

I_2 was estimated graphically to be $1{\cdot}31/\sqrt{G\rho}$. To estimate I_1 , the approximate behaviour of v near $x = -\frac{1}{2}d$ is required. We can get this by putting $x = -\frac{1}{2}d + \delta$ in the differential equation for v^2 . This equation becomes

$$\frac{d(v^2)}{d\delta} + \frac{1}{2}\,\frac{\pi\rho d^2}{m_0}\,v^2 = \frac{2Gm_0}{d^2}$$

of which the solution is

$$v^2 = \frac{2Gm_0}{d}\cdot\frac{\delta}{d}, \tag{5.10}$$

where m_0 is the initial mass of a condensation. The true expression for v^2 is greater than this, so using (5.10) to estimate I_1 will not give an underestimate of the time. Using (5.10) and the fact that $m_0 = \rho d^3(\alpha - \frac{\pi}{24})$ we obtain $I_1 = 7{\cdot}81/\sqrt{G\rho}$ so that the total time of the encounter is $9{\cdot}12/\sqrt{G\rho}$. For $\rho = 10^{-22}$ gm./c.c. this is $1{\cdot}12 \times 10^8$ years and for $\rho = 10^{-20}$ gm./c.c., it is $1{\cdot}12 \times 10^7$ years.

Conclusion.

We may summarize the results of the present investigation by saying that our theory of binary formation applied at an early stage of stellar evolution is consistent with the accretion theory and under suitable circumstances, systems with three or four components can also be formed. If the situation investigated is in any way related to what occurred when the stars first condensed we may draw the following conclusions: The possibility of a given condensation becoming a component of a binary will depend on the mass which it can attain before becoming attracted particularly to one of its neighbours. If it is attracted early in its life it has a high chance of becoming a binary component. In this case the mass of the resulting component is practically independent of the mass of the original condensation and depends only on the interstellar material density and on its original distance from its prospective partner (but, of course, the mass of the component may subsequently be altered by further accretion). But if the mass of a condensation increases past a certain value before becoming particularly attracted to a neighbour, there is no chance of it becoming a binary component except by a three body encounter.

A Simple Statistical Theory of Binary Star Formation.

According to the theory developed above, once a condensation has had an encounter with another, which does not result in capture, there is no chance of the condensation becoming a component of a binary by a further encounter. This was not realised in the early stages of the development of this theory and consequently a simple statistical theory was worked out on the assumption that the chances of a star having a binary forming encounter does not vary with time. As this theory is mathematically interesting, it is included here.

We consider the stars to be contained in a number of "star systems", each system containing r stars, where r may be any positive integer. Thus if $r = 1$, the system is a single star. If $r = 2$, the system is a binary star and so on.

Let $N(t) =$ number of star systems per unit volume at time t

$N_r(t) =$ number of star systems, containing r stars, per unit volume at time t,

so
$$N = \sum_{r=1}^{\infty} N_r.$$

We assume that close encounters occur between the star systems and that such encounters may result in the two systems becoming bound together to form a single system. Since the number of close encounters per unit volume per unit time will be proportional to N^2 and since a fixed proportion of these close encounters may be expected to result in captures, we may take αN^2 as the number of captures per unit volume per unit time where α is a constant. At such a capture, two systems combine to form one so that the total number of systems is reduced by one at each capture. Hence

$$dN = -\alpha N^2 dt$$

$$\therefore \frac{dN}{dt} = -\alpha N^2.$$

The probability that a given capture is between two single stars is $(N_1/N)^2$ since N_1/N is the probability that each system is a single star. The probability that a given capture is between a single star and a binary is $2N_1N_2/N^2$ and so on. Thus

$$dN_1 = -\alpha N^2\left[2\left(\frac{N_1}{N}\right)^2 + \frac{2N_1N_2}{N^2} + \frac{2N_1N_3}{N^2} + \dots\right]dt.$$

The coefficient 2 of $(N_1/N)^2$ is due to the fact that each capture between two single stars removes two single stars.

$$\therefore \frac{dN_1}{dt} = -\alpha N^2 \frac{2}{N^2} N_1 N$$

$$= -2\alpha N_1 N$$

Similarly we obtain

$$\frac{dN_2}{dt} = \alpha\left(N_1^2 - 2NN_2\right)$$

$$\frac{dN_3}{dt} = \alpha\left(2N_1N_2 - 2NN_3\right)$$

$$\frac{dN_4}{dt} = \alpha\left(2N_1N_3 + N_2^2 - 2NN_4\right)$$

Let $N_{4(1+3)}$ be the number of 4-systems formed by collisions between a single star and a 3-system and let $N_{4(2+2)}$ be the number of 4-systems formed by collisions between two binaries. Then

$$\frac{dN_{4(1+3)}}{dt} = \alpha\left(2N_1N_3 - 2NN_{4(1+3)}\right)$$

$$\frac{dN_{4(2+2)}}{dt} = \alpha\left(N_2^2 - 2NN_{4(2+2)}\right)$$

Putting $N_1 = n,\ N_r = 0\ (r = 2, 3, \ldots)$ at $t = 0$ we obtain the numbers of systems per unit volume at $t = T$ as in the second column of the accompanying table. The present observed value of N_2 in the neighbourhood of the sun is about a

r	$N_r(T)$	$N_r(T)$ with $n\xi=2$	Average age when $n\xi=2$
	$N(T)=\left(\frac{1}{n}+\alpha T\right)^{-1}=\frac{1}{\xi}$, say.	$\frac{1}{2}n$	
1	$\frac{1}{n\xi^2}$	$\frac{1}{4}n$	
2	$\frac{1}{n\xi^2}-\frac{1}{n^2\xi^3}$	$\frac{1}{8}n$	$\frac{1}{n\alpha}(2-2\ln 2)$
3	$\frac{1}{n\xi^2}-\frac{2}{n^2\xi^3}+\frac{1}{n^3\xi^4}$	$\frac{1}{16}n$	$\frac{1}{n\alpha}(6-8\ln 2)$
4		$\frac{1}{32}n$	$\frac{1}{n\alpha}(17-24\ln 2)$
$4(1+3)$	$\frac{2}{3n\xi^2}-\frac{2}{n^2\xi^3}+\frac{2}{n^3\xi^4}-\frac{2}{3n^4\xi^5}$	$\frac{1}{48}n$	$\frac{1}{n\alpha}(17-24\ln 2)$
$4(2+2)$	$\frac{1}{3n\xi^2}-\frac{1}{n^2\xi^3}+\frac{1}{n^3\xi^4}-\frac{1}{3n^4\xi^5}$	$\frac{1}{96}n$	$\frac{1}{n\alpha}(17-24\ln 2)$

half of the value of N_1. So putting

$$\frac{1}{2} = \frac{N_2}{N_1} = 1 - \frac{1}{n\xi}$$

where $\xi = \frac{1}{n} + \alpha T$, we get $\frac{1}{n\xi} = \frac{1}{2}$.

At the value of T satisfying this, the numbers of systems per unit volume are given in the third column of the table.

