Group Cohomology and Quantum Fields

A thesis presented by

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ABSTRACT

In this thesis it is shown how the 1-Cohomology of groups can be used to classify certain representations of the Canonical Commutation Relations.

First, the algebra of the Canonical Commutation Relations is described in the framework of C^*-algebras. The Fock and displaced-Fock representations are defined.

A unitary representation of a connected Lie group is introduced into the complex pre-Hilbert space, over which the C.C.R. algebra is built. This group action induces an automorphism of the C.C.R. algebra, and the automorphism is shown to be unitarily implemented in the Fock representation. The question of unitary implementability in the displaced-Fock representation leads to the study of 1-cohomology of groups.

The cohomology of the Poincare group is studied, for various representations of the Poincare group. Also the cohomology with values in the Hilbert-Schmidt operators of the one-particle space is calculated to be trivial.

The results obtained then determine whether or not there do exist representations of the C.C.R. which are inequivalent to the Fock representation and which have a group automorphism unitarily implemented.

To my Mother, and to the Memory of my Father

Acknowledgements

Doctoral theses, like good deeds in a naughty world, need no apology, wrote Felix Raab in his book "The English Face of Machiavelli". I adhere to this dictum and therefore take pleasure only in acknowledging my debts.

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Contents

Introduction

The original motivation for the work presented in this thesis was the occurrence of non-Fock representations in the algebraic theory of the free quantum field in 1+1 space-time dimensions [20]. These non-Fock representations are of the kind which are named displaced Fock representations, and they correspond to the addition of a classical field to the free quantum field.

One can describe a representation of the Canonical Commutation Relations in terms of a family of unitary operators on a Hilbert space, indexed by a complex pre-Hilbert space, τ. The set $\{W(f): f \in \tau\}$ must satisfy the relation

$$W(f)W(h) = e^{i\mathrm{Im}(f;h)}W(f+h)$$

where $W(f)$ is a unitary operator on a Hilbert space for each $f \in \tau$. The Fock representation is that representation in which the Hilbert space possesses a cyclic vector, and this cyclic vector is annihilated by an operator constructed from the Weyl operators $W(f)$. Displaced Fock representations can then be defined on the Fock space by unitary operators

$$W_F(f) = e^{i\mathrm{Im}F(f)}W_0(f)$$

where $W_0(f)$ is the Fock version of the representation, and $F \in \tau^X$, the algebraic dual of τ. Although the map $f \to W_0(f)$ is strongly continuous, from τ to the Fock space, it is not necessarily the case for the map $f \to W_F(f)$. One can show that this leads to the result that W_F has a vacuum vector if and only if F is a continuous linear functional on τ, and so when F is not continuous, the representation W_F corresponds to a theory with no vacuum.

When a connected Lie group, G, is represented unitarily in τ, so that it can be thought of as the one-particle symmetry group, it is known that, in the Fock representation, there is a unitary operator in Fock space with

$$W_0(U_g f) = V_g W_0(f) V_g$$

where U_g represents g in τ, and V_g is the representative of g in Fock space. Further, $g \to V_g$ defines a strongly continuous representation. One then asks for the same conditions to hold for some of the representations W_F. They are found to hold if and only if

$$M_g F - F \in \mathcal{K}$$

where $\mathcal{K}$ is the Hilbert space completion of τ, and M_g is defined by

$$(M_g F)(f) = F(U_g^{-1} f) \quad \text{and} \quad f \in \tau$$

This is the point of entry of 1-cohomology into the problem.

These results are described in more detail in Chapter 1.

In the second chapter some theorems about cohomology are presented. The first part introduces definitions, and the second part describes results of Pinczon and Simon [15], with the restriction to unitary representations. The proofs given are expansions of those given in [15], with a new proof given for Proposition 2.2.1. A full proof is given for Lemma 2.2.3, as this is useful in the third section, and since it appears only as a comment in [15]. The third part of the chapter presents two new results which are necessary underpinnings for the calculations in the third and fourth chapters. In the final section of this chapter, results due to Araki and new results are presented together.

The third chapter treats the cocycles for physical representations of the Poincaré groups $P_+^\uparrow(s+1)$, s = 1,3. The Poincare group of 2+1 space-time dimensions is not mentioned, as this is dealt with in an article by the author of this thesis [3]. An example of a cocycle for a subgroup of $P_{+}^{\uparrow}(3+1)$ is given, and it is shown to be a cocycle, with the help of Lemma 2.3.1.

In the fourth chapter, cocycles for direct product representations of the Poincare group are examined. The results are then applied to Hilbert-Schmidt valued cohomology for the Poincare group. It turns out that the ideas of the previous chapters are useful in the determination of these problems. A corollary to these calculations is the result that certain representations of the C.A.R. and the C.C.R., having the Poincaré group unitarily implemented, must be unitarily equivalent to the Fock representations involved.

It is interesting to note that as early as 1938, group 1-cohomology was considered by Wigner, in his famous paper about the unitary representations of the Poincaré group. This occurred when Wigner reduced the multiplier to ±1. The relevant reference is Section C, p.174 of Wigner's paper.

CHAPTER 1

Characterization of Displaced Fock Representations of the Canonical Commutation Relations

1. The Canonical Commutation Relations (C.C.R.)

The quantum theory of a boson with one degree of freedom is governed by the equation:

$$PQ - QP = -i\,1$$

Where P, Q are self-adjoint operators acting in a complex Hilbert space $\mathcal{H}$ with a common, densely defined domain of vectors, D, in $\mathcal{H}$. This equation is not easy to deal with, as we have the following theorem due to Wielandt:

Theorem [21]. If P, Q are defined on the same dense domain D in some complex Hilbert space H, and they satisfy

$$PQ - QP = -i\,1 \tag{1.1.1}$$

on D, then P and Q cannot both belong to the bounded operators of $\mathcal{H}$. It follows from this theorem that at least one of these operators is unbounded, and in general both are. A way out of this seeming impasse is to form the one-parameter unitary families $V(t) = \exp(-itQ)$ and $U(s) = \exp(-isP)$ with $s,t \in \mathbb{R}$. Then for each $t \in \mathbb{R}$, $V(t)$ is unitary and for each $s \in \mathbb{R}$, $U(s)$ is unitary. The equation (1.1.1) is replaced by

$$V(t)V(s) = e^{-ist}U(s)V(t)$$

for all $s,t \in R$ and defined for any vector in $\mathcal{H}$. Equation (1.1.2) is called the Weyl Form of the C.C.R. for one degree of freedom. The operators P and Q can be recovered by using Stone's Theorem. Now the problem with quantum field theory is that we must deal with an infinite number of degrees of freedom. We must therefore reflect this in the mathematical structure which we erect. Further, if we are to be somewhat general, it is necessary to include, as a special case, the above system. We now make a first step towards this aim.

Definition 1.1.1. A Weyl system over the test-function space τ is the structure defined by

(1) τ is a complex pre-Hilbert space with inner product $(f;g)$ for $f,g \in \tau$.

(2) For each $f \in \tau$ there is a unitary operator $W(f)$ which acts in a Hilbert space $\mathcal{H}$, called the representation space, and $\mathcal{H}$ is common to the set $\{W(f): f \in \tau\}$.

(3) The map $f \mapsto W(f)$ satisfies

$$W(f)W(g) = W(f+g)e^{i\mathrm{Im}(f;g)}.$$

(4) The map $s \mapsto W(sf)$ from $\mathbb{R}$ to the unitaries on $\mathcal{H}$ defines a one-parameter weakly continuous family of unitaries for each $f \in \tau$.

(5) There is a dense linear manifold D in $\mathcal{H}$ which is stable under the application of the generators of all the unitary groups $\{W(tf): t \in \mathbb{R}\}$ with $f \in \tau$. We require these generators to be essentially self-adjoint on D.

Remark 1.1.1. In the case of the particle with one degree of freedom we had two unitary operators $U(s)$ and $V(t)$. We have this situation in the infinite dimension case: decompose τ into a direct sum of two real linear subspaces $\tau = \tau_{\mathbb{R}} \oplus \tau_{\mathbb{R}}$. Each $f \in \tau$ can be written as $f = f_1 + if_2$ where $f_1, f_2 \in \tau_{\mathbb{R}}$. Then we have

$$U(f) = W(f) \qquad f \in \tau_{\mathbb{R}}$$

$$V(g) = W(ig) \qquad g \in \tau_{\mathbb{R}}$$

i is defined as a real-linear operator on $\tau_{\mathbb{R}}$ which becomes multiplication by $\sqrt{-1}$ in τ. In this situation, the 1-parameter group

$\{U(sf): f \in \tau_{\mathbb{R}}, s \in \mathbb{R}\}$ gives us the momentum and $\{V(tg): g \in \tau_{\mathbb{R}}, t \in \mathbb{R}\}$ gives us the field operator. So we write

$$U(sf) = e^{isP(f)}$$

and

$$V(tg) = e^{itQ(g)}$$

By applying our demands on W(f), W(ig), we obtain, as equations defined in the dense domain D,

$$[Q(f), Q(g)] = [P(f), P(g)] = 0$$

and

$$[Q(f), P(g)] = i(f;g)\,1 \qquad \text{for } f,g \in \tau_{\mathbb{R}}$$

We construct annihilation and creation operators, defined of course on D, as follows:

$$a^*(g) = \frac{1}{\sqrt{2}}\{Q(g) + iP(g)\}$$

and

$$a(g) = \frac{1}{\sqrt{2}}\{Q(g) - iP(g)\} \qquad \text{for all } g \in \tau_{\mathbb{R}}$$

These operators are necessary in the construction of the Fock representation. They also satisfy commutation properties on D:

$$[a^*(f), a^*(g)] = [a(f), a(g)] = 0$$

and

$$[a(f), a^*(g)] = (f;g)\,1 \qquad \text{for } f,g \in \tau_{\mathbb{R}}$$

In order to construct such a system we must erect a C*-algebra framework which gives us all our results.

Remark 1.1.2. Conditions (3) and (4) of Definition 1.1.1 are equivalent to condition (3) together with

(4') The map $f \longmapsto W(f)$ is weakly continuous on finite-dimensional subspaces of τ.

2. C*-algebras, States and Representations

We recall that a C*-algebra is a normed algebra $\mathfrak{A}$ over $\mathbb{R}$ or $\mathbb{C}$ with a map $*: \mathfrak{A} \to \mathfrak{A}$. Further * satisfies

$$||A^*|| = ||A||$$

and $(A^*)^* = A$ for all $A \in \mathfrak{A}$. The algebra $\mathfrak{A}$ is complete in $||\cdot||$ and the C*-condition holds, namely the condition $||A^*A|| = ||A||$ for each $A \in \mathfrak{A}$.

A state on $\mathfrak{A}$ is a linear functional $\phi: \mathfrak{A} \to \mathbb{C}$ (we suppose our field of scalars to be $\mathbb{C}$) such that $\langle\phi; A^*A\rangle \geqq 0$ for each $A \in \mathfrak{A}$. The symbol $\langle\phi; A\rangle$ is the evaluation of ϕ at the element $A \in \mathfrak{A}$. The set of states on $\mathfrak{A}$ is written $S(\mathfrak{A})$.

A representation of a C*-algebra $\mathfrak{A}$ is a pair $(\pi, \mathcal{H}_\pi)$ where π is the map from $\mathfrak{A}$ into the bounded operators on the Hilbert space $\mathcal{H}_\pi$. The map π obeys the exigencies

$$\pi(\lambda A+\mu B) = \lambda\pi(A) + \mu\pi(B)$$

for $\lambda, \mu \in \mathbb{C}$ and for $A, B \in \mathfrak{A}$. Moreover, $\pi(A^*) = \pi(A)^*$. Here we make no notational distinction between the *-map on $\mathfrak{A}$ and the adjoint map on the algebra of operators on $\mathcal{H}_\pi$.

The representation π of $\mathfrak{A}$ on $\mathcal{H}_\pi$ is said to be cyclic in $\mathcal{H}_\pi$ if there is at least one vector $\Omega \in \mathcal{H}_\pi$ such that the set $\{\pi(A)\Omega: A \in \mathfrak{A}\}$ is total in $\mathcal{H}_\pi$ i.e. finite linear combinations of elements of this set form a dense set in $\mathcal{H}_\pi$. Cyclic representations and states of $\mathfrak{A}$ go together. This is the result of the Gelfand-Naimark-Segal Theorem:

Theorem 1.2.1 (Gelfand-Naimark-Segal)

Let $\mathfrak{A}$ be a complex C*-algebra and let ϕ be a state on $\mathfrak{A}$. Then there exists a Hilbert space $\mathcal{H}$, a representation π and a vector $\Omega \in \mathcal{H}$ so that Ω is cyclic for π and

$$\langle\phi; A\rangle = (\Omega, \pi(A)\Omega) \qquad \text{for any } A \in \mathfrak{A}.$$

Further, if π_1 is another representation of $\mathcal{A}$, on a Hilbert space $\mathcal{H}_1$ with cyclic vector Ω_1 such that

$$<\phi;\ A> = (\Omega_1,\ \pi_1(A)\Omega_1) \qquad \text{for any } A \in \mathcal{A},$$

then there exists a unitary map $V\colon \mathcal{H} \to \mathcal{H}_1$ such that

$$\Omega_1 = V\Omega \quad \text{and} \quad \pi_1(A) = V\pi(A)V^{-1}$$

We now proceed to construct a C*-algebra which we will represent on a Hilbert space to give a Weyl system.

3. The Algebra of the C.C.R.

We choose a complex pre-Hilbert space τ. For each element $f \in \tau$ we define a symbol $W(f)$. A map * is defined on the set of these symbols by the formula $W(f)^* = W(-f)$. This collection of symbols, with the map *, is denoted $\Delta_0(\tau)$. Next we define $\Delta(\tau)$ as the collection of all finite (complex) linear combinations of elements of $\Delta_0(\tau)$ i.e. if $R \in \Delta(\tau)$, there is a number $n \geqq 1$, a collection $\{f_1,\ldots,f_n\} \subset \Delta_0(\tau)$ with

$$R = \sum_{k=1}^{n} z_k W(f_k)$$

where $\{z_1,\ldots,z_n\} \subset C$. $\Delta(\tau)$ is a linear space. We make $\Delta(\tau)$ into an algebra by imposing the multiplication law

$$W(f)W(g) = e^{i\mathrm{Im}(f;g)}.W(f+g)$$

on $\Delta_0(\tau)$ and extending it to $\Delta(\tau)$. Together with the * map, $\Delta(\tau)$ becomes a *-algebra. We must choose a norm on $\Delta(\tau)$ to make it into a C*-algebra. To this end, we prove the following technical Lemma.

Proposition 1.3.1. $\Delta(\tau)$ is a simple algebra i.e. if I is any linear space in $\Delta(\tau)$ satisfying $R.I \subset I$ for $R \in \Delta(\tau)$ and $IR \subset I$ then either $I = \{0\}$ or $I = \Delta(\tau)$.

Proof

First we show that $I \cap \Delta_0(\tau) = \phi$. Assume that $I \neq \Delta(\tau)$ and

$W(f) \in I$ for some $f \in \tau$. Then $W(-f)W(f) = 1 \in I$ where 1 is the unit in $\Delta(\tau)$. From this it follows that $I = \Delta(\tau)$. This contradicts the assumption that $I \neq \Delta(\tau)$, so we conclude that $I \cap \Delta_0(\tau) = \phi$.

Now define the quotient space $\Delta(\tau)/I$. Denote by p the quotient map. Because of the above discussion, it follows that $p(\Delta_0(\tau))$ is an isomorphic copy of $\Delta_0(\tau)$, in $\Delta(\tau)/I$. Now $\Delta(\tau)/I$ is another linear space and $\Delta(\tau)$ is the smallest complex linear space which contains $\Delta_0(\tau)$, so $\Delta(\tau)/I$ must be isomorphic to a space which contains $\Delta(\tau)$. From this it follows that $\Delta(\tau)/I$ is isomorphic to $\Delta(\tau)$. So p is invertible and hence $I = \{0\}$. This then establishes the result.

Corollary. Every non-zero representation π of $\Delta(\tau)$ on a Hilbert space $\mathcal{H}_\pi$, is faithful i.e. $\ker\pi = \{0\}$.

Proof

$\ker\pi = \{R \in \Delta(\tau) : \pi(R) = 0\}$ satisfies the conditions of I in the above proposition. Hence, as π is non-zero, $I = \{0\}$.

We consider certain types of representations of $\Delta(\tau)$.

Definition 1.3.1. (1) A non-degenerate representation of $\Delta(\tau)$ is a map π into the bounded operators of the Hilbert space $\mathcal{H}_\pi$ such that the set $\{\pi(R)\Psi : R \in \Delta(\tau), \Psi \in \mathcal{H}_\pi\}$ is total in $\mathcal{H}_\pi$.

(2) $P(\tau)$ is the collection of all non-degenerate representations π of $\Delta(\tau)$ such that for any $\Psi, \Phi \in \mathcal{H}_\pi$ and for any $f \in \tau$ the map from $\mathbb{R}$ to $\mathbb{C}$ defined by

$$\lambda \to (\Psi, \pi[W(\lambda f)]\Phi)$$

is continuous in λ.

Proposition 1.3.2. $\Delta(\tau)$ completed in the norm

$$||R|| := \sup_{\pi \in P(\tau)} ||\pi(R)||_\pi$$

becomes a C*-algebra. $\overline{\Delta(\tau)}$ is the C*-algebra obtained.

Proof. We merely check the C*-condition: we have

$$\|R^*R\| = \sup_{\pi\in P(\tau)} \|\tau(R^*R)\|_\pi = \sup_{\pi\in P(\tau)} \|\pi(R)\|_\pi^2 = \|R\|^2$$

Since the condition holds for bounded operators on a Hilbert space. From the simplicity of $\Delta(\tau)$, it follows that $\|R\| = 0 \Leftrightarrow R = 0$. The rest is quite easy to prove.

Remark. The C*-algebra $\overline{\Delta(\tau)}$ is in fact a C*-inductive limit.

We call the algebra $\overline{\Delta(\tau)}$ the C*-algebra of the Canonical Commutation Relations. It is interesting to note that $\overline{\Delta(\tau)}$ is also simple: this follows from the simplicity of $\Delta(\tau)$. Further, any representation $\pi \in P(\tau)$ is automatically a continuous map, as $\|\pi(R)\|_\pi \leq \|R\|$. Each $\pi \in P(\tau)$ is also faithful, so that the norm defined on $\Delta(\tau)$ satisfied the condition $\|R-T\| = 0 \Leftrightarrow R = T$.

4. Representation of the C.C.R.

Definition 1.4.1. For each $\pi \in P(\tau)$ we define the set

$$W_\pi(\tau) = \{W_\pi(f) := \pi(W[f]) \text{ and } f \in \tau\}.$$

Remark 1.4.1. For each representation $\pi \in P(\tau)$, $W_\pi(\tau)$ is a Weyl system over τ, acting on $\mathcal{H}_\pi$.

Lemma 1.4.1. π is a cyclic representation of $\overline{\Delta(\tau)}$ if and only if $W_\pi(\tau)$ is cyclic.

Proof. If π is cyclic in $\mathcal{H}_\pi$ then there is a vector $\Omega \in \mathcal{H}_\pi$ such that $(\Psi, \pi(R)\Omega) = 0$ for all $R \in \overline{\Delta(\tau)}$ implies that $\Psi = 0$. Because $\Delta(\tau)$ is dense in $\overline{\Delta(\tau)}$, the same implication holds if $(\Psi, \pi(R)\Omega) = 0$ for all $R \in \Delta(\tau)$. Hence if $(\Psi, W_\pi(f)\Omega) = 0$ for all $f \in \tau$, by linearity of π, we have $(\Psi, \pi(R)\Omega) = 0$ for $R \in \Delta(\tau)$. Therefore $\Psi = 0$. So if π is cyclic in $\mathcal{H}_\pi$, so is W_π. The converse holds, as can be seen, using the same sort of analysis.

This Lemma is used in the proof of the next Theorem. It says that if π is got from the G.N.S. construction, then W_π is automatically cyclic.

Theorem 1.4.1

Suppose $\hat{\phi}$ is a function from τ to $\mathbb{C}$ which satisfies the conditions

(1) $\hat{\phi}(0) = 1$

(2) For each fixed $f \in \tau$, the function $\lambda \to \phi(\lambda f)$ is a continuous function of $\lambda \in \mathbb{R}$.

(3) For each finite sequence of pairs $\{(z_k, f_k) \in \mathbb{C} \times \tau, k = 1,\ldots,n\}$ we have

$$\sum_{j,k=1}^{n} \bar{z}_k z_j \hat{\phi}(f_j - f_k).\exp -\left[\frac{i}{2}\mathrm{Im}(f_k;\ f_j)\right] \geqq 0$$

i.e. $\hat{\phi}$ is a positive definite function.

Then $\hat{\phi}$ determines a Weyl system $W_{\hat{\phi}}(\tau)$ acting on a Hilbert space $\mathcal{H}_{\hat{\phi}}$. This system is cyclic and it is unique up to unitary equivalence. Therefore, if $\hat{\psi}$ is another function which satisfies the above, and $\hat{\psi}(f) = \hat{\phi}(f)$ for each $f \in \tau$, then there exists a unitary operator $M: \mathcal{H}_{\hat{\phi}} \to \mathcal{H}_{\hat{\psi}}$ such that

$$MW_{\hat{\phi}}(f) = W_{\hat{\psi}}(f)M \qquad \text{for each } f \in \tau,$$

and $\Omega_{\hat{\psi}} = M\Omega_{\hat{\phi}}$ where $\Omega_{\hat{\psi}}$ and $\Omega_{\hat{\phi}}$ are the cyclic vectors in the two representations.

Remark

Lemma 1.4.1 says that the representation of $\overline{\Delta(\tau)}$ obtained by the GNS construction from $\hat{\phi}$, forces $W_{\hat{\phi}}$ to be cyclic as $W_{\hat{\phi}}(f) = \pi_{\hat{\phi}}[W(f)]$, $\pi_{\hat{\phi}}$ being the representation of $\overline{\Delta(\tau)}$ got from $\hat{\phi}$. This lemma also implies the converse of this theorem.

A function $\hat{\phi}$ such as the one in Theorem 1.4.1, is called an expectation functional and it determines, in the language of Segal, a regular state ϕ on $\overline{\Delta(\tau)}$. An expectation functional on a complex pre-Hilbert space τ determines a cyclic Weyl system over τ. The converse holds, and it is not difficult to see that

$$f \longmapsto (\Omega_{\hat{\phi}},\ W_{\hat{\phi}}(f)\Omega_{\hat{\phi}})$$

determines an expectation functional for each cyclic Weyl system. Thus $(\Omega_{\hat{\phi}},\ W_{\hat{\phi}}(f)\Omega_{\hat{\phi}})$ is typical of the system and determines it uniquely, up to a unitary isomorphism.

5. The Fock Representation

We define the Fock representation as that cyclic system W_0, with Hilbert space $\mathcal{H}_0$ and cyclic vector Ω, such that $a(g)\Omega = 0$ for all $g \in \tau_{\mathbb{R}}$. $\tau_{\mathbb{R}}$ is the real space whose complexification is τ, and $a(g)$ is the annihilation operator defined in §1. Given this condition, it can be shown that the expectation functional is

$$(\Omega,\ W_0(f)\Omega) = \exp\left[-\frac{1}{4}\|f\|^2\right] \quad \text{for any } f \in \tau.$$

Hence the Fock representation exists and is (up to unitary isomorphism) unique. We now prove a well-known property of the Fock representation. This property is necessary in the ensuing discussion.

