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STRONG TRANSVERSALS IN HYPERGRAPHS AND DOUBLE TOTAL DOMINATION IN GRAPHS*

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Abstract. Let H be a 3-uniform hypergraph of order n and size m, and let T be a subset of vertices of H. The set T is a strong transversal in H if T contains at least two vertices from every edge of H. The strong transversal number $\tau_s(H)$ of H is the minimum size of a strong transversal in H. We show that $7\tau_s(H) \le 4n + 2m$, and we characterize the hypergraphs that achieve equality in this bound. In particular, we show that the Fano plane is the only connected 3-uniform hypergraph H of order $n \ge 6$ and size m that achieves equality in this bound. A set S of vertices in a graph G is a double total dominating set of G if every vertex of G is adjacent to at least two vertices in S. The minimum cardinality of a double total dominating set of G is the double total domination number $\gamma_{\times 2,t}(G)$ of G. Let G be a connected graph of order n with minimum degree at least three. As an application of our hypergraph results, we show that $\gamma_{\times 2,t}(G) \le 6n/7$ with equality if and only if G is the Heawood graph (equivalently, the incidence bipartite graph of the Fano plane). Further if G is not the Heawood graph, we show that $\gamma_{\times 2,t}(G) \le 11n/13$, while if G is a cubic graph different from the Heawood graph, we show that $\gamma_{\times 2,t}(G) \le 5n/6$, and this bound is sharp.

Key words. total domination, transversals, cubic graphs, hypergraphs

AMS subject classifications. 05C65, 05C69

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1. Introduction.

In this paper, we continue the study of transversals in hypergraphs and their interplay with total domination in graphs. A *hypergraph* $H = (V, E)$ is a finite set V of elements, called *vertices*, together with a finite multiset E of arbitrary subsets of V, called *edges*. Two edges in a hypergraph are said to be *overlapping* if they intersect in at least two vertices. A k-edge in H is an edge of size k in H. The hypergraph H is said to be k-uniform if every edge of H is a k-edge. The number of i-edges (of size i) in H is denoted by $e_i(H)$. We denote the number of edges in H of size at least 3 by $e_{\ge 3}(H)$.

The *degree* of a vertex v in H, denoted $d_H(v)$ or simply by $d(v)$ if H is clear from the context, is the number of edges of H which contain v. A vertex of degree k is called a *degree-k vertex*. The minimum degree (resp., maximum degree) among the vertices of H is denoted by $\delta(H)$ (resp., $\Delta(H)$). The *open neighborhood* of v in H is $N(v) = \{u \in V \mid \{u, v\} \subseteq e \text{ for some } e \in E\}$. Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq e$. Further, x and y are *connected* if there is a sequence $x = v_0, v_1, v_2, \dots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i = 1, 2, \dots, k$. A *connected hypergraph* is a hypergraph in which every pair of vertices are connected. A *(connected) component* of a hypergraph H is a maximal connected subhypergraph of H.

A subset T of vertices in a hypergraph H is a *transversal* in H if T has a nonempty intersection with every edge of H, while T is a *strong transversal* in H if T is a transversal in H that contains at least two vertices from every k-edge in H where $k \ge$

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3. The *transversal number* $\tau(H)$ and the *strong transversal number* $\tau_s(H)$ of H are the minimum cardinalities of a transversal and strong transversal in H, respectively. A strong transversal of H of size $\tau_s(H)$ is called a $\tau_s(H)$-set.

In this paper, we also continue the study of total domination in graphs, which is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [8, 9]. A recent survey of total domination in graphs can be found in [10]. Let G be a graph with no isolated vertex, and let S be a subset of vertices of G. The set S is a *total dominating set* of G if every vertex of G is adjacent to a vertex in S (other than itself), while S is a *double total dominating set* (DTDS) of G if every vertex of G is adjacent to at least two vertices in S. The minimum cardinalities of a total dominating set and a DTDS in G are the *total domination number* $\gamma_t(G)$ and the *double total domination number* $\gamma_{\times 2,t}(G)$ of G, respectively. A DTDS of G of cardinality $\gamma_{\times 2,t}(G)$ is called a $\gamma_{\times 2,t}(G)$-set.

We remark that a DTDS is also called a 2-tuple total dominating set in the literature. The more general concept of a *k-tuple dominating set* S, where every vertex is either in S and has at least $k-1$ neighbors in S or is not in S and has at least k neighbors in S, has been studied by several authors (see, for example, [4, 5, 6, 7, 16, 17] and elsewhere), while the analogous concept of a *k-tuple total dominating set* S, where every vertex has at least k neighbors in S, is studied in [11] and elsewhere.

For notation and graph theory terminology we in general follow [8]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$, and the *closed neighborhood of* v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$ and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. If $Y \subseteq V$, then the set S is said to *totally dominate* the set Y if $Y \subseteq N(S)$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. We denote the degree of v in G by $d_G(v)$ or simply by $d(v)$ if the graph G is clear from context. The minimum degree (resp., maximum degree) among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$).

Two edges in a graph G are *independent* if they are not adjacent in G. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum matching of G is called the *matching number* of G, which we denote by $\alpha'(G)$.

Much of the recent interest in total domination in graphs arises from the fact that total domination in graphs can be translated to the problem of finding transversals in hypergraphs. For a graph $G = (V, E)$, we denote by H_G the open neighborhood hypergraph (ONH) of G; that is, H_G is the hypergraph with vertex set $V(H_G) = V$ and with edge set $E(H_G) = \{N_G(x) \mid x \in V(G)\}$ consisting of the open neighborhoods of vertices of V in G. We observe that $\gamma_t(G) = \tau(H_G)$. This idea of using transversals in hypergraphs to obtain results on total domination in graphs first appeared in a paper by Thomassé and Yeo [18] and subsequently in several other papers, including [13, 14, 15].

Chvátal and McDiarmid [3] and Tuza [19] independently established that if H is a hypergraph on n vertices and m edges with all edges of size at least three, then $4\tau(H) \le n+m$. Thomassé and Yeo [18] proved that if H is a hypergraph on n vertices and m edges with all edges of size at least four, then $21\tau(H) \le 5n + 4m$.

In this paper, we obtain upper bounds on the strong transversal number of a hypergraph along the lines of the Chvátal–McDiarmid and Tuza upper bound and the Thomassé–Yeo upper bound on the transversal number. We then study the interplay between strong transversals in hypergraphs and double total domination in graphs. We shall need the following key observation.

Observation 1. If G is a graph with $\delta(G) \geq 3$, then $\gamma_{\times 2,t}(G) = \tau_s(H_G)$.

Let H be a connected 3-uniform hypergraph on $n \geq 6$ vertices and m edges. We remark that there are many pairs of numbers (a, b) such that the inequality $\tau_s(H) \leq an + bm$ holds; clearly, both $(1, 0)$ and $(0, 2)$ work. By Observation 1, this inequality implies that if G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_{\times 2,t}(G) \leq (a + b)/n$ since the ONH H_G of G has size $m = n$. We wish to choose a and b so that their sum is as small as possible, as this would be interesting for double total domination purposes. This is the same approach used for total domination where $\tau(H) \leq n/4 + m/4$ is used when H is 3-uniform (see [3, 19]) and $\tau(H) \leq 5n/21 + 4m/21$ is used when H is 4-uniform (see [18]). The whole idea is to minimize $a + b$, and the big problem is often deciding which values of a and b to use.

Applying probabilistic arguments used by Alon [1], we note that by choosing vertices at random with probability p and then picking one or two more vertices from edges that had only one or zero vertices picked randomly, we have $\tau_s(H) \leq pn + (3p(1 - p)^2 + 2(1 - p)^3)m$ for all $0 \leq p \leq 1$. Hence, (a, b) can be chosen to be the pair $(p, 3p(1 - p)^2 + 2(1 - p)^3)$ for any value of p with $0 \leq p \leq 1$. However, for such a choice of a and b, we note that $a + b = 2 - 2p + p^3$, which attains its minimum nonnegative value when $p = \sqrt{2/3}$. Thus for all such choices of a and b, we note that $a + b > 0.9113$. To improve this trivial probabilistic lower bound on the sum of a and b, a much more detailed and intricate analysis is needed.

In this paper we determine that $a = 4/7$ and $b = 2/7$ work in our case. Furthermore we note that $a + b$ cannot be less than $6/7$ due to the Fano plane. Of course it would also be interesting to determine other values of a and b, but they cannot improve the $6/7$ bound we get for double total domination.

1.1. The hypergraph family $\mathcal{H}$.

In order to state our main results, we first define a hypergraph family $\mathcal{H}$.

DEFINITION 1. *Let $\mathcal{H} = \{F_4, F_5, F_6, F_7, H_3, H_4, H_5, T_5\}$ be a family of eight hypergraphs shown in Figure* 1. *We remark that the Fano plane F_7 is obtained from F_6 by adding a new vertex v and expanding the three* 2*-edges in F_6 to three* 3*-edges that contain v.*

2. Main results.

We shall prove the following two hypergraph results. A proof of Theorem 1 is presented in section 3.6, while a proof of Theorem 2 can be found in section 3.8.

THEOREM 1. *If H is a hypergraph with only* 2*-edges and* 3*-edges, then*

$$14\tau_s(H) \leq 8|V(H)| + 4e_3(H) + 2e_2(H)$$

with equality if and only if every component of H belongs to $\mathcal{H}$.

THEOREM 2. *If H is a connected* 3*-uniform hypergraph on $n \geq 6$ vertices and m edges, then $7\tau_s(H) \leq 4n + 2m$ with equality if and only if H is the Fano plane.*

In order to state our main graph theory results, let G_{14} be the Heawood graph (or, equivalently, the incidence bipartite graph of the Fano plane) on 14 vertices shown in Figure 2.

Using our earlier hypergraph results, we shall prove the following results about double total domination in graphs.

