There Are No $R^3 \times S^1$ Vacuum Gravitational Instantons

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Gravitational instantons, solutions to the Euclidean Einstein equations, with topology $R^3 \times S^1$ arise naturally in finite-temperature quantum gravity. It is shown here that all such instantons must have the same asymptotic structure as the Schwarzschild instanton. From this follows that if the Ricci tensor of such a manifold is non-negative it must be flat. Hence there is no nontrivial vacuum gravitational instanton on $R^3 \times S^1$. This places a significant restriction on the instabilities of hot flat space. Another consequence is that any static vacuum Lorentzian Kaluza-Klein solution is flat.

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Instantons, solutions to the classical Euclideanized field equations, play a prominent role in quantum field theory. One situation in which they arise is in computing a transition amplitude in the standard, Lorentzian, signature. The transition amplitude may be dominated by a classical solution to the field equations. If such does not exist, it may be possible to deform the contour of integration into a region of imaginary time and find that the integral may be dominated by a solution to the Euclidean field equations, an instanton. Such an instanton may be interpreted as a tunneling solution.

Another use of instantons, of much more relevance to this Letter, is in finite-temperature quantum field theory. It can be shown that the partition function at some given temperature is equivalent to a transition amplitude in which the time is made both imaginary and periodic, with period $\tau = \beta = 1/T$, where T is the temperature [1]. Again, the partition function may be dominated by classical solutions, but now the classical solutions which one considers are periodic as well as being Euclidean.

Finite-temperature quantum gravity has been intensively studied ever since the discovery of black hole thermodynamics by Beckenstein [2] and Hawking [3]. As part of this investigation, people have tried to find gravitational instantons, especially those with a periodic character. It is clear that one can identify flat slices of flat Euclidean four-space to give a flat instanton. Further, it was realized that if one took the standard Schwarzschild solution and Euclideanized it by $t \rightarrow i\tau$, one got a regular vacuum instanton if one simultaneously made it periodic with period $\tau_0 = 8\pi M$ [4].

The Schwarzschild instanton has topology $R^2 \times S^2$. It is widely assumed that there is no vacuum gravitational instanton with topology $R^3 \times S^1$ (except rolled up flat space, of course). Such an instanton would signal an instability of hot flat space [5]. Here we will give a proof that no such instanton exists. Witten [6] has already shown that there is no nontrivial vacuum gravitational instanton on R^4. The technique we use can be thought of as an adaptation of the Witten proof.

We assume that the metric is asymptotically flat in the R^3 directions. We further assume that the manifold has a constant period (τ_0) near infinity, but we do *not* assume that the period remains constant in the interior. In other words, we are assuming a constant temperature at infinity but we do not care what happens in the interior.

The first point to be resolved is the asymptotic behavior of Ricci-flat Riemannian metrics on $R^3 \times S^1$. Since the metric is asymptotically flat, we can write the Einstein equation in the "Lorentz" gauge [7]. This means that we consider

$$h_{\mu\nu} = g_{\mu\nu} - \delta_{\mu\nu} \tag{1}$$

and reverse its trace by defining

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu} g^{\alpha\beta} h_{\alpha\beta} \,. \tag{2}$$

The "Lorentz gauge" means making a coordinate transformation so that $\tilde{h}$ satisfies

$$\tilde{h}_{\mu\nu,\nu} = 0 \,. \tag{3}$$

Such a transformation can always be made. The linearized Einstein equation in the Lorentz gauge is simply

$${}^{(4)}\nabla^2 \tilde{h}_{\mu\nu} = 0 \,. \tag{4}$$

Equation (4) holds only in the weak-field region, but this is the part of the space where we assume that we have a constant period, τ_0. We can write $\tilde{h}_{\mu\nu}$ as a Fourier series in u, the periodic coordinate,

$$\tilde{h}_{\mu\nu} = \sum_{n=-\infty}^{\infty} \phi_{\mu\nu n}(\mathbf{r}) e^{in\omega u} \,, \tag{5}$$

where $\omega = 2\pi/\tau_0$ and $\mathbf{r}$ is the spatial radius vector. We substitute Eq. (5) into Eq. (4) to get

$$\begin{aligned} {}^{(4)}\nabla^2 \tilde{h}_{\mu\nu} &= {}^{(3)}\nabla^2 \tilde{h}_{\mu\nu} + \frac{\partial^2 \tilde{h}_{\mu\nu}}{\partial u^2} && (6) \\ &= \sum_{n=-\infty}^{\infty} [{}^{(3)}\nabla^2 \phi_{\mu\nu n}(\mathbf{r}) - n^2\omega^2 \phi_{\mu\nu n}(\mathbf{r})] e^{in\omega u} && (7) \\ &= 0 \,. && (8) \end{aligned}$$

Since the Fourier components are linearly independent, Eq. (8) implies that each mode must satisfy

$$^{(3)}\nabla^2\phi_{\mu\nu n}(\mathbf{r}) - n^2\omega^2\phi_{\mu\nu n}(\mathbf{r}) = 0 \,, \quad \forall\, n \,. \tag{9}$$

It can be shown that [8] the solutions of Eq. (9) for $n \neq 0$ decay exponentially at infinity. Therefore, the asymptotic behavior is dominated by the $n = 0$ mode which satisfies

$$^{(3)}\nabla^2\phi_{\mu\nu 0} = 0 \,. \tag{10}$$

This means that the $n = 0$ mode is determined by the harmonic functions of the flat-space three-Laplacian. Hence the leading term is of the form C_0/r, where C_0 is a constant. More precisely

$$\tilde{h}_{\mu\nu} \simeq C_{\mu\nu}/r \tag{11}$$

near infinity, where $C_{\mu\nu}$ are ten constants and r is the three-dimensional radial distance. All the terms with u dependence fall off exponentially.