Proposition 1.5.1. The map $f \longmapsto W_0(f)$ is strongly continuous in the Fock representation. Namely, for each vector $\Psi \in \mathcal{H}_0$

$$\|W_0(f)\Psi - W_0(h)\Psi\|_{\mathcal{H}_0} \to 0 \quad \text{as} \quad \|f-h\|_\tau \to 0.$$

Proof

Consider the vectors of the form $W_0(g)\Omega$. Then

$$(W_0[h]\Omega,\ W_0[f]W_0[g]\Omega) = (\Omega,\ W_0[-h]W_0[f]\ W_0[g]\Omega) =$$

$$\exp\frac{i}{2}\{\mathrm{Im}(f;g) + \mathrm{Im}(-h;f+g)\}.(\Omega,\ W_0[f+g-h]\Omega) =$$

$$\exp\frac{i}{2}\{\mathrm{Im}(f;g) + \mathrm{Im}(-h;f+g)\}.\exp\left[-\frac{1}{4}\|f+g-h\|^2\right]$$

and so, as $f \to 0$ in τ (in the norm), the last expression becomes

$$\exp\frac{i}{2}\{\mathrm{Im}(-h;g)\}\exp\left[-\frac{1}{4}\|g-h\|^2\right] = (W_0[h]\Omega,\ W_0[g]\Omega)$$

Therefore we have established that

$$(W_0[h]\Omega,\ W_0[f]W_0[g]\Omega) \to (W_0[h]\Omega,\ W_0[g]\Omega)$$

as $f \to 0$ in τ.

Using the fact that the collection of all finite sums of the form $\sum_{j=1}^{n} z_j W_0[g_j]\Omega$ is a dense set in $\mathcal{H}_0$, it follows quite easily that $||W_0[f]\Psi-\Psi|| \to 0$ as $f \to 0$ in τ, where $\Psi = \sum_{j=1}^{n} z_j W_0[g_j]\Omega$. Then using the fact that any $\Psi \in \mathcal{H}_0$ can be approximated by elements of the form $\sum_{j=1}^{n} z_j W_0(f_j)\Omega$, and that the map $f \longmapsto W_0(f)$ is uniformly bounded, as the $W_0(f)$ are unitary, it follows that $||W_0(f)\Psi - \Psi|| \to 0$ as $f \to 0$ in τ for any $\Psi \in \mathcal{H}_0$.

Now if $||f-h||_\tau \to 0$ then we have for any $\Psi \in \mathcal{H}_0$

$$||W_0(f)\Psi - W_0(h)\Psi|| = ||e^{\frac{i}{2}\mathrm{Im}(-h;f)} W_0(f-h)\Psi-\Psi||$$

$$\leq ||W_0(f-h)\Psi-\Psi|| + |e^{\frac{i}{2}\mathrm{Im}(-h;f)} -1|.||\Psi||$$

and the last two terms tend to zero as $f \to h$ in τ. This then establishes the result.

6. The Displaced Fock Representation and Manuceau's Lemma

We now define displaced Fock representations.

Definition 1.6.1. Let $W_0(\tau)$ be the Fock representation over τ, on the Hilbert space $\mathcal{H}_0$, and suppose F is a linear functional $F: \tau \to \mathbb{C}$. Then we say that $W_F(\tau)$ defined through

$$W_F(f) = e^{i\mathrm{Im}F(f)} W_0(f)$$

is a displaced Fock representation of the C.C.R. over τ.

Remark. It is clear that $f \longmapsto W_F(f)$ is strongly continuous if and only if $f \longmapsto \mathrm{Im}F(f)$ is continuous. Further, we have that $W_F(f)W_F(g) = e^{\frac{i}{2}\mathrm{Im}(f;g)} W_F(f+g)$. Further we have that $W_F(f)W_F(g) = e^{\frac{i}{2}\mathrm{Im}(f;g)} W_F(f+g)$. The expectation functional is $e^{i\mathrm{Im}F(f)} e^{-\frac{1}{4}||f||^2}$.

We know that the Fock representation is unique up to isomorphism and are entitled to ask about the conditions of uniqueness of any given displaced Fock representation. To this end, we are able to give a very useful result, known as Manuceau's Lemma. We present the proof, which is due to Roepstroff [17].

Theorem 1.6.1. (Manuceau's Lemma)

Suppose F_1 and F_2 are elements of $\tau^\times$, the set of complex-valued linear functionals on τ. Then there is a unitary operator $M: \mathcal{H}_0 \to \mathcal{H}_0$ such that:

$$MW_{F_1}(f) = W_{F_2}(f)M$$

if and only if there exists a constant $C \to 0$ so that

$$|(F_1-F_2)[f]| \leqq C.\|f\|$$

i.e. if F_1-F_2 is a continuous linear functional, so that $F_1-F_2 \in \tau^*$, the topological dual of τ.

Proof

Assume that such a unitary operator exists. Then we obtain the equation

$$(MW_0[f],\ W_0[f]M\Omega) = \exp i\,[\mathrm{Im}F_1(f)-\mathrm{Im}F_2(f)]$$

Now, we know from Proposition 1.5.1 that the left-hand side is continuous in f, so the right-hand side must also be continuous in f (in the norm topology on τ). Hence $\mathrm{Im}(F_1-F_2)(f)$ defines a continuous linear functional on τ. Whence we deduce that $(F_1-F_2)(f)$ defines a continuous linear functional on τ i.e. there is a constant $C > 0$ such that $|(F_1-F_2)(f)| \leqq C.\|f\|$.

Now we assume this to be true and prove the existence of a unitary operator $M: \mathcal{H}_0 \to \mathcal{H}_0$, satisfying the intertwining condition. It is clear that F_1-F_2 extends to a continuous linear functional on the Hilbert space completion of τ. We denote this Hilbert space by $\mathcal{K}$ and the extension of $F_1 - F_2$ by F. Then the celebrated Riesz Theorem implies the existence of a vector $\xi \in \mathcal{K}$ such that

$$F(f) = (\xi; f) \qquad \text{for } f \in \mathcal{K}.$$

Moreover, $(F_1 - F_2)(f) = (\xi; f)$ for $f \in \tau$. Now write $\Phi = W_0(\xi)\Omega$ where Ω is the Fock cyclic vector. We have the following:

$$(\Phi, W_{F_2}[f]\Phi) = (\Omega, W_0(-\xi)W_{F_2}[f]W_0(\xi)\Omega) = (\Omega, W_{F_1}[f]\Omega)$$

We can define $W_0(\xi)$ for $\xi \in \mathcal{K}$ as $f \longmapsto W_0(f)$ is strongly continuous in f, with respect to the norm in τ, and so we can extend $W_0(\xi)$ to $\xi \in \mathcal{K}$.

It follows from this that W_{F_1} and W_{F_2} are unitarily equivalent. Indeed, we have

$$W_0(\xi)W_{F_1}(f) = W_{F_2}(f)W_0(\xi)$$

for any $f \in \tau$. This proves the theorem.

We call any collection of displaced Fock representations, which are pairwise equivalent in the sense that there is a unitary operator which satisfies the condition given, a <u>sector</u>. Clearly, the theorem implies that W_F is equivalent to the Fock representation if and only if F is a continuous linear functional on τ. For this reason we call the Fock representation the zero sector.

7. Symmetry Groups and Implementability

In the relativistic theory of boson systems, we represent the Poincaré group on the space of wave-functions by a unitary representation. The present setting has τ playing the role of the wave functions. We allow a generalization in the choice of group: we take any connected Lie group G. Further, we ask that G be represented by unitary operators on τ so that for each $g \in G$ there is a unitary U_g with

$$U_g : \tau \to \tau \qquad \text{for each } g \in G$$

$$(U_g f; U_g h) = (f;h) \qquad \text{for all } f,h \in \tau$$

and

$$U_g U_k = U_{gk} \qquad \text{for any } g,k \in G$$

We also demand that $g \mapsto U_g f$ be strongly continuous at the identity. Namely, given an open set $\mathcal{O}$ containing the identity of G, there is an $\varepsilon > 0$ so that for all $g \in \mathcal{O}$, the vectors $U_g f$ belong to the set $N(f, \varepsilon) = \{h: h \in \tau \text{ and } \|f-h\| < \varepsilon\}$. This is equivalent to demanding the condition $\|U_g f - U_k f\| \to 0$ as $g \to k$ in G. This is just strong continuity anywhere in G.

Now we consider the group G acting on τ and how this is reflected in the Fock representation.

Theorem 1.7.1. Let U be a strongly continuous unitary representation of the connected Lie group G on the complex pre-Hilbert space τ. Then, in the Fock representation of the C.C.R., there exists a strongly continuous unitary representation V of G in $\mathcal{H}_0$ such that for each $f \in \tau$ and for each $g \in G$

$$W_0(U_g f) = V_g W_0(f) V_g^{-1}$$

and $V_g \Omega = \Omega$ where Ω is the Fock vacuum vector.

Proof

Define $\hat{\phi}_g(f) := (\Omega, W_0(U_g f)\Omega)$. Then $\hat{\phi}_g(f) = \exp(-\frac{1}{4}\|U_g f\|^2) = \exp(-\frac{1}{4}\|f\|^2)$. Hence $\hat{\phi}_g$, for each $g \in G$, agrees with the expectation functional of the Fock representation. Theorem 1.4.1 then assures of the existence of a unitary operator $V_g: \mathcal{H}_0 \to \mathcal{H}_0$ satisfying

$$W_0(U_g f) = V_g W_0(f) V_g^{-1} \text{ and } V_g \Omega = \Omega$$

Strong continuity must be established separately. We consider only the identity of G and strong continuity there. Proposition 1.5.1 assures us that $f \mapsto W_0(f)$ is strongly continuous from τ (in the norm topology) to $W_0(f)$. We then have, on the total set defined by $\{W_0(f)\Omega: f \in \tau\}$, the following calculation

$$||V_g W_0[f]\Omega - W_0[f]\Omega|| = ||V_g W_0[f]V_g^{-1}\Omega - W_0[f]\Omega|| = ||W_0[U_g f]\Omega - W_0[f]\Omega||$$

and this last term tends to zero as $g \overset{G}{\to} e$, by strong continuity of U_g in τ and strong continuity of the map $f \longmapsto W_0(f)$. This establishes strong continuity of V_g on a total set. If $\Psi = \sum_{j=1}^{n} z_j W_0(f_j)\Omega$ then

$$||V_g\Psi - \Psi|| \leqq \sum_{j=1}^{n} |z_j| . ||V_g W_0[f_j]\Omega - W_0[f_j]\Omega|| \to 0 \text{ as } g \overset{G}{\to} e.$$

Hence, as the set of such vectors Ψ is dense in $\mathcal{H}_0$, V_g is strongly continuous on a dense set. Since the unitary operators V_g are uniformly bounded by 1, and since any $\Psi \in \mathcal{H}_0$ is the limit of a sequence of elements $\{\Psi_n\}$ in the dense set, it follows that V_g is strongly continuous on the whole of $\mathcal{H}_0$. This proves the theorem.

This is a very useful property for a representation to have, so we devote a definition to it.

<u>Definition 1.7.1.</u> A representation π of the C.C.R. over τ on a Hilbert space $\mathcal{H}_\pi$ is said <u>to implement the group G</u> (and G is said to be <u>implementable</u> in π) if there is a unitary operator V, representing G in $\mathcal{H}_\pi$, such that

$$\pi[\sigma_g(A)] = V_g \pi(A) V_g^{-1}$$

where A is an element of the C.C.R. algebra and σ_g is the automorphism of the algebra defined on the unitary elements as

$$\sigma_g(W[f]) = W(U_g f) \qquad \text{for } f \in \tau.$$

We say, further, that π is <u>a G-covariant representation</u> if G is implementable in π and the implementing representation V is strongly continuous in g.

In the new language, Theorem 1.7.1 says that the Fock representation is G-covariant. Let us then see what the condition is for any displaced Fock representation to implement G. We have the calculation

$$W_F(U_g f) = e^{i\text{Im}F(U_g f)} W_0(U_g f) = e^{i\text{Im}F(U_g f)} V_g W_0(f) V_g^{-1}$$

Therefore $g \longmapsto W_F(U_g f)$ is implemented if and only if the map

$$g \longmapsto e^{i\mathrm{Im}F(U_g f)} W_0(f)$$

is implemented. So for each $g \in G$ we seek a unitary operator, T_g say, so that $T_g: \mathcal{H}_0 \to \mathcal{H}_0$ and

$$e^{i\mathrm{Im}F(U_g f)} W_0(f) = T_g e^{i\mathrm{Im}F(f)} W_0(f) T_g^{-1}$$

Now, both $f \longmapsto e^{i\mathrm{Im}F(f)} W_0(f)$ and $f \longmapsto e^{i\mathrm{Im}F(U_g f)} W_0(f)$, for each $g \in G$, are displaced Fock representations. We can apply Manuceau's Lemma (Theorem 1.6.1) and we then obtain the result that T_g exists for each $g \in G$, if and only if

$$f \longmapsto F(f) - F(U_g f)$$

defines a continuous linear functional on τ, for each $g \in G$.

Let us now denote, as in Theorem 1.6.1, by $\tau^\times$ the set of all linear functionals on τ. The subset of $\tau^\times$, consisting of the continuous linear functionals, is of course defined by the space $\mathcal{K}$, the Hilbert space completion of τ in thenorm of τ. This is a corollary of Riesz's Theorem. Also, we denote by $U_g^\times$ the dual of U_g, acting on $\tau^\times$ as

$$(U_g^\times F)(f) := F(U_g f) \quad \text{for } f \in \tau,\ g \in G,\ F \in \tau^\times.$$

Now it is easy to see that $U_g^\times$ does not define a representation of G since we have the calculation

$$(U_g^\times U_k^\times F)(f) = (U_k^\times F)(U_g f) = F(U_k U_g f) = F(U_{kg} f) = (U_{kg}^\times F)(f)$$

for $k, g \in G$ and for all $f \in \tau$, i.e. $U_g^\times U_k^\times = U_{kg}^\times$ on $\tau^\times$.

We remedy this by defining a representation M, of G, on $\tau^\times$ by

$$M_g F = U_{g^{-1}}^\times F$$

Then we have

$$(M_g M_k F)(f) = (U^{\times}_{g^{-1}} U^{\times}_{k^{-1}} F)(f) = F(U_{k^{-1}} U_{g^{-1}} f)$$

$$= F(U_{[gk]^{-1}} f) = (M_{gk} F)(f)$$

for all $g, k \in G$, $f \in \tau$ and $F \in \tau^{\times}$. Taking all this into account, our result is that $F \in \tau^{\times}$ defines a displaced Fock representation W_F in which G is unitarily implemented if and only if

$$F - M_g F \in \mathcal{K} \qquad \text{for each } g \in G.$$

Moreover, if $F \in \mathcal{K}$ then M_g coincides with U_g and so M can be regarded as an extension of U to $\tau^{\times}$.

8. The Cocycle Condition and Classification

Define a function ψ_F: $G \to \mathcal{K}$ by the formula

$$\psi_F(g) = F - M_g F \qquad \text{where } F \in \tau^{\times}.$$

Then we have $M_g \psi_F(k) = \psi_F(gk) - \psi_F(g)$. This is called the 1-cocycle condition. The result of the last section is that a G-covariant displaced Fock representation gives rise to a 1-cocycle of G with values in $\mathcal{K}$ in the representation M of G. Now we ask whether any 1-cocyle gives a G-covariant displaced Fock representation. First, however, a word about 1-cocycles.

Given any group G and a Hilbert space K and a map ψ: $G \to K$ satisfying

$$M_g \psi(k) = \psi(gk) - \psi(g)$$

it is not necessarily true that ψ is of the form $F - M_g F$ where F lies either in $\mathcal{K}$, or just outside of $\mathcal{K}$. Thus the 1-cocylces ψ_F given above are particular examples of 1-cocycles of G with values in $\mathcal{K}$.

We say that ψ_F is a trivial 1-cocyle or a 1-coboundary if F lies in $\mathcal{K}$. Two cocycles ψ_{F_1} and ψ_{F_2} are said to be cohomologous if and $\psi_{F_1} - \psi_{F_2}$ is a 1-coboundary.

Now let us suppose that W_{F_1} and W_{F_2} are equivalent displaced Fock representations. Then we know that $F_1 - F_2 \in \mathfrak{K}$. This follows from Theorem 1.6.1. Then we have

$$\psi_{F_1}(g) - \psi_{F_2}(g) = F_1 - F_2 - M_g(F_1 - F_2)$$

Since M_g restricted to $\mathfrak{K}$ is equal to $U_{g^{-1}}$ and since $U_g \mathfrak{K} = \mathfrak{K}$ for each $g \in G$, then we have that ψ_{F_1} and ψ_{F_2} are cohomologous.

Next, suppose ψ_{F_1} and ψ_{F_2} differ by a coboundary. Further, assume that all G-invariant functionals on τ, vanish on τ. Namely if $M_g F = F$ then $F(f) = 0$ for every $f \in \tau$. Then we know, from the cohomology of ψ_{F_1} with ψ_{F_2} that there exists a vector $\xi \in \mathfrak{K}$ with

$$\psi_{F_1}(g) - \psi_{F_2}(g) = \xi - M_g \xi$$

It follows from the form of ψ_{F_1} and ψ_{F_2} that

$$F_1 - F_2 - M_g(F_1 - F_2) = \xi - M_g \xi$$

from which we deduce that

$$F_1 - F_2 - \xi = M_g[F_1 - F_2 - \xi]$$

By assumption, all G-invariant functionals on τ, vanish on τ, so

$$F_1 - F_2 - \xi = 0$$

Therefore $F_1 - F_2 = \xi \in \mathfrak{K}$. Hence, ψ_{F_1} and ψ_{F_2} being cohomologous implies that W_{F_1} and W_{F_2} must be unitarily equivalent displaced Fock representations.

All this argument can then be summed up in the following Classification Theorem.

<u>Theorem 1.8.1.</u> If all G-invariant functionals vanish on τ, then there is a one-to-one correspondence between the equivalence classes (sectors) of displaced Fock representations in which G is implemented, and 1-cocycles of G, with values in $\mathfrak{K}$, of the form

$$\psi_F(g) = F - M_g F \qquad \text{where } F \in \tau^{\times} \text{ and } g \in G.$$

9. The Group Representation in the Displaced Fock Sectors

The discussions of §7 and §8 have given us a tool with which we can "enumerate" all the G-covariant displaced Fock representations. We shall exploit this in later sections for various examples of G. However, we continue here a discussion begun, but not completed, in §7. This is the implementability problem in displaced Fock sectors. We arrived at

$$W_F(U_g f) = V_g e^{i\mathrm{Im}F(U_g f)} W_0(f) V_g^{-1}$$

and we seek an operator T_g, say, so that

$$W_F(U_g f) = V_g T_g W_F(f) T_g^{-1} V_g^{-1}$$

and T_g is to be unitary on $\mathcal{H}_0$. Then we have the following formula for T_g:

$$T_g = W_0[-\psi_F(g^{-1})]$$

Now $\psi_F(g)$, for $g \in G$, lies in $\mathcal{K}$ in general, not in τ. However, in view of the strong continuity of $f \mapsto W_0(f)$, for the Fock sector, we can give a good meaning for $W_0(f)$ if $f \in \mathcal{K}$. Therefore we have

$$V_g W_0[-\psi_F(g^{-1})]\, e^{i\mathrm{Im}F(f)} W_0(f) W_0[\psi_F(g^{-1})] V_g^{-1}$$

$$= V_g e^{i\mathrm{Im}F(f)} e^{i\mathrm{Im}(-\psi_F(g^{-1});f)} e^{\frac{i}{2}\mathrm{Im}(-\psi_F(g^{-1})+f;\psi_F(g^{-1}))} W_0(f) V_g^{-1}$$

$$= V_g e^{i\mathrm{Im}F(f)} e^{i\mathrm{Im}(-\psi_F(g^{-1});f)} W_0(f) V_g^{-1}$$

$$= V_g e^{i\mathrm{Im}F(f)} e^{-i\mathrm{Im}F(f)} e^{i\mathrm{Im}(M_{g^{-1}}F)(f)} W_0(f) V_g^{-1}$$

$$= V_g e^{i\mathrm{Im}F(U_g f)} W_0(f) V_g^{-1} = e^{i\mathrm{Im}F(U_g f)} W_0(U_g f) = W_F(U_g f)$$

Therefore the map $g \mapsto W_F(U_g f)$ is implemented by the unitary operator

$$V_g W_0(-\psi_F(g^{-1}))$$

This, however, does not, in general, give an ordinary group representation: it defines a projective representation of G. To see this we complete as follows:

$$V_g W_0[-\psi_F(g^{-1})] V_k W_0[-\psi_F(k^{-1})]$$

$$= V_g V_k V_{k^{-1}} W_0[-\psi_F(g^{-1})] V_k W_0[-\psi_F(k^{-1})]$$

$$= V_g V_k W_0[-M_{k^{-1}}\psi_F(g^{-1})] W_0[-\psi_F(k^{-1})]$$

$$= V_{gk} W_0[-M_{k^{-1}}\psi_F(g^{-1}) - \psi_F(k^{-1})] e^{\frac{i}{2}\mathrm{Im}(M_{k^{-1}}\psi_F(g^{-1});\psi_F(k^{-1}))}$$

$$= V_{gk} W_0[-\psi_F\ (gk)^{-1}]\ e^{\frac{i}{2}\mathrm{Im}(\psi_F(g^{-1});M_k\psi_F(k^{-1}))}$$

$$= V_{gk} W_0[-\psi_k([gk]^{-1})] e^{-\frac{i}{2}\mathrm{Im}(\psi_F(g^{-1});\psi_F(k))}$$

where we have used the fact that U_g is implemented in W_0 by V_g, and $U_g\xi = M_g\xi$ for each vector $\xi \in K$. Further, we have used the unitarity of M_g and the relation

$$M_k\psi_F(k^{-1}) = M_k(F - M_{k^{-1}}F) = M_k F - F = -\psi_F(k)$$

If we denote by V_g^F the operator $V_g W_0[-\psi_F(g^{-1})]$, then we have the following two results:

(1) $\quad V_g^F W_F(f) V_{g^{-1}}^F = W_F[U_g f] \quad$ for $f \in \tau$

and

(2) $\quad V_g^F$ is a projective unitary representation of G with multiplier

$$\omega(g, k) = e^{-\frac{i}{2}\mathrm{Im}(\psi_F[g^{-1}];\psi_F[k])}$$

From the form of V_g^F, we see that if $g \mapsto \psi_F(g)$ is continuous in g then

$\psi_F(g) \to \psi_F(k)$ in norm as $g \overset{G}{\to} k$, whence $V_g^F \to V_k^F$ strongly on $\mathcal{H}_0$ as V_g does so already. The converse of this also holds since if $V_g^F \to V_k^F$ strongly, we have that

$$(\Omega, V_g^F\Omega) = (\Omega, W_0[-\psi_F(g^{-1})]\Omega) = e^{-\frac{1}{4}\|\psi_F(g^{-1})\|^2}$$

so as $g \to e$, $\psi_F(g^{-1})$ must tend to 0 in norm. Moreover, we note that $V_g^F\Psi \to V_k^F\Psi$ for each $\Psi \in \mathcal{H}_0$ if and only if $V_{g^{-1}}^F V_k^F\Psi \to \Psi$ for each $\Psi \in \mathcal{H}_0$. Using this, we arrive at the conclusion that

$$\|\psi_F(g)-\psi_F(k)\| = \|M_kF-M_gF\| = \|\psi_F(g^{-1}k)\| \to 0$$

as $g \overset{G}{\to} k$. This now gives us the following theorem.

