

Topological order and topological entropy in classical systems

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We show that the concept of topological order, introduced to describe ordered quantum systems which cannot be classified by broken symmetries, also applies to classical systems. We discuss some of the fundamental properties of this type of classical order and propose how to expose it via a generalized topological entropy. Starting from a specific example, we show how to use (quantum) pure state density matrices to construct corresponding (classical) thermally mixed ones that retain precisely half of the original topological entropy, a result that we generalize to a whole class of systems.

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I. INTRODUCTION

Over the past two decades, the notion of topological order has been developed to describe quantum systems exhibiting exotic properties such as a ground state (GS) degeneracy that cannot be lifted by any local perturbations^{1,2} and fractionalized degrees of freedom.³ These systems have been shown to exhibit a type of order that defies the canonical classification *à la* Landau-Ginzburg, and that was named topological order.⁴ Levin and Wen,⁵ and Kitaev and Preskill⁶ recently proposed the idea that an appropriately defined topological entropy could be used to measure the presence of topological order in the GS of a quantum system. Such entropy can be obtained as a linear combination of von Neumann entanglement entropies of different bipartitions of the system,

$$S_{\text{topo}} = \lim_{r, R \rightarrow \infty} (-S_{1_A} + S_{2_A} + S_{3_A} - S_{4_A}), \quad (1)$$

aimed at removing all bulk and surface terms to uncover the sole topological contribution. A particular choice of the four bipartitions, as used in Ref. 5, is illustrated in Fig. 1.

Despite all the efforts to understand topological order so far, the research focused exclusively on zero-temperature quantum systems, and the possibility of realizing topological order in classical systems has been left open.⁷

In this paper, we explicitly show that the concept of topological order is not a pure zero-temperature property but that it also applies to classical systems. Starting from a known example of quantum topological order in a toric code, the Kitaev model,⁸ we show how an appropriate coupling to a thermal bath can, in a certain limit of the coupling constants, bring the system into a totally mixed state that exhibits a nonvanishing topological entropy, therefore suggesting that topological order is capable of surviving thermal dephasing. We discuss one of the fundamental properties of classical topologically ordered systems, namely, the peculiar structure of phase space divided into sectors that are connected exclusively by extensive rearrangements of the microscopic degrees of freedom. These sectors are not distinguishable under any local measurement, and therefore lead to what ought to be called *topological ergodicity breaking*. A display of glassy behavior is expected to accompany classical topological order (and possibly vice versa). We briefly comment on other properties of classical topological order, although a thorough characterization is beyond the scope of the present paper.

Finally, we reformulate the definition of topological entropy by using, instead of the von Neumann entropy, a boundary entropy that is based on the mutual information entropy defined in information theory. This allows for a natural and more symmetric extension to classical as well as quantum systems. We then generalize the results on the Kitaev model to a whole class of quantum systems with GS wave functions whose amplitudes are positive and factorizable when expressed in terms of some (local) microscopic degrees of freedom. We argue that loss of half the topological entropy between pure states and corresponding mixed states occurs whenever such a correspondence can be established.

II. TOPOLOGICAL ENTROPY IN THE KITAEV MODEL

Consider the Kitaev model in Ref. 8. It consists of spin-1/2 degrees of freedom on the bonds of a square lattice with periodic boundary conditions. The Hamiltonian of the model can be written in terms of plaquette (p) and star (s) operators as

$$H = -\lambda_A \sum_s A_s - \lambda_B \sum_p B_p, \quad (2)$$

where λ_A, λ_B are real, positive parameters; $B_p = \prod_{j \in p} \sigma_j^z$ where j labels the four spins belonging to plaquette p , and $A_s = \prod_{j \in s} \sigma_j^x$, where j labels the four spins belonging to the star centered on vertex s .

Using the fact that all A_s and B_p operators have eigenvalues ± 1 , that they all commute, and that they are subject to the constraint $\prod_s A_s = 1 = \prod_p B_p$, one can show that the GS of the model is degenerate and the dimension of the GS manifold is precisely 2^2 . Furthermore, one can prove that this degeneracy is not lifted by any local perturbation in the thermodynamic limit, and the system exhibits topological order.