THEOREM 3. *If G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_{\times 2,t}(G) \leq 6n/7$ with equality if and only if G is the Heawood graph G_{14}.*

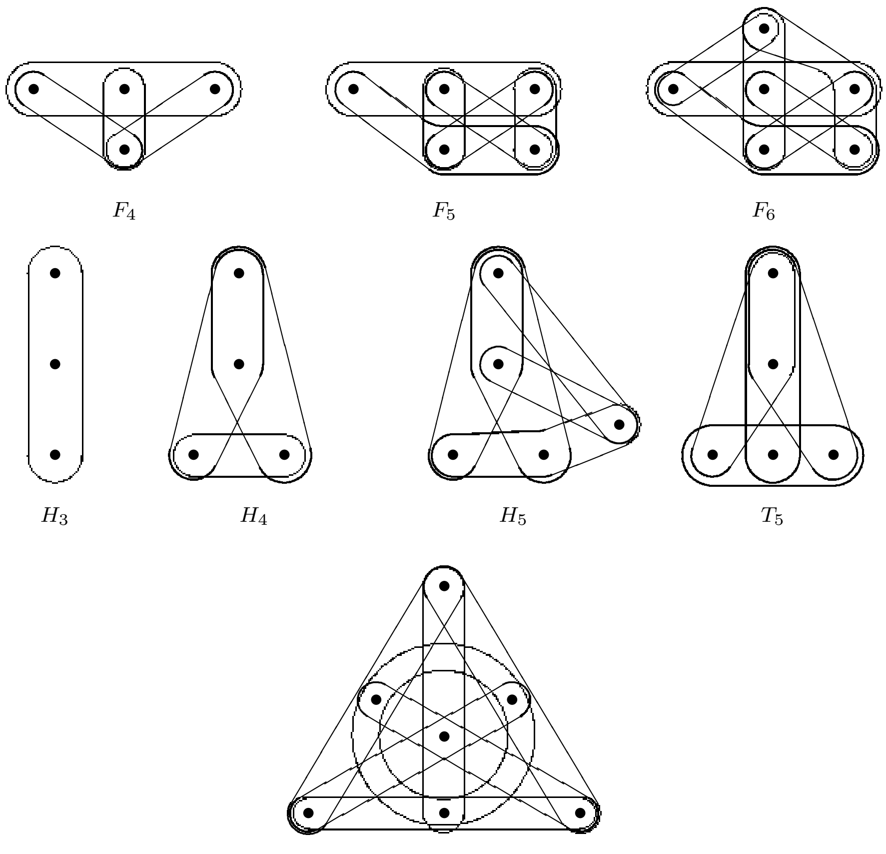

FIG. 1. *The hypergraphs F_4, F_5, F_6, H_3, H_4, H_5, T_5, and F_7.*

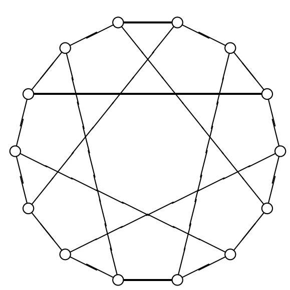

FIG. 2. *The Heawood graph G_{14}.*

THEOREM 4. *If $G \neq G_{14}$ is a connected cubic graph of order n, then $\gamma_{\times 2,t}(G) \leq 5n/6$, and this bound is sharp.*

THEOREM 5. *If $G \neq G_{14}$ is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_{\times 2,t}(G) \leq 11n/13$.*

Proofs of Theorems 3, 4, and 5 can be found in sections 4.3, 4.1, and 4.2, respectively. Our proof techniques demonstrate an interplay between strong transversals in hypergraphs and double total domination in graphs.

3. Proof of hypergraph results. In order to prove our two main hypergraph results, namely, Theorems 1 and 2, we need to prove two key results. However, in order to state these results, we shall need certain hypergraph definitions.

3.1. Hypergraph definitions. In the definitions that follow, we assume throughout that H is a hypergraph with all edges of size at least 2.

DEFINITION 2. *By* shrinking *an edge e of H we mean either deleting the edge e if e is a 2-edge or replacing e by an arbitrary 2-edge e', where $e' \subset e$ if e is an edge of size at least 3. Further, we define $e^* = e$ if e is a 2-edge and $e^* = e'$, where e' is an arbitrary selected 2-element subset of e if e is a 3-edge. If we shrink an edge in H, then we say that the resulting hypergraph is obtained from H by an* edge-change.

DEFINITION 3. *For a subset X of vertices in H, let $H - X$ denote a hypergraph obtained from H by*

- *deleting the vertices in X,*
- *deleting all 2-edges incident with X,*
- *deleting all edges of size at least 3 that intersect X in at least two vertices,*
- *shrinking all edges of size at least 3 that intersect X in exactly one vertex to a 2-edge that contains no vertex of X, and*
- *deleting the resulting set of isolated vertices, if any.*

If $X = \{v\}$, then we simply denote $H - X$ by $H - v$. We note that if T' is a strong transversal in $H - X$, then $T' \cup X$ is a strong transversal in H.

DEFINITION 4. *If F is a set of edges in H, then we define $H - F$ to be the hypergraph obtained from H by deleting the edges in F. If $F = \{e\}$, then we simply write $H - e$ rather than $H - \{e\}$. If F is the set of all edges in H of size at least 4, then we denote the hypergraph $H - F$ by $H_{\le 3}$.*

DEFINITION 5. *We define a component of H to be a* bad component *if it belongs to the family $\mathcal{H}$. We define $b(H)$ to be the number of bad components in H. Further, we define $b_{\le 3}(H)$ to be the number of bad components in $H_{\le 3}$. Thus, $b_{\le 3}(H) = b(H_{\le 3})$.*

DEFINITION 6. *If H_1 is a bad component of a hypergraph that can be obtained from a hypergraph H by an edge-change but H_1 is not a bad component of H itself, then we say that H_1 is an* edge-bad component *of H. We define* $\mathrm{eb}(H)$ *to be the maximum number of vertex-disjoint edge-bad components in H. We note that an edge-bad component of H may not be a subhypergraph of H.*

DEFINITION 7. *If H_2 is a bad component of a hypergraph that can be obtained from a hypergraph H by two edge-changes resulting from two distinct edges in H but H_2 is neither a bad component nor an edge-bad component of H, then we say that H_2 is an* edge-bad 2-component *of H.*

DEFINITION 8. *We define* $\mathrm{ve}(H)$ *(standing for "vertex-edge contribution") and* $\mathrm{bc}(H)$ *(standing for "bad contribution") by*

$$\begin{aligned} \mathrm{ve}(H) &= 6|V(H)| + 4e_3(H) + 2e_2(H), \\ \mathrm{bc}(H) &= 2b(H) + \mathrm{eb}(H), \\ \phi(H) &= \mathrm{ve}(H) + \mathrm{bc}(H). \end{aligned}$$

DEFINITION 9. *For every edge e in H, we define the* cost *of the edge e to be*

$$c(e) = \begin{cases} 2 & \text{if } e \text{ is a 2-edge,} \\ 4 & \text{if } e \text{ is a 3-edge.} \end{cases}$$

3.2. Key results needed to prove Theorems 1 and 2. We are now in a position to state two key results we shall need to prove our main results, namely, Theorems 1 and 2.

THEOREM 6. *If H is a hypergraph with only 2-edges and 3-edges, then the following holds:* (a) $12\tau_s(H) \le 6|V(H)| + 4e_3(H) + 2e_2(H) + 2b(H) + \mathrm{eb}(H)$.
(b) $13\tau_s(H) \le 7|V(H)| + 4e_3(H) + 2e_2(H) + b(H)$.

Proofs of Theorems 6(a) and 6(b) are presented in sections 3.4 and 3.5, respectively. We next prove a slight strengthening of Theorem 6(b), a proof of which can be found in section 3.7.

THEOREM 7. *If H is a hypergraph with all edges of size at least 2, then* $13\tau_s(H) \le 7|V(H)| + 4e_{\ge 3}(H) + 2e_2(H) + b_{\le 3}(H)$.

3.3. Properties of hypergraphs in $\mathcal{H}$. In this section, we list properties of hypergraphs in the family $\mathcal{H}$ that we will need in the subsequent proofs of our main results. We begin with the following observation. We omit the proof of Observation 2 since it is a routine exercise to verify that these properties hold for each of the eight hypergraphs in the family $\mathcal{H}$. A vertex v in a hypergraph H is said to *cover* H if v is adjacent to every vertex of H.

Observation 2. If $H \in \mathcal{H}$, then H has the following properties:

(a) Every two vertices of H belong to a common edge, and so every vertex covers H.
(b) $\tau_s(H) = |V(H)| - 1$.
(c) Every set of $|V(H)| - 1$ vertices in H is a $\tau_s(H)$-set.
(d) $12\tau_s(H) = \mathrm{ve}(H) + 2$.
(e) $\mathrm{ve}(H) = 12(|V(H)| - 1) - 2$.
(f) $12\tau_s(H) = \phi(H)$.
(g) If e and f are distinct edges in H, then $e \not\subseteq f$.
(h) If $H \ne H_3$, then H is 2-edge connected.
(i) For any two 2-edges e_1 and e_2 in H, there exists a vertex in $V(H) \setminus (e_1 \cup e_2)$.
(j) For any vertex v and 2-edge e in H, there exists a vertex in $V(H) \setminus (\{v\} \cup e)$.
(k) For any two vertices v_1 and v_2 in H, there exists a vertex in $V(H) \setminus \{v_1, v_2\}$.

The following characterization of the family $\mathcal{H}$ will prove to be useful.

LEMMA 1. *Let H be a hypergraph with only 2-edges and 3-edges. Then, $H \in \mathcal{H}$ if and only if* $\mathrm{ve}(H) \le 12(|V(H)| - 1) - 2$ *and every vertex covers H.*

Proof. The necessity follows immediately from Observations 2(a) and 2(e). To prove the sufficiency, for the sake of contradiction, let $H = (V, E)$ be a counterexample such that $|V| + |E|$ is a minimum. Thus, $\mathrm{ve}(H) \le 12(|V| - 1) - 2$, and every two distinct vertices of H belong to a common edge, but $H \notin \mathcal{H}$. Clearly, $|V| \ge 3$. Let v be a vertex of maximum degree in H.

Suppose that $d_H(v) \ge 4$ and consider the hypergraph $H' = H - v$. We note that if $\{v, x, y\}$ is a 3-edge in H, then $\{x, y\}$ is a 2-edge in H'. Hence since every two distinct vertices of H belong to a common edge, every two distinct vertices of H' belong to a common edge. Thus every vertex of H' covers H'. We also note that $\mathrm{ve}(H') = \mathrm{ve}(H) - 6 - 2d_H(v) \le \mathrm{ve}(H) - 14 \le 12(|V| - 1) - 2 - 14 = 12(|V(H')| - 1) - 4 < 12(|V(H')| - 1) - 2$. By the minimality of H, we have that $H' \in \mathcal{H}$. But then by Observation 2, $\mathrm{ve}(H') = 12(|V(H')| - 1) - 2$, contradicting our earlier observation that $\mathrm{ve}(H') < 12(|V(H')| - 1) - 2$. Hence, $d_H(v) \le 3$. Since v covers H, we have that $|V| \le 7$.

Suppose $|V| = 7$. Then every vertex in H has degree 3, and there are no overlapping edges in H. Hence, H is a 3-regular 3-uniform hypergraph with the property that every two distinct vertices of H belong to a common edge. But then H is the Fano plane $F_7 \in \mathcal{H}$, contradicting the fact that H is a counterexample. Hence, $|V| \le 6$.

Suppose $|V| = 6$. By assumption, $\mathrm{ve}(H) \le 12(|V| - 1) - 2 = 58$. If $e_2(H) = 0$, then H is 3-regular, and so $e_3(H) = 6$ and $\mathrm{ve}(H) = 60$, a contradiction. Hence,

$e_2(H) \geq 1$. Let $V = \{v, v_1, v_2, \dots, v_5\}$. Renaming vertices if necessary, we may assume that v is incident with a 2-edge and that $\{\{v, v_1, v_2\}, \{v, v_3, v_4\}, \{v, v_5\}\} \subset E$. If $\{v_1, v_2, v_5\} \in E$, then $\{v_3, v_4, v_5\} \in E$ since v_5 covers H. But then $\{v_1, v_3, v_4\} \in E$ and $\{v_2, v_3, v_4\} \in E$, and so $d_H(v_3) \geq 4$, contradicting the fact that $\Delta(H) = 3$. Thus, $\{v_1, v_2, v_5\} \notin E$. Similarly, $\{v_3, v_4, v_5\} \notin E$. Since v_5 covers H, we may therefore assume that $\{\{v_1, v_3, v_5\}, \{v_2, v_4, v_5\}\} \subset E$. But then $\{\{v_1, v_4\}, \{v_2, v_3\}\} \subset E$, and so $H = F_6$, a contradiction. Hence, $|V| \leq 5$.

