BREDON COHOMOLOGICAL FINITENESS CONDITIONS FOR GENERALISATIONS OF THOMPSON GROUPS

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ABSTRACT. We define a family of groups that generalises Thompson's groups T and G, and also those of Higman, Stein and Brin. For groups in this family we describe centralisers of finite subgroups and show, that for a given finite subgroup Q, there are finitely many conjugacy classes of finite subgroups isomorphic to Q. We consider a slightly weaker property, quasi-$\underline{F}_\infty$, to that of a group possessing a finite type model for the classifying space for proper actions $\underline{E}G$, and give criteria for the T versions of our groups to be of type quasi-$\underline{F}_\infty$. We also generalise some well-known properties of ordinary cohomology to Bredon cohomology.

1. INTRODUCTION

Thompson's groups F, T and G (also denoted V), which can be defined as certain homeomorphism groups of the unit interval, the circle and the Cantor-set, respectively, have received a large amount of attention in recent years. There are many interesting generalisations of these groups, such as the Higman-Thompson groups $F_{n,r}$, $T_{n,r}$, $G_{n,r}$ (recall that $T = T_{2,1}$ and $G = G_{2,1}$), the T- and G-groups defined by Stein [24] and the higher dimensional Thompson groups $sV = sG_{2,1}$ defined by Brin [4]. All these groups contain free abelian groups of infinite rank, are finitely presented and with the exception of sV for $s \geq 4$, are known to be of type FP_∞ [5, 24, 12, 9]. Furthermore, the G- and T-groups contain finite groups of arbitrarily large orders. In this paper we consider automorphism groups of certain Cantor algebras which include Higman-Thompson, Stein and Brin's groups.

As in the original exposition by Higman [10] and in Brown's proof [5] that $F_{n,r}$, $T_{n,r}$ and $G_{n,r}$ are of type FP_∞, we consider a Cantor algebra $U_r(\Sigma)$ on a so called valid set of relations Σ and define groups $G_r(\Sigma)$ as follows: the elements of $G_r(\Sigma)$ are bijections between certain subsets of $U_r(\Sigma)$, which we call admissible. One can show that these groups are finitely generated, see [18]. Provided that the relations in Σ are order preserving we can also define the groups $T_r(\Sigma)$, which are given by cyclic order preserving bijections. One can also define generalisations of $F_{n,r}$.

The admissible subsets of $U_r(\Sigma)$ form a poset, and the groups $T_r(\Sigma)$ and $G_r(\Sigma)$ act on the geometric realisation $|\mathfrak{A}_r(\Sigma)|$ of this poset (for the original Thompson-Higman algebras this was already used by Brown in [5]).

Let G be either $T_r(\Sigma)$ or $G_r(\Sigma)$. For every finite subgroup Q we consider the fixed point sets $\mathfrak{A}_r(\Sigma)^Q$. The Q-set structure of every admissible subset $Y \in \mathfrak{A}_r(\Sigma)^Q$ is determined by its decomposition into transitive Q-sets. We show (Theorem 4.3) that there are finitely many conjugacy classes in G of subgroups isomorphic to Q. Furthermore we show (Theorem 4.4) that there is an extension

$$K \rightarrowtail C_{G_r(\Sigma)}(Q) \twoheadrightarrow G_{r_1}(\Sigma) \times \ldots \times G_{r_t}(\Sigma)$$

with locally finite kernel, where the $r_1, ..., r_t$ are integers uniquely determined by Q. We also get the analogous result (Theorem 4.8) for the groups $T_r(\Sigma)$ (if defined) that, for a certain l also determined by Q, there is a central extension

$$Q \rightarrowtail C_{T_r(\Sigma)}(Q) \twoheadrightarrow T_l(\Sigma).$$

This generalises a result of Matucci [19, Theorem 7.1.5] for the original Thompson group T.

Recently a variant of the Eilenberg-Mac Lane space, the classifying space with respect to a family $\mathfrak{X}$ of subgroups, has been well researched. Let X be a G-CW-complex. X is said to be a model for $E_{\mathfrak{X}}G$, the classifying space with isotropy in the family $\mathfrak{X}$ if X^K is contractible for $K \in \mathfrak{X}$ and X^K is empty otherwise. The classifying space X for a family satisfies the following universal property: whenever there is a G-CW-complex Y with isotropy lying in the family $\mathfrak{X}$, there is a G-map $Y \to X$, which is unique up to G-homotopy. In particular, $E_{\mathfrak{X}}G$ is unique up to G-homotopy equivalence.

For the family $\mathfrak{F}$ of finite subgroups we denote $E_{\mathfrak{F}}G$ by $\underline{E}G$, the classifying space for proper actions. We say a group is of type $\underline{F}_\infty$ if it admits a finite type model for $\underline{E}G$. We show:

Theorem 3.1. $|\mathfrak{A}_r(\Sigma)|$ is a model for $\underline{E}G_r(\Sigma)$.

Obviously, this model has infinite dimension. Since these groups contain free abelian groups of infinite rank, they cannot possess any finite dimensional model. Exactly as ordinary classifying spaces yield free resolutions which can be used to define ordinary group cohomology, classifying spaces with isotropy in a family produce free resolutions in a functor category, which are used to define Bredon cohomology. We shall review properties of Bredon cohomology in Section 2. Many notions from ordinary cohomology have a Bredon analogue. For example, we say a group G is of type Bredon-FP$_\infty$ if there is a Bredon-projective resolution of the constant Bredon-module $\mathbb{Z}(-)$ by finitely generated Bredon-projective modules. The connection to classifying spaces and to ordinary cohomology is given by the following two results:

Theorem 1.1. [17, Theorem 0.1] *A group G has a finite type model for a classifying space with isotropy in a family if and only if the group is of type Bredon-FP$_\infty$ and there is a model for the classifying space with finite 2-skeleton.*

In particular we say a group is of type $\underline{FP}_\infty$ if it is of type Bredon-FP$_\infty$ for the family of finite subgroups.

Theorem 1.2. [16, Theorem 4.2] *A group G admits a finite type model for $\underline{E}G$ if and only G has finitely many conjugacy classes of finite subgroups and for each finite subgroup K of G the centraliser $C_G(K)$ is of type FP$_\infty$ and finitely presented.*

Equivalently, G admits a finite type model for $\underline{E}G$ if and only it is of type $\underline{FP}_\infty$ and centralisers of finite subgroups are finitely presented. The purpose of this paper is to study the possible finiteness conditions a model for $\underline{E}G$ for the groups $G_r(\Sigma)$ and $T_r(\Sigma)$ can satisfy. Since the groups we are considering do not have a bound on the orders of their finite subgroups, we need to weaken the condition on the number of conjugacy classes. We consider the property quasi-$\underline{FP}_\infty$, which has the same condition on the centralisers of finite subgroups as $\underline{FP}_\infty$ but just requires that for each finite subgroup Q of G, there are only finitely many conjugacy classes of subgroups isomorphic to Q. Note, that for groups with a bound on the orders of their finite subgroups both properties coincide. In [14] it was shown that there are examples of groups of type FP$_\infty$, which have a bound on the orders of the finite subgroups, yet are not of type $\underline{FP}_\infty$. These examples are virtually torsion free, admit a finite dimensional model for $\underline{E}G$ and can be constructed to have either

infinitely many conjugacy classes of finite subgroups, or to have centralisers of finite subgroups not of type FP_∞. There are a number of classes of groups of type FP_∞ admitting cocompact models for $\underline{E}G$ including Gromov hyperbolic groups [20], $Out(F_n)$ [25] or elementary amenable groups of type FP_∞ [13].
Using our results on centralisers and conjugacy classes of finite subgroups we show:

Theorem 7.2. $T_r(\Sigma)$ is of type quasi-$\underline{\mathrm{FP}}_\infty$ if and only if $T_l(\Sigma)$ is of type FP_∞ for any $1 \leq l \leq d$ such that $\gcd(l,d) \mid r$.

We also consider the geometric analogue, to be of type quasi-$\underline{\mathrm{F}}_\infty$, and the corresponding version of 7.2. We conjecture that similar results hold true for the groups $G_r(\Sigma)$.

The paper is structured as follows: In Section 2 we define the Cantor algebras and the corresponding generalisations of Thompson's groups G and T. We then use this Cantor algebra to build a model for $\underline{E}G$ in Section 3. In Section 4 we prove the results on centralisers and conjugacy classes of finite subgroups that will be used later.

In Section 5 we collect all necessary background on Bredon cohomology with respect to an arbitrary family, and on Bredon cohomological finiteness conditions for modules. We prove an analogue to the Bieri-Eckmann criterion for property FP_n for modules. In Section 6 we specialise to the case of the family of finite subgroups and define what it means for a group to be quasi-$\underline{\mathrm{FP}}_\infty$ and quasi-$\underline{\mathrm{F}}_\infty$. Finally, the main results are proven in Section 7.

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2. Generalisations of Thompson-Higman groups

As mentioned in the introduction, the generalised Thompson-Higman groups can be viewed as certain automorphisms groups of Cantor algebras. We shall begin by defining these algebras. We use the notation of [12], Section 2. In particular, we consider a finite set $\{1, \ldots, s\}$ whose elements are called colours. To each colour i we associate an integer $n_i > 1$ which is called its arity. We say that U is an Ω-algebra, if, for each colour i, the following operations (we let all operations act on the right) are defined in U (for detail, see [7] and [12]):

   i) One n_i-ary operation λ_i:

$$\lambda_i : U^{n_i} \to U.$$

   We call these operations ascending operations, or contractions.
   ii) n_i 1-ary operations $\alpha_i^1, \ldots, \alpha_i^{n_i}$:

$$\alpha_i^j : U \to U.$$

   We call these operations 1-ary descending operations.
We denote $\Omega = \{\lambda_i, \alpha_i^j\}_{i,j}$. For each colour i we also consider the map $\alpha_i : U \to U^{n_i}$ given by

$$v\alpha_i := (v\alpha_i^1, v\alpha_i^2, \ldots, v\alpha_i^{n_i})$$

for any $v \in U$. These maps are called descending operations, or expansions. For any subset Y of U, a simple expansion of colour i of Y is obtained by substituting some element $y \in Y$ by the n_i elements of the tuple $y\alpha_i$. A simple contraction of colour i of Y is the set obtained by substituting a certain collection of n_i distinct elements of Y, say $\{a_1, \ldots, a_{n_i}\}$, by $(a_1, \ldots, a_{n_i})\lambda_i$. We also use the term operation to refer to the effect of a simple expansion, respectively contraction on a set .

For any set X there is an Ω-algebra, free on X, which is called the Ω-word algebra on X and is denoted by $W_\Omega(X)$. An admissible subset $A \subseteq W_\Omega(X)$ is a subset that can be obtained after finitely many expansions or contractions from the set X.