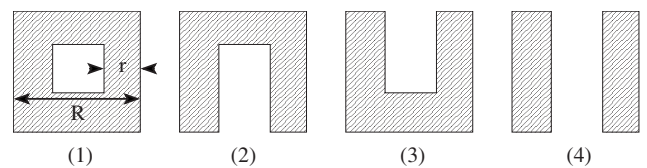


FIG. 1. Illustration of the four bipartitions used to compute the topological entropy in Ref. 5.

Let us first rederive the topological entropy of the model with a new approach that can be immediately applied to both classical and quantum systems. We are going to work within one specific topological sector. This choice will not affect the results as Hamma *et al.* showed that all sectors exhibit the same entanglement entropy.⁹ In the z basis for the spins, where all the B_p operators are diagonal, the GS is given by the equal amplitude superposition of all states where the product of the four σ^z components around each plaquette is positive. Notice that the action of a star operator A_s in the z basis does not violate the plaquette constraint above. One can show that any state in the equal amplitude superposition (within a single topological sector) is uniquely specified by a product g of star operations required to obtain it from a given reference state $|0\rangle$ that satisfies as well the plaquette constraint (and lies in the same topological sector). The GS can thus be written as

$$|\Psi\rangle = \frac{1}{|G|^{1/2}} \sum_{g \in G} g|0\rangle, \quad (3)$$

where G is the (Abelian) group of all possible products of star operators, and $|G|=2^{n/2-1}$ is the number of elements in the group (n being the total number of spin degrees of freedom). At zero temperature, the system is described by the density matrix

$$\rho = \frac{1}{|G|} \sum_{g, g' \in G} g|0\rangle\langle 0|g'. \quad (4)$$

As elegantly shown in Ref. 9 by Hamma *et al.*, the von Neumann entropy of any bipartition (A, B) of the system prepared in such GS is given by

$$S_A = -\text{Tr}(\rho_A \log_2 \rho_A) = \log_2 |G| - \log_2 (d_A d_B) (=S_B). \quad (5)$$

Here, ρ_A is obtained by taking the partial trace over subsystem B ($\rho_A = \text{Tr}_B \rho$) and d_A (d_B) is the number of elements in the subgroup $G_A \subset G$ ($G_B \subset G$) of transformations acting solely on A (B) and leaving B (A) unchanged. Moreover, Hamma *et al.* showed that whenever subsystem A is connected, then $d_A = 2^{\Sigma_A}$ and $d_B = 2^{\Sigma_B}$, where Σ_A and Σ_B are the number of single star operators acting solely on A and B , respectively.

Since we are interested in computing the topological entropy [Eq. (1)], we need to extend the above results to the case of bipartitions involving multiple disjoint components. Let us consider first the case when set A is composed of two disjoint regions A_1 and A_2 , each of them connected, as in bipartition 4 in Fig. 1. While all the operators in G that act solely on A can be obtained as products of star operators acting solely on A , the converse is no longer true for subsystem B . In fact, the product of *all* star operators acting on A_1 (i.e., both those stars acting solely on spins in A_1 and those boundary stars acting on spins in A_1 and on spins in B simultaneously) flips all σ^z in A_1 two times, therefore leaving them unchanged, while it has a nontrivial action on B . Namely, this product gives rise to an operator that flips a string of spins in B along the boundary of A_1 . Similarly, we consider subsystem A_2 and construct the product of all star

operators acting on A_2 . The composition of these two operators gives the product of all star operators acting on A (i.e., both those acting solely on spins in A and those at the boundary acting simultaneously on A and on B), which is equivalent to the product of all star operators acting solely on B . Therefore, out of the two additional operators, only one of them is a new operator on B that cannot be obtained as a product of star operators acting only on B . In this specific example then, $d_A = 2^{\Sigma_A}$ while $d_B = 2^{\Sigma_B+1}$.