Theorem 1.9.1 (Implementability Theorem)

Suppose $\psi_F(g) = F-M_gF$ is a cocycle for the action M of G on $\tau^\times$, with values in $\mathcal{K}$, and suppose U is the action of G on τ, where M has been defined by

$$(M_gF)(f) = F(U_{g^{-1}}f)$$

for each $g \in G$, $f \in \tau$ and $F \in \tau^\times$. Further, suppose V_g is the implementing operator of G in the Fock representation, W_0, of the Canonical Commutation Relations. Define

$$V_g^F = V_gW_0[-\psi_F(g^{-1})]$$

Then

(1) V_g^F implements G in the displaced Fock representation $W_F(f) = e^{i\text{Im}F(f)}W_0(f)$ for each $g \in G$, for each $f \in \tau$,

(2) V_g^F defines a multiplier (or projective) representation of G with multiplier $\omega(g, k) = \exp\{-\frac{i}{2}\text{Im}(\psi_F(g^{-1}); \psi_F(k))\}$, and

(3) $g \mapsto V_g^F$ is strongly continuous on $\mathcal{H}_0$ if and only if $g \mapsto \psi_F(g)$ is norm continuous in $\mathcal{K}$.

This completes the initial discussion of group covariance in displaced Fock representations. The results presented so far generalize to arbitrary connected topological groups. However, Lie groups are of most importance to physics, so we have restricted our attention to those.

Theorem 1.9.1 part (2) gives us a projective representation of G. The projective representation appears because of the factor $W_0(-\psi_F(g^{-1}))$ in V_g^F, and we know that the map $f \longmapsto W_0(f)$ satisfies the multiplication law $W_0(f)W_0(g) = e^{\frac{1}{2}\mathrm{Im}(f;g)}W_0(f+g)$ which defines a projective unitary representation of τ (as a group) on $\mathcal{H}_0$. If G is abelian, it is well-known [2] that G has as many non-trivial equivalence classes of multipliers as there are skew-symmetric bilinear forms on the Lie algebra $\underset{\sim}{G}$. For other groups, the situation is in general less clear. If $G = P_+^{\uparrow}(3+1)$, the Poincaré group in 3+1 space-time dimensions, these are trivial. If $G = P_+^{\uparrow}(1+1)$, the Poincaré group in 1+1 space-time dimensions, these multipliers are not necessarily trivial.

We pursue, in the following chapters, the problem of identifying the cocycles and we take various examples for G.

Sections 8 and 9 are simple generalizations of work presented in [26]

CHAPTER 2

MATHEMATICAL RESULTS

§1. Preliminary Definitions

We have established, in Theorem 1.8.1, that we must look at certain 1-cocycles with values in a Hilbert space $\mathcal{K}$, the 1-particle Hilbert space. The theorems we now present are typical of Lie groups: we establish a relationship between the 1-cohomology of the group and the 1-cohomology of the Lie algebra.

Our setting is the Hilbert space $\mathcal{K}$ on which a connected Lie group, G, acts through a continuous unitary representation U.

We say that the function $\psi: G \to \mathcal{K}$ is a (continuous) 1-cocycle of G for the representation U if ψ is a continuous function on G, in the norm topology on $\mathcal{K}$, and if ψ satisfies the 1-cocycle law

$$\psi(gk) = U_g\psi(k) + \psi(g)$$

for all $g,k \in G$. Further, a 1-cocycle $\psi: G \to \mathcal{K}$ is said to be a 1-coboundary if there exists a vector $\xi \in \mathcal{K}$ such that for each $g \in G$

$$\psi(g) = U_g\xi - \xi.$$

We also say that ψ is a trivial cocycle if it is a 1-coboundary.

Two 1-cocycles, ψ_1 and ψ_2, are cohomologous if they differ by a 1-coboundary. Namely, if there exists a vector $\xi \in \mathcal{K}$ such that for each $g \in G$

$$\psi_1(g)-\psi_2(g) = U_g\xi - \xi$$

The set of 1-cocycles forms a group under the binary operation of pointwise addition, and the set of 1-coboundaries forms a subgroup of the 1-cocycles. $Z^1(G,\mathcal{K})$ denotes the set of 1-cocycles and $B^1(G,\mathcal{K})$ denotes the set of 1-coboundaries.

The first cohomology group of G with values in $\mathcal{K}$ is defined by the quotient

$$H^1(G,\mathcal{K}) = Z^1(G,\mathcal{K})/B^1(G,\mathcal{K})$$

This is well-defined since the relationship defined by cohomology of 1-cocycles is an equivalence relationship.

The Lie algebra of G will be denoted by $\underline{G}$. Each element X of $\underline{G}$ can be represented by a densely-defined self-adjoint operator, $\pi(X)$. Indeed, we have the following relation

$$U(e^{tX}) = e^{it\pi(X)}$$

The operator $\pi(X)$ is obtained from Stone's Theorem.

One can show (see [9]) that if $\{X_1,\ldots,X_n\}$ forms a basis of the Lie algebra $\underline{G}$ then the operators $\{\pi(X_1),\ldots,\pi(X_n)\}$ have a common dense, invariant domain on which they are defined. Furthermore, each vector in this domain is an analytic vector i.e. if $X_j \in \{X_1,\ldots,X_n\}$ and ξ is a vector in the invariant domain, then there exists a $t > 0$ such that

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} ||\pi(X_j)^n \xi||$$

converges. It follows, quite easily, that if $X \in \underline{G}$, then a vector which is analytic for all operators $\{\pi(X_1),\ldots,\pi(X_n)\}$, is also analytic for $\pi(X)$.

A vector $\xi \in \mathcal{K}$ is said to be an analytic vector for the representation U of G, if the function $g \longrightarrow U(g)\xi$ is (strongly) analytic at the identity of G. The set of analytic vectors for U is denoted by $\mathcal{K}_\omega$. One can show that $\mathcal{K}_\omega$ is dense in $\mathcal{K}$.

A function η: $\underline{G} \rightarrow \mathcal{K}$ is called a 1-cocycle of $\underline{G}$ for the representation π if η is linear, and η satisfies the relation

$$\eta([X,Y]) = \pi(X)\eta(Y) - \pi(Y)\eta(X)$$

with $X,Y \in \underline{G}$ and $[X,Y]$ the Lie bracket of X and Y. η is a 1-coboundary of $\underline{G}$ if there exists a vector $\xi \in \mathcal{K}$ such that for every $X \in \underline{G}$ we have $\eta(X) = \pi(X)\xi$.

Two cocycles are said to be cohomologous if their difference is a coboundary.

We use $Z^1(\underline{G},\mathcal{K})$ to denote the 1-cocycles of the Lie algebra $\underline{G}$, with values in $\mathcal{K}$, and we use $B^1(\underline{G},\mathcal{K})$ to denote the 1-coboundaries of the Lie algebra $\underline{G}$.

The 1-cohomology group of $\underline{G}$ with values in $\mathcal{K}$ is then defined as the quotient

$$H^1(\underline{G},\mathcal{K}) = Z^1(\underline{G},\mathcal{K})/B^1(\underline{G},\mathcal{K})$$

§2 The relation between $H^1(G,\mathcal{K})$ and $H^1(\underline{G},\mathcal{K}_\omega)$

It is usual in the theory of Lie groups to perform differentiation along 1-parameter groups in order to pass to the Lie algebra. This "infinitesimal" method is taken over into cohomology theory. However, to differentiate a continuous cocycle is not always proper - but this can be remedied by the result which follows. It says that each (continuous) cocycle is equivalent (i.e. cohomologous) to a cocycle which is analytic at the identity. First, a result which allows us to prove this.

Lemma 2.2.1 [9]. Suppose $p: G \to \mathbb{R}$ is a continuous function from a connected Lie group G to the real numbers. Further, suppose p satisfies the inequality

$$p(gk) \leqq p(g) + p(k) \qquad \text{for all } g,k \in G.$$

Then $p(g) \leqq c.\rho(g)+c$ for all $g \in G$. Here $\rho(g)$ is metric function defined on G by the formula

$$\rho(g) = \inf_{\gamma} \int d\mu(g)$$

and γ varies through the paths joining g to the identity, $d\mu(g)$ is the invariant measure on G. The constant c is the greatest lower bound of p on the unit ball of G, in the metric $\underset{\sim}{\rho}$.

Proof

Let $S_j = \{g \in G: \rho(g) \leqq j\}$ where j is a natural number. Since (G,ρ) is a metric space we know that $G = \bigcup_{j=1}^{\infty} S_j$.

Now, given $g \in S_{m+1} - S_m$, we can find a sequence $\{g_j \in S_j: j = 1,\ldots,m\}$ such that

$$\rho(g_{j+1}^{-1}.g_j) \leqq 1 \quad \text{and} \quad \rho(g_m^{-1}.g) \leqq 1$$

From this follows the calculation

$$\begin{aligned} p(g) &= p(g_1(g_1^{-1}g_2)\ldots(g_m^{-1}g)) \\ &\leqq p(g_1)+p(g_1^{-1}g_2)+\ldots+p(g_m^{-1}g) \\ &\leqq c(m+1) = cm+c \leqq c\rho(g)+c \end{aligned}$$

The last inequality follows from the fact that the element g lies in S_{m+1} and outside of S_m, so $\rho(g) \geqq m$ in this case. This proves the Lemma.

We are now ready to prove the next result.

Proposition 2.2.1. Suppose $\psi: G \to \mathcal{K}$ is a cocycle of G. Then there exists a cocycle $\psi': G \to \mathcal{K}$ such that $\psi'(g)$ is analytic at the identity and ψ' is cohomologous to ψ.

Proof

The function $f(g) = \exp(-[\rho(g)]^2)$ is analytic at the identity of G, and is of rapid decrease with respect to the metric $\rho(g)$. From this it follows that any integral of the type

$$\int f(k^{-1})\psi(k)\,d\mu(k)$$

exists because $||\psi(k)||$ is continuous in k, as a consequence of $\psi(k)$ being continuous in k. Further, we note that $||\psi(g)||$ satisfies the following inequality

$$||\psi(gk)|| = ||U_g\psi(k)-\psi(g)|| \leqq ||\psi(g)|| + ||\psi(k)||.$$

Therefore, the function $||\psi(\cdot)||$ satisfies the conditions of Lemma 2.2.1. Hence $||\psi(g)|| \leqq c.\rho(g)+c$ for each $g \in G$, and it follows now that the integrals exist.

Now define the function ψ' on G by the equation

$$\psi'(g) = \int_G f(k^{-1}g)\psi(k)d\mu(k) - \int_G f(k^{-1})\psi(k)d\mu(k)$$

The measure μ defined on G is the left-invariant measure which normalises f i.e. we have

$$\int f(k)d\mu(k) = 1.$$

It now follows, using left invariance of μ and mapping k to gk, that

$$\psi'(g) = \int_G f(k^{-1})\psi(gk)d\mu(k) - \int_G f(k^{-1})\psi(k)d\mu(k)$$

$$= \int_G f(k^{-1})[U_g\psi(k)+\psi(g)]d\mu(k) - \int_G f(k^{-1})\psi(k)d\mu(k)$$

$$= \psi(g) + (U_g - 1)\int_G f(k^{-1})\psi(k)d\mu(k)$$

We have used the cocycle law and the condition that $\int f(k)d\mu(k) = 1$.

Now the expression $\int_G f(k^{-1})\psi(g)d\mu(k)$ is a vector in $\mathcal{K}$, so we conclude that $\psi'(g) - \psi(g)$ is a coboundary, which proves that ψ' is cohomologous to ψ.

To show that $\psi'(g)$ is analytic at the identity, we note that f(g) is analytic at the identity and the mapping $(k,g) \rightarrow k^{-1}g$ is analytic at the identity for any fixed $k \in G$. It follows, then, that $\psi'(g)$ is analytic at the identity. This proves the proposition.

We define $Z^1(G,\mathcal{K}_\omega)$ to be the set of cocycles of G with values in the analytic vectors $\mathcal{K}_\omega$, such that each cocycle is analytic at the identity.

The set of coboundaries which are analytic at the identity is defined by the relation $B^1(G,\mathcal{K}_\omega) = Z^1(G,\mathcal{K}_\omega) \cap B^1(G,\mathcal{K})$. We define the corresponding cohomology group as

$$H^1(G,\mathcal{K}_\omega) := Z^1(G,\mathcal{K}_\omega)/B^1(G,\mathcal{K}_\omega)$$

Corollary 2.2.1. $H^1(G,\mathcal{K}) = H^1(G,\mathcal{K}_\omega)$.

Proof

Proposition 2.2.1 shows that any cocycle $\psi \in Z^1(G,\mathcal{K})$ has a representative $\psi' \in Z^1(G,\mathcal{K}_\omega)$. Hence the equivalence class $[\psi] \in H^1(G,\mathcal{K})$ corresponds uniquely to the equivalence class $[\psi'] \in H^1(G,\mathcal{K}_\omega)$. This correspondence is bijective. This proves the result.

This establishes the relation between the "continuous cohomology" and the "analytic cohomology" of the group. We next proceed from $H^1(G,\mathcal{K}_\omega)$ and embed this into the cohomology group $H^1(\underline{G},\mathcal{K}_\omega)$ for the Lie algebra $\underline{G}$ of the group G.

Suppose $\psi \in Z^1(G,\mathcal{K}_\omega)$. Then define the function Δ on $\underline{G}$, with values in $\mathcal{K}$, by the equation

$$\Delta(X) = \frac{d}{dt}\psi(e^{tX})\Big|_{t=0} \qquad \text{for } X \in \underline{G}$$

The derivative exists, as ψ is analytic at the identity.

Lemma 2.2.1. $\Delta: \underline{G} \to \mathcal{K}$ is a linear map and $\Delta(X) \in \mathcal{K}_\omega$ for each $X \in \underline{G}$.

Proof

We note that $\Delta(X) = \Delta(\frac{d}{dt}e^{tX}\Big|_{t=0})$. First we show that $\Delta(\lambda X) = \lambda\Delta(X)$ for $\lambda \in \mathbb{R}$. We have

$$\Delta(\lambda X) = \frac{d}{dt}\psi(e^{t\lambda X})\Big|_{t=0} = \frac{ds}{dt}\cdot\frac{d}{ds}\psi(e^{sX})\Big|_{s=0}$$

where we have put $t\lambda = s$. Now $\frac{ds}{dt} = \lambda$ and it therefore follows that $\Delta(\lambda X) = \lambda\Delta(X)$.

Now we show $\Delta(X+Y) = \Delta(X)+\Delta(Y)$. To prove this we use the formula

$$e^{tX}e^{tY} = e^{t(X+Y)+O(t^2)}$$

for small enough t. This formula can be found, for instance, in [6].

Using this formula, we obtain

$$\Delta(X+Y) = \frac{d}{dt}\psi(e^{t(X+Y)+O(t^2)})\Big|_{t=0}$$

$$= \frac{d}{dt}\psi(e^{tX}e^{tY})\Big|_{t=0}$$

Now $\psi(e^{tX}e^{tY}) = U(e^{tX})\psi(e^{tY})+\psi(e^{tX})$. Writing $\pi(X)$ for the self-adjoint generator of the 1-parameter unitary group $U(e^{tX})$, we obtain

$$\frac{d}{dt}\psi(e^{tX}e^{tY}) = i\pi(X)U(e^{tX})\psi(e^{tY})+U(e^{tX})\frac{d}{dt}\psi(e^{tY})$$

$$+\frac{d}{dt}\psi(e^{tX})$$

Hence $\frac{d}{dt}\psi(e^{tX}e^{tY})\Big|_{t=0} = \frac{d}{dt}\psi(e^{tX})\Big|_{t=0} + \frac{d}{dt}\psi(e^{tY})\Big|_{t=0}$

$$= \Delta(X)+\Delta(Y).$$

It now follows that $\Delta(X+Y) = \Delta(X)+\Delta(Y)$ for any pair $X,Y \in \underline{G}$. This proves the linearity of the map Δ. Δ is also a continuous mapping of $\underline{G}$ into $\mathcal{K}$.

We note also that the function $(g,t) \longrightarrow U_g\psi(e^{tX})$ is analytic at the point $(e,0) \in G\times\mathbb{R}$. Hence its derivative at $t=0$ is analytic at $e \in G$, and this derivative is none other than $U_g\Delta(X)$. Therefore, $g \longrightarrow U_g\Delta(X)$ is analytic at the identity for any $X \in \underline{G}$, so that for each element X of $\underline{G}$, $\Delta(X) \in \mathcal{K}_\omega$. This proves the Lemma.

Lemma 2.2.2. $\Delta: \underline{G} \to \mathcal{K}_\omega$ is a 1-cocycle for the representation π of the Lie algebra $\underline{G}$.

Proof

We have shown that $\Delta: \underline{G} \to \mathcal{K}_\omega$ is a linear map, so we must show that $\Delta([X,Y]) = \pi(X)\Delta(Y)-\pi(Y)\Delta(X)$. For this we need the equations

$$e^{ad(g)Y} = g e^{Y} g^{-1} \qquad \text{for } g \in G \text{ and } Y \in \underline{G}$$

and

$$[X,Y] = \frac{d}{dt} ad(e^{tX})Y \Big|_{t=0} \qquad \text{for } X,Y \in \underline{G}.$$

These equations can be found in [6].

Using the linearity of the map Δ, we obtain $\Delta([X,Y]) = \frac{d}{dt} \Delta(ad(e^{tX})Y) \Big|_{t=0}$. Moreover, we note that $ad(g)Y = \frac{d}{ds}(ge^{sY}g^{-1}) \Big|_{s=0}$. Hence we obtain

$$\begin{aligned}\Delta([X,Y]) &= \frac{d}{dt} \Delta(ad(e^{tX})Y) \Big|_{t=0} \\ &= \frac{\partial^2}{\partial t \partial s} \cdot \psi(e^{tX}e^{sY}e^{-tX}) \Big|_{s,t=0}\end{aligned}$$

Further, we have

$$\begin{aligned}\psi(e^{tX}e^{sY}e^{-tX}) &= U(e^{tX})\psi(e^{sY}e^{-tX})+\psi(e^{tX}) \\ &= U(e^{tX})U(e^{sY})\psi(e^{-tX})+U(e^{tX})\psi(e^{sY})+\psi(e^{tX})\end{aligned}$$

From this it follows that

$$\Delta([X,Y]) = \pi(X)\Delta(Y)-\pi(Y)\Delta(X).$$

This proves the Lemma.

Lemma 2.2.3. If $\psi \in Z^1(G,\mathcal{K}_\omega)$ such that $\frac{d}{dt} \psi(e^{tX}) \Big|_{t=0} = \Delta(X) \in B^1(\underline{G},\mathcal{K}_\omega)$, then it follows that $\psi \in B^1(G,\mathcal{K}_\omega)$.

Proof

Let us remark that we have the following expression for the derivative of $\psi(e^{tX})$:

$$\frac{d}{dt}\psi(e^{tX}) = \lim_{h\to 0}\frac{\psi(e^{(t+h)X})-\psi(e^{tX})}{h}$$

$$= \lim_{h\to 0}\frac{U(e^{tX})\psi(e^{hX})+\psi(e^{tX})-\psi(e^{tX})}{h}$$

$$= \lim_{h\to 0} U(e^{tX})\frac{\psi(e^{hX})}{h} = U(e^{tX})\Delta(X).$$

Hence if $\Delta(X) \in B^1(\underline{G},\mathcal{K}_\omega)$, there exists a vector $\xi \in \mathcal{K}_\omega$ such that $\Delta(X) = i\pi(X)\xi$ for all $X \in \underline{G}$. We then have, from the above, that

$$\frac{d}{dt}\psi(e^{tX}) = U(e^{tX}).i.\pi(X)\xi = \frac{d}{dt}[U(e^{tX})\xi].$$

It now follows that $\psi(e^{tX})$ and $U(e^{tX})\xi$ differ by a vector, ξ_0. We have $\psi(e^{tX}) = U(e^{tX})\xi+\xi_0$. Since $\psi(e^{tX})$ vanishes when t=0, it follows that $\psi(e^{tX}) = U(e^{tX})\xi-\xi$.

Now we evaluate $\psi(g)$. Since G is connected, we can find elements $X_1,\ldots,X_n \in \underline{G}$ such that $g = e^{X_1}\ldots e^{X_n}$. Using this result, we obtain the following calculation

$$\psi(g) = \psi(e^{X_1}\ldots e^{X_n}) = U(e^{X_1})\psi(e^{X_2}\ldots e^{X_n})+\psi(e^{X_1})$$

$$= \sum_{j=2}^{n-1} U(e^{X_1}\ldots e^{X_j})\psi(e^{X_{j+1}})+U(e^{X_1})\psi(e^{X_2})+\psi(e^{X_1})$$

$$= \sum_{j=2}^{n-1} U(e^{X_1}\ldots e^{X_j})[U(e^{X_{j+1}})\xi-\xi]+U(e^{X_1})[U(e^{X_2})\xi-\xi]+U(e^{X_1})\xi-\xi$$

$$= U(e^{X_1}\ldots e^{X_{n-1}})U(e^{X_n})\xi-\xi$$

$$= U(e^{X_1}\ldots e^{X_n})\xi-\xi$$

$$= U(g)\xi-\xi.$$

So for any $g \in G$, $\psi(g) = U(g)\xi-\xi \in B^1(G,\mathcal{K}_\omega)$. Hence ψ is a coboundary. This proves the Lemma.

Remark 2.2.1. Notice that the assumed connectedness of G is important in the derivation of Lemma 2.2.3. We shall meet this argument, in a more general situation, later on.

We also remark that if $\psi \in B^1(G,\mathcal{K}_\omega)$, then Δ maps ψ into $B^1(\underline{G},\mathcal{K}_\omega)$.

Proposition 2.2.2. If G is connected, then Δ induces a canonical imbedding of $H^1(G,\mathcal{K}_\omega)$ into $H^1(\underline{G},\mathcal{K}_\omega)$. Thus Δ is injective.

Proof

Lemmas 2.2.1 and 2.2.2 show that Δ maps cocycles of G to cocycles of $\underline{G}$. Lemma 2.2.3 shows that Δ is an injective mapping with kernel $B^1(G,\mathcal{K}_\omega)$. If $\psi \in Z^1(G,\mathcal{K}_\omega)$ then Δ maps the equivalence class of ψ to the corresponding equivalence class in $Z^1(\underline{G},\mathcal{K}_\omega)$. This establishes the proposition.