Descending operations can be visualised by tree diagrams, see the following example with $X = \{x\}$, $s = 1$ and $n_1 = 2$:

$$
\begin{array}{c}
x \\
\alpha^1 \diagup\diagdown \alpha^2 \\
\alpha^1 \diagup\diagdown \alpha^2
\end{array}
$$
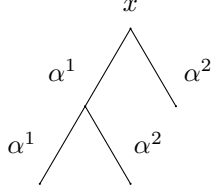
The set $A = \{x\alpha^1\alpha^1, x\alpha^1\alpha^2, x\alpha^2\}$ is an admissible subset. In pictures we often omit the maps and label the nodes by positive integers as follows:

$$
\begin{array}{c}
\\
3 \\
1 \quad 2
\end{array}
$$
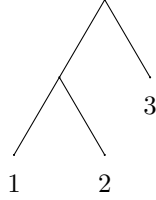
From now on we fix the set X and assume it is finite. We consider the variety of Ω-algebras satisfying a certain set of identities as follows:

Definition 2.1. Let Σ be the following set of laws in the alphabet X.

i) For any $u \in W_\Omega(X)$, any colour i, and any n_i-tuple $(u_1, \ldots, u_{n_i}) \in W_\Omega(X)^{n_i}$,

$$u\alpha_i\lambda_i = u,$$

$$(u_1, \ldots, u_{n_i})\lambda_i\alpha_i = (u_1, \ldots, u_{n_i}).$$

The set of all these relations is denoted Σ_1

ii) A certain set

$$\Sigma_2 = \bigcup_{1 \le i < i' \le s} \Sigma_2^{i,i'}$$

such that each $\Sigma_2^{i,i'}$ is either empty or consists of all the laws of the following form: Consider first i and fix a map $f : \{1, \ldots, n_i\} \to \{1, \ldots, s\}$. For each $1 \le j \le n_i$, we see $\alpha_i^j \alpha_{f(j)}$ as a set of length 2 sequences of descending operations and let $\Lambda_i = \cup_{j=1}^{n_i} \alpha_i^j \alpha_{f(j)}$. Do the same for i' (with a corresponding map f') to get $\Lambda_{i'}$ and now fix a bijection $\phi : \Lambda_i \to \Lambda_{i'}$. Then $\Sigma_2^{i,i'}$ is the set of laws

$$u\nu = u\phi(\nu) \quad \nu \in \Lambda_i, u \in W_\Omega(X).$$

(Note that by an abuse of notation we omit the $u \in W_\Omega(X)$ when we specify Σ_2 in the examples below).

When factoring out the fully invariant congruence $\mathfrak{q}$ generated by Σ, we obtain an Ω-algebra $W_\Omega(X)/\mathfrak{q}$ satisfying the identities in Σ. For detail of the construction the reader is referred to [12, Section 2].

Definition 2.2. Let $r = |X|$ and Σ as in Definition 2.1. Then the algebra $W_\Omega(X)/\mathfrak{q} = U_r(\Sigma)$ is called a Cantor-Algebra.

Moreover, there is an epimorphism of Ω-algebras

$$W_\Omega(X) \twoheadrightarrow U_r(\Sigma)$$

$$A \mapsto \bar{A}.$$

As in [12] we say that Σ is *valid* if for any admissible $Y \subseteq W_\Omega(X)$, we have $|Y| = |\bar{Y}|$. This condition implies that $U_r(\Sigma)$ is a free object on X in the class of those Ω-algebras which satisfy the identities Σ above.

If the set Σ used to define $U_r(\Sigma)$ is valid, we also say that $U_r(\Sigma)$ is valid.

Example 2.3. Higman [10] defined an algebra $V_{n,r}$ with $|X| = r$, $s = 1$ and arity n as above with Σ_2 being empty. This algebra, which we call Higman algebra, is used in the original construction of the Higman-Thompson-groups $G_{n,r}$. For detail see also [5]. In particular, these algebras are valid [10, Section 2].

Example 2.4. Higman's construction for arity $n = 2$ can be generalised as follows [12, Section 2]: Let $s \geq 1$ and $n_i = 2$ for all $1 \leq i \leq s$. Hence we consider the set of s colours $\{1, \ldots, s\}$, all of which have arity 2, together with the relations: $\Sigma := \Sigma_1 \cup \Sigma_2$ with

$$\Sigma_2 := \{\alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l \mid 1 \leq i \neq j \leq s; l, t = 1, 2\}.$$

Then Σ is valid (see [12] Lemma 2.9).

Furthermore one can also consider s colours, all of arity $n_i = n$, for all $1 \leq i \leq s$. Let

$$\Sigma_2 := \{\alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l \mid 1 \leq i \neq j \leq s; 1 \leq l, t \leq n\}.$$

Using the same arguments as in [12, Section 2] one can show that the Σ obtained in this way is also valid.

We call the resulting Cantor algebras $U_r(\Sigma)$ Brin algebras.

The following tree-diagram visualises the relations in Σ_2. Here $r = 1, s = 2$ and $n = 2$. We express an expansion of colour 1 with dotted lines and an expansion of colour 2 by solid lines. The leaves with the same label are identified.
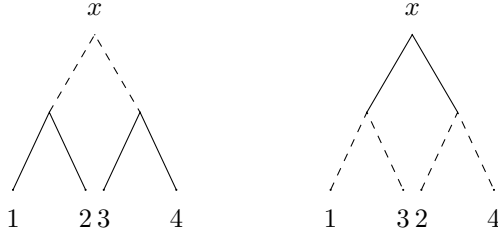
Definition 2.5. Let Σ be valid and consider $Y, Z \subseteq U_r(\Sigma)$. If Z can be obtained from Y by a finite number of simple expansions then we say that Z is a *descendant* of Y and denote

$$Y \leq Z.$$

Conversely, Y is called an *ascendant* of Z and can be obtained after a finite number of simple contractions. Note that this implies that if either of the sets Y or Z is admissible, then so is the other. In fact, the set of admissible subsets of $U_r(\Sigma)$ is a poset with respect to the partial order $\leq$. This poset is denoted by $\mathfrak{A}_r(\Sigma)$.

It is easy to prove that any admissible subset is a basis of $U_r(\Sigma)$ (see [12] Lemma 2.5).

Remark 2.6. Let Σ be valid and assume that we have s colours of arities $\{n_1, \ldots, n_s\}$. Let r be a positive integer. Observe that the cardinality of any admissible subset of $U_r(\Sigma)$ must be of the form $m \equiv r \bmod d$ for

$$d := \gcd\{n_i - 1 \mid i = 1, \ldots, s\}.$$

Moreover, for any $m \equiv r \bmod d$, there is some admissible subset of cardinality m. And as admissible subsets are bases, we get $U_r(\Sigma) = U_m(\Sigma)$.

Definition 2.7. Let B, C be admissible subsets of $U_r(\Sigma)$. We say that T is the unique *least upper bound* of B and C if $B \leq T$, $C \leq T$ and for all admissible sets S such that $B \leq S$ and $C \leq S$ we have $T \leq S$.
We say, by abusing notation a little, that $U_r(\Sigma)$ is *bounded* if for all admissible subsets B, C such that there is some admissible A with $A \leq B, C$ there is a unique least upper bound of B and C.

One can also define greatest lower bounds, but this places a stronger restriction on the algebra, see [12]. Moreover, note that a priori we require the existence of an upper bound only when our sets have a lower bound (A) but this turns out to be not too restrictive:

Lemma 2.8. *Let $U_r(\Sigma)$ be valid and bounded. Then any two admissible subsets have some (possibly not unique) common upper bound.*

Proof. Use the same proof as in [12, Proposition 3.4]. □

Example 2.9. The Brin algebras defined in Example 2.4 are valid and bounded. The existence of a unique least upper bound for $n = 2$ is shown in [12, Lemma 3.2]. The general case is analogous.

Example 2.10. Let $P \subseteq \mathbb{Q}_{>0}$ be a finitely generated multiplicative group. Then by a result of Brown, see [24, Proposition 1.1], P has a basis of the form $\{n_1, \ldots, n_s\}$ with all $n_i \geq 0$ $(i = 1, ..., s)$. Now consider Ω-algebras on s colours of arities $\{n_1, \ldots, n_s\}$ and let $\Sigma = \Sigma_1 \cup \Sigma_2$ with Σ_2 the set of identities given by the following order preserving identification:

$$\{\alpha_i^1 \alpha_j^1, \ldots, \alpha_i^1 \alpha_j^{n_j}, \alpha_i^2 \alpha_j^1, \ldots, \alpha_i^2 \alpha_j^{n_j}, \ldots, \alpha_i^{n_i} \alpha_j^1, \ldots, \alpha_i^{n_i} \alpha_j^{n_j}\} =$$

$$\{\alpha_j^1 \alpha_i^1, \ldots, \alpha_j^1 \alpha_i^{n_i}, \alpha_j^2 \alpha_i^1, \ldots, \alpha_j^2 \alpha_i^{n_i}, \ldots, \alpha_j^{n_j} \alpha_i^1, \ldots, \alpha_j^{n_j} \alpha_i^{n_i}\},$$

where $i \neq j$ and $i, j \in \{1, ..., s\}$.
The Cantor algebras $U_r(\Sigma)$ thus obtained will be called Brown-Stein algebras.

Note that, as $\{n_1, \ldots, n_s\}$ is a basis for P, the n_i are all distinct. Hence, when visualising the identities in Σ_2 for the Brown-Stein algebra, it suffices to only use one colour, as the arity of an expansion already determines the colour. In the following example let $r = 1, s = 2, n_1 = 2$ and $n_2 = 3$.
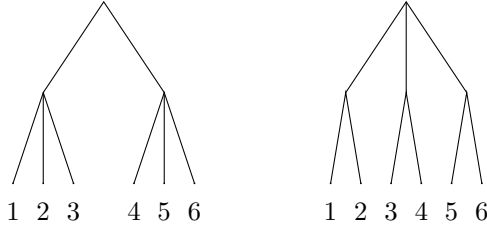
Lemma 2.11. *The Brown-Stein algebras are valid and bounded.*

Proof. This is Proposition 1.2 (due to K. Brown) in [24]. □

In fact, in [24] Lemma 2.8 is proven directly, i.e that any two admissible subsets have some common upper bound.

We can now define the generalised Thompson-Higman groups. Recall, that in a valid Cantor algebra $U_r(\Sigma)$, admissible subsets are bases.

Definition 2.12. Let $U_r(\Sigma)$ be a valid Cantor algebra. We define $G_r(\Sigma)$ to be the group of those Ω-algebra automorphisms of $U_r(\Sigma)$, which are induced by a map $V \to W$, where V and W are admissible subsets of the same cardinality.

Example 2.13. If $U_r(\Sigma)$ is a Higman algebra as in Example 2.3, we retrieve the original Higman-Thompson-groups $G_{n,r}$. Let $U_r(\Sigma)$ be a Brin algebra on s colours of arity 2 as in Example 2.4. Then the groups constructed are Brin's [4] generalisations sV of Thompson's group $V = G_{2,1}$. The description of sV as automorphism groups of a Cantor algebra can be found in [12]. Finally, the groups $G_r(\Sigma)$, when $U_r(\Sigma)$ is a Brown-Stein algebra as in Example 2.10, were considered in [24].

Remark 2.14. It is conceivable that in fact $G_r(\Sigma)$ equals the full group of Ω-algebra automorphisms of $U_r(\Sigma)$. This would follow if one could prove that any finite basis of $U_r(\Sigma)$ is an admissible subset (this is the case for the Higman algebra, see [10] Corollary 1).

We go back to the case of an arbitrary valid Cantor algebra $U_r(\Sigma)$ and assume that the set X is ordered. It can be seen that this order is inherited by certain subsets of $W_\Omega(X)$ including all admissible subsets, see for example [5] or [10]. If the relations in Σ_2 preserve that ordering, in the sense that the bijection ϕ in Definition 2.1 do, then we also have an inherited order on the admissible subsets of $U_r(\Sigma)$. We shall call this the induced ordering.