In the generic case of $A = A_1 \cup \dots \cup A_{m_A}$ and $B = B_1 \cup \dots \cup B_{m_B}$, with $A_1 \dots A_{m_A}$ and $B_1 \dots B_{m_B}$ disjoint and connected, we obtain

$$d_A = 2^{\Sigma_A + m_B - 1}, \quad d_B = 2^{\Sigma_B + m_A - 1}. \quad (6)$$

Finally, we can compute the topological entropy of the system prepared in its GS [Eq. (3)] using Eq. (1) and the four different bipartitions defined in Fig. 1. From Eqs. (5) and (6) and from the fact that

$$\Sigma_{1_A} + \Sigma_{4_A} = \Sigma_{2_A} + \Sigma_{3_A},$$

$$m_{1_B} = m_{4_A} = 2 \quad (\text{and all others} = 1),$$

we obtain $S_{\text{topo}} = 2 = \log_2(D^2)$, where $D=2$ is the so-called topological dimension of the system, in agreement with the result by Levin and Wen.⁵

III. CLASSICAL TOPOLOGICAL ENTROPY

Let us consider now what happens when the system in question is coupled to a thermal bath that allows for dephasing and thermalization within the lowest lying eigenstates of the plaquette operators B_p . This is the case in the limit of $\lambda_B \rightarrow \infty$, i.e., acting as a *local* hard constraint, with $\lambda_A/T \rightarrow 0$. Later on, we discuss how the physics of this hard-constrained system is relevant to the soft case where the coupling constants are finite, and $\lambda_B \gg T \gg \lambda_A$. In the hard-constrained regime, the system is described by a totally mixed density matrix:¹⁰

$$\rho = \frac{1}{|G|} \sum_{g \in G} g|0\rangle\langle 0|g. \quad (7)$$

Following the same arguments as for the pure state [Eq. (4)] in Ref. 9, we show that

$$\rho_A = \frac{d_B}{|G|} \sum_{g \in G/G_B} g_A |0_A\rangle\langle 0_A|g_A, \quad (8)$$

where G/G_B is the quotient group, and g_A and $|0_A\rangle$ are given by the generic tensor product decompositions of $|0\rangle = |0_A\rangle \otimes |0_B\rangle$ and $g = g_A \otimes g_B$. We can then show that

$$\rho_A^n = \left(\frac{d_B}{|G|} \right)^{n-1} \rho_A \quad (9)$$

and use the identity

$$-\text{Tr}(\rho_A \log_2 \rho_A) = -\frac{1}{\ln 2} \lim_{n \rightarrow 1} \frac{\partial}{\partial n} (\text{Tr} \rho_A^n) \quad (10)$$

to finally obtain the von Neumann entropy,

$$S_A = \log_2 |G| - \log_2 d_B \quad (11)$$

$$= S_A^{(\text{pure state})} + \log_2 d_A \neq S_B. \quad (12)$$

We will comment more on this result below. From Eq. (12), we can compute the topological entropy [Eq. (1)]:

$$\begin{aligned} S_{\text{topo}}^{(\text{mixed state})} &= S_{\text{topo}}^{(\text{pure state})} + \lim_{r,R \rightarrow \infty} \left(\log_2 \frac{d_{2A} d_{3A}}{d_{1A} d_{4A}} \right) \\ &= S_{\text{topo}}^{(\text{pure state})} - 1 = 1. \end{aligned} \quad (13)$$

Notice that, while the bulk and boundary contributions present in the classical von Neumann entropy in Eq. (12) precisely cancel in the expression for the topological entropy, the topological contributions give a nonvanishing result, therefore suggesting that *topological order may survive thermal mixing* (at least in 2D hard-constrained systems).

Notice also that the topological entropy of the classical model obtained upon suppressing the off-diagonal elements in the density matrix of the Kitaev model differs from the quantum case by a factor of 1/2, namely, $\log_2 D$ instead of $\log_2(D^2)$. The physical meaning of this factor is that each of the underlying gauge structures of the Kitaev model (the so-called electric and magnetic loops) is independently responsible for half of the zero-temperature topological entropy, as we recently showed in Ref. 12.