It is now clear that, in general, $H^1(G,\mathcal{K})$ is determined by a part of $H^1(\underline{G},\mathcal{K}_\omega)$ - provided, of course, that G is connected. However, it is not always the case that there is an isomorphic correspondence between the two cohomology groups. To achieve this, we must impose an extra condition. The following result can be obtained. We do not present the proof.

Proposition 2.2.3 [15]. If G is simply connected, then there is an isomorphism between the group $H^1(G,\mathcal{K}_\omega)$ and $H^1(\underline{G},\mathcal{K}_\omega)$.

We are able to obtain a result similar to this, for the case when G is connected, but not necessarily simply connected. This result is presented in the next section.

§3. Quasi-coboundaries

We saw in Chapter 1 that our interest lies in the study of cocycles of the form $\psi_F(g) = M_g F - F$. Here $F \in \tau^X$, the algebraic dual of the pre-Hilbert space τ. The completion of τ is $\mathcal{K}$. The group G is represented by unitary operators U_g in $\mathcal{K}$, with which the operators M_g agree when the M_g are restricted to $\mathcal{K}$. Such cocycles we call quasi-coboundaries and F is called a cocycle function.

Suppose $F \in \tau^X$ and that for each $X \in \underline{G}$ we have $\pi(X)F \in \mathcal{K}_\omega$. π is the representation of $\underline{G}$ got by differentiating M_g along 1-parameter groups. $\mathcal{K}_\omega$ is the set of analytic vectors for the representation U. The function $\Delta_F(X) := \pi(X)F$ is a cocycle for $\underline{G}$ with values in $\mathcal{K}_\omega$. In this case Δ_F is a quasi-coboundary for the Lie algebra and F is, again, a cocycle function.

We define $Z^1_Q(G,\mathcal{K})$ to be the set of quasi-coboundaries with values in $\mathcal{K}$. Clearly, $B^1(G,\mathcal{K}) \subseteq Z^1_Q(G,\mathcal{K})$. Further $Z^1_Q(G,\mathcal{K}_\omega)$ denotes those quasi-coboundaries which are analytic at the identity. The corresponding cohomology groups are defined as usual:

$$H^1_Q(G,\mathcal{K}) = Z^1_Q(G,\mathcal{K})/B^1(G,\mathcal{K})$$

and

$$H^1_Q(G,\mathcal{K}_\omega) = Z^1_Q(G,\mathcal{K}_\omega)/B^1(G,\mathcal{K}_\omega).$$

For the quasi-coboundaries for $\underline{G}$ we use the symbol $Z^1_Q(\underline{G},\mathcal{K})$ to denote those quasi-coboundaries with values in $\mathcal{K}$. Again, we have $B^1(\underline{G},\mathcal{K}_\omega) \subseteq Z^1_Q(\underline{G},\mathcal{K}_\omega)$. We also have

$$H^1_Q(\underline{G},\mathcal{K}_\omega) = Z^1_Q(\underline{G},\mathcal{K}_\omega)/B^1(\underline{G},\mathcal{K}_\omega).$$

The next proposition shows that the map Δ, defined in the previous section, gives rise to an isomorphism between $H^1_Q(G,\mathcal{K}_\omega)$ and $H^1_Q(\underline{G},\mathcal{K}_\omega)$. The assumption that G is connected, but not necessarily simply connected, is important in this connection.

Lemma 2.3.1. Suppose that $F \in \tau^X$ and that $R(t)$ is a representation of the real numbers in τ^X, which is unitary when acting in $\mathcal{K}$. The following conditions are equivalent

(1) $R(t)F-F \in \mathcal{K}$ for all $t \in \mathbb{R}$

(2) there exists a number $\varepsilon > 0$ such that for all

$|t| < \varepsilon$, $R(t)F-F \in \mathcal{K}$

Proof

If (1) is true, then (2) is a clear consequence. Suppose, then, that (2) is true. We have, writing $\psi(t) = R(t)F-F$,

$$R(t)F-F = R\left(\frac{t}{2}\right)\{R\left(\frac{t}{2}\right)F-R\left(\frac{-t}{2}\right)F\}$$

$$= R\left(\frac{t}{2}\right)\{\psi\left(\frac{t}{2}\right) - \psi\left(\frac{-t}{2}\right)\}$$

Proceeding in this way, we obtain, for any $n \geq 1$,

$$R(t)F-F = A(t)\{\psi\left(\frac{t}{2^n}\right) - \psi\left(\frac{-t}{2^n}\right)\}$$

where $A(t)$ is a sum of operators, which is bounded when restricted to $\mathcal{K}$.

Given any $t \in \mathbb{R}$, we can choose n so that $\frac{|t|}{2^n} < \varepsilon$. It follows that for this n, $\psi\left(\frac{t}{2^n}\right) \in \mathcal{K}$ and $\psi\left(\frac{-t}{2^n}\right) \in \mathcal{K}$. Now it follows that, for any $t \in \mathbb{R}$, $\psi(t) \in \mathcal{K}$. This proves the Lemma.

Proposition 2.3.1. $H^1_Q(G,\mathcal{K}) = H^1_Q(G,\mathcal{K}_\omega) = H^1_Q(\underline{G},\mathcal{K}_\omega)$.

Proof

If $\psi \in Z^1_Q(G,\mathcal{K})$ then there exists a $\psi' \in Z^1_\omega(G,\mathcal{K}_\omega)$ such that

$$\psi'(g) = \psi(g) + U_g\xi-\xi$$

for some vector $\xi \in \mathcal{K}$. This follows from Proposition 2.2.1. Since ψ is of the form $\psi(g) = M_gF-F$ for $F \in \tau^X$, where M_g agrees with U_g when acting in $\mathcal{K}$, then we conclude that $\psi'(g) = M_g(F+\xi) - (F+\xi) \in Z^1_Q(G,\mathcal{K}_\omega)$. From this we conclude that $H^1_Q(G,\mathcal{K}) = H^1_Q(G,\mathcal{K}_\omega)$.

If $\psi \in Z^1_Q(G,\mathcal{K}_\omega)$ then the map Δ from $\underline{G}$ to $\mathcal{K}_\omega$ is given by $\Delta(X) = i\pi(X)F$, where $F \in \tau^X$. This establishes that $Z^1_Q(G,\mathcal{K}_\omega)$ is mapped injectively into $Z^1_Q(\underline{G},\mathcal{K}_\omega)$. The injectivity follows from Lemma 2.2.3.

Suppose, now, that for each $X \in \underline{G}$, $F \in \tau^X$ satisfies the condition $\pi(X)F \in \mathcal{K}_\omega$. Since $\pi(X)F$ is an analytic vector for the representation U (and of M, as $M|\mathcal{K} = U$) we conclude that for any $n \geq 1, \pi(X)^n F \in \mathcal{K}_\omega$. Further, there exists a number $\varepsilon > 0$ such that for any $X \in \underline{G}$, the series

$$\sum_{n=1}^{\infty} \frac{(it)^n}{n!} \pi(X)^n F$$

converges in $\mathcal{K}$, for at $t \in \mathbb{R}$ such that $|t| < \varepsilon$. Namely, for $|t| < \varepsilon$ we have the inclusion

$$e^{it\pi(X)}F - F \in \mathcal{K}$$

i.e. $$M(e^{tX})F - F \in \mathcal{K} \quad \text{for} \quad |t| < \varepsilon.$$

We now apply Lemma 2.3.1 and obtain that, for any $t \in \mathbb{R}$ and for any $X \in \underline{G}$,

$$M(e^{tX})F - F \in \mathcal{K}$$

Clearly we have $M(e^{tX})F - F \in \mathcal{K}_\omega$. This follows from the fact that $\pi(X)F \in \mathcal{K}_\omega$.

Since G is connected, any $g \in G$ can be written as a product $g = e^{X_1} \ldots e^{X_n}$ for elements $X_1, \ldots, X_n \in \underline{G}$. We then apply the argument used in Lemma 2.2.3 to establish that $M_g F - F \in \mathcal{K}_\omega$ for each $g \in G$.

We have therefore proved that any quasi-coboundary of G, with values in $\mathcal{K}_\omega$, gives rise to a quasi-coboundary of $\underline{G}$, with values in $\mathcal{K}_\omega$. Further, any quasi-coboundary of $\underline{G}$ gives rise to a quasi-coboundary for G. Therefore Δ gives us a bijection between $H^1_Q(G,\mathcal{K}_\omega)$ and $H^1_Q(\underline{G},\mathcal{K}_\omega)$. We have established

$$H^1_Q(G,\mathcal{K}) = H^1_Q(G,\mathcal{K}) = H^1_Q(\underline{G},\mathcal{K}_\omega).$$

This proves the proposition.

This section shows that, provided G is connected, we obtain a bijection between the quasi-coboundaries of G and those of $\underline{G}$. In general, we do not expect there to be an isomorphism between the full cohomology group of G and the cohomology group of $\underline{G}$. In order to achieve our result we had to consider only certain types of cocycles. For a given G, there may be cocycles which are not quasi-coboundaries. An example is SL(2,C). One can prove that SL(2,C) has no quasi-coboundaries, but that the principal series gives rise to a cocycle which is not a coboundary. For details of this see [8].

We are now ready to analyse the structure of cocycles; we do this in the next section.

§4. The Origin of Cocycles

The results which are quoted here are due mainly to Araki [1]. Some elementary consequences are drawn. To begin with, we have the following.

Definition 2.4.1 [1]. Let $h \in \mathcal{D}(\mathbb{R})$ such that its transform

$$\tilde{h}(\lambda) = \int e^{it\lambda} h(t)dt$$

satisfies $\tilde{h}(0) = 1$, $1 \geqq \tilde{h}(\lambda) \geqq 0$, and $\tilde{h}(\lambda) \neq 1$ if $\lambda \neq 0$, and $\tilde{h}''(0) \neq 0$. Further, let $\{X_1,\ldots,X_n\}$ be a linearly independent basis of $\underline{G}$, the Lie algebra of the connected group G, and define for each X_j the operator

$$R(X_j) = \mathbb{1} - \int U(e^{tX_j})h(t)dt$$

Here, U is a unitary representation of G in the Hilbert space $\mathcal{K}$.

Next define $R = \sum_{j=1}^{n} R(X_j)$. It is clear that R is bounded, self-adjoint and positive. Then we have the following spectral decomposition

$$R = \int_0^\infty \lambda \, dE(\lambda)$$

We define $R^{-\frac{1}{2}}$ to be the inverse of the operator $R^{\frac{1}{2}}$ from $[1 - E(0)]\mathcal{K}$ into $\mathcal{K}$. $D^{\pm}$ is the range of $R^{\mp\frac{1}{2}}$ in $\mathcal{K}$. On D^+ a new topology can be defined as follows $\|\xi\|_+ := \|R^{\frac{1}{2}}\xi\|$. $\overline{D^+}$ denotes the completion of D^+ in this topology. $\overline{D^-}$ is closure of D^- in the norm $\|\eta\|_- := \|R^{-\frac{1}{2}}\eta\|$.

It can be shown that the precise choice of the function h(t) is immaterial. Provided that h(t) satisfies the basic demands, the topologies on D^+ and D^- are unique.

The spaces D^+ and D^- are dual to each other with respect to the inner product (;) on $\mathcal{K}$. Indeed, we have the following result.

Lemma 2.4.1 [1]. If $\xi \in D^+$ then $R\xi \in D^-$. The form $(\xi;\eta)$ for $\xi \in D^+$ and $\eta \in D^-$ can be extended to $\xi \in \overline{D^+}$.

Proof

We have $\|\xi\|_+^2 = \|R^{\frac{1}{2}}\xi\|^2 = \|R^{-\frac{1}{2}}R\xi\|^2 = \|R\xi\|_-^2$. This proves the first part. Notice that R is defined on a dense set D^+ in $\overline{D^+}$ and is isometric from D^+ to D^-. RD^+ is dense in D^-. We can extend R to be a unitary operator from $\overline{D^+}$ onto D^-. Given any vector $\eta \in D^-$ there is a vector $\eta_0 \in \overline{D^+}$ such that $R\eta_0 = \eta$. Using this result we see that $(\xi;\eta) = (\xi;R\eta_0)$ and this is bounded since $|(\xi;R\eta_0)| = |(R^{\frac{1}{2}}\xi;R^{\frac{1}{2}}\eta_0)| \leq \|R^{\frac{1}{2}}\xi\|.\|R^{\frac{1}{2}}\eta_0\| = \|\xi\|_+.\|\eta_0\|_+$. This proves the duality of $\overline{D^+}$ and $\overline{D^-}$.

Because of this duality, we are able to extend the representation U of G on $\mathcal{K}$ to a representation M of G on $\overline{D^+}$ as follows: each $\xi \in \overline{D^+}$ defines a functional on D^- and for each $g \in G$ we define $M_g\xi$ with the identification

$$(M_g\xi;\eta) = (\xi;U_{g^{-1}}\eta)$$

for all $\eta \in D^-$. The right-hand side is well defined, since one has for any $\zeta \in \mathcal{K}$ that $U_g\zeta-\zeta \in D^-$ for all $g \in G$. Hence $U_g\eta-\eta \in D^-$ and since $\eta \in D^-$ by assumption, it follows that $U_g\eta \in D^-$ for any $g \in G$.

It is clear that $\mathcal{K} \subseteq \overline{D^+}$ and it follows from this that M agrees with U on $\mathcal{K}$.

The next two results are useful for our choices of G.

Theorem 2.4.1 [4]. Suppose $\psi \in Z^1(G,\mathcal{K})$ and that G contains an abelian normal subgroup N. ψ can be decomposed into two parts as follows

$$\psi(g) = M_g\Omega-\Omega + \psi_1(g)$$

where Ω is a vector in $\overline{D^+}(N)$, the space $\overline{D^+}$ which is constructed with the Lie algebra of N, and $\psi_1(g)$ takes values in the subspace of vectors which are invariant under the action of N.

Proposition 2.4.1. If G contains a compact subgroup K, then any cocycle ψ is cohomologous to one which vanishes on K. In particular, for any quasi-coboundary, we may assume that the cocycle function is invariant under the action of K.

Proof

Define $\xi = \int_K \psi(k)dk$, where dk is the normalised Haar measure on K. $\xi \in \mathcal{K}$ as K is a compact space and dk is totally finite. Consider then

$$\psi'(g) = \psi(g) + (U_g\xi-\xi)$$

This is a cocycle which is cohomologous to ψ. Further,

$$\begin{aligned}\psi'(k) &= \psi(k) + (U_k\xi-\xi)\\ &= \psi(k) + \int_K [U_k\psi(k_1)-\psi(k_1)]dk_1\\ &= \psi(k) + \int_K \psi(kk_1)dk_1 - \int_K \psi(k_1)dk_1 - \int_K \psi(k)dk_1\\ &= \psi(k) - \psi(k) + \int_K \psi(k_1)dk_1 - \int_K \psi(k_1)dk_1 = 0\end{aligned}$$

We have made use of the cocycle identity, the normalization of dk on K and the left-invariance of dk. This proves the first result.

If ψ is a quasi-coboundary, $\psi(g) = M_g F - F$, say, where $F \in \overline{D^+}$, then the above result implies that there exists $F_1 \in \overline{D^+}$ such $F - F_1 \in \mathcal{K}$ and that $M_g F_1 - F_1$ vanishes on the compact subgroup K. This proves the second assertion.

A useful result, which we shall need, is the following.

<u>Proposition 2.4.2.</u> Suppose that G satisfies the conditions of Theorem 2.4.1. Further, suppose that one element of the Lie algebra of the subgroup N can be represented by a self-adjoint operator whose spectrum is bounded away from 0. Then the space $\overline{D^+}(N) = \mathcal{K}$ and the quasi-coboundaries are all coboundaries.

<u>Proof</u>

Suppose $X \in \underline{N}$, the Lie algebra of N, has the property that its representative $\pi(X)$ has spectrum in $\mathbb{R}\backslash(-\delta,\varepsilon)$ where $\delta, \varepsilon > 0$. Then we have

$$R(X) = 1 - \int U(e^{tX})h(t)\,dt = 1 - \int e^{it\pi(X)}h(t)\,dt$$

$$= 1 - \int_{\mathbb{R}}\int_{\mathbb{R}\backslash(-\delta,\varepsilon)} e^{it\lambda}h(t)\,dt\,dE(\lambda)$$

Here we have used the spectral decomposition of $\pi(X)$. Hence

$$R(X) = 1 - \int_{\mathbb{R}\backslash(-\delta,\varepsilon)} \tilde{h}(\lambda)\,dE(\lambda)$$

Now on $\mathbb{R}\backslash(-\delta,\varepsilon)$, $\tilde{h}(\lambda)$ is bounded above by a number $c < 1$. Therefore $R(X) \geqq 1-c > 0$. From this it follows that $R \geqq 1-c > 0$ and so R is bounded away from zero. Moreover, R is bounded above, by 2. It now follows that $R^{-\frac{1}{2}}$ exists as a bounded operator on $\mathcal{K}$, from which we obtain the following sequence of inequalities

$$\|\xi\|_+ = \|R^{\frac{1}{2}}\xi\| \leqq C.\|\xi\| = C.\|R^{-\frac{1}{2}}R^{\frac{1}{2}}\xi\| \leqq C.C_1.\|R^{\frac{1}{2}}\xi\| = C.C_1\|\xi\|_+$$

where C is the bound of $R^{\frac{1}{2}}$ and C_1 is the bound of $R^{-\frac{1}{2}}$. It follows that $\|\cdot\|_+$ and $\|\cdot\|$ are equivalent norms, and so $\overline{D^+} = \mathcal{K}$. Hence all quasi-coboundaries are true coboundaries. This proves the proposition.

As a direct consequence of this proposition, we have the following result concerning the Poincare group $\mathcal{P}_+^\uparrow(s+1)$ for s+1 dimensions of space-time, with $s \geqq 1$.

Proposition 2.4.3. For any representation of $\mathcal{P}_+^\uparrow(s+1)$ on the one-particle Hilbert space $L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\sqrt{\underline{p}^2+m^2}})$ where $m > 0$ is the mass, all cocycles are coboundaries.

Proof

$\mathcal{P}_+^\uparrow(s+1)$ contains the space-time translation subgroup $\mathbb{R}^{s+1}$, which is normal. Therefore any cocycle is of the form

$$\psi(g) = M_g\Omega - \Omega + \psi_1(g)$$

where $\Omega \in \overline{D^+}(\mathbb{R}^{s+1})$ and $\psi_1(g)$ takes on values in those vectors invariant under the action of $\mathbb{R}^{s+1}$. In $L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\sqrt{\underline{p}^2+m^2}})$ the only such vector is the zero vector, so $\psi_1(g) = 0$. Hence $\psi(g) = M_g\Omega - \Omega$.

Since $m > 0$, the time-translation generator $\sqrt{\underline{p}^2+m^2} \geqq m > 0$ and so the conditions of Proposition 2.4.2 are obtained. It follows that $\overline{D^+}(\mathbb{R}^{s+1}) = L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\sqrt{\underline{p}^2+m^2}})$, and this establishes the proposition.

A further result in this vein occurs when the physical theory we look at only contains "hard photons". Namely, the mass $m = 0$ and the energy $|\underline{p}| \geqq \omega$ where $\omega > 0$. In this case we again apply the Proposition 2.4.2 and find that all cocycles are trivial.

The last few results show that under certain conditions one only obtains Fock representations of the C.C.R. Moreover, we can see that cocycles can only occur when the mass is zero and photons of low momentum are allowed i.e. only in the presence of the infra-red problem do we hope to see other representations of the C.C.R. This is the famous problem of obtaining an infinite number of photons of low energy in a given state, thus causing a "condensate" to occur.

In the succeeding chapters we will see that the dimension of the momentum space is also an important factor in deciding whether or not we obtain non-trivial cohomology.

CHAPTER 3

Cocycles and Representations for the Poincaré Group and its subgroups

§1 The space $\overline{D^+}$ for $P_+^\uparrow(s+1)$

We consider the Poincaré group $P_+^\uparrow(s+1)$ in (s+1) space-time dimensions, for s = 1 and s = 3.

As a result of Proposition 2.4.3, we may restrict our attention to the massless case: for m > 0, the cohomology is trivial. We consider only irreducible representations on the Hilbert space $L^2(\mathbb{R}^s, \frac{d^s p}{\omega})$ where s = 1,3 and $\omega = |\underline{p}|$. When s = 3 we can define both zero-spin and non-zero spin representations on the same space, and this allows us to solve our problems more easily. In each representation the translations are represented by

$$(U_{(a,\mathbb{1})}f)(\underline{p}) = e^{ia_0\omega}.e^{-i\underline{a}.\underline{p}}f(\underline{p})$$

where $a = (a_0,\underline{a})$ is the translation.

All our representations, as already remarked, are irreducible, and no $P_+^\uparrow(s+1)$-invariant vector, other than the zero vector, exists in $L^2(\mathbb{R}^s, \frac{d^s p}{\omega})$. From this remark, we may deduce that any cocycle of $P_+^\uparrow(s+1)$ in the above representation is a quasi-coboundary. This is because of the result contained in Theorem 2.4.1. Our first result is then expressed as: $H^1(P_+^\uparrow(s+1),\mathcal{K}) = H_Q^1(P_+^\uparrow(s+1)\mathcal{K})$.

Before we proceed further to the calculation of $H_Q^1(P_+^\uparrow(s+1),\mathcal{K})$, we construct $\overline{D^+}(\mathbb{R}^s)$ by deriving a sufficient and necessary condition for a function to belong to $\overline{D^+}(\mathbb{R}^s)$. Moreover, we show that all cocycle functions of τ^x belong to $\overline{D^+}(\mathbb{R}^s)$.

Our choice of the vector space τ, over which we build representations of the C.C.R., is governed by the condition in Theorem 1.8.1, which states that all $P_+^\uparrow(s+1)$-invariant functionals must vanish on τ. In the coordinate space version of the representation, these $P_+^\uparrow(s+1)$-invariant functionals are all constants so that in momentum space these functionals take the form

$$F = c.\delta(\underline{p}) \qquad \text{where} \qquad c \in \mathbb{R}.$$

If $f \in \tau$ then we require $(F;f) = 0$ and this gives us $(\delta;f) = 0$, namely $f(0) = 0$. A further condition on τ is that it must be dense in $L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\omega})$.

Now we must choose a candidate for τ. We nominate the space $\tilde{\mathcal{D}}_0 = \{\omega\tilde{f}: f \in \mathcal{D}(\mathbb{R}^s)\}$. The tilde denotes the Fourier transform. From the definition of the space $\tilde{\mathcal{D}}_0$, it is obvious that $\tilde{\mathcal{D}}_0 \subset L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\omega})$ and it is not hard to see that the unitary representation of $P_+^\uparrow(s+1)$ maps $\tilde{\mathcal{D}}_0$ into $\tilde{\mathcal{D}}_0$. Another important property is that $\tilde{\mathcal{D}}_0$ is dense in $L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\omega})$. It seems that this is known, but the proof is not to be found in the literature, so we present it here.

Lemma 3.1.1. The space $\tilde{\mathcal{D}}_0$ is dense in $L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\omega})$.