Definition 2.15. Suppose we have a Cantor algebra $U_r(\Sigma)$ where Σ preserves the induced ordering. We may define subgroups $F_r(\Sigma)$ and $T_r(\Sigma)$ of $G_r(\Sigma)$ as follows. We let $F_r(\Sigma)$ be the group of order preserving automorphisms between ordered admissible subsets of the same cardinality and $T_r(\Sigma)$ the group of cyclic order preserving automorphisms between ordered admissible subsets of the same cardinality.

Example 2.16. For $U_r(\Sigma)$ a Higman algebra of Example 2.3 the definition above yields the groups $F_{n,r}$ and $T_{n,r}$ as in [5]. Recall that Thompson's groups are $F = F_{2,1}$ and $T = T_{2,1}$.
Let $U_r(\Sigma)$ be a Brown-Stein algebra as in Example 2.10 In this case, Σ is order preserving, so we may define the groups $F_r(\Sigma)$ and $T_r(\Sigma)$, which are considered in [24].
Since Σ_2 in the definition of the Brin algebra of Example 2.4 is not order-preserving, it there is no obvious way to define the groups $F_r(\Sigma)$ or $T_r(\Sigma)$ for this algebra.

Remark 2.17. Note, that if definable, the groups $F_r(\Sigma)$ are torsion-free. In both cases mentioned in Example 2.16, the resulting groups $F_r(\Sigma)$ are known to be of type FP_∞ and finitely presented [5, 24].
Since, for torsion-free groups ordinary and Bredon cohomological finiteness conditions are identical, we will not consider these groups further.

3. A MODEL FOR $\underline{E}G$ FOR GENERALISED THOMPSON GROUPS

From now on we fix a valid Σ and a finite positive integer r. Also assume that the Cantor algebra $U_r(\Sigma)$ is bounded. In this section we give a quite elementary proof of the following result.

Theorem 3.1. *The geometric realisation of the poset of admissible subsets is a model for $\underline{E}G_r(\Sigma)$.*

We fix an admissible subset $X \subseteq U_r(\Sigma)$ of cardinality r.

Lemma 3.2. *For any finite $Q \leq G_r(\Sigma)$ there exists some admissible subset Z such that $ZQ = Z$. Moreover we may assume $X \leq Z$.*

Proof. For every $q \in Q$ choose a common upper bound T_q of X and Xq. Then put $Z_q := T_q q^{-1}$ and let Y be an upper bound of

$$\{Z_q \mid q \in Q\}.$$

Note that $X \leq Z_1 = T_1$ and for any $q \in Q$,

$$X \leq T_q = Z_q q \leq Yq.$$

Therefore we may choose Z the least upper bound of $\{Yq \mid q \in Q\}$. By definition of unique least upper bound we get $ZQ = Z$. $\square$

Proposition 3.3. *Any two elements in $\mathfrak{A}_r(\Sigma)^Q$ have an upper bound in $\mathfrak{A}_r(\Sigma)^Q$.*

Proof. Let $Y, Z \in \mathfrak{A}_r(\Sigma)^Q$. We begin by showing that there are admissible sets $Y_1, Z_1 \in \mathfrak{A}_r(\Sigma)^Q$ such that Y_1 is an upper bound of X and Y and Z_1 is an upper bound of X and Z. It suffices to prove the existence of Y_1. Take an upper bound $Y_2 \in \mathfrak{A}_r(\Sigma)$ of X and Y and consider

$$\{Y_2 q^{-1} \mid q \in Q\}.$$

Let $Y_3 \in \mathfrak{A}_r(\Sigma)$ be an upper bound of this set. Then, for any $q \in Q$,

$$Y_2 \leq Y_3 q.$$

Therefore $X \leq Y_3 q$. This implies that we may choose Y_1 to be the least upper bound of

$$\{Y_3 q \mid q \in Q\}.$$

Clearly, $Y, X \leq Y_1$. Again, the definition of least upper bound implies that $Y_1 \in \mathfrak{A}_r(\Sigma)^Q$.

Now, let T be the least upper bound of Y_1 and Z_1. Then for any $q \in Q$

$$Y_1 = Y_1 q \leq Tq,$$
$$Z_1 = Z_1 q \leq Tq$$

so we get $T \in \mathfrak{A}_r(\Sigma)^Q$. $\square$

Proof. (of Theorem 3.1) Lemmas 3.2 and 3.3 imply that for any finite subgroup $Q \leq G_r(\Sigma)$ the poset $\mathfrak{A}_r(\Sigma)^Q$ is non-empty and directed, thus $|\mathfrak{A}_r(\Sigma)|^Q = |\mathfrak{A}_r(\Sigma)^Q| \simeq *$. Moreover for any $V \in \mathfrak{A}_r(\Sigma)$,

$$\mathrm{Stab}_{G_r(\Sigma)}(V) = \{g \in G_r(\Sigma) \mid Vg = V\}$$

is contained in the group of permutations of the finite set V, thus it is finite. This implies that for any $H \leq G_r(\Sigma)$, $\mathfrak{A}_r(\Sigma)^H = \varnothing$ unless H is finite. $\square$

This model is not of finite type, but there is a filtration of $|\mathfrak{A}_r(\Sigma)^Q|$ by finite type subcomplexes, exactly as in the construction in [5, Theorem 4.17]:

Proposition 3.4. *For any finite $Q \leq G_r(\Sigma)$ there is a filtration of $|\mathfrak{A}_r(\Sigma)^Q|$*

$$\ldots \subset |\mathfrak{A}_r(\Sigma)^Q|_{h-1} \subset |\mathfrak{A}_r(\Sigma)^Q|_h \subset |\mathfrak{A}_r(\Sigma)^Q|_{h+1} \subset \ldots$$

such that each $|\mathfrak{A}_r(\Sigma)^Q|_h / C_{G_r(\Sigma)}(Q)$ is finite.

Proof. Let

$$|\mathfrak{A}_r(\Sigma)^Q|_h := \{Y \in \mathfrak{A}_r(\Sigma)^Q \mid |Y| \leq h\}.$$

Consider $Y, Z \in \mathfrak{A}_r(\Sigma)^Q$ with $|Y| = |Z|$ and isomorphic as Q-sets. This means that there is a Q-bijection

$$\sigma : Y \to Z.$$

Let $g \in G_r(\Sigma)$ be the element given by $yg = y\sigma$ for each $y \in Y$. Then for any $q \in Q$, $(yq)g = (yq)\sigma = y\sigma q = ygq$. This means that the commutator $[g, q]$ acts as the identity on the admissible set Y and therefore $[g, q] = 1$. Hence $g \in C_{G_r(\Sigma)}(Q)$. As for any $m \leq h$ there are finitely many possible Q-sets of cardinality m, the result follows. $\qquad\square$

Remark 3.5. Provided that Σ is order-preserving, Theorem 3.1 and Proposition 3.4 can be restated replacing $G_r(\Sigma)$ with $T_r(\Sigma)$.

Remark 3.6. The filtration of Proposition 3.4 is used by Brown [5, Theorem 4.17] with $Q = 1$, to show that the Higman-Thompson groups $G_{n,r}, T_{n,r}$ and $F_{n,r}$ are of type FP$_\infty$. The approach used by Brown is as follows: Fix an admissible subset Y. Show that if $|\mathfrak{A}_{n,r}|_{<Y}$ denotes the set of admissible subsets which are contractions of Y, then the connectivity of $|\mathfrak{A}_{n,r}|_{<Y}$ grows with the cardinality of Y. Then, show that this implies that the connectivity of the pair $(|\mathfrak{A}_{r,n}|_{h+1}, |\mathfrak{A}_{r,n}|_{h+1})$ tends to ∞, which in turn yields that $G_{n,r}, T_{n,r}$ and $F_{n,r}$ are all of type FP$_\infty$. Key to this approach is understanding the complex $|\mathfrak{A}_{n,r}|_{<Y}$. In the case of the Higman algebra, Brown shows ([5, Lemmas 4.18; 4.19]), that any two simple contractions Y_1, Y_2 of Y have a common lower bound if and only if the contracted vertices are disjoint, which allows him to show that $|\mathfrak{A}_{n,r}|_{<Y}$ is homotopy equivalent to a much simpler complex. However, this is no longer true if we work with a more general Cantor algebra $U_r(\Sigma)$: Consider for example a Brown-Stein algebra as in Example 2.10 with arities 2 and 3. Let Y be any admissible set with 6 elements labeled 1, 2, 3, 4, 5 and 6. Let Y_1 be the simple contraction of arity 2 of the elements 3 and 4 and Y_2 the simple contraction of arity 3 of the elements 1, 2 and 3. Then the sets of contracted vertices are not disjoint, however there is a common lower bound $Z \leq Y_1, Y_2$ as the picture before Lemma 2.11 shows. Stein used a different method to the one described here to prove that the groups of [24] are of type FP$_\infty$.

Similar problems were encountered when Kochloukova and the authors considered Brin's groups [12]. In general, the same difficulty applies to the groups $G_r(\Sigma)$, as well as to $T_r(\Sigma), F_r(\Sigma)$ where definable. It is conceivable, however, that Brown's approach can be applied more generally using an analogue of Brown's connectivity result, see for example [12] where it is used to show that Brin's groups for $r = 1$, $s = 2, 3$ are of type FP$_\infty$.

4. CENTRALISERS AND CONJUGACY CLASSES OF FINITE SUBGROUPS FOR $G_r(\Sigma)$ AND $T_r(\Sigma)$.

Let $Q \leq G_r(\Sigma)$ be a finite subgroup. In this section we give a more detailed analysis of the poset $\mathfrak{A}_r(\Sigma)^Q$ to describe $C_{G_r(\Sigma)}(Q)$ and the number of conjugacy classes of subgroups isomorphic to Q. In case $T_r(\Sigma)$ is defined, we also derive the corresponding results. This will be used later when we prove our main result on the cohomological finiteness properties of these groups.

Let $\{w_1, \ldots, w_t\}$ be the set of lengths of all the possible transitive permutation representations of Q. Any $Y \in \mathfrak{A}_r(\Sigma)^Q$ is a finite Q-set so it is determined by its decomposition into transitive Q-sets. If we take one of those sets and apply the operation α_i for a fixed colour i to each of its elements, we obtain a new admissible subset which is also fixed by Q. We say that this is a simple Q-expansion of Y. More explicitly, the admissible set obtained from Y is:

$$Y \smallsetminus \{yq \mid q \in Q\} \cup \{yq\alpha_i^j \mid q \in Q, 1 \leq j \leq n_i\}$$

for a certain $y \in Y$. We also use the term Q-expansion to refer to a chain of simple Q-expansions.

Conversely, if we choose n_i different orbits of the same type (i.e., corresponding to the same permutation representation) in Y, then we may contract them to a

single orbit (of the same type). We call this a simple Q-contraction. Simple Q-contractions are more complicated to handle than simple Q-expansions: we may contract an element of the first of the orbits with any of the elements on the others. Hence, even if the orbits to be contracted are determined, there are many possibilities to perform the explicit contraction. Note that the admissible subsets obtained this way will lie in $\mathfrak{A}_r(\Sigma)^Q$.

Large parts of the next three results can be found in [10, Section 6]. We shall, for the reader's convenience, recall the arguments.