What are the physical properties of such a classical topologically ordered system? First, it follows from the density matrix [Eq. (7)] that the expectation value for any product of spins (z component, as these are thought as classical now) is zero unless the spins belong to a closed, nonwinding loop. The system is thus a featureless “spin liquid” in a classical way (much as a paramagnet). However, it does have order, topological order, in the sense that there are different such paramagnets distinguished by the product of spins over a winding loop on the torus. When the constraint $\prod_{j \in p} \sigma_j^z = +1$ is fully enforced ($\lambda_B/T \rightarrow \infty$), classically changing the topological sector requires the concomitant flipping of a number of spins that is of the order of the linear size L of the system. The probability for such event is of order q^L , where $q^{\text{length}(\ell)}$ is the independent probability of the combined flipping of all the spins along a loop ℓ (equivalent to a quantum tunneling event). This leads to the breakdown of phase space into disconnected topological sectors (broken ergodicity), and the associated time scales grow exponentially in the size of the system, a typical signature of glassiness.¹¹ Notice the topological nature of such ergodicity breaking process: the disconnected sectors cannot be distinguished by any local measurements.

Other properties observed in quantum topologically ordered systems, such as fractionalized excitations, lead to less intuitive classical counterparts to be found, for example, in the behavior of defects. In our classical version of the Kitaev model, they appear in pairs connected by fluctuating strings, and they are likely to give rise to unusual response and relaxation processes. While the topological entropy vanishes identically, in the thermodynamic limit, as soon as the constraint is softened,^{12,13} the topological nature of the defects will persist at small enough temperatures ($\lambda_B \gg T \gg \lambda_A$), very

much as the fractionalized excitations do in the quantum case ($\lambda_B, \lambda_A \gg T$). In fact, at such low temperatures, the defects are sparse and the constraint is satisfied everywhere in between. Thus the defect creation, motion, and annihilation are locally dictated by the same “vacuum” structure that would lead to topological order were it enforced throughout the system.

IV. CONSTRUCTING A “BOUNDARY ENTROPY”

From Eq. (12), one can see that the von Neumann entropy of a classical system is not a good measure of the boundary entropy of a bipartition of the system because of the explicit dependence on the bulk entropy of one of the two partitions. One can obviate the problem by using the mutual information entropy of a bipartition, defining the boundary entropy to be

$$S_{\text{boundary}} = \frac{1}{2}(S_A + S_B - S_{A \cup B}). \quad (14)$$

Clearly, $S_{\text{boundary}} = S_A = S_B$ when the system is prepared in a pure state. In the case of a thermally mixed state, all bulk entropy contributions cancel in S_{boundary} , contrarily to the von Neumann entropies S_A and S_B , and only boundary terms are retained.

We propose here to use S_{boundary} in Eq. (14) as an alternative definition of the von Neumann entanglement entropy, applicable to quantum as well as classical systems. With this definition, Eqs. (5) and (12) imply that the entanglement entropy stored in the boundary of a bipartition of a system prepared in the pure equal amplitude superposition state [Eq. (4)] is twice as large as the entropy stored in the classical counterpart [Eq. (7)]:

$$S_{\text{boundary}}^{(\text{mixed state})} = \frac{1}{2} S_{\text{boundary}}^{(\text{pure state})}. \quad (15)$$

At least half the entanglement entropy in the Kitaev model has actually a classical origin.¹⁴

Accordingly, we redefine S_{topo} in Eq. (1) using Eq. (14),

$$\tilde{S}_{\text{topo}} = \lim_{r,R \rightarrow \infty} (-S_{\text{boundary}}^{(1)} + S_{\text{boundary}}^{(2)} + S_{\text{boundary}}^{(3)} - S_{\text{boundary}}^{(4)}). \quad (16)$$

For the Kitaev model, we immediately obtain

$$\tilde{S}_{\text{topo}}^{(\text{mixed state})} = \frac{1}{2} \tilde{S}_{\text{topo}}^{(\text{pure state})}, \quad (17)$$

in agreement with our direct calculation in Eq. (13). We argue that Eq. (16) is the proper reformulation of the topological entropy by Levin and Wen in order to extend it to classical and quantum systems alike. Together with Eq. (15), which we show hereafter to hold for a wide class of GS wave functions, the new definition of topological entropy implies that Eq. (17) also holds true within the same class. For this type of systems, at least half of the quantum topological order has therefore a classical origin. (Whether part of the remaining half of the topological entropy could be further ascribed to classical correlations, in the sense of Ref. 14, is an interesting problem.)