Proof. Since $\tilde{\mathcal{D}} = \{\tilde{f}: f \in \mathcal{D}\}$ is dense in $L^2(\mathbb{R}^s, d^s\underline{p})$ it follows that $\{\sqrt{\omega}\,\tilde{f}: \tilde{f} \in \tilde{\mathcal{D}}\}$ is dense in the space $L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\omega})$. Hence we have that $L^2(\mathbb{R}^s, d^s\underline{p}) \cap L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\omega})$ is dense in $L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\omega})$.

Now suppose that $\tilde{h} \in L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\omega}) \cap L^2(\mathbb{R}^s, d^s\underline{p})$ and that

$$\int \overline{\tilde{h}(\underline{p})}\tilde{g}(\underline{p})\frac{d^s\underline{p}}{\omega} = 0$$

for all $\tilde{g} \in \tilde{\mathcal{D}}_0$. Because $\tilde{g} = \omega\tilde{f}$ with $\tilde{f} \in \tilde{\mathcal{D}}$, we obtain

$$\int \overline{\tilde{h}(\underline{p})}\tilde{f}(p)\,d^s\underline{p} = 0$$

for all $\tilde{f} \in \tilde{\mathcal{D}}$, and this implies $\tilde{h} = 0$, since $\tilde{\mathcal{D}}$ is dense in $L^2(\mathbb{R}^s, d^s\underline{p})$. It must follow that $\tilde{\mathcal{D}}_0$ is dense in $L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\omega})$.

Having made a choice for τ, we turn to the construction of the space $\overline{D^+}$.

The space $\overline{D^+}$ can be constructed from the Lie algebra of the space-time translations. This follows from Theorem 2.4.1. We obtain $\overline{D^+}$ by completing the range of the operator $R^{-\frac{1}{2}}$ in the topology defined by

$$\|f\|_+ := \|R^{\frac{1}{2}}f\|$$

where R is defined by

$$R = \sum_{j=1}^{n} R(x_j)$$

Here, X_j is a typical basis element of the Lie algebra, and $R(X_j)$ is defined by

$$R(X_j) = \mathbb{1} - \int U(e^{tX_j})h(t)\,dt.$$

The choice of function h is immaterial, provided it satisfies the requirements of Definition 2.4.1. We may even relax the requirement $h \in \mathcal{D}(\mathbb{R})$ and ask for $h \in \mathcal{S}(\mathbb{R})$. Let us choose

$$h(t) = \frac{1}{2}e^{-|t|}$$

This means that

$$\int_{-\infty}^{\infty} e^{it\alpha}h(t)\,dt = \frac{1}{1+\alpha^2} \qquad \text{for} \qquad \alpha \in \mathbb{R}$$

Using this integral, we find

$$R = \sum_{\beta=0}^{3} R(p_\beta) = \frac{\Phi(\underline{p})\,\omega^2}{1+\omega^2}$$

where $\Phi(\underline{p})$ is a function with

$$2 \leqq \Phi(\underline{p}) \leqq 4$$

From this it follows that F is an element of $\overline{D^+}$ if and only if

$$\int_{\mathbb{R}^3} \frac{\omega|F(\underline{p})|^2}{1+\omega^2}\, d^s\underline{p} < \infty$$

Hence we are justified in calling F a function as we have

$$\overline{D^+} = L^2(\mathbb{R}^s, \frac{\omega d^s p}{1+\omega^2}) \qquad s = 1,3$$

This constructs the space $\overline{D^+}$.

Let us note that if $F \in \tau^x$ and F is a cocycle function giving rise to an analytic cocycle, we have the relation

$$\omega^n F \in L^2(\mathbb{R}^s, \frac{d^s p}{\omega}) \qquad \text{for} \quad n \geqq 1$$

from which we deduce

$$\int_{\mathbb{R}^3} \frac{\omega|F(\underline{p})|^2}{1+\omega^2}\, d^s\underline{p} < \infty$$

i.e. $F \in \overline{D^+}$. It is now apparent that we may refer to a cocycle function as coming from either $\overline{D^+}$ or τ^x, the two coinciding for cocycle functions.

§2 The Free Wave Equation

Now we present results concerning the free wave equation, and point out how elements of $L^2(\mathbb{R}^s, \frac{d^s p}{\omega})$ can be identified with Cauchy data for the free wave equation. The function $f(\underline{x},t)$ defined by

$$f(\underline{x},t) = \frac{1}{2}\cdot(2\pi)^{-s/2}\int \{\phi(\underline{p})\,e^{-i\underline{p}.\underline{x}+i\omega(\underline{p})t} + \overline{\phi(\underline{p})}\,e^{i\underline{p}.\underline{x}-i\omega(\underline{p})t}\}\,\frac{d^s p}{\omega(\underline{p})}$$

with $\phi \in L^2(\mathbb{R}^s, \frac{d^s p}{\omega})$, is a solution to the free wave equation. Indeed, we obtain

$$f(\underline{x}) = \frac{1}{2}\cdot(2\pi)^{-s/2}\int\{\phi(\underline{p})e^{-i\underline{p}\cdot\underline{x}} + \overline{\phi(\underline{p})}e^{i\underline{p}\cdot\underline{x}}\}\frac{d^s\underline{p}}{\omega(\underline{p})}$$

and $$g(\underline{x}) = \frac{i}{2}\cdot(2\pi)^{-s/2}\int\{\phi(\underline{p})e^{-i\underline{p}\cdot\underline{x}} - \overline{\phi(\underline{p})}e^{i\underline{p}\cdot\underline{x}}\}\, d^s\underline{p}$$

where $f(\underline{x}) = f(\underline{x},0)$ and $g(\underline{x}) = g(\underline{x},0)$ and $g(\underline{x},t) \doteq \frac{\partial f}{\partial t}(\underline{x},t)$.

It is quite clear that f and g are real-valued functions.

Using the functions f and g, we can recover the corresponding ϕ as follows

$$\phi(f) = (2\pi)^{-s/2}\int\{\omega(\underline{p})f(\underline{x}) - ig(\underline{x})\}e^{i\underline{p}\cdot\underline{x}}d^s\underline{x}$$

In order to make these formulae rigorously true, we may choose f and g as follows:

(1) $f \in \mathcal{D}(\mathbb{R}^3)$ and $g \in \mathcal{D}(\mathbb{R}^3)$ when $s = 3$

(2) $f \in \mathcal{D}(\mathbb{R})$ and $g \in \mathcal{D}_0(\mathbb{R}) = \{\frac{d\theta}{dx} : \theta \in \mathcal{D}(\mathbb{R})\}$ when $s = 1$

Here we assume $\mathcal{D}$ and $\mathcal{D}_0$ to consist of real-valued functions. Hence, to each ϕ we associate a pair from $\mathcal{D}(\mathbb{R}^3) \oplus \mathcal{D}(\mathbb{R}^3)$ (when $s = 3$) or a pair from $\mathcal{D}(\mathbb{R}^3) \oplus \mathcal{D}_0(\mathbb{R}^3)$ (when $s = 1$). In either case, we complete the spaces in the topology given by the inner product

$$\langle\theta_1;\theta_2\rangle = \int\{f_1(\underline{x})(\mu f_2)(\underline{x}) + g_1(\underline{x})(\mu^{-1}g_2)(\underline{x})\}d^s\underline{x}$$
$$+ i\int\{f_2(\underline{x})g_1(\underline{x}) - f_1(\underline{x})g_2(\underline{x})\}d^s\underline{x}$$

where $\theta_\ell = f_\ell \oplus g_\ell$, $\ell = 1,2$, and $\mu = (-\Delta)^{\frac{1}{2}}$. One can easily compute the equality

$$-\mathrm{Im}(\phi_1;\phi_2) = \mathrm{Im}\langle\theta_1;\theta_2\rangle$$

where ϕ_ℓ corresponds to θ_ℓ, $\ell = 1,2$.

It is through these formulae that we identify elements of of $L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\omega})$ with Cauchy data of the free wave equation. We do the same for certain cocycle functions.

§3. Cocycles for $P_+^\uparrow(1+1)$

Now we take the special case of $s = 1$. It is in this context that the first cocycles were seen [20]. In this case, it is easy to show that there are non-trivial cocycles. To classify the cohomology is not so easy, and it is only possible to classify a part of this cohomology.

The representation space is $L^2(\mathbb{R}, \frac{dp}{|p|})$ and the unitary action is given by

$$(U_{(a,\Lambda)}f)(p) = e^{i|p|a_0} e^{-ipa_1} f(\underline{\Lambda^{-1}p})$$

where $a = (a_0, a_1)$ and $\underline{\Lambda^{-1}p}$ is the space part of the vector $\Lambda^{-1}\begin{pmatrix} p_0 \\ \underline{p} \end{pmatrix}\Bigg|_{p_0=|\underline{p}|}$

We know that $H^1(P_+^\uparrow(1+1), \mathcal{K}) = H_Q^1(P_+^\uparrow(1+1), \mathcal{K}_\omega)$.

It follows from Proposition 2.3.1, that we need only check whether a function gives a quasi-coboundary for the Lie algebra $p_+^\uparrow(1+1)$ of $P_+^\uparrow(1+1)$, since we have

$$H^1(P_+^\uparrow(1+1), \mathcal{K}) = H_Q^1(P_+^\uparrow(1+1), \mathcal{K}_\omega) = H_Q^1(p_+^\uparrow(1+1), \mathcal{K}_\omega)$$

Theorem 3.3.1. $H^1(P_+^\uparrow(1+1), \mathcal{K}) \neq \{0\}$

i.e. there exist non-trivial cocycles for $P_+^\uparrow(1+1)$.

Proof

The Lie algebra generators are

$$p, \quad |p| \quad \text{and} \quad |p|\frac{d}{dp}$$

They can be defined on a dense set. We choose the space

$$[\tilde{\mathcal{D}}_0 \cup S\tilde{\mathcal{D}}_0]$$

where $[\quad]$ denotes linear span, and where S is the sign operator.

Let us consider now the set $\tilde{\mathfrak{D}}$. If $\theta \in \tilde{\mathfrak{D}}$ and $\theta(0) \neq 0$ then

$\theta \notin L^2(\mathbb{R}, \frac{dp}{|p|})$. Moreover, we have $p\theta \in \tilde{\mathcal{D}}_0$, $|p|\theta \in S\tilde{\mathcal{D}}_0$ and $|p|\frac{d\theta}{dp} \in S\tilde{\mathcal{D}}_0$.

From this it follows that $H^1_\Omega(P_+^\uparrow(1+1), \mathcal{K}_\omega) \neq \{0\}$. Therefore, from the remark before the theorem,

$$H^1(P_+^\uparrow(1+1), \mathcal{K}) \neq \{0\}$$

This proves the result.

This theorem merely tells us that there are non-trivial cocycles. We will see that there are more cocycles which are not equivalent to those given by functions in $\tilde{\mathcal{D}}$.

Suppose, now, that $\theta \in \mathcal{D}_r \oplus \mathcal{D}_{0r}$ where the subscript r refers to the reality of the functions involved. Then we can define

$$\{\Phi, \theta\} = \int [\Phi(x)\theta_2(x) - \Pi(x)\theta_1(x)]dx$$

where Φ is the quantum field at time zero, and Π is the canonically conjugate momentum at time zero, and

$$\theta = \theta_1 \oplus \theta_2 \in \mathcal{D}_r \oplus \mathcal{D}_{0r}.$$

The mapping given by

$$\theta \longrightarrow W(\theta) = \exp i\{\Phi, \theta\}$$

defines a Weyl system over $\mathcal{D}_r \oplus \mathcal{D}_{0r}$.

Now consider the set

$$\mathcal{D}_{1r} = \{\xi : \frac{d\xi}{dx} \in \mathcal{D}_r\}$$

and form the direct sum $\mathcal{D}_{1r} \oplus \mathcal{D}_r$. If $\Xi = \xi_1 \oplus \xi_2 \in \mathcal{D}_{1r} \oplus \mathcal{D}_r$ then

$$\frac{d\Xi}{dx} = \frac{d\xi_1}{dx} \oplus \frac{d\xi_2}{dx} \in \mathcal{D}_r \oplus \mathcal{D}_{0r}$$

We have the following useful result

Lemma 3.3.1. The map $\theta \longrightarrow \{\Xi, \theta\}$ defined by

$$\{\Xi, \theta\} = \int \Big(\xi_1(x)\theta_2(x) - \xi_2(x)\theta_1(x)\Big)dx$$

for $\Xi = \xi_1 \oplus \xi_2 \in \mathcal{D}_{1r} \oplus \mathcal{D}_r$ and $\theta = \theta_1 \oplus \theta_2 \in \mathcal{D}_r \oplus \mathcal{D}_{0r}$ defines a real linear functional on $\mathcal{D}_r \oplus \mathcal{D}_{0r}$ which is not continuous in the inner product structure on $\mathcal{D}_r \oplus \mathcal{D}_{0r}$.

Proof.

The map is clearly real-linear. We prove the non-continuity as follows.

To each $\Xi \in \mathcal{D}_{1r} \oplus \mathcal{D}_r$ we associate a function in $\tilde{\mathcal{D}}$ by the mapping

$$\Xi = \xi_1 \oplus \xi_2 \leftrightarrow p\tilde{\xi}_1(p) + i\tilde{\xi}_2(p) = \tilde{\Xi}$$

The same mapping takes $\mathcal{D}_r \oplus \mathcal{D}_{0r}$ onto a dense subset of $L^2(\mathbb{R}, \frac{dp}{|p|})$.

Using this mapping, it follows that

$$\{\Xi, \xi\} = \mathrm{Im}(\tilde{\Xi}; \tilde{\theta})$$

The inner product structure on $\mathcal{D}_r \oplus \mathcal{D}_{0r}$ is unitarily equivalent to that on $\tilde{\mathcal{D}}_0 \subset L^2(\mathbb{R}, \frac{dp}{|p|})$. Hence $\theta \rightarrow \{\Xi, \theta\}$ is continuous in the inner product structure on $\mathcal{D}_r \oplus \mathcal{D}_{0r}$ if and only if the map $\tilde{\theta} \rightarrow \mathrm{Im}(\tilde{\Xi}; \tilde{\theta})$ is continuous in the inner product structure of $L^2(\mathbb{R}, \frac{dp}{|p|})$. Because in general $\tilde{\Xi} \in \tilde{\mathcal{D}}$, and $\tilde{\mathcal{D}} \not\subset L^2(\mathbb{R}, \frac{dp}{|p|})$, this mapping is not in general a continuous mapping. Therefore $\theta \rightarrow \{\Xi, \theta\}$ is not, in general, a continuous linear functional.

This proves the result.

Corollary 3.1.1. $\theta \rightarrow e^{i\{\Xi, \theta\}} W(\theta)$ defines a displaced Fock representation of the C.C.R. over $\mathcal{D}_r \oplus \mathcal{D}_{0r}$.

Lemma 3.3.2. The functions F_α defined by

$$F_\alpha(P) = \left[\ln \frac{1}{|p|}\right]^{\alpha/2} \quad \text{for} \quad |p| \leqq 1$$

$$= e^{-p^2}(p^2-1) \quad \text{for} \quad |p| > 1$$

for $0 < \alpha < 1$ define cocycle functions for $P_+^\uparrow(1+1)$ and they give rise to displaced Fock representations of the C.C.R. over the space $\tilde{\mathcal{D}}_0$. Moreover, F_α is inequivalent to any cocycle function from $\tilde{\mathcal{D}}$. F_α and F_β are inequivalent if $\alpha \neq \beta$.

Proof.

If $\theta \in \tilde{\mathfrak{D}}_0$ then $\theta(p) = p\theta_1(p)$ with $\theta_1 \in \tilde{\mathfrak{D}}$. Then

$$(F_\alpha;\theta) = \int_{|p|\leq 1} \left(\ell n \frac{1}{|p|}\right)^{\alpha/2} p\theta_1 \frac{dp}{|p|} + \int_{|p|>1} e^{-p^2}(p-1)\theta\frac{dp}{|p|}$$

whence we obtain

$$|(F_\alpha;\theta)| \leq c_1 \cdot \|\theta_1\|_\infty + c_2\|\theta\|_\infty$$

where $c_2 = \int_{|p|>1} e^{-p^2}(p^2-1)\frac{dp}{|p|}$ and

$$c_1 = \left(\left(\int_{|p|\geq 1} |p|^{1-\alpha/2}\,\frac{dp}{p}\right) \cdot \left(\sup_{|p|\leq 1} |p|\,\ell n\frac{1}{|p|}\right)^{/2}\right)$$

Moreover, $|p|F\ (p)$ is square-integrable with respect to the measure $\frac{dp}{|p|}$, and this implies that $pF_\alpha(p)$ is also square-integrable with respect to this measure. It is easy to see that

$$|p|^n F_\alpha,\ p^n F_\alpha \in L^2(\mathbb{R}, \frac{dp}{|p|}) \quad \text{for} \quad n \geq 1$$

Also we have

$$|p|\ \frac{d}{dp} F_\alpha(p) = -\frac{\alpha}{2}(\ell n \frac{1}{p})^{\frac{\alpha}{2}-1} \cdot \frac{|p|}{p} \text{ for } |p| \leq 1$$

and this is square-integrable with respect to the measure $\frac{dp}{|p|}$. Therefore we conclude that

$$(|p|\frac{d}{dp})^n F_\alpha \in L^2(\mathbb{R}, \frac{dp}{|p|}) \quad \text{for} \quad n \geq 1.$$

This is enough to show that the functions F_α do define a displaced Fock representation. Clearly, $F_\alpha \notin L^2(\mathbb{R}, \frac{dp}{|p|})$ and if $f \in \tilde{\mathfrak{D}}$ we obtain

$$f-F_\alpha \notin L^2(\mathbb{R}, \frac{dp}{|p|})$$

so that the displaced Fock sectors defined by $f \in \tilde{\mathfrak{D}}$ and the F_α are always inequivalent. This now establishes the lemma.

Lemma 3.3.2 says that there are more cocycles for $P_+^{\uparrow}(1+1)$ than those defined by elements of $\tilde{\mathfrak{D}}$.

Proposition 3.3.1. Suppose F is a cocycle function which defines a displaced Fock representation

$$W_F(f) = e^{i\,\mathrm{Im}(F;f)}W(f)$$

over $\tilde{\mathfrak{D}}_0 \subset L^2(\mathbb{R}, \frac{dp}{p})$. Then F is equivalent to a cocycle function which is analytic at p = 0 if and only if F corresponds to Cauchy data $\Xi \in \mathfrak{D}_{1r} \oplus \mathfrak{D}_r$ which defines a displaced Fock representation

$$\theta \longrightarrow e^{i\{\Xi,\theta\}}W(\theta)$$

over $\mathfrak{D}_r \oplus \mathfrak{D}_{0r}$.

Proof.

Let us assume that F is already analytic at p = 0. If not, and F_A is analytic at p = 0 with

$$F - F_A \in L^2(\mathbb{R}, \frac{dp}{|p|})$$

then

$$F - F_A = f$$

so that f corresponds to Cauchy data in the Hilbert space completion of $\mathfrak{D}_r \oplus \mathfrak{D}_{0r}$, and this defines a continuous linear functional on $\mathfrak{D}_r \oplus \mathfrak{D}_{0r}$, which means, ultimately, that F and F_A define equivalent displaced Fock representations over $\mathfrak{D}_r \oplus \mathfrak{D}_{0r}$. This is just a reflection of their equivalence in the momentum-space formulation.

Using the assumed analyticity at p = 0, let $c \in \mathbb{C}$ with

$$F(0) = c$$

There exists a function $f \in \mathfrak{D}$ with

$$F(0) = c = f(0)$$

so $F - f \in L^2(\mathbb{R}, \frac{dp}{|p|})$, whence F and f define unitarily equivalent displaced

Fock representations. We write

$$F = f + f_1$$

with $f_1 \in L^2(\mathbb{R}, \frac{dp}{|p|})$. Using the argument at the beginning of the proof, we assume that we also have

$$F \in \tilde{\mathfrak{D}}.$$

Now define $\xi_1 \in \mathfrak{D}_{1r}$ and $\xi_2 \in \mathfrak{D}_r$ by

$$\tilde{\xi}_1(p) = \frac{F(p) + \overline{F(-p)}}{2p}$$

and

$$\tilde{\xi}_2(p) = \frac{F(p) - \overline{F(-p)}}{2i}$$

The above formulae provide the inverse to the mapping

$$\xi_1 \oplus \xi_2 \leftrightarrow p\tilde{\xi}_1(p) + i\tilde{\xi}_2(p)$$

Through this, we associate $F \in \tilde{\mathfrak{D}}$ with $\Xi = \xi_1 \oplus \xi_2 \in \mathfrak{D}_{1r} \oplus \mathfrak{D}_r$. If $F \notin \tilde{\mathfrak{D}}$, but $F - f \in L^2(\mathbb{R}, \frac{dp}{|p|})$ with $f \in \tilde{\mathfrak{D}}$, then we have

$$F = f + f_1$$

with $f_1 \in L^2(\mathbb{R}, \frac{dp}{|p|})$. If f_1 corresponds to Cauchy data θ in the Hilbert space completion of $\mathfrak{D}_r \oplus \mathfrak{D}_{0r}$, we obtain that

$$F \text{ corresponds to } \Xi + \theta$$

This proves one direction of the proposition. The other direction is proved quite easily, and requires only a slight reversal of the arguments we have used.

The reason for picking out $\mathfrak{D}_{1r} \oplus \mathfrak{D}_r$ is that any function from this space defines a localised automorphism of the local algebras of the quantum field in two dimensions of space-time [20]. Moreover, any localised automorphism, which is defined by a displaced Fock representation, can be defined in terms of a function from the space $\mathcal{D}_{1r} \oplus \mathcal{D}_{r}$.

Proposition 3.3.1 says that that the group of cohomology classes, defined by functions analytic at p = 0, is isomorphic to $\mathbb{R}^2$ as an abelian group.

§4. Cohomology for $P_+^{\uparrow}(3+1)$

In this section we prove that $H^1(P_+^{\uparrow}(3+1), \mathcal{K}) = \{0\}$ i.e. the cohomology is trivial. We do this for massless representations for the cases of discrete and continuous spin. The case for space-like representations (also known as representations of imaginary mass) is dealt with in Chapter 4 of this thesis.

Let us consider discrete spins first. For this we need the following result, due to Redheffer [24]. Indeed, we quote a modified version of Theorem 2 of [24], suitable for our purposes.

Redheffer's Theorem. Let $K(p) \geqq 0$ for $p \in (0, \infty)$ and suppose that K and u satisfy the requirements

$$\lim_{p\to 0_+} K(p) = 0, \quad K'(p) \geqq 0, \quad \liminf_{p\to\infty} u(p) = 0$$

Further let r > 0 be a constant and let

$$\int_0^{\infty} |u'(p)|^2 p^{1+r} K(p)\,dp < \infty$$

Then it follows that

$$\int_0^{\infty} |u'(p)|^2 p^{1+r} K(p)\,dp \geqq r \int_0^{\infty} |u(p)|^2 p^{r} K'(p)\,dp$$

We call the last inequality Redheffer's Inequality. A corollary of Redheffer's Theorem is the following result, which we will have occasion to use.