Lemma 4.1. *Let* $Y, Z \in \mathfrak{A}_r(\Sigma)^Q$ *with* $Y < Z$ *and assume there is no admissible subset* $C \in \mathfrak{A}_r(\Sigma)^Q$ *with* $Y \leq C \leq Z$. *Then* Z *is a simple* Q-*expansion of* Y. *Hence* Y *is a simple* Q-*contraction of* Z.

Proof. We may choose a chain of simple expansions

$$Y < Y_1 < \ldots < Y_r < Z.$$

Let $w \in Y$ be the vertex expanded in the first simple expansion $Y < Y_1$ and $W \subseteq Y$ be the Q-orbit with $w \in W$. Assume also that this first expansion corresponds to the colour i. Then as Z contains certain descendants of $\{w\alpha_i\}$ and it is Q-invariant it must also contain the analogous descendants of $\{u\alpha_i \mid u \in W\}$. Therefore if C denotes the simple Q-expansion consisting of expanding W by α_i, then $Y < C \leq Z$. As $C \in \mathfrak{A}_r(\Sigma)^Q$, we deduce by the hypothesis that $C = Z$. $\qquad\square$

Proposition 4.2. *For any finite subgroup* $Q \leq G_r(\Sigma)$, *there is a uniquely determined set of integers* $\pi(Q) := \{r_1, \ldots, r_t\}$ *with* $0 \leq r_j \leq d$ *and*

$$\sum_{j=1}^t r_j w_j \equiv r \bmod d$$

such that there is an admissible subset $Y \in \mathfrak{A}_r(\Sigma)^Q$ *with* $|Y| = \sum_{j=1}^t r_j w_j$.
Moreover, any other element in $\mathfrak{A}_r(\Sigma)^Q$ *can be obtained from* Y *by a finite sequence of simple* Q-*expansions or* Q-*contractions.*

Proof. First, note that by 3.2, $\mathfrak{A}_r(\Sigma)^Q \neq \varnothing$. Now choose some $Z \in \mathfrak{A}_r(\Sigma)^Q$ and decompose it as a disjoint union of transitive Q-sets. Let k_j be the number of transitive sets in this decomposition which are of type j, i.e which correspond to the same permutation representation. Observe that whenever we apply simple Q-contractions or Q-expansions to Z, if the set thus obtained has m_j transitive Q-sets of type j, then $m_j \equiv k_j \bmod d$. Note also that

$$|Z| = \sum_{j=1}^t k_j w_j \equiv r \bmod d.$$

Let

$$r_j = \begin{cases} 0, & \text{if } k_j = 0 \\ d, & \text{if } 0 \neq k_j \equiv 0 \bmod d \\ l \text{ with } 0 < l < d \text{ and } l \equiv k_j \bmod d, & \text{otherwise.} \end{cases}$$

By successively performing simple Q-contractions or Q-expansions of Z we may get an admissible set Y such that the number of transitive Q-sets of type j in Y is exactly r_j. Observe that the r_j are uniquely determined, whereas Y is not. Finally, 3.3 implies that for any other $C \in \mathfrak{A}_r(\Sigma)^Q$, there is an upper bound, say D, of Y and C with $D \in \mathfrak{A}_r(\Sigma)^Q$ which means that

$$Y \leq D \geq C.$$

By Lemma 4.1 we may choose chains

$$Y = D_0 < D_1 < \ldots < D_{l_1} = D = C_0 > C_1 > \ldots > C_{l_2} = C$$

such that each step consists of a simple Q-expansion/Q-contraction and we are done. □

Theorem 4.3. *Let $Q_1, Q_2 \leq G_r(\Sigma)$ be finite subgroups with $Q_1 \cong Q_2$. Then Q_1 and Q_2 are conjugate in $G_r(\Sigma)$ if and only if $\pi(Q_1) = \pi(Q_2)$.*
In particular, there are only finitely many conjugacy classes of subgroups isomorphic to Q_1.

Proof. Fix an isomorphism $\phi : Q_1 \to Q_2$. Assume first that $\pi(Q_1) = \pi(Q_2)$. Then there are admissible subsets V_1, V_2 with $V_i \in \mathfrak{A}_r(\Sigma)^{Q_i}$ having the same number of elements and moreover the same structure as Q_i-sets, which means that there is a bijection between them which we denote g such that for any $q \in Q_1$ and $v \in V_1$, $(vq)g = vgq^\phi$. This yields an element $g \in G_r(\Sigma)$ with $g^{-1}qg = q^\phi$.

Conversely, assume $Q_2 = g^{-1}Q_1g$ with $g \in G_r(\Sigma)$. Then for any $V_1 \in \mathfrak{A}_r(\Sigma)^{Q_1}$, $V_1g \in \mathfrak{A}_r(\Sigma)^{Q_2}$. Moreover, g induces an isomorphism as Q_i-sets so the orbit structure of the minimal elements of $\mathfrak{A}_r(\Sigma)^{Q_1}$ and $\mathfrak{A}_r(\Sigma)^{Q_1}$ has to be the same.

□

Theorem 4.4. *Let $Q \leq G_r(\Sigma)$ be a finite subgroup and $\pi(Q) = \{r_1, \ldots, r_t\}$ as in Proposition 4.2. Then $C_{G_r(\Sigma)}(Q) \cong H_{r_1} \times \ldots \times H_{r_t}$ where each of the H_{r_i} fits into the following split group extension*

$$K_i \rightarrowtail H_{r_i} \twoheadrightarrow G_{r_i}(\Sigma)$$

with K_i locally finite.

Proof. Choose an admissible $Y \in \mathfrak{A}_r(\Sigma)^Q$ as in Proposition 4.2. We begin by proving the result in the special case when there are exactly l Q-orbits all of the same type in the Q-set Y. In other words, we assume that in Proposition 4.2 for some k, $l := r_k$ and all the others $r_j = 0$. Let $w := w_k$ be the length of those Q-orbits, and for each $i = 1, ..., l$, choose an orbit representative y_i. We call the subset $\{y_1, ..., y_l\} \subseteq Y$ thus obtained the set of marked elements. We consider any Q-expansion of Y as marked, by marking precisely the descendants of marked elements in Y. And we say that a Q-contraction is marked if marked elements are contracted only with marked elements and result in the marked elements of the new subset. Note that this implies that elements of the form yq with $q \in Q$ and y marked can only be contracted with elements $y'q$ for the same $q \in Q$ and y' marked. We now define:

$$\mathcal{M} = \{M \mid M \in \mathfrak{A}_r(\Sigma)^Q \text{admissible and obtained from } Y \text{ by marked}$$
$$Q\text{-expansions and marked } Q\text{-contractions}\}.$$

$\mathcal{M}$ is the set of marked admissible subsets of $\mathfrak{A}_r(\Sigma)^Q$ and can also be seen as the diagonal subposet:

$$\mathcal{M} \subseteq \underbrace{\mathfrak{A}_l(\Sigma) \times \ldots \times \mathfrak{A}_l(\Sigma)}_{w}.$$

Now fix an admissible subset $X = \{x_1, \ldots, x_l\} \in \mathfrak{A}_l(\Sigma)$ and fix a bijection $\iota_{X,Y} : x_i \mapsto y_i$ from X to the marked elements of Y. From $X \mapsto Y$ we get a poset map

$$\iota^{\mathfrak{A}} : \mathfrak{A}_l(\Sigma) \to \mathfrak{A}_r(\Sigma)^Q,$$

which commutes with expansions and contractions in $\mathfrak{A}_l(\Sigma)$, and with Q-expansions and marked Q-contractions in $\mathfrak{A}_r(\Sigma)^Q$. This is well defined since X is a basis of the algebra used to construct $\mathfrak{A}_l(\Sigma)$. The fact that we only contract marked elements allows us to avoid ambiguities. Observe that $\mathrm{Im}(\iota^{\mathfrak{A}}) = \mathcal{M}$. Moreover, whenever $\iota^{\mathfrak{A}}(X_1) = Y_1$, there is a well defined bijection ι_{X_1,Y_1} between X_1 and the set of

marked elements in Y_1. For convenience we let $\iota_{X,Y}$ act on the right. We use this to define a group homomorphism

$$\iota : G_l(\Sigma) \to C_{G_r(\Sigma)}(Q),$$

as follows: Let g be given by a map $g : X_1 \to X_2$ for $X_1, X_2 \in \mathfrak{A}_r(\Sigma)$ and put $Y_1 = \iota^{\mathfrak{A}}(X_1)$, $Y_2 = \iota^{\mathfrak{A}}(X_2)$. Then $\iota(g) : Y_1 \to Y_2$ is the unique map which commutes with the Q-action and such that $g\iota_{X_2,Y_2} = \iota_{X_1,Y_1}\iota(g)$ (recall that the marked elements are representatives of the Q-orbits). Obviously $\iota(g) \in C_{G_r(\Sigma)}(Q)$.
Next, we define a second poset map

$$\tau^{\mathfrak{A}} : \mathfrak{A}_r(\Sigma)^Q \to \mathfrak{A}_l(\Sigma)$$

such that $\tau^{\mathfrak{A}}\iota^{\mathfrak{A}} = id_{\mathfrak{A}_l(\Sigma)}$. To do this, put $\tau^{\mathfrak{A}}(Y) = X$, identify all the elements in the Q-orbit of each y_i with x_i and extend using the corresponding operations on both sides. Proposition 4.2 and the fact that Y is admissible, imply that $\mathfrak{A}_r(\Sigma)^Q$ is also free on Y, hence $\tau^{\mathfrak{A}}$ is well defined. In an analogous way as before, there is also an explicit bijection between the Q-orbits in any Y_1 and the elements of $\tau^{\mathfrak{A}}(Y_1)$ which can be used to define a group homomorphism

$$\tau : C_{G_r(\Sigma)}(Q) \to G_l(\Sigma).$$

Observe that whenever $g \in C_{G_r(\Sigma)}(Q)$ and $Y_1 \in \mathfrak{A}_r(\Sigma)^Q$, then $Y_1 g \in \mathfrak{A}_r(\Sigma)^Q$.

In particular, $\tau\iota = id_{G_l(\Sigma)}$, giving us the desired split group extension. We now proceed to describe $K := \mathrm{Ker}\tau$. To begin we observe that K consists precisely of those $h \in C_{G_r(\Sigma)}(Q)$ such that for any $A \in \mathfrak{A}_r(\Sigma)^Q$, $\tau^{\mathfrak{A}}(Ah) = \tau^{\mathfrak{A}}(A)$ and h fixes the Q-orbits of A setwise.

We claim that for any $h \in K$ there is some Q-expansion of Y, $Z \in \mathfrak{A}_r(\Sigma)^Q$ with $Zh = Z$. To see this, using Proposition 3.3, take $Z \in \mathfrak{A}_r(\Sigma)^Q$ to be an upper bound of Y, Yh^{-1}. Then Lemma 4.1 implies that Z and Zh are both Q-expansions of Y and therefore they are marked. Thus $Z, Zh \in \mathcal{M} = \mathrm{Im}\iota^{\mathfrak{A}}$. As h lies in K, we have $\tau^{\mathfrak{A}}(Z) = \tau^{\mathfrak{A}}(Zh)$. So the fact that $\tau^{\mathfrak{A}}$ is injective when restricted to $\mathrm{Im}\iota^{\mathfrak{A}}$ implies the claim. In particular, K is the union of its subgroups of the form

$$K_Z := \{h \in C_{G_r(\Sigma)}(Q) \mid Zh = Z, h \text{ fixes the } Q\text{-orbits setwise}\}$$

where Z is a Q-expansion of Y. As each K_Z is finite, using Proposition 3.3 we see that K is locally finite, thus proving the special case.