Suppose $|V| = 5$. By assumption, $\mathrm{ve}(H) \leq 46$. Let $V = \{v, v_1, v_2, v_3, v_4\}$. Since every vertex covers H, every vertex is contained in at least one 3-edge. Thus, $e_3(H) \geq 2$. Suppose $e_3(H) = 2$. Then we may assume that $\{\{v, v_1, v_2\}, \{v, v_3, v_4\}\} \subset E$. But then $\{\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}\} \subset E$, whence $H = F_5$, a contradiction. Hence, $e_3(H) \geq 3$. Then H contains two overlapping 3-edges. We may assume that $\{\{v, v_1, v_2\}, \{v, v_1, v_3\}\} \subset E$. Suppose $e_3(H) = 3$. We note that $\mathrm{ve}(H) = 6|V| + 4e_3(H) + 2e_2(H) = 42 + 2e_2(H)$. By assumption, $\mathrm{ve}(H) \leq 46$. Hence, $e_2(H) \leq 2$. This implies that $\{v_2, v_3, v_4\}$ is the third 3-edge and that $\{\{v, v_4\}, \{v, v_1\}\} \subset E$. Thus, $H = H_5$, a contradiction. Hence, $e_3(H) \geq 4$. We note then that $\mathrm{ve}(H) = 6|V| + 4e_3(H) + 2e_2(H) = 46 + 2e_2(H)$. By assumption, $\mathrm{ve}(H) \leq 46$. Hence, $e_3(H) = 4$ and $e_2(H) = 0$. It follows that $\{\{v, v_1, v_4\}, \{v_2, v_3, v_4\}\} \subset E$, whence $H = T_5$, a contradiction. Hence, $|V| \leq 4$.

Suppose $|V| = 4$. By assumption, $\mathrm{ve}(H) \leq 34$. Let $V = \{v, v_1, v_2, v_3\}$. If $e_3(H) = 0$, then every two vertices in H are joined by a 2-edge, and so $e_2(H) = 6$ and $\mathrm{ve}(H) = 6|V| + 4e_3(H) + 2e_2(H) = 24 + 12 = 36$, a contradiction. If $e_3(H) \geq 3$, then $\mathrm{ve}(H) \geq 36 + 2e_2(H) \geq 36$, a contradiction. Hence either $e_3(H) = 1$ or $e_3(H) = 2$. If $e_3(H) = 1$, then $H = F_4$, while if $e_3(H) = 2$, then $H = H_4$. In both cases, $H \in \mathcal{H}$, a contradiction. Hence, $|V| = 3$.

By assumption, $\mathrm{ve}(H) \leq 22$. If $e_3(H) = 0$, then every two vertices in H are joined by a 2-edge, and so $e_2(H) = 3$ and $\mathrm{ve}(H) = 6|V| + 4e_3(H) + 2e_2(H) = 24$, a contradiction. Hence, $e_3(H) \geq 1$. But then $\mathrm{ve}(H) = 22 + 2e_2(H)$, implying that $e_3(H) = 1$ and $e_2(H) = 0$, and so $H = H_3 \in \mathcal{H}$, a contradiction. ∎

3.4. Proof of Theorem 6(a). Before presenting a proof of Theorem 6(a), we shall need the following result about matchings in cubic graphs which first appeared in a paper by Biedl et al. [2]. We remark that a short proof of this result can be found in [12].

THEOREM 8 (see [2]). *If G is a connected cubic graph G of order n, then $\alpha'(G) \geq 4(n-1)/9$.*

For $n \geq 4$, we note that $\lceil \frac{4}{9}(n-1) \rceil \geq \lceil \frac{7}{16} n \rceil$. Hence we have the following immediate corollary of Theorem 8.

COROLLARY 1. *If G is a cubic graph G of order n, then $\alpha'(G) \geq 7n/16$.*

Using the hypergraph terminology defined in section 3.1, Theorem 6(a) can be restated as follows.

THEOREM 6(a). *If H is a hypergraph with only 2-edges and 3-edges, then $12\tau_s(H) \leq \phi(H)$.*

Proof. For the sake of contradiction, among all counterexamples, let $H = (V, E)$ be one such that $|V| + |E|$ is a minimum. We proceed further with a series of claims that we may assume the hypergraph H satisfies.

CLAIM A. *The following properties hold in H:*

(a) *$H \notin \mathcal{H}$.*

(b) *H is connected.*

(c) $b(H) = 0$.

(d) *If* e *and* f *are distinct edges in* H*, then* $e \not\subseteq f$*.*
(e) $\delta(H) \geq 2$.
(f) $\mathrm{eb}(H) = 0$.
(g) *There is no edge-bad* 2*-component in* H*.*
(h) $\Delta(H) \leq 3$.
(i) H *is* 2*-regular.*
(j) H *is* 3*-uniform.*
(k) *There are no overlapping edges in* H*.*

Proof. (a) If $H \in \mathcal{H}$, then, by Observation 2(f), $12\tau_s(H) = \phi(H)$, contradicting the fact that H is a counterexample to the theorem.

(b) Suppose H is disconnected. By the minimality of H, we have that $12\tau_s(H') \leq \phi(H')$ for each component of H. Since $\tau_s(H) = \sum \tau_s(H')$ and $\phi(H) = \sum \phi(H')$ where the sum is taken over all components H' in H, we have that $12\tau_s(H) \leq \phi(H)$, a contradiction.

(c) If $b(H) > 0$, then, by part (b), $H \in \mathcal{H}$, contradicting part (a).

(d) Suppose e and f are distinct edges in H but $e \subseteq f$. Then, e must be a 2-edge since H has only 2-edges and 3-edges. Further, every strong transversal in $H - e$ in a strong transversal in H, and so $\tau_s(H) \leq \tau_s(H - e)$. By the minimality of H, we have that $12\tau_s(H - e) \leq \phi(H - e)$. Now $\mathrm{ve}(H - e) = \mathrm{ve}(H) - c(e) = \mathrm{ve}(H) - 2$. If there is a new bad component in $H - e$, then it contains the edge f and by (c) $b(H - e) = 1$. Furthermore $\mathrm{eb}(H - e) \leq \mathrm{eb}(H)$. Thus, $\mathrm{bc}(H - e) \leq \mathrm{bc}(H) + 2$. Thus, $12\tau_s(H - e) \leq \phi(H - e) \leq (\mathrm{ve}(H) - 2) + (\mathrm{bc}(H) + 2) = \phi(H)$, contradicting the fact that H is a counterexample to the theorem. Hence, $b(H - e) = 0$. If there is a new edge-bad component in $H - e$ but not in H, then $b(H - e) = 0 = b(H)$ and $\mathrm{eb}(H-e) \leq \mathrm{eb}(H)+3$. If $\mathrm{eb}(H-e) \leq \mathrm{eb}(H)+2$, then once again $12\tau_s(H - e) \leq \phi(H)$, a contradiction. Hence, $\mathrm{eb}(H - e) = \mathrm{eb}(H) + 3$ and $\mathrm{bc}(H - e) = \mathrm{bc}(H) + 3$. But then e is a 3-edge, a contradiction.

(e) Suppose that $d_H(v) = 1$ for some vertex v in H, and let e be the edge of H containing v. Suppose first that e is a 2-edge. Let $e = \{u, v\}$. If $d_H(u) = 1$, then $V = \{u, v\}$ and $E = \{e\}$. But then $\tau_s(H) = 1$ and $\phi(H) = 14$, and so $12\tau_s(H) < \phi(H)$, a contradiction. Hence, $d_H(u) \geq 2$. We now consider the hypergraph $H' = H - u$. Let U denote the set of isolated vertices resulting from deleting u from H. We note that $v \in U$, and so $|U| \geq 1$. Then, $\mathrm{ve}(H') = \mathrm{ve}(H) - 6|\{u\}| - 6|U| - 2d_H(u) \leq \mathrm{ve}(H) - 12 - 2d_H(u)$. Further we note that the number of new bad-components or vertex-disjoint edge-bad components that are created in H' is at most $d_H(u) - 1$. Hence, $b(H')+\mathrm{eb}(H') \leq b(H)+\mathrm{eb}(H)+(d_H(u)-1)$. However, every new bad component in H' (that was not a bad component in H) is an edge-bad component of H or is intersected by two or more edges incident with u, implying that $\mathrm{bc}(H') \leq \mathrm{bc}(H) + (d_H(u) - 1)$. Hence, $\phi(H') = \mathrm{ve}(H') + \mathrm{bc}(H') \leq (\mathrm{ve}(H) - 12 - 2d_H(u)) + (\mathrm{bc}(H) + d_H(u) - 1) = \phi(H) - 13 - d_H(u) \leq \phi(H) - 15$. Since H' is not a counterexample to our theorem, we have that $12\tau_s(H') \leq \phi(H')$. Hence, $12\tau_s(H) \leq 12(\tau_s(H') + 1) \leq (\phi(H) - 15) + 12 = \phi(H) - 3$, a contradiction.

Therefore, e is a 3-edge. Let $e = \{v, v_1, v_2\}$. We may assume that $d_H(v_2) \geq d_H(v_1)$. If $E = \{e\}$, then $H = H_3 \in H$, contradicting part (a). Hence, $d_H(v_2) \geq 2$. Let $X = \{v_1, v_2\}$ and consider the hypergraph $H' = H - X$. Let Y denote the set of isolated vertices resulting from deleting X from H. We note that $|Y| \geq 1$ since $v \in Y$. Then $\mathrm{ve}(H') = \mathrm{ve}(H) - 6|X| - 6|Y| - 2d_H(v_1) - 2d_H(v_2) \leq \mathrm{ve}(H) - 18 - 2d_H(v_1) - 2d_H(v_2)$. Further we note that $\mathrm{bc}(H') \leq \mathrm{bc}(H) + (d_H(v_1) - 1) + (d_H(v_2) - 1)$. Hence, $\phi(H') = \mathrm{ve}(H') + \mathrm{bc}(H') \leq \phi(H) - 20 - d_H(v_1) - d_H(v_2)$. Since H' is not a counterexample to our theorem, we have that $12\tau_s(H') \leq \phi(H')$. Further,

$\tau_s(H) \le \tau_s(H') + 2$. Hence, $12\tau_s(H) \le 12(\tau_s(H')+2) \le \phi(H) + 4 - d_H(v_1) - d_H(v_2)$. If $d_H(v_1) + d_H(v_2) \ge 4$, then $12\tau_s(H) \le \phi(H)$, a contradiction. Hence, $d_H(v_1) = 1$ and $d_H(v_2) = 2$. But then the hypergraph with vertex set $\{v, v_1, v_2\}$ and edge set consisting of the 3-edge e is an edge-bad component of H, and therefore contributes 1 to $\mathrm{eb}(H)$. Thus, since this edge-bad component is deleted from H when constructing H', we have that $\mathrm{bc}(H') \le \mathrm{bc}(H) - 1 + (d_H(v_1) - 1) + (d_H(v_2) - 1)$, implying that $12\tau_s(H) \le \phi(H) + 3 - d_H(v_1) - d_H(v_2) \le \phi(H)$, a contradiction. Hence, $\delta(H) \ge 2$.