V. BEYOND THE KITAEV MODEL

All the calculations, as well as Eqs. (3)–(5) and (7)–(12), generalize straightforwardly to any quantum system whose GS is given by the equal amplitude superposition of a given set of states.¹⁵

On the other hand, extending our results to generic wave functions $|\Psi\rangle = |Z|^{-1/2} \sum_{g \in G} a(g) |g\rangle$ is highly nontrivial because $g = g_A \otimes g_B$ does not necessarily imply that $a(g) = a(g_A) a(g_B)$. Supporting results in this direction come from considering the case of non-negative, *local* wave functions, as defined by the condition that in the continuum limit, $a(g) \rightarrow \alpha(\phi) = \alpha^A(\phi^A) \alpha^\partial(\phi^\partial) \alpha^B(\phi^B)$, with $\alpha^\partial(\phi^\partial)$ depending only on the boundary ∂ between subsystems A and B where the two field configurations ϕ^A and ϕ^B match (and equal ϕ^∂).¹⁶ This is the case, for example, of scale invariant GS wave functions, such as those of systems at a quantum critical point.¹⁷

Consider the GS wave function(al) in the continuum limit

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \int D\phi e^{-\beta E(\phi)/2} |\phi\rangle, \quad (18)$$

where $Z = \int D\phi e^{-\beta E(\phi)}$, and $E(\phi)$ satisfies $E(\phi) = E^A(\phi^A) + E^B(\phi^B) + E^\partial(\phi^A, \phi^B)$ with $E^\partial(\phi^A, \phi^B)$ dependent only on the boundary ∂ between subsystem A and subsystem B for any bipartition (A, B) .

As discussed in Ref. 17, one can start from the corresponding pure state density matrix

$$\rho = \frac{1}{Z} \int D\phi D\phi' e^{-\beta[E(\phi) + E(\phi')]/2} |\phi\rangle \langle \phi'| \quad (19)$$

and obtain the elements of the reduced density matrix $\rho_A = \text{Tr}_B \rho$,

$$\begin{aligned} \langle \phi_1^A | \rho_A | \phi_2^A \rangle &= \frac{1}{Z} \int D\phi^B e^{-\beta[E^A(\phi_1^A) + E^B(\phi^B) + E^\partial(\phi_1^A, \phi^B)]/2} \\ &\quad \times e^{-\beta[E^A(\phi_2^A) + E^B(\phi^B) + E^\partial(\phi_2^A, \phi^B)]/2}. \end{aligned} \quad (20)$$

One can then evaluate

$$\begin{aligned} \text{Tr} \rho_A^n &= \frac{1}{Z^n} \int \prod_{i=1}^n D\phi_i e^{-\beta[\sum_i E^A(\phi_i^A) + \sum_i E^B(\phi_i^B)]} \\ &\quad \times e^{-\beta[\sum_i E^\partial(\phi_i^A, \phi_{i+1}^B) + \sum_i E^\partial(\phi_{i+1}^A, \phi_i^B)]/2} \prod_{i=1}^n \delta(\phi_i^\partial, \phi_{i+1}^\partial), \end{aligned} \quad (21)$$

where, by construction, $\phi_{n+1}^\partial \equiv \phi_1^\partial$ and $\phi_{n+1}^A \equiv \phi_1^A$. The term $\prod_{i=1}^n \delta(\phi_i^\partial, \phi_{i+1}^\partial)$ is the boundary constraint arising from the product of ρ_A with itself (which enforces the boundaries to all match, because ϕ_i^A matches ϕ_i^B , which then matches ϕ_{i+1}^A , which, in turn matches ϕ_{i+1}^B , and so on). If we factorize the integral over $\prod_{i=1}^n D\phi_i$ as $\prod_{i=1}^n D\phi_i^A D\phi_i^B D\phi_i^\partial$, we can simplify the integrals over boundaries and the product of boundary delta functions to obtain