To obtain the average ages of the various types of system at $t = T$, proceed as follows: The number of r-systems lost in time $d\tau$ is given by $-2N\alpha . N_r d\tau$. Let a certain set of r-systems in existence at time t be marked and let n_{r0} be the number of them at time t and let n_r be the number of them subsequently, then

$$dn_r = \frac{n_r}{N_r}\left(-2N\alpha . N_r d\tau\right)$$

$$\therefore \frac{dn_r}{d\tau} = -2\alpha N n_r$$

$$\therefore \log n_r = -2\alpha \int N d\tau$$

$$\log\left(\frac{n_r}{n_{r0}}\right) = -2\alpha \int_t^T N d\tau$$

$$= -2\log\left(\frac{\frac{1}{n} + \alpha T}{\frac{1}{n} + \alpha t}\right)$$

$$\frac{n_r}{n_{r0}} = \left(\frac{\frac{1}{n} + \alpha t}{\frac{1}{n} + \alpha T}\right)^2 = \frac{N^2(T)}{N^2(t)}.$$

The average age of the r-systems in existence at time T is

$$\frac{1}{N_r(T)}\int_0^T (T-t)\left(\frac{N^2(T)}{N^2(t)}\right) dN_r^*$$

where $N_r^*(t) =$ number of r-systems per unit volume at time t, including those which are incorporated in systems of larger r. The connection between N_r and N_r^* is

$$\frac{d}{dt}\left(N_r^* - N_r\right) = 2\alpha N N_r.$$

Using this, the average age becomes

$$\frac{N^2(T)}{N_r(T)}\int_0^T \frac{N_r(t)\,dt}{N^2(t)}.$$

This is given in the fourth column of the table when $n\xi = 2$.

Chapter VI: Interactions between Binary Stars and Interstellar Material.

In problems involving accretion by a single star, we have seen that there is essentially only one centre of gravitation, namely the star itself. When we come to consider accretion by binary stars, there are two such centres and this fact renders an exact mathematical treatment extremely difficult. It is simple to show, however, that the accretion mechanism must be considerably modified in the case of binary stars. In this chapter, these modifications are considered and an attempt is made to estimate the dynamical effects of accretion on binary stars by means of a somewhat simplified treatment.

Consider a binary star consisting of two stars A and B each of mass m, rotating in a common circular orbit of radius r with its centre at rest relative to the interstellar material. Then the equation of motion is

$$\frac{Gmm}{(2r)^2} = \frac{mv^2}{r}$$

where v is the velocity of either star. Thus

$$v = \sqrt{\frac{Gm}{4r}}. \tag{6.1}$$

Consequently

$$\frac{2Gm}{v^2} = 8r.$$

In order for the accretion rate by each component star to be that given in equation (1.36), it is necessary for an accretion column to extend to a distance $8r$ behind the star. This is clearly impossible, for neither star can capture the material at the centre O of the binary orbit since this is equally attracted by each star. Material originally nearer to A than O may be captured by A. Let the tangent to the orbit of A at A be AC. Ignoring the other star let us find where material originally near O would hit AC. For this we use equation (1.10), putting $\sigma = r$ and $\mu/v^2 = 4r$. Hence $x = r/8$, x being the distance from A to the point where the said material hits AC. The rate of accretion is therefore much less than that for a single star moving with the same velocity. This calculation shows, however, that all the material that can possibly be captured by A, hits AC very close to the star and provided the collisions are inelastic, all this material will certainly be gravitationally bound to the star. Thus, as the stars revolve, they will each tunnel out a section of radius r So unless there is some means of replacing the material captured, the region of space occupied by the binary will soon be cleared of interstellar material.

In the case of a binary of large separation, of the order of a quarter to one-half of the mean interstellar distance, the possibility of replacing the material captured will depend on the motion of the binary relative to the interstellar material and relative to its neighbours. In the simplest case for a binary at rest relative to both the interstellar material and its neighbours, the gravitation of the binary as a whole will draw in the material from the surrounding space. But there is a limit to this process owing to the cut-off effect of the neighbouring stars. Thus, the accretion process can only occur for a limited period and will then cease. In the more realistic cases where the binary is moving relative to the interstellar material and relative to its neighbours, the accretion process can be continued. This is partly due to the fact that the binary will be continually moving into fresh regions of interstellar material. There is also the possibility that the sphere of influence of the binary may be extended due to the neighbours moving more rapidly relative to the interstellar material and therefore being less capable of capturing it .

In the case of binaries of small separation, they may be considered as a single star as far as distant material is concerned. An accretion system will then form as for a single star. It is only when we consider what happens to the material flowing in along the axis towards the binary that we need to take account of the fact that there are really two stars. This material will be heading for the centre of mass of the binary, but since there will be nothing solid there to stop it, it will continue to move past the centre of mass under its momentum. As soon as it has passed the centre of mass, it will be slowed down and pulled back. The result will be that a steady state is established in which a dense cloud is formed round the stars. These will continuously tunnel out material from this cloud but it will be replaced by the material flowing in along the axis. There are thus two accretion stages: the large scale one in which the binary is considered as a single star and the small scale one in which each star captures material from a cloud which forms round the binary. It will be noted that the centre of mass of the binary is at rest in this cloud. If the interstellar material is gas, the assumption of inelastic collisions is equivalent to assuming the gas cools quickly so that pressure does not develop to any appreciable extent. The material forming the cloud round the binaries will therefore be in a similar condition of temperature and pressure as that in undisturbed space. It will, however, be more dense.

Before going on to discuss the effects of accretion in more detail, we shall mention the conditions which must be satisfied in order that a binary may be considered to be "of small separation" in the sense used above. The separation must be small compared with

(1) Half the local mean interstellar distance;

(2) The distance from the binary to the neutral point on the accretion column of the large scale accretion process (i.e. $2\mu/v^2$, μ being the total mass of the binary $\times$ constant of gravitation and v being the velocity of the mass centre of the binary relative to the interstellar material.)

The definition of "small" is somewhat arbitrary but from the geometry of the problem, the mechanism appears to work if the separation does not exceed 5 per cent of the lesser of (1) and (2). So in the recommended units, the maximum separation for a binary of small separation is the lesser of $\frac{1}{40}$ (supposing the mean interstellar distance to be a parsec) and $M/10v^2$, M being the total mass of the binary. For example, consider a binary consisting of two stars each of unit mass. If the velocity of the binary relative to the interstellar material is less than about 0.3 km./sec., the maximum separation is $\frac{1}{40}$ (= 5000 astronomical units). If the velocity is 50 units (about 4 km./sec.) the maximum separation is only $\frac{1}{12500}$ (= 16 astronomical units). The small values of v are most interesting as they give maximum accretion effects and it is for small values of v that the maximum allowable separation is greatest.

The Sharing of Accreted Material by the Unequal Components of a Binary of Small Separation.

Consider a binary of small separation consisting of components of masses m_1, m_2 describing circular orbits of radii r_1, r_2 respectively about their mass centre with velocities v_1, v_2 and let s be the separation, i.e.

$$s = r_1 + r_2$$

We have

$$\frac{v_1}{v_2} = \frac{r_1}{r_2} = \frac{m_2}{m_1}. \tag{6.2}$$

Let R_1, R_2 be the distances of the components from the point D where the attractions of the components are equal, then

$$\frac{Gm_1}{R_1^2} = \frac{Gm_2}{R_2^2} \quad \text{and} \quad R_1 + R_2 = s \tag{6.3}$$

As previously explained, we consider a cloud to form around the binary and each component sweeps up the material in circular sections of radii R_1, R_2. Thus if ρ_c is the density of the cloud,

$$\frac{\text{accretion by } m_1}{\text{accretion by } m_2} = \frac{\pi R_1^2 \rho_c v_1}{\pi R_2^2 \rho_c v_2} = \frac{R_1^2}{R_2^2} \cdot \frac{v_1}{v_2}$$

from (6.2), (6.3)

$$= \frac{m_1}{m_2} \cdot \frac{m_2}{m_1} = 1.$$

Thus the accreted material is shared equally between the components. This statement breaks down if there is a great difference between the masses of the components because in this case, the centre of mass of the system will be close to or within the star of greater mass. Thus the material from the large-scale accretion system will fall directly on to this star and no cloud will form. So it appears that for binaries of small separation, the accretion process tends to equalise the masses of the components if these were originally nearly equal and it tends to increase the disparity of the masses if these were originally considerably different.