(f) Suppose $\mathrm{eb}(H) > 0$. Let H_1 be an edge-bad component of H that results from shrinking the edge e of H. Then $|e \cap V(H_1)| \ge 1$. Suppose that $e \subseteq V(H_1)$. Then $V(H_1) = V$. If e is a 2-edge in H, then by Observation 2(a), there exists an edge f of H_1 such that $e \subseteq f$, contradicting part (d) above. Hence, e is a 3-edge, and so H_1 contains a 2-edge implying that $|V(H)| \ge 4$. Let $v \in V \setminus e$. By Observation 2(c), the set $V \setminus \{v\}$ is a $\tau_s(H_1)$-set. Since $V \setminus \{v\}$ is a strong transversal in H, $\tau_s(H) \le \tau_s(H_1) = |V| - 1$. We note that $\mathrm{ve}(H') = \mathrm{ve}(H) - 2$. Further, we note that $b(H) = 0$, $b(H') = 1$, $\mathrm{eb}(H) = 1$, and $\mathrm{eb}(H') = 0$, and so $\mathrm{bc}(H) = 1$ and $\mathrm{bc}(H') = 2$. Thus, $\mathrm{bc}(H') = \mathrm{bc}(H) + 1$ and $\phi(H') = \phi(H) - 1$. By Observation 2, $\phi(H') = 12\tau_s(H') = 12(|V(H_1)| - 1) = 12(|V| - 1)$. Thus, $\phi(H) = \phi(H') + 1 = 12(|V| - 1) + 1 > 12\tau_s(H)$, contradicting the fact that H is a counterexample to our theorem. Hence, $e \not\subseteq V(H_1)$, and so $V \setminus V(H_1) \ne \emptyset$.

We now consider the hypergraph $H' = H - V(H_1)$. We note that $\mathrm{ve}(H') = \mathrm{ve}(H) - \mathrm{ve}(H_1) - 2$ and that H' is connected. By Observation 2(e), $\mathrm{ve}(H_1) = 12(|V(H_1)| - 1) - 2$, and so $\mathrm{ve}(H') = \mathrm{ve}(H) - 12(|V(H_1)| - 1)$. If $H' \in \mathcal{H}$, then H' is an edge-bad component of H, and so $b(H) = 0$, $b(H') = 1$, $\mathrm{eb}(H) = 2$, and $\mathrm{eb}(H') = 0$, whence $\mathrm{bc}(H') = \mathrm{bc}(H)$. If $H' \notin \mathcal{H}$, then $b(H) = b(H') = 0$ and $\mathrm{eb}(H') \le \mathrm{eb}(H)$, and so $\mathrm{bc}(H') \le \mathrm{bc}(H)$. In both cases, we have $\mathrm{bc}(H') \le \mathrm{bc}(H)$, and so $\phi(H') \le \phi(H) - 12(|V(H_1)| - 1)$. Since H' is not a counterexample to our theorem, we have that $12\tau_s(H') \le \phi(H')$. Let $x \in V(H_1) \setminus e$, and let $X = V(H_1) \setminus \{x\}$. We note that $\tau_s(H) \le \tau_s(H') + |X| = \tau_s(H') + (|V(H_1)| - 1)$. Hence, $12\tau_s(H) \le 12\tau_s(H') + 12(|V(H_1)| - 1) \le \phi(H') + 12(|V(H_1)| - 1) \le \phi(H)$, contradicting the fact that H is a counterexample to our theorem. Hence, $\mathrm{eb}(H) = 0$.

(g) Suppose that H_1 is an edge-bad 2-component of H that results from shrinking the (distinct) edges e and f of H. Then $|e \cap V(H_1)| \ge 1$ and $|f \cap V(H_1)| \ge 1$. If e is a 2-edge and $e \subset V(H_1)$, then by Observation 2(a), there exists an edge e' in H_1 such that $e \subseteq e'$, contradicting part (d) above. Hence we note that if $e \subseteq V(H_1)$, then e is a 3-edge. Similarly, if $f \subseteq V(H_1)$, then f is a 3-edge.

Suppose $e \subseteq V(H_1)$ and $f \subseteq V(H_1)$. Then $V(H_1) = V$ and $V \setminus \{v\}$ is a strong transversal in H for any vertex $v \in V$. Thus, $\tau_s(H) \le |V| - 1 = \tau_s(H_1)$. We note that $\mathrm{ve}(H_1) = \mathrm{ve}(H) - 4$. Further, we note that $b(H) = 0$, $b(H_1) = 1$, and $\mathrm{eb}(H) = \mathrm{eb}(H_1) = 0$, and so $\mathrm{bc}(H) = 0$ and $\mathrm{bc}(H_1) = 2$. Thus, $\mathrm{bc}(H_1) = \mathrm{bc}(H) + 2$ and $\phi(H_1) = \phi(H) - 2$. By Observation 2, $\phi(H_1) = 12(|V| - 1)$. Thus, $\phi(H) = \phi(H_1) + 2 = 12(|V| - 1) + 2 > 12\tau_s(H)$, a contradiction. Hence, $e \not\subseteq V(H_1)$ or $f \not\subseteq V(H_1)$, and so $V \setminus V(H_1) \ne \emptyset$.

We now consider the hypergraph $H' = H - V(H_1)$. We note that $\mathrm{ve}(H') = \mathrm{ve}(H) - \mathrm{ve}(H_1) - 4$. By Observation 2(e), $\mathrm{ve}(H_1) = 12(|V(H_1)| - 1) - 2$, and so $\mathrm{ve}(H') = \mathrm{ve}(H) - 12(|V(H_1)| - 1) - 2$. If there is a bad component in H', then since $\mathrm{eb}(H) = 0$ by part (f) above, this bad component is H' itself (and both e and f intersect H'), implying that $b(H) = 0$, $b(H') = 1$, and $\mathrm{eb}(H) = \mathrm{eb}(H') = 0$, whence $\mathrm{bc}(H') = \mathrm{bc}(H) + 2$. If there is no bad component in H', then $b(H) = \mathrm{eb}(H) = b(H') = 0$ and $\mathrm{eb}(H') \le 2$, and so $\mathrm{bc}(H') \le \mathrm{bc}(H) + 2$. In both cases, we have $\mathrm{bc}(H') \le \mathrm{bc}(H) + 2$, and so $\phi(H') \le \phi(H) - 12(|V(H_1)| - 1)$. Since H'

is not a counterexample to our theorem, we have that $12\tau_s(H') \le \phi(H')$. We note that either e is a 2-edge, in which case $|e \cap V(H_1)| = 1$, or e is a 3-edge, in which case $|e \cap V(H_1)| = 1$ or e^* is a 2-edge in H_1 (see Definition 2). A similar statement holds for the edge f. We note therefore that by Observations 2(i), 2(j), and 2(k) there exists a vertex $x \in V(H_1) \setminus (e^* \cup f^*)$. Let $X = V(H_1) \setminus \{x\}$. We note that $\tau_s(H) \le \tau_s(H') + |X| = \tau_s(H') + (|V(H_1)| - 1)$. Hence, $12\tau_s(H) \le 12\tau_s(H') + 12(|V(H_1)| - 1) \le \phi(H') + 12(|V(H_1)| - 1) \le \phi(H)$, a contradiction. Therefore, there is no edge-bad 2-component in H.

(h) Suppose that $\Delta(H) \ge 4$. Let x be a vertex of maximum degree $\Delta(H)$ in H, and consider the hypergraph $H' = H - x$. Since H' is not a counterexample to our theorem, we have that $12\tau_s(H') \le \phi(H')$. Let Y denote the set of isolated vertices, if any, resulting from deleting x from H. Then $\mathrm{ve}(H') = \mathrm{ve}(H) - 6|\{x\}| - 6|Y| - 2d_H(x) \le \mathrm{ve}(H) - 6 - 2d_H(x)$. By parts (c) and (f) above, we note that $\mathrm{bc}(H) = 0$.

If $b(H') + \mathrm{eb}(H') \le 1$, then $\mathrm{bc}(H') \le 2$, and so $\mathrm{bc}(H') \le \mathrm{bc}(H) + 2$, whence $\phi(H') \le \phi(H) - 4 - 2d_H(x) \le \phi(H) - 12$. Thus, $12\tau_s(H) \le 12(\tau_s(H') + 1) \le \phi(H') + 12 \le (\phi(H) - 12) + 12 \le \phi(H)$, a contradiction. Hence, $b(H') + \mathrm{eb}(H') \ge 2$.

By parts (f) and (g) above, we note that if there is a bad component in H', then at least three edges incident with x intersect such a component, while if there is an edge-bad component in H', then at least two edges incident with x intersect such a component. Hence, $3b(H') + 2\mathrm{eb}(H') \le d_H(x)$, implying that $\mathrm{bc}(H') = 2b(H') + \mathrm{eb}(H') \le d_H(x) - b(H') - \mathrm{eb}(H') \le d_H(x) - 2 = \mathrm{bc}(H) + d_H(x) - 2$, whence $\phi(H') \le \phi(H) - 8 - d_H(x) \le \phi(H) - 12$. Thus, $12\tau_s(H) \le 12(\tau_s(H') + 1) \le \phi(H') + 12 \le \phi(H)$, a contradiction. Hence, $\Delta(H) \le 3$.

(i) Suppose that H is not 2-regular. Then, by parts (e) and (h), we have that $\Delta(H) = 3$. Let x be a vertex of maximum degree $\Delta(H)$ in H and consider the hypergraph $H' = H - x$. As shown in the previous part (h), $3b(H') + 2\mathrm{eb}(H') \le d_H(x) = 3$, and so $b(H') + \mathrm{eb}(H') \le 1$. Further, $12\tau_s(H') \le \phi(H')$ and $\mathrm{ve}(H') = \mathrm{ve}(H) - 12$. If $b(H') + \mathrm{eb}(H') = 0$, then $\mathrm{bc}(H') = \mathrm{bc}(H) = 0$, and so $\phi(H') \le \phi(H) - 12$ and $12\tau_s(H) \le 12(\tau_s(H') + 1) \le \phi(H') + 12 \le \phi(H)$, a contradiction. Hence, $b(H') + \mathrm{eb}(H') = 1$. Thus either $b(H') = 1$ and $\mathrm{eb}(H') = 0$ or $b(H') = 0$ and $\mathrm{eb}(H') = 1$.

Suppose that $b(H') = 1$ and $\mathrm{eb}(H') = 0$. Then $\mathrm{bc}(H') = 2 = \mathrm{bc}(H) + 2$. Hence since $\mathrm{ve}(H') = \mathrm{ve}(H) - 12$, we have that $\phi(H') = \phi(H) - 10$. By Observation 2, $\mathrm{ve}(H') = 12(|V(H')| - 1) - 2 = 12(|V| - 2) - 2$, and so $\mathrm{ve}(H) = \mathrm{ve}(H') + 12 = 12(|V| - 1) - 2$. Further, $\phi(H') = 12(|V| - 2)$, and so $\phi(H) = 12(|V| - 2) + 10$. Since $b(H') = 1$, all three edges incident with x intersect H', and so $V = V(H') \cup \{x\}$. If there is a vertex v not covered by x, then $V \setminus \{v, x\}$ is a strong transversal in H, and so $12\tau_s(H) \le 12(|V| - 2) = \phi(H) - 10 < \phi(H)$, a contradiction. Hence, x covers H. By Observation 2(a), every vertex of H' covers H'. Therefore every vertex of H covers H. As observed earlier, $\mathrm{ve}(H) = 12(|V| - 1) - 2$. Hence, by Lemma 1, $H \in \mathcal{H}$, a contradiction. Thus, $b(H') = 0$ and $\mathrm{eb}(H') = 1$.