Corollary. Suppose u, K, r are as in Redheffer's Theorem, except that we do not assume

$$\lim_{p\to 0_+} K(p) = 0$$

Assume that there is a function K_1 which does fulfil all the requirements of Redheffer's Theorem and that, in addition, we have

$$K(p) \geqq K_1(p) \geqq 0 \quad \text{and} \quad K_1'(p) \geqq K'(p) \geqq 0$$

Then

$$\int_0^\infty |u'(p)|^2 p^{1+r} K(p)\,dp \geqq r\int_0^\infty |u(p)|^2 p^r K'(p)\,dp$$

Proof

$$\int_0^\infty |u'(p)|^2 p^{1+r} K(p)\,dp \geqq \int_0^\infty |u'(p)|^2 p^{1+r} K_1(p)\,dp$$

Now apply Redheffer's Inequality, and we obtain

$$\int_0^\infty |u'(p)|^2 p^{1+r} K(p)\,dp \geqq r\int_0^\infty |u(p)|^2 p^r K_1'(p)\,dp$$

$$\geqq r\int_0^\infty |u(p)|^2 p^r K'(p)\,dp$$

The last inequality follows from the fact $K_1'(p) \geqq K'(p) \geqq 0$ by assumption.

This proves the corollary to Redheffer's Theorem.

Having prepared the ground, we may begin the work.

Proposition 3.4.1. Let U be a strongly continuous representation of $P_+^\uparrow(3+1)$, in a Hilbert space $\mathcal{K}$, which corresponds to mass zero and non-zero discrete spin. If F is a cocycle function with

$$\psi_F(g) = U_g F - F \in \mathcal{K} \text{ for all } g \in P_+^\uparrow(3+1)$$

then $F \in \mathcal{K}$, and hence the cohomology is trivial.

Proof

Given ψ_F, we can find ψ_F' so that $\psi_F - \psi_F'$ is a coboundary and ψ_F' is analytic at the identity, with values in $\mathcal{K}_\omega$, the analytic vectors of U. Now take ψ_F', and find ψ_F'' which vanishes on the compact subgroup, SO(3), of $P_+^\uparrow(3+1)$. Using the formulae of Proposition 2.2.1 and Proposition 2.4.1, we see that ψ_F'' is analytic at the identity and vanishes on the subgroup SO(3) of $P_+^\uparrow(3+1)$. Hence we may assume that F is such that ψ_F already satisfies the requirements.

It is shown in [4] and [13] that we may realize $\mathcal{K}$ as $L^2(\mathbb{R}^3, \frac{dp}{|\underline{p}|})$, and in this case the generators of the rotation group are given by

$$J_1 = -i(\underline{p} \times \nabla)_1 + S$$

$$J_2 = -i(\underline{p} \times \nabla)_2 + \frac{p_2}{p+p_1} S$$

$$J_3 = -i(\underline{p} \times \nabla)_3 + \frac{p_3}{p+p_1} S$$

where $p = |\underline{p}|$ and S is a number. Since we assume non-zero discrete spin, we have that

$$S \in \{n, \frac{2n+1}{2}: n \in \mathbb{Z}, n \neq 0\}$$

Since ψ_F vanishes on the compact subgroup of rotations, we obtain

$$J_1F = J_2F = J_3F = 0$$

so that $$\underline{J}F = \begin{pmatrix} J_1F \\ J_2F \\ J_3F \end{pmatrix} = 0$$

A small calculation shows that $\underline{p}.\underline{J}F = pS\underline{F} = 0$. Since $S \neq 0$, it follows that $pF = 0$ and this means that

$$F(\underline{p}) = \kappa\delta^3(\underline{p})$$

Applying J_1 to F, we have that $\kappa S\delta^3(\underline{p}) = 0$, and this means that $\kappa = 0$, since $S \neq 0$. It follows, then, that $F = 0$.

The statement of the result is now seen to hold, so that the Proposition has been established.

<u>Proposition 3.4.2.</u> Let F be a cocycle for the irreducible representation U of $P_+^\uparrow(3+1)$ which corresponds to mass zero and spin zero. Then $F \in \mathcal{K}$, the Hilbert space of the representation, and so the cocycles are all coboundaries.

Proof.

The preliminary remarks of Proposition 3.4.1 apply also to this case (they are, indeed, true for all connected Lie groups), and so we assume that the cocycle

$$\psi_F(g) = U_g F - F$$

obeys the necessary requirements.

The Hilbert space is $\mathcal{K} = L^2(\mathbb{R}^3, \frac{d^3\underline{p}}{|\underline{p}|})$ and the generators for rotations are

$$J_\beta = -i(\underline{p}\times\nabla)_\beta \qquad \beta = 1,2,3$$

We require that

$$J_\beta F = 0 \qquad \beta = 1,2,3$$

From this it follows that F depends only upon $p = |\underline{p}|$.

The generators for space-time translations are the multiplication operators p_1, p_2, p_3, p, and the Lorentz boosts are given by

$$\mathcal{J}_\beta = ip\frac{\partial}{\partial p_\beta} \qquad \beta = 1,2,3$$

If X represents any one of these generators, we have that $X^n F \in \mathcal{K}_\omega$, the analytic vectors of U in $\mathcal{K}$, for $n \geqq 1$. In particular, we have

$$p^n F \in L^2(\mathbb{R}^3, \frac{d^3\underline{p}}{|\underline{p}|}) \quad \text{for} \quad n \geqq 1$$

and this implies that $\lim_{p\to\infty} F(p) = 0$.

Since F depends only on p, it follows that

$$\mathcal{J}_\beta F = ip_\beta \frac{dF}{dp} \qquad \beta = 1,2,3$$

Moreover, $\mathcal{J}_\beta F \in L^2(\mathbb{R}^3, \frac{d^3\underline{p}}{|\underline{p}|})$ for $\beta = 1,2,3$. Hence

$$\sum_{\beta=1}^{3}\int|\mathcal{Y}_\beta F(\underline{p})|^2 \frac{d^3\underline{p}}{|\underline{p}|} = \sum_{\beta=1}^{3}\int|\frac{dF}{dp}(p)|^2 p_\beta^2 \frac{d^3\underline{p}}{|\underline{p}|}$$

$$= \int|\frac{dF}{dp}(p)|^2|\underline{p}|^2\frac{d^3\underline{p}}{|\underline{p}|}$$

$$= 4\pi\int_0^\infty |F'(p)|^2 p^2.p.dp$$

Now apply Redheffer's Theorem with $K(p) = p$. All the requirements are fulfilled, as can easily be seen, and we obtain, with $r = 1$,

$$\int_0^\infty |F'(p)|^2p^2.p.dp \geqq \int_0^\infty |F(p)|^2 p dp = \int_0^\infty |F(p)|^2p^2 \frac{dp}{p}$$

It now follows that

$$\sum_{\beta=1}^{3}\int|\mathcal{Y}_\beta F(\underline{p})|^2 \frac{d^3\underline{p}}{|\underline{p}|} \geqq \int|F(\underline{p})|^2 \frac{d^3\underline{p}}{|\underline{p}|}$$

Since the left-hand side is finite, so is the right-hand side, and it follows that $F \in L^2(\mathbb{R}^3, \frac{d^3\underline{p}}{|\underline{p}|})$.

We see quite readily, now, that the Proposition has been established.

Now we consider the case of the continuous spin representations.

Proposition 3.4.3. Let U be a strongly continuous unitary representation of $P_+^\uparrow(3+1)$ on a Hilbert space $\mathcal{K}$, which corresponds to mass zero and continuous spin. Then, if F is a cocycle function with

$$\psi_F(g) = U_gF - F \in \mathcal{K}$$

it follows that $F \in \mathcal{K}$ and hence all cocycles are coboundaries.

Proof.

Again we assume that F is such that ψ_F has all the properties we need.

The space $\mathcal{K}$ can be realized as $L^2(\mathbb{R}^3; \ell^2; \frac{d^3\underline{p}}{|\underline{p}|})$, where $\ell^2 = \{(a_n): n \in \mathbb{Z},\ a_n \in \mathbb{C} \text{ and } \sum_{n=-\infty}^{\infty}|a_n|^2 < \infty\}$. The inner product is given by

$$(f;h) = \sum_{n=-\infty}^{\infty}\int \overline{f(\underline{p},n)}h(\underline{p},n)\frac{d^3\underline{p}}{|\underline{p}|}$$

In this space, the generators for the rotations are

$$J_1 = -i(\underline{p}\times\nabla)_1 + S$$

$$J_2 = -i(\underline{p}\times\nabla)_2 + \frac{p_2}{p+p_1} S$$

$$J_3 = -i(\underline{p}\times\nabla)_2 + \frac{p_3}{p+p_1} S$$

The operator S is a matrix operator, operating in ℓ^2 i.e. on the variable n. S is given by its matrix elements

$$S_{nn'} = (n+\alpha)\delta_{nn'}$$

and α is a number, with

$\alpha = 0$ for a single-valued representation

$\alpha = \frac{1}{2}$ for a double-valued representation.

The Lorentz boosts are of the form

$$\mathcal{J}_\beta = ip\nabla_\beta + (\underline{p} \times \underline{\Theta})_\beta$$

where $\beta = 1,2,3$, and $\underline{\Theta}$ is a vector expression which acts on the ℓ^2-variables, and contains S, and T_1 and T_2, which are the generators of translations of the two-dimensional Euclidean group. The actual action of T_1 and T_2 is not relevant to the analysis which we present, so we omit expressing it. Details can be found in [13].

We obtain, as usual, that

$$J_\beta F = 0 \quad \text{i.e.} \quad (J_\beta F)_n = 0, \; n \in \mathbb{Z}, \quad \beta = 1,2,3$$

and $\mathcal{J}_\beta F \in \mathcal{K}$ i.e. $(\mathcal{J}_\beta F)_n \in \mathcal{K}, \; n \in \mathbb{Z}, \quad \beta = 1,2,3$

It follows that $\underline{p}.\underline{J}F = 0$ and this implies that

$$p\,SF = 0 \quad \text{i.e.} \quad p(SF)_n = 0, \; n \in \mathbb{Z}$$

Now, if $\alpha = \frac{1}{2}$, this means that

$$p \sum_{n'=-\infty}^{\infty} (n+\tfrac{1}{2})\delta_{nn'}F_{n'} = 0 \qquad n \in \mathbb{Z}$$

from which we obtain the result

$$p(n+\tfrac{1}{2})F_n = 0 \qquad n \in \mathbb{Z}.$$

Since $n+\frac{1}{2} \neq 0$ when $n \in \mathbb{Z}$, it follows that

$$pF_n = 0$$

and, thus, that $F_n(\underline{p}) = \kappa_n \overset{3}{\delta}(\underline{p})$ with $\kappa_n \in \mathbb{C}$.

Applying J_1 to F, we obtain that

$$\kappa_n(n+\tfrac{1}{2})\overset{3}{\delta}(\underline{p}) = 0$$

and, since $n+\frac{1}{2} \neq 0$ for $n \in \mathbb{Z}$, it follows that $\kappa_n = 0$, $n \in \mathbb{Z}$. This means that $F_n = 0$ and hence $F = (F_n) = 0$.

Now assume $\alpha = 0$, and in this case we obtain from

$$p(SF)_n = 0 \qquad n \in \mathbb{Z}$$

the result

$$pnF_n = 0 \qquad n \in \mathbb{Z}$$

If $n \neq 0$, we obtain $F_n = \kappa_n \overset{3}{\delta}(\underline{p})$ with $\kappa_n \in \mathbb{C}$, and applying J_1 to F we obtain

$$(J_1F)_n = -i(\underline{p}\times\nabla)_1 F_n + (SF)_n = 0 \qquad n \in \mathbb{Z}$$

For $n \neq 0$, we have $-i(\underline{p}\times\nabla)_1 F_n = 0$ so that

$$\kappa_n n\overset{3}{\delta}(\underline{p}) = 0 \quad \text{for} \quad n \neq 0$$

and this means that $\kappa_n = 0$ when $n \neq 0$. However, when $n = 0$ we obtain the result that $(SF)_0 = 0$ and this implies that $(J_\beta F)_0 = -i(\underline{p}\times\nabla)_\beta F_\beta = 0$ for $\beta = 1,2,3$. Therefore F_0 can only depend upon $p = |\underline{p}|$. We write

$$F(\underline{p}) = u(p)(\delta_{0n})$$

i.e. $F(\underline{p})$ is equal to a scalar function u, depending only upon $p = |\underline{p}|$, multiplied by the vector $(\delta_{0n}) \in \ell^2$, the vector with zero in all entries, except for the entry 1, where n = 0.

Since $\mathfrak{J}_\beta F \in \mathfrak{K}$ for $\beta = 1,2,3$, we obtain that

$$\hat{\underline{p}}.\underline{\mathfrak{J}}F \in \mathfrak{K}$$

where $\hat{\underline{p}} = \underline{p}/|\underline{p}|$. From the form of $\underline{\mathfrak{J}} = (\mathfrak{J}_\beta)$, it follows that $\hat{\underline{p}}.\underline{\mathfrak{J}}F = i(\underline{p}.\nabla)F$, and since $F(\underline{p}) = u(p)(\delta_{0n})$, we obtain

$$\hat{\underline{p}}.\underline{\mathfrak{J}}F = ip\,\frac{du}{dp}.(\delta_{0n})$$

The operator $\hat{\underline{p}}.\underline{\mathfrak{J}}$ is diagonal with respect to the ℓ^2 variables, and F consists of only one component, so we obtain

$$\|\hat{\underline{p}}.\underline{\mathfrak{J}}F\|^2 = \int |\hat{\underline{p}}.\underline{\mathfrak{J}}F|^2 \frac{d^3\underline{p}}{|\underline{p}|} = \int |u'(p)|^2 p^2 \frac{d^3\underline{p}}{|\underline{p}|}$$

Using Schwartz's inequality, we obtain the result

$$\int |\hat{\underline{p}}.\underline{\mathfrak{J}}F|^2 \frac{d^3\underline{p}}{|\underline{p}|} \leqq \sum_{\beta=1}^{3} \int |\mathfrak{J}_\beta F|^2 \frac{d^3\underline{p}}{|\underline{p}|}$$

since $\mathfrak{J}_\beta F \in \mathfrak{K}$ for $\beta = 1,2,3$. Further, we use Redheffer's theorem and obtain, as in the case of zero spin, that

$$\int |u'(p)|^2 p^2 \frac{d^3\underline{p}}{|\underline{p}|} \geqq \int |u(p)|^2 \frac{d^3\underline{p}}{|\underline{p}|} = \int |F(\underline{p})|^2 \frac{d^3\underline{p}}{|\underline{p}|}$$

Hence we obtain that, for the case of continuous spin,

$$\|F\|^2 \leqq \sum_{\beta=1}^{3} \|\mathfrak{J}_\beta F\|^2 < \infty$$

The proposition is now seen to be true.

This last Proposition completes the discussion of the massless cases. In each case, we have shown that the cocycle function is actually an element of the Hilbert space, so that the cocycle is always a coboundary. This means that, in each case, the cohomology is trivial. Therefore the only representations of displaced Fock type are the Fock representations, up to a unitary equivalence.

We now proceed to exhibit a case of highly non-trivial cohomology for a subgroup of the Poincaré group.

§5. Non-trivial Cohomology for a subgroup of $P_+^\uparrow(3+1)$

The subgroup which we study will be called G. It consists of

1) the space-time translations

2) the rotations about the p_3-axis

3) the boosts along the p_3-axis.

Of course, we choose a massless representation of $P_+^\uparrow(3+1)$, and the spin is discrete. This representation is restricted to G. The fact that G has only one Lorentz boost is highly significant. First we consider the spin-zero case, and then use this to give us answers for non-zero spin.

Let $J_n = \{\underline{p} \in \mathbb{R}^3 : \varepsilon_{n+1} \leqq p \leqq \varepsilon_n$ and $p^2 = p_1^2 + p_2^2 \leqq \varepsilon_{n+1}\}$, where $\{\varepsilon_n : n \in \mathbb{N}\}$ is a sequence defined by recursion, as follows:

$$\varepsilon_1 = 1 \text{ and } \varepsilon_{n+1} = e^{-n^2}\varepsilon_n \qquad (n > 1)$$

$\{J_n : n \in \mathbb{N}\}$ gives us a sequence of cylinders. All these cylinders have their axes on the p_3-axis.

Now define a sequence of functions $\{f_n : n \in \mathbb{N}\}$ as follows

$$f_n(\underline{p}) = 1 \quad \text{if} \quad \underline{p} \in J_n$$

$$f_n(\underline{p}) = 0 \quad \text{if} \quad \underline{p} \notin J_n \qquad \text{for } n \in \mathbb{N}.$$

With $\mathcal{K} = L^2(\mathbb{R}^3, \frac{d^3\underline{p}}{|\underline{p}|})$, we obtain $f_n \in \mathcal{K}$ for $n \in \mathbb{N}$. Therefore $\{\hat{f}_n : \hat{f}_n = f_n/\|f_n\|\}$ is an orthonormal sequence in $L^2(\mathbb{R}^3, \frac{d^3\underline{p}}{|\underline{p}|})$. The orthonormality arises from the disjunction of the J_n.

Define a function $\hat{f}$ as follows

$$\hat{f} := \sum_{n=1}^{\infty} \hat{f}_n$$

It is easy to see that $\|\hat{f}\| = \infty$. Moreover, $\hat{f}(\underline{p})$ is finite for each $\underline{p} \in \mathbb{R}^3$.

We write

$$(U_{(a,\Lambda)}\hat{f})(\underline{p}) = e^{i|\underline{p}|a_0}e^{-i\underline{p}.\underline{a}}\hat{f}(\underline{\Lambda^{-1}p})$$

for $(a,\Lambda) \in G$. Therefore, if R is a rotation about the p_3-axis, we obtain

$$(U_R\hat{f})(\underline{p}) = (U_{(0,R)}\hat{f})(\underline{p}) = \hat{f}(R^{-1}\underline{p}) = \hat{f}(\underline{p})$$

This is because the individual functions f_n, $n \in \mathbb{N}$, are all rotation-invariant. Hence

$$U_R\hat{f}-\hat{f} \in \mathcal{K}$$

for all rotations R. Therefore, $\hat{f}$ is a rotation cocycle.

Now consider the Lorentz boosts in G. These are the matrix family

$$\left\{ L = \begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

We write U_L for the representation of an element, L, of the boosts.

The boosts in G are a one-parameter subgroup of G, parametrized by $\lambda \in \mathbb{R}$. Therefore $\{U_L$: L is a boost in G$\}$ is a unitary representation of $\mathbb{R}$ in $\mathcal{K}$, and to show that $\hat{f}$ is a cocycle function for the boosts, we need only show

$$U_L\hat{f}-\hat{f} \in \mathcal{K}$$

for the parameter $\lambda \in (-\delta,\delta)$ where $\delta > 0$. This is the implication of Lemma 2.3.1. Moreover, we need only show that

$$U_L\hat{f}-\hat{f} \in \mathcal{K} \quad \text{for} \quad \lambda \in (-\delta,0]$$

Since, if $\psi(\lambda) = U_L\hat{f}-\hat{f} \in \mathcal{K}$, it follows that

$$\psi(\lambda) \in \mathcal{K} \text{ if and only if } \psi(-\lambda) \in \mathcal{K} .$$

If $\lambda \in (-\delta, 0]$ for some $\delta > 0$, then

$$\underline{L^{-1}p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \cosh\alpha + |\underline{p}| \sinh\alpha \end{pmatrix}$$

where $\alpha = -\lambda > 0$.

Now we calculate a condition which will prove that $\hat{f}$ is a cocycle function for the boosts. Indeed we have the following

$$||\hat{f} - U_L \hat{f}||^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (\hat{f}_n - U_1 \hat{f}_n ; \hat{f}_m - U_L \hat{f}_m)$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \{ (\hat{f}_n ; \hat{f}_m) + (\hat{f}_n ; \hat{f}_m) - (U_L \hat{f}_n ; \hat{f}_m) - (\hat{f}_n ; U_L \hat{f}_m) \}$$

We have exploited the fact that $\hat{f}_n \in \mathcal{K}$ for $n \in \mathbb{N}$. The functions $\hat{f}_n$ and $U_L \hat{f}_n$ are real, so we obtain

$$||\hat{f} - U_L \hat{f}||^2 = 2 \sum_{n=1}^{\infty} \{ 1 - \sum_{m=1}^{\infty} (\hat{f}_n ; U_L \hat{f}_m) \}$$

Our task is now to prove that the right hand side converges. A problem we encounter is the evaluation of the cross-terms $(\hat{f}_n ; U_L \hat{f}_m)$. If L is represented by a parameter λ, and $\lambda \in (-\delta, 0]$ for a sufficiently small $\delta > 0$, we might expect that the contribution to the sum

$$\sum_{m=1}^{\infty} (\hat{f}_n ; U_L \hat{f}_m)$$

is due only to the overlap between the cylinder J_n and a part of itself, and the overlap between J_n and a part of J_{n+1} which has been shifted. This can be arranged, as we now see.

If L is represented by $\lambda \in (-\delta, 0]$ and $\alpha = -\lambda > 0$ then

$$(U_L \hat{f}_n)(\underline{p}) = \hat{f}_n(\underline{L^{-1}p})$$

$$= \hat{f}_n(p_1, p_2, p_3 \cosh\alpha + |\underline{p}| \sinh\alpha)$$

Under the action of this L, the base of J_n, given by $\{\underline{p}: p_1^2+p_2^2 \leqq \varepsilon_{n+1}^2, p_3 = \varepsilon_{n+1}\}$ is shifted so that it becomes the set $\{\underline{p}: p_1^2+p_2^2 \leqq \varepsilon_{n+1}^2, p_3 = \varepsilon'_{n+1}\}$ and

$$\varepsilon'_{n+1} = \varepsilon_{n+1}\cosh\alpha + (\varepsilon_{n+1}^2+s^2)^{\frac{1}{2}}\sinh\alpha$$

where $0 \leqq s = (p_1^2+p_2^2)^{\frac{1}{2}} \leqq \varepsilon_{n+1}$.

Similarly, the base of J_{n+1}, given by $\{\underline{p}: p_1^2+p_2^2 \leqq \varepsilon_{n+2}^2, p_3 = \varepsilon_{n+2}\}$ is shifted so that it becomes the set $\{\underline{p}: p_1^2+p_2^2 \leqq \varepsilon_{n+2}^2, p_3 = \varepsilon'_{n+2}\}$ and

$$\varepsilon'_{n+2} = \varepsilon_{n+2}\cosh\alpha + (\varepsilon_{n+2}^2+t^2)^{\frac{1}{2}}\sinh\alpha$$

where $t = (p_1^2+p_2^2)^{\frac{1}{2}} \leqq \varepsilon_{n+2}$.