$$\begin{aligned} \text{Tr} \rho_A^n &= \frac{1}{Z^n} \int D\phi^\partial e^{-\beta n E^\partial} \int \prod_{i=1}^n D\phi_i^A \int \prod_{i=1}^n D\phi_i^B \\ &\quad \times e^{-\beta[\sum_i E^A(\phi_i^A) + \sum_i E^B(\phi_i^B)]}, \end{aligned} \quad (22)$$

where we explicitly used the assumption that all the $E^\partial(\phi_i^A, \phi_i^B)$ terms depend solely on the respective boundaries ϕ_i^∂ . Now that the delta functions are no longer present, we can address the remaining integrals as follows. The condition that the n fields ϕ_i^A and the n fields ϕ_i^B agree at the boundary is equivalent to having one replica free to take any value and the remaining $n-1$ to be pinned, with fixed (Dirichlet) boundary conditions. One thus obtains¹⁸

$$\text{Tr} \rho_A^n = \left(\frac{Z_D^A Z_D^B}{Z} \right)^{n-1} \frac{1}{Z} \int D\phi^\partial D\phi^A D\phi^B e^{-\beta n E^\partial - \beta E^A(\phi^A) - \beta E^B(\phi^B)}, \quad (23)$$

where $Z_D^A = \int_D D\phi^A e^{-\beta E^A(\phi^A)}$ is given by the integral over subsystem A with Dirichlet boundary conditions, and, equivalently, for Z_D^B .

Finally, using identity (10), we can compute the von Neumann entropy,

$$\begin{aligned} S_A &= -\log_2 \frac{Z_D^A Z_D^B}{Z} + \frac{1}{Z} \int D\phi \frac{\beta}{\ln 2} E^\partial e^{-\beta E(\phi)} \\ &= -\log_2 \frac{Z_D^A Z_D^B}{Z} + \frac{\beta}{\ln 2} \langle E^\partial \rangle. \end{aligned} \quad (24)$$

The case of the totally mixed state,

$$\rho = \frac{1}{Z} \int D\phi e^{-\beta E(\phi)} |\phi\rangle \langle \phi|, \quad (25)$$

involves similar calculations, where the term $\prod_{i=1}^n \delta(\phi_i^\partial, \phi_{i+1}^\partial)$ in Eq. (21) gets replaced by $\prod_{i=1}^n \delta(\phi_i^\partial, \phi_{i+1}^\partial) \delta(\phi_i^A, \phi_{i+1}^A)$. As a result,

$$\text{Tr} \rho_A^n = \left(\frac{Z_D^B}{Z} \right)^{n-1} \frac{1}{Z} \int D\partial D\phi^A D\phi^B e^{-\beta n E^\partial - \beta n E^A(\phi^A) - \beta E^B(\phi^B)}, \quad (26)$$

and

$$\begin{aligned} S_A &= -\log_2 \frac{Z_D^B}{Z} + \frac{1}{Z} \int D\phi \frac{\beta}{\ln 2} [E^\partial + E^A] e^{-\beta E(\phi)} \\ &= -\log_2 \frac{Z_D^B}{Z} + \frac{\beta}{\ln 2} (\langle E^\partial \rangle + \langle E^A \rangle). \end{aligned} \quad (27)$$

Notice that, while for an equal amplitude superposition, the totally mixed state could be reached via an appropriate high-temperature limit (i.e., via coupling to a thermal bath), this is no longer true for Eqs. (19) and (25), which should be interpreted only as a recipe to construct an associated classical system, given the quantum one.

In conclusion, we obtain

$$S_A^{\text{(pure state)}} = \beta F_A + \beta F_B - \beta F_{A \cup B} + \frac{\beta}{\ln 2} \langle E^\partial \rangle,$$

$$S_A^{(\text{mixed state})} = \beta F_B - \beta F_{A \cup B} + \frac{\beta}{\ln 2} \langle E^\partial \rangle + \frac{\beta}{\ln 2} \langle E^A \rangle,$$

where $\beta F_A = -\log_2 Z_D^A$, $\beta F_B = -\log_2 Z_D^B$, and $\beta F_{A \cup B} = -\log_2 Z$.