To finish our proof, we now prove the general case when Y has Q orbits of different types. Let $\pi(Q) = \{r_1, \ldots, r_t\}$ and $w_1, \ldots, w_t$ be as in Proposition 4.2. Let $Y = \bigcup_{i=1}^{t} Y_i$ with Y_i the union of the r_i Q-orbits of type i in Y. Then Q acts on each Y_i. Note that a single action might not be faithful, but the intersection of the kernels must be trivial. Also note, that in $\mathfrak{A}_r(\Sigma)^Q$, Q-contractions can not mix elements belonging to orbits of different type. This implies that we have a direct product of posets

$$\mathfrak{A}_r(\Sigma)^Q \cong \mathfrak{A}_{w_1 r_1}(\Sigma)^Q \times \ldots \times \mathfrak{A}_{w_t r_t}(\Sigma)^Q,$$

where we let the group Q act on each poset $\mathfrak{A}_{w_i r_i}(\Sigma)$ using its action on Y_i and extending via extensions and contractions. This action yields also a group homomorphism $\phi_i : Q \to G_{w_i r_i}(\Sigma)$. The direct product of posets above implies that $C_{G_r(\Sigma)}(Q)$ decomposes as the direct product of the centralisers of $\phi_i(Q) \leq G_{w_i r_i}(\Sigma)$. For each of these we can apply the case of a single type of orbit, and we are done. $\square$

Remark 4.5. In an analogous way, one can prove that there is also a group epimorphism

$$N_{G_r(\Sigma)}(Q) \twoheadrightarrow G_{r_1}(\Sigma) \times \ldots \times G_{r_t}(\Sigma)$$

with locally finite kernel.

Remark 4.6. In [3], there is a description of centralisers of elements g in the Higman groups $G_{n,1}$ associated to $U_r(\Sigma) = V_{n,1}$. Whenever $Q = < g >$ has finite order in those groups, this coincides with ours.

Remark 4.7. With little more effort we can give a description of the kernel K appearing in the single type of orbit case in the proof of Theorem 4.4: Let S_w be the symmetric group of degree w and choose a permutation representation $\phi : Q \to S_w$ associated to the Q action on the orbits of Y. Denote $L := C_{S_w}(\phi(Q))$. Then if Z is a Q-expansion of Y with $|Z| = mw$, we have the following isomorphism:

$$\{h \in C_{G_r(\Sigma)}(Q) \mid Zh = Z, h \text{ fixes the } Q\text{-orbits setwise}\} \cong L^m := \underbrace{L \times \ldots \times L}_{m}.$$

Consider now a simple Q-expansion $Z \leq Z_1$ with $|Z_1| = m_1 w$, consisting of applying a descending operation of arity n_j to a vertex z_0. We get a group homomorphism $L^m \to L^{m_1}$ given by mapping the copy of L corresponding to z_0 to the product of copies of L corresponding to the descendants of z_0 via the diagonal map $L \to L^{n_j}$, and leaving the remaining factors intact. This gives, in an obvious way, a direct system of groups and hence K is the directed limit of the system thus obtained.

We shall now consider the groups $T_r(\Sigma)$ whenever they are defined, i.e. whenever Σ preserves the induced order in the admissible subsets of our Cantor-Algebra $U_r(\Sigma)$. In this case, centralisers of finite subgroups have an easier structure.

A first observation is that any finite subgroup $Q \leq T_r(\Sigma)$ is cyclic. Moreover following the argument of Proposition 4.2 we see that by writing the transitive permutation representations of Q to have the faithful representation first, i.e. $w_1 = |Q|$, we obtain $\pi(Q) = \{r_1, 0, \ldots, 0\}$ and $r_1 w_1 \equiv r \bmod d$. To see this, take for example $Y \in \mathfrak{A}_r(\Sigma)^Q$ the admissible subset obtained following the argument of Proposition 4.2 and assume that certain $g \in Q$ fixes some $y_0 \in Y$. The condition that g preserves cyclically the order, implies that g fixes Y pointwise, thus $g = 1$. As a consequence, $|Y| = r_1 w_1$.

Theorem 4.8. *Let $U_r(\Sigma)$ be a Cantor-Algebra with order preserving Σ and $Q \leq T_r(\Sigma)$ a finite subgroup. Then there is only one conjugacy class of finite subgroups of order $|Q|$ and for a certain $0 < l \leq d$, depending on Q there is a central extension*

$$Q \rightarrowtail C_{T_r(\Sigma)}(Q) \twoheadrightarrow T_l(\Sigma).$$

Proof. For the first assertion observe that any two cyclic groups of the same order are isomorphic and they only have one faithful permutation representation. Hence it suffices to choose a cyclic order preserving h between the corresponding admissible subsets Y. Note that they have the same cardinality.

For the second assertion, embed $T_r(\Sigma)$ in $G_r(\Sigma)$ and let ι, τ be the group homomorphisms of Theorem 4.4, we use the same notation as there. The result will follow once we check that $\tau(C_{T_r(\Sigma)}(Q)) = T_l(\Sigma)$ and that $Q = \mathrm{Ker}\tau \cap T_r(\Sigma)$.

Note first that we may choose the map $\iota_{X,Y}$ to be order preserving. The fact that the action of Q cyclically preserves the order on Y, implies that we may assume that if for any basis Y_1 the marked elements are $\{y_1, \ldots, y_m\}$, then the elements of Y_1 are ordered as

$$y_1 < \ldots < y_m < y_1 q < \ldots < y_m q < \ldots < y_1 q^i < \ldots y_m q^i < \ldots,$$

for certain (fixed) q generating Q (note that the marked elements of elements of Y can be chosen so that Y is ordered this way). If $g \in C_{T_r(\Sigma)}(Q)$ represents a map between two such sets cyclically preserving that ordering, then it is obvious that the corresponding map $\tau(g)$ also does. In fact, if we denote $y_1' < \ldots < y_m' < y_1' q < \ldots < y_m' q < \ldots < y_1' q^i < \ldots y_m' q^i < \ldots$ the elements of $Y_1 g$ and choose the index

j such that $y_j g = y'_m q^a$. Then if $j < m$, $y_{j+1} g = y'_1 q^{a+1}$ and if $j = m$, $y_1 g = y'_1 q^a$. Here $0 \le a \le |Q| - 1$. This implies $\tau(C_{T_r(\Sigma)}(Q)) \subseteq T_l(\Sigma)$.

Conversely, take $g \in T_l(\Sigma)$. Then g is determined by its action on a pair of ordered admissible subsets $X_1 : x_1 < \ldots < x_m$, $X_2 = X_1 g : x'_1 < \ldots < x'_m$. Put $Y_1 := \iota^{\mathfrak{A}}(X_1)$, $Y_2 := \iota^{\mathfrak{A}}(X_2)$ and denote their elements as before. Now, $\iota(g)$ as defined in Theorem 4.4 does not cyclically preserve the order between Y_1 and Y_2. Let j be the subindex such that $x_1 g = x'_j$. We construct $k \in K = \mathrm{Ker}\tau$ as follows:

$$y'_i q^a k = \begin{cases} y'_i q^{a+1} \text{ for } 1 \le i < j \\ y'_i q^a \text{ for } j \le i \le m, \end{cases}$$

where, as before, $0 \le a \le |Q| - 1$. A routine check shows that this is well defined and that $\iota(g)k : Y_1 \to Y_2$ cyclically preserves the order between Y_1 and Y_2, in other words, that $\iota(g)k \in T_r(\Sigma)$. From this we deduce that $\tau : C_{T_r(Q)}(Q) \twoheadrightarrow T_l(\Sigma)$ is an epimorphism.

Finally, recall that by the proof of Theorem 4.4 and Remark 4.7, $\mathrm{Ker}\tau$ is the union of its subgroups of the form

$$K_Z = \{h \in C_{G_r(\Sigma)}(Q) \mid Zh = Z \text{ and } h \text{ fixes the } Q\text{-orbits setwise}\} \cong L^m$$

for each Q-expansion Z of Y. Moreover, as Q is transitive and regular, $L = C_{S_{w_1}}(\phi(Q)) \cong Q$. The observation before this Theorem implies that the finite group $K_Z \cap T_r(\Sigma)$ must in fact act in the same way as Q acts on Z. Hence

$$K_Z \cap T_r(\Sigma) = Q.$$

Note, that under the isomorphism $K_Z \cong Q^m$, this corresponds to the diagonal subgroup of Q^m. That the extension is central now follows immediately. $\qquad\square$

Remark 4.9. For $U_r(\Sigma) = V_{2,1}$, the Higman algebra and $T_r(\Sigma) = T$, the original Thompson-group T this reproves [19, Theorem 7.1.5].

5. Finiteness conditions in Bredon cohomology

In this section we collect all necessary background on Bredon cohomological finiteness conditions and also prove an analogue to Bieri's criterion for FP_n.

Let $\mathcal{X}$ denote a family of subgroups of a given group G. In Bredon cohomology, the group G is replaced by the orbit category $\mathcal{O}_\mathcal{X} G$. The category $\mathcal{O}_\mathcal{X} G$ has as objects the transitive G-sets with stabilisers in $\mathcal{X}$. Morphisms in $\mathcal{O}_\mathcal{X} G$ are G-maps between those G-sets. Modules over the orbit category, called $\mathcal{O}_\mathcal{X} G$-modules are contravariant functors from the orbit category to the category of abelian groups. Exactness is defined pointwise: a sequence

$$A \to B \to C$$

of $\mathcal{O}_\mathcal{X} G$-modules is exact at B if and only if

$$A(\Delta) \to B(\Delta) \to C(\Delta)$$

is exact at $B(\Delta)$ for every transitive G-set Δ.

The category $\mathcal{O}_\mathcal{X} G\text{-}\mathfrak{Mod}$ of $\mathcal{O}_\mathcal{X} G$-modules has enough projectives, which are constructed as follows: For any G-sets Δ and Ω, denote by $[\Delta, \Omega]$ the set of G-maps from Δ to Ω. Let $\mathbb{Z}[\Delta, \Omega]$ be the free abelian group on $[\Delta, \Omega]$. One now obtains an $\mathcal{O}_\mathcal{X} G$-module $\mathbb{Z}[-, \Omega]$ by fixing Ω and letting Δ range over the transitive G-sets with stabilisers in $\mathcal{X}$. A Yoneda-type argument, see [21], yields that these modules are free. In particular, the modules $P^K(-) = \mathbb{Z}[-, G/K]$ for $K \in \mathcal{X}$ are free and can be viewed as the building blocks for free $\mathcal{O}_\mathcal{X} G$-modules. Projective modules are now defined analogously to the ordinary case. The trivial $\mathcal{O}_\mathcal{X} G$-module, denoted $\mathbb{Z}(-)$, is the constant functor $\mathbb{Z}$ from $\mathcal{O}_\mathcal{X} G$ to the category of abelian groups.