It must be made clear that the above discussion is only intended to be approximate, as for example, when it is said that the components of the binary sweep out circular sections of radii R_1, R_2. Such a picture of the process is obviously not exact since the circle of larger radius will be rotated about a line which intersects it. Consequently the statement, that the accreted material is shared equally between the components can only be expected to apply accurately when the components are almost equal, just as the statement that all of the material is captured by one of the components is only true when the disparity in the component masses is very great.

We may here obtain an expression for the density ρ_c of the cloud which forms around the binary. The equations of motion for the components (ignoring accretion effects) are

$$\frac{m_1 v_1^2}{r_1} = \frac{m_2 v_2^2}{r_2} = \frac{G m_1 m_2}{s^2} \qquad (6.4)$$

Hence $$v_1^2 = G m_2 r_1 / s^2$$

but from (6.2) $$\frac{r_1}{r_2} = \frac{m_2}{m_1} \qquad \therefore \frac{r_1}{r_1 + r_2} = \frac{m_2}{m_1 + m_2}$$

i.e. $$\frac{r_1}{s} = \frac{m_2}{m_1 + m_2}$$

$$\therefore \ v_1^2 = G m_2^2 / s(m_1 + m_2). \tag{6.5}$$

To find ρ_c, the accretion by m_1 is

$$\pi R_1^2 \rho_c v_1 = A/2,$$

A being the total accretion by the binary per unit time, from the large scale system. From (6.3),

$$\frac{R_1}{R_2} = \sqrt{\frac{m_1}{m_2}}$$

$$\therefore \frac{\sqrt{m_1}}{\sqrt{m_1} + \sqrt{m_2}} = \frac{R_1}{R_1 + R_2} = \frac{R_1}{s} \tag{6.6}$$

$$\therefore \ \frac{A}{2} = \pi R_1^2 \rho_c v_1$$

$$= \frac{\pi s^2 m_1}{(\sqrt{m_1} + \sqrt{m_2})^2} \rho_c \sqrt{\frac{G m_2^2}{s(m_1 + m_2)}}$$

$$\rho_c = \frac{A(\sqrt{m_1} + \sqrt{m_2})^2 \sqrt{m_1 + m_2}}{2\pi \sqrt{G}\, s^{3/2} m_1 m_2}.$$

The cloud density ρ_c is in general substantially greater than that of the undisturbed interstellar material ρ. For a binary of given total mass and moving with a given velocity, ρ_c is least when the masses of the components are equal. ρ_c of course depends on the material A arriving from the large scale accretion system and this in turn depends on the velocity of the binary as a whole relative to the interstellar material. Consider for example a binary consisting of equal components of unit mass and at a separation $s = 1/12500$ parsec. Then, when the velocity of the binary is 2.8 (in the recommended units), $\rho_c = 1{\cdot}97 \times 10^6 \rho$

but when the velocity is 50, $\rho_c = 3{\cdot}58 \times 10^2 \rho.$

ρ_c is less, the greater the separation s. If the velocity of the binary is 2.8, then when

$s = 1/12500, \quad \rho_c = 1{\cdot}97 \times 10^6 \rho$

but when $s = 1/40, \quad \rho_c = 3{\cdot}58 \times 10^2 \rho.$

<u>The Rate of Reduction of the Separation of the Components of an Accreting Binary.</u>

Suppose that the binary with components m_1, m_2 considered in the last section is involved in an accretion process such that the components accrete at rates M_1, M_2 and experience forces F_1, F_2 in a direction opposite to their motion and tangential to their orbits, respectively. We wish to know the rate at which the separation of the components is reduced by the process. It is assumed that the time required for the process to have an appreciable effect is larger than the period of revolution of the binary.

Referring to the angular momentum of the binary about its mass centre as Φ , we use the fact that the rate of change of angular momentum of the binary is equal to the moment of the forces acting, i.e.

$$\frac{d\Phi}{dt} = -F_1 r_1 - F_2 r_2 .$$

Now $\Phi = m_1 r_1 v_1 + m_2 r_2 v_2$

$$= m_1 \cdot \frac{s m_2}{m_1 + m_2} \cdot \frac{\sqrt{G}\, m_2}{\sqrt{s(m_1+m_2)}} + m_2 \cdot \frac{s m_1}{m_1 + m_2} \cdot \frac{\sqrt{G}\, m_1}{\sqrt{s(m_1+m_2)}}$$

$$= \frac{\sqrt{Gs}\; m_1 m_2}{(m_1 + m_2)^{\frac{1}{2}}} .$$

So
$$\frac{d\Phi}{dt} = \sqrt{G}\left\{\frac{1}{2}\frac{1}{\sqrt{s}}\frac{ds}{dt}\cdot\frac{m_1 m_2}{(m_1+m_2)^{\frac{1}{2}}} + \frac{\sqrt{s}\, m_2}{(m_1+m_2)^{\frac{1}{2}}}\frac{dm_1}{dt} + \frac{\sqrt{s}\, m_1}{(m_1+m_2)^{\frac{1}{2}}}\cdot\frac{dm_2}{dt}\right.$$

$$\left. - \frac{1}{2}\frac{\sqrt{s}\, m_1 m_2}{(m_1+m_2)^{3/2}}\left(\frac{dm_1}{dt} + \frac{dm_2}{dt}\right)\right\} = -F_1 r_1 - F_2 r_2 . \qquad \textbf{(6.7)}$$

Now
$$\frac{dm_1}{dt}\left(\frac{\sqrt{Gs}\; m_2}{(m_1+m_2)^{\frac{1}{2}}} - \frac{\sqrt{Gs}\; m_1 m_2}{2(m_1+m_2)^{3/2}}\right)$$

$$= M_1 \frac{\sqrt{Gs}\; m_2}{(m_1+m_2)^{\frac{1}{2}}}\left(1 - \frac{m_1}{2(m_1+m_2)}\right)$$

$$= M_1 \frac{\sqrt{Gs}\; m_2}{2(m_1+m_2)^{3/2}}(m_1 + 2m_2)$$

$$= M_1 \frac{\sqrt{Gs}\; m_1 m_2}{2(m_1+m_2)^{3/2}} + M_1 r_1 v_1$$

Similarly for the terms in dm_2/dt . So (6.7) becomes

$$\frac{\sqrt{G}}{2}\frac{1}{\sqrt{s}}\frac{ds}{dt}\frac{m_1 m_2}{(m_1+m_2)^{\frac{1}{2}}} = -\left(F_1 + M_1 v_1\right)v_1 - \left(F_2 + M_2 v_2\right)v_2 - \frac{(M_1+M_2)\sqrt{Gs}\,m_1 m_2}{2(m_1+m_2)^{3/2}}$$

$$\therefore -\frac{1}{s}\frac{ds}{dt} = \frac{2s}{G m_1 m_2}\left[v_1\left(F_1 + M_1 v_1\right) + v_2\left(F_2 + M_2 v_2\right)\right] + \frac{M_1 + M_2}{m_1 + m_2} \qquad (6.8)$$

The approximate time for a 10 per cent reduction of the separation is $\delta t = \frac{1}{10}\Big/ -\frac{1}{s}\frac{ds}{dt}$. If $-\frac{1}{s}\frac{ds}{dt}$ is in the recommended units,

$$\delta t = \frac{1{\cdot}23 \times 10^6}{\left(-\frac{1}{s}\frac{ds}{dt}\right)} \text{ years.} \qquad (6.9)$$

We shall now consider the application of (6.8) and (6.9) to a number of cases: The results of the calculations are given in the accompanying table in the case of $\rho = 10^{-22}$ gm./c.c. Results are given for a binary in which the components are both of unit mass in the recommended units and for one in which the components are of masses 1 and 2. Results are given for separations of $\frac{1}{2}$ and $\frac{1}{5}$ parsec for binaries of large separation. For binaries of small separation, the results are independent of the separation as we shall see.