We would like $\delta > 0$ so that if $\alpha = \lambda$ and $\lambda \in (-\delta,0]$, then

$$\varepsilon'_{n+2} < \varepsilon_{n+1} \quad \text{and} \quad \varepsilon'_{n+1} < \varepsilon_n$$

for every $n \in \mathbb{N}$. Choosing $0 \leqq \delta < 0.3$, this is seen to be true (after some calculation, of course). Under these conditions, we obtain

$$\sum_{m=1}^{\infty}(\hat{f}_n;U_L\hat{f}_m) = 2\pi\int_0^{\varepsilon_{n+2}} sds\int_{\varepsilon_{n+1}}^{\varepsilon'_{n+L}} dp_3.(p_3^2+s^2)^{-\frac{1}{2}}.(\|f_n\|.\|f_{n+1}\|)^{-1}$$

$$+ 2\pi\int_0^{\varepsilon_{n+1}} sds\int_{\varepsilon'_{n+1}}^{\varepsilon_n} dp_3(p_3^2+s^2)^{-\frac{1}{2}}\|f_n\|^{-2}$$

$$> 1-2\pi\int_{\varepsilon_{n+2}}^{\varepsilon_{n+1}} sds\int_{\varepsilon_{n+1}}^{\varepsilon'_{n+1}} dp_3(p_3^2+s^2)^{-\frac{1}{2}}\|f_n\|^{-2}$$

This inequality arises from the fact that $\|f_{n+1}\| < \|f_n\|$ and so $\|f_n\|^{-1} < \|f_{n+1}\|^{-1}$, and a little arrangement. It now follows that

$$\|\hat{f}-U_L\hat{f}\|^2 < \sum_{n=1}^{\infty} 2\pi\int_{\varepsilon_{n+2}}^{\varepsilon_{n+1}} sds\int_{\varepsilon_{n+1}}^{\varepsilon'_{n+1}} dp_3(p^2+s^2)^{-\frac{1}{2}}\|f_n\|^{-2}$$

We obtain, after performing standard integration and some routine estimation,

$$\|f_n\|^2 \geqq \pi\varepsilon_{n+1}^2(n^2-0.5)$$

and

$$\int_{\varepsilon_{n+2}}^{\varepsilon_{n+1}} sds \int_{\varepsilon_{n+1}}^{\varepsilon'_{n+1}} dp_3 (p_3^2+s^2)^{-\frac{1}{2}} \leqq \varepsilon_{n+1} h(\alpha)$$

The last estimate depends upon taking a fine approximation, and is spoiled when the estimate on ε'_{n+1} is too crude. We take the estimate

$$\varepsilon'_{n+1} \leqq \varepsilon_{n+1} (\cosh\alpha + \sqrt{2}\sinh\alpha).$$

$h(\alpha)$ is a function of α, and is continuous in α, and $\lim_{\alpha\to 0} h(\alpha) = 0$, and $h(\alpha)$ is bounded on $\mathbb{R}$.

Combining all these estimates, we obtain

$$\|\hat{f}-U_L\hat{f}\|^2 < \sum_{n=1}^{\infty} \frac{2h(\alpha)}{(n^2-0.5)} < \infty$$

and this proves that $\hat{f}$ is a cocycle function for the Lorentz boosts.

Now we come to the space-time translations. We have

$$\begin{aligned}
\| |\underline{p}|\hat{f}\|^2 &= \sum_{n=1}^{\infty} \| |\underline{p}|\hat{f}_n\|^2 \\
&= \sum_{n=1}^{\infty} \left(\iint_{J_n} |\underline{p}|^2 \frac{d^3\underline{p}}{|\underline{p}|}\right) \cdot \|f_n\|^{-2} \\
&= \sum_{n=1}^{\infty} \left(\int_0^{\varepsilon_{n+1}} sds \int_{\varepsilon_{n+1}}^{\varepsilon_n} (p_3^2+s^2)^{\frac{1}{2}} dp_3\right) \|f_n\|^{-2} \\
&\leqq \sum_{n=1}^{\infty} \left(\int_0^{\varepsilon_{n+1}} sds\right) . (\varepsilon_n - \varepsilon_{n+1}) \|f_n\|^{-2} \\
&= \sum_{n=1}^{\infty} \frac{\varepsilon_{n+1}^2}{2} (\varepsilon_n - \varepsilon_{n+1}) \|f_n\|^{-2} \\
&\leqq \sum_{n=1}^{\infty} \frac{\varepsilon_{n+1}^2 (\varepsilon_n - \varepsilon_{n+1})}{2\varepsilon_{n+1}^2 (n^2-0.5)} = \sum_{n=1}^{\infty} \frac{(\varepsilon_n - \varepsilon_{n+1})}{2(n^2-0.5)} = c
\end{aligned}$$

where c is a constant number in $\mathbb{R}$. From this calculation the following estimates are obtained

$$\| |\underline{p}|^m \hat{f} \|^2 = \sum_{n=1}^{\infty} \| |\underline{p}|^m \hat{f} \|^2 = \sum_{n=1}^{\infty} \left(\int_{J_n} |\underline{p}|^{2m} \frac{d^3 p}{|\underline{p}|} \right) \cdot \| f_n \|^{-2}$$

and when $\underline{p} \in J_n$, for any $n \in \mathbb{N}$, $|\underline{p}|^{2m} \leqq |\underline{p}|^2$ for all $m \in \mathbb{N}$, so that we obtain

$$\| |\underline{p}|^m \hat{f} \|^2 \leqq \| |\underline{p}| \hat{f} \|^2 \leqq c \qquad \text{for all } m \in \mathbb{N}.$$

Moreover, we also have

$$\| p_j^m \hat{f} \|^2 \leqq \| |\underline{p}|^m \hat{f} \|^2 \leqq \| |\underline{p}| \hat{f} \|^2 \leqq c$$

for all $m \in \mathbb{N}$, and $j = 1,2,3$. It now follows from these estimates that

$$\| \hat{f} - e^{i\not{p}a} \hat{f} \| < \infty$$

where $\not{p} = (|\underline{p}|, \underline{p})$ is the generator of space-time translations, $a = (a_0, \underline{a})$ is any translation in $\mathbb{R}^4$, and $\not{p}a = |\underline{p}| a_0 - \underline{p}.\underline{a}$.

This establishes that $\hat{f}$ is a cocycle function for the space-time translations.

Any element of our group G can be written as $g = (a, LR)$ where $a \in \mathbb{R}^4$, L is a boost in the p_3-direction and R is a rotation about the p_3-axis. Then we have the following sequence of equalities

$$\begin{aligned} \hat{f} - U_g \hat{f} &= f - e^{i\not{p}a} U_L U_R \hat{f} \\ &= \hat{f} - e^{i\not{p}a} U_L \hat{f} \\ &= \hat{f} - e^{i\not{p}a} \hat{f} + e^{i\not{p}a} [\hat{f} - U_L \hat{f}] \end{aligned}$$

and these establish that $\hat{f}$ is a cocycle function for our group G.

Having exhibited a non-trivial cocycle function, and hence constructed a cocycle, for the case of spin equal to zero, we turn to a construction of a cocycle for spin different from zero.

We write $F_0(|\underline{p}|, \phi, \theta) = \hat{f}(\underline{p})$. F is regarded as the function $\hat{f}$ in polar coordinates. In these coordinates, the generator of rotations about the p_3-axis is, for spin S (not necessarily equal to zero)

$$J_3^S = -i(\underline{p}\times\nabla)_3 + S = -i\frac{\partial}{\partial\phi} + S$$

where S is a number. This is the discrete spin case, and was mentioned in Proposition 3.4.1. The representation space is still $L^2(\mathbb{R}^3, \frac{d^3\underline{p}}{|\underline{p}|})$.

Now define

$$F_S(|\underline{p}|,\phi,\theta) = e^{-i\phi S}F_0(|\underline{p}|,\phi,\theta)$$

Then we have

$$J_3^S F_S = -i\frac{\partial F_S}{\partial\phi} + SF_S = 0$$

because F_0 is actually independent of ϕ, the angle of rotation about the p_3-axis. It follows from this that

$$e^{i\phi J_3^S}F_S = F_S$$

where ϕ parametrises the rotations about the p_3-axis, and so

$$F_S - U_R^S F_S = 0$$

where U^S is the representation of G for spin S, and R is a rotation. Then we obtain the following

$$F_S - U_g^S F_S \in \mathcal{K} \qquad \text{for all } g \in G$$

The value of S can be any element of the set $\{n, \frac{2n+1}{2}: n \in \mathbb{Z}\}$. If $z \in \mathbb{C}$, then it follows that

$$F_S - zF_S \notin \mathcal{K} \quad \text{only if } z \neq 1.$$

Therefore $\{zF_S: z \in C, z \neq 0\}$ is a collection of cocycle functions such that any pair, corresponding to different values of $z \in C$, give inequivalent cocycles for our group G, for the given value of spin. When the spin is integer-valued, the cocycle functions give rise to displaced Fock representations of the C.C.R. which are inequivalent amongst themselves, and which are not equivalent to the Fock representation. This result is summed up in the following theorem.

Theorem 3.5.1 [27] Let G be the group defined in this section. Then

$$H^1(G,\mathcal{K}) \neq \{0\}$$

i.e. the cohomology is non-trivial. This is true for all values of discrete spin, and only for the massless representations.

When the spin is an integer, then there exist uncountable many inequivalent non-trivial displaced Fock representations of the Canonical Commutation Relations in which the group G is unitarily implemented by a projective unitary group representation on the Fock space.

Proof

Combine the calculations of section 5 together with Theorem 1.8.1 and Theorem 1.9.1. The implementing operators form a strongly continuous projective representation of G in the Fock space.

The function $\hat{f}$ is a version of a function given in [12].

§6. The Spectrum Condition

In this last section of chapter 3, we consider conditions under which the spectrum condition holds. It is known that, in the Fock representation, the spectrum condition holds, namely

$$H \geqq 0$$

and

$$H^2 \geqq \underline{P}^2$$

where H is the generator of time-translations, and $\underline{P}$ is the generator of space-translations.

As a preliminary result, we have the following lemma.

Lemma 3.6.1. Suppose F is a cocycle function which gives an analytic cocycle, and that $F \in L^2(\mathbb{R}^3, d^s\underline{p})$ s = 1,3. Then

$$\mathrm{Im}(F;\psi_t)$$

is finite, where

$$\psi_t = F - e^{i|\underline{p}|t}F \quad \text{or} \quad \psi_t = F - e^{-ip_\beta t}F, \quad \beta = 1,\dots,s$$

Proof

Suppose $\psi_t = F - e^{i|\underline{p}|t}F$ then we have

$$(F;\psi_t) = \int \bar{F}(\underline{p})F(\underline{p})(1-e^{i|\underline{p}|t})\frac{d^s\underline{p}}{|\underline{p}|}$$

$$= -\int \bar{F}(\underline{p})F(\underline{p})e^{i|\underline{p}|t/2}[e^{i|\underline{p}|t/2} - e^{-i|\underline{p}|t/2}]\frac{d^s\underline{p}}{|\underline{p}|}$$

$$= -i\int |F(\underline{p})|^2.e^{i|\underline{p}|t/2}\,2\sin(\frac{|\underline{p}|t}{2}).\frac{d^s\underline{p}}{|\underline{p}|}$$

Therefore

$$\mathrm{Im}(F;\psi_t) = -\int |F(\underline{p})|^2\,2\cos(\frac{|\underline{p}|t}{2})\sin(\frac{|\underline{p}|t}{2})\frac{d^s\underline{p}}{|\underline{p}|}$$

$$= -\int |F(\underline{p})|^2.\frac{\sin|\underline{p}|t}{|\underline{p}|}.d^s\underline{p}$$

It now follows that for each $t \in \mathbb{R}$, $\mathrm{Im}(F;\psi_t)$ exists if and only if $F \in L^2(\mathbb{R}^s, d^s\underline{p})$.

If $\psi_t = F - e^{-ip_\beta t}F$, we obtain

$$(F;\psi_t) = \int |F(\underline{p})|^2 e^{-ip_\beta t/2}[e^{ip_\beta t/2} - e^{-ip_\beta t/2}]\frac{d^s\underline{p}}{|\underline{p}|}$$

$$= i\int |F(\underline{p})|^2 e^{-ip_\beta t/2}.2.\sin(\frac{p_\beta t}{2})\frac{d^s\underline{p}}{|\underline{p}|}$$

so that we obtain

$$\mathrm{Im}(F;\psi_t) = \int |F(\underline{p})|^2\,2\cos(\frac{p_\beta t}{2}).\sin(\frac{p_\beta t}{2}).\frac{d^s\underline{p}}{|\underline{p}|}$$

$$= \int |F(\underline{p})|^2.\frac{p_\beta}{|\underline{p}|}.\frac{\sin(p_\beta t)}{p_\beta}.d^s\underline{p}$$

Again we obtain that $\mathrm{Im}(F;\psi_t)$ is finite for each $t \in \mathbb{R}$, if and only if $F \in L^2(\mathbb{R}^s, d^s\underline{p})$.

This proves the Lemma.

It is necessary to consider this, because we use the expression $\mathrm{Im}(F;\psi_t)$ in order to make the projective representation of the space-time translations into a unitary representation. To extend this unitarity to the rest of the group is not necessarily possible. In fact, it is not known whether it can be done. However, we restrict down to the translations of space-time, in order to determine the positivity of the Hamiltonian, considered as a quadratic form.

Lemma 3.6.2. Suppose F is as in Lemma 3.6.1, then if $g \in \mathbb{R}^{s+1}$, an element of the translation group, we have that

$$U_g^F = V_g^F \exp \frac{i}{2} \mathrm{Im}(F;\psi[g])$$

is a unitary representation of the translation group. Here,

$$\psi[g] \equiv F - M_g F$$

and $\mathrm{Im}(F;\psi[g])$ is finite for $g \in \mathbb{R}^{s+1}$, this result following from Lemma 3.6.1.

Proof

$$U_g^F U_h^F = V_{gh}^F \exp \frac{i}{2}[-\mathrm{Im}(\psi[g^{-1}];\psi[h]) + \mathrm{Im}(F;\psi[g]) + \mathrm{Im}(F;\psi[h])$$

$$= V_{gh}^F \exp \frac{i}{2} \mathrm{Im}(F;\psi[gh]) = U_{gh}^F$$

where $g,h \in \mathbb{R}^{s+1}$. The last line follows after some routine calculation, using the fact that $\mathrm{Im}(F;\psi[g])$ is finite for each $g \in \mathbb{R}^{s+1}$, and using the cocycle law.

It is not difficult to see that U_g^F implements the action of $g \in \mathbb{R}^{s+1}$ in W_F; indeed, we have

$$U_g^F W_F(f) U_{g^{-1}}^F = V_g^F W_F(f) V_{g^{-1}}^F$$

$$= W_F(U_g f)$$

since $\exp \frac{i}{2} \mathrm{Im}(F;\psi[g])$ is merely a numerical factor.

Now we turn to the calculation of the generators of the 1-parameter subgroups of $\mathbb{R}^{s+1}$ in the representation U_g^F. We have for each one-parameter subgroup

$$t \to U_t^F$$

the following

$$U_t^F = V_t W_0(-\psi[-t]) e^{\frac{i}{2} \operatorname{Im}(F;\psi[t])}$$

where $\psi[t] = F - M(e^{tX})F$ and $\{e^{tX}: t \in \mathbb{R}\}$ is the one-parameter group generated by the Lie algebra element X.

We take strong derivatives, at t = 0, on a suitable dense domain in Fock space. The actual construction of Fock space which we take is the usual one, on which we define annihilation and creation operators $a(\underline{p})$ and $a^*(\underline{p})$. Then we have

$$W_0(f) = \exp \frac{i}{\sqrt{2}} \overline{[a(f) + a^*(f)]}$$

where

$$a(f) = \int a(\underline{p}) \overline{f(\underline{p})} \frac{d^s\underline{p}}{\sqrt{\omega}}$$

and $$a^*(f) = \int a^*(\underline{p}) f(\underline{p}) \frac{d^s\underline{p}}{\sqrt{\omega}}$$

where $f \in L^2(\mathbb{R}^s, \frac{d^s\underline{p}}{\omega})$. The bar in $\overline{[a(f)+a^*(f)]}$ denotes operator closure.

Let us first consider the time-translation subgroup. In this case, the derivative $\frac{1}{i}\frac{d}{dt} U_t^F \Big|_{t=0}$ (defined, as remarked, on a dense domain by $\frac{1}{i}\frac{d}{dt} U_t^F \xi \Big|_{t=0}$ for some ξ) has three parts. The first part is the free Hamiltonian

$$H_0 = \int \omega(\underline{p}) a^*(\underline{p}) a(\underline{p}) d^s\underline{p}$$

H_0 is defined as a quadratic form in Fock space. We take this dense space of definition as $\mathfrak{M} \times \mathfrak{M}$ where $\mathfrak{M}$ is the dense subspace got from $\tilde{\mathfrak{D}}_0$ in the one-particle space.

The second part is got by differentiating $W_0(-\psi[-t])$. We obtain for this, bearing in mind the definition in terms of annihilation and creation operators,

$$-\frac{1}{\sqrt{2}}\left[\int a(\underline{p})\, i\omega\bar{F}(\underline{p})\frac{d^s\underline{p}}{\sqrt{\omega}} - \int a^*(\underline{p})\, i\omega F(\underline{p})\frac{d^s\underline{p}}{\sqrt{\omega}}\right]$$

$$= -\int a(\underline{p})\sqrt{\omega}\,.\,\frac{i\bar{F}(\underline{p})}{\sqrt{2}}d^s\underline{p} + \int a^*(\underline{p})\,.\sqrt{\omega}.i.\frac{F(\underline{p})}{\sqrt{2}}d^s\underline{p}$$

The last part is got by differentiating $e^{\frac{i}{2}\mathrm{Im}(F;\psi[t])}$ from which we obtain

$$-\frac{1}{2}\int |F(\underline{p})|^2 d^s\underline{p}$$

Gathering all these terms together, we obtain for H^F, the generator of the time-translations in the representation U^F of the space-time translations,

$$H^F = \int \omega(\underline{p})a^*(\underline{p})a(\underline{p})d^s\underline{p} + \int a^*(\underline{p})\sqrt{\omega}.i.\frac{F(\underline{p})}{\sqrt{2}}\,d^s\underline{p}$$

$$- \int a(\underline{p})\,.\sqrt{\omega}.i.\frac{\bar{F}(\underline{p})}{\sqrt{2}}\,d^s\underline{p} - \int \frac{\overline{F(\underline{p})}F(\underline{p})}{2}\,d^s p$$

$$= \int \omega(\underline{p})\,(a^*(\underline{p}) - \frac{i\,\overline{F(\underline{p})}}{\sqrt{2}.\sqrt{\omega}}(a(\underline{p}) + \frac{i\,\overline{F(\underline{p})}}{\sqrt{2}.\sqrt{\omega}})d^s\underline{p}$$

$$- \int |F(\underline{p})|^2 d^s\underline{p}$$

Writing $b(f) = a(\underline{p}) + \dfrac{i\,F(\underline{p})}{\sqrt{2}.\sqrt{\omega}}$, and so $b^*(\underline{p}) = a^*(\underline{p}) - \dfrac{i\,\overline{F(\underline{p})}}{\sqrt{2}.\sqrt{\omega}}$, we obtain

$$H^F = \int \omega(\underline{p})b^*(\underline{p})b(\underline{p})d^s\underline{p} - \int |F(\underline{p})|^2 d^s\underline{p}$$

Since $\int \omega(\underline{p})b^*(\underline{p})b(\underline{p})d^s\underline{p} \geqq 0$ on the dense domain of H_0, it follows that

$$H^F \geqq -\|F\|_2^2$$

Namely, H^F is bounded below provided $F \in L^2(\mathbb{R}^s, d^s\underline{p})$.

The condition $F \in L^2(\mathbb{R}^s, d^s\underline{p})$ derives from the assumption that F defines a one-particle state with finite energy i.e.

$$|(F;\ \omega F)| < \infty$$

This means $\int \overline{F(\underline{p})}\omega F(\underline{p})\frac{d^s\underline{p}}{|p|} = \int |F(\underline{p})|^2 d^s\underline{p} < \infty$.

We have shown the following result to be true.

Theorem 3.6.1. Suppose F is a cocycle function which defines a one-particle state with finite energy, then the spectrum of the Hamiltonian in the displaced Fock representation defined by F, is bounded below by

$$-\|F\|_2^2 = -\int |F(\underline{p})|^2 d^s\underline{p}$$

Similar calculations give us

$$P_\beta^F = \int p_\beta b^*(\underline{p})b(\underline{p})d^s\underline{p} = \int |F(\underline{p})|^2\frac{p_\beta d^s\underline{p}}{\omega}$$

for $\beta = 1,\ldots,s$. The integral

$$\int |F(\underline{p})|^2 p_\beta \frac{d^s\underline{p}}{\omega}$$

exists, since $\left|\frac{p_\beta}{\omega}\right| \leq 1$ for $\beta = 1,\ldots,s$ and we assume that $F \in L^2(\mathbb{R}^s,\ d^s\underline{p})$.

Let us "renormalize" the Hamiltonian H^F:

$$H_R^F := H^F + \|F\|_2^2$$

and let us "renormalize" the momenta

$$P_{R_\beta}^F = P_\beta^F - \int |F(\underline{p})|^2\frac{p_\beta d^s\underline{p}}{\omega}$$

Because the relevant integrals exist, the renormalizations correspond to unitary transformations of the Fock space onto itself. We obtain the following result.

Theorem 3.6.2. Suppose that F is a cocycle function which defines a one-particle state with finite energy. Then $(H_R^F,\ \underline{P}_R^F)$ has its spectrum in the forward light cone. The one-parameter groups $e^{itH_R^F}$ and $e^{itP_{R_\beta}^F}$

implement the time and space translations in the displaced Fock representation of F.

This concludes the present chapter. We have exhibited non-trivial and trivial cocycles. Moreover, in the last section, a condition on the cocycle function has been isolated, which ensures the validity of the spectrum condition.

At the end of Chapter 2, we claimed that the dimension of momentum space is an important factor in deciding whether or not we obtain non-trivial 1-cohomology. When we consider the full Poincaré group of the space, this is true. However, section 5 of this chapter forces us to modify this claim. In fact, it is the number of Lorentz generators which give us an indication of the absence or presence of cohomology - and hence of non-Fock displaced Fock representations. This is pointed out by a comparison of the construction in section 5, where only one of the Lorentz boosts was used, and the theorems of section 4, where all of the Lorentz boosts were used. It may be that a similar situation arises for Lie groups which have a structure similar to the Poincaré group. Namely, groups of the form

$$T \circledS N$$

where T is a normal abelian non-compact group, and N is a group with both compact and non-compact subgroups. In such cases, $T \circledS N$ would not have any 1-cohomology, but $T \circledS N_1$ where N_1 contains only one parameter of the non-compact subgroups. This is, however, only a speculative statement.

CHAPTER 4

Cohomology of Direct Products and the Hilbert-Schmidt Cohomology

§1 Introduction

In this chapter, we prove the triviality of the 1-cohomology of the Poincaré group $P_+^\uparrow(3+1)$ in the representation $V \otimes U$, where V and U are arbitrary irreducible representations of $P_+^\uparrow(3+1)$, neither belonging to the vanishing of four-momentum. $V \otimes U$ is the direct product representation i.e. the usual tensor product representation of $G \times G$ restricted to the diagonal subset of $G \times G$, namely the set $\{(g,h) \in G \times G: g = h\}$. It is because of this restriction that $V \otimes U$ is not irreducible. We therefore use the theorems on reduction to give us our answers.

The representation $V \otimes U$ is unitarily equivalent to a direct integral of irreducible representations. The irreducible representations which take part in the direct integral decomposition are dependent upon the representations V and U, but those corresponding to massless particles occur with measure equal to zero, despite the possibility that both V and U correspond to massless particles. This is shown by Manfred Schaaf in [18]. Before we list all possible cases, we begin with some preparatory results. First we have a result about unitarily equivalent representations.