Bieri [1] gives criteria for a $\mathbb{Z}G$-module to be of type FP_n involving certain Ext- and Tor-functors commuting with exact colimits and direct products respectively. In this section we prove that those criteria can also be used for Bredon cohomology. The Bredon cohomology functors $\text{Ext}_{\mathfrak{X}}^*(M, -)$ are defined as derived functors of $\text{Hom}_{\mathfrak{X}}(M, -)$. In particular, let $M(-) \in \mathcal{O}_{\mathcal{X}}G\text{-}\mathfrak{Mod}$ be a contravariant $\mathcal{O}_{\mathcal{F}}G$-module admitting a projective resolution $P_*(-) \twoheadrightarrow M(-)$. Then, for each $N(-) \in \mathcal{O}_{\mathcal{X}}G\text{-}\mathfrak{Mod}$,

$$\text{Ext}_{\mathfrak{X}}^*(M, N) = H_*(mor(P_*, N)).$$

One can also define Bredon homology functors $\text{Tor}_*^{\mathfrak{X}}(-, M)$. In particular, by analogy with the contravariant case, one can define covariant $\mathcal{O}_{\mathcal{X}}G$-modules, or just comodules for short. The category of covariant $\mathcal{O}_{\mathcal{X}}G$-modules, denoted $\mathfrak{Mod}\text{-}\mathcal{O}_{\mathcal{X}}G$, behaves just as expected. For example, we have short exact sequences and enough projectives as above. In particular, the building blocks for projective modules in $\mathfrak{Mod}\text{-}\mathcal{O}_{\mathcal{X}}G$ are the covariant functors $P_K(-) = \mathbb{Z}[G/K, -]$ for subgroups $K \in \mathfrak{X}$. Let $M(-) \in \mathcal{O}_{\mathcal{X}}G\text{-}\mathfrak{Mod}$ be as before. Then Bredon homology functors are the derived functors of $- \otimes_{\mathfrak{X}} M$, i.e., for any $L(-) \in \mathfrak{Mod}\text{-}\mathcal{O}_{\mathcal{X}}G$,

$$\text{Tor}_*^{\mathfrak{X}}(L, M) = H_*(L \otimes_{\mathfrak{X}} P_*).$$

For detail on these definitions including the categorical tensor product and Yoneda-type isomorphism the reader is referred to [22]. In particular, $\text{Tor}_*^{\mathfrak{X}}(-, M)$ can be calculated using flat resolutions of $M(-)$.

The category of $\mathcal{O}_{\mathcal{X}}G$-modules, as an abelian category, has well defined colimits and limits and in particular coproducts and products. We say a functor

$$T : \mathcal{O}_{\mathcal{X}}G\text{-}\mathfrak{Mod} \to \mathcal{A}b$$

commutes with exact colimits, denoted here by $\varinjlim$, if, for every directed system $(M_\lambda)_{\lambda \in \Lambda}$ of $\mathcal{O}_{\mathcal{X}}G$-modules, the natural map

$$\varinjlim T(M_\lambda) \to T(\varinjlim M_\lambda)$$

is an isomorphism. Analogously, we say a functor

$$S : \mathfrak{Mod}\text{-}\mathcal{O}_{\mathcal{X}}G \to \mathcal{A}b$$

commutes with exact limits, denoted here by $\varprojlim$, if, for every inverse system $(N_\lambda)_{\lambda \in \Lambda}$ of $\mathcal{O}_{\mathcal{X}}G$-comodules, the natural map

$$S(\varprojlim N_\lambda) \to \varprojlim S(N_\lambda)$$

is an isomorphism.

We say an $\mathcal{O}_{\mathcal{X}}G$-module $M(-)$ is finitely generated if there is a finitely generated free module mapping onto it. In particular, there is a G-finite G-set Δ such that $\mathbb{Z}[-, \Delta] \twoheadrightarrow M(-)$ (here we are extending the notation $\mathbb{Z}[-, \Delta]$ to non transitive sets in the obvious way).

Lemma 5.1. *Let M be an $\mathcal{O}_{\mathcal{X}}G$-module. Then M is the direct colimit of its finitely generated submodules.*

Proof. This follows from [15, §9.19]. $\square$

The notions of type Bredon-FP, Bredon-FP_n and Bredon-FP_∞ are defined in terms of projective resolutions over $\mathcal{O}_{\mathcal{X}}G$ analogously to the classical notions of type FP, FP_n and FP_∞.

Proposition 5.2. *Let A be an $\mathcal{O}_{\mathcal{X}}G$-module of type Bredon-$\text{FP}_n$, $0 \le n \le \infty$. Then*

(i) *For every exact limit, the natural homomorphism*

$$\operatorname{Tor}_k^{\mathfrak{X}}(\varprojlim N_*, A) \to \varprojlim \operatorname{Tor}_k^{\mathfrak{X}}(N_*, A)$$

is an isomorphism for all $k \leq n - 1$ and an epimorphism for $k = n$.

(ii) *For every exact colimit, the natural homomorphism*

$$\varinjlim \operatorname{Ext}_{\mathfrak{X}}^k(A, M_*) \to \operatorname{Ext}_{\mathfrak{X}}^k(A, \varinjlim M_*)$$

is an isomorphism for all $k \leq n - 1$ and a monomorphism for $k = n$.

Proof. The proof goes completely analogously to that of Bieri [1, Proposition 1.2]. It relies on the Yoneda isomorphisms, i.e that $N \otimes_{\mathfrak{X}} \mathbb{Z}[-, G/K] \cong N(G/K)$ and $\operatorname{Hom}_{\mathfrak{X}}(\mathbb{Z}[-, G/K], M) \cong M(G/K)$, the fact that $\varprojlim$ and $\operatorname{Hom}_{\mathfrak{X}}(-, M)$ commute with finite direct sums and that $\varprojlim$ and $\varinjlim$ are exact and hence commute with the homology functor. $\square$

Bieri's argument can be carried through completely for Bredon-Ext and Bredon-Tor functors.

Theorem 5.3. *Let A be an $\mathcal{O}_{\mathcal{X}}G$-module. Then the following are equivalent:*

(i) *A is of type Bredon-FP_n.*

(ii) *For every exact colimit, the natural homomorphism*

$$\varinjlim \operatorname{Ext}_{\mathfrak{X}}^k(A, M_*) \to \operatorname{Ext}_{\mathfrak{X}}^k(A, \varinjlim M_*)$$

is an isomorphism for all $k \leq n - 1$ and a monomorphism for $k = n$.

(iii) *For the direct limit of any directed system of $\mathcal{O}_{\mathcal{X}}G$-modules M_* with $\varinjlim M_* = 0$, one has $\varinjlim \operatorname{Ext}_{\mathfrak{X}}^k(A, M_*) = 0$, for all $k \leq n$.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are either obvious or follow from Proposition 5.2. Every $\mathcal{O}_{\mathcal{X}}G$-module is the directed colimit of finitely generated submodules, Lemma 5.1, and hence $(iii) \Rightarrow (i)$ is proved completely analogously to [1, Theorem 1.3 $(iiib) \Rightarrow (i)$]. $\square$

Theorem 5.4. *Let A be an $\mathcal{O}_{\mathcal{X}}G$-module. Then the following are equivalent:*

(i) *A is of type Bredon-FP_n.*

(ii) *For every exact limit, the natural homomorphism*

$$\operatorname{Tor}_k^{\mathfrak{X}}(\varprojlim N_*, A) \to \varprojlim \operatorname{Tor}_k^{\mathfrak{X}}(N_*, A)$$

is an isomorphism for all $k \leq n - 1$ and an epimorphism for $k = n$.

(iii) *For any $K \in \mathfrak{X}$ consider any arbitrary direct product $\prod_{\Lambda_K} \mathbb{Z}[G/K, -]$. Then the natural map*

$$\operatorname{Tor}_k^{\mathfrak{X}}(\prod_{K \in \mathfrak{X}} \prod_{\Lambda_K} \mathbb{Z}[G/K, -], A) \to \prod_{K \in \mathfrak{X}} \prod_{\Lambda_K} \operatorname{Tor}_k^{\mathfrak{X}}(\mathbb{Z}[G/K, -], A)$$

is an isomorphism for all $k \leq n - 1$ and an epimorphism for $k = n$.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are again either obvious or consequence of Proposition 5.2.

$(iii) \Rightarrow (i)$: The proof is in the same spirit as Bieri's proof. We begin by letting $n = 0$ and claim that A is finitely generated as an $\mathcal{O}_{\mathcal{X}}G$-module. As an index set we take $\prod_{K \in \mathfrak{X}} A(G/K)$ and consider $\prod_{K \in \mathfrak{X}} \prod_{a \in A(G/K)} \mathbb{Z}[G/K, -]$. By (iii), the natural map

$$\mu : (\prod_{K \in \mathfrak{X}} \prod_{A(G/K)} \mathbb{Z}[G/K, -]) \otimes_{\mathfrak{X}} A(-) \to \prod_{A(G/K)} A(G/K)$$

is an epimorphism. Let c be the element with $\mu(c) = \prod_{K \in \mathfrak{X}} \prod_{a \in A(G/K)} a$. Then c is of the form

$$c = \sum_{i=1}^{l} (\prod_{K \in \mathfrak{X}} \prod_{A(G/K)} f_i^{a,K}) \otimes b_i,$$

for certain subgroups $H_1, \ldots, H_l \in \mathfrak{X}$ and elements $b_i \in A(G/H_i)$. Here, $f_i^{a,K} \in \mathbb{Z}[G/K, G/H_i]$. Now we claim that there is an epimorphism

$$\tau : \bigoplus_{i=1}^{l} \mathbb{Z}[-, G/H_i] \twoheadrightarrow A$$

given by $\tau(f) := f^*(b_i) \in A(G/K)$ whenever $f \in \mathbb{Z}[G/K, G/H_i]$. Observe that this is well defined. In particular it is functorial. To prove the claim, take any $K \in \mathfrak{X}$ and any $a \in A(G/K)$. Note that

$$\mu(c) = \sum_{i=1}^{l} \prod_{K \in \mathfrak{X}} \prod_{a \in A(G/K)} (f_i^{a,K})^*(b_i) = \prod_{K \in \mathfrak{X}} \prod_{a \in A(G/K)} \sum_{i=1}^{l} (f_i^{a,K})^*(b_i)$$

so the fact that c maps onto the diagonal means that

$$a = \sum_{i=1}^{l} (f_i^{a,K})^*(b_i) = \tau(\sum_{i=1}^{l} f_i^{a,K}).$$

The case $n \geq 1$ is now done analogously to [1, Theorem 1.3] using a diagram chase. $\qquad\square$

Remark 5.5. For $n \geq 1$, condition (iii) is equivalent to the following, which in ordinary homology is often referred to as the Bieri-Eckmann criterion for FP_n : For every subgroup $K \in \mathfrak{X}$ consider an arbitrary direct product $\prod_{\Lambda_K} \mathbb{Z}[G/K, -]$. Then the natural map

$$(\prod_{K \in \mathfrak{X}} \prod_{\Lambda_K} \mathbb{Z}[G/K, -]) \otimes_{\mathfrak{X}} A(-) \to \prod_{K \in \mathfrak{X}} \prod_{\Lambda_K} A(G/K)$$

is an isomorphism and $\mathrm{Tor}_k^{\mathfrak{X}}(\prod_{K \in \mathfrak{X}} \prod_{\Lambda_K} \mathbb{Z}[G/K, -], A) = 0$, for all $1 \leq k \leq n-1$. We call this condition the global Bieri-Eckmann criterion for Bredon homology.

We say a group satisfies the local Bieri-Eckmann criterion for Bredon cohomology if, for any K and direct product as before, the natural map

$$(\prod_{\Lambda_K} \mathbb{Z}[G/K, -]) \otimes_{\mathfrak{X}} A(-) \to \prod_{\Lambda_K} A(G/K)$$

is an isomorphism and $\mathrm{Tor}_k^{\mathfrak{X}}(\prod_{\Lambda_K} \mathbb{Z}[G/K, -], A) = 0$ for all $1 \leq k \leq n-1$.