We first notice the top two rows of the table. These give the velocities of the components relative to the mass centre of the binary, using formula (6.5).

Rates of Reduction of Component Separation of Binary Stars.

		$m_1 = 1,\ m_2 = 1$		$m_1 = 1,\ m_2 = 2$	
		$s = \frac{1}{2}$	$s = \frac{1}{5}$	$s = \frac{1}{2}$	$s = \frac{1}{5}$
	v_1	$0.71\ s^{-\frac{1}{2}}$		$1.15\ s^{-\frac{1}{2}}$	
	v_2	$0.71\ s^{-\frac{1}{2}}$		$0.58\ s^{-\frac{1}{2}}$	
Case 1.	v	$2\sqrt{2}$		$2\sqrt{3}$	
	$M_1 = M_2$	1.11		1.36	
	δt	3.67×10^5 yr.		3.89×10^5 yr.	
Case 2.	$M_1 = M_2$	2.01×10^{-4}		4.52×10^{-4}	
	δt	2.03×10^9 yr.		1.17×10^9 yr.	
Case 3(a).	$M_1 = M_2$	0.20	0.050	0.22	0.056
	δt	2.04×10^6 yr.	8.16×10^6 yr.	2.40×10^6 yr.	9.44×10^6 yr.
Case 3(b).	F_1	0.38	0.15	0.63	0.25
	F_2	0.38	0.15	0.36	0.14
	δt	3.22×10^6 yr.	1.29×10^7 yr.	1.86×10^6 yr.	7.46×10^6 yr.
Case 4.	M_1	1.01×10^{-4}	1.01×10^{-4}	1.01×10^{-4}	1.01×10^{-4}
	M_2	1.01×10^{-4}	1.01×10^{-4}	4.02×10^{-4}	4.02×10^{-4}
	F_1	5.69×10^{-4}	7.53×10^{-4}	8.90×10^{-4}	1.17×10^{-3}
	F_2	5.69×10^{-4}	7.53×10^{-4}	1.68×10^{-3}	2.18×10^{-3}
(a)	δt	4.04×10^9 yr.	4.04×10^9 yr.	2.82×10^9 yr.	2.82×10^9 yr.
(b)	δt	1.08×10^9 yr.	1.29×10^9 yr.	8.71×10^8 yr.	1.05×10^9 yr.
(c)	δt	8.50×10^8 yr.	9.75×10^8 yr.	6.65×10^8 yr.	7.67×10^8 yr.

Case 1.: A binary of small separation whose mass centre is moving relative to the interstellar material with a velocity which is such that the accretion is greatest. This velocity is given by (1.35) where μ is now the total mass of the binary $\times$ constant of gravitation. Thus, in the recommended units, (1.35) becomes

$$v^2 = 4(m_1 + m_2)$$

taking $\Sigma = \frac{1}{2}$. The total large scale accretion is given by (1.36) or (1.37), i.e. $\pi v/4$.

Consequently, since the accreted material is shared equally we have

$$M_1 = M_2 = \pi v/8.$$

We consider the diameter of the cloud round the binary to be about twice the separation of the binary components. Consequently, within this cloud, there is insufficient material for a resistive force-producing mechanism to operate. Hence

$$F_1 = F_2 = 0$$

These values of M_1, M_2, F_1 and F_2 are substituted in (6.8) to obtain the values of δt given in the table. It will be noticed, using the values of v_1 and v_2 given in the table, that the s disappears from (6.8) when $F_1 = F_2 = 0$ and consequently, the rate of reduction of the separation is independent of the separation.

Case 2.: A binary of small separation whose mass centre is moving relative to the interstellar material with a velocity 50 (about 4 Km./sec.). The only difference from Case 1, is that the accretion rate is given by (1.36), i.e.

$$\frac{4\pi(m_1+m_2)^2}{(50)^3}.$$

So
$$M_1 = M_2 = 2\pi(m_1+m_2)^2/(50)^3.$$

Case 3.: A binary of large separation whose mass centre is almost at rest in the interstellar material. It is assumed that the binary moves through the medium with just sufficient speed for the effective interstellar material density to remain constant. The mathematical model is taken to be a binary having its mass centre at rest in the medium which is maintained at a constant density. We obtain values for δt on two different assumptions:

(a) We assume that the components sweep out circular sections of radii R_1, R_2 given by (6.6). It has already been shown that in this case the accretion rates by the components are equal. It must be remembered, however, that in this case, the material swept up is the original interstellar material, so that in using the formula (1.37) we put $\Sigma = R_1$ or R_2 and ρ is unity in the recommended units. Hence the accretion rate for either component is

$$M_1 = M_2 = \pi R_1^2 v_1,$$

v_1 being given at the top of the table and R_1 being given by

(6.6). It is assumed that $F_1 = F_2 = 0$.

(b) We assume that no accretion occurs but that the material in the circular sections of radii R_1 , R_2 exerts resistive forces on the components. To obtain this force we use (1.26) with $\Sigma = R_1$ or R_2 . Hence

$$M_1 = M_2 = 0,$$
$$F_1 = \frac{2\pi m_1^2}{v_1^2} \ln\left(1 + \frac{v_1^4 R_1^2}{m_1^2}\right),$$
$$F_2 = \frac{2\pi m_2^2}{v_2^2} \ln\left(1 + \frac{v_2^4 R_2^2}{m_2^2}\right).$$

Case 4.: A binary of large separation whose mass centre is moving with velocity 50 relative to the interstellar material in a direction perpendicular to the plane of the binary orbit. Each component has its own accretion system. The accretion is given by (1.36), i.e.

$$M_1 = \frac{4\pi m_1^2}{(50)^3}, \quad M_2 = \frac{4\pi m_2^2}{(50)^3}.$$

For the force we use (1.33) with $\Sigma = R_1$ or R_2. Thus

$$\mathcal{F}_1 = \frac{2\pi m_1^2}{(50)^2} \ln\frac{1}{5}\left(1 + \frac{(50)^4 R_1^2}{m_1^2}\right)$$

$$\mathcal{F}_2 = \frac{2\pi m_2^2}{(50)^2} \ln\frac{1}{5}\left(1 + \frac{(50)^4 R_2^2}{m_2^2}\right)$$

approcimately. (Accurately, we should put $\sqrt{(50)^2 + v_2^2}$ for the velocity.) However, these forces act almost perpendicular to the plane of the binary orbit. The values of F_1, F_2 are the components of $\mathcal{F}_1$, $\mathcal{F}_2$ in the plane of the orbit. Thus

$$F_1 = \frac{v_1}{\sqrt{(50)^2 + v_1^2}}\mathcal{F}_1 \simeq \frac{v_1}{50}\mathcal{F}_1$$

and
$$F_2 \simeq \frac{v_2}{50}\mathcal{F}_2.$$

Values of δt are given in the table for three cases:

(a) When the resistive force is absent, i.e. M_1, M_2 as above and $F_1 = F_2 = 0$.