Let R be the edge-bad component of H' resulting from shrinking the edge e. By Observation 2, $\mathrm{ve}(R) = 12(|V(R)| - 1) - 2$. By part (g), at least two edges incident with x intersect R.

Suppose $V = V(R) \cup \{x\}$. Then all three edges incident with x intersect R and $e \subseteq V(R)$. We note that $\mathrm{ve}(R) = 12(|V| - 2) - 2$ and $\mathrm{ve}(R) = \mathrm{ve}(H) - 6 - 2d_H(x) - 2 = \mathrm{ve}(H) - 14$, and so $\mathrm{ve}(H) = \mathrm{ve}(R) + 14 = 12(|V| - 1)$. Thus, since $\mathrm{bc}(H) = 0$ we have that $\phi(H) = \mathrm{ve}(H) + \mathrm{bc}(H) = 12(|V| - 1)$. However, the set $V \setminus \{x\}$ is a strong transversal in H, and so $12\tau_s(H) \le 12(|V| - 1)$. Consequently, $12\tau_s(H) \le \phi(H)$, a contradiction. Therefore, $V \setminus (V(R) \cup \{x\}) \ne \emptyset$.

We now consider the hypergraph $H^* = H - (V(R) \cup \{x\})$. We note that $\mathrm{ve}(H^*) \leq \mathrm{ve}(H) - \mathrm{ve}(R) - 6 - 2d_H(x) - 2 = \mathrm{ve}(H) - 12|V(R)|$, and so $\mathrm{ve}(H) \geq \mathrm{ve}(H^*) + 12|V(R)|$. By parts (f) and (g) above, we also note that $b(H^*) = 0$ and $\mathrm{eb}(H^*) \leq 1$, and so $\mathrm{bc}(H^*) \leq 1 = \mathrm{bc}(H) + 1$.

Suppose $\mathrm{eb}(H^*) = 0$. Then $\mathrm{bc}(H^*) = \mathrm{bc}(H) = 0$ and $\phi(H) \geq \phi(H^*) + 12|V(R)|$. Therefore since H^* is not a counterexample to our theorem, we have that $12\tau_s(H^*) \leq \phi(H^*) = \phi(H) - 12|V(R)|$. We note that if $R = H_3$, then the edge e intersects R in exactly one vertex (otherwise R would have a 2-edge), while if $R \neq H_3$, then $|V(R)| \geq 4$. Hence there exists a vertex v in R not contained in e. Every $\tau_s(H^*)$-set can be extended to a strong transversal in H by adding to it the set $(V(R) \cup \{x\}) \setminus \{v\}$ of size $|V(R)|$, and so $12\tau_s(H) \leq 12(\tau_s(H^*) + |V(R)|) \leq \phi(H)$, a contradiction. Hence, $\mathrm{eb}(H^*) = 1$.

Since $\mathrm{eb}(H^*) = 1$ and there is no edge-bad 2-component in H, the edge e intersects H^* as does one edge, f, say, incident with x. Let R_1 be the edge-bad component of H^* resulting from shrinking the edge g. By Observation 2, $\mathrm{ve}(R_1) = 12(|V(R_1)| - 1) - 2$. We note that the edges e, f, and g all intersect R_1.

If $g \subseteq V(R_1)$, then g is a 3-edge and $V = V(R_1) \cup V(R_2) \cup \{x\}$. Let $y \in V(R) \cap e$, and let $z \in V(R_1) \setminus (e \cup g)$. Then $V \setminus \{y, z\}$ is a strong transversal in H, and so $12\tau_s(H) \leq 12(|V| - 2)$. We note that $\mathrm{ve}(H) = \mathrm{ve}(R) + \mathrm{ve}(R_1) + 6|\{x\}| + 2d_H(x) + c(g) = \mathrm{ve}(R) + \mathrm{ve}(R_1) + 16 = (12(|V(R)| - 1) - 2) + (12(|V(R_1)| - 1) - 2) + 16 = 12(|V| - 2) \geq 12\tau_s(H)$, a contradiction. Hence, $g \not\subseteq V(R_1)$, and so $V(H^*) \setminus V(R_1) \neq \emptyset$.

Let $H'' = H^* - V(R_1)$. If $b(H'') > 0$, then $\mathrm{eb}(H) > 0$, a contradiction. If $\mathrm{eb}(H'') > 0$, then there would be an edge-bad 2-component in H, a contradiction. Hence, $b(H'') = \mathrm{eb}(H'') = 0$, and so $\mathrm{bc}(H'') = \mathrm{bc}(H) = 0$. We note further that $\mathrm{ve}(H'') = \mathrm{ve}(H) - \mathrm{ve}(R) - \mathrm{ve}(R_1) - 16 = \mathrm{ve}(H) - (12(|V(R)| - 1) - 2) - (12(|V(R_1)| - 1) - 2) - 16 = \mathrm{ve}(H) - 12(|V(R)| + |V(R_1)| - 1)$. Thus, $\phi(H'') = \phi(H) - 12(|V(R)| + |V(R_1)| - 1)$. By Observations 2(i), 2(j), and 2(k), there exists vertices $y \in V(R) \setminus N(x)$ and $z \in V(R_1) \setminus (e \cup g)$. Any $\tau_s(H'')$-set can be extended to a strong transversal in H by adding to it the set $(V(R) \cup V(R_1) \cup \{x\}) \setminus \{y, z\}$, and so $12\tau_s(H) \leq 12(\tau_s(H'') + |V(R)| + |V(R_1)| - 1)$. Since H'' is not a counterexample to our theorem, we have that $12\tau_s(H'') \leq \phi(H'')$. Consequently, $12\tau_s(H) \leq \phi(H'') + 12(|V(R)| + |V(R_1)| - 1) = \phi(H)$, a contradiction.

(j) Suppose that H contains a vertex v that is incident with a 2-edge $e = \{v, x\}$. By part (i), H is 2-regular. Let f be the other edge incident with v.

Suppose f is a 2-edge. Let $f = \{v, y\}$. Suppose there is an edge g containing x and y. If g is a 2-edge, then $g = \{x, y\}$, and $V = \{v, x, y\}$ and $E = \{e, f\}$. But then $\tau_s(H) = 2$ and $\phi(H) = 24$, and so $12\tau_s(H) = \phi(H)$, a contradiction. Hence, g is a 3-edge. We now consider the hypergraph $H' = H - \{x, y\}$, and note that the resulting isolated vertex v is deleted when constructing H'. By parts (f) and (g) above, we note that $b(H') = \mathrm{eb}(H') = 0$, and so $\mathrm{bc}(H') = \mathrm{bc}(H) = 0$. Also, $\mathrm{ve}(H') = \mathrm{ve}(H') - 26$, and so $\mathrm{ve}(H) = \mathrm{ve}(H') + 26$. Thus, $\phi(H) = \phi(H') + 26$. Therefore since H' is not a counterexample to our theorem, we have that $12\tau_s(H') \leq \phi(H') = \phi(H) - 26$. However, every $\tau_s(H')$-set can be extended to a strong transversal in H by adding to it the set $\{x, y\}$, and so $12\tau_s(H) \leq 12(\tau_s(H') + 2) \leq \phi(H') + 24 < \phi(H)$, a contradiction.

Thus, $\{\{v, x\}, \{v, y\}\} \subseteq E$, and there is no edge containing both x and y. We will show that $12\tau_s(H) \leq \phi(H) - 2$. By part (i), H is 2-regular. Let H^* be the hypergraph obtained from H by deleting the edges e and f and identifying the vertices x and y to produce a new vertex w. We say that H^* is obtained from H by contracting x and y. Since H is 2-regular, so too is H^*. We note that $b(H^*) + \mathrm{eb}(H^*) \leq 1$, and so $\mathrm{bc}(H^*) \leq 2 = \mathrm{bc}(H) + 2$. Also, $\mathrm{ve}(H^*) = \mathrm{ve}(H) - 16$, and so $\mathrm{ve}(H) = \mathrm{ve}(H^*) + 16$.

Thus, $\phi(H) \geq \phi(H^*)+14$. Since H^* is not a counterexample to our theorem, we have that $12\tau_s(H^*) \leq \phi(H^*) \leq \phi(H) - 14$. Let T^* be a $\tau_s(H^*)$-set, and so $|T^*| = \tau_s(H^*)$. If $w \notin T^*$, let $T = T^* \cup \{v\}$. If $w \in T^*$, let $T = (T^* \setminus \{w\}) \cup \{x, y\}$. In both cases, T is a strong transversal in H of size $\tau_s(H^*) + 1$. Hence, $12\tau_s(H) \leq 12(\tau_s(H^*) + 1) \leq (\phi(H) - 14) + 12 = \phi(H) - 2$, as claimed. Thus, $12\tau_s(H) < \phi(H)$, a contradiction. Hence, f must be a 3-edge.

Let $f = \{v, y, z\}$. If $\{x, y, z\}$ is an edge of H, then $H = H_4$, a contradiction. Hence, renaming y and z if necessary, we may assume that there is no edge containing both x and y. We now consider the hypergraph $H_z = H - z$. By parts (f) and (g) above, we note that $b(H_z) = 0$ and $\mathrm{eb}(H_z) \leq 1$, and so $\mathrm{bc}(H_z) \leq \mathrm{bc}(H) + 1$. Also, $\mathrm{ve}(H_z) = \mathrm{ve}(H) - 10$, and so $\mathrm{ve}(H) = \mathrm{ve}(H_z) + 10$. Thus, $\phi(H) \geq \phi(H_z) + 9$. By construction, $\{v, x\}$ and $\{v, y\}$ are the two edges in H_z containing v, and there is no edge containing both x and y. A similar proof as shown in the previous paragraph shows that $12\tau_s(H_z) \leq \phi(H_z) - 2$. Thus, $12\tau_s(H_z) \leq \phi(H) - 11$. If $12\tau_s(H_z) \leq \phi(H) - 12$, then $12\tau_s(H) \leq 12(\tau_s(H_z) + 1) \leq \phi(H)$, a contradiction. Consequently, $12\tau_s(H_z) = \phi(H) - 11$. Thus, $\mathrm{eb}(H_z) = 1$, and both edges incident with z intersect H_z. In particular, the edge-bad component of H_z contains v or y. Let H^* be the hypergraph obtained by contracting x and y in H_z. Then $b(H^*) + \mathrm{eb}(H^*) \leq 1$, and so $\mathrm{bc}(H^*) \leq 2 = \mathrm{bc}(H_z) + 1$. Thus the bound $\mathrm{bc}(H^*) \leq \mathrm{bc}(H) + 2$ which we use to establish the upper bound $12\tau_s(H) \leq \phi(H) - 2$ is improved to $\mathrm{bc}(H^*) \leq \mathrm{bc}(H_z) + 1$, which in turn implies that $12\tau_s(H_z) \leq \phi(H_z) - 3$. But then $12\tau_s(H) \leq 12(\tau_s(H_z) + 1) \leq (\phi(H_z) - 3) + 12 \leq \phi(H)$, a contradiction. This completes the proof of part (j).