For the boundary entropy defined in Eq. (14), we find that

$$\begin{aligned} S_{\text{boundary}}^{(\text{mixed state})} &= \frac{1}{2} \left[\beta F_B - \beta F_{A \cup B} + \frac{\beta}{\ln 2} \langle E^\partial \rangle + \frac{\beta}{\ln 2} \langle E^A \rangle + \beta F_A \right. \\ &\quad \left. - \beta F_{A \cup B} + \frac{\beta}{\ln 2} \langle E^\partial \rangle + \frac{\beta}{\ln 2} \langle E^B \rangle + \beta F_{A \cup B} \right. \\ &\quad \left. - \frac{\beta}{\ln 2} \langle E^{A \cup B} \rangle \right] \\ &= \frac{1}{2} \left[\beta F_A + \beta F_B - \beta F_{A \cup B} + \frac{\beta}{\ln 2} \langle E^\partial \rangle \right] \\ &= \frac{1}{2} S_{\text{boundary}}^{(\text{pure state})}. \end{aligned} \quad (28)$$

Thus, once again, for the boundary entropy defined in Eq. (14), we find that Eq. (15) [and therefore Eq. (17)] *still holds*. For any quantum system whose GS wave function is local and with positive amplitudes, there exists an associated classical system which exhibits precisely half the topological entropy. The configurations of the classical system have as Boltzmann weights the squares of the amplitudes in the associated quantum GS.

VI. CONCLUSIONS

In this paper, we show how the concept of topological order applies to classical systems. We propose a generalization of the definition of topological entropy^{5,6} that applies to

classical and quantum systems alike. We use this topological entropy to identify topologically ordered phases at the classical level, and we discuss some of the properties of such classical systems. We also show how to construct thermally mixed (classical) density matrices that retain precisely half the topological entropy of associated pure state density matrices for a special class of quantum systems. Our results imply that for quantum systems whose GS wave function in a given basis can be gauged so as to have non-negative (local) amplitudes, quantum topological order is present if and only if the corresponding classical system obtained by removing all off-diagonal matrix elements of the quantum density matrix is also topologically ordered, as defined above.

The result that topological order applies to classical systems could possibly provide a way to classify classical orders without an obvious order parameter. We end with a speculative note on the possibility that this might be the case in glassy systems. No order parameter can be constructed in glasses from equal-time correlation functions (as opposed to the two-time Edwards-Anderson order parameter) of physical degrees of freedom (as opposed to replicated variables). This fact is suggestive that, much as in quantum topologically ordered systems such as spin liquids, no local order parameter can detect the underlying order of the glassy states, and the hidden order could indeed be topological. In this case, our finding that topological order and topological entropy can be defined in classical systems may have implications to the understanding of the physics of glassy systems.

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- ¹⁶A similar notion of separability of the wave function amplitudes has been introduced by Hamma *et al.*, [A. Hamma, R. Ionicioiu, and P. Zanardi, Phys. Rev. A **72**, 012324 (2005)].
- ¹⁷E. Fradkin and J. E. Moore, Phys. Rev. Lett. **97**, 050404 (2006).
- ¹⁸The boundary condition of the $n-1$ pinned replicas is determined by the field configuration (at the boundary) of the single replica that has free boundary conditions. The fixed boundary condition for the $n-1$ replicas should be averaged. One case in which it is not necessary to average over boundary conditions is that of free bosons, in which case, a rotation $\tilde{\phi}_1^A = \frac{1}{\sqrt{n}} \sum_i \phi_i^A$, $\tilde{\phi}_i^A = \phi_i^A - \phi_{i-1}^A / \sqrt{2}$, $i=2, \dots, n$, makes it clear that $\tilde{\phi}_1$ is free while all others have Dirichlet (i. e., vanishing) boundary conditions.