(b) When the accretion is absent, i.e. ~~M_1, M_2~~ F_1, F_2 as above and $M_1 = M_2 = 0$.

(c) When both accretion and the resistive force are present, i.e. F_1, F_2, M_1, M_2 as above.

It will be seen from the table that the values of δt vary between 10^5 and 10^9 years and consequently binary stars can be appreciably affected by the accretion process in astronomically reasonable periods of time. For higher densities of the interstellar material, shorter times are required, the time δt being inversely proportional to the square root of the density.

<u>Note on the Effect of a Resistive Force on the Eccentricity of a Central Orbit.</u>

We have been unable to arrive at any general conclusions about the effect of accretion on the eccentricity of an elliptic binary orbit. In order to ascertain the effect, it would be necessary to do a detailed integration of the effects over a complete revolution of the binary. We can show this to be the case as follows: Consider a particle of unit mass moving under a central force μ/r^2. The eccentricity of the orbit is given by (1.8), i.e.

$$e = \sqrt{1 + \frac{2Eh^2}{\mu^2}} \,. \tag{6.10}$$

When at a distance r from the centre, let the velocity of the particle be v in a line which is a perpendicular distance p from the centre. We consider the effect on e of a small reduction of v at this point. From (6.10),

$$\frac{de^2}{dv} = \frac{2}{\mu^2}\frac{d(Eh^2)}{dv}$$

but $E = \frac{1}{2}v^2 - \mu/r$, $h = pv$,

$$\therefore \frac{de^2}{dv} = \frac{2}{\mu^2}\frac{d}{dv}\left(p^2\left[\frac{1}{2}v^4 - \frac{\mu}{r}v^2\right]\right)$$

$$= \frac{2}{\mu^2}p^2\left(4\cdot\frac{1}{2}v^3 - \frac{2\mu}{r}v\right)$$

$$= \frac{4vp^2}{\mu^2}\left[\frac{1}{2}v^2 + E\right].$$

The effect of the eccentricity therefore depends on the sign of $\frac{1}{2}v^2 + E$. Consider a circular orbit of radius R and let the velocity be v_0. Then

$$\frac{v_0^2}{R} = \frac{\mu}{R^2}$$

$$\therefore v_0^2 = \mu/R$$

$$\therefore E = \tfrac{1}{2}v_0^2 - \frac{\mu}{R} = -\frac{\mu}{2R}$$

$$\therefore \tfrac{1}{2}v_0^2 + E = 0.$$

Consider now an elliptic orbit with the same value of E. At points on this elliptic orbit such that $r < R$, we have $\frac{\mu}{R} < \frac{\mu}{r}$ but

$$\tfrac{1}{2}v^2 = E + \frac{\mu}{r}$$

so at such points, $v > v_0$. Hence

$$\tfrac{1}{2}v^2 + E > 0.$$

But for $r > R$, $\frac{\mu}{R} > \frac{\mu}{r}$ $\quad \therefore \tfrac{1}{2}v^2 + E < 0.$

Hence, a resistive force tends to reduce the eccentricity of an orbit if it acts while the particle is within the circular orbit of equal energy but tends to increase the eccentricity if it acts while the particle is outside the circular orbit. It is not, therefore, possible to say that a resistive force always reduces the eccentricity. It is necessary to integrate the effect over the whole orbit to determine whether any particular resistive force reduces the eccentricity.

Chapter VII: A Numerical Determination of the Torque experienced by a Binary Star in a Cloud of Particles.

If a binary star is in motion through an interstellar medium, there is in general an interchange of angular momentum between the binary and the medium, with consequent effects upon the orbital elements of the binary. A full mathematical treatment of the aerodynamical problem is quite beyond the range of existing techniques. In order to estimate the effects, use was made in Chapter VI of known results for simpler problems. In this simplified treatment, the interstellar material is supposed to consist of particles which move under the gravitational influence of the components of the binary but do not influence one another. This supposition is equivalent to assuming that the interstellar material is a gas at zero absolute temperature. On this supposition, if the orbits of the binary components are circular, the determination of the motion of a given particle of interstellar material is reduced to the classical "restricted three body problem". A considerable amount of computation on this problem was performed by E. Strömgren [22] and his collaborators between 1919 and 1931 in order to determine the forms of periodic orbits of the bodies. These calculations were almost entirely concerned with the motion of the bodies in a plane.

The orbits (for various initial conditions) of a particle of negligible mass in the field of a binary star have recently been determined by numerical integration on the Manchester University Electronic Computer. The binary was taken to consist of a pair of particles of equal mass, pursuing a common circular orbit. This chapter is an account of this work and its application.

It should be stated explicitly here that the particle-orbits are not restricted to plane motion but are in three dimensions.

Numerical Method.

We wish to determine the motion of a particle C of negligible mass in the gravitational field of two other particles A and B which are of equal mass and are revolving under their mutual gravitational attraction in a common circular orbit of radius a, the centre of which is at rest in a Newtonian frame of reference. The two particles A and B are always at the ends of a diameter of the common orbit and each moves with the velocity

$$V = (\mu/4a)^{\frac{1}{2}}$$

where $\mu =$ mass of A (or B) $\times$ constant of gravitation.

If we place a frame of reference $Oxyz$ with its origin O at the centre of the circular orbit and such that this orbit is in the x,y plane, the equations of motion for the particle C are

$$\dot{u} = -\frac{\mu}{R_1^3}(x - a\cos\theta) - \frac{\mu}{R_2^3}(x + a\cos\theta),$$

$$\dot{v} = -\frac{\mu}{R_1^3}(y - a\sin\theta) - \frac{\mu}{R_2^3}(y + a\sin\theta),$$

$$\dot{w} = -\frac{\mu}{R_1^3}z - \frac{\mu}{R_2^3}z,$$

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = w, \quad \theta = Vt/a,$$

where (x, y, z) are the coordinates and (u, v, w) the velocity components of C at time t and

$$R_1^2 = (x - a\cos\theta)^2 + (y - a\sin\theta)^2 + z^2;$$

$$R_2^2 = (x + a\cos\theta)^2 + (y + a\sin\theta)^2 + z^2.$$

In the calculations by Strömgren the bodies A and B were reduced to rest by a suitable change of variables but this would be of no help for our purpose. So the equations of motion were integrated as they stand, except for a transformation to make the variables dimensionless. This was obtained by measuring distances in terms of a, velocities in terms of $(\mu/a)^{\frac{1}{2}}$ and time in terms of $(a^3/\mu)^{\frac{1}{2}}$. The variables were also scaled down so as to be accommodated in the number range of the machine in the manner customary in this type of computation.