Lemma 4.1.1. Let V and U be unitarily equivalent representations of a connected Lie group G, so that

$$V_g S = S U_g$$

for each $g \in G$ and S is a unitary map between the Hilbert spaces $\mathfrak{H}_V$ and $\mathfrak{H}_U$, upon which V and U act. Then

$$H^1(G,\mathfrak{H}_V) \cong H^1(G,\mathfrak{H}_U)$$

Proof

If $\psi \in Z^1(G, \mathcal{H}_U)$ then ψ_S, defined by $\psi_S(g) = S\psi(g)$, is an element of $Z^1(G, \mathcal{H}_V)$.

It now follows that we may as well assume that $V \otimes U$ is actually equal to the direct integral decomposition, because the above lemma says that unitarily equivalent representations have isomorphic cohomology groups.

We now have to solve another problem: to find the relation between cocycles for a direct integral decomposition and the cocycles for the irreducible representations which take part in the decomposition. To this end, we quote the following result [14], which solves the question.

Theorem 4.1.1. [14]. Let U be a continuous unitary representation of a connected Lie group G, and let

$$U_g = \int_\Omega^\oplus U_g^\alpha d\mu(\alpha)$$

where Ω is a standard Borel space and μ a standard Borel measure on Ω. Each U^α is an irreducible representation of G on a space $\mathcal{H}^\alpha$ so that

$$\mathcal{H} = \int_\Omega^\oplus \mathcal{H}^\alpha d\mu(\alpha)$$

Suppose $\psi: G \to \mathcal{H}$ is a continuous cocycle for U, then we can write

$$\psi(g) = \int_\Omega^\oplus \psi(\alpha;g)\,d\mu(\alpha)$$

where, for each fixed $\alpha \in \Omega$, $\psi(\alpha;\ ): G \to \mathcal{H}^\alpha$ is a (continuous) cocycle for U^α.

Of course, the cocycles $\psi(\alpha;\ )$, which correspond to representations carrying zero mass, do not occur in the analysis, as these representations occur with measure zero in the direct integral decomposition.

§2. Estimates on Cocycles

We will prove that for the representation $V \otimes U$, under the conditions indicated, of $P_+^\uparrow(3+1)$, the cocycles ψ are all coboundaries. To do this, we show that they are quasi-coboundaries, and then go inside the direct integral and prove a certain growth estimate on the cocycle functions, which then proves that the global cocycle function is an element of the Hilbert space, thus proving the triviality of the cocycle for $V \otimes U$.

Proposition 4.2.1. Suppose that F is a cocycle function for an irreducible representation belonging to mass $m > 0$. Of the Poincaré group $P_+^\uparrow(3+1)$. Further suppose

$$\psi_F(g) = F - U_g F$$

is analytic at the identity, and vanishes on SO(3). Then

$$\|F\|^2 \leq \sum_{\beta=1}^{\infty} \|\mathfrak{J}_\beta F\|^2$$

where $\{\mathfrak{J}_\beta : \beta = 1,2,3\}$ is the set of the Lorentz boost generators for the representations.

Proof

It follows from Proposition 2.4.3 that $F \in L^2(\mathbb{R}^3; \mathbb{C}^{2s+1}; \frac{d^3\underline{p}}{\sqrt{\underline{p}^2+m^2}})$, the representation space of U.

Since $\psi_F(k) = F - U_k F = 0$ for every element $k \in SO(3)$, we obtain

$$J_\beta F = 0 \qquad \beta = 1,2,3$$

where J_β is the rotation generator about the β-direction. Writing $\underline{J} = (J_1, J_2, J_3)$ we have (see [4]) the following form for $\underline{J}$:

$$\underline{J} = -i(\underline{p} \times \nabla) + S_3(p+p_3)^{-1}(p_1, p_2, p+p_3)$$

where we have $p = |\underline{p}|$. S_3 is the diagonal Hermitian spin operator corresponding to the spin about the third axis. This spin operator has at most one eigenvector $e_0 \in \mathbb{C}^{2s+1}$, such that

$$S_3 e_0 = 0$$

From the relation $\underline{J}\,F = 0$, we obtain the equations

$$-i(\underline{p} \times \nabla)_1 F_j + \frac{p_1}{p+p_3}\lambda_j F_j = 0$$

$$-i(\underline{p} \times \nabla)_2 F_j + \frac{p_2}{p+p_3}\lambda_j F_j = 0$$

$$-i(\underline{p} \times \nabla)_3 F_j + \lambda_j F_j = 0$$

where $\{\lambda_j\colon j = 1,\ldots,2s+1\}$ is the set of eigenvalues of S_3 and $F = (F_1,\ldots,F_{2s+1})$ is a vector of functions in $\mathbb{C}^{2s+1}$. For all the non-zero eigenvalues in $\{\lambda_j\colon j = 1,\ldots,2s+1\}$ the above equations imply $F_j = 0$ for all except at most one of the j's. To see this, we need only apply the method of Proposition 3.4.1, when $s \neq 0$. When $s = 0$, the above equations reduce to

$$-i(\underline{p} \times \nabla)F = 0.$$

Thus, we see that F is a vector with a non-zero component in one place only, and therefore we may write

$$F(\underline{p}) = u(p).e_0$$

u depends only upon $p = |\underline{p}|$, since we have

$$-i(\underline{p} \times \nabla)_\beta u(\underline{p}) = 0 \qquad \text{for } \beta = 1,2,3.$$

We will write $F(\underline{p}) = u(p)e_0$ for the cases $s = 0$ and $s \neq 0$, on the understanding that $e_0 = 1$ for $s = 0$.

In the same representation, the vector $\underline{\mathfrak{K}}$ of boosts, $\underline{\mathfrak{K}} = (\mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_3)$ can be written as

$$\underline{\mathfrak{K}} = ip_0\nabla + \underline{p} \times \underline{\textcircled{w}}$$

where $\underline{\textcircled{w}}$ is a vector expression containing the spin matrices. $p_0 = \sqrt{\underline{p}^2+m^2}$ is the energy of the state. From the assumed analyticity of the cocycle defined by F, we obtain

$$\mathfrak{Y}_\beta F \in L^2(\mathbb{R}^3;\ C^{2s+1};\ \frac{d^3\underline{p}}{\sqrt{\underline{p}^2+m^2}}$$

and writing $\hat{\underline{p}} = \underline{p}/p$, we obtain

$$\hat{\underline{p}}.\mathfrak{Y}F = (ip_0 \frac{du}{dp}).e_0$$

It follows from Schwartz's inequality that

$$||\hat{\underline{p}}.\mathfrak{Y}F||^2 \leqq \sum_{\beta=1}^{3} ||\mathfrak{Y}_\beta F||^2$$

Furthermore, the norm squared of $(ip_0 \frac{du}{dp}).e_0$ is equal to

$$\int |u'|^2 p^2 p\ dp\ d\sigma(\theta,\phi)$$

where $d\sigma(\theta,\phi)$ is the measure on the unit sphere in $\mathbb{R}^3$. We apply the Corollary to Redheffer's Theorem (see Chapter 3) with $p_0 = \sqrt{p^2+m^2} = K(p)$ and $K_1(p) = p$. All the requirements are fulfilled, and we obtain

$$\int |u'|^2 p_0 p^2 dp d\sigma(\theta,\phi) \geqq \int |u|^2 \frac{p^2 dp}{p_0} d\sigma(\theta,\phi)$$

$$= \int |F(\underline{p})|^2 \frac{d^3\underline{p}}{\sqrt{\underline{p}^2+m^2}} = ||F||^2$$

Combining all these calculations, we obtain

$$||F||^2 \leqq \sum_{\beta=1}^{3} ||\mathfrak{Y}_\beta F||^2$$

This is the promised result.

Proposition 4.2.2. Suppose that F is a cocycle function for an irreducible representation U of $P_+^\uparrow(3+1)$ corresponding to the mass parameter m satisfying $m^2 < 0$. Also suppose that F gives rise to a quasi-coboundary which is analytic at the identity and which vanishes on SO(3). Then

$$||F||^2 \leqq \sum_{\beta=1}^{3} ||\mathfrak{Y}_\beta F||^2$$

Proof

The proof is along the same lines as that of Proposition 4.2.1. The exceptions are that the Hilbert space is

$$L^2(\mathbb{R}^3;\ M;\ \frac{\theta(\underline{p}^2-\beta^2)\,d^3\underline{p}}{\sqrt{\underline{p}^2-\beta^2}})$$

where M is some Hilbert space, possibly finite dimensional, where $\beta^2 = -m^2 > 0$; Redheffer's inequality can be applied readily, without the need for the corollary to be used. The vector of rotation generators, $\underline{J}$, has exactly the same expression as in Proposition 4.2.1, and the vector of Lorentz boosts has the same form as before.

Apart from these qualifications, the argument is the same.

A useful result is that, for irreducible, space-like representations of $P_+^\uparrow(3+1)$, the 1-cohomology is trivial, since the only cocycles are quasi-coboundaries, and these are true coboundaries. This remark then completes the classification of cocycles for all irreducible representation of $P_+^\uparrow(3+1)$ which do not correspond to momentum equal to zero.

§3 The Reduction of V ⊗ U

We present here the results of Schaaf [18]. The representations V and U are assumed to be irreducible and not to correspond to vanishing four-momentum. ε_V and ε_U are the signs of the energy of V and U respectively, and we have $\varepsilon_V = \pm 1$, $\varepsilon_U = \pm 1$. The masses of V and U are written as m_V, m_U. We list the irreducible representations taking part in the decomposition of V ⊗ U by the mass parameter m. The analysis of Schaaf gives very fine results, but we do not need all the fineness, so we do not give anything other than a gross characterization of the representations which do occur in the direct integral decomposition.

We now have the following result.

Theorem 4.3.1. Suppose V ⊗ U is a direct product of irreducible representations of the Poincaré group $P_+^\uparrow(3+1)$, defined by any of the

Table 4.3.1

V	U	Types of irreducible representations taking part in the decomposition of $V \otimes U$
$m^2 > 0 \quad \varepsilon_V = 1$	$m_U^2 > 0, \quad \varepsilon_U = 1$	$m^2 > 0$
$m_V^2 > 0 \quad \varepsilon_V = 1$	$m_U = 0 \quad \varepsilon_U = 1$	$m^2 > 0$
$m_V^2 > 0 \quad \varepsilon_V = 1$	$m_U^2 < 0 \quad \varepsilon_U = 1$	$m^2 > 0$ and $m^2 < 0$
$m_V^2 > 0 \quad \varepsilon_V = 1$	$m_U = 0 \quad \varepsilon_U = -1$	$m^2 > 0$ and $m^2 < 0$
$m_V^2 > 0 \quad \varepsilon_V = 1$	$m_U^2 > 0 \quad \varepsilon_U = -1$	$m^2 > 0$ and $m^2 < 0$
$m_V = 0 \quad \varepsilon_V = 1$	$m_U = 0 \quad \varepsilon_U = 1$	$m^2 > 0$
$m_V = 0 \quad \varepsilon_V = 1$	$m_U^2 < 0 \quad \varepsilon_U = 1$	$m^2 > 0$ and $m^2 < 0$
$m_V = 0 \quad \varepsilon_V = 1$	$m_U = 0 \quad \varepsilon_U = -1$	$m^2 < 0$
$m_V^2 < 0 \quad \varepsilon_V = 1$	$m_U^2 < 0 \quad \varepsilon_U = 1$	$m^2 > 0$

combinations given in Table 4.3.1. Then the cocycles of $V \otimes U$ are all coboundaries.

<u>Proof</u>.

If ψ is any cocycle for $V \otimes U$, then we may write, as in Theorem 4.1.1

$$\psi(g) = \int_{\Omega}^{\oplus} \psi(\alpha;g)\,d\mu(\alpha)$$

If we assume analyticity at the identity, and vanishing on SO(3), for ψ, these properties are reflected in the $\psi(\alpha;\ )$ for almost all $\alpha \in \Omega$. The $\psi(\alpha;\ )$ are cocycles for unitary irreducible representations of $P_+^\uparrow(3+1)$ corresponding to $m^2 > 0$ and $m^2 < 0$ (the representations of mass $m = 0$ occur with μ-measure zero), and hence each $\psi(\alpha;\ )$ is a coboundary, as is seen in Propositions 4.2.1 and 4.2.2. So we write

$$\psi(\alpha;g) = F^\alpha - U_g^\alpha F^\alpha$$

for each $g \in P_+^\uparrow(3+1)$ and for almost all $\alpha \in \Omega$. Let us write

$$F = \int_{\Omega}^{\oplus} F^\alpha d\mu(\alpha)$$

Then we have

$$\psi(g) = F - V_g \otimes U_g F = \int_{\Omega}^{\oplus} (F^\alpha - U_g^\alpha F^\alpha)\,d\mu(\alpha)$$

and the integral is convergent. Having assumed differentiability of the cocycle ψ, we obtain

$$\mathfrak{J}_\beta F = \int_{\Omega}^{\oplus} \mathfrak{J}_\beta^\alpha F^\alpha d\mu(\alpha) \qquad \beta = 1,2,3$$

for the boosts $\mathfrak{J}_\beta$ of the representation $V \otimes U$ and the boosts $\mathfrak{J}_\beta^\alpha$ for U^α. The integral must converge, as we demand that $\mathfrak{J}_\beta F \in \mathcal{H}$, the Hilbert space of $V \otimes U$, for $\beta = 1,2,3$.

From Proposition 4.2.1 and Proposition 4.2.2, we obtain

$$||F^{\alpha}||^2 \leqq \sum_{\beta=1}^{3} ||\mathfrak{J}_{\beta}^{\alpha}F||^2 \qquad \text{for almost all } \alpha$$

so that

$$\int_{\Omega} ||F^{\alpha}||^2 d\mu(\alpha) \leqq \sum_{\beta=1}^{3} \int_{\Omega} ||\mathfrak{J}_{\beta}^{\alpha}F^{\alpha}||^2 d\mu(\alpha) = \sum_{\beta=1}^{3} ||\mathfrak{J}_{\beta}F||^2$$

and hence we have proved that $F \in \mathfrak{H}$ since the direct integral giving F is convergent.

This proves the stated result.

§5 Cocycles for $P_+^{\uparrow}(3+1)$ with values in the Hilbert-Schmidt Operators

Now we prove the triviality of cocycles

$$\psi_A(g) = V_g A V_g^{-1} - A \in B(\mathcal{K})_2$$

where A is either a linear or anti-linear operator in the Hilbert space $\mathcal{K}$, upon which the irreducible unitary representation V of $P_+^{\uparrow}(3+1)$ acts. $B(\mathcal{K})_2$ is the space of Hilbert-Schmidt operators on $\mathcal{K}$.

The space $\mathcal{K}$ is always of the form $\mathcal{K} = L^2(M;S;d\mu)$ where $M = \mathbb{R}^3$ for $P_+^{\uparrow}(3+1)$, and S is a complex Hilbert space (either finite dimensional or equal to ℓ^2). M has a measure μ defined on it, and μ is invariant under the group G which acts irreducibly on $\mathcal{K}$. Using the realization of $B(\mathcal{K})_2$ as $L^2(M \times M;\ S \times S;\ d\mu \otimes d\mu)$ it follows that the action $B \to V_g B V_g^{-1}$, for $B \in B(\mathcal{K})_2$ and $g \in G$, can be written as one of the following actions

1) $\bar{V} \otimes V$ if B is linear, where $\bar{V} = CVC$ and C is a conjugation on $\mathcal{K}$

2) $V \otimes V$ if B is antilinear.

The following result now follows.

Proposition 4.5.1. The group of one cohomology classes for cocycles $\psi: G \to B(\mathcal{K})_2$, such that $\psi(g)$ is a linear (respectively, antilinear) operator, in the representation $V(\cdot)V^{-1}$ of G, is isomorphic to the group of one cohomology classes for the representation $\bar{V} \otimes V$ (respectively, $V \otimes V$).

Proof

Suppose $\psi(g)$ is a linear operator in $B(\mathcal{K})_2$, then we can define K_g by

$$(\psi(g)f;h) = K_g(Cf \otimes h)$$

for $f,h \in \mathcal{K}$ and C a conjugation on $\mathcal{K}$. K_g defines a kernel in $\mathcal{K} \otimes \mathcal{K}$. Moreover, K_g satisfies a cocycle condition. Indeed, we have

$$(\psi(gk)f;h) = (V_g\psi(k)V_g^{-1}f;h) + (\psi(g)f;h)$$

and this implies

$$K_{gk}(Cf \otimes h) = K_k(CV_g^{-1}f \otimes V_g^{-1}h) + K_g(Cf \otimes h)$$

$$= K_k(\bar{V}_g^{-1}Cf \otimes V_g^{-1}h) + K_g(Cf \otimes h)$$

where $\bar{V}_g = CV_gC$. Hence we obtain

$$K_{gk}(Cf \otimes h) = (\bar{V}_g \otimes V_g)K_k(Cf \otimes h) + K_g(Cf \otimes h)$$

We write $\bar{V}_g \otimes V_g$ for the dual action, on K_k, of the group.

Since K_g, for each $g \in G$, defines a Hilbert-Schmidt kernel in $\mathcal{K} \otimes \mathcal{K}$, we have $K_g \in \mathcal{K} \otimes \mathcal{K}$. Also K_g is weakly continuous and locally bounded, as a function of $g \in G$, from which it follows that K_g is strongly continuous. Here we use the result, due to Araki [1], that a weakly continuous, locally bounded cocycle in a separable Hilbert-space is strongly continuous.

For antilinear operators, we define K_g by

$$(\psi(g)f;h) = K_g(f \otimes h)$$

and we obtain

$$K_{gk}(f \otimes h) = (V_g \otimes V_g)K_k(f \otimes h) + K_g(f \otimes h)$$

Again, K_g turns out to be a strongly continuous cocycle, as a consequence of the same result due to Araki.

It is not hard to see that the cohomology classes of ψ are in a one to one correspondence with those of the corresponding K.

This establishes the result.

Theorem 4.5.1. The cocycles ψ: $P_+^\uparrow(3+1) \to B(\mathcal{K})_2$ for the action $V(\cdot)V^{-1}$, where V does not correspond to vanishing four-momentum, are all coboundaries.

Proof

Combine the result of Proposition 4.5.1 with Table 4.3.1 and Theorem 4.3.1. This proves the result.

The reason for considering Hilbert-Schmidt valued cohomology is that it arises in the construction of non-Fock (quasi-free) representations of the C.A.R. and the C.C.R. It follows from our results that non-Fock quasi-free representations of the C.A.R. and C.C.R. in which the full Poincaré group is to be implemented, must automatically be of Fock type. Furthermore, to obtain non-Fock representations we must abandon Poincaré covariance, and make do with a subgroup of the Poincaré group, or look at reducible representations.

In the works of Kraus and Streater [12] and Polley, Reents and Streater [16], this is the attitude taken.

It is worthwhile remarking that the results of Chapter 3 and Chapter 4 allow us to prove the following, general, result.

Theorem 4.5.2. If V is any unitary representation of $P_+^\uparrow(3+1)$ on a Hilbert space $\mathcal{K}$, and if V contains neither the trivial representation nor the representation corresponding to vanishing four-momentum, then all cocycles for V are true coboundaries.

Theorem 4.5.1 has been proved independently by W.J.M.A. Hochstenbach [11], using global methods due to G. uichardet [10]. However, the approach of G.uichardet can only establish existence proofs for cohomology, and cannot give us detailed estimates nor can it distinguish between quasi-coboundaries and algebraic cocycles. Indeed, the infinitesimal method of Pinczon and Simon is ideally suited for our needs.

This chapter is based upon work written in [25].

Conclusion

In this final section of the thesis, we take the opportunity to sum up the work presented, and to point out possible avenues of further work.

The motivating theme is the study of displaced Fock representations This leads to a study of the 1-cohomology of groups, with values in a Hilbert space. Solutions to the problem of "counting" displaced Fock representations are presented in terms of solutions to the 1-cohomology problem. A by-product is the application of the infinitesimal method of cohomology for Lie groups, as expounded by Pinczon and Simon [15]. Extra results on this method are presented, in order to provide a sharper basis for the calculations involved.

Connected to the displaced Fock representations of the C.C.R., are the symplectically transformed Fock representations, defined by Weyl operators

$$W_T(f) = W_0(Tf)$$

$f \in \mathcal{K}$, the one-particle space, T a symplectic operator, W_0 the Fock representation of the Weyl operators. These turn out to be associated to the condition

$$(V_g \otimes V_g)F - F \in \mathcal{K} \otimes \mathcal{K}$$

where F is a functional on a dense subset of $\mathcal{K} \otimes \mathcal{K}$. A similar condition arises from consideration of pure, gauge-invariant, quasi-free, non-Fock representations of the C.A.R. For this we obtain

$$(\bar{V}_g \otimes V_g)F - F \in \mathcal{K} \otimes \mathcal{K}$$

where F is related to the projection operator defining the type of CAR representation, and $\bar{V}_g = CV_gC$, where C obeys $C^2 = \mathbb{1}$, C is anti-linear.

These last two conditions are reminiscent of displaced Fock representations. A possible interesting avenue of further work is the connection of physical conditions on F (e.g. finite energy) with physical conditions on the representation of the C.C.R. or C.A.R. (e.g. spectrum condition). Furthermore, when does F correspond to

1) a sympletically transformed representation of the C.C.R.

2) a pure, gauge-invariant, quasi-free, non-Fock representation of the C.A.R.?

All the work done in this thesis is concerned with free and quasi-free fields in Minkowski space. A further point of interest is the relation of cocycles for $P_+^\uparrow(3+1)$ with cocycles, in Euclidean field theory, for $SO(4) \circledS \mathbb{R}^4$. It is known that there are uncountably many non-trivial cocycles for $SO(4) \circledS \mathbb{R}^4$, and there are none of these for $P_+^\uparrow(3+1)$, for physical representations. A question which arises naturally is: which cocycles for $SO(4) \circledS \mathbb{R}^4$ can one analytically continue over to Minkowski cocycles?

One other feature arises out of the work presented. In the case of the two-dimensional Poincaré group, infinitely many non-trivial cocycles exist, thus giving infinitely many inequivalent sectors for the field. This is tied up with the fact that the quantum field $\phi(x)$ is not an operator-valued distribution. However, if $f \in \mathcal{D}(\mathbb{R})$ and $g \in P_+^\uparrow(1+1)$ then

$$f \to \phi(U_g f - f)$$

defines an operator-valued distribution. Formally, we may write

$$\phi(U_g f - f) = V_g \phi(f) V_g^{-1} - \phi(f)$$

This leads us to conclude that the object

$$V_g \phi(x) V_g^{-1} - \phi(x)$$

defines an operator-valued distribution in $\mathcal{D}'_{op}$, whereas $\phi(x)$ does not. The result is

$$H^1(P_+^\uparrow(1+1), \mathcal{D}'_{op}) \neq \{0\}$$

We may then ask about the triviality, or non-triviality, of

$$H^1(P_+^\uparrow(s+1), \mathcal{D}'_{op}) \qquad \text{for } s = 2,3.$$

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