6. Classifying spaces with finite isotropy

In this section we shall restrict ourselves to the family $\mathcal{F}$ of all the finite subgroups of G.

To stay in line with notation previously used, we say a module is of type $\underline{\mathrm{FP}}_\infty$ if it is of type Bredon-FP_∞ with respect to $\mathcal{F}$. The notions of $\underline{\mathrm{FP}}_n$ and $\underline{\mathrm{FP}}$ are defined analogously. For Bredon cohomology with respect to $\mathcal{F}$ there is a good algebraic description for modules of type $\underline{\mathrm{FP}}_n$. For the original approach via classifying spaces, see [16].

Theorem 6.1. [13] *Let G be a group having finitely many conjugacy classes of finite subgroups. Then an $\mathcal{O}_{\mathcal{F}}G$-module $M(-)$ is of type $\underline{\mathrm{FP}}_n$ if and only if $M(G/K)$ is of type FP_n as a $\mathbb{Z}(WK)$-module for each finite subgroup K of G.*

It was also shown, [13], that a group G is of type $\underline{FP}_0$ if and only if G has finitely many conjugacy classes of finite subgroups. Hence we have the following corollary:

Corollary 6.2. [13] *A group G is of type $\underline{FP}_n$ if and only if G has finitely many conjugacy classes of finite subgroups and $C_G(K)$ is of type FP_n for every finite subgroup K of G.*

Recall that we say a group G is of Bredon-type FP_n if the trivial module $\mathbb{Z}(-)$ is of type FP_n as an $\mathcal{O}_{\mathcal{X}}G$-module. We can, of course rephrase Theorems 5.3 and 5.4 in terms of Bredon-cohomology and Bredon-homology replacing the module $A(-)$ with $\mathbb{Z}(-)$, $\mathrm{Ext}^*_{\mathfrak{X}}(A,-)$ with $\mathrm{H}^*_{\mathfrak{X}}(G,-)$ and $\mathrm{Tor}^{\mathfrak{X}}_*(-,A)$ with $\mathrm{H}^{\mathfrak{X}}_*(G,-)$.

We shall now weaken the hypothesis on the conjugacy classes of finite subgroups:

Definition 6.3. We say a group is of type quasi-$\underline{FP}_n$ if, for each finite subgroup K of G there are finitely many conjugacy classes of subgroups isomorphic to K and the Weyl-groups $WK := N_G(K)/K$ are of type FP_n.

Note that a group of type quasi-$\underline{FP}_n$ with a bound on the orders of the finite subgroups is of type $\underline{FP}_n$.
Let k be a positive integer. We denote by $\mathbb{Z}_k(-)$ the $\mathcal{O}_{\mathcal{F}}G$-module defined by

$$\mathbb{Z}_k(G/H) = \begin{cases} \mathbb{Z} \text{ if } |H| \leq k \\ 0 \text{ otherwise,} \end{cases}$$

together with the obvious morphisms.

Lemma 6.4. *A group G is of type quasi-$\underline{FP}_0$ if and only if, for each $k \geq 1$, the module $\mathbb{Z}_k(-)$ is finitely generated. Moreover, in that case, the finite G-set Δ_k with $\mathbb{Z}[-,\Delta_k] \twoheadrightarrow \mathbb{Z}_k(-)$ can be chosen to have stabilisers of order bounded by k.*

Proof. Suppose G is of type quasi-$\underline{FP}_0$. Take

$$\Delta_k = \bigsqcup_{|H| \leq k, \text{up to } G\text{-conj.}} G/H.$$

This is a G-finite G-set with stabilisers of order bounded by k and $\mathbb{Z}[-,\Delta_k] \twoheadrightarrow \mathbb{Z}_k(-)$. For the converse, we need to show that, for each finite subgroup K, there are only finitely many conjugacy classes of subgroups of order bounded by $k = |K|$. Let Δ_k be the finite G-set with $\mathbb{Z}[-,\Delta_k] \twoheadrightarrow \mathbb{Z}_k(-)$ and take any finite subgroup H of G with $|H| \leq k$. Hence $\mathbb{Z}_k(G/H) \cong \mathbb{Z} \neq 0$. Since the map $\mathbb{Z}[G/H, \Delta] \twoheadrightarrow \mathbb{Z}_k(G/H)$ is onto, it follows that $\mathbb{Z}[G/H, \Delta] \neq 0$ and hence H has to be subconjugated to one of the finitely many stabilisers of Δ. $\square$

Note that finitely generated $\mathcal{O}_{\mathcal{F}}G$-modules are precisely those of type $\underline{FP}_0$. Fix an integer $k \geq 1$ and let $M_k(-)$ be an $\mathcal{O}_{\mathcal{F}}G$-module such that $M_k(G/L) = 0$ whenever $|L| > k$. Suppose $M_k(-)$ is finitely generated. Then there exists a G-finite G-set Δ with stabilisers of order $\leq k$ and a short exact sequence of $\mathcal{O}_{\mathcal{F}}G$-modules

$$N_k(-) \rightarrowtail \mathbb{Z}[-,\Delta] \twoheadrightarrow M_k(-)$$

with the property that $N_k(G/L) = 0$ for all finite subgroups L with $|L| > k$.

Proposition 6.5. *A group G is of type quasi-$\underline{FP}_n$ if and only if, for each integer $k \geq 1$, the $\mathcal{O}_{\mathcal{F}}G$-module $\mathbb{Z}_k(-)$ is of type $\underline{FP}_n$.*

Proof. The "if"-direction follows from Lemma 6.4, Theorem 6.1 and the definition as $\mathbb{Z}_{|K|}(G/H)$ is of type FP_n as a WH-module for each $|H| \leq |K|$.
Now suppose G is of type quasi-$\underline{FP}_n$. For each $k \geq 1$ we construct a projective resolution of $\mathbb{Z}_k(-)$ which is finitely generated in dimensions up to n; note that we

may assume $n > 0$. By Lemma 6.4 and the above remark we have a short exact sequence

$$C_0(-) \rightarrowtail \mathbb{Z}[-, \Delta_0] \twoheadrightarrow \mathbb{Z}_k(-)$$

with Δ_0 a G-finite G-set and $C_0(G/L) = 0$ for all $|L| > k$. We claim that $C_0(-)$ is a finitely generated $\mathcal{O}_\mathcal{F}G$-module.

We know that there are finitely many conjugacy classes of subgroups of order bounded by k. Let H be one of those. As Δ_0 is G-finite, the WH-module $\mathbb{Z}[G/H, \Delta_0]$ is of type FP$_\infty$. This is a consequence of the fact that for any K, $\mathbb{Z}[G/H, G/K]$ is a sum of exactly $|\{x \in N_G(H)\backslash G/K \mid H^{x^{-1}} \leq K\}|$ WH-modules, which are of type FP$_\infty$. As K is finite, this sum must also be finite. So evaluating the previous short exact sequence at G/H, we see that the WH-module $C_0(G/H)$ is of type FP$_{n-1}$ and in particular, finitely generated. Fix a finite WH-generating set X_H for $C_0(G/H)$. Then the $\mathcal{O}_\mathcal{F}G$-set formed by the union of all those X_H where $H \in \mathrm{Stab}\Delta_0$, generates C_0.

We can now proceed to construct the desired resolution by using the remark before Proposition 6.5. $\qquad\square$

Theorem 6.6. *Let G be of type quasi-$\underline{\mathrm{FP}}_n$, where $n \geq 1$. Then G satisfies the local Bieri-Eckmann criterion for Bredon homology.*

Proof. It follows from the definition of the modules $\mathbb{Z}_k(-)$ that

$$\mathbb{Z}(-) = \varinjlim_{k \in \mathbb{N}} \mathbb{Z}_k(-).$$

In the category of $\mathcal{O}_\mathcal{F}G$-modules the construction of a free module mapping onto a given one is functorial. Hence, we can get a direct colimit of free resolutions $\varinjlim_{k \in \mathbb{N}}(F_{*,k}(-) \twoheadrightarrow \mathbb{Z}_k(-)) = F_*(-) \twoheadrightarrow \mathbb{Z}(-)$, which gives us a flat resolution of $\mathbb{Z}(-)$. For details the reader is referred to [22, Lemma 3.4]. Hence

$$\mathrm{H}^{\mathcal{F}}_k(G, \prod_\Lambda \mathbb{Z}[G/K, -]) = \mathrm{H}_*(\prod_\Lambda \mathbb{Z}[G/K, -] \otimes_\mathcal{F} F_*(-))$$

$$= \mathrm{H}_k(\prod_\Lambda \mathbb{Z}[G/K, -] \otimes_\mathcal{F} \varinjlim_{k \in \mathbb{N}} F_{*,k}(-))$$

$$= \varinjlim_{k \in \mathbb{N}} \mathrm{H}_k(\prod_\Lambda \mathbb{Z}[G/K, -] \otimes_\mathcal{F} F_{*,k}(-))$$

$$= \varinjlim_{k \in \mathbb{N}} \mathrm{Tor}_k(\prod_\Lambda \mathbb{Z}[G/K, -], \mathbb{Z}_k(-)) = 0 \quad,$$

where the last line follows from Proposition 6.5 and Theorem 5.4. The first assertion follows by a similar argument.

$\qquad\square$

For each $k \geq 1$ we consider the family $\mathcal{F}_k$ and the orbit category $\mathcal{O}_{\mathcal{F}_k}G$. For a given positive integer k the family $\mathcal{F}_k$ consists of all subgroups H of G with $|H| \leq k$. By using the arguments of the proofs of Lemma 6.4 and Proposition 6.5 we can show:

Proposition 6.7. *A group is of type quasi-$\underline{\mathrm{FP}}_n$ if and only if it is of type Bredon-FP$_n$ over $\mathcal{O}_{\mathcal{F}_k}G$ for each k.*

We can also rephrase Theorems and 5.3 and 5.4:

Corollary 6.8. *Let G be a group. Then the following are equivalent:*
   (i) *G is of type quasi-$\underline{\mathrm{FP}}_n$.*
   (ii) *For every exact colimit and any k, the natural homomorphism*

$$\varinjlim \mathrm{H}^l_{\mathfrak{F}_k}(G, M_*) \to \mathrm{H}^l_{\mathfrak{F}_k}(G, \varinjlim M_*)$$

   is an isomorphism for all $l \leq n - 1$, and a monomorphism for $l = n$.

(iii) *For any k and any $K \in \mathfrak{F}_k$ consider an arbitrary direct product $\prod_{\Lambda_K} \mathbb{Z}[G/K, -]$. Then the natural map*

$$\mathrm{H}_l^{\mathfrak{F}_k}(\prod_{K \in \mathfrak{F}_k} \prod_{\Lambda_K} \mathbb{Z}[G/K, -], A) \to \prod_{K \in \mathfrak{F}_k} \prod_{\Lambda_K} \mathrm{H}_l^{\mathfrak{F}_k}(\mathbb{Z}[G/K, -], A)$$

is an isomorphism for all $l \leq n - 1$ and an epimorphism for $l = n$.