(k) Suppose that e and f are two overlapping edges in H. Let $e = \{a, b, x\}$ and $f = \{a, b, y\}$. If $\{x, y\}$ is a edge of H, then $H = H_4$, a contradiction. If there is a 3-edge containing x and y, then the hypergraph with vertex set $\{a, b, x, y\}$ and edge set $\{e, f, \{x, y\}\}$ is an edge-bad component of H, and so $\mathrm{eb}(H) \geq 1$, contradicting part (f). Hence, there is no edge containing x and y. We now consider the hypergraph $H_a = H - a$ and note that $\phi(H) = \phi(H_a) + 10$. By construction, $\{b, x\}$ and $\{b, y\}$ are the two edges in H_a containing v, and there is no edge in H_a containing x and y. A similar proof as shown in the third paragraph of part (j) shows that $12\tau_s(H_a) \leq \phi(H_a) - 2$. Hence, $12\tau_s(H) \leq 12(\tau_s(H_a) + 1) \leq (\phi(H_a) - 2) + 12 = \phi(H)$, a contradiction. This completes the proof of Claim A. $\square$

We now return to the proof of Theorem 6(a). By Claim A, we have that H is a 2-regular 3-uniform hypergraph with no overlapping edges. Let G be the incidence bipartite graph of the hypergraph $H = (V, E)$; that is, G has partite sets V and E, where every vertex e in E is joined to the three vertices in V that belong to the edge e in H. We note that $d_G(v) = 2$ if $v \in V$, while $d_G(v) = 3$ if $v \in E$. Further, since H has no overlapping edges, we note that in the graph G every two vertices in E have at most one common neighbor.

Let F be the graph with vertex set $V(F) = E$ and where two vertices are adjacent in F if they have a common neighbor in G. We note that F is a cubic graph of order $|E|$ and that G is obtained from F by subdividing every edge of F exactly once. Let M be a maximum matching in F. By Corollary 1, we have that $|M| = \alpha'(F) \geq 7|E|/16$. For each edge e in M, let v_e be the (unique) vertex in V whose neighbors in G are the two ends of e. Let $V_M = \cup\{v_e\}$, where the union is taken over all edges $e \in M$. Then $|V_M| = |M| \geq 7|E|/16$. Let $T = V \setminus V_M$. Then T is a strong transversal in H, and so $\tau_s(H) \leq |T| = |V| - |V_M| \leq |V| - 7|E|/16$. Since $2|V| = 3|E|$, we therefore have that $\tau_s(H) \leq 17|E|/16$. We also note that $\mathrm{ve}(H) = 6|V| + 4e_3(H) + 2e_2(H) = 6|V| + 4|E| = 13|E|$ and $\mathrm{bc}(H) = 0$, and so $\phi(H) = \mathrm{ve}(H) + \mathrm{bc}(H) = 13|E|$. Thus,

$12\tau_s(H) \le 204|E|/16 = 12.75|E| < \phi(H)$, a contradiction. This completes the proof of Theorem 6(a). ∎

3.5. Proof of Theorem 6(b). Recall the statement of Theorem 6(b).

THEOREM 6(b). *If H is a hypergraph with only 2-edges and 3-edges, then $13\tau_s(H) \le 7|V(H)| + 4e_3(H) + 2e_2(H) + b(H)$.*

Proof. We show first that $b(H) + \mathrm{eb}(H) \le |V(H)| - \tau_s(H)$. Let T be a $\tau_s(H)$-set. If R is a bad component of H, then by Observation 2(b), there is a vertex in R that does not belong to T, and therefore the component R contributes 1 to the difference $|V(H)| - \tau_s(H)$. Suppose that R is an edge-bad component of H resulting from shrinking the edge e. Let v be a vertex in R not contained in e. If $V(R) \subseteq T$, then $T \setminus \{v\}$ is a strong transversal in H, contradicting the minimality of T. Hence, by Observation 2(b), there is exactly one vertex u in R that does not belong to T, and therefore the component R contributes 1 to the difference $|V(H)| - \tau_s(H)$. Hence, $b(H) + \mathrm{eb}(H) \le |V(H)| - \tau_s(H)$, as claimed. Hence, by Theorem 6(a), we note that $13\tau_s(H) \le 7|V(H)| + 4e_3(H) + 2e_2(H) + b(H)$. ∎

3.6. Proof of Theorem 1. Recall the statement of Theorem 1.

THEOREM 1. *If H is a hypergraph with only 2-edges and 3-edges, then*

$$14\tau_s(H) \le 8|V(H)| + 4e_3(H) + 2e_2(H)$$

with equality if and only if every component of H belongs to $\mathcal{H}$.

Proof. If the desired result holds for each component of H, then the result holds for the hypergraph H itself. Hence we may assume that H is connected. In particular, $b(H) \le 1$. As shown in the proof of Theorem 6(b), we note that $b(H) \le |V(H)| - \tau_s(H)$. By Theorem 6(b), we therefore note that $14\tau_s(H) \le 8|V(H)| + 4e_3(H) + 2e_2(H)$. Further, if $14\tau_s(H) = 8|V(H)| + 4e_3(H) + 2e_2(H)$, then $b(H) = |V(H)| - \tau_s(H)$. Consequently, since $1 \ge b(H)$ and $|V(H)| - \tau_s(H) \ge 1$, we deduce that $b(H) = 1$ and $H \in \mathcal{H}$. Conversely, if $H \in \mathcal{H}$, then by Observation 2, we have that $\tau_s(H) = |V(H)| - 1$ and $8|V(H)| + 4e_3(H) + 2e_2(H) = \mathrm{ve}(H) + 2|V(H)| = 12(|V(H)| - 1) - 2 + 2|V(H)| = 14(|V(H)| - 1)$. Thus, $14\tau_s(H) = 8|V(H)| + 4e_3(H) + 2e_2(H)$. ∎

3.7. Proof of Theorem 7. Recall the statement of Theorem 7.

THEOREM 7. *If H is a hypergraph with all edges of size at least 2, then $13\tau_s(H) \le 7|V(H)| + 4e_{\ge 3}(H) + 2e_2(H) + b_{\le 3}(H)$.*

Proof. Define $\xi(H) = 7|V(H)| + 4e_{\ge 3}(H) + 2e_2(H) + b_{\le 3}(H)$. We show that $13\tau_s(H) \le \xi(H)$. We proceed by induction on $|E(H) \setminus E(H_{\le 3})| + \sum_{x \in V(H)} d_H(x)$. Assume that there exists a k-edge, e, with $k > 4$. Let H' be obtained from H by replacing e by a 4-edge e' where $e' \subset e$. We note that $\xi(H) = \xi(H')$ and that $\tau_s(H) \le \tau_s(H')$. Applying the induction to H', we have that $13\tau_s(H) \le 13\tau_s(H') \le \xi(H') = \xi(H)$, as desired. Therefore we may assume that all edges in H have size 2, 3, or 4.

Assume that $e = \{x_1, x_2, x_3, x_4\}$ is a 4-edge intersecting a 2- or 3-edge e'. If $e' \subseteq e$ and e' is a 3-edge, then we may remove e and use the induction hypothesis. Hence we may assume that $|e' \cap e| \le 2$. Renaming vertices, if necessary, we may assume that $e' \cap e \subseteq \{x_1, x_2\}$. Let H' be the hypergraph obtained from H by replacing e by the 3-edge $e \setminus \{x_4\}$. We note that any strong transversal in H' is also a strong transversal in H, and so $\tau_s(H) \le \tau_s(H')$. If $b(H'_{\le 3}) \le b(H_{\le 3})$, then $\xi(H') \le \xi(H)$, and we are done by induction. Therefore we may assume that R is a bad component in $H'_{\le 3}$, which was not a bad component in $H_{\le 3}$. We note that $\{x_1, x_2, x_3\}$ is an edge of R.

Suppose $x_4 \in V(R)$. Let H'' be obtained from H by removing the edge e and adding the two 2-edges $\{x_1, x_2\}$ and $\{x_3, x_4\}$. We note that any strong transversal in H'' is also a strong transversal in H. Furthermore since $R \in \mathcal{H}$, Observation 2(a) implies that some edge in R contains the vertices $\{x_3, x_4\}$, and so this 2-edge may be removed from H'', and we can use induction on the remaining hypergraph. Hence we may assume that $x_4 \notin V(R)$. We note that since the edge e' intersects $\{x_1, x_2\}$, the bad component $R \neq F_3$.

We now let H^* be obtained from H by removing the edge e and adding the 3-edge $e^* = e \setminus \{x_3\}$. We note that $b(H^*_{\leq 3}) \leq b(H_{\leq 3})$, as the edge e^* separates x_1 from x_4 in $H^*_{\leq 3}$ and therefore does not belong to a bad component. We are therefore done by induction.

Therefore we may assume that no 4-edge intersects any 2- or 3-edge in H. We now delete all 4-edges in H and replace each deleted 4-edge with two vertex disjoint 2-edges whose union is the original 4-edge. This does not create any bad components, as all bad components contain at least one 3-edge. We are therefore done by induction if there exists any 4-edge at all. If no 4-edge exists in H, then we are done by Theorem 6(b). $\square$

3.8. Proof of Theorem 2. Recall the statement of Theorem 2.

THEOREM 2. *If H is a connected 3-uniform hypergraph on $n \geq 6$ vertices and m edges, then $7\tau_s(H) \leq 4n + 2m$ with equality if and only if H is the Fano plane.*

Proof. We note that $e_3(H) = m$ and $e_2(H) = 0$. We note further that if $H \in \mathcal{H}$, then $H = F_7$, as the Fano plane F_7 is the only 3-uniform hypergraph on at least six vertices in $\mathcal{H}$. Thus, by Theorem 1, $7\tau_s(H) \leq 4n + 2m$ with equality if and only if H belongs to $\mathcal{H}$. $\square$

4. Proof of graph theory results.

Next we provide proofs of our three main graph theory results, namely, Theorems 3, 4, and 5. We begin with a proof of Theorem 4.

4.1. Proof of Theorem 4. We shall need the following result in [14].

LEMMA 2 (see [14]). *If G is a connected bipartite graph, then the open neighborhood hypergraph H_G of G contains exactly two components (which are induced by the two partite sets of G). If G is a connected nonbipartite graph, then H_G contains exactly one component.*

Recall the statement of Theorem 4.

THEOREM 4. *If $G \neq G_{14}$ is a connected cubic graph of order n, then $\gamma_{\times 2,t}(G) \leq 5n/6$, and this bound is sharp.*

Proof. Let $G \neq G_{14}$ be a connected cubic graph of order n, and let $H = H_G$. We note that H is a 3-regular, 3-uniform hypergraph of order n and size m. Suppose that $b(H) > 0$. The only 3-regular, 3-uniform hypergraph in $\mathcal{H}$ is the Fano plane. Thus each bad component of H is a copy of the Fano plane. If G is a nonbipartite graph, then, by Lemma 2, H contains exactly one component, implying that $H = F_7$. But then G is a cubic graph of order 7, which is impossible. Hence, G is a bipartite graph. Thus, by Lemma 2, H contains exactly two components which are induced by the two partite sets of G. Since at least one of these components is the Fano plane, we must have that G is the Heawood graph G_{14}, a contradiction. Hence, $b(H) = 0$.