However, we must simultaneously satisfy the Lorentz condition [Eq. (3)]. This implies

$$\frac{\partial}{\partial x^\mu}\left(\frac{C_{\mu\nu}}{r}\right) = 0 \,. \tag{12}$$

Therefore all the $C_{\mu\nu}$'s must vanish except C_{00}, because r depends only on the three asymptotically flat coordinates (x, y, z). Hence near infinity we have

$$\tilde{h}_{\mu\nu} \simeq \frac{C_{00}}{r}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,. \tag{13}$$

We can reverse Eq. (3) to give

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}g^{\alpha\beta}\tilde{h}_{\alpha\beta} \,. \tag{14}$$

When we substitute (13) in (14) we get

$$h_{\mu\nu} \simeq \frac{C_{00}}{2r}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \,. \tag{15}$$

Finally, using $-4M = C_{00}$, we get

$$g_{\mu\nu} \simeq \begin{pmatrix} 1 - \frac{2M}{r} & 0 & 0 & 0 \\ 0 & 1 + \frac{2M}{r} & 0 & 0 \\ 0 & 0 & 1 + \frac{2M}{r} & 0 \\ 0 & 0 & 0 & 1 + \frac{2M}{r} \end{pmatrix} + \mathcal{O}\left(\frac{1}{r^2}\right) \,. \tag{16}$$

We should recognize Eq. (16) as being just the leading part of the Schwarzschild instanton [4]. This should come as no surprise because all we have been doing is determining the asymptotic behavior of the gravitational instanton, and $R^3 \times S^1$ is indistinguishable from $R^2 \times S^2$ near infinity. This result holds even if the Ricci tensor of the manifold is nonzero; all we require is that it fall off sufficiently rapidly at infinity.

This means that if we wish to obtain a nonexistence result we need a global argument; a "local near infinity" argument will never get us anything. This is why we try to mimic the Witten R^4 proof [6].

Following Witten, we seek a solution to the equation

$$^{(4)}\nabla^2\phi^0 = 0 \,, \quad \phi^0 \to u \text{ near infinity.} \tag{17}$$

Let us write

$$\phi^0 = u + \phi_1^{(0)} \,. \tag{18}$$

Equation (17) can be written as

$$^{(4)}\nabla^2\phi_1^{(0)} = -{}^{(4)}\nabla^2 u \,, \quad \phi_1^{(0)} \to 0 \text{ near infinity.} \tag{19}$$

We have

$$^{(4)}\nabla^2 u = -g^{\alpha\beta}\Gamma^u_{\alpha\beta} = \mathcal{O}(1/r^3) \,. \tag{20}$$

We assume that the given (curved) metric can be smoothly distorted back to flat space. It is easy to show that the four-Laplacian has no harmonic function θ that vanishes at infinity anywhere along this sequence; just integrate by parts $\theta^{(4)}\nabla^2\theta$ over the manifold, and throw away the surface term to leave the integral of $(\nabla\theta)^2$. This is zero, so θ itself must also be zero. The Laplacian on flat $R^3 \times S^1$ is an isomorphism; we can explicitly write down the Green function. The method of continuity [9] now guarantees that the isomorphism property holds along the whole sequence of metrics. This means that a solution to (19) exists which decays at infinity at least as fast as $1/r$, and hence we have a solution to (17).

The following identity is now used:

$$\nabla_\mu(\nabla_\nu\phi^0\nabla^\nu\nabla^\mu\phi^0) = (\nabla_\nu\nabla_\mu\phi^0)^2 + \nabla_\nu\phi^0\nabla_\mu\nabla^\nu\nabla^\mu\phi^0 \,. \tag{21}$$

Let us add and subtract $\nabla_\nu\phi^0\nabla^\nu\nabla_\mu\nabla^\mu\phi^0$ [which is identically zero from (17)] to Eq. (21) to give

$$\nabla_\mu(\nabla_\nu\phi^0\nabla^\nu\nabla^\mu\phi^0) = (\nabla_\nu\nabla_\mu\phi^0)^2 + \nabla_\nu\phi^0\nabla_\mu\phi^0 R^{\mu\nu} \,. \tag{22}$$