One may also add the statements analogous to 5.3 ii) and 5.4 ii). Note also that for $n \geq 1$ the above is equivalent to:

(iv) *For any k, any $K \in \mathfrak{F}_k$ and any arbitrary direct product $\prod_{\Lambda_K} \mathbb{Z}[G/K, -]$, the natural map*

$$\mathbb{Z}_k(-) \otimes_{\mathfrak{F}_k} \prod_{K \in \mathfrak{F}_k} \prod_{\Lambda_K} \mathbb{Z}[G/H, -] \to \prod_{K \in \mathfrak{F}_k} \prod_{\Lambda_K} \mathbb{Z}_k$$

is an isomorphism and $\mathrm{H}_l^{\mathfrak{F}_k}(G, \prod_{K \in \mathfrak{F}_k} \prod_{\Lambda_K} \mathbb{Z}[G/H, -]) = 0$ for all $1 \leq l \leq n - 1$.

Definition 6.9. We say a group G is of type quasi-$\underline{\mathrm{F}}_\infty$ if for all positive integers k, G admits a finite type model for $E_{\mathfrak{F}_k} G$.

Analogously to the algebraic case, any group of type quasi-$\underline{\mathrm{F}}_\infty$, which has a bound on the orders of the finite subgroups, is of type $\underline{\mathrm{F}}_\infty$.

Lück's Theorem [16, Theorem 4.2] (Theorem 1.2) goes through for arbitrary families of finite subgroups. Hence combining Theorems 1.1 and 1.2 yields:

Proposition 6.10. *A group G is of type quasi-$\underline{\mathrm{F}}_\infty$ if and only if G is of type quasi-$\underline{\mathrm{FP}}_\infty$ and G and all centralisers $C_G(K)$ of finite subgroups are finitely presented.* □

We can now prove what is largely equivalent to Proposition 6.7:

Theorem 6.11. *A group G is of type quasi-$\underline{\mathrm{F}}_\infty$ if and only if it admits a model for $\underline{E}G$, which is the mapping telescope of finite type models for $E_{\mathfrak{F}_k} G$ for each $K \in \mathcal{F}$.*

Proof. The "if'-direction follows directly from the definition. Now suppose we have finite type models X_K for $E_{\mathfrak{F}_k} G$ for all $K \in \mathcal{F}$. For each $H \leq K$ the universal property for classifying spaces for a family yields G-maps $\nu_H^K : X_H \to X_K$. Now the mapping telescope yields a G-CW-complex X, for which X^K is contractible for all $K \in \mathcal{F}$ and empty otherwise.   □

7. BREDON COHOMOLOGICAL FINITENESS PROPERTIES FOR GENERALISED THOMPSON-HIGMAN GROUPS

We can now prove

Theorem 7.1. *Let $U_r(\Sigma)$ be a Cantor-Algebra with order preserving Σ. Then the following conditions are equivalent for $1 \leq r \leq d$:*

  i) *$T_r(\Sigma)$ is quasi-$\underline{\mathrm{FP}}_\infty$.*
  ii) *$T_l(\Sigma)$ is of (ordinary) type FP_∞ for any $1 \leq l \leq d$ such that $gcd(l, d) \mid r$.*

Proof. Assume that i) holds and take $1 \leq l \leq d$ with $gcd(l, d) \mid r$. This condition implies that there is some positive integer w with $lw \equiv r \bmod d$. Then we may choose an admissible subset $A \subseteq U_r(\Sigma)$ of cardinality lw and consider the subgroup Q of $T_r(\Sigma)$ defined by cyclic permutations of A on l orbits all of length w. By Theorem 4.8, $C_{T_r(\Sigma)}(Q)$ is an extension of a finite group by $T_l(\Sigma)$ so this last group must be of type FP_∞.

Now assume ii). Observe, that for any finite subgroup Q of cardinality w and any admissible subset Y_1 fixed by Q, the observation before Theorem 4.8 implies that

for certain l, $|Y_1| = lw \equiv r \mod d$ thus $\gcd(l, d) \mid r$. This together with Theorem 4.8 implies that $T_r(\Sigma)$ is quasi-$\underline{FP}_\infty$.

$\square$

We also have the same result for finiteness conditions on classifying spaces

Theorem 7.2. *Let $U_r(\Sigma)$ be a Cantor-Algebra with order preserving Σ. Then the following conditions are equivalent:*
  i) $T_r(\Sigma)$ *is quasi-$\underline{F}_\infty$.*
  ii) $T_l(\Sigma)$ *is of (ordinary) type* F_∞ *for any* $1 \le l \le d$ *such that* $\gcd(l, d) \mid r$. $\square$

Proof. This follows from Theorem 4.8 exactly as Theorem 7.1. $\square$

Corollary 7.3. *Let $U_r(\Sigma)$ be a Higman algebra. Then $T_{n,r} = T_r(\Sigma)$ is quasi-$\underline{F}_\infty$.*

Proof. This follows directly from [5] and Theorem 7.2. $\square$

Corollary 7.4. *Let $U_r(\Sigma)$ be a Brown-Stein algebra. Then $T = T_r(\Sigma)$ is quasi-$\underline{F}_\infty$.*

Proof. This is a consequence of Theorem 7.2 and [24, Theorem 2.5] where it is proven that $F_r(\Sigma)$ is finitely presented and of type FP_∞ for any r. Stein's argument carries over to G and T, [24]. $\square$

Conjecture 7.5. *Let $U_r(\Sigma)$ be a Cantor-algebra. Then*
  (i) $G_r(\Sigma)$ *is quasi-$\underline{FP}_\infty$ if and only if $G_l(\Sigma)$ is of (ordinary) type FP_∞ for any $1 \le l \le d$.*
  (ii) $G_r(\Sigma)$ *is quasi-$\underline{F}_\infty$ if and only if $G_l(\Sigma)$ is of (ordinary) type F_∞ for any $1 \le l \le d$.*

Remark 7.6. Our description of centralisers of finite subgroups implies the "only if" part of this conjecture. To see this, assume that $G_r(\Sigma)$ is of type quasi-$\underline{FP}_\infty$ and choose for any $1 \le l \le d$ positive integers n, s with $n \ge 3$ such that $ln + s \equiv r \mod d$. Then there is some admissible subset $Y \in \mathfrak{A}_r(\Sigma)$ of cardinality precisely $ln + s$ and we may consider the finite group $Q \cong S_n \le G_r(\Sigma)$ defined by the action on l orbits of n elements as the natural representation of S_n and acting trivially on the remaining s elements of Y. Then Theorem 4.4 implies, using the same notation here, that

$$C_{G_r(\Sigma)}(Q) \cong H_l \times H_s.$$

Hence both H_l and H_s are of type FP_∞. Moreover, H_l is an extension

$$K_1 \rightarrowtail H_l \twoheadrightarrow G_l(\Sigma),$$

where by Remark 4.7 K_1 is a direct limit of products of $L = C_{S_n}(S_n) = 1$. Thus, in this case, $L = K_1 = 1$ implying that $G_l(\Sigma)$ is of type FP_∞.

Remark 7.7. By [5], Proposition 4.1, $F_{n,r} \cong F_{n,s}$, for any r, s. However, this is false for the groups G, in fact $G_{n,r} \cong G_{n,s}$ implies $\gcd(n-1, r) = \gcd(n-1, s)$ ([10] Theorem 6.4). Recently, Pardo founded [23] that the converse also holds true (see also [8]).

REFERENCES

[1] R. Bieri. *Homological dimension of discrete groups.* Queen Mary College Mathematical Notes. Queen Mary College Department of Pure Mathematics, London, second edition, 1981.
[2] R. Bieri, B. Renz, *Valuations on free resolutions and higher geometric invariants of groups,* Comment. Math. Helv. 63 (3) (1988), 464–497.
[3] C. Bleak, H. Bowman, A. Gordon, G. Graham, J. Hughes, F. Matucci, and E. Sapir, *Centralizers in the R. Thompson group V_n,* http://arxiv.org/abs/1107.0672
[4] M. G. Brin, *Higher dimensional Thompson groups,* Geom. Dedicata, 108 (2004), 163–192.
[5] K. S. Brown, *Finiteness properties of groups, Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985),* J. Pure Appl. Algebra 44 (1987), 45–75.

[6] J. Burillo and S. Cleary, *Metric properties of higher-dimensional Thompson's groups,* Pacific Journal of Mathematics, 248 (1) (2010), 49–62.

[7] P. M. Cohn, *Universal Algebra. Mathematics and its Applications,* 6, D. Reidel Pub. Company, 1981.

[8] W. Dicks and C. Martínez-Pérez, *Isomorphisms of Brin-Higman-Thompson groups,* http://arxiv.org/abs/1112.1606

[9] J. Hennig and F.Matucci, *Presentations for the higher dimensional Thompson groups,* http://arxiv.org/abs/1105.3714

[10] G. Higman, *Finitely presented infinite simple groups,* Notes on Pure Mathematics 8, Australian National University, Canberra 1974.

[11] D. H. Kochloukova, C. Martínez-Pérez, and B. E. A. Nucinkis, *Fixed points of finite groups acting on generalised Thompson groups,* to appear in Israel J. of Math.

[12] D. H. Kochloukova, C. Martínez-Pérez, and B. E. A. Nucinkis, *Cohomological finiteness properties of the Brin-Thompson-Higman groups 2V and 3V,*, http://arxiv.org/abs/1009.4600.

[13] P. Kropholler, C. Martínez-Pérez, and B. E. A. Nucinkis, *Cohomological finiteness conditions for elementary amenable groups,* J. reine angew. Math. 637 (2009), 49–62.

[14] I. J. Leary and B. E. A. Nucinkis, *Some groups of type VF,* Invent. Math., 151(1) (2003), 135–165.

[15] W. Lück. *Transformation groups and algebraic K-theory,* volume 1408 of *Lecture Notes in Mathematics.* Mathematica Gottingensis. Springer-Verlag, Berlin, 1989.

[16] W. Lück, *The type of the classifying space for a family of subgroups,* J. Pure Appl. Algebra, 149(2) (2000), 177–203.

[17] W. Lück and D. Meintrup, *On the universal space for group actions with compact isotropy,* Grove, Karsten (ed.) et al., Geometry and topology, Aarhus. Proceedings of the conference on geometry and topology, Aarhus, Denmark, August 10-16, 1998. Providence, RI: American Mathematical Society (AMS). Contemp. Math. 258, 293-305 (2000).

[18] C. Martínez-Pérez, *A finite generating system for $G_r(\Sigma)$,* in preparation.

[19] F. Matucci, *Algorithms and classification in groups of piecewise linear homeomorphisms* Ph.D Thesis, Cornell University, arXiv:0807.2871v1, (2008).

[20] D. Meintrup and T. Schick, *A model for the universal space for proper actions of a hyperbolic group,* New York J. Math., 8 (electronic) (2002) 1–7.

[21] G. Mislin, *Equivariant K-homology of the classifying space for proper actions,* Notes on an Advanced Course on Proper Group Actions, CRM Barcelona (2001), 7–86.

[22] B. E. A. Nucinkis, *On dimensions in Bredon homology,* Homology, Homotopy Appl. 6(1) (2004), 33–47.

[23] E. Pardo, *The Isomorphism Problem for Higman-Thompson groups,* arXiv:1006.1759v1, (2010).

[24] M. Stein, *Groups of piecewise linear homeomorphisms,* Transactions of the A.M.S. 332 (2), (1992) 477–514.

[25] K. Vogtmann, *Automorphisms of free groups and outer space,* Geom. Dedicata 94 (2002), 1–31.

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