The method of integration used was a modified form of the Runga-Kutta process which has been developed by S. Gill[23] especially for use in electronic machines. We used this process in the following form: Let the independent variable be y_0 and the dependent variable be $y_i(y_0), (i = 1, 2, \ldots, n)$ and the differential equations be

$$\frac{dy_i}{dy_0} = f_i(y_0, y_1, \ldots, y_n).$$

Let Y be the initial value of y_0 so that $y_{i,0} = y_i(Y)$ are given. Let h be the integration step in y_0 then we calculate the following quantities with $\alpha = 0$,

$$k_{i,\alpha} = h f_i(y_{0,\alpha}, y_{1,\alpha}, \cdots, y_{n,\alpha})$$

$$r_{i,\alpha+1} = A_\alpha (k_{i,\alpha} - q_{i,\alpha}) - B_\alpha q_{i,\alpha}$$

$$y_{i,\alpha+1} = y_{i,\alpha} + r_{i,\alpha+1}$$

$$q_{i,\alpha+1} = q_{i,\alpha} + 3 r_{i,\alpha+1} - (A_\alpha + 2B_\alpha) k_{i,\alpha}$$

where $q_{i,0} = 0$ and A_α, B_α are given in the following table

α	A_α	B_α
0	$\frac{1}{2}$	0
1	$1 - \sqrt{\frac{1}{2}}$	0
2	$1 + \sqrt{\frac{1}{2}}$	0
3	$\frac{1}{6}$	$\frac{1}{6}$

From the values for $\alpha = 0$, the values for $\alpha = 1, 2$ and 3 are calculated. Then

$$y_{i,4} = y_i(Y+h)$$

The process is repeated, starting with $y_i(Y+h)$ to obtain $y_i(Y+2h)$ but the values taken for $q_{i,0}$ will be the same as $q_{i,4}$ of the previous step. The error involved in this process is of order h^5 . To obtain the values of the variables at a given point, the method only requires a knowledge of the variables at the previous point and not at several previous points. This removes the need for storing previous values in the machine and also the need for shifting these values after each step. Another advantage is that no special starting procedure is necessary. The quantities $q_{i,4}$ are used to give an indication of the previous behaviour of the variables but they are usually small and there is practically no loss of accuracy if they are taken to be zero at any point. This enables changes in the magnitude of h to be made without the need for any special calculations. Although the calculations described here were performed with a fixed integration step h , it is possible to make the machine automatically adjust the size of the step so as to keep a specified accuracy. It is also obvious that this integration method on the machine can be used for the general problem of three bodies or in fact of any number of bodies subject to the storage limitations in the machine but the time required for each integration step would be increased.

The cosines and sines in the equations of motion were calculated directly from the power series using terms up to that involving Θ^{12} where Θ is such that

$$\theta = \pi n \pm \Theta \qquad (n \text{ being any integer})$$

and $$0 \leqslant \Theta \leqslant \pi/2.$$

The machine took 11 seconds to advance the integration by one step. This, however, involved two actual calculations and a comparison of the results to eliminate the risk of "random" machine errors. At intervals of two or four steps, the values of x, y, z, u, v, w, t and $(x^2+y^2+z^2)$ were printed out. Each printing required 10 seconds and involved six decimal digits and a sign for each variable.

If the particle C were to approach too closely to either A or B, the rapid variation in the gravitational field would considerably affect the accuracy of the solutions and this might not be noticed by examining the printed results. To remove this risk, it was arranged for the machine to stop if R_1 or R_2 became less than $\frac{1}{8}a$. This did not occur very often so that few orbits were lost on this account.

<u>The Problem.</u>

The main problem consisted of finding the angular momentum gained by the particle C when it is projected parallel to the z-axis towards the revolving bodies A and B. More precisely, the initial conditions were

$$x = R\cos\phi, \quad y = R\sin\phi, \quad z = 4a,$$

$$u = 0, \qquad v = 0, \qquad \omega = -W,$$

where various values of R and ϕ were taken; ϕ being measured from a direction parallel to the x -axis. Orbits were obtained for one value of W only. This was $1{\cdot}6\,(\mu/a)^{\frac{1}{2}}$. This value of the velocity is of the right order to give a result for comparison with our other methods of estimation. The particular factor $1{\cdot}6$ was selected from the relevant range because, after the variables have been scaled down, it gives a convenient number in the scale of two, for use in the machine. The orbits were computed in most cases until the z coordinate of C was less than $-4a$. The initial and final values of z (i.e. $\pm 4a$) were arbitrary but were taken to be sufficiently large so that beyond these limits the angular momentum of C would not have been appreciably changed.

For each of the orbits obtained, the final value of the angular momentum per unit mass of C about O (i.e. $xv - yu$) was then calculated on a desk machine. The initial conditions were, of course, such that the initial angular momentum about O was zero in all orbits.

In the accompanying table, the angular momentum per unit mass gained by C is given in the dimensionless form.

Table of $(xv - yu)(a\mu)^{-\frac{1}{2}}$

	R=0.5a	R=0.75a	R=a	R=1.25a	R=1.5a	R=2a	R=3a	R=4a
0°	+0.3000	+0.5170	+0.6308	+0.6265	+0.5518	+0.3594	+0.1260	+0.0481
15°		+0.7710	+1.0000	+1.0070	+0.8742	+0.5504		
30°	+0.4577	+1.0187	+1.4981	+1.5183	+1.2370	+0.6902	+0.2232	+0.0876
45°		+1.1260	+1.8744	+2.4128	+1.5078	+0.6712		
60°	+0.2458	+0.7457	[+1.90]	[+2.90]	+0.8266	+0.3272	+0.0978	+0.0410
75°		-0.0502	[-0.25]	[-0.70]	-0.8070	-0.2821		
90°	-0.1382	-0.4189	-0.9052	-1.3243	-1.4007	-0.6559	-0.1536	-0.0502
105°		-0.4642	-0.7987	-1.0435	-1.0385	-0.6574		
120°	-0.1674	-0.3426	-0.5140	-0.6193	-0.6270	-0.4813	-0.2007	-0.0845
135°		-0.1552	-0.2278	-0.2776	-0.2653	-0.2673		
150°	+0.0404	+0.0549	+0.0461	+0.0203	-0.0099	-0.0532	-0.0579	-0.0338
165°		+0.2784	+0.3241	+0.3118	+0.2647	+0.1556		
f(R)	+0.1230	+0.2567	+0.38	+0.40	+0.0928	+0.0297	+0.0058	+0.0014

To obtain these orbits, a step of $0{\cdot}25$ in dimensionless time was used. About 20 steps were required for each orbit. Most of the values given are correct to within one per cent. This limit of accuracy was sufficient for our requirements but the machine could be made to work to a considerably higher accuracy. Orbits were not obtained for the four cases: $R = a, 1{\cdot}25a$ and $\phi = 60°, 75°$, owing to the close approach of C to either A or B resulting in the machine being stopped as mentioned above. The values given in the table (in square brackets) are obtained by graphical interpolation between the numbers in the columns. The values in the range $\phi = 180°$ to $360°$ are a repetition of those given in the table. Inspection of the numbers in any column shows that they lie approximately on a sine curve and the oscillation is about a mean value that is small compared with the amplitude of the oscillation. This situation makes it difficult to obtain an accurate value for the mean. Since the function is necessarily periodic (period π) the best method (24) is ~~simplø~~ simply to take the arithmetic mean of the values in each column. The mean values, $f(R)$, are given in the bottom row of the table. These values are accurate to about 5 per cent except for the cases $R = a$ and $1{\cdot}25a$ where only 10 per cent accuracy is claimed.

<u>Calculation of the Torque experienced by a Binary.</u>

From the results of the work described in the last section, it is possible to calculate the torque experienced by a binary moving in a certain way through a cloud of interstellar material. For consider the binary consisting of A and B to be moving in the direction Oz with velocity W through a cloud of particles which are at rest at a great distance from the binary. By giving the entire system a velocity $-W$, we have the situation described above, neglecting the effect of the gravitational field upon the angular momentum outside $z = \pm 4a$. The mass of material which crosses the plane $z = 4a$ between the circles

$$x^2 + y^2 = R^2 \quad \text{and} \quad x^2 + y^2 = (R + dR)^2$$

in unit time is

$$2\pi R dR . W\rho$$

where ρ is the mass of particles per unit volume in the undisturbed cloud. The angular momentum gained by this mass in its encounter with the revolving system is

$$2\pi R dR . W\rho \, f(R)$$

so the total angular momentum gained by the material in unit time is

$$2\pi W\rho \int_0^\infty R \, f(R) \, dR .$$

This is the rate of loss of angular momentum by the rotating system and so is equal to the torque acting on it. This expression was evaluated for $W = 1 \cdot 6 \, (\mu/a)^{\frac{1}{2}}$ by plotting $R\,f(R)$ and obtaining the integral by counting squares.