Since we assume that the manifold is Ricci flat, we can throw away the last term in Eq. (22). Let us now integrate (22) over the whole manifold to give

$$\int \nabla_\mu(\nabla_\nu\phi^0\nabla^\nu\nabla^\mu\phi^0)\sqrt{g}d^4x = \int (\nabla_\nu\nabla_\mu\phi^0)^2\sqrt{g}d^4x \,. \tag{23}$$

The left-hand side of (23) can be turned into a surface integral at infinity to give

$$\oint_\infty \nabla_\nu\phi^0\nabla^\nu\nabla^\mu\phi^0\hat{n}_\mu\sqrt{g}d^3S = \int (\nabla_\nu\nabla_\mu\phi^0)^2\sqrt{g}d^4x \,. \tag{24}$$

Any term in the integrand of the surface integral that falls off faster than $1/r^2$ can be neglected because the "area" of the "surface at infinity" blows up like $4\tau_0\pi r^2$, where τ_0 is the period in the u direction. This allows us to ignore the contribution from the $\phi_1^{(0)}$ term. The only term that remains is the connection from $\nabla^\nu\nabla^\mu\phi^0$. It is easy to show that the surface integral reduces to

$$-4\pi\tau_0\Gamma^u_{ur} = -4\pi M\tau_0 \ . \tag{25}$$

Hence (24) gives us

$$\int (\nabla_\nu\nabla_\mu\phi^0)^2\sqrt{g}d^4x = -4\pi M\tau_0 \ . \tag{26}$$

Therefore M, the analog of the Schwarzschild mass, must be negative.

However, ϕ^0 is not the only natural harmonic function we could define on this manifold. Another candidate is

$$^{(4)}\nabla^2\phi^1 = 0 \ , \quad \phi^1 \to x \text{ near infinity.} \tag{27}$$

We repeat the calculation following (17) and write

$$\phi^1 = x + \phi_1^{(1)} \ , \tag{28}$$

and we again have

$$^{(4)}\nabla^2 x = -g^{\alpha\beta}\Gamma^x_{\alpha\beta} = \mathcal{O}(1/r^3) \ . \tag{29}$$

The connection, in general, falls off like $1/r^2$, but a cancellation occurs in the particular combination in (29) so that the leading terms cancel. In other words, the standard coordinates on any manifold which is asymptotically Schwarzschildian are "almost harmonic."

We can use exactly the same identity as before, just substituting ϕ^1 for ϕ^0. Now we get, instead of (26),

$$\int (\nabla_\nu\nabla_\mu\phi^1)^2\sqrt{g}d^4x = -4\pi r^2\Gamma^x_{xr} = +4\pi M\tau_0 \ . \tag{30}$$

Now we get $M > 0$. But we have already shown $M < 0$.

The only way that (30) can be compatible with (26) is that we really have $M = 0$, and this implies

$$\nabla_\nu\nabla_\mu\phi^0 = \nabla_\nu\nabla_\mu\phi^1 = 0 \ . \tag{31}$$

The existence of these functions (and their equivalents ϕ^2 and ϕ^3) whose double derivatives vanish is sufficient to show that the four-space is flat. This nonexistence argument extends to the case where we have a nonzero Ricci tensor if the Ricci tensor has non-negative eigenvalues (see [10] for a similar result in the R^4 case).

The undoubted existence of the Schwarzschild instanton does not contradict this nonexistence result. It is very easy to repeat this calculation in the Schwarzschild case because the standard time coordinate t in the Schwarzschild instanton is harmonic due to the static nature of the metric. However, the length of the vector $\nabla_\nu t$ becomes unboundedly large as one approaches $r = 2M$. When we evaluate expression (24), it turns out that the integral of $\nabla_\nu t\nabla^\nu\nabla^\mu t\, \hat{n}_\mu$ actually diverges on any surface that shrinks to $r = 2M$. Thus the negative term at infinity is more than compensated for by a positive interior term. Such behavior will not occur on a manifold which is topologically $R^3 \times S^1$.

Over the years, a number of "static + vacuum implies trivial" theorems have been derived. (The only counterexample to date has been the Einstein-Yang-Mills system [11].) The result obtained here, that a Ricci-flat Riemannian manifold with topology $R^3 \times S^1$ is flat, can be used to show that "static + vacuum implies trivial" is valid for standard (Lorentzian) Kaluza-Klein theory. In Kaluza-Klein theory we consider a manifold with topology $R^4 \times S^1$ where the S^1 and three of the four directions in R^4 are spacelike. Let us consider a static, vacuum Kaluza-Klein manifold. By static we mean that there exists a timelike, surface-forming Killing vector. Obviously, the surface orthogonal to the Killing vector, M^4, is a Riemannian manifold with topology $R^3 \times S^1$. The "Einstein" equations in this case reduce to

$$NR_{\mu\nu} = \nabla_\mu\nabla_\nu N, \ \ ^{(4)}\nabla^2 N = 0 \ , \tag{32}$$

where $R_{\mu\nu}$ is the Ricci tensor of the four-manifold, and N is the norm of the Killing vector. The second equation in (32) tells us that N must be constant, and the first equation then tells us that the Ricci tensor must vanish. Therefore M^4 must be flat and hence the five-manifold must be trivial.

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