Suppose that $\mathrm{eb}(H) \geq 1$. Recall that H is a 3-regular, 3-uniform hypergraph. Let R be an edge-bad component of H that results from shrinking the edge e of H. Then $|e \cap V(R)| \geq 1$ and every vertex of R, except for possibly one vertex, has degree 3 in R. If $|e \cap V(R)| \geq 2$, then R is 3-regular and contains exactly one 2-edge. But there

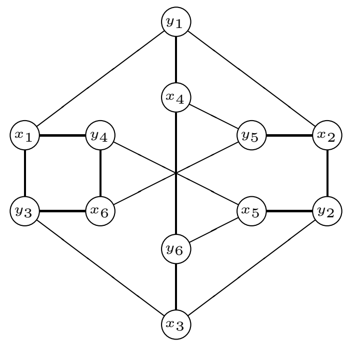

FIG. 3. *The graph G_{12}.*

is no such hypergraph in the family $\mathcal{H}$, a contradiction. Hence, $|e \cap V(R)| = 1$. Let v be the vertex of R that belongs to the edge e. Then R is a 3-uniform hypergraph with one vertex of degree 2 and all other vertices of degree 3. Once again, there is no such hypergraph in the family $\mathcal{H}$, a contradiction. Hence, $\mathrm{eb}(H) = 0$, and so $2b(H)+\mathrm{eb}(H) = 0$. Therefore, by Theorem 6(a), $12\tau_s(H) \le 6|V(H)|+4e_3(H) = 10n$. Hence, by Observation 1, $\gamma_{\times 2,t}(G) = \tau_s(H_G) = 5n/6$. ∎

To see that the bound of Theorem 4 is sharp, consider the cubic graph G_{12} of order $n = 12$ shown in Figure 3 that satisfies $\gamma_{\times 2,t}(G) = 10 = 5n/6$, due to the following. Since every pair of vertices in $X = \{x_1, x_2, \ldots, x_6\}$ have a common neighbor and G is cubic, we note that every DTDS in G must contain at least five vertices from this set. Analogously it must also contain at least five vertices from $Y = \{y_1, y_2, \ldots, y_6\}$, which implies that $\gamma_{\times 2,t}(G) \ge 10 = 5n/6$. It is easy to see that equality holds, as we can take any five vertices from X and any five vertices from Y to obtain a DTDS of size 10.

4.2. Proof of Theorem 5. Recall the statement of Theorem 5.

THEOREM 5. *If $G \ne G_{14}$ is a connected graph of order n with $\delta(G) \ge 3$, then $\gamma_{\times 2,t}(G) \le 11n/13$.*

Proof. Let $G \ne G_{14}$ be any connected graph of order n with $\delta(G) \ge 3$, and let $H = H_G$. Let $B_1, \ldots, B_k$ denote the bad components in $H_{\le 3}$. For each $i = 1, \ldots, k$, let W_i be defined such that $w \in W_i$ if and only if $N_G(w) \in E(B_i)$. We will first prove the following claim that we may assume the graph G satisfies.

CLAIM 1. *We may assume that the graph G has the following properties:*

(a) *If $e = uv$ is an edge in G such that $d(u), d(v) \ge 4$, then $G - e$ contains a component isomorphic to G_{14}.*

(b) *$V(B_i) \cap V(B_j) = \emptyset$ for all $1 \le i < j \le k$.*

(c) *$V(B_i) \cap W_i = \emptyset$ for all $1 \le i \le k$.*

(d) *Removing any bridge of G creates at most one copy of G_{14}.*

(e) *$V(B_i)$ is an independent set in G.*

Proof. (a) If there exists an edge $e = uv$ in G, such that $d(u), d(v) \ge 4$, and $G - e$ contains no component isomorphic to G_{14}, then we may consider the components in $G - e$ instead of G, as $\gamma_{\times 2,t}(G - e) \ge \gamma_{\times 2,t}(G)$. Therefore we may assume that no such edge exists.

(b) This is immediate since no two components can intersect.

(c) To prove part (c), we note that every edge in H has size at least 3. Thus the bad components in $H_{\le 3}$ contain no 2-edges, and therefore each bad component belongs to the family $\{H_3, T_5, F_7\}$. Let $i \in \{1, \ldots, k\}$, let $B = B_i$ and $W = W_i$, and let $X = W \cap V(B)$. Since B is 3-uniform, all vertices in W have degree three in G. For the sake of contradiction, assume that there exists a vertex $u \in X$. Thus, $e_u = N_G(u)$ is a 3-edge in B. We note that $d_B(u)$ of the vertices in e_u belong to W and that they all belong to $V(B)$ (as $e_u \in E(B)$), and so $d_B(u)$ vertices in e_u belong to X.

If $B = F_3$, then as $u \in W$, we have that $u \in V(B) = N_G(u)$, which is impossible since no vertex belongs to its own open neighborhood. Thus, $B \neq F_3$. If $B = F_7$, then $N_G(u) \subseteq X$, and analogously all vertices connected to u by a path also belong to X. This implies that $V(B) = W$ induces a 3-regular graph on seven vertices, which is impossible as no such graph exists. Thus, $B \neq F_7$.

Hence, $B = T_5$. Name the vertices of T_5 so that x_1, x_2, and x_3 are the three vertices of degree 2 and x_4 and x_5 are the two vertices of degree 3. Let $e = \{x_1, x_2, x_3\}$ and for $i = 1, 2, 3$, let $e_i = \{x_i, x_4, x_5\}$. Then, $E(B) = \{e, e_1, e_2, e_3\}$. Suppose $x_1 \in X$. Since $x_1 \notin N_G(x_1)$, we note that $N_G(x_1)$ is either the edge e_2 or the edge e_3. Similarly, if $x_2 \in X$, $N_G(x_2)$ is either e_1 or e_3, while if $x_3 \in X$, $N_G(x_3)$ is either e_1 or e_2. In particular, we note that if $x_j \in X$, then as two vertices in $N_G(x_j)$ belong to X, at least one of x_4 and x_5 belong to X. If $u \in \{x_4, x_5\}$, then, renaming the vertices x_4 and x_5, if necessary, we may assume that $x_4 \in X$. If $u \in \{x_1, x_2, x_3\}$, then at least one of x_4 and x_5 belong to X, and once again we may assume that $x_4 \in X$. This implies that $N_G(x_4) = e$. As $d_B(x_4) = 3$, we note that $\{x_1, x_2, x_3\} \subseteq X$. As observed earlier, $N_G(x_1)$ is either the edge e_2 or the edge e_3. Renaming x_2 and x_3, if necessary, we may assume that $N_G(x_1) = e_2$. Thus, $x_1x_2 \in E(G)$. Since $N_G(x_2)$ is either e_1 or e_3, we therefore have that $N_G(x_2) = e_1$. But then $N_G(x_3)$ is neither e_1 or e_2, a contradiction. This completes the proof of part (c).

(d) Suppose G consists of two vertex disjoint copies of G_{14} connected by an edge uv. Let H_{uv} denote the ONH of $G - uv$, and let N_u and N_v denote the neighborhoods of u and v, respectively, in $G - uv$. We note that H_{uv} consists of four disjoint copies of the Fano plane F_7, where one copy F_u contains the vertex u, one copy $F_{N(u)}$ contains the 3-edge N_u, one copy F_v contains the vertex v, and the last copy $F_{N(v)}$ contains the 3-edge N_v. Let T_u be a strong transversal in F_u, containing u, of size 6, and let T_v be a strong transversal in F_v, containing v, of size 6. Let $T_{N(u)}$ be any five vertices in $F_{N(u)}$ such that $|T_{N(u)} \cap V(N_u)| = 1$, and let $T_{N(v)}$ be any five vertices in $F_{N(v)}$ such that $|T_{N(v)} \cap V(N_v)| = 1$. It is not difficult to see that $T_u \cup T_v \cup T_{N(u)} \cup T_{N(v)}$ is a strong transversal in H, implying that $\tau_s(H) \leq 22$. We note that $b_{\leq 3}(H) = 2$ and $|V(H)| = |E(H)| = |e_{\geq 3}(H)| = n$, and so $11n + 2 = 7|V(H)| + 4e_{\geq 3}(H) + 2e_2(H) + b(H_{\leq 3}) = 7 \times 28 + 4 \times 28 + 2 = 310 \geq 13\tau_s(H) + 11$. Thus, by Observation 1, $\gamma_{\times 2,t}(G) = \tau_s(H) \leq (11n - 9)/13 < 11n/13$, as desired. Therefore we may assume that removing any bridge of G creates at most one copy of G_{14}, which completes the proof of part (d).

(e) Assume that $e = uv$ is an edge in G and $u, v \in V(B_i)$. As $u, v \notin W_i$ by part (c) and B_i is a component in $H_{\leq 3}$, we note that $d_G(u), d_G(v) \geq 4$. By part (a), $G - e$ contains a component isomorphic to G_{14}. However, as u and v are connected by a path in $G - e$ (by just using edges in G between $V(B_i)$ and W_i), we note that G is isomorphic to $G_{14} + e$. However, it is not difficult to see that this implies that $\gamma_{\times 2,t}(G) = 11 \leq 11 \times 14/13$ (by letting T be a DTDS of $G - e$ of size 12 such that $u \in T$ and $|T \cap N_{G-e}(v)| = 2$ and noting that $T \setminus \{w\}$ is a DTDS of G for any $w \in N_{G-e}(v) \cap T$). $\square$

We now return to the proof of Theorem 5. For $i = 1, 2, \ldots, k$, let $G_i = G[V(B_i) \cup W_i]$. We now proceed with a number of definitions. If for some i, $1 \leq i \leq k$, the induced subgraph $G_i = G_{14}$ and there is a bridge separating G_i from the rest of G, then we say that B_i is a *special* F_7. Let ℓ be the number of bad components in H, none of which is a special F_7. Renaming indices, if necessary, we may assume that no B_i is a special F_7 for any $i \le \ell$ and all B_i with $i > \ell$ are special F_7's.

Let $N_i = N_G(V(B_i)) \setminus V(G_i)$. Let $Q = \{q_1, q_2, \dots, q_r\}$ such that $Q = \cup_{i=1}^{\ell} N_i$. We now consider four copies of Q. Let $Q' = \{q'_1, q'_2, \dots, q'_r\}$, $Q'' = \{q''_1, q''_2, \dots, q''_r\}$, $Q''' = \{q'''_1, q'''_2, \dots, q'''_r\}$, and let $Q^* = Q \cup Q' \cup Q'' \cup Q'''$. For $j = 1, \dots, r$, let $Q^*_j = \{q_j, q'_j, q''_j, q'''_j\}$.

We now define a bipartite graph F as follows. Let the partite sets of F be the set Q^* and the set $P = \{p_1, p_2, \dots, p_\ell\}$, where the vertex p_i corresponds to the graph G_i. Let the edge set of F be defined as follows. Add an edge from p_i to each vertex in Q^*_j if and only if $q_j \in N_i$. Let M be a maximum matching in F.

For each $q_i \in Q$, let e_i be the hyperedge in H such that $e_i = N_G(q_i)$. As $q_i \notin W_i$, we note that $e_i \notin E(B_i)$, and therefore e_i is a hyperedge of size at least four. We now construct a sequence of new hypergraphs $H^0, H^1, \dots, H^r$ as follows. Initially, we let $H^0 = H$.