The result was

$$2\pi \rho a^2 \mu \times 0{\cdot}6. \tag{7.1}$$

The function $Rf(R)$ rises and falls steeply near its maximum value. Consequently, the value of the integral is not appreciably affected by an inaccuracy in the maximum value of the function. The maximum value of $Rf(R)$ was estimated from the graph to be 0.5 at $R = 1{\cdot}25a$. A 20 per cent error in this maximum value would only have caused a 5 per cent error in the value of the integral but as we have already stated, we do not expect this maximum value to be in error by more than 10 per cent. [As an alternative to the graphical method, an attempt was made to fit a curve of the type $Ax^2 \exp(-Bx)$. This was fitted to two of the accurately known values; namely at $R = 0{\cdot}75a$ and $R = 1{\cdot}5a$. The values of A and B were 1.89 and 2.27 respectively. The result of the integration was then 13 per cent less than that obtained by graphical integration but as the curve did not fit very well, the result from the graphical method is considered to be the more accurate.] After this discussion of various errors we consider that 10 per cent accuracy can be claimed for the final result (7.1).

A question arises as to whether the effect of particles passing very near the revolving bodies is large compared with the effect of the remaining particles. In the immediate neighbourhood of one of the bodies, a particle will behave almost as if it were influenced by this body alone. The effect of the particles on a single star has been examined in Chapter I and equation (1.26) gives the force on the star due to the particles passing within a given distance of the star. It is seen that this force tends to zero as the given distance tends to zero. It follows that particles, passing very close to either component of the revolving system considered here, do not contribute appreciably to the total effect.

It will be noticed that we have neglected any change in the motion of A and B. This is justified because, as we shall see later, the orbital elements of the binary are only appreciably affected by the torque after a period which is several times the period of revolution of the binary.

Comparison of the Result with a Theoretical Formula.

It is possible to obtain a theoretical formula for the torque in cases where W/V is large compared with unity. From equation (1.26), the force exerted on a single star due to its passage through a cloud of interstellar material is

$$2\pi\rho\mu^2 U^{-2} \ln\left(1+\Sigma^2 U^4 \mu^{-2}\right) \qquad (7.2)$$

where ρ is the density of the undisturbed cloud, U is the velocity of the star relative to the cloud and μ = mass of star × constant of gravitation. Σ is a "cut-off distance" which is of the order of half the distance to the nearest neighbouring star. The force acts in the opposite direction to the velocity. If we assume that this formula holds for each component of a binary star which is moving perpendicular to the plane of its orbit, we put

$$\Sigma = a \quad \text{and} \quad U^2 = W^2 + V^2$$

in (7.2) to obtain the force on each component. The force on each component can be resolved into a force parallel to the direction of motion of the binary relative to the cloud and into a force tangential to the common circular orbit. The forces of the latter type, acting on the two components, together form a torque of value

$$T = 2\pi \rho a^{\frac{1}{2}} \mu^{\frac{5}{2}} W^{-3} \ln\left(1 + a^2 W^4 \mu^{-2}\right) \qquad (7.3)$$

if W/V is large compared with unity. This condition is imposed by the mechanism which produces the force (7.2). Each star "tunnels" through the interstellar material and this condition ensures that one star does not tunnel through a region which has already been disturbed by the other star.

Putting $W = 1{\cdot}6(\mu/a)^{\frac{1}{2}}$ we obtain

$$T = 2\pi \rho a^2 \mu \times 0{\cdot}49$$

so that the formula is in satisfactory agreement with the numerical result (7.1). In this case, $W = 3{\cdot}2V$ so this agreement is in the region of W where we might hardly expect the formula to hold accurately. It is also to be noted that the formula does not overestimate the torque.

It should be noted that this agreement between the formula (7.3) and the numerical result is only offered as additional support for the formula. Formula (7.3) rests on the reasoning given above and so its application is not limited to the value of W used in the numerical work.

Rate of Reduction of Separation of Binary Components.

If we neglect any variation in the masses of the components of the binary, the effect of the torque is to reduce the separation of these components. To determine the rate of reduction of this separation, let E be the total energy of the binary at time t, i.e.

$$E = \tfrac{1}{2}mV^2 + \tfrac{1}{2}mV^2 - \tfrac{1}{2}Gm^2/r = -Gm^2/4r$$ where m = mass of A (or B), G = constant of gravitation and r = the radius of the orbit of the binary at time t. In a time interval dt, the loss of energy is

$$-dE = Gm^2 dr/4r^2.$$

But this also equals the work done by the torque T in dt, i.e.

$$TV dt/r$$

Hence

$$\frac{1}{r}\frac{dr}{dt} = \frac{2T}{m(Gmr)^{1/2}}$$

so that a 10 per cent reduction of the separation will occur in the time $m(Gma)^{\frac{1}{2}}/20T$, approximately.

We may illustrate this formula by a specific example, using the value of T given by (7.1). Consider a binary consisting of two equal stars, each of 5 solar masses and separated by a distance of 0.1 parsec. If this binary moves through a cloud of interstellar material of density 10^{-22}gm./c.c. in a direction perpendicular to the plane of its orbit with velocity 1.05 km./sec., the separation will be reduced by 10 per cent in about 2.68×10^7 yrs. The period of revolution of the binary is 9.36×10^5 yrs.

Another Set of Orbits.

A set of less accurate integrations were performed to find the orbits of the particle C after being released from rest at various points on the sphere

$$x^2 + y^2 + z^2 = (4a)^2.$$

For some positions of release, the particle C moved once through the circular orbit of the bodies A and B and then receded to a distance greater than $4a$ from the origin. For other positions of release, the particle C receded to a distance of less than $4a$ from the origin and then returned for one or more encounters with the revolving system before escaping from the sphere. One of the more complicated orbits was computed accurately and projections of it on the planes $y = 0$ and $z = 0$ are shown in Fig. 7.1. In one of these, the positions of C are marked when AB has turned through various multiples of $\frac{1}{2}\pi$. The accuracy of the curves is not guaranteed beyond the point D where the particle has a close encounter with one of the

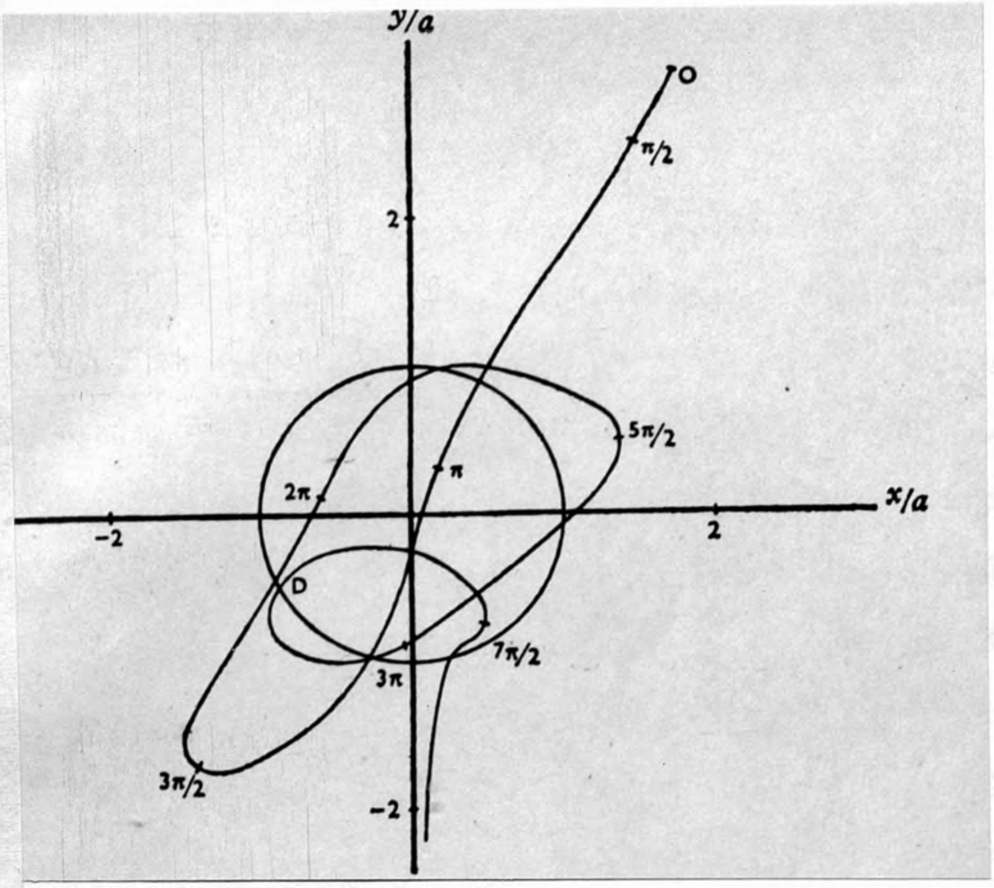


Fig. 7.1.

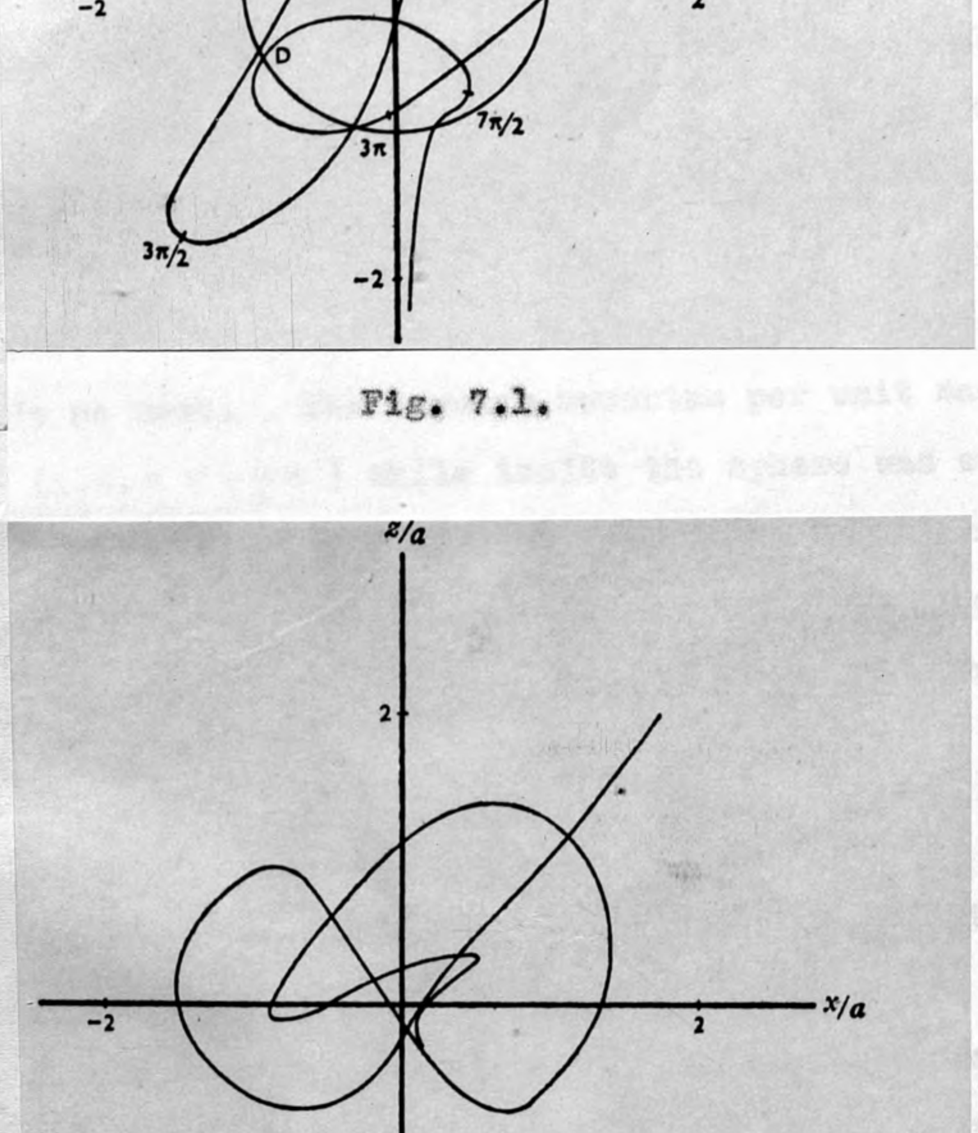


Fig. 7.2.

revolving bodies. In the orbits which were computed, the particle C eventually escaped from the sphere and it seems certain that this would always occur. For, it is impossible for C to settle into a periodic orbit around the revolving system; otherwise, by reversing the time coordinate, it would be possible for periodic orbits of C to become non-periodic. When C escapes from the sphere of radius $4a$, it had gained energy from the revolving system. It will also have gained a proportional amount of angular momentum. This follows from the fact that the revolving system as a whole has no translatory motion and so any forces (as opposed to torques) acting on it during the encounter with the particle can do no work. The angular momentum per unit mass gained by C (i.e. $xv - yu$) while inside the sphere was evaluated for the various orbits and was considered as a function of the point of release of C. Then by a crude integration over the surface of the sphere, a mean value was obtained. This was

$$0.58(\mu a)^{\frac{1}{2}}$$

This result can be used to obtain the torque exerted on the rotating system when material falls steadily from the surface of the sphere and there are no particle collisions. We may compare this result with that obtained on the assumption that the particles collide and are eventually all captured by one or other of the revolving bodies. In this case the angular momentum per unit mass gained by a particle C is aV (V being the velocity of A or of B $=\frac{1}{2}(\mu/a)^{\frac{1}{2}}$) which is

$$\frac{1}{2}(\mu a)^{\frac{1}{2}}.$$

We thus see that these results, and hence the torques, are of the same order when all the material is captured as when no material is captured. It would then appear that the torque will be of the same order whatever fraction of the material is captured by the revolving bodies.

Concluding Remarks.

Although, in the cases we have considered, the interaction of non-revolving material with a binary causes a reduction in the angular momentum of the latter, we do not claim to have shown that this is always the case. In fact, there appears to be no dynamical principle which would be violated if such were not the case.

The work described in the present chapter was undertaken with the object of checking the mathematical basis of the work in Chapter VI, but the problem of angular momentum exchange in the gravitational interaction of a particle with a revolving system is in itself a problem of purely dynamical interest and may be worthy of further investigation when high-speed computing facilities become more generally available.

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