For $i \ge 1$, we define H^i as follows. Let a_i denote the number of vertices in Q^*_i that are M-matched to a vertex p_j in the graph F, where B_j is still a bad component in $H^{i-1}_{\le 3}$. If $a_i = 0$, let $H^i = H^{i-1}$. If $a_i = 4$, let H^i be obtained from H^{i-1} by deleting the edge e_i and adding two vertex disjoint 2-edges e'_i and e''_i such that each of the four bad components in $H^{i-1}_{\le 3}$ corresponding to the vertices of P that are M-matched with a vertex in Q^*_i now intersect either e'_i or e''_i. If $1 \le a_i \le 3$, let H^i be obtained from H^{i-1} by deleting the edge e_i and adding a 3-edges e'_i such that each of the a_i bad components in $H^{i-1}_{\le 3}$ corresponding to the vertices of P that are M-matched with a vertex in Q^*_i intersect e'_i. We note that whenever we change the edge e_i when constructing $H^i_{\le 3}$, the new edge(s) are always cut-edges in $H^i_{\le 3}$ or completely lie within $V(B_j)$ for some j, $1 \le j \le \ell$. Thus, no new bad components are created. That is, $b(H^i_{\le 3}) \le b(H^{i-1}_{\le 3}) - a_i$. Furthermore if p_j is M-matched with a vertex in Q^*_i, then B_j is not a bad component in $H^i_{\le 3}$, as it is incident with a new 2- or 3-edge. Finally we note that a strong transversal in H^i is also a strong transversal in H^{i-1} and $7|V(H^i)| + 4e_{\ge 3}(H^i) + 2e_2(H^i) = 7|V(H^{i-1})| + 4e_{\ge 3}(H^{i-1}) + 2e_2(H^{i-1})$.

Let $H' = H^r$. Then $\tau_s(H) \le \tau_s(H')$ and $7|V(H')| + 4e_{\ge 3}(H') + 2e_2(H') = 7|V(H)| + 4e_{\ge 3}(H) + 2e_2(H) = 11|V(H)|$. Further, $b(H'_{\le 3}) \le b(H_{\le 3}) - |M|$.

Let $U \subseteq P$ denote those vertices which are M-unmatched in the bipartite graph F. Let S be the set of all vertices in P which are reachable by an M-alternating path starting from a vertex in U. Then $U \subseteq S \subseteq P$ and $|S| = |N_F(S)| + |U|$. We note that each vertex in $S \setminus U$ is M-matched in F to a vertex in $N_F(S)$ and that the edges of M incident with vertices in S form a perfect matching in the induced subgraph $F[N_F(S) \cup (S \setminus U)]$. By construction of F, we also note that either $|V(Q^*_j) \cap N_F(S)|$ is equal to four or zero for all $j = 1, \dots, r$. We now define T such that $q_j \in T$ if and only if $|V(Q^*_j) \cap N_F(S)| = 4$, and we note that $4|T| = |N_F(S)|$.

Let $p_i \in S$ be arbitrary, where $1 \le i \le \ell$. As observed earlier, each bad component belongs to the family $\{H_3, T_5, F_7\}$. Assume that $B_i = H_3$ and note that each vertex in B_i has exactly one edge to W_i in G and no edge to $V(B_i)$, due to Claim 1(e). Therefore each of the three vertices has two edges to N_i, which implies that there are at least six edges in G with exactly one end in $V(G_i)$. If $B_i = T_5$, then analogously there are at least three edges in G with exactly one end in $V(G_i)$. If $B_i = F_7$, then since $i \le \ell$ there are at least two edges in G with exactly one end in $V(G_i)$. In all three cases, we let e'_i and e''_i be two distinct edges in G with exactly one end in $V(G_i)$. If e'_i and e''_i are adjacent to a common vertex in B_i, then add this vertex to a set S_1; otherwise, add the two distinct endpoints of e_i' and e_i'' in B_i to a set S_2. Let $E_{ST} = \cup_{p_i \in S}\{e_i', e_i''\}$ and note that $2|S_1| + |S_2| = |E_{ST}| = 2|S| = 2(|N_F(S)| + |U|) = 8|T| + 2|U|$.

Let $H'' = H' - T$ and note that all hyperedges of the form $N_G(u)$ where $u \in S_1$ will be deleted from H' and all hyperedges of the form $N_G(v)$ where $v \in S_2$ will either be deleted from H' or will be shrunk to a 2-edge. Therefore $4e_{\geq 3}(H'') + 2e_2(H'')$ is at least $4|S_1| + 2|S_2| = 2|E_{ST}|$ smaller than $4e_{\geq 3}(H') + 2e_2(H')$. Assume that it is in fact $2|E_{ST}| + 2c$ smaller, where we note that $c \geq 0$ is the number of edge-changes which we have not counted above. In this case, $7|V(H'')| + 4e_{\geq 3}(H'') + 2e_2(H'') = 11|V(G)| - 7|T| - 2|E_{ST}| - 2c$. However, each time we remove or shrink an edge the number of bad components may increase by one. As $b(H'_{\leq 3}) \leq (k - \ell) + |U|$, we note that $b(H''_{\leq 3}) \leq b(H'_{\leq 3}) + |E_{ST}| + c \leq (k - \ell) + |U| + |E_{ST}| + c$.

For all B_i with $\ell < i \leq k$, there exists a bridge y_iu_i in G such that $N_i = \{y_i\}$ and $u_i \in V(B_i)$. Let v_i and w_i be any two vertices in $N_G(u_i) \cap W_i$. Let $T_i = W_i \setminus \{v_i, w_i\}$ and note that $T_i \cup \{y_i\}$ is a double dominating set for all vertices in $V(B_i)$. Let $T' = \cup_{i=\ell+1}^k T_i$ and let $T'' = \cup_{i=\ell+1}^k \{y_i\}$.

Let $H''' = H'' - T' - T''$. We note that all vertices in W_i and all hyperedges touching W_i are removed from H''' when $i > \ell$. Apart from the removal of the hyperedges intersecting the W_is ($i > l$), assume that we have c' additional edge-changes in order to get from H'' to H'''. This implies the following, as $|T'| = 5(k-\ell)$ and $|T'' \setminus T| \leq (k - \ell)$:

$$\begin{aligned}
&7|V(H''')| + 4e_{\geq 3}(H''') + 2e_2(H''') \\
&\quad \leq 11|V(G)| - 7|T| - 2|E_{ST}| - 2c - 7(7(k-\ell)) - 4(7(k-\ell)) - 7|T'' \setminus T| - 2c' \\
&\quad = 11|V(G)| - 7|T| - 2|E_{ST}| - 2c - 13|T'| - 12(k-\ell) - 7|T'' \setminus T| - 2c'.
\end{aligned}$$

Analogously to the above we note that $b(H'''_{\leq 3}) \leq b(H''_{\leq 3}) + c' \leq (k - \ell) + |U| + |E_{ST}| + c + c'$. Combining the two formulas we get the following:

$$\begin{aligned}
&7|V(H''')| + 4e_{\geq 3}(H''') + 2e_2(H''') + b(H'''_{\leq 3}) \\
&\quad \leq 11|V(G)| - 7|T| - |E_{ST}| - c - 13|T'| - 11(k-\ell) - 7|T'' \setminus T| - c' + |U| \\
&\quad \leq 11|V(G)| - 7|T| - (8|T| + 2|U|) - 13|T'| - 11(k-\ell) - 7|T'' \setminus T| + |U| \\
&\quad \leq 11|V(G)| - 15|T| - 13|T'| - 11|T'' \setminus T| - 7|T'' \setminus T| - |U| \\
&\quad \leq 11|V(G)| - 15|T| - 13|T'| - 18|T'' \setminus T| - |U|.
\end{aligned}$$

Let S^* be a $\tau_s(H''')$-set. By Theorem 7 and our upper bound on $7|V(H''')| + 4e_{\geq 3}(H''') + 2e_2(H''') + b(H'''_{\leq 3})$, we note that $|S^*| \leq (11|V(G)| - 15|T| - 13|T'| - 18|T'' \setminus T| - |U|)/13 \leq \frac{11}{13}|V(G)| - |T| - |T'| - |T'' \setminus T|$. However, we note that $S^* \cup T \cup T' \cup (T'' \setminus T)$ is a strong transversal of H. Thus, by Observation 1, $\gamma_{\times 2,t}(G) = \tau_s(H) \leq |S^*| + |T| + |T'| + |T'' \setminus T| \leq 11n/13$, as desired. □

4.3. Proof of Theorem 3. Recall the statement of Theorem 3.

THEOREM 3. *If G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_{\times 2,t}(G) \leq 6n/7$ with equality if and only if G is the Heawood graph G_{14}.*

Proof. Suppose $G = G_{14}$. The ONH of the Heawood graph, $H_{G_{14}}$, consists of two disjoint copies of the Fano plane F_7, which implies by Observation 1 and Theorem 1 that $\gamma_{\times 2,t}(G) = 2\tau_s(F_7) = 2(6) = 6|V(G_{14})|/7$. If $G \neq G_{14}$, then by Theorem 5, $\gamma_{\times 2,t}(G) < 6n/7$. □

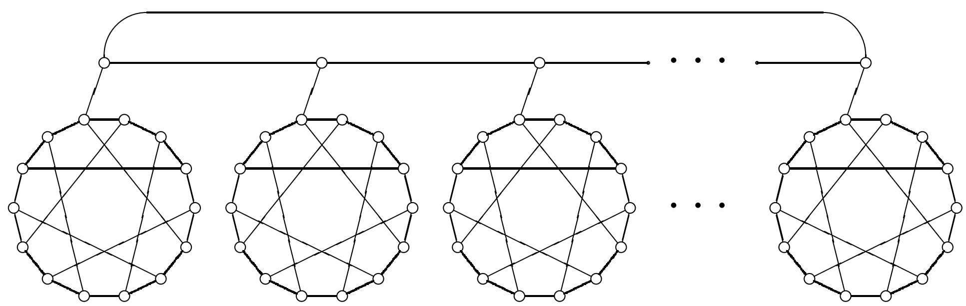

FIG. 4. *A graph $G \in \mathcal{F}$ of order n with $\delta(G) \geq 3$ and $\gamma_{\times 2,t}(G) = 4n/5$.*

5. Closing conjectures.

We close with the following two conjectures.

CONJECTURE 1. *If $G \neq G_{14}$ is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_{\times 2,t}(G) \leq 5n/6$.*

CONJECTURE 2. *If $G \neq G_{14}$ is a connected graph of sufficiently large order n with $\delta(G) \geq 3$, then $\gamma_{\times 2,t}(G) \leq 4n/5$.*

We remark that Conjecture 1 is true for cubic graphs (see Theorem 4). We also remark that if Conjecture 2 is true, then the bound is sharp, as may be seen by considering the following family $\mathcal{F}$ of all graphs that can be obtained as follows: Take a connected graph F with $\delta(F) \geq 2$ and for each vertex v of F, add a copy of the Heawood graph G_{14} and join v to one vertex in that copy of G_{14}. A graph G in the family $\mathcal{F}$ is illustrated in Figure 4 (here, F is a cycle). Each graph G of order n in the family $\mathcal{F}$ is a connected graph with $\delta(G) \geq 3$ satisfying $\gamma_{\times 2,t}(G) = 4n/